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FROBENIUS' THEOREM FOR DIFFERENTIAL FORMS ON ANALYTIC SPACES

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Let be $A = \mathbb{C}\{Z\}_{k/\mathfrak{a}}$, $B = \mathbb{C}\{X\}_{m/\mathfrak{b}}$, $C = \mathbb{C}\{Y\}_{n/\mathfrak{c}}$, where $\mathbb{C}\{.\}$ is the ring of all convergent power series. The images of Z_χ, X_μ, Y_ν in A, B, C will be denoted by z_χ, x_μ, y_ν. $\Omega^1(A)$ is the finite differential module of A over $\mathbb{C}$. The module $\overset{r}{\Lambda}\,\Omega^1(A)$ of all differential forms of degree r is denoted by $\Omega^r(A)$. $d : \Omega^r(A) \to \Omega^{r+1}(A)$ is the natural derivation.

The rings B, C are called a decomposition of A if A is the analytic tensor product of B and C :

$$A = B \hat{\otimes} C = \mathbb{C}\{X,Y\}/\mathbb{C}\{X,Y\}.(\mathfrak{b}, \mathfrak{c}).$$

Let M_o, N_o be germs of complex analytic varieties and let $B = \mathcal{O}(M_o)$, $C = \mathcal{O}(N_o)$ be the (reduced) structure rings of the germs, then we have

$$B \hat{\otimes} C = \mathcal{O}(M_o \times N_o).$$

If B, C are a decomposition of A, the module $\Omega^r(A)$ is a direct sum :

$$\Omega^r(A) = \sum_{p+q=r} \Omega^{p,q}(B,C),$$

where $\Omega^{p,q}(B,C)$ is the module generated by

$$\{dx_{\mu_1} \wedge \ldots \wedge dx_{\mu_p} \wedge dy_{\nu_1} \wedge \ldots \wedge dy_{\nu_q} \ : \ 1 \leqslant \mu_1 < \ldots < \mu_p \leqslant m, 1 \leqslant \nu_1 < \ldots < \nu_q \leqslant n\}.$$

We have $d = d_B + d_C$, where d_B, d_C are the derivations relatively to B resp. C, and we have a differential sequence

$$0 \longrightarrow \Omega^p(B) \xrightarrow{\varepsilon} \Omega^{p,o}(B,C) \xrightarrow{d_C} \Omega^{p,1}(B,C) \longrightarrow \ldots,$$

where ε is the natural injection. If C is contractible, then this sequence is exact. Il A is reduced, the sequence

$$0 \to B \to A \to \Omega^{o,1}(B,C)$$

is exact, and we have

$$B = d^{-1}(\Omega^{1,o}(B,C)), C = d^{-1}(\Omega^{o,1}(B,C)).$$

In this case the sum $\Omega^1(A) = \Omega^{1,o}(B,C) + \Omega^{o,1}(B,C)$ determines the rings B, C.

We now will study the following problem : Given a direct sum $\Omega^1(A) = \Omega'+\Omega''$, can it be obtained from a decomposition of A ?

THEOREM 1. Let A be a domain and let the summands Ω' , Ω'' of the direct sum $\Omega^1(A) = \Omega'+\Omega''$ be generated by elements df , $f \in A$. Then there are rings B , C such that $A = B \hat{\otimes} C$, $\Omega' = \Omega^{1,0}(B,C), \Omega'' = \Omega^{0,1}(B,C)$.-

Proof. We have

$$A = \mathbb{C}\{X,Y\}/\alpha, \Omega' = A.(dx_1,\dots,dx_m),$$
$$\Omega'' = A.(dy_1,\dots,dy_n).$$

If A is regular, then there is an isomorphism $\varphi : A \to R$, where R is a ring of power series. By φ we have an isomorphism $\varphi^1 : \Omega^1(A) \to \Omega^1(R)$. We may suppose, that

$$R = \mathbb{C}\{U,V\}, \varphi^1(\Omega') = R.(dU_1,\dots,dU_p),$$
$$\varphi^1(\Omega'') = R.(dV_1,\dots,dV_q).$$

If φ is given by the substitution of $\Phi = (\Phi',\Phi'')$, $\Phi' = (\Phi'_1,\dots,\Phi'_m)$, $\Phi'' = (\Phi''_1,\dots,\Phi''_n)$, we have $d\Phi'_\mu = d\varphi(x_\mu) \in R.(dU_1,\dots,dU_p)$, $\Phi'_\mu \in \mathbb{C}\{U\}$ and $\Phi''_\nu \in \mathbb{C}\{V\}$. Then Φ' , Φ'' are biholomorphic mappings of the germs $\mathbb{C}^p_o, \mathbb{C}^q_o$ onto germs $M_o \subset \mathbb{C}^m_o$, $N_o \subset \mathbb{C}^n_o$. We have $A = \mathcal{O}(M_o \times N_o)$.

In the general case A is the structure ring of an irreducible germ K_o . Let K represent K_o in an open neighbourhood $W = W' \times W''$, $W' \subset \mathbb{C}^m$, $W'' \subset \mathbb{C}^n$, of 0 . We use the following notations.

$\mathcal{O}$, $\tilde{\mathcal{O}}$ are the structure sheaves of W resp. K , $\mathfrak{J}$ is the ideal sheaf of K , $\tilde{\Omega}^1$ is the sheaf of differential forms of degree 1 on K. We set $\tilde{\Omega}' := \tilde{\mathcal{O}}.(dx_1,\dots,dx_m)$, $\Omega'' := \tilde{\mathcal{O}}.(dy_1,\dots,dy_n)$.

