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BANACH SPACES

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Introduction.

Although this paper is meant as a survey on Banach spaces it contains some 'new' results and many new proofs of old results. An example of the latter is (3.6) and (3.10) where one proves that every closed subspace of a free Banach space is itself free.

Most of section 7, Differential equations, is new. In this section one constructs primitive functions for continuous functions and rediscovers a formula of D. Treiber. Subsequently differential equations are solved. A more detailed study of primitive functions shows that any function which is the pointwise limit of a sequence of continuous functions and whose image is relatively compact has a primitive function.

Section 5 makes the well known connection betwenn Banach space and modules over a valuation ring explicit. Some problems and results of earlier sections are phrased in terms of modules. The first five sections contain standard material enriched with a set of open problems.

This survey together with A.F. Monna's contribution to the proceedings of this conference gives a fairly complete summary of the theory of ultrametric Banach spaces.

§.1. Examples of Banach spaces and notations.

The field K we are working with is supposed to be complete with respect to a non-trivial, non-archimedian valuation. Its valuation ring $\{\lambda \in K \,\|\, |\lambda| \leqslant 1\}$ is denoted by V, the maximal ideal of V by $m = \{\lambda \in K \mid |\lambda| < 1\}$ its residue field $V/_{\underline{m}}$ by k. The value group of K will be denoted by $|K^*|$. For constructions etc. we often choose $\pi \in K$ with $0 < |\pi| < 1$. If the valuation of K is discrete we suppose that $|\pi|$ generates $|K^*|$ i.e. $|K^*| = \{|\pi|^n \mid n \in \mathbb{Z}\}$.

(1.1) Let I be a set and $\mu : I \rightarrow \{r \in \mathbb{R} \mid r > 0\}$. Then $l^\infty(I,\mu,K) = l^\infty(I,\mu)$ will denote the Banach space of all functions $f : I \rightarrow K$ satisfying $\sup|f(i)|\mu(i) < \infty$. The norm is given by $\|f\| = \sup|f(i)|\,\mu(i)$. For any $i \in I$, e_i stands for the element of $l^\infty(I,\mu)$ given by $e_i(j) = 0$ if $j \neq i$, $e_i(i) = 1$.

The closed subspace $c_o(I,\mu,K) = c_o(I,\mu)$ of $l^\infty(I,\mu)$ is defined by : $f : I \rightarrow K$ belongs to $c_o(I,\mu)$ if $\lim |f(i)|\,\mu(i) = 0$. It is clear that $l^\infty(I,\mu)$ is isomorphic to $l^\infty(I,\mu')$ if $\mu(i)\,\mu'(i)^{-1} \in |K^*|$ for all i. The same holds for $c_o(I,\mu)$. So we can normalize μ such that $0 < \inf \mu(i) \leqslant \sup \mu(i) < \infty$. For normalized μ one defines the subspace $c(I,\mu,K) = c(I,\mu)$ by : $f : I \rightarrow K$ belongs to $c(I,\mu)$ if $\lim f(i)$ exists. So $c_o(I,\mu) \subset c(I,\mu) \subset l^\infty(I,\mu)$. If μ has the property $\mu(I) = \{1\}$ then we abbreviate $l^\infty(I,\mu)$ (resp. $c(I,\mu)$ and $c_o(I,\mu)$) by $l^\infty(I)$ (resp. $c(I)$ and $c_o(I)$).

(1.2) Let E be a Banach space (or just a topological space) and X a topological space then $C(X \rightarrow E)$ denotes the set of all continuous functions of $X \rightarrow E$. If E is a Banach space and X is compact then $C(X \rightarrow E)$ is a Banach space under the norm $\|f\| = \sup \{\|f(x)\| \mid x \in X\}$.

For the space $C(X \rightarrow K)$ we sometines use the abbrevation $C(X)$.

(1.3) Let E and F be Banach spaces then $\mathcal{L}(E,F) = \{l : E \rightarrow F \mid l$ is K-linear and continuous$\}$ is a Banach space under the norm $\|l\| = \sup\{\|l(x)\|\,\|x\|^{-1} \mid x \in E, x \neq 0\}$. The dual $\mathcal{L}(E,K)$ of E is denoted by E'.

(1.4) Let $\{E_i\}_{i \in I}$ be a family of Banach spaces. The Banach spaces ΠE_i and ΣE_i are defined as follows :

$$\Pi E_i = \{(e_i)_{i \in I} \in \underset{i \in I}{\mathrm{X}} E_i \mid \sup \|e_i\| < \infty\}$$

$$\Sigma E_i = \{(e_i)_{i \in I} \in \underset{i \in I}{\mathrm{X}} E_i \mid \lim \|e_i\| = 0\}$$

Both vector spaces are normed by $\|(e_i)_{i\in I}\| = \sup \|e_i\|$.

(1.5) Let E be a Banach space and F a closed subspace of E. Then the vector space $E/_F$ is again a Banach space under the quotient-norm given by $\|t\| = \inf \{\|e\| \mid e\in E,\ \rho(e) = t\}$, where ρ denotes the canonical map $\rho\colon E \to E/_F$. Let $E \xrightarrow{\alpha} G$ be a continuous map between Banach spaces. We will say that α induces the norm on G if the induced map $E/_{\ker(\alpha)} \to G$ is bijective and isometric.

(1.6) For a Banach space E we denote the sphere $\{x\in E \mid \|x-a\|\leqslant\rho\}$ by $B(a,\rho)$.

§.2. Injective Banach spaces.

(2.1) Définition. A Banach space E (over K) is called injective if for every diagram

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r, "\alpha"] \arrow[d, "\phi_0"'] & B \\ & E & \end{tikzcd}$$

with α isometric and ϕ_0 bounded, there exists $\phi : B \to E$ such that $\|\phi\| = \|\phi_0\|$ and $\phi\alpha = \phi_0$.

(2.2) Theorem. The following conditions are equivalent :

(1) E is injective,

(2) Every $\phi_0 : c_0(\mathbb{N},\mu) \to E$ has an extension $\phi : c(\mathbb{N},\mu) \to E$ with $\|\phi\| = \|\phi_0\|$.

(3) E is maximally complete (i.e. every set $\{B_i\}$ of spheres in E, with the property $B_i \cap B_j \neq \emptyset$ for all i and j, has a non-empty intersection).

Proof. (1) $\Rightarrow$ (2) is clear ; (2) $\Rightarrow$ (3). Let $\{B_i\}$ be a set of spheres such that $B_i \cap B_j \neq \emptyset$ for every $i \neq j$. The strong triangle inequality yields that $B_i \subseteq B_j$ or $B_j \subseteq B_i$. Hence we can find a countable subset of spheres $B(a_n,\rho_n)$ with : $a_o = 0$, $B(a_n,\rho_n) \supset B(a_{n+1},\rho_{n+1})$ for all n, $\rho_0 > \rho_1 > \rho_2 > \ldots$ such that

$$\cap B_i = \cap B(a_n,\rho_n).$$

Define $\mu : \mathbb{N} \to \mathbb{R}_{>0}$ by $\mu(i) = \|a_i - a_{i-1}\|$ $(i \geqslant 1)$ and define $\phi_0 : c_0(\mathbb{N},\mu) \to E$ by $\phi_0(e_i) = a_i - a_{i-1}$ $(i \geqslant 1)$. There is a map $\phi_0 : c(\mathbb{N},\mu) \to E$ extending ϕ_0 such that $\|\phi\| = \|\phi_0\| = 1$. The element

$a = \phi(1,1,1,\dots)$ belongs to every $B(a_n, \rho_n)$ since

$$\|a-a_n\| = \|\phi(0,\dots0,1,1,1\dots)\| \leqslant \|(0,\dots0,1,1,\dots)\| =$$

$$= \sup_{i > n} \mu(i) \leqslant \rho_n .$$

(3) ⇒ (1). Using Zorn's lemma one sees that it suffices to consider the situation

$$\begin{tikzcd} A \arrow[r, hook] \arrow[d, "\phi_0"'] & B, \\ E & \end{tikzcd}$$

where $B = A+Kx$ for some $x \in B$.

Every extension ϕ of ϕ_0 is determined by $e = \phi(x)$. The condition $\|\phi\| = \|\phi_0\|$ is equivalent to : for all $a \in A$, $|\phi(x-a)| = |e-\phi_0(a)| \leqslant \|\phi_0\| \, \|x-a\|$, and also to $e \in \bigcap_{a \in A} B(\phi_0(a), \|\phi_0\| \, \|x-a\|) = Y$.

For any $a,a' \in A$ we have $B(\phi_0(a), \|\phi_0\| \|x-a\|) \cap B(\phi_0(a'), \|\phi_0\| \, \|x-a'\|) \neq \emptyset$ since $\|\phi_0(a) - \phi_0(a')\| \leqslant \|\phi_0\| \, \|a-a'\| \leqslant \max(\|\phi_0\| \, \|x-a\| , \|\phi_0\| \, \|x-a'\|)$. Since E is supposed to be maximally complete it follows that $Y \neq \emptyset$ and e and ϕ can be chosen such that $\|\phi\| = \|\phi_0\|$.

(2.3) Corollary. The field K is an injective Banach space if and only if K is maximally complete in the sense of Krull ([3]).

(2.4) Proposition. Every quotient of an injective Banach space is injective. Every product of injective Banach spaces is injective.

Proof. Let E be injective and F a quotient of E, $\pi : E \rightarrow F$ the canonical map. Consider a sequence of spheres $B(a_n, \rho_n) = B_n$ in F with the property $B_n \supsetneq B_{n+1}$ for all n. By induction one constructs a sequence $\{b_n\}$ in F such that $B(b_n, \rho_{n-1}) \supset B(b_{n+1}, \rho_n)$ and $\pi(b_n) = a_n$ for all n. (Induction step: $a_{n+1}-a_n = \pi(c)$ for some $c \in E$, since $|a_{n+1}-a_n| \leqslant \rho_n$ one can suppose $|c| < \rho_{n-1}$. Put $b_{n+1} = b_n+c$) Any $e \in \bigcap B(b_n, \rho_{n-1})$ has the property $\pi(e) \in \bigcap B_n$. The second statement of (2.4) has analogous proof.

(2.5) Proposition. Let $\{E_n\}$ be a sequence of Banach spaces. The Banach space $\prod E_n / \sum E_n$ is injective.

Proof. Analogous to (2.4). See [5] .

Notation. If $E_n = E$ for all n, we write $l^\infty(E)$ for ΠR_n , $c_o(E)$ for ΣE_n and $c(E)$ for the subspace of $l^\infty(E)$ of all sequences having a limit in E. The map $E \to l^\infty(E)$ given by $e \mapsto (e,e,\dots)$ induces an isometry $\Delta_E : E \mapsto l^\infty(E)/_{c_0(E)}$. And we find for every E a *canonical injective resolution*

$$0 \to E \xrightarrow{\Delta_E} l^\infty(E)/_{c_0(E)} \longrightarrow l^\infty(E)/_{c(E)} \to 0 .$$

(2.6) *Theorem*. E *is injective if and only if the map* 'lim' : $c(E) \to E$ *has an extension with norm* 1 *to* $l^\infty(E) \to E$.

Proof. E is injective if and only if Δ_E has a left-inverse $P : l^\infty(E)/_{c_0(E)} \to E$ of norm 1 ; this follows from (2.2), (2.4) and (2.5). The existence of P means the existence of a map $\phi : l^\infty(E) \to E$ with $\phi = 1$, $\phi \mid c(E) = \underset{n\to\infty}{\text{"lim"}}$.

(2.7) *Definition*. E is called *weakly injective* if for every diagram $\begin{tikzcd} 0 \arrow[r] & A \arrow[r, "\alpha"] \arrow[d, "\phi_0"'] & B \\ & E & \end{tikzcd}$ with α isometry, $\|\phi_0\| < \infty$, there exists a $\phi : B \to E$ such that $\phi\alpha = \phi_0$ and $\|\phi\| < \infty$

(2.8) *Corollary*. *If* E *is weakly injective there exists a constant* $C \geqslant 1$ *and for every diagram* $\begin{tikzcd} 0 \arrow[r] & A \arrow[r, "\alpha"] \arrow[d, "\phi_0"'] & B \\ & E & \end{tikzcd}$ *with* α *isometry and* $\|\phi_0\| < \infty$ *a map* $\phi : B \to E$ *satisfying* $\phi\alpha = \phi_0$ *and* $\|\phi\| \leqslant C\|\phi_0\|$.

Proof. Δ_E has a left inverse P with $\|P\| = C < \infty$. The map P induces a norm on E which makes E injective and has the property $\| \quad \|^\infty \leqslant \| \quad \| \leqslant C\| \quad \|^*$.

(2.9) *Definitions*. A K-linear isometry $E \hookrightarrow F$ is called *essential* (or F an *essential extension* of E) if for all $f \in F$ there exists $e \in E$ with $\|f-e\| < \|f\|$. A K-linear isometry $E \subset F$ is a *maximal completion* if F is injective and $E \hookrightarrow F$ is essential.

(2.10) *Proposition*. *Every Banach space* E *has a maximal completion* (*denoted by* $\check{E}$) *which is unique up to* (*non-canonical*) *isomorphism.*

Proof. Take for $\check{E}$ a maximal essential extension of $\Delta_E(E)$ in the Banach space $l^\infty(E)/_{c_0(E)}$. By definition $\Delta_E : E \subset \check{E}$ is essential and since $l^\infty(E)/_{c_0(E)}$ is maximally complete also E is maximally complete. The unicity follows easily from (2.2).

(2.11) In the last proof there was a choice of a maximal essential extension of a subspace F inside an injective space G. The next lemma clarifies this situation.

Lemma. *Let* F *be a closed subspace of an injective space* G *and let* F_i = $(i=1,2)$ *denote maximal essential extensions of* F *inside* G. *Then*

(i) F_1 *and* F_2 *are injective and there exists a* K*-linear bijective isometric* $\sigma : G \to G$ *such* $\sigma|F = \mathrm{id}$ *and* $\sigma(F_1) = F_2$.

(ii) *If* $F \hookrightarrow G$ *is not essential and* F *is not injective then* F *has many different maximal extensions in* G.

Proof.(i) If F_i is not injective then there exists a set of spheres $\{B(a_n,\rho_n)\}$ in G with $a_n \in F_i$ for all n and such that $\cap B(a_n,\rho_n) \neq \emptyset$ and $\cap B(a_n,\rho_n) \cap F_i = \emptyset$. Choose $e \in \cap B(a_n,\rho_n)$. Then, as one easily sees, $F_i + Ke$ is an essential extension, contrary to the assumption that F_i is maximal. Hence F_i is injective. Let H be a subspace of G which is maximal with respect to the property $\|f+h\| = \max(\|f\|,\|h\|)$ for all $f \in F$, $h \in H$. (We express this sometimes by $H \perp F$). Then it is easily seen that H is injective, $H \oplus F_1 = H \oplus F_2 = E$. By (2.10) there is a bijective isometric map $\tau : F_1 \to F_2$ with $\tau|E = \mathrm{id}$. Then $\sigma = \mathrm{id}_H \oplus \tau$ has the required properties.

(ii) Let a maximal extension F_1 of F inside G be given. Choose $x \in F_1/F$ and an element $y \in G$ with $Ky \perp F_1$, $y \neq 0$, $\|y\| < \inf\{\|x-f\| \mid f \in F\}$. Then $F \subset F + Kz$, where $z = x+y$, is an essential extension contained in a maximal extension F_2 . Clearly $F_1 \neq F_2$ since $y \notin F_1$.

(2.12) *Remark*. Let the complete field $L \supset K$ be an essential field extension in the sense of Kaplansky ([3]).

Then L as K-Banach space is an essential extension of K and by (2.10) isomorphic to a subspace of $\check{K}$. Hence $\mathrm{card}(L) \leq \mathrm{card}(\check{K})$ and the class of all essential field extensions of K is in fact a set. The lemma of Zorn applied to this set yields the existence of a maximal complete field $L \supset K$ which is an essential extension of K. Again (2.10) yields L is isomorphic to $\check{K}$ as a Banach space. Kaplansky has shown that K might have non isomorphic maximal complete field extensions L_1 , L_2 . As Banach spaces L_1 and L_2 are isomorphic.

Examples.

(2.13) $c_0(I,\mu)$ *is not injective if* $\mu(I)$ *contains a sequence* $a_1 > a_2 > a_3 > \dots$ *with* $a_i > 0$.

Proof. Let $\mathbb{N} \cong J \subset I$ be the subset corresponding to the given sequence. Since $c_0(J,\mu|J)$ is a direct summand of $c_0(I,\mu)$ an application of (2.4) shows that it is enough to consider the case $c_0(\mathbb{N},\mu)$ and $\mu(1) > \mu(2) > \dots$, $\lim \mu(n) > 0$. If $c_0(\mathbb{N},\mu)$ were injective then there exists a map $\phi : c(\mathbb{N},\mu) \to c_0(\mathbb{N},\mu)$ with $\|\phi\| = 1$ and $\phi | c_0(\mathbb{N},\mu) = \mathrm{id}$.

