Bulletin
de la SOCIÉTÉ MATHÉMATIQUE DE FRANCE

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Tome 137
Fascicule 2

2009

SOCIÉTÉ MATHÉMATIQUE DE FRANCE
Publié avec le concours du Centre national de la recherche scienti que
pages 185-223

Bull. Soc. math. France
137 (2), 2009, p. 185–223

QUASI-SEMI-STABLE REPRESENTATIONS

BY XAVIER CARUSO & TONG LIU

ABSTRACT. — Fix K a p-adic field and denote by G_K its absolute Galois group. Let K_∞ be the extension of K obtained by adding p^n-th roots of a fixed uniformizer, and $G_\infty \subset G_K$ its absolute Galois group. In this article, we define a class of p-adic torsion representations of G_∞, called *quasi-semi-stable*. We prove that these representations are "explicitly" described by a certain category of linear algebraic objects. The results of this note should be considered as a first step in the understanding of the structure of quotient of two lattices in a crystalline (resp. semi-stable) Galois representation.

RÉSUMÉ (*Représentations quasi-semi-stables*). — Soient K un corps p-adique et G_K son groupe de Galois absolu. Soit K_∞ l'extension de K obtenue en ajoutant les racines p^n-ièmes d'une uniformisante fixée. Notons $G_\infty \subset G_K$ le groupe de Galois absolu de K_∞. Dans cet article, on définit une classe de représentations p-adiques de torsion du groupe G_∞, que l'on appelle *quasi-semi-stables*. Nous montrons que ces représentations sont « explicitement » décrites *via* une certaine catégories d'objets d'algèbre linéaire. Les résultats dans cette note doivent être considérés comme une première étape dans l'étude de la structure des représentations qui apparaissent comme quotients de deux réseaux d'une représentation galoisienne cristalline (resp. semi-stable).

Texte reçu le 17 septembre 2007, révisé le 23 mai 2008 et le 15 octobre 2008, accepté le 21 octobre 2008

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2000 Mathematics Subject Classification. — 11F85, 11S20, 11S23.

Key words and phrases. — Torsion Galois representations, semi-stable representations, norm field theory.

Introduction

Let p be a prime number and k a perfect field of characteristic p. Put $W = W(k)$, the ring of Witt vectors with coefficients in k, and $K_0 = \operatorname{Frac} W$. Denote by σ the Frobenius on k, W and K_0. Let K be a totally ramified extension of K_0 of degree e and $\mathcal{O}_K$ its ring of integers. Fix π an uniformizer of $\mathcal{O}_K$. We denote by $\bar{K}$ an algebraic closure of K, by $\mathcal{O}_{\bar{K}}$ its ring of integers and by G_K its absolute Galois group. Fix a sequence (π_n) of elements of $\bar{K}$ satisfying $\pi_0 = \pi$ and $\pi_{n+1}^p = \pi_n$. Put $K_n = K(\pi_n)$, $K_\infty = \bigcup_{n\in\mathbb{N}} K_n$ and denote by $G_\infty \subset G_K$ the absolute Galois group of K_∞.

We wish to study representations which can be written as a quotient of two lattices in a crystalline or semi-stable representation. For technical reasons we have to make an assumption on Hodge-Tate weights, namely, they all belong to $\{0,\dots,r\}$ for a nonnegative integer $r < p-1$. The theory of Breuil modules then gives a description of these lattices in term of linear algebra: there exists a category $\operatorname{Mod}_{/S}^{r,\phi,N}$ that is dually equivalent to the category whose objects are these lattices. By mimicking the definition of $\operatorname{Mod}_{/S}^{r,\phi,N}$, one can construct a category of torsion objects $\operatorname{Mod}_{/S_\infty}^{r,\phi,N}$ equipped with a contravariant functor, T_{st}, which takes values in the category of Galois representations. When $er < p-1$, we can prove that $\operatorname{Mod}_{/S_\infty}^{r,\phi,N}$ is an abelian category and T_{st} is fully faithful (see [7]). However, these assertions are false if the assumption $er < p-1$ is removed. In this article, we draw a picture of this structure in a slightly different situation. More precisely, we remove the operator N (that appears in the subscript $\operatorname{Mod}_{/S}^{r,\phi,N}$) and study a new category $\operatorname{Mod}_{/S}^{r,\phi}$. It is endowed with a functor T_{qst} with values in a certain category of G_∞-representations, that we call *quasi-semi-stable*. We define a full subcategory $\operatorname{Max}_{/S_\infty}^{r,\phi}$ and a functor $\operatorname{Max}^r : \operatorname{Mod}_{/S_\infty}^{r,\phi} \to \operatorname{Max}_{/S_\infty}^{r,\phi}$, which is a retraction (and a left adjoint) of the natural inclusion $\operatorname{Max}_{/S_\infty}^{r,\phi} \hookrightarrow \operatorname{Mod}_{/S_\infty}^{r,\phi}$ and which commutes with T_{qst}. We then prove the following (see Theorem 3.7.1 for a more complete statement).