We may suppose, that the sum $\tilde{\Omega}^1 = \tilde{\Omega}' + \tilde{\Omega}''$ is direct and that $\mathfrak{J}$ is generated by holomorphic functions $h_1,\dots,h_t$ on U .

If $w^o \in K$ is a regular point, we have

$$\mathfrak{J}_{w^o} = \mathcal{O}_{w^o}.(h_1,\dots,h_t) = \mathcal{O}_{w^o}.(f_1,\dots,f_r,g_1,\dots,g_s) , f_\varrho \in \mathbb{C}\{X\}, g_\sigma \in \mathbb{C}\{Y\} .$$

Setting

$$M := \{w' \in W' : h_\tau(w',w^o_{m+1},\dots,w^o_{m+n}) = 0 , \tau = 1,\dots,t\},$$
$$N := \{w'' \in W'' : h_\tau(w^o_1,\dots,w^o_m,w'') = 0 , \tau = 1,\dots,t\}$$

we get $K_{w^o} = (M \times N)_{w^o}$. Then K_o must be an irreducible component of $(M\times N)_o$.-

STORCH has given an algebraic proof for theorem 1 ([3]). By STORCH's proof theorem 1 is valid in the complete case too.

If A is regular, the theorem of FROBENIUS gives a condition for Ω' being generated by elements df , $f \in A$:

Let A be regular and let $\Omega^1(A) = \Omega' + \Omega''$ be a direct sum. Then we have : Ω' is generated by elements df , $f \in A$ iff $d\Omega' \subset \Omega^1 \wedge \Omega'$.–

In the singular case we have :

THEOREM 2. Let A and the direct sum $\Omega^1(A) = \Omega' + \Omega''$ satisfy the following conditions : A is a domain, $\Omega^1(A)$ is torsionless, there is a contraction vector field v on A such that $v(\Omega') = 0$, emdim $A/v(\Omega^1(A)) = \dim \Omega'/\mathfrak{m}\Omega'$. Then we have :
Ω' is generated by elements df , $f \in A$ iff $d\Omega' \subset \Omega^1 \wedge \Omega'$.–

An A-module M is called torsionless if the natural mapping $M \to M^{**}$ (M^{**} bidual module) is injective. For a reduced complete intersection the following are equivalent :

(i) $\Omega^1(A)$ is torsionless.

(ii) The codimension of the singular locus of A is ≥ 2 .

(iii) A is normal.

A contraction vector field v on A is a vector field on A , which in an appropriate coordinate system $Z_1,\ldots,Z_k$ can be represented by a vector field $\sum m_\varkappa Z_\varkappa \, \partial/\partial Z_\varkappa$, $m_\varkappa \geq 0$ integer. For the embedding dimension of $A_v := A/v(\Omega^1(A))$ we have the formula emdim A_v = emdim A – rank dv , where dv is the linear mapping in the tangent space given by the matrix

$$\begin{bmatrix} m_1 & 0 & \ldots & 0 \\ & \cdot & & \\ & & \cdot & \\ 0 & \ldots & 0 & m_k \end{bmatrix} .$$

For the proof of theorem 2, see [2].

We give an application ([1], [2]).

The analytic ring $A = \mathbb{C}\{U\}/\mathfrak{a}$ is called real if we have $\bar{f} := \sum \bar{a}_\alpha U^\alpha \in \mathfrak{a}$ for all $f = \sum a_\alpha U^\alpha \in \mathfrak{a}$. The morphisms in the category of real analytic rings are given by substitutions of real power series.

If $K_0 \subset \mathbb{R}^k_0$ is the germ of a real analytic variety, the ring $A = \mathcal{R}(K_0)$ of all germs of complex-valued real analytic functions on K_0 is a real analytic ring. We have $A = \mathcal{O}(\tilde{K}_0)$, where $\tilde{K}_0$ is the complexification of K_0 .

A direct sum $\Omega^1(A) = \Omega' + \Omega''$ is called an almost holomorphic structure on the real analytic ring A if we have $\overline{\Omega'} = \Omega''$. The quasi-local ring $H(A) := d^{-1}(\Omega')$ is called the ring of almost holomorphic functions. In general $H(A)$ is no analytic ring.

The germ $K_o \subset \mathbb{K}^k_o$ ($\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$) of a $\mathbb{K}$-analytic variety is called a cone if there is a coordinate system such that the ideal of K_o in this coordinate system is homogeneous of a type $(m_1,\dots,m_\ell)$, $m_\lambda > 0$.

We have ([2]) :

Let $K_o \subset \mathbb{C}^k_o$ be an irreducible germ of a complex analytic variety with an isolated singularity. Then K_o is a complex cone iff K_o is a real cone.-

Hereby and by theorem 2 we have ([2]) :

THEOREM 3. Let $K_o \subset \mathbb{R}^k_o$ be an irreducible real cone with an isolated singularity and let $\Omega^1(A)$, $A := \mathcal{R}(K_o)$, be torsionless. Then for an almost holomorphic structure $\Omega^1(A) = \Omega' + \Omega''$ the following are equivalent :

(i) K_o is complex analytic with holomorphic structure ring $H(A)$.

(ii) We have $d\Omega' \subset \Omega^1 \wedge \Omega'$, and there is a contraction vector field v on A such that $v(\Omega') = 0$, rank $dv = 1/2$ emdim A.-

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