Then $x = (\lambda_1, \lambda_2, \dots) = \phi(1,1,1,\dots) \in c_0(N,\mu)$ has the property

$$\|x - (1,\dots,1,0,0,0,\dots)\| = \|\phi(0,\dots,0,1,1,1,\dots)\| \leq$$

$$\leq \|(0,\dots,0,1,1,1,\dots)\| = \mu(n+1).$$ Hence $|\lambda_n - 1| < 1$ for all n; this contradicts $\lim \lambda_n = 0$.

(2.14) *Let* E *be a Banach space such that every strictly decreasing sequence in* $|E|$ *has limit zero. Then* E *is injective.*
(Note that the existence of such $E \neq 0$ implies that the valuation of K is discrete).

Proof. Let $\{B_n\}$ be a sequence of spheres in E such that $B_n \supset B_{n+1}$ for all n. We may suppose that all radii ρ_n lie in $|E|$ and that $\rho_n > \rho_{n+1}$ for all n. Then $\lim \rho_n = 0$ and the completeness of E implies $\cap B_n \neq \emptyset$.

(2.16) *Let* I *be an infinite set and* μ *a map* : $I \to \mathbb{R}_{>0}$. *The Banach space* $c_0(\mathbb{N},\mu)$ *is injective if and only if the valuation of* K *is discrete and every strictly decreasing sequence in* $\mu(I)$ *has limit zero.*

Proof. If the valuation of K is dense then $c_0(\mathbb{N},\mu) \cong c_0(\mathbb{N},\mu')$, where μ' can be chosen such that $\mu'(I)$ contains a strictly decreasing sequence with positive limit. Hence the condition is necessary. Also sufficient because $|K^*|$ discrete and every strictly decreasing sequence in $\mu(I)$ has limit 0 implies that every strictly decreasing sequence in $\|c_0(I,\mu)\|$ has limit zero. Apply now (2.14).

(2.16) *If* K *is maximally complete then* $l^\infty(I,\mu)$ *is injective for every* I *and* μ

Proof. (2.4)

(2.17) *Suppose that the valuation of* K *is discrete and* $\mu : \mathbb{N} \to \mathbb{R}_{>0}$ *satisfies* $\mu(1) > \mu(2) > \dots$ $\lim \mu(i) > 0$. *Then* $l^{\infty}(\mathbb{N}, \mu)$ *is the maximal completion of* $c_0(\mathbb{N}, \mu)$.

Proof. By (2.15) all we have to show is that for any $f = (f_1, f_2, \dots) \in l^{\infty}(\mathbb{N}, \mu)$ there exists $e \in c_0(\mathbb{N}, \mu)$ with $\|f-e\| < \|f\|$. The discreteness of $|K^{*}|$ and the properties of μ imply that the set $\{n \in \mathbb{N} \mid \|f\| = |f_n| \mu(n)\}$ is non-empty and finite. Let n_0 be the last integer with $\|f\| = |f_{n_0}| \mu(n_0)$. Then $e = (f_1, \dots, f_{n_0}, 0, 0, \dots)$ has the required property.

(2.18) An extension of (2.17) is the following :

Suppose that the valuation of K *is discrete and consider* $E = c_0(I, \mu)$, *where* μ *is normalized by* $|\pi| < \mu(i) \leq 1$ *for all* i. *A subset* J *of* I *will be called decreasing if every sequence* $j_1, j_2, \dots$ *in* J *such that* $\mu(j_1) \leq \mu(j_2) \leq \mu(j_3) \leq \dots$ *is finite.*
Then $\check{E}$ *is the subspace of* $l^{\infty}(I, \mu)$ *given by*
$\check{E} = \{f \in l^{\infty}(I, \mu)\}$ *for every* $\varepsilon > 0$ *the set* $\{j \in I \mid |f(j)| \geq \varepsilon\}$ *is decreasing*.

Proof. We note that a finite union of decreasing sets is again decreasing. It follows that the subspace $l^{\infty}(I, \mu)$ given in the statement is equal to

$F = \bigcup \{ l^{\infty}(J, \mu \mid J) \mid J \subset I \text{ decreasing} \}$. As in (2.17), for any decreasing set J the inclusion $c_0(J, \mu/J) \subset l^{\infty}(J, \mu \mid J)$ is essential. Hence F is an essential extension of $c_0(I, \mu)$. Consider an extension $F \subset F + Ke$ with $e \notin F$. In order to show that F is injective, we have to show that this extension is not essential. Put $d(e,F) = \inf \{\|e-f\| \mid f \in F\} > 0$. Choose a sequence $\alpha_1 > \alpha_2 \dots$ in $\mathbb{R}$ with $\lim \alpha_n = d(e,F)$. For any $n \geq 1$ the set $J_n = \{i \subset I \mid |e(i)| \mu(i) \geq \alpha_n\}$ is decreasing and one easily sees that also $J = \bigcup J_n$ is decreasing. Let $f \in F$ be the element given by $f(i) = 0$ if $i \notin J$ and $f(i)$ if $i \in J$. Then $d(e,F) = \|e-f\|$ and for any $f' \in F$ we have $\|(e-f)-f'\| \geq \|e-f\|$. Hence $F \subset F + Ke$ is not essential.

(2.19) *Suppose that the valuation of* K *is discrete. Let* n *be a positive integer. For any Banach space* E *over* K *there exists a norm* $\| \ \|^{*}$ *on* E *such that* $|\pi|^{1/n} \| \ \| \leq \| \ \|^{*} \leq \| \ \|$ *and* $\|E\|^{*} \subseteq \Gamma = \{|\pi|^{m/n} \mid m \in \mathbb{Z}\} \cup \{0\}$.

Proof. Take $\|x\|^{*} = \sup \{t \in \Gamma \mid t \leq \|x\|\}$.

(2.20) *Suppose that the valuation of* K *is discrete. Then any Banach space* E *over* K *is weakly injective and moreover* $\inf\{C \in \mathbb{R} \mid \Delta_E$ *has a left-inverse of norm* $\leq C\} = 1$.

Proof. (2.19) and (2.14).

(2.21) *Problems.*

(i) Do there exist weakly injective Banach spaces E such that $\inf\{C \in \mathbb{R} \mid \Delta_E$ has a left-inverse of norm $\geq C\} > 1$?

(ii) Let K be a maximally complete field, with dense valuation. Can one give an explicit description of a maximal completion of $c_o(\mathbb{N},K)$ inside $l^\infty(\mathbb{N},K)$?

(iii) Suppose that K is not maximally complete ; can one describe K explicitly as a subspace of $l^\infty(\mathbb{N},K)/c_o(\mathbb{N},K)$?

§3. *Projective Banach spaces.*

(3.1) *Définitions*. A (bounded linear) map $\phi : E \to F$ is called a *strict surjection* if for any $f \in F$ we have $\|f\| = \min\{\|e\| \mid e \in E,\ \phi(e) = f\}$. (i.e. the surjective map ϕ induces the norm on F and for every $f \in F$ there exists $e \in \phi^{-1}(f)$ with $\|e\| = \|f\|$).

A Banach space E is called *projective* (resp. *weakly-projective*) if for every diagram

$$\begin{tikzcd} B \arrow[r, "\alpha"] & C \arrow[r] & 0 \\ & E \arrow[u, "\phi_o"'] & \end{tikzcd}$$

with α a strict surjection and $\|\phi_o\| < \infty$, there exists a $\phi : E \to B$ such taht $\|\phi\| = \|\phi_o\|$ (resp. $\|\phi\| < \infty$).

A Banach space E is called *free* (or is said to have an *orthogonal base*) if $E \cong c_o(I,\mu)$ (isometric) for some I and $\mu : I \to \mathbb{R}_{>o}$.

Remarks.

(3.2) If the condition " α is a strict surjection" in the definition of projective is replaced by " α is surjective and induces the norm on F" then the field K is not projective.

(3.3) Every free Banach space is projective.

(3.4) Let E be a Banach space. Put $I = E/\{0\}$ and define $\mu : I \to \mathbb{R}_{>0}$ by $\mu(x) = \|x\|$. Then the map $\pi_E : c_0(I,\mu) \to E$, given by $\pi_E(f) = \sum_{x \in I} f(x)x$, is a strict surjection.

(3.5) <u>Proposition</u>. <u>A Banach space</u> E <u>is projective if and only if</u> E <u>is a direct summand of a free Banach space</u>.

<u>Proof</u>. $E \subset F$ is called a direct summand if there exists a projection $P : F \to E$ with $\|P\| = 1$.

" $\Rightarrow$ " Since E is projective $\pi_E : c_0(I,\mu) \to E$ has a right-inverse ρ of norm 1. Hence E is isomorphic to the direct summand $\rho(E)$ of $c_0(I,\mu)$.

" $\Leftarrow$ " Let E be a direct summand of the free space F ; $P : F \to E$ a projection of norm 1 ; $B \overset{\alpha}{\to} C$ a strict surjection ; $\phi_0 : E \to C$ a bounded map. Then $\phi_0 P : F \to C$ can be lifted $\psi : F \to B$ with $\|\psi\| = \|\phi_0 P\| = \|\phi\|$ and $\phi = \psi/E : E \to B$ has the required property.

(3.6) <u>Proposition</u>. <u>Every closed subspace of a projective space is projective</u>.

<u>Proof</u>. Let a diagram

$$\begin{tikzcd} B \arrow[r, "\alpha"] & C \arrow[r] & 0 \\ & E \arrow[u, "\phi_0"'] & \end{tikzcd}$$

α strict, $\|\phi_0\| < \infty$ be given.

We complete this diagram to a commutative one in the following way :

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r, "\beta"] \arrow[d, "1"] & B \arrow[r, "\alpha"] \arrow[d, "\gamma"] & C \arrow[r] \arrow[d, "\delta"] & 0 & \\ 0 \arrow[r] & A \arrow[r, "\varepsilon"] & \check{B} \arrow[r, "\chi"] & D \arrow[r] & 0 & E \arrow[ul, "\phi_0"'] \arrow[d, hook, "i"'] \\ & & & & & F \arrow[ulll, dashed, "\psi"] \arrow[ull, dashed, "\psi_0"'] \end{tikzcd}$$

$A = \ker \alpha$; $\check{B}$ is a maximal completion of B with canonical map $\gamma : B \to \check{B}$; $\varepsilon = \gamma \circ \beta$; $D = \check{B}/A$ with canonical projection $\chi : \check{B} \to D$;

the map $\chi \circ \gamma : B \to D$ has kernel A and induces an isometry $\delta : C \to D$.
By (2.4), D is injective. Let $D_0 \subset D$ be a maximal essential extension of $\delta\phi_0(E)$.
Then D_0 is maximally complete, (since D is maximally complete) and there exists a map $\psi_0 : F \to D_0$ with $\|\psi_0\| = \|\phi_0\|$ and $\delta\phi_0 = \psi_0 i$.
We claim that for any $d_0 \in D_0$ there exists $\check{b} \in \check{B}$ with $\chi(\check{b}) = d_0$ and $\|\check{b}\| = \|d_0\|$.
Indeed, there exists $c \in C$ with $\|\delta(c) - d_0\| < \|d_0\|$ and $b \in B$, with $\alpha(b) = c$, $\|b\| = \|c\| = \|d_0\|$. Hence $\|\chi\gamma(b) - d_0\| < \|d_0\|$ and there exists $b' \in \check{B}$ with $\|b'\| < \|d_0\|$ and $\chi(b') = d_0 - \chi\gamma(b)$.
Now $\check{b} = b+b'$ has the required properties.
By (3.5) F may be supposed to be free, and the existence of a map $\psi : F \to \check{B}$ with $\|\psi\| = \|\psi_0\|$, $\chi\psi = \psi_0$ now follows.
The map ψi maps E in fact into $\gamma(B)$. Indeed, for any $e \in E$ and $b \in B$ with $\alpha(b) = \phi_0(e)$ we have $\chi\psi i(e) = \psi_0 i(e) = \delta\phi_0(e) = \delta\alpha(b) = \chi\gamma(b)$.
So $\chi(\psi i(e) - \gamma(b)) = 0$ and $\psi i(e) - \gamma(b) \in \ker \chi = A \subset (B)$.
So there exists a map $\phi : E \to B$ with $\|\phi\| = \|\psi i\| = \|\phi_0\|$ and $\gamma\phi = \psi i$. Also $\alpha\phi = \phi_0$ and the proof is finished.

(3.7) Before giving the proof that every projective Banach space is in fact free, we turn to Banach spaces of countable type.

Definition. A Banachspace E is of countable type if it has a countable subset which generates a dense linear subspace of E.

Remarks.

The definition above is the analogous of "separable Banach space over ℝ or ℂ". The condition E is separable would be too restrictive since the base field K need not be separable. Further we note that subspaces and quotient spaces of an E of countable type are also of countable type.

Definition. Let E be a Banach space over K, A a subset of E and $\alpha \in \mathbb{R}$, $0 < \alpha \leq 1$. The set A is called α *-orthogonal* if for every finite (or convergent) linear combination $\sum_{a \in A} \lambda_a a$ the inequality $\|\sum \lambda_a a\| \geq \alpha \max |\lambda_a| \, \|a\|$ holds.

A is said to be an α *-orthogonal base* of E if moreover every $x \in E$ can be written as a convergent sum $x = \sum \lambda_a a$.

Remark.

E has an α-orthogonal base if and only if there exists a bijective linear map $\phi : E \to c_0(I, \mu)$ (for some I and μ) with $\|\phi\| < 1$, $\|\phi^{-1}\| \leq \alpha^{-1}$. In particular, E has an *orthogonal base* (i.e. an 1-orthogonal base) if and only if E is free.

(3.8) *Theorem*. (*Existence of bases*)

1) *If* E *is a Banach space of countable type then* E *has for every* α, $0 < \alpha < 1$, an α *-orthogonal base*.

2) *If* E *is a Banach space of countable type and* K *is maximally complete then* E *has an orthogonal base*.

3) *I* E *is a subspace of* $c_0(\mathbb{N}, \mu)$ *then* E *has an orthogonal base*.

4) *If every strictly decreasing sequence in* $\|E\|$ *has limit zero then* E *has an orthogonal base*.

4) *If the valuation of* K *is discrete and* E *is a Banachspace over* K *then for every* α, $0 < \alpha < 1$, E *has an* α *-orthogonal base.*

Proof. 1) Assume for notational convenience that dim E = ∞. Choose a sequence $\{E_n\}$ of subspace of E such that $E_n \subset E_{n+1}$, $\overline{\bigcup E_n} = E$, dim $E_n = n$. Choose further a sequence $\{\alpha_n\} \subset \mathbb{R}$, $0 < \alpha_n < 1$, with $\prod_{n=1}^{\infty} \alpha_n \geq \alpha$.

Take an element $y_n \in E_n \setminus E_{n-1}$ and $z_n \in E_{n-1}$ with

$\|y_n - z_n\| \leq \alpha_n^{-1} \inf \{ \|y_n - z\| \mid z \in E_{n-1} \}$. Put $x_n = y_n - z_n$. We claim that $\{x_n\}$ is an α-orthogonal base of E.

(a) x_n has the property $\|\lambda x_n + y\| \geq \alpha_n \max(\|\lambda x_n\|, \|y\|)$ for $y \in E_{n-1}$.

Proof of (a). We may suppose $\lambda = -1$. If $\|x_n - y\| \leq \alpha_n \max(\|x_n\|, \|y\|)$ then $\|y_n - z_n - y\| \leq \alpha_n \|y_n - z_n\| \leq \inf\{\|y_n - z\| \mid z \in E_{n-1}\}$. This is a contradiction.

(b) For every $n \geq 1$, $\| \sum_{i=1}^{n} \lambda_i x_i \| \geq \prod_{i=1}^{n} \alpha_i \max(\| \lambda_i x_i \|)$.

Proof of (b). The formula is correct for $n = 1$. If $n > 1$ then by (a) we have $\| \sum_{i=1}^{n} \lambda_i x_i \| \geq \alpha_n \max(\| \lambda_n x_n \| , \| \sum_{i=1}^{n-1} \lambda_i x_i \|)$ and, by induction hypothesis again, $\geq \prod_{i=1}^{n} \alpha_i \max(\| \lambda_i x_i \|)$.

Hence we did prove that $\{x_n\}$ is α-orthogonal. It is an α-orthogonal base of the closed subspace F generated by the set $\{x_n\}$. But F contains every E_n and must be equal to E.