THEOREM 1. — *The category* $\operatorname{Max}_{/S_\infty}^{r,\phi}$ *is abelian and artinian. Moreover, the restriction of* T_{qst} *to* $\operatorname{Max}_{/S_\infty}^{r,\phi}$ *is exact and fully faithful.*

Of course, using duality, we can define the category $\operatorname{Min}_{/S_\infty}^{r,\phi}$ and the functor $\operatorname{Min}^r : \operatorname{Mod}_{/S_\infty}^{r,\phi} \to \operatorname{Max}_{/S_\infty}^{r,\phi}$; they satisfy analogous properties as those stated in Theorem 1. In § 3.6, assuming k to be algebraically closed, we also provide a complete description of simple objects of $\operatorname{Max}_{/S_\infty}^{r,\phi}$, and by duality of $\operatorname{Min}_{/S_\infty}^{r,\phi}$.

If $r = 1$, quasi-semi-stable representations are linked with geometry. In this case, the category $\operatorname{Mod}_{/S_\infty}^{r,\phi}$ is dually equivalent to the category of finite flat

group schemes over $\mathcal{O}_K$ killed by a power of p (see [3]). Under this equivalence, the functor Min^r (resp. Max^r) corresponds to the maximal (resp. minimal) models defined by Raynaud in [15]. The following result is then a direct consequence of Theorem 1.

THEOREM 2. — *The category of minimal (resp. maximal) finite flat group schemes over $\mathcal{O}_K$ killed by a power of p is abelian.*

Finally, in the case $r = 1$, we can derive from our results a new proof of the following theorem.

THEOREM 3. — *Let $\mathcal{G}$ and $\mathcal{G}'$ be two finite flat group schemes over $\mathcal{O}_K$ killed by a power of p. Put $T = \mathcal{G}(\bar{K})$, $T' = \mathcal{G}'(\bar{K})$ and consider $f : T \to T'$ a G_∞-equivariant map. Then f is G_K-equivariant.*

Unfortunately, if $r > 1$, quasi-semi-stable representations no longer have a geometric interpretation. Then it is difficult to derive concrete results from Theorem 1 in general. Actually, Theorem 1 should be seen as a preliminary study of the more interesting category $\mathrm{Mod}^{r,\phi,N}_{/S_\infty}$; a first part of this work is achieved in [8].

Now, we detail the content of the article. First, we recall definitions of categories of Breuil modules. This allows us to explain more precisely and more clearly our motivations and results. In the second section, we introduce the category $\mathrm{Mod}^{r,\phi}_{/\mathfrak{S}_\infty}$ and we prove that it is equivalent to the category $\mathrm{Mod}^{r,\phi}_{/S_\infty}$. This result is interesting because it will be easier to work with objects of $\mathrm{Mod}^{r,\phi}_{/\mathfrak{S}_\infty}$. Section 3 is devoted to the study of the structure of $\mathrm{Mod}^{r,\phi}_{/\mathfrak{S}_\infty} = \mathrm{Mod}^{r,\phi}_{/S_\infty}$: essentially we give a proof of Theorem 1. Then, we assume $r = 1$ and show how the previous results easily imply Theorem 3. The paper ends with some perspectives and open questions.