2) and 3). One has to show that the construction in part 1) can be carried out with $\alpha_n = 1$ for all n. For this it suffices to show that for subspaces $F_1 \subset F_2 \subset E$ with $\dim F_2 = \dim F_1 + 1 < \infty$ there exists a projection $p : F_2 \to F_1$ with norm 1.

Case 2) We prove a more general result : "Every finite-dimensional F over a maximally complete K is free (and hence injective by 2.4))"

If $\dim F = 1$ this is clear. If $\dim F > 1$, F has a subspace F_1 with $0 < \dim F_1 < \dim F$. By induction F_1 is free and hence by (2.4) a direct summand of F. Write $F = F_1 \oplus F_2$. Again by induction F_2 is free and so F is free.

Case 3) Suppose $F_1 \subset F_2 \subset c_0(\mathbb{N}, \mu)$, $\dim F_2 = \dim F_1 + 1 < \infty$.
Take $x \in F_1$, $x \neq 0$ and let $n_0 \in \mathbb{N}$ be such that $\mu(n_0) | x_{n_0} | = \| x \|$.
We may assume that $x_{n_0} = 1$. The map $A : c_0(\mathbb{N}, \mu) \to c_0(\mathbb{N}, \mu)$ given by $A(e_i) = e_i$ if $i \neq n_0$ and $A(e_{n_0}) = x$ is bijective and isometric. So after applying A we may assume $e_{n_0} \in F_1$. Then $F_i = Ke_{n_0} \oplus \tilde{F}_i (i=1,2)$, $\tilde{F}_1 \subset \tilde{F}_2$ where $\tilde{F}_i = F_i \cap \{y \in c_0(\mathbb{N}, \mu) \mid y_{n_0} = 0\}$. By induction on the dimension there exists a projection $p : F_2 \to F_1$ with $\|p\| = 1$ given by $p(e_{n_0}) = e_{n_0}$.

4) Take a maximal orthogonal subset A of E and let $F \subset E$ be the closed subspace spanned by it. Then F is free and F is injective according to (2.13). There exists a projection $p : E \to F$ with $\|p\| = 1$. If $E \neq F$ then $(1-p)E \neq 0$ and for any $b \neq 0$, $b \in (1-p)(E)$ the set $\{b\} \cup A$ is also orthogonal. This contradicts the maximality of A.

5) For every α, $0 < \alpha < 1$, E has a norm $\| \;\|^*$ with $\alpha\| \;\| < \| \;\|^* < \| \;\|$ such that $(E, \| \;\|^*)$ is free. (Apply (2.17) and (2.13)).

Remark

The property familiar for complex Hilbert-spaces : "Every maximal orthogonal subset is an orthogonal base" is in general not true for free Banach spaces over K as will be shown in the next proposition. Criteria for maximal orthogonal subsets to be an orthogonal base are provided in

(3.9) Proposition. Let E be a Banach space over K. The following conditions are equivalent.

(1) Every maximal orthogonal subset of E is an orthogonal base

(2) E satisfies one of the following two conditions

a) $\dim E < \infty$ and E has an orthogonal base.

b) every strictly decreasing sequence in $\|E\|$ has limit zero.

Proof. (2) $\Rightarrow$ (1) Case a). Let F_1 be the linear subspace of $E = c_0(I, \mu)$ (card $I < \aleph_0$) spanned by a maximal orthogonal subset A of E. If $F_1 \neq E$ then there there exists F_2 with $F_1 \subsetneqq F_2 \subset E$, $\dim F_2 = \dim F_1 + 1$. According to case 3) of (3.8) a projection $p : F_2 \to F_1$ with norm 1 exists. For any $b \neq 0$, $b \in (1-p)F_2$ the set $A \cup \{b\}$ is orthogonal. Contradiction.

Case b). This is in fact proved in part 4) of (3.8).

(1) $\Rightarrow$ (2). $E \cong c_0(I, \mu)$ for some I and μ. If E does not satisfy (2) then I is infinite and we can choose μ such that the set (I) contains a strictly decreasing sequence with positive limit.

So it suffices to give a maximal orthogonal subset of $c_0(\mathbb{N}, \mu)$, where $\mu(1) > \mu(2) > \ldots$ and $\lim \mu(i) > 0$, which is not an orthogonal base. Put $f_n = e_n + e_{n+1}$ $(n \geq 1)$.

Since $\|f_n - e_n\| < \|f_n\| = \|e_n\|$ for all n, the set $\{f_n\}$ is a maximal orthogonal subset of $c_0(\mathbb{N}, \mu)$. It is not an orthogonal base since e_1 cannot be expanded as a convergent sum $\sum_{n=1}^{\infty} \lambda_n f_n$.

Indeed $e_1 = \sum_{n=1}^{\infty} \lambda_n f_n$ with $\lim \lambda_n \mu(n) = 0$ would imply

$e_1 = \lambda_1 e_1 + \sum_{2}^{\infty} (\lambda_n + \lambda_{n-1}) e_n$. Hence $\lambda_n = (-1)^n$ contradicting $\lim \lambda_n \mu(n) = 0$.

(3.10) Theorem. Every projective Banach space is free.

Proof. Let E be a projective Banach space. By (3.5) E can be represented by a direct summand of some $c_0(I, \mu)$. Choose a projection $p : c_0(I, \mu) \to E$ with norm 1. A subset J of I is called stable if the subspace $c_0(J, \mu/J)$ of $c_0(I, \mu)$ is invariant under p. Consider the collection X of all pairs (J,B) where J is a stable subset of I and B is an orthogonal base of $E(J) = E \cap c_0(J, \mu/J) = p(c_0(J, \mu/J))$. The set X is ordered by $(J,B) \leq (J',B')$ if $J \subsetneq J'$ and $B \subsetneq B'$. We will show that this order is inductive ; indeed, let $\{(J_i, B_i)\}$ be a totally ordered subset of X. Then $J^* = \bigcup J_i$, is again stable and it suffices to prove that $B^* = \bigcup B_i$ is an orthogonal base of $E(J^*)$. Clearly B^* is orthogonal. Let F be the closed subspace of $E(J^*)$ generated by B^*, clearly $F \supset E(J_i)$ for all i. Let $x \in E(J^*)$ and $\varepsilon > 0$. There is $y \in c_0(J_i, \mu/J_i)$ for some i such that $\|x-y\| \leq \varepsilon$. Then also $\|x-p(y)\| = \|p(x-y)\| \leq \varepsilon$ and $p(y) \in E(J_i) \subsetneq F$. So $F = E(J^*)$ and B^* is an orthogonal base of $E(J^*)$.
Zorn's lemma asserts the existence of a maximal element $(J,B) \in X$. If $J \neq I$, choose $i \in I \setminus J$. The smallest stable set J' containing $\{i\}$ is at most countable. Then also $J^* = J \cup J'$ is stable. The natural projection $\pi : c_0(J^*, \mu/J^*) \to c_0(J, \mu/J)$ induces a projection $p \circ \pi$, with the norm 1, of $E(J^*)$ onto $E(J)$. Hence $E(J^*) = E(J) \oplus F$, where F is isomorphic to a subspace of $c_0(J'', \mu/J), J'' = J^* \setminus J$. By (3.8) part 3) it follows that F has an orthogonal base B' and that $B^* = B \cup B'$ is an orthogonal base of $E(J^*)$. Contradiction with the maximality of (J,B).

(3.11) Theorem. (Change of base). Let B be a maximal orthogonal subset of $c_0(I, \mu)$. There exists a map $\rho : B \to c_0(I, \mu)$ such that $\|\rho(b)\| < \|b\|$ for all b and $\{b + \rho(b) \mid b \in B\}$ is an orthogonal base of $c_0(I, \mu)$.

Proof. A subset J of I is called stable if $B \cap c_0(J, \mu/J)$ is a maximal orthogonal subset of $c_0(J, /J)$. Consider the set X of all pairs (J, ρ) with J stable and

$\rho : B \cap c_0(J, \mu/J) \to c_0(J, \mu/J)$ such that $\{b + \rho(b) \mid b \in c_0(J, \mu/J)\}$ is an

orthogonal base of $c_0(J, \mu/J)$. By Zorn's lemma there is a maximal pair (J', ρ) (in the obvious ordering of X). Suppose $J' \neq I$.

Since B is maximal every e_i can be written as $e_i = \sum \lambda_{ib} b + R_i$ with $\|R_i\| < \|e_i\|$ It follows that every $i \in I$ is contained in a stable countable subset of I. By the

same trick, there exists a stable J^* with $J' \subset J^* \subset I$ and $J^* \setminus J'$ is at most countable.
But $B^* = B \cap c_o(J^*,\mu/J^*)$ and $B' = B \cap c_o(J',\mu/J\ )$. It suffices to find a map $\rho^*: B^* \setminus B' \to c_o(J^*,\mu/J^*)$ such that the image of $\{b+\rho^*(b) \mid b \in B^* \setminus B'\}$ under the canonical projection $\pi: c_o(J^*,\mu/J^*) \to c_o(J^* \setminus J', \mu/J^* \setminus J')$ is an orthogonal base of the latter. So we are reduced to the countable case of (3.11) : $I = \mathbb{N}$.

Proof : Since B is a maximal orthogonal set in $c_o(\mathbb{N},\mu)$ every $x \neq 0$ can be written as $x = \sum \lambda(x,b)b + R(x)$ where $\|\lambda(x,b)b\|$ is either $\|x\|$ or 0 and $\|R(x)\| < \|x\|$. With this notation we proceed as follows : $e_1 = \sum \lambda(e_1,b)b + R(e_1)$; number the set of b's such that $\lambda(x,b) \neq 0$ as $b_1, \dots, b_{n_1}$ and change them into $b_1^* = b_1 + \lambda(e_1,b_1)^{-1} R(e_1)$, $b_i^* = b_i$ for $i = 2,\dots,n_1$. Write $B_1 = B \setminus \{b_1,\dots,b_{n_1}\}$. Then $e_1 \in Kb_1^* + \dots + Kb_{n_1}^* = E_1$. One easily concludes form (3.8) that there exists a projection p_1 with norm 1 of $c_o(\mathbb{N},\mu)$ onto E_1.
Now if $x = e_2 - p_1(e_2)$ is non-zero then it equals $\sum_{b \in B_1} \lambda(x,b)b + R(x)$.
Number $\{b \in B_1 \mid \lambda(x,b) \neq 0\}$ as $b_{n_1+1}, \dots, b_{n_2}$; change them into $b^*_{n_1+1} = b_{n_1+1} + \lambda(x,b_{n_1+1})^{-1} R(x)$ and $b_i^* = b_i$ for $i = n_1+2,\dots,n_2$.
Then $e_2 \in Kb_1^* + \dots + Kb_{n_2}^*$. With induction one easily completes this proof.

Definitions. An orthogonal set (resp. -base) is called an orthonormal set (resp. -base) if all its elements have norm 1. A subring R (containing 1) of $V = \{\lambda \in K \mid |\lambda| \leq 1\}$ is called discrete if $\sup\{|r| \mid r \in R, |r| < 1\} < 1$.

(3.12) Theorem. Let B be a maximal orthonormal subset of $c_o(I)$.
Put $b = \sum \lambda_{b,i} e_i$ for every $b \in B$. If there exists for every countable subset $B' \subset B$ a discrete ring R such that $R \supset \{\lambda_{b,i} \mid b \in B'\}$ then B is an orthonormal base of $c_o(I)$.

Proof. The method of (3.10) and (3.11) yields that it suffices to show (3.12) in the case $I = \mathbb{N}$. We will use the following notations : F = the closed subspace of $E = c_o(\mathbb{N})$ generated by B ; m is the maximal ideal of V; $k = V/m$ in the residue field of V. R_o is a discrete ring containing all the coefficients $\lambda_{b,i}$; $R = S^{-1} R_o$ with $S = \{a \in R_o \mid |a| = 1\}$ is also discrete ; $\pi \in V$ satisfies

$1 > |\pi| \geqslant \sup\{|r| \;\; |r \in R, \;\; |r| < 1\}$. The image of R in ${}^{V}/_{\pi V}$ is a field I which can be identified with a subfield of k by means of the map ${}^{V}/_{\pi V} \to {}^{V}/_{m} = k$. Consider the ${}^{V}/_{\pi V}$-module .

$M_1 = \{x \in E| \;\; \|x\| \leqslant 1\} / \{x \in E| \;\; \|x\| \leqslant |\pi|\}$ and the

k-vector space $M_2 = \{x \in E| \;\; \|x\| \leqslant 1\} / \{x \in E| \;\; \|x\| < 1\}$.

The image of elements t in M_1 or ${}^{V}/_{\pi V}$ will be denoted by $\bar{t}$ and images in M_2 or ${}^{V}/_{m}$ by $\bar{\bar{t}}$.

M_1 is a free ${}^{V}/_{\pi V}$-module with base $\{\bar{e}_i\}$ and $\bar{b} = \sum \bar{\lambda}_{b,i}\bar{e}_i$ with $\bar{\lambda}_{b,i} \in 1$ for all b and i. And M_2 is a vector space over k with base $\{\bar{\bar{e}}_i\}$. The maximality of B implies that $\{\bar{\bar{b}} \mid b \in B\}$ is also a base. Hence there are $\mu_{i,b} \in 1$ with $\sum \mu_{i,b}\bar{\bar{b}} = \bar{\bar{e}}_i$ for all i. Choose $\rho_{i,b} \in R$ with $\bar{\bar{\rho}}_{i,b} = \mu_{i,b}$. Then $\bar{e}_i = \bar{\rho}_{i,b}\bar{b}$ holds in M_1. So in E one has $\|e_i - \sum \rho_{i,b} b\| \leqslant |\pi|$ for all i and that easily implies that F = E. It follows that B is an orthonormal base.

(3.13) <u>Problem.</u> Can (3.12) be extended to the case $c_o(I,\mu)$ where $\mu(I) \not\subset |K^*|$?

<u>Examples, corollaries and problems</u>.

(3.14) For every field K there are non-free Banach spaces and there are non-injective Banach-spaces.

(3.15) <u>The following conditions are equivalent</u> :

(a) <u>The valuation of</u> K <u>is discrete</u>.

(b) <u>Every Banach space over</u> K <u>is weakly injective</u>.

(c) <u>Every Banach space over</u> K <u>is weakly projective</u>.

<u>Proof</u>. Of course (b) and (c) are equivalent ; (a) $\Longrightarrow$ (c) is proved in (2.17). Now (b) $\Longrightarrow$ (a). First of all weakly injective and injective are the same for the Banach space K. So K is maximally complete. Consider $c_o(\mathbb{N}) = E$. For some norm $\| \; \|^*$ on E(equivalent with the usual norm), E is injective. By (3.8) $(E, \| \; \|^*) \cong c_o(\mathbb{N}, \mu)$

for some μ . By (2.15) K is discrete.

(3.16) Let Ω_p denote the completion of the algebraic closure of $\mathbb{Q}_p$, the field of p-adic integers. Then Ω_p <u>is not maximally complete</u>.

<u>Proof</u>. Ω_p is a Banach space of countable type over $\mathbb{Q}_p$, hence by (3.8) is isomorphic to $c_o(\mathbb{N}, \mu, \mathbb{Q}_p)$ and by (2.15) not maximally complete.

(3.17) <u>Description of</u> $\check{\Omega}_p$:

Let K denote the complete subfield of Ω_p which has the properties : $|K^*| = \{|p|^n \mid n \in \mathbb{Z}\}$ and k=algebraic closure of $\mathbb{F}_p$. Let ϑ_n denote a sequence of elements in Ω_p satisfying $\vartheta_{n+1}^{n+1} = \vartheta_n$ and $\vartheta_1 = p$. Then $\Omega_p = \overline{\bigcup_n K(\vartheta_n)}$ and the set $\{\vartheta^{(\alpha)} \mid \alpha \in T\}$ is an orthogonal base of Ω_p over K where $T = \{\alpha \in \mathbb{Q} \mid 0 \leqslant \alpha < 1\}$; $\vartheta^{(\alpha)} = \vartheta_n^{n!}$ with n such that $n!\,\alpha \in \mathbb{N}$.

Following (2.18) $\check{\Omega}_p$ consists of all formal expressions $f = \sum a_\alpha \vartheta^{(\alpha)}$ satisfying.

(i) $a_\alpha \in K$; $\sup |a_\alpha| < \infty$

(ii) for every $\varepsilon > 0$ the set $\{\alpha \in T \mid |a_\alpha| \geqslant \varepsilon\}$ is decreasing.

This describes $\check{\Omega}_p$ as a Banach space. Now the multiplication on $\check{\Omega}_p$: for $f = \sum a_\alpha \vartheta^{(\alpha)}$, $g = \sum b_\beta \vartheta^{(\beta)}$ we define $fg = \sum_\gamma \big(\sum_{\alpha+\beta=\gamma} a_\alpha b_\beta + p \sum_{\alpha+\beta=\gamma+1} a_\alpha b_\beta \big) \vartheta^{(\gamma)}$. Then condition (ii) on f and g implies that the sums converge. Further, this multiplication clearly extends the multiplication on Ω_p. Showing that $\check{\Omega}_p$ is in fact a field presents no difficulties.

(3.18) $\check{\Omega}_p$ <u>is not of countable type over</u> Ω_p.

<u>Proof</u>. Indeed, $\check{\Omega}_p$ is not of countable type over K (or $\mathbb{Q}_p$) according to (3.8) and (2.15). Since Ω_p is a Banach space of countable type over $\mathbb{Q}_p$, the assertion follows.

(3.19) Suppose that E is not injective and $\check{E}/_E$ is of countable type, then

a) K is maximally complete.

b) If K is not discrete then dim $\check{E}/_E < \infty$.

(3.20) Proof. Since $\check{E}/_E$ is of countable type it has a continuous linear map l : $\check{E}/_E \to K$, $l \neq 0$. Hence K is (weakly) injective and $\check{E}/_E$ is isomorphic to $c_o(I,\mu,K)$ with card I $\leq \aleph_o$. By (2.15) the set I is finite if K is not discrete.

(3.20) Suppose that E is an injective Banach space over K and let $\check{K}$ be a valued field which is a maximal completion of K. Then E has a structure of Banach space over $\check{K}$ compatible with its structure as K-Banach space.

Proof. Let E_o be the closed subspace of E generated by a maximal orthogonal subset. Then $E_o = c_o(I,\mu,K)$. The $\check{K}$-space $E_1 = c_o(I,\mu,\check{K})$ is an essential extension of E_o. The maximal completion E_2 of the $\check{K}$-Banach space E_1 is again a essential extension of E_o.

So E_2 and E are both maximal completions of E_o , hence K-isomorphic by (2.10). The isomorphism with the K-Banach space E_2 induces a $\check{K}$-Banach space structure on E.

(3.21) Proposition. Let X be a compact set. Then C(X $\to$ K) has an orthonormal base consisting of characteristic functions.

Proof. Let P denote a discrete complete subfield of K. By (3.8) part 4 and (3.9) it follows that C(X $\to$ P) has an orthonormal base consisting of characteristic functions (of necessarily open and closed subsets of X). One easily checks that it remains an orthonormal base of C(X $\to$ K).

(3.22) Problems.

(i) Suppose that E has an α-orthogonal base for some $\alpha < 1$. Does it follow that E has a β-orthogonal base for every $\beta < 1$?

(ii) Let A_i (i=1,2) be subsets of E and $0 < \alpha_1 < \alpha_2 \leq 1$ such that A_i is maximal α_i-orthogonal. Is card A_1 = card A_2 ?

(iii) Let the valuation of K be dense and E a Banach space on K. Does E have an equivalent norm $\|\ \ \|^*$ for which $\|E\|^* = |K|$? As a testcase one could try

$E = l^\infty(\mathbb{N},K)$.

(iv) Suppose that E has the property : every $e \in E$ lies in an injective subspace of E. Does E have the structure of a $\check{K}$-Banach space ? If E itself is injective the answer is "yes" to (3.20).

(v) Let E be a Banach space over K. Is the center of $\mathcal{L}(E,E)$ equal to $K\mathrm{id}_E$?

§.4. Duality.

In this section we study the duals of Banach spaces E and the canonical map $\phi = \phi_E : E \longrightarrow E''$. A Banach space E is called reflexive if ϕ_E is bijective and isometric.

(4.1) Proposition. Suppose that K is maximally complete, then for any E, ϕ_E is isometric. Further ϕ_E is bijective if and only if $\dim E < \infty$.

Proof. If the sequence $0 \to E_1 \overset{\alpha}{\to} E_2 \overset{\beta}{\to} E_3 \to 0$ (i.e. α isometric $\|\beta\| = 1$ and β induces the norm on E_3) then by (2.2) and (2.3) the induced sequence $0 \to E_3' \to E_2' \to E_1' \to 0$ is also exact. So ϕ_E is isometric for all E. Further if E_2 is reflexive then also E_1 is reflexive since we have a commutative diagram, with exact rows :

$$\begin{tikzcd}
0 \arrow[r] & E_1 \arrow[r] \arrow[d, "\phi_1"] & E_2 \arrow[r] \arrow[d, "\phi_2"] & E_3 \arrow[r] \arrow[d, "\phi_3"] & 0 \\
0 \arrow[r] & E_1'' \arrow[r] & E_2'' \arrow[r] & E_3'' \arrow[r] & 0
\end{tikzcd}$$

Since ϕ_2 is bijective and ϕ_3 is injective it follows that ϕ_1 is surjective and hence E_1 is reflexive.

Suppose that there exists a reflexive Banach space E with $\dim E = \infty$. According to (3.8) E has a closed subspace isomorphic to $c_0(\mathbb{N},\mu)$. Hence $c_0(\mathbb{N},\mu) = F$ would be reflexive. But $F' = l^\infty(\mathbb{N},\mu)$ and there exists a bounded K-linear $\phi, \phi \neq 0$, $\phi : l^\infty(\mathbb{N},\mu)/c_0(N,\mu) \to K$, contradicting the reflexivity of F.

In the sequel of this section we suppose that K is not maximally complete.

(4.2) Proposition. If E is a Banach space over K such that every $e \in E$ lies in an injective subspace of E (in particular if E itself is injective) then $E' = 0$.

Proof. If $l : E \to K$ with $l \neq 0$ exists then for some injective $F \subset E$ we have $(F) = K$. So K is weakly injective and hence injective, contrary to our assumption.

(4.3) Theorem. $c_0(\mathbb{N})$ is reflexive.

Proof. $l^\infty(\mathbb{N})$ is the dual of $c_0(\mathbb{N})$. Let a bounded linear $\rho : l^\infty(\mathbb{N}) \to K$ be given. Since $(l^{\infty(\mathbb{N})}/c_0(\mathbb{N}))' = 0$ by (4.2) and (2.5) it follows that ρ is determined by $\{\rho(e_i) \mid i \in \mathbb{N}\}$. It suffices to show $\lim \rho(e_i) = 0$ because then $\rho \in \mathrm{im}(c_0(\mathbb{N}) \to c_0(\mathbb{N})'')$. Suppose the contrary, then there is a bounded linear $\mu : l^\infty(\mathbb{Z}) \to l(\mathbb{N})$ such that $\rho\mu(e_i) = 1$ for all $i \in \mathbb{Z}$. Let $\tau : l^\infty(\mathbb{Z}) - l^\infty(\mathbb{Z})$ denote the translation over 1, then $\rho\mu = \rho\mu\tau$ on $1\ (\mathbb{Z})$ since this holds on $c_0(\mathbb{Z})$.

Consider the element $f \in l^\infty(\mathbb{Z})$ given by $f_i = 0$ if $i < 0$, $f_i = 1$ if $i \geqslant 0$. Then $e_0 = f - \tau f$ and $\rho\mu(e_0) = \rho\mu(f) - \rho\mu\tau(f) = 0$. This contradicts $\rho\mu(e_0) = 1$.

(4.4) Corollary. Let I be a set with non-measurable cardinal number. Then $c_0(I,\mu)$ is reflexive.

Proof. The map $\phi : c_0(I,\mu) \to c_0(I,\mu)''$ is clearly isometric. The method of (4.3) can be applied in this case if one shows $(l^\infty(I,\mu)/c_0(I,\mu))' = 0$. For this (and further information on reflexivity) we refer to [6] .

(4.5) Example. Consider on $\mathbb{N}$ the Fréchet filter $\mathcal{F}_0 = \{A \subset \mathbb{N} \mid \mathbb{N}\setminus A \text{ is finite}\}$. For any filter $\mathcal{F} \supset \mathcal{F}_0$ we consider the subspace $E(\mathcal{F})$ of $l^\infty(\mathbb{N})$ consisting of all $x \in l^\infty(\mathbb{N})$ with $\lim_{\mathcal{F}} x = 0$. For notational purposes we allow a filter to contain the empty set. The filter containing ϕ will be called $\mathcal{A}$.

Or equivalently $E(\mathcal{F}) = \overline{\bigcup \{l^\infty(A) \mid \mathbb{N}\setminus A \in \mathcal{F}\}}$. It follows from (4.3) that $E(\mathcal{F})' = E(\mathcal{F}^+)$ where $\mathcal{F}^+$ is the filter $\{A \subset \mathbb{N} \mid A \cup B \in \mathcal{F}_0 \text{ for all } B \in \mathcal{F}\}$. One checks that $\mathcal{F}^+ = \mathcal{F}^{+++}$, hence $E(\mathcal{F})'$ is reflexive for all $\mathfrak{F}$. In general $\mathfrak{F} \neq \mathfrak{F}^{++}$ e.g. let $\mathfrak{F}$ be a free ultrafilter on $\mathbb{N}$ then $\mathfrak{F}^{+} = \mathfrak{F}_{o}$ and $\mathfrak{F}^{++} = \mathcal{N}$.

(4.6) Problems.

(i) Is the dual E' of any Banach space E (with card E non-measurable) reflexive ?

(ii) Suppose that $\phi_E : E \to E''$ is bijective. Does it follow that ϕ_E is isometric ?

(iii) Suppose that $E' = 0$ and let $l \in \mathcal{L}(E,\check{K})$, $l \neq 0$. Is $\overline{l(E)} = \check{K}$? In particular let E be a closed subspace of K, with $E \neq 0$ and $E' = 0$. Does it follow that $E = \check{K}$?

iv) A weaker version of (iii) is the question : Does $\check{K}$ have non-trivial topological direct summands ?

§.5. Tensor products.

Let E and F be Banach spaces over K. On $E \underset{s}{\otimes} F$ we introduce the semi-norm $\| \ \|$ given by $\|a\| = \inf\{\max_{1\leq i\leq s} \|e_i\|\,\|f_i\| \mid a = \sum_{i=1}^{s} e_i \otimes f_i\}$. Put $T = (E \otimes F, \| \ \|)$.

(5.1) Lemma. T has the following universal property :

For every Banach space G over K and every bounded bilinear map $t : E \times F \to G$ the corresponding linear map $t' : E \otimes F \to G$ has the property $\|t\| = \|t'\|$.

Proof. First we note that $\|t\|$ is defined to be the supremum of $\{\|e\|^{-1}\,\|f\|^{-1}\,\|t(e,f)\| \mid e \in E,\ f \in F\}$. Let $a = \sum e_i \otimes f_i \in E \otimes F$. Then $\|t'(a)\| = \|\sum t(e_i,f_i)\| \leq \max_i \|t(e_i,f_i)\| \leq \|t\| \max_i \|e_i\|\,\|f_i\|$.

Consequently $\|t'(a)\| \leq \|t\|\,\|a\|$ and so $\|t'\| \leq \|t\|$. On the other hand $\|t(e\,,\,f)\| = \|t'(e \otimes f)\| \leq \|t'\|\,\|e\otimes f\| \leq \|t'\|\,\|e\|\,\|f\|$. So $\|t\| \leq \|t'\|$.

(5.2) Lemma.

1) Take $\alpha \in \mathbb{R}$, $0 < \alpha \leq 1$. If $\{e_i \mid 1 \leq i \leq s\} \subset E$ is α-orthogonal then for all $f_1,\ldots,f_s \in F$, $\|\sum_{i=1}^{s} e_i \otimes f_i\| \geq \alpha \max(\|e_i\|\,\|f_i\|)$.

2) The semi-norm on $E \otimes F$ is a norm and satisfies $\|e \otimes f\| = \|e\|\,\|f\|$.

(3) *For any subspace* E_1 *of* E *and* F_1 *of* F *the map* $(E_1 \otimes F_1, \| \ \|) \to (E \otimes F, \| \ \|)$ *is isometric*.

(4) *If every finite dimensional subspace of* E *has an orthogonal base then every* $a \in E \otimes F$ *can be written as* $a = \sum e_i \otimes f_i$ *where* $\|a\| = \max(\|e_i\| \, \|f_i\|)$

Proof.(1) Let G be a spherically complete field containing K. Define $t_1 : Ke_1 + \ldots + Ke_s \to G$ by $t_1(e_1) = 1$ and $t_1(e_j) = 0$ if $j \neq i$.

Define $t_2 : Kf_i \to G$ by $t_2(f_i) = 1$ (we suppose here, as we may, that $f_i \neq 0$). Extend both mappings to the whole of E, resp. F, with values in G and without increasing their norms.

Consider $t : E \times F \to G$, $t(e,f) = t_1(e)t_2(f)$ and let $t' : E \otimes F \to G$ be the corresponding K-linear map. Then $t'(a) = 1$ and

$\|t'\| = \|t\| = \|t_1\| \, \|t_2\| \leqslant \alpha^{-1} \|e_i\|^{-1} \|f_i\|^{-1}$. So $\|a\| \geqslant \alpha \|e_i\| \, \|f_i\|$.

Alternative proof. (after T.A. Springer). Let $x = \sum_{i=1}^{a} e_i \otimes f_i$ and let $x = \sum_{j=1}^{b} e'_j \otimes f'_j$ be another representation of x. We have to show

$$\max \|e'_j\| \, \|f'_j\| \geqslant \alpha \max \|e_i\| \, \|f_i\|.$$

Take $\beta \in \mathbb{R}$, $0 < \beta < 1$, and let $g_1, \ldots, g_c$ be an β-orthogonal base of the vector space $Kf'_1 + \ldots + Kf'_b$. (For every β, $0 < \beta < 1$, such a base exists!). Then $f'_j = \sum_{k=1}^{c} \lambda_{jk} g_k$ with

$\|f'_j\| \geqslant \beta \max_k (|\lambda_{jk}| \, \|g_k\|)$. Further $x = \sum_j e'_j \otimes f'_j = \sum_k (\sum_j \lambda_{jk} e'_j) \otimes g_k$.

Since the $\{g_1, \ldots, g_c\}$ are linearly independent we have $\sum_j \lambda_{jk} e'_j = \sum_{i=1}^{a} \mu_{ki} e_i$ and $f_i = \sum_k \mu_{ki} g_k$, for some $\mu_{ki} \in K$.

Now $\max_j \|e'_j\| \, \|f'_j\| \geqslant \beta \max_{j,k} \|e'_j\| \, |\lambda_{jk}| \, \|g_k\| \geqslant \beta \max_k \|\sum_j \lambda_{jk} e'_j\| \, \|g_k\|$

$\geqslant \alpha\beta \max_{i,j} |\mu_{ki}| \, \|e_i\| \, \|g_k\| \geqslant \alpha\beta \max_i \|e_i\| \, \|f_i\|$.

Since $\beta \in \mathbb{R}$, $0 < \beta < 1$, was arbitrary, it follows that "$\max \| e_j' \| \| f_j' \| \geqslant \alpha \max \| e_i \| \| f_i \|$".

(2) Take $a \in E \otimes F$, $a \neq 0$. Write $a = \sum e_i \otimes f_i$ where $\{e_1, \ldots, e_s\}$ is linearly independent over K. Then for some α, $0 < \alpha < 1$, $\{e_1, \ldots, e_s\}$ is α-orthogonal. According to (1), $\| x \| \neq 0$. Hence $\| \quad \|$ is a norm. The equality $\| e \otimes f \| = \| e \| \, \| f \|$ follows directly from (1).

(3) The norm on $E_1 \otimes F_1$ will be denoted by $\| \quad \|_1$. Clearly $\| x_1 \| \geqslant \| x \|$ for all x in $E_1 \otimes F_1$. On the other hand : for $x \in E_1 \otimes F$ for $x \in E_1 \otimes F_1$ and $\alpha \in \mathbb{R}$, $0 < \alpha < 1$, there are $e_1, \ldots, e_s$ in E and $f_1, \ldots, f_s$ in F such that $e_1, \ldots, e_s$ is α-orthogonal and $x = \sum e_i \otimes f_i$.

Hence (1) yields $\| x \| \geqslant \alpha \max(\| e_i \| \| f_i \|) \geqslant \alpha \| x_1 \|$. Since $\alpha \in \mathbb{R}$, $0 < \alpha < 1$, was arbitrary we may conclude $\| x \|_1 \leqslant \| x \|$.

(4) Take $x \in E \otimes F$. Then $x = \sum_{i=1}^{s} e_i \otimes f_i$. Choose an orthogonal base $\{e_i'\}$ of $Ke_1 + \ldots + Ke_s$. Then x can also be expressed as $\sum e_i' \otimes f_i'$ $(f_i' \in F)$. According to (1) we have $\| x \| = \max \| e_i' \| \| f_i' \|$.

<u>Definition</u>. The completion of $E \otimes F$ with respect to the norm on the tensor product is denoted by $E \hat{\otimes} F$.

(5.3) <u>Proposition</u>. <u>$\hat{\otimes} F$ is an exact functor for every Banach space</u> F.