1. Motivations and settings

In the rest of the paper, we will make an intensive use of Breuil modules, so we gather below all basic definitions about it. The reader may skip it in a first time and come back after when objects are really used.

1.1. Breuil modules. — *Fix a nonnegative integer $r < p-1$.* Recall that π is a fixed uniformizer. Denote by S the p-adic completion of the PD-envelope of $W[u]$ with respect to the kernel of the surjection $W[u] \to \mathcal{O}_K$, $u \mapsto \pi$ (and compatible with the canonical divided powers on $pW[u]$). This ideal is principal generated by $E(u)$, the minimal polynomial of π over K_0. The ring S is endowed with the canonical filtration associated to the PD-envelope and with two endomorphisms:

- a Frobenius ϕ: it is the unique continuous map σ-semi-linear which sends u to u^p
- a monodromy operator N: it is the unique continuous map W-linear that sends u to $-u$ and satisfies $N(xy) = N(x)y + xN(y)$ for all x and y in S (Leibniz rule).

They satisfy $N\phi = p\phi N$. We have $\phi(\mathrm{Fil}^r S) \subset p^r S$ (recall $r < p-1$) and we define $\phi_r = \frac{\phi}{p^r} : \mathrm{Fil}^r S \to S$. Set $c = \phi_1(E(u))$: it is a unit in S.

First, we define a "big" category $\text{`Mod}_{/S}^{r,\phi,N}$ whose objects are the following data:

1. a S-module $\mathcal{M}$;
2. a submodule $\mathrm{Fil}^r\mathcal{M} \subset \mathcal{M}$ such that $\mathrm{Fil}^r S\,\mathcal{M} \subset \mathrm{Fil}^r\mathcal{M}$;
3. a ϕ-semi-linear map $\phi_r : \mathrm{Fil}^r\mathcal{M} \to \mathcal{M}$;
4. a W-linear map $N : \mathcal{M} \to \mathcal{M}$ such that:
   - (Leibniz condition) $N(sx) = sN(x) + N(s)x$ for all $s \in S$, $x \in \mathcal{M}$
   - (Griffiths transversality) $E(u)N(\mathrm{Fil}^r\mathcal{M}) \subset \mathrm{Fil}^r\mathcal{M}$
   - the following diagram is commutative:

$$\begin{array}{ccc} \mathrm{Fil}^r\mathcal{M} & \xrightarrow{\phi_r} & \mathcal{M} \\ {\scriptstyle E(u)N}\downarrow & & \downarrow{\scriptstyle cN} \\ \mathrm{Fil}^r\mathcal{M} & \xrightarrow{\phi_r} & \mathcal{M} \end{array}$$

Morphisms in $\text{`Mod}_{/S}^{r,\phi,N}$ are $S$-linear maps compatible with $\mathrm{Fil}^r$, $\phi_r$ and $N$. There exists in $\text{`Mod}_{/S}^{r,\phi,N}$ a notion of exact sequence: a sequence $0 \to \mathcal{M}' \to \mathcal{M} \to \mathcal{M}'' \to 0$ is said exact if both sequences $0 \to \mathcal{M}' \to \mathcal{M} \to \mathcal{M}'' \to 0$ and $0 \to \mathrm{Fil}^r\mathcal{M}' \to \mathrm{Fil}^r\mathcal{M} \to \mathrm{Fil}^r\mathcal{M}'' \to 0$ are exact as sequences of S-modules.

Now, we are ready to define full subcategories of $\text{`Mod}_{/S}^{r,\phi,N}$. The first one is the category of *strongly divisible modules*, denoted by $\mathrm{Mod}_{/S}^{r,\phi,N}$: it consists of objects $\mathcal{M} \in \text{`Mod}_{/S}^{r,\phi,N}$ satisfying the following conditions:

- the module $\mathcal{M}$ is free of finite rank over S;
- the quotient $\mathcal{M}/\mathrm{Fil}^r\mathcal{M}$ has no p-torsion;
- the image of ϕ_r generates $\mathcal{M}$ (as an S-module).

The second category is $\mathrm{Mod}^{r,\phi,N}_{/S_1}$: these objects are the $\mathcal{M} \in \text{`}\mathrm{Mod}^{r,\phi,N}_{/S}$ such that

- the module $\mathcal{M}$ is free of finite rank over $S_1 = S/pS$;
- the image of ϕ_r generates $\mathcal{M}$ (as an S-module).

Finally, let $\mathrm{Mod}^{r,\phi,N}_{/S_\infty}$ be the smallest subcategory of $\text{`}\mathrm{Mod}^{r,\phi,N}_{/S}$ containing $\mathrm{Mod}^{r,\phi,N}_{/S_1}$ and stable under extensions (*i.e.* if $0 \to \mathcal{M}' \to \mathcal{M} \to \mathcal{M}'' \to 0$ is an exact sequence in $\text{`}\mathrm{Mod}^{r,\phi,N}_{/S}$ and if $\mathcal{M}'$ and $\mathcal{M}''$ are objects of $\mathrm{Mod}^{r,\phi,N}_{/S_\infty}$, then $\mathcal{M}$ is also).

The three former categories are equipped with a contravariant functor T_{st} which takes values in the category of $\mathbb{Z}_p$-representations of G_K. On $\mathrm{Mod}^{r,\phi,N}_{/S}$, it is defined by the formula

$$T_{\mathrm{st}}(\mathcal{M}) = \mathrm{Hom}_{\text{`}\mathrm{Mod}^{r,\phi,N}_{/S}}(\mathcal{M}, \hat{A}_{\mathrm{st}})$$

where $\hat{A}_{\mathrm{st}}$ is a certain period ring, object of $\text{`}\mathrm{Mod}^{r,\phi,N}_{/S}$ endowed with an action of G_K. We refer to [1], § 3.1.1 for the precise definition of $\hat{A}_{\mathrm{st}}$. On the category $\mathrm{Mod}^{r,\phi,N}_{/S_\infty}$ it is defined by

$$T_{\mathrm{st}}(\mathcal{M}) = \mathrm{Hom}_{\text{`}\mathrm{Mod}^{r,\phi,N}_{/S}}(\mathcal{M}, \hat{A}_{\mathrm{st}} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p/\mathbb{Z}_p).$$

We define similarly $\text{`}\mathrm{Mod}^{r,\phi}_{/S}$, $\mathrm{Mod}^{r,\phi}_{/S}$, $\mathrm{Mod}^{r,\phi}_{/S_1}$ and $\mathrm{Mod}^{r,\phi}_{/S_\infty}$ by forgetting the operator N. The three latest categories are equipped with a functor T_{qst} with values in the category of $\mathbb{Z}_p$-representations of G_∞ [(1)] (defined in the introduction): definitions are obtained by replacing the period ring $\hat{A}_{\mathrm{st}}$ by A_{cris}. We have a collection of forgetful functors, and if $\mathcal{M}$ is an object of $\mathrm{Mod}^{r,\phi,N}_{/S}$ (resp. $\mathrm{Mod}^{r,\phi,N}_{/S_\infty}$), there is a canonical and functorial G_∞-equivariant isomorphism

$$T_{\mathrm{st}}(\mathcal{M}) \simeq T_{\mathrm{qst}}(\mathcal{M}) \tag{1}$$

(see Lemma 2.3.1.1 of [2]).

1.2. Aim of the paper. — Semi-stable $\mathbb{Q}_p$-representations of G_K are classified by (weakly) admissible filtered (φ, N)-modules (see [10]). Our motivation is to find a description of quotients of two lattices in such representations, in term of some linear algebraic data. If Hodge-Tate weights of the semi-stable representations are in $\{0, \dots, r\}$, such a description exists for lattices (stable by G_K):

(1) $T_{\mathrm{qst}}(\mathcal{M})$ is not endowed with an action of G_K since this group does not act trivially on $u \in A_{\mathrm{cris}}$.