<u>Proof</u>. Let $0 \to E_1 \xrightarrow{\alpha} E_2 \xrightarrow{\beta} E_3 \to 0$ be an exact sequence of Banach spaces (i.e the sequence is exact as a sequence of vector spaces over K, α is isometric, $\| \beta \| = 1$ and β induces the norm on E_3). We have to show the derived sequence $0 \to E_1 \hat{\otimes} F \xrightarrow{\alpha'} E_2 \hat{\otimes} F \xrightarrow{\beta'} E_3 \hat{\otimes} F \to 0$ is exact. The most difficult part, "α', is isometric", follows firectly from (5.2) part (3). The rest is left to the rearder.

<u>Remarks and examples</u>.

(5.4) The tensorproduct-norm as defined above corresponds with the "classical" π-tensorproduct topology. The classical ε-topology on $E \otimes F$ is given by the (semi-) norm

$\|z\|_\varepsilon = \sup\{ \, | l \otimes m(z) | \, \|l\|^{-1} \|m\|^{-1} \mid 0 \neq l \in E', \ 0 \neq m \in F' \}$ where $z \in E \otimes F$. Of course this is not very meaningful if $E' = F' = 0$. However we will show :

If $E \to E''$ and $F \to F''$ are isometric then $\| \ \|_\varepsilon = \| \ \|_\pi$.

Proof. (i) For finite dimensional E and F this follows from the existence of an α-orthogonal base for every $\alpha, 0 < \alpha < 1$.

(ii) If E_1 is a finite-dimensional subspace of E and $\rho > 1$ then there exists a projection $p : E \to E_1$ with $\|p\| \leq \rho$.

Indeed; since $E \to E''$ is isometric we have $E_1 \subset E \subset l^\infty(I)$ for some index set I. So it suffices to make a projection on $p : l^\infty(I) \to E_1$ with $\|p\| \leq \rho$. For $\dim E_1 = 1$ such a p exists and easy induction proves the general case.

(iii) For finite-dimensional $E_1 \subset E$ and $F_1 \subset F$ the map$(E_1 \otimes F_1, \| \ \|_\varepsilon) \to (E \otimes F, \| \ \|_\varepsilon)$ is isometric.

This follows from (ii) since $l_1 \in E_1'$, $m_1 \in F_1'$ with $\|l_1\| < 1$, $\|m_1\| < 1$ can be extended to $l \in E'$, $m \in E'$ with $\|l\| < 1$ and $\|m\| < 1$.

(iv) The assertion now follows since also $(E_1 \otimes F_1, \| \ \|_\pi) \to (E \otimes F, \| \ \|_\pi)$ is isometric.

Corollary. For locally convex spaces E and F over a maximally complete field the ε-topology and π-topology on $E \otimes F$ coincide. Every locally convex space over a maximally complete fields is nuclear.

(5.5) For compact sets X,Y and complete locally convex E over K we have $C(X \to E) \cong C(X \to K) \hat{\otimes} E$ and $C(X \times Y \to K) \cong C(X \to K) \hat{\otimes} C(Y \to K)$. And for sets I and J we have $c_0(I) \hat{\otimes} c_0(J) \cong c_0(I \times J)$.

(5.6) Problem.

Does there exist another complete tensor product, say $\check{\otimes}$, of Banach spaces which has the property $l^\infty(I) \check{\otimes} l^\infty(J) \cong l^\infty(I \times J)$?

(5.7) Related with tensor products is the theory of nuclear maps and the Fredholm theory. We will only sketch this and refer to [1] for more details.

Let E be a locally convex space over K and $A \subset E$ a V-submodule. ($V = \{\lambda \in K \mid |\lambda| \leq 1\}$). Then A is called precompact if there exists for every open V-module B of E a finite V-submodule of $E/_B$ which contains $A/B \cap A$.

Let E and F be Banach spaces, then the canonical map $E' \hat{\otimes} F \to \mathcal{L}(E,F) = \{l : E \to F \mid l \text{ is K-linear and continuous}\}$ is isometric as one easily deduces from (5.2) and (5.3). The image $\mathcal{C}(E,F)$ is called the space of nuclear maps.

For any $t \in \mathcal{L}(E,F)$ the following conditions are equivalent :

(i) $t \in \mathcal{C}(E,F)$

(ii) t is the uniform limit of elements in $\mathcal{L}(E,F)$ of finite rank.

(iii) $t(\{x \in E \mid \|x\| \leq 1\})$ is a precompact subset of F.

Proof. See [1] ; We will call elements of $\mathcal{C}(E,F)$ completely continuous maps.

(5.8) The notion of a precompact set seems to be an useful one. For Banach spaces E we will show the connection with ordinary compactness :

Let E be a Banach space and A a V-submodule of E. Then A is precompact if and only if there exists a compact set $T \subset E$ such that $A \subset \mathrm{conv}(T)$ = the closed convex hull.

Proof. "⇐" is trivial "⇒". For every $n \geq 1$ there exists a finite set say $b_1^{(n)}, \ldots, b_{s_n}^{(n)}$ such that $A \subset Vb_1^{(n)} + \ldots + Vb_{s_n}^{(n)} + \{x \in E \mid \|x\| \leq \frac{1}{n}\}$.

Hence A lies in the closed subspace of countable type generated by $\{b_i^{(n)} \mid n \geq 1 \; ; \; 1 \leq i \leq s_n\}$. So we may suppose that $E = c_0(\mathbb{N})$. We choose a sequence $\alpha_1, \alpha_2, \ldots$ in K such that, in case the valuation is dense $0 < |\alpha_i| < 1$ and $\pi |\alpha_i| \geq |\pi| > 0$ for some $\pi \in K$ and, in case the valuation is discrete we take $\alpha_i = 1$ for all i. Choose $a_1 \in A$ with $\|a_1\| \geq |\alpha_1| \sup \|A\|$ and let

$a_1 = \sum a_{1,i} e_i$ with $\|a_1\| = |a_{1,1}|$. Then $\alpha_1 A \subset Va_1 + A_1$ where

$$A_1 = A \cap \{x \in c_0(\mathbb{N}) \mid x = \sum_{i=2}^{\infty} x_i e_i\} .$$

We proceed in the same way with A_1, then $\alpha_2 A_1 \subseteq Va_2 + A_2$ where $A_2 = A \cap \{x \in c_o(\mathbb{N}) \mid x = \sum_{i=3}^{\infty} x_i e_i\}$ and $a_2 \in A_1$. By induction one finds a sequence $\{a_1\} \subset A$ which is orthogonal by construction. Since $\sum Va_i \subset A$ is also precompact it follows that $\lim \|a_i\| = 0$, $\lim\sup \|A_i\| = 0$ and

$$A \subseteq \overline{\mathrm{conv}(\{\pi^{-1} a_i\} \cup \{0\})}.$$

Problem. Let E be a locally convex space and A precompact subset. Does there exist a compact set T with $A \subset \overline{\mathrm{conv}(T)}$? (For locally convex spaces with a countable base for the neighbourhoods of 0 the proof given above works).

Remark. Suppose that $V \subset K$ is compact and let A be a V-submodule of a complete locally convex space. Then A is precompact if and only if $\bar{A}$ is compact.

(5.9) Another property of precompact sets is given in [1] :

Suppose that K *is maximally complete and let* A *be a* V-*submodule of a separated locally convex space*. *Then* A *is precompact if and only if* A *is bounded and linearly compact in its induced topology.* The module A is called linearly compact if every filter $\mathcal{F}$ generated by translates of open submodules of A has the property $\cap \mathcal{F} \neq \emptyset$. This property is also called c-compact by some authors.

(5.10) A curious result of J. Hily is the following :

Let K *be a maximally complete field with dense valuation. Then any* K-*linear bounded* $\phi : l^{\infty}(I) \to c_o(\mathbb{N})$ *is completely continuous.*

Proof. (See [2] section 3 for more general results). As in the proof of (5.8) one can show that $A = \{\phi(x) \mid x \in l^{\infty}(I), \ \|x\| \leq 1\}$ has the property :

There is an orthogonal sequence $a_1, a_2, \ldots$ in A with

$$\pi A \subset \{\sum \lambda_i a_i \mid \lambda_i \in V, \lim \lambda_i = 0\} \subset A.$$

If $\lim \|a_i\| = 0$ then ϕ is completely continuous. If for some $\varepsilon > 0$ the set $\{i \in \mathbb{N} \mid \|a_i\| \geq \varepsilon\}$ is infinite then one can find a map $\rho : c_o(\mathbb{N}) \to c_o(\mathbb{N})$ such that $\rho\phi : l^{\infty}(I) \to c_o(\mathbb{N})$ is surjective.

But this would imply that $c_o(\mathbb{N})$ is weakly injective which is false according to (3.8) part2) and (2.15).

§.6. Categorical aspects of Banach spaces.

(6.1) There are two natural categories of Banach spaces over K namely

(i) $\mathcal{B} = \mathcal{B}/K$; the objects are the Banach spaces over K and $\mathrm{Hom}(E,F) = \mathcal{L}(E,F) = \{ l : E \to F \mid l \text{ is K-linear and } \|l\| < \infty \}$.

(ii) $\mathcal{B}^1 = \mathcal{B}^1/K$; the objects are the Banach spaces over K and $\mathrm{Hom1}(E,F) = \{ l : E \to F \mid l \text{ is K-linear and } \|l\| \leqslant 1 \}$.

Neither category is abelian. Using a method of A. Heller [8] one gives the categories a structure of exact category by a choice of a suitable set of short exact sequences. Natural choices are :

The set of all sequences $0 \to A \xrightarrow{\alpha} B \xrightarrow{\beta} C \to 0$ satisfying

(A) $\alpha, \beta \in \mathcal{B}$ and the sequence is exact as a sequence of linear spaces over K.

(B) $\alpha, \beta \in \mathcal{B}^1$; α isometric ; β induces the norm on C and the sequence is exact as a sequence of linear spaces over K.

(C) $\alpha, \beta \in \mathcal{B}^1$; α isometric ; β induces the norm on C ; for all $c \in C$ there is a $b \in B$ with $\beta(b) = c$, and $\|b\| = \|c\|$; and the sequence is exact as a sequence of linear spaces over K.

An objet E is called projective (resp. injective) if the functor Hom(E,.) (resp. Hom(.,E)) is exact on the given class of exact sequences. In sections 2 and 3 we found :

category	projective objets	injective objets	
$\mathcal{B}$ with (A)	weakly projective	weakly injective	has injective and projective resolutions
$\mathcal{B}^1$ with (B)	none	injective	has only injective resolutions
$\mathcal{B}^1$ with (C)	projective	?	has only projective resolutions

The resolutions are of course those considered in (2.5) and (3.4). We will denote them by $0 \to E \to q_0E \to q_1E \to 0$ and $0 \to p_1E \to p_0E \to E \to 0$.

For left- (or right-) exact, co- (or contra-) variant functors T of $\mathcal{B}_{/K}$ or $\mathcal{B}^1_{/K}$ into any abelian category one defines left- or right derived functors L^nT or R^nT, $n \geq 0$ as usual. It follows of course that $L^nT = R^nT = 0$ for $n > 1$.

Examples of derived functors.

(6.2) The functor Hom(E,.) : $\mathcal{B}$ with (A) → (Vector spaces over K) is covariant and left-exact. Its derived functor is denoted by $\mathrm{Ext}_A(\ ,.)$ and we have for every $F \in \mathcal{B}$ the exact sequence $0 \to \mathrm{Hom}(E,F) \to \mathrm{Hom}(E,q_0F) \to \mathrm{Hom}(E,q_1F) \to \mathrm{Ext}_A(E,F) \to 0$. As usual $\mathrm{Ext}_A(E,F)$ can be interpreted as the set of isomorphy classes of extensions of E with F.

(6.3) The functor Hom(.,F) : $\mathcal{B}$ with (A) – (Vector spaces over K) is contravariant and right-exact. its derived functor applied to $E \in \mathcal{B}$ is equal to $\mathrm{Ext}_A(E,F)$ as defined in (6.2). So we are justified in denoting the left-derived functor of Hom(.,F) by $\mathrm{Ext}_A(.,F)$. Further one has the exact sequence

$0 \to \mathrm{Hom}(E,F) \to \mathrm{Hom}(p_0E,F) \to \mathrm{Hom}(p_1E,F) \to \mathrm{Ext}(E,F) \to 0.$

(6.4) The right-derivate of $\mathrm{Hom}^1(E,.)$: $\mathcal{B}^1$ with (B) → (Modules over V) is denoted by $\mathrm{Ext}_B(E,.)$. The left-derivate of $\mathrm{Hom}^1(.,F)$: $\mathcal{B}^1$ with (C) → Modules over V) is denoted by $\mathrm{Ext}_C(.,F)$.

(6.5) *Lemma*. *There exists for any* $E,F \in \mathcal{B}^1$ *a canonical injective map*

$\alpha : \mathrm{Ext}_C(E,F) \to \mathrm{Ext}_B(E,F)$ *with coker* α *is a vectorspace over* k.

Further $\mathrm{Ext}_C(E,F) \otimes {}_VK \cong \mathrm{Ext}_B(E,F) \otimes {}_VK \cong \mathrm{Ext}_A(E,F)$.

Proof. The sequence $0 \to p_1E \to p_0E \to E \to 0$ (in class (C) hence also in class (B) and (A)) induces exact sequences :

(a) $\to \mathrm{Hom}(p_0E,F) \to \mathrm{Hom}(p_1E,F) \to \mathrm{Ext}_A(E,F) \to 0$

(b) $\to \mathrm{Hom}^1(p_0E,F) \to \mathrm{Hom}^1(p_1E,F) \to \mathrm{Ext}_B(E,F) \to \mathrm{Ext}_B(p_0E,F) \to \dots$

(c) $\to \mathrm{Hom}^1(p_0E,F) \to \mathrm{Hom}^1(p_1E,F) \to \mathrm{Ext}_C(E,F) \to 0.$

This implies the existence of a canonical injective map $\alpha : \mathrm{Ext}_C(E,F) \to \mathrm{Ext}_B(E,F)$. After applying $\otimes_V K$ to the sequence (c) and comparing with (a) one finds $\mathrm{Ext}_C(E,F) \otimes_V K \cong \mathrm{Ext}_A(E,F)$. So the proof will be finished as we have shown that $\mathrm{Ext}_{B_1}(p_0E,F)$ is a vector-space over k. Let P be a projective Banach space then $\to \mathrm{Hom}^1(p,q_0F) \xrightarrow{\beta} \mathrm{Hom}^1(P,q_1F) \to \mathrm{Ext}_B(P,F) \to 0$ is exact. Using the ortohonormal base of P one finds $\mathrm{im}\,\beta \supseteq \{l : P \to q_1F \mid \|l(x)\| < \|x\| \text{ for all } x\}$. Hence $\mathrm{im}\,\beta \supseteq \underline{m}\,\mathrm{Hom}^1(P,q_1F)$ and $\mathrm{Ext}_B(P,F)$ is a vector-space over k.

(6.6) For every Banach space E we form $\mathrm{gr}(E) = \sum_{\rho \in \mathbb{R},\, \rho > 0} \{x \in E \mid \|x\| \leq \rho\} / \{x \in E \mid \|x\| < \rho\}$

This is a graded module over gr(K).

The graded ring gr(K) can be described as follows : Let G be the value group of K written as an additive group and let $\phi : G \to K^*$ be a map satisfying $|\phi(g)| = e^{-g}$ for all $g \in G$. The map 0 induces a symmetric 2-cocycle $\xi : G \times G \to K^*$ (where G acts trivially on k^*) by the formula $\xi(g,h)$ = the residue class of $\phi(g)\,\phi(h)\,\phi(g+h)^{-1}$ in k^*. Then gr(K) is isomorphic to $k[G,\xi]$ = the group algebra of G over k twisted by ξ. In particular if the valuation of K is discrete then $\mathrm{gr}(K) = k[\mathbb{Z}] \cong k[t,t^{-1}]$.

Let Gr(K) = Gr denote the category of all graded gr(K)-modules whose morphisms are the homogeneous gr(K)-linear maps of degree 0. Then Gr is an abelian category. We remark that $\alpha \in \mathcal{B}_1$ is isometric (resp. essential) if and only if $\mathrm{gr}(\alpha)$ is injective (resp. bijective). The functor $\mathrm{gr} : \mathcal{B}^1$ with $(B) \to \mathrm{Gr}$ is left-exact and covariant and its derived functor will be denoted by R'(gr).

(6.7) Let E be a Banach space over K. A <u>hole</u> in E is free filter $\mathcal{F}$ on E generated by spheres. The <u>diameter of</u> $\mathcal{F}$ is the infimum over all radii of spheres belonging to $\mathcal{F}$. Two holes $\mathcal{F}$ and $\mathcal{F}^*$ are said to be equivalent if there exists $e \in E$ with $e + \mathcal{F} = \mathcal{F}^*$.