THEOREM 1.2.1 (Liu, [14]). — *The functor T_{st} from $\mathrm{Mod}^{r,\phi,N}_{/S}$ to the category of lattices in semi-stable representations with Hodge-Tate weights in $\{0,\dots,r\}$ is an anti-equivalence.*

Furthermore, we have the following lemma:

LEMMA 1.2.2. — *Let $\mathcal{M}'\subset\mathcal{M}$ be two strongly divisible modules such that $\mathcal{M}'\otimes_{\mathbb{Z}_p}\mathbb{Q}_p\simeq\mathcal{M}\otimes_{\mathbb{Z}_p}\mathbb{Q}_p$ and $\mathrm{Fil}^r\mathcal{M}'\otimes_{\mathbb{Z}_p}\mathbb{Q}_p\simeq\mathrm{Fil}^r\mathcal{M}\otimes_{\mathbb{Z}_p}\mathbb{Q}_p$. Then $\mathcal{M}/\mathcal{M}'$ is an object of $\text{`}\mathrm{Mod}^{r,\phi,N}_{/S}$ and the following sequence of G_K-representations:*

$$0\to T_{\mathrm{st}}(\mathcal{M})\to T_{\mathrm{st}}(\mathcal{M}')\to \mathrm{Hom}_{\text{`}\mathrm{Mod}^{r,\phi,N}_{/S}}(\mathcal{M}/\mathcal{M}',\hat{A}_{\mathrm{st}}\otimes_{\mathbb{Z}_p}\mathbb{Q}_p/\mathbb{Z}_p)\to 0$$

is exact.

Proof. — The argument is the same as in Lemma V.4.2.4 of [6]. □

Based on the above results, we can draw a plan to study our representations:

1. recognize objects in $\text{`}\mathrm{Mod}^{r,\phi,N}_{/S}$ that can be written as a quotient of two divisible modules as in Lemma 1.2.2;
2. study the functor $\mathrm{Hom}_{\text{`}\mathrm{Mod}^{r,\phi,N}_{/S}}(—,\hat{A}_{\mathrm{st}}\otimes_{\mathbb{Z}_p}\mathbb{Q}_p/\mathbb{Z}_p)$ on this subcategory.

The aim of this article is to explain how the previous plan can be achieved for categories $\text{`}\mathrm{Mod}^{r,\phi}_{/S}$ and $\mathrm{Mod}^{r,\phi}_{/S}$ (instead of $\text{`}\mathrm{Mod}^{r,\phi,N}_{/S}$ and $\mathrm{Mod}^{r,\phi,N}_{/S}$). Precisely we prove that the category of torsion quotients of two objects of $\mathrm{Mod}^{r,\phi}_{/S}$ is exactly the category $\mathrm{Mod}^{r,\phi}_{/S_\infty}$, and then Theorem 1.

We can imagine that a representation arising from an object of $\mathrm{Mod}^{r,\phi}_{/S}$ should be just a lattice in a crystalline representation, but unfortunately the situation is much more complicated. Lattices in crystalline representations correspond to objects of $\mathrm{Mod}^{r,\phi,N}_{/S}$ for which $N(\mathcal{M})\subset(uS+\mathrm{Fil}^1S)\mathcal{M}$. Let's call $\mathrm{Mod}^{r,\phi,(N)}_{/S}$ their subcategory. One can easily prove that a N satisfying the above condition is necessary unique. However, the following lemma shows that it does not exist in general.

LEMMA 1.2.3. — *Assume $r\geqslant 2$ and consider $\mathcal{M}$ the object of $\mathrm{Mod}^{r,\phi}_{/S}$ defined by the following equations:*

1. $\mathcal{M}=Se_1\oplus Se_2$;
2. $\mathrm{Fil}^r\mathcal{M}=E(u)^{r-2}e_1S+E(u)^re_2S+\mathrm{Fil}^pS\,\mathcal{M}$;
3. $\phi(e_1)=p^2(e_1+ue_2)$ *and* $\phi(e_2)=ue_1+e_2$.

Then it is impossible to equip $\mathcal{M}$ with a monodromy operator N.

Proof. — For simplicity, we assume $e > 1$ (the proof is little more technical when $e = 1$ and is left to the reader in this case). Assume by contradiction that such an $N : \mathcal{M} \to \mathcal{M}$ exists. Put $x_1 = N(e_1)$ and $x_2 = N(e_2)$. The relation $N\phi = p\phi N$ implies the following equalities:

$$(S) : \begin{cases} px_1 + pux_2 = \phi(x_1) + pue_2 \\ ux_1 + x_2 = p\phi(x_2) + ue_1. \end{cases}$$

For all integer n, denote by J_n the topological closure of the ideal of S generated by all $\frac{u^i}{q(i)!}$ for $i \geqslant n$. Here $q(i)$ is the quotient in the Euclidean division of i by e. The first equation of (S) implies $\phi(x_1) \equiv px_1 \pmod{J_1}$. Since $S/J_1 \simeq W$, this congruence proves that $x_1 \in J_1\mathcal{M}$ and then $\phi(x_1) \in \phi(J_1) \subset J_p$. By the same way, it follows from the second equation of (S) that $x_2 \equiv p\phi(x_2) \pmod{J_1}$, and then $\phi(x_2) \in J_p$. Resolving (S), we get:

$$x_1 \equiv -\frac{u^2}{1-u^2}e_1 + \frac{u}{1-u^2}e_2 \pmod{J_p\mathcal{M}}$$

which gives $\phi(x_1) \equiv u^p e_2 \pmod{J_{p+1}\mathcal{M}}$. Hence, $\phi(x_1)$ is not divisible by p in S (here, we use $e > 1$). But, on the other hand, the first equation of (S) shows directly that $\phi(x_1)$ have to be divisible by p. This is a contradiction. □

Briefly, we have an inclusion $\mathrm{Mod}_{/S}^{r,\phi,(N)} \subset \mathrm{Mod}_{/S}^{r,\phi}$ but it is always strict if $r > 1$. We call G_∞-representations arising from objects of $\mathrm{Mod}_{/S}^{r,\phi}$ *quasi-semi-stable* representations. Note that if V is a lattice in a semi-stable representation of G_K, its restriction to G_∞ is quasi-semi-stable [(2)].

2. The category $\mathrm{Mod}_{/\mathfrak{S}_\infty}^{r,\phi}$

The case of quasi-semi-stable representations is simpler because we may use an alternative category (defined by Breuil and studied by Kisin) to describe them. In this section, we give definitions and basic properties of this category and we prove that it is equivalent to the category of Breuil modules.

2.1. Definitions and basic properties. — When dealing with $\mathrm{Mod}_{/\mathfrak{S}_\infty}^{r,\phi}$, we may relax the condition $r < p-1$ and assume only $r \in \{0, 1, 2, 3, \ldots, \infty\}$. So, from now on, except if the contrary is explicitely mentionned, we work in this more general setting.

(2) The converse is not true in general. In fact, there exists a full subcategory of $\mathrm{Mod}_{/S}^{r,\phi}$, whose objects are called *quasi-strongly divisible lattices*, which is anti-equivalent to the category of G_∞-lattices in semi-stable representations. See [14] for details.

Linear algebraic objects. — Set $\mathfrak{S} = W[[u]]$ and endow it with a Frobenius $\phi : \mathfrak{S} \to \mathfrak{S}$ defined by:

$$\phi\left(\sum_{n=0}^{\infty} a_n u^n\right) = \sum_{n=0}^{\infty} \sigma(a_n) u^{pn}.$$

Put $\mathfrak{S}_1 = \mathfrak{S}/p\mathfrak{S} = k[[u]]$. As in § 1.1, we define some categories of modules over $\mathfrak{S}$. First, the "big" category $'\mathrm{Mod}^{r,\phi}_{/\mathfrak{S}}$: if r is finite, its objects are the $\mathfrak{S}$-modules $\mathfrak{M}$ equipped with a ϕ-semi-linear endomorphism $\phi : \mathfrak{M} \to \mathfrak{M}$ such that