Proposition. *There is a bijective correspondence between the homogeneous elements* ($\neq 0$) *of* $R^1(gr)\,E$ *of degree* $\rho(\rho \in \mathbb{R},\ \rho > 0)$ *and the equivalence classes of holes of diameter* ρ *in* E.

Proof. The injective resolution $0 \to E \xrightarrow{\Delta} l^\infty(E)/c_0(E) \xrightarrow{\pi} l^\infty(E)/c(E) \to 0$

of E induces the exact sequence $0 \to gr(E) \to gr(l^\infty(E)/c_0(E)) \to gr(l^\infty(E)/c(E)) \to$ $\to R^1(gr)(E) \to 0$.

Let ξ be a homogeneous element ($\neq 0$) of degree ρ in $R^1(gr)E$. Then ξ has a representative $x \in l^\infty(E)/c(E)$ with $\|x\| = \rho$. Choose $x_0 \in l^\infty(E)/c_0(E)$ such that $\pi(x_0) = x$. The collection of spheres $\{y \in E \mid \|\Delta(y) - x_0\| \leq \rho'\}(\rho' > \rho)$ generates a hole $\mathcal{F}$ of diameter ρ. Another choice of x does not affect $\mathcal{F}$ and another choice of x_0 translates $\mathcal{F}$. So we have assigned to ξ a class of holes of diameter ρ.

For a hole $\mathcal{F}$ in E of diameter ρ generated by $\{B(a_n,\rho_n)\}_{n \geq 1}$ one choose $x_0 \in \bigcap_{n=1}^{\infty} B(\Delta(a_n),\rho_n)$. The element $x = \pi(x_0)$ has norm ρ and gives rise to a homogeneous ξ ($\neq 0$) of degree ρ in $R^1(gr)E$. It is easily seen that the two maps described above are each other's inverses.

Relations with the category Mod(V) *of all* V-*modules*.

(6.8) First we shall recall some properties of modules over a valuation ring.

Lemma. (Fleischer) *A module* M *over* V *is injective if and only if* M *is divisible and every filter* $\mathcal{F}$ *on* M *generated by sets of the type* $m_0 + \{m \in M \mid \pi m = 0\}$ (π *an element of* V) *has a non-empty intersection*.

Proof. $\Leftarrow$ M is injective if Hom(V,M $\to$ Hom(I,M) is surjective for every ideal I of V. Let $\phi : I \to M$ be given, I is generated by a sequence of elements $\lambda_1, \lambda_2, \ldots$ with $|\lambda_i| \leq |\lambda_{i+1}|$ for all i.

The map ϕ can be extended to $V \to M$ if there exists $x \in M$ with $\lambda_i x = \phi(\lambda_i)$ for all i. Since M is divisible there are elements x_i satisfying $\lambda_i x_i = \phi(\lambda_i)$. Hence x must be an element in $\cap(x_i + \{m \in M \mid \lambda_i m = 0\})$. By assumption this intersection is non-empty.

$\Rightarrow$ Analogous.

(6.9) *Corollary*. *If* M *is an injective* V-*module and* N *a divisible submodule of* M *then* M/N *is injective.*

Proof. It is clear that M/N is also divisible and inherits the filter property from M.

(6.10) *Corollary*. *Every module* M *has* inj. dim $M \leq 2$. *If* M *is divisible then* inf. dim $M \leq 1$. *The global homological dimension of* Mod(V) *is* ≤ 2 *and* $= 1$ *if and only if the valuation ring* V *is discrete. Further* inj. dim $V = 1$ *if and only if* K *is maximally complete.*

Proof. For any module N let $\mathcal{E}(N)$ denote the injective envelope of N. For N we make the exact sequence $0 \to M \to \mathcal{E}(M) \to M_1 \to 0$ and $0 \to M_1 \to \mathcal{E}(M_1) \to M_2 \to 0$. The module M_1 is divisible and by (6.8) this yields that M_2 is injective. So inj. dim $M \leq 2$. If M is already divisible then M_1 is injective and inj. dim ≤ 1. For discrete V it is well known that global dim(Mod(V)) = 1. If V is non-discrete then inj. dim $V^{(\mathbb{N})} = 2$. ($A^{(I)}$ means the direct sum of I copies of A). Indeed, $0 \to V^{(\mathbb{N})} \to K^{(\mathbb{N})} \to K/V^{(\mathbb{N})} \to 0$ is exact, $K^{(\mathbb{N})}$ is not injective, and we have to show that $K/V^{(\mathbb{N})} = M$ is not injective. Choose a sequence $\lambda_1, \lambda_2, \ldots$ in K with $|\lambda_1| > |\lambda_2| > \cdots > 1$ and consider the subsets $(\bar{\lambda}_1, \bar{\lambda}_2, \ldots, \bar{\lambda}_{n-1}, 0, 0 \ldots) + \{m \in M \mid \lambda_n^{-1} M = 0\}$ of M. (here $\bar{\lambda}$ means the image of λ in K/V). The filter generated by them has an empty intersection. According to (6.8) we see that $K/V^{(\mathbb{N})}$ is not injective.

Further, inj. dim $V = 1$ if and only if K/V has the "filter property". This filter property is easily seen to be equivalent with maximally completeness.

(6.11) The counterpart for projective dimensions is :

Proposition. *Every module has projective dimension* ≤ 2. *If* M *is flat (equivalent to torsion free) then* proj. dim $M \leq 1$. *Every projective module is free*.

Proof. The last statement is a special case of Kaplansky's "big projectives are free". The proposition will be proved is we can show : any full submodule M of a free module P (i.e. P/M has no torsion) is itself projective. For this one can imitate the proof of 3.6).

We will exclude in the sequel of this section the trivial case of a discrete valuation ring.

(6.12) The relation between Banach spaces over K and modules over V can be expressed by various functors e.g.

$B : \mathcal{B}^1 \to \mathrm{Mod}(V)$ given by $B(e) = \{x \in E \mid \|x\| < 1\}$

$Q : \mathcal{B}^1 \to \mathrm{Mod}(V)$ given by $Q(E) = E/B(E)$.

Proposition. (i) B *and* Q *are exact w.r.t. both* (B) *and* (C) ;
(ii) $\mathrm{Hom}^1(E,F) \cong \mathrm{Hom}_V(BE,BF) \cong \mathrm{Hom}_V(QE,QF)$;
(iii) E *is injective if and only if* QE *is injective* ;
(iv) $\mathrm{Ext}_B(E,F) \cong \mathrm{Ext}_V^1(BE,BF) \cong \mathrm{Ext}_V^1(QE,QF)$.

Proof. (i) is obvious and (iii) follows from (6.8). To prove (ii) we use a *lemma* : *Let* M *be a torsion-free* V*-module*. *Then* M *is complete* (i.e. $M = \varprojlim M/\pi^n M$ *for some* $\pi \in V$, $0 < |\pi| < 1$) if and only if M is a *cotorsion-module* (i.e. $\mathrm{Hom}(K,M) = \mathrm{Ext}_V^1(K,M) = 0$).

Proof : "⇒" If M is complete then $\bigcap \pi^n M = 0$ and so $\mathrm{Hom}(K,M) = 0$.

For K we have a free resolution $0 \to V^{(\mathbb{N})} \xrightarrow{\alpha} V^{(\mathbb{N})} \xrightarrow{\beta} K \to 0$;
given by $\beta(\lambda_1, \lambda_2, \lambda_3, \dots) = \sum \lambda_i \pi^{-i}$ and

$$\alpha(\lambda_1, \lambda_2, \lambda_3, \dots) = (\lambda_1, \lambda_2 - \pi\lambda_1, \lambda_3 - \pi\lambda_2, \dots).$$

$\mathrm{Ext}_V^1(K,M)$ is the cokernel of the induced map $\mathrm{Hom}(V^{(\mathbb{N})}), M \xrightarrow{\alpha^*} \mathrm{Hom}(V^{(\mathbb{N})}, M)$.

Let $\phi : V^{(\mathbb{N})} \to M$ be given by $\phi(\lambda_1, \lambda_2, \lambda_3, \dots) = \sum \lambda_i m_i$ then the map ψ given by $\psi(\lambda_1, \lambda_2, \dots) = \sum \lambda_i \left(\sum_{n=0}^{\infty} \pi^n m_{i+m}\right)$ satisfies $\alpha^*(\psi) = \phi$.
Hence $\mathrm{Ext}_V^1(K,M) = 0$. "⇐" Analogous.

Proof of (ii). $\mathrm{Hom}^1(E,F) \cong \mathrm{Hom}_V(BE,BF)$ and the injectivity of $\mathrm{Hom}^1(E,F) \to \mathrm{Hom}_V(QE,QF)$ are obvious. Take $t \in \mathrm{Hom}_V(QE,QF)$ and let s be the map $E \to QE \xrightarrow{t} QF$. We have to show that s can be lifted to a map $E \to F$ or that $\mathrm{Hom}_V(E,F) \to \mathrm{Hom}_V(E,QF)$ is surjective. The cokernel of the latter is $\mathrm{Ext}_V^1(E,BF)$. Since E is a direct sum of copies of K and BF is complete the lemma yields $\mathrm{Ext}_V^1(E,BF) = 0$.

Proof of (iv). The injective resolution $0 \to F \to q_0F \to q_1F \to 0$ yields exact sequences:

$$\to \mathrm{Hom}^1(E,q_0F) \to \mathrm{Hom}^1(E,q_1F) \to \mathrm{Ext}_B(E,F) \to 0$$

$$\to \mathrm{Hom}(BE, Bq_oF) \to \mathrm{Hom}(BE, Bq_1F) \to \mathrm{Ext}_V^1(BE, BF) \to \mathrm{Ext}_V^1(BE, Bq_oF) \to \dots$$

$$\to \mathrm{Hom}(QE, Qq_oF) \to \mathrm{Hom}(QE, Qp_1F) \to \mathrm{Ext}_V^1(QE, QF) \to \mathrm{Ext}_V^1(QE, Qq_oF) \to \dots$$

By (ii) it suffices to show that $\mathrm{Ext}_V^1(BE, Bq_oF) = 0 = \mathrm{Ext}_V^1(QE, Qq_oF)$.

The last statement follows from (iii) and the first one from the exact sequence $0 = \mathrm{Ext}_V^1(E, Bq_oF) \to \mathrm{Ext}_V^1(BE, Bq_oF) \to \mathrm{Ext}_V^2(QE, Bq_oF) \cong \mathrm{Ext}_V^1(QE, Qq_oF) = 0$.

(6.13) *Proposition. Let* M *be a* V-*module.*

(i) $M \cong B(E)$ *for some Banach space* E *if and only if* M *is a torsion-free, cotorsion module and* $mM = M$.

(ii) $M \cong Q(E)$ *for some Banach space* E *if and only if* M *is a divisible torsion module such that* $\mathrm{Ann}(x) = \{\lambda \in V \mid \lambda x = 0\}$ *is non-principal for any* $x \in M$.

Proof.

(i) "⇒" follows from the lemma in (6.12). "⇐". Choose for $E = M \otimes_V K \supset M$ with norm given by $\| x \| = \inf \{ |\lambda| \mid \lambda \in K \text{ and } x \in \lambda M \}$.

(ii) "⇒" clear. "⇐" Take $\pi \in K$, $0 < |\pi| < 1$, and let $m \in M$, $m \neq 0$ be given. There are elements $m = m_o, m_1, m_2, \dots$ in M such that $\pi m_{i+1} = m_i (i \geqslant 0)$. Hence there is a V-linear map $\phi : K \to M$ satisfying $\phi(\pi^{-n}) = m_n$ for all $n \geqslant 0$. As a consequence there exists a surjective $\alpha : L - M$ where L is a vector space over K. The kernel L_o of α may be supposed to have no divisible submodule. On L we introduce a norm by $\| x \| = \inf \{ |\lambda| \mid \lambda \in K \text{ and } x \in \lambda \ L_o \}$. Let L denote the completion of L with respect to this norm and $\bar{L}_o$ the closure of L_o in L. Then $M \cong L/L_o$. The extra condition Ann(x) is non-principal for every $x \in M$ implies that $L_o = BL$. Hence $M \cong Q\hat{L}$.

(6.14) *Consequences*. Using (6.12) and (6.13) one can translate properties, constructions etc. of Banach spaces into properties etc. of V-modules. *Examples*.

(i) Let $0 \to E_1 \to E_2 \to E_3 \to 0$ be an exact sequence of Banach spaces of type (B) and let E and F denote Banach spaces. Then $0 \to BE_1 \to BE_2 \to BE_3 \to 0$ is exact and since BF_1 is flat also $0 \to BE_1 \otimes_V BF \to BE_2 \otimes_V BF \to BE_3 \otimes_V F \to 0$ is exact.

Further as one easily sees $\{ x \in E \otimes F \mid \| x \| < 1 \} = BE \otimes BF$. This yields (5.3) : $0 \to E_1 \hat{\otimes} F \to E_2 \hat{\otimes} F \to E_3 \hat{\otimes} F \to 0$ is an exact sequence of type (B).

(ii) (6.12) part(iii) combined with (6.9) proves that the quotient of an injective Banach space is again injective.

(iii) (6.11) is the counterpart of (3.6) : every closed subspace of a projective Banach space is projective. Further Kaplansky's theorem "Projective modules over a local ring are free" is the counterpart of (3.10) : every projective Banach space is free.

(iv) Let E be a Banach space and $\mathcal{E}(QE)$ the injective envelope of QE. By (6.13) part (ii), $\mathcal{E}(QE) \cong QF$ for some F and as one easily sees F is a maximal completion (see (2.10)) of E.

(v) The problem on reflexive Banach spaces (4.6) part (i) is equivalent to the following problem : Let K be a non-maximally complete field. Is the dual M^* (= Hom(M,V)) of any V-module reflexive (i.e. $M^* \to M^{***}$ bijective)?

Or, using the functor Q instead of B the problem is equivalent with : Let M' denote $\mathrm{Hom}_V(M,K/V)$ for any V-module M. Is $M' \to M'''$ bijective for any divisible torsion module M?

Remark. In comparing $\mathcal{B}'$ with Mod(V) as we did, one often has the disregard modules over k. So it seems more appropriate to compare $\mathcal{B}'$ with Mod(V)/Mod(k) = the quotient of Mod(V) the Serre-subcategory of Mod(k), all modules over k.

§.7. Differential Equations.

The first step in solving differential equations is the construction of a primitive function for every continuous function. This is done by approximating a continuous function, say $f : K \to K$, by locally constant functions. Any locally constant has a primitive function. A good choice for a primitive function of the characteristic function ξ of a sphere $B(a,\rho) \subset K$ is $F(t) = (t-a)\,\xi(t)$. The function F has the additional property $|F(t+h) - F(t) - h\,\xi(t)| \leq |h|$ for all t and h. To show this processin detail we consider first a simple case :

(7.1) Proposition. Let X be a compact subset of K which has no isolated points and let E be a Banach space over K. There exists (for every $\varepsilon > 0$) a bounded linear $P : C(X \to E) \to C(X \to E)$ (with $\|P\| \leq \varepsilon$) satisfying :

(a) $P(f)' = f$ and $\lim_{y \to x} (y-x)^{-1}(P(f)(y) - P(f)(x)) = f(x)$ uniformly on X.

(b) For any $f \in C(X \to E)$ and any $x,y \in X$ the following inequality holds : $\|P(f)(y) - P(f)(x) - (y-x)\,f(x)\| \leq |y-x|\,\|f\|$.

(c) <u>If</u> $\dim E < \infty$ <u>then</u> P <u>is completely continuous</u>.

<u>Proof</u>. Since X is compact we have $C(X \to E) \cong C(X \to K) \hat{\otimes} E$. It suffices to construct $P : C(X \to K) \to C(X \to K)$ with the required properties because $P \hat{\otimes} 1_E : C(X \to E) \to C(X \to K)$ has an orthonormal base $\{\xi_i \mid i \geq 0\}$ consisting of characteristic functions ξ_i of spheres $B(a_i, \rho_i) \subset X$. Define P by $P(\xi_i)(t) = (t-a_i)\,\xi_i(t)$ and extend P by linearity and continuity to $P : C(X \to K) \to C(X \to K)$. Clearly $\|P\| = \sup \|P(\xi_i)\| = \sup\{\rho_i \mid i \geq 0\}$. So for given $\varepsilon > 0$ the base $\{\xi_i\}$ can be chosen such that $\|P\| \leq \varepsilon$. Further P is completely continuous (i.e. the uniform limit of bounded linear maps with finite-dimensional range) since $\lim \|Pe_i\| = \lim \rho_i = 0$.

Let $f \in C(X \to K)$ have the expansion $f = \sum \lambda_i \xi_i$, $\lim \lambda_i = 0$. Then

$$|P(f)(y) - P(f)(x) - (y-x)f(x)| = |\sum \lambda_i\, P(\xi_i)(y) - P(\xi_i)(x) - (y-x)\,\xi_i(x)| \leq |y-x| \max\{|\lambda_i| \mid \rho_i < |x-y|\}.$$

Hence (a) and (b) follow.

(7.2) <u>Example</u>. Let $X = \mathbb{Z}_p = \{x \in \mathbb{Q}_p \mid |x| \leq 1\}$, K a field containing $\mathbb{Q}_p$, the field of p-adic numbers. The characteristic function of $\{t \in \mathbb{Z}_p \mid |t-n| < \frac{1}{n}\}$ will be denoted by $\phi_n (n \geq 1)$ and $\phi_0 = 1$. The set $\{\phi_n \mid n \geq 0\}$ is an orthonormal base of $C(\mathbb{Z}_p \to K)$. Indeed, as one easily sees $\{\phi_n \mid 0 \leq n < p^k\}$ is an orthonormal base of $\{f \in C(\mathbb{Z}_p \to K) \mid f$ is constant on spheres of radii $p^{-k}\}$ and further the space of locally constant functions is dense in $C(\mathbb{Z}_p \to K)$. So every $f \in C(\mathbb{Z}_p \to K)$ has an expansion $f = \sum \lambda_n \phi_n$. ($\lambda_n \in K$, $\lim \lambda_n = 0$).

The coefficients λ_n can be calculated in the following way :
On $\mathbb{N} \cup \{0\}$ we introduce a partial ordering $n \trianglelefteq m$ as follows

(i) $0 \trianglelefteq m$ for all m.

(ii) if $n \neq 0$, $n = a_0 + a_1 p + \ldots + a_k p^k$; $0 \leq a_i < p$; $a_k \neq 0$ then $n \trianglelefteq m$
if $m = b_0 + b_1 p + \ldots + b_l p^l$ with $l \geq k$ and $a_i = b_i$ for all $i = 0, \ldots, k$.

This ordering satisfies $n \trianglelefteq m$ if and only if $\phi_n(m) = 1$. For $n \neq 0$, $n = a_0 + a_1 p + \ldots + a_k p^k$, $0 \leq a_i < p$, $a_k \neq 0$ we put $n_- = n - a_k p^k$ or in other words n_- is the largest integer satisfying $n_- \neq n$ and $n_- \trianglelefteq n$.

Then for any continuous function f we have $f = \sum_{n=1}^{\infty} (f(n) - f(n_-))\, \phi_n + f(0)\phi_0$.

It is enough to check this formula for integral values l :

$$\sum_{n=1}^{\infty} (f(n) - f(n_-))\phi_n(l) + f(0)\phi_o(l) = \sum_{0 \neq n \leq l} (f(n) - f(n_-)) + f(0) = f(l).$$

Further $P(f)(t) = \sum_{n=1}^{\infty} (t-n)(f(n) - f(n_-))\phi_n(t) + f(0)t$. Let

$t = \sum_{i=0}^{\infty} a_i p^i$; $0 \leqslant a_i < p$; then $P(f)(t) = \sum_{k=0}^{\infty} a_{k+1} p^{k+1} f(\sum_{i=0}^{k} a_i p^i)$.

Another implication of the expansion $f = \sum (f(n) - f(n_-))\phi_n + f(0)\phi_o = \sum_{n=0}^{\infty} \lambda_n \phi_n$ is the following : $\lim_{y \to x} \frac{f(y) - f(x)}{y-x} = 0$ uniformly on $\mathbb{Z}_p$ ($f' = 0$ uniformly, for short)

is equivalent with $\lim n|\lambda_n| = 0$. Of course a sequence $a_n \in K$ with $\lim a_n p^{-n} = 0$ and $\lim |a_n p^{-2n}| = \infty$; define $f : \mathbb{Z}_p \to K$ by $f(x) = a_n$ if $|x-p^n| < p^{-2n}$ and

$f(x) = 0$ for all other values of x. Then $f' = 0$ and $\left|\frac{f(p^k+p^{2k}) - f(p^k)}{2k}\right| = |a_k| p^{-2k}$

is unbounded. So $f' = 0$ not uniformly.

(7.3) <u>A more general case</u>. Let X be a subset of K which has no isolated points and let E be a Banach space over K (or if necessary a locally convex space over K). We want to construct a (continuous) linear $P : C(X \to E) \to C(X \to E)$ which satisfies $P(f)' = f$ for all $f \in C(X \to E)$.

We will show that it suffices to give a primitive function of just one continuous map, namely the map : $X \to \mathcal{M}(X)$. Here $\mathcal{M}(X)$ denotes the vectorspace over K of all measures on W with compact support. (i.e. $\mu \in \mathcal{M}(X)$ if there exists a compact $T \subset X$ and a bounded linear $\mathbf{l} : C(T \to K) \to K$ with $\mu : C(X \to K) \xrightarrow{\rho} C(T \to K) \xrightarrow{\mathbf{l}} K$

and where ρ is the restriction map). The topology on $\mathcal{M}(X)$ is the locally convex topology generated by the sets $\{O(f_1, \dots, f_s) | s \geqslant 1$;

$f_1, \dots, f_s \in C(X \longrightarrow K)\}$ in which $O(f_1, \dots, f_s) = \{\mu \in \mathcal{M}(X) | |\mu(f_i)| < 1$ for all $i\}$. The continuous map $\delta : X \to \mathcal{M}(X)$ is given by $\delta(x)(f) = f(x)$ for all $x \in X$, $f \in C(X \to K)$.

(7.4) <u>Lemma</u>. <u>There exists</u> $\Delta : X \to \mathcal{M}(X)$ <u>with</u> $\Delta' = \delta$.

Proof. In order to approximate δ by locally constant functions on X we introduce some terminology. Let $\pi \in K$, $0 < |\pi| < 1$, and let $\underset{n}{\sim}$ denote the equivalence relation on X given by $x \underset{n}{\sim} y$ if $|x-y| \leq |\pi|^n$. Choose $x_0 \in X$ and for every $n \geqslant 1$ a map $\rho_n : X/\underset{n}{\sim} \to X$ such that $X/\underset{n}{\sim} \xrightarrow{\rho_n} X \to X/\underset{n}{\sim} = \mathrm{id}$ and $\rho_n(x_0) = x_0$ for all n.

Define ρ_0 by $\rho_0(X) = \{x_0\}$ and let $R_n : X \to X/\underset{n}{\sim} \xrightarrow{\rho_n} X$ for all $n \geqslant 0$.

Let $\delta_n : X \to \mathcal{M}(X)$ be given by $\delta_n(x) = \delta(R_n x)$. Then $\delta = \lim \delta_n$ and each δ_n is locally constant. Hence $\delta = \delta_0 + \sum_{n=1}^{\infty} (\delta_n - \delta_{n-1})$. Each $g_n = \delta_n - \delta_{n-1}$ is constant on spheres of radii $|\pi|^n$ and has an obvious primitive function G_n given by $G_n(x) = (x-R_n(x))g_n(x)$.

Define $\Delta : X \to \mathcal{M}(X)$ by $\Delta(x) = (x-x_0)\delta_0 + \sum G_n(x)$. Clearly $\Delta(x) \in \mathcal{M}(X)$ and has support in the compact set $\{x\} \cup \{R_n x \mid n \geqslant 1\}$. Further $\frac{1}{y-x}(\Delta(y) - \Delta(x)) - \delta(x) = $
$= \sum_{n=1}^{\infty} (\frac{1}{y-x}(G_n(y)-G_n(x))-g_n(x))$. In order to show that $\lim_{y \to x}$ of this expression is zero it suffices to prove for any $f \in C(X \to K)$ that

$$\lim_{y\to x} \sum_{n=1}^{\infty} (\frac{1}{y-x}(G_n(y) - G_n(x)) - (\delta_n(x) - \delta_{n-1}(x)))(f) = 0.$$

Choose $\varepsilon > 0$ and n_0 such that $|f(x) - f(y)| \leq \varepsilon$ whenever $|x-y| \leq |\pi|^{n_0-1}$.

Then for $n \geqslant n_0$:

$$\left[\frac{1}{y-x}(G_n(y) - G_n(x)) - (\delta_n(x) - \delta_{n-1}(x))\right](f) = \frac{R_n(x) - R_n(y)}{y-x}(f(R_n x) - f(R_{n-1}x)) +$$
$$+ \frac{y-R_n(y)}{y-x}(f(R_n y) - f(R_n x) + f(R_{n-1}) - f(R_{n-1}y)).$$ Hence if $|y-x| \leq |\pi|^{n_0-1}$ this expression has absolute value $\leq \varepsilon$ and "$\lim_{y\to x}$" is equal to zero.

This completes the proof.

(7.5) Remarks. (i) A compact subset T of X is called full if $R_n(T) \subseteq T$ for all $n \geqslant 0$. Any compact set T lies in a full compact set. For a full compact set T we have support $(\Delta(x)) \subset T$ for all $x \in T$. So we can restrict δ and Δ to t ; $\tilde{\delta} = \delta/T : T \to C(T \to K)'$ and $\tilde{\Delta} = \Delta/T : T \to C(T \to K)'$. With the usual norm on $C(T \to K)'$ we have $\|\tilde{\Delta}(y) - \tilde{\Delta}(x) - (y-x)\tilde{\delta}(x)\| = \varepsilon(x,y)\,|y-x|$ with $\varepsilon(x,y) < 1$ for all $x,y \in T$ and $\lim_{y\to x} \varepsilon(x,y) = 0$ for all x.

(ii) The map Δ can be written in a slightly different form :

$$\Delta(x) = (x-x_0)\,\delta(x_0) + \sum_{n=1}^{\infty}(x-R_n(x))(\delta_n(x) - \delta_{n-1}(x)) = \sum_{n=0}^{\infty}(R_{n+1}x-R_nx)\,\delta(R_nx).$$

(7.6) *Proposition.* (Treiber) *Let* X *be a subset of* K *which has no isolated points and let* E *be a Banach space over* K. *Let* $\varepsilon > 0$. *There exists a linear* $P : C(X \to E) \to C(X \to E)$ *satisfying* :

(i) $(Pf)' = f$ *and on any compact set* $\lim\limits_{y\to x}\frac{1}{y-x}(P(f)(y) - P(f(x))) = f(x)$ *uniformly*.

(ii) *For every full compact set* T, *the restriction of* P *to* T *has norm* $\leq \varepsilon$ *and* $\| P(f)(y) - P(f)(x) - (y-x)f(x)\| \leq \|y-x\|\,\|f\|_T$ *for all* $x,y \in T$; $f \in C(X \to E)$.

(iii) *If* $\dim E < \infty$ *then* P *restricted to any compact full* T *is completely continuous*.

Proof. Every $\mu \in \mathcal{M}(X)$ induces a map $\tilde{\mu} : C(X \to E) \to E$. Indeed ; let $T \subset X$ be a compact set such that $\mu : C(X \to K) \xrightarrow{\rho} C(T \to K) \xrightarrow{1} K$, then $\tilde{\mu}$ is defined by $C(X \to E) \xrightarrow{\rho} C(T \to E) \overset{\sim}{=} C(T \to K) \hat{\otimes} E \xrightarrow{1\otimes 1_E} K \otimes E = E$. Define P by the formula $P(f)(x) = \widetilde{\Delta(x)}(f)$. A change of Δ into $\Delta^*(x) = \sum_{n=k}^{\infty}(R_{n+1}x-R_nx)\,\delta(R_nx)$ changes P into P^* with $\|P^*\|_T \leq |\pi|^k$. The other properties of P (or P^*) follow directly from the corresponding properties of Δ (or Δ^*).

(7.7) *Proposition.* (Treiber). *Let* X *be a subset of a Banach space* E *such that for every* $x \in X$ *and* $h \in E$ *the element* 0 *is non-isolated in* $\{t \in K \mid x+th \in X$. *Let* F *be another Banach space and* $\omega : X \to \mathcal{L}(E,F) = \{ l : E \to F \mid$ is K-linear and continuous$\}$ *a continuous map*. *Then there exists* $\Omega : X \to F$ *with* $d\Omega = \omega$.

Proof. First we solve the "universal problem" $\delta : X \to \mathcal{L}(E, \mathcal{M}(X) \hat{\otimes} E)$. Here $\mathcal{M}(X) \hat{\otimes} E$ is the completion of $\mathcal{M}(X) \otimes E$ which has the topology derived from the semi-norms on $\mathcal{M}(X)$, the norm on E and the tensor product (semi)-norm construction of §.5. As in (7.4) one defines maps $R_n : X \to X$ $(n \geq 0)$ with the properties:

(i) $R_0(x) = \{x_0\}$ and $R_n(x_0) = x_0$ for all $n \geq 1$;

(ii) $R_n(x) = R_n(y)$ if and only if $\|x-y\| \leq |\pi|^n$. Then $\delta = \delta R_0 + \sum_{n=1}^{\infty}(\delta R_n - R_{n-1})$.

One defines $\Delta : X \to \mathcal{M}(X) \hat{\otimes} E$ by "term by term integration" of this infinite sum :

$$\Delta(x) = \delta(R_o x) \otimes (x-x_o) + \sum_{n=1}^{\infty} (\delta R_n x - \delta R_{n-1} x) \otimes (x - R_n x).$$

It is easily seen that $d\Delta = \delta$ and $\Delta(x) = \sum_{n=0}^{\infty} \delta(R_n(x)) \otimes (R_{n+1}(x) - R_n(x))$.

Further, to return to ω, any $\tau \in \mathcal{M}(X) \hat{\otimes} E$ induces a map $\tilde{\tau}$, $\tilde{\tau} : C(X) \to \mathcal{L}(E,F)) \to F$ in an obvious way. Then a solution Ω of $d\Omega = \omega$ is $\Omega(x) = \widetilde{\Delta(x)}(\omega)$.

Remarks.

(i) The solution Ω in (7.7) can also be written in the form $\Omega(x) = \sum_{n=0}^{\infty} \omega(R_n x)(R_{n+1} x - R_n x)$. The case $X = \{x \in E \mid \|x\| \leqslant 1\}$ (or $X = E$) is considered by D. Treiber [7]. The choice of the R_n's is done as follows : Let A be a set of representations (containing 0) of $X/\{x \in E \mid \|x\| \leqslant |\pi|\}$. Then every element x in X has a unique expansion $x = \sum_{n=0}^{\infty} \pi^n a_n$ with $a_n \in A$ for all n. Put $R_n(x) = \sum_{k=0}^{n-1} {}^{k}a_k$ $(n \geqslant 1)$ and $R_o(x) = 0$. Then our formula for Ω reduces to the one given by Treiber [7] section 10.

(ii) As a corollary of (7.7) one finds that every continuous k-form (closed or not) is exact. In particular there is a function $f : Z_p^2 \longrightarrow Q_p$ with $df = ydx$. So $\frac{\partial f}{\partial x} = y$, $\frac{\partial f}{\partial y} = 0$ and $\frac{\partial f}{\partial x \partial y} \neq \frac{\partial f}{\partial x \partial y}$.

An explicit formula for f is given by the following :

$$f\left(\sum_{i=0}^{\infty} a_i p^i , \sum_{j=0}^{\infty} b_j p^j\right) = \sum_{i > j \geqslant 0} a_i b_j p^{i+j} \text{ , where } 0 \leqslant a_i < p \text{ ; } 0 \leqslant b_j < p.$$

(iii) The example (7.2) gives a primitive function for which one has derived the formula $P(f)\left(\sum_{n=0}^{\infty} a_k p^k\right) = \sum_{k=0}^{\infty} a_{k+1} p^{k+1} f\left(\sum_{i=0}^{k} a_i p^i\right)$.

This operator P could also be obtained from (7.6) where $R_n : \mathbf{Z}_p \to \mathbf{Z}_p$ is defined by (i) $R_o(\mathbf{Z}_p) = \{0\}$; $R_n(0) = 0$ for all n ; (ii) $R_n(\sum a_i p^i) = \sum_{i=0}^{n-1} a_i p^i$ with $0 \leqslant a_i < p$ for all p.

(iv) Solving differential equations is an exercice after (7.6). To be complete we will solve the exercise.

(7.8) *Proposition*. *Let* X *be a subset of* K *which has no isolated points*, E *a Banach space over* K *and* L *a (not necessarily linear)* map : $C(X \to E) \to C(X \to E)$ *which satisfies the Lipschitz-condition* : *There exists a constant* ρ *such that for any compact full* $T \subset X$, *any* $f,g \in C(X \to E)$ *the inequality* $\|L(f) - L(g)\|_T \leqslant \rho \|f-g\|_T$ *holds. Then there exists a bijective and, for every* $\| \;\|_T$, *with* T *full compact, isometric map* $\tau : \{h \in C(X \to E) \mid h' = 0\} \to \{f \in C(X \to E) \mid f' = L(f)\}$.