$$E(u)^r \mathfrak{M} \subset \langle \mathrm{im}\, \phi \rangle \tag{2}$$

where $\langle \mathrm{im}\, \phi \rangle$ denotes the $\mathfrak{S}$-submodule of $\mathfrak{M}$ generated by the image of ϕ. If $\phi^\star\mathfrak{M} = \mathfrak{S} \otimes_{(\phi),\mathfrak{S}} \mathfrak{M}$, the previous condition is equivalent to ask the cokernel of $\mathrm{id} \otimes \phi : \phi^\star\mathfrak{M} \to \mathfrak{M}$ to be killed by $E(u)^r$. If $r = \infty$, we ask condition (2) for a non fixed integer r: in this way, $'\mathrm{Mod}^{\infty,\phi}_{/\mathfrak{S}}$ is just the union (in an obvious sense) of all categories $'\mathrm{Mod}^{r,\phi}_{/\mathfrak{S}}$ for r finite. Morphisms in $'\mathrm{Mod}^{r,\phi}_{/\mathfrak{S}}$ are just $\mathfrak{S}$-linear morphisms commuting with Frobenius.

Now, we define full subcategories of $'\mathrm{Mod}^{r,\phi}_{/\mathfrak{S}}$. The category $\mathrm{Mod}^{r,\phi}_{/\mathfrak{S}}$ (resp. $\mathrm{Mod}^{r,\phi}_{/\mathfrak{S}_1}$) gathers all objects $\mathfrak{M} \in {}'\mathrm{Mod}^{r,\phi}_{/\mathfrak{S}}$ free of finite rank over $\mathfrak{S}$ (resp. over $\mathfrak{S}_1$), whereas $\mathrm{Mod}^{r,\phi}_{/\mathfrak{S}_\infty}$ is the smallest subcategory of $'\mathrm{Mod}^{r,\phi}_{/\mathfrak{S}}$ containing $\mathrm{Mod}^{r,\phi}_{/\mathfrak{S}_1}$ and stable under extensions [(3)]. For simplicity, we also define the category $'\mathrm{Mod}^{r,\phi}_{/\mathfrak{S}_\infty}$ as the full subcategory of $'\mathrm{Mod}^{r,\phi}_{/\mathfrak{S}}$ gathering all objects killed by a power of p. Obviously $\mathrm{Mod}^{r,\phi}_{/\mathfrak{S}_\infty} \subset {}'\mathrm{Mod}^{r,\phi}_{/\mathfrak{S}_\infty}$. The following proposition summarizes basic properties of these modules.

PROPOSITION 2.1.1. — (i) *Let* $\mathfrak{M} \in \mathrm{Mod}^{r,\phi}_{/\mathfrak{S}_\infty}$. *Then* $\mathrm{id} \otimes \phi : \phi^\star\mathfrak{M} \to \mathfrak{M}$ *is injective.*

(ii) *Let* $\mathfrak{M}$ *be an object of* $'\mathrm{Mod}^{r,\phi}_{/\mathfrak{S}}$. *Then* $\mathfrak{M}$ *is in* $\mathrm{Mod}^{r,\phi}_{/\mathfrak{S}_\infty}$ *if and only if it is of finite type over* $\mathfrak{S}$, *it has no* u*-torsion and it is killed by a power of* p.

(iii) *The category* $\mathrm{Mod}^{r,\phi}_{/\mathfrak{S}_\infty}$ *is stable under kernels and images.*

Proof. — See [13], § 2.3. □

The relation between $\mathrm{Mod}^{r,\phi}_{/\mathfrak{S}_\infty}$ and $'\mathrm{Mod}^{r,\phi}_{/S}$ is given by the functor $M_{\mathfrak{S}_\infty} : \mathrm{Mod}^{r,\phi}_{/\mathfrak{S}_\infty} \to {}'\mathrm{Mod}^{r,\phi}_{/S}$ defined as follows. Let $\mathfrak{M}$ be an object of $\mathrm{Mod}^{r,\phi}_{/\mathfrak{S}_\infty}$. As an S-module, $M_{\mathfrak{S}_\infty}(\mathfrak{M}) = S \otimes_{(\phi),\mathfrak{S}} \mathfrak{M}$ where the subscript "(ϕ)" means that S is considered as a $\mathfrak{S}$-module *via* the composite $\mathfrak{S} \to S \to S$, the first map