Proof. Let k be such that $|\pi|^k \rho < 1$. The map P given by the formula $P(f)(t) = \sum_{n=k}^{\infty} (R_{n+1}(t) - R_n(t)) f(R_n t)$ has the property $(Pf)' = f$ and $\|P(f)\|_T \leqslant |\pi|^k \|f\|_T$ for every full compact $T \subset X$. Take $h \in C(X \to E)$ with $h' = 0$.

The map $f \mapsto h + PL(f)$ of $C(X \to E)$ into itself is a strict contraction with respect to every $\| \;\|_T$. Hence there exists a unique $f = \tau(h)$ satisfying $f = h + PL(f)$. Clearly τ is isometric with respect to $\| \;\|_T$ and also surjective since $f' = L(f)$ implies $P(f') = PL(f)$ and $h = f - p(f')$ has derivate zero.

(7.9) *Corollary*. *(Linear equations) Let* X *be a subset of* K *which has no isolated points*, E *a Banach space over* K *and* $A : X \to \mathcal{L}(E,E)$ *a continuous and bounded map. Then there exists a continuous* $B : X \to \mathcal{L}(E,E)$ *such that* $B : \{h \in C(X \to E) \mid h' = 0\} \to$ $\to \{f \in C(X \to E) \mid f' = Af\}$ *is linear bijective and isometric*.

Proof. Consider $L : C(X \to \mathcal{L}(E,E)) \to C(X \to \mathcal{L}(E,E))$ given by $L(B) = AB$. Then as in (7.8) there exists a solution B of $B'(t) = A(t)B(t)$ with $\|B(t) - I\| \leqslant \rho < 1$ for all $t \in X$. Clearly if $h \in C(X - E)$ satisfies $h' = 0$ then $(Bh)' = A(Bh)$. Further if f satisfies $f' = Af$ then $(B^{-1}f)' = 0$.

(7.10) *Example*. For any differential equation $f^{(n)}(t) + a_{n-1}(t)f^{(n-1)}(t) + \ldots$ $\ldots + a_0(t)f(t) = g(t)$, $g, a_i : K \to K$ continuous and bounded, there are functions $y, x_i : K \to K$ $(i=1,\ldots,n)$ such that every solution of the differential equation has the unique form $y + \sum_{i=1}^{n} h_i x_i$, where $h_1' = \ldots = h_n' = 0$. This is a special case of (7.9).

(7.11) *Remarks*. (i) It is likely that a more detailed study of the "primitivation" P will show that the Lipschitz-conditions in (7.8) and (7.9) can be weakened.

(ii) Another interesting question is : which functions $f : X \subset K \to K$ are the derivative of other functions. An obvious necessary condition is that f is the pointwise limit of a sequence of continuous functions. In the last part of this section we will show that this condition is also sufficient, provided that $\overline{f(X)}$ is a compact subset of K.

We introduce the following notations : let X be any topological space, then $C_p(X \to K)$ is the Banach algebra of all continuous functions $f : X \to K$ such that $\overline{f(X)}$ is compact. Further X_d denotes the set X provided with the discrete topology.

(7.12) *Proposition*. *Let* X *be a subset of* K *which has no isolated points and let* $f : X \to K$ *be a function such that* $\overline{f(X)}$ *is compact. The following conditions are equivalent* :

(i) *There exists* $F : X \to K$ with $F' = f$.

(ii) *There exists a sequence* $\{F_n\} \subset C(X \to K)$ *such that for every* $x \in X$, $\lim F_n(x) = f(x)$.

Proof. (1) $\Longrightarrow$ (2) is trivial. The implication (2) $\Longrightarrow$ (1) will be proved using some lemma.

(7.13) *Lemma*. *The algebra* $R = \{f \in C_p(X_d \to K) \mid f$ is pointswise-limit of continuous functions$\}$ *is a closed subalgebra of* $C_p(X_d \to K)$.

Proof. It suffices to show that for any sequence $\{f_n\} \subset R$ with $\lim \|f_n\| = 0$, the sum $F = \sum f_n$ belongs to R.

Write $f_n =$ p-lim $f_{n,k}$, where "p-lim" means point-wise-limit and all $f_{n,k} \in C(X \to K)$. We may assume that $\|f_{n,k}\| \leqslant \|f_n\|$ for all k. Then $F_k = \sum_{n=1}^{\infty} f_{n,k} \in C(X \to K)$ since the sum is uniformly convergent on X. We claim $F =$ p-lim F_k. Indeed take $x \in X$, $\varepsilon > 0$, $N(\varepsilon) \in \mathbb{N}$ such that $\|f_n\| \leqslant \varepsilon$ whenever $n \geqslant N(\varepsilon)$ and take $k_o(x,\varepsilon) \in \mathbb{N}$ such that for all $k \geqslant k_o(x,\varepsilon)$ the inequality

$|f_n(x) - f_{n,k}(x)| \leq \varepsilon$; $n = 1,\dots,N(\varepsilon)$ holds. Then for $k \geq k_0(x,\varepsilon)$,

$$|F(x) - F_k(x)| = |\sum_{1}^{\infty} (f_n(x) - f_{n,k}(x)| \leq \max(\varepsilon, \sum_{n=1}^{N(\varepsilon)} (f_n(x) - f_{n,k}(x))|) = \varepsilon .$$

(7.14) *Lemma. Let Z be a topological space and A a closed subalgebra (containing 1) of $C_p(Z \to K)$. Then there is a compact 0-dimensional Y such that $A \cong C(Y \to K)$. In particular A has an orthonormal base consisting of characteristic functions.*

Proof. Let $\phi : A \to K$ be a K-algebra-homomorphism. Then $\phi(f) \in f(Z)$. Indeed if $\phi(f) \notin f(Z)$ then one can normalize f such that $\phi(f) = 1$ and $\sup |f(Z)| = \rho < 1$. Hence f-1 is invertible in R contradicting $\phi(f-1) = 0$. We take for Y the set of all K-algebra-homomorphisms $\phi : A \to K$. The canonical map ; $Y \to \prod_{f \in R} f(X)$, given by $(\tau(\phi))_f = \phi(f)$, is injective and has a closed image. We identify Y with its compact image $\tau(Y)$ and regard R as a subalgebra of $C(Y \to K)$. This subalgebra closed, separates the points of Y and contains 1.

According to the Stone-Weierstrass theorem (see [4]) $R \cong C(Y \to K)$.

Remarks. Combining (7.13) and (7.14) we see that R as defined in (7.13) has an orthonormal base $\{\chi_i\}_{i \in I}$ consisting of characteristic functions. Our next step will be to characterize sets $T \subset X$ for which the characteristic function χ_T belongs to R and to find a suitable primitive function for χ_T.

(7.15) *Lemma. The characteristic function of a subset T of $X \subset K$ belongs to R if and only if T is both the countable union of closed sets ans the countable intersection of open sets.*

Proof. "$\Leftarrow$". Write $T = \bigcup_{n=1}^{\infty} F_n = \bigcap_{n=1}^{\infty} O_n$ with all F_n closed and all O_n open. We may suppose $O_n \supset O_{n+1}$ and $F_n \subset F_{n+1}$ for all n. Let C_n be a closed and open subset such that $O_n \supset C_n \supset F_n$ and let χ_n be the characteristic function of C_n. Then χ_n is continuous and $\chi_T = \text{p-lim}\ \chi_n$ belongs to R.

"$\Rightarrow$". Suppose that $\chi_T = \text{p-lim}\ f_n$ with $\{f_n\} \subset C(X \to K)$. Let χ_n denote the characteristic function of $\{t \in X \mid |f_n(t)| = 1\} = C_n$ then C_n is both closed and open. Further $\chi_T = \text{p-lim}\ \chi_n$. Put $O_n = \bigcup_{k \geq n} C_k$ and $F_n = \bigcup_{k \geq n} C_k$. Then $\bigcap_{n=1}^{\infty} O_n = \bigcup_{n=1}^{\infty} F_n = T$.

(7.16) Example. $X = \mathbb{Z}_p$; $T = \mathbb{N}$ and $K \supset \mathbb{Q}_p$. Then $\chi_{\mathbb{N}} \notin R$.

Proof. Suppose that $\mathbb{N} = \bigcap_{n=1}^{\infty} O_n$ with O_n open for all n. We may assume that $O_n \supset \bigcup_{n=1}^{\infty} B(m, r_m^{(n)})$ and $r_m^{(n)} > r_m^{(n+1)}$ for all n and m.

Put $s_n = r_n^{(n)}$. Then $\bigcap_{n=1}^{\infty} O_n \supset \bigcap_{n=1}^{\infty} \bigcup_{m=n}^{\infty} B(m,s_m) = T$. In order to establish a contradiction we will show that T is uncountable. We may assume that $s_m > s_{m+1}$ and $1,\dots,m-1 \notin B(m,s_m)$ for all m. The map, which assigns to $x \in T$ the subset $\{n \in \mathbb{N} \mid x \in B(n,s_n)\}$, is injective since this subset is infinite and $\lim s_n = 0$. Further we note that every sphere $B(m,s_m)$ contains infinitely many spheres $B(n,s_n)$ with $n > m$. For each m we choose a bijection ϕ_m of $\mathbb{N}$ onto $\{n \in \mathbb{N} \mid n > m$ and $B(n,s_n) \subset B(m,s_m)\}$. Now we are ready to make an injective map $\tau : \mathbb{N}^{\mathbb{N}} \longrightarrow T$. Given $f : \mathbb{N} \to \mathbb{N}$ we make a sequence of spheres $B(m_k, s_{m_k})$ as follows :

$$m_A = f(1),\ m_2 = \phi_{m_1}(f(2)), \dots,\ m_{k+1} = \phi_{m_k}(f(k))$$

Define $\tau(f) = \bigcap B(m_k, s_{m_k})$. It follows easily that τ is injective and hence T is uncountable.

(7.17) Lemma. Let T be a subset of X such that $\chi_T \in R$. There exists $F : X \to K$ with $F' = \chi_T$ and $|F(y) - F(x) - (y-x)\chi_T(x)| \leq |y-x|$ for all $x,y \in X$ and $\|F\| \leq 1$.

Proof. As in (7.15) we put $\chi_T = \text{p-lim}\ \chi_{C_n}$ where $\{C_n\}$ is a collection of open and closed sets. Put $O_n = \bigcup_{k \geqslant n} C_k$, $F_n = \bigcap_{k \geqslant n} C_k$, $X_n = O_n \setminus F_n$ open and ∂X_n denotes its boundary. Further $T = \bigcap O_n = \bigcup F_n$ and $\bigcap X_n = \emptyset$.

$$\text{Then } \chi_T = \chi_{C_1} + \sum_{n=1}^{\infty} (\chi_{C_{n+1}} - \chi_{C_n}) = \chi_{C_1} + \sum_{n=1}^{\infty} (\chi_{C_{n+1} \setminus C_n} - \chi_{C_n \setminus C_{n+1}})$$

For each term in this infinite sum we construct a primitive function. Write C_1 as a disjoint union of spheres $B(a_i, \rho_i)$ and define a primitive function F_{C_1} of χ_{C_1} by $F_{C_1}(t) = (t-a_i)$ if $t \in B(a_i, \rho_i)$ for some index i, and $F_{C_1}(t) = 0$ otherwise.

Write $C_{n+1}\setminus C_n$ as a disjoint union of spheres $B(b_j,r_j)$ such that for each j, $(r_j)^{1/2} \leqslant \min(\frac{1}{n}$, distance of b_j to $\partial X_n)$. This is meaningful since the set $C_{n+1}\setminus C_n$ is contained in X_n. Define a primitive function $F_{C_{n+1}\setminus C_n}$ of $\chi_{C_{n+1}\setminus C_n}$ by $F_{C_{n+1}\setminus C_n}(t) = 0$ if $t \notin C_{n+1}\setminus C_n$ and $F_{C_{n+1}\setminus C_n}(t) = (t-b_j)$ if $t \in B(b_j,r_j)$. We construct in the same way $F_{C_n\setminus C_{n+1}}$, here also $C_n\setminus C_{n+1} \subset X_n$. We claim that $F = F_{C_1} + \sum_{n=1}^{\infty} (F_{C_{n+1}\setminus C_n} - F_{C_n\setminus C_{n+1}})$ has the required properties.

First of all this sum is uniformly convergent since $\|F_{C_{n+1}\setminus C_n}\| \leqslant \frac{1}{n^2}$ and $\|F_{C_n\setminus C_{n+1}}\| \leqslant \frac{1}{n^2}$. The inequality

$|F(y) - F(x) - (y-x)\chi_T(x)| \leqslant |y-x|$ follows directly from the inequality $|(y-a)\chi_B(y) - (x-a)\chi_B(x) - (y-x)\chi_B(x)| \leqslant |y-x|$ where $B = B(a,\rho)$ is any sphere.

We want to show $F'(t) = \chi_T(t)$. Let $t \notin X_k$ then this is equivalent to $G = \sum_{n\geqslant k}(F_{C_{n+1}\setminus C_n} - F_{C_n\setminus C_{n+1}})$ satisfies $G'(t) = 0$. We consider two cases :

(a) $t \notin \sum_{n=1}^{\infty} X_n$. Then $t \notin \overline{X}_t$ for some k and, for small h also, $t+h \notin X_k$. Since G has support in X_k one has $G(t+h) = G(t) = 0$.

(b) $t \in \sum_{n=1}^{\infty} \overline{X}_n$, then $t \in \bigcap_{n\geqslant k} \partial X_n$ and $t \notin X_k$ for some k.

Choose h with $t+h \in X$. Then $\frac{1}{h}(G(t+h) - G(t)) = \frac{1}{h}G(t+h)$ since G has support in X_k. If for some $n \geqslant k$ the term $F_{C_{n+1}\setminus C_n}(t+h) \neq 0$ then $t+h \in C_{n+1}\setminus C_n$ and so $t+h \in B(b_i,r_i)$ with $r_i^2 \leqslant d(b_i, \partial X_n) \leqslant |h|$.
Hence $|F_{C_{n+1}\setminus C_n}(t+h)| \leqslant |h|^2$. The same reasoning holds for $F_{C_n\setminus C_{n+1}}(t+h)$ and we find $\frac{1}{h}G(t+h)| \leqslant |h|$. Hence $\lim_{h\to 0} \frac{1}{h}G(t+h) = 0$.

<u>Conclusion of the proof of</u> (7.12). Let $f \in R$ then $f = \sum_{i\in I} \lambda_i \chi_i$ with $\lim \lambda_i = 0$ and $\{\chi_i | i \in I\}$ an orthonormal base of R consisting of characteristic functions. For each χ_i there exists according to (7.16) a primitive function F_i such that

$|F_i(y) - F_i(x) - (y-x)\chi_i(x)| \leqslant |y-x|$. Then $F = \sum \lambda_i F_i$ satisfies $F'(x) = f(x)$ for all $x \in X$. Indeed , take $\varepsilon > 0$ and put $I(\varepsilon) = \{i \in I \mid |\lambda_i| > \varepsilon\}$. Since $I(\varepsilon)$ is finite there exists $\delta > 0$ such that for all $|y-x| \leqslant \delta$ and $i \in I(\varepsilon)$ the inequality $|\lambda_i(\frac{1}{y-x}(F_i(y) - F_i(x)) - \chi_i(x))| \leqslant \varepsilon$ holds.

Then $| \frac{1}{y-x}(F(y) - F(x - f(x)| \leqslant \varepsilon$ for $|y-x| \leqslant \delta$.

Problem. Does (7.12) remain valid if the condition f(X) is compact is omitted ?

BIBLIOGRAPHIE

[1] GRUSON, L. Théorie de Fredholm p-adique. Bull. Soc. math. France t. 94, 1966, p. 67-96.

[2] GRUSON, L. Catégories d'espaces de Banach ultramétriques. Bull. Soc. math. France, 94, 1966, p. 287-299.

[3] KAPLANSKY, I. Maximal fields with valuations. Duke. J., vol. 9, 1942, p. 303-321.

[4] KAPLANSKY, I. The Weierstrass theorem fields with valuations. Proc. Am. math. Soc. 1950, p. 356-357.

[5] VAN DER PUT, M. Espaces de Banach non Archimédiens. Bull. Soc. math. France t. 97, 1969, p. 309-320.

[6] VAN DER PUT, M. Reflexive non-Archimedean Banach spaces. Indag. Math., 32, No. 4, 1970, 341-352.

[7] TREIBER, D. Beiträge zur nicht-archimedischen Funktional-analysis. Dissertation an der Universität Köln, 1971.

[8] HELLER, A. Homological algebra in abelian categories. Annals of Math. Series 2, t. 68, 1958, p. 484-525.

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