(3) An sequence of objects of $'\mathrm{Mod}^{r,\phi}_{/\mathfrak{S}}$ is said exact if it is exact as a sequence of $\mathfrak{S}$-modules.

being the canonical map and the second the Frobenius ϕ. The Frobenius on $\mathfrak{M}$ induces a S-linear map $\mathrm{id}\otimes\phi : \mathcal{M} \to S\otimes_{\mathfrak{S}}\mathfrak{M}$. We then define $\mathrm{Fil}^r\mathcal{M}$ by the formula

$$\mathrm{Fil}^r\mathcal{M} = \{x\in\mathcal{M},\ (\mathrm{id}\otimes\phi)(x)\in \mathrm{Fil}^r S\otimes_{\mathfrak{S}}\mathfrak{M}\subset S\otimes_{\mathfrak{S}}\mathfrak{M})\}\,.$$

The map ϕ_r is given by the following composite:

$$\mathrm{Fil}^r\mathcal{M}\xrightarrow{\mathrm{id}\otimes\phi}\mathrm{Fil}^r S\otimes_{\mathfrak{S}}\mathfrak{M}\xrightarrow{\phi_r\otimes\mathrm{id}}\mathcal{M}.$$

Identical constructions give rise to an other functor $M_{\mathfrak{S}} : \mathrm{Mod}^{r,\phi}_{/\mathfrak{S}} \to \text{`}\mathrm{Mod}^{r,\phi}_{/S}$.

Proposition 2.1.2. — *The functor $M_{\mathfrak{S}_\infty}$ (resp. $M_{\mathfrak{S}}$) takes values in $\mathrm{Mod}^{r,\phi}_{/S_\infty}$ (resp. $\mathrm{Mod}^{r,\phi}_{/S}$). Moreover, both functors are exact and fully faithful.*

Proof. — The case $r=1$ is done in proposition 1.1.11 of [12]. The same proof works for any r. □

Proposition 2.1.3. — *Let $\mathfrak{M}'\subset\mathfrak{M}$ be two objects of $\mathrm{Mod}^{r,\phi}_{/\mathfrak{S}}$ such that $\mathfrak{M}'\otimes_{\mathbb{Z}_p}\mathbb{Q}_p\simeq\mathfrak{M}\otimes_{\mathbb{Z}_p}\mathbb{Q}_p$. Then the quotient $\mathfrak{M}''=\mathfrak{M}/\mathfrak{M}'$ is an object of $\mathrm{Mod}^{r,\phi}_{/\mathfrak{S}_\infty}$. Moreover, the sequence*

$$0\to M_{\mathfrak{S}}(\mathfrak{M}')\to M_{\mathfrak{S}}(\mathfrak{M})\to M_{\mathfrak{S}_\infty}(\mathfrak{M}'')\to 0$$

is exact.

Proof. — The first statement is proved in Proposition 2.3.2 of [13]. For the second one, the proof is the same as for the exactness of $M_{\mathfrak{S}_\infty}$. □

Functors to Galois representations. — We recall the construction of the functor $\text{`}T_{\mathfrak{S}_\infty}$ from $\text{`}\mathrm{Mod}^{r,\phi}_{/\mathfrak{S}_\infty}$ to the category of $\mathbb{Z}_p$-representations of G_∞. First, we define several rings. Put $R=\varprojlim\mathcal{O}_{\bar K}/p$ where the transition maps are given by Frobenius. There is a unique surjective map $\theta : W(R)\to\widehat{\mathcal{O}_{\bar K}}$ to the p-adic completion $\widehat{\mathcal{O}_{\bar K}}$ of $\mathcal{O}_{\bar K}$, which lifts the projection $R\to\mathcal{O}_{\bar K}/p$ onto the first factor. Recall that we have fixed a sequence $(\pi_n)_{n\geqslant 0}$ of compatible p^n-th root of π. It defines an element of R and we denote by $[\underline{\pi}]$ its Teichmüller representative. We have an embedding $\mathfrak{S}\to W(R)$, $u\mapsto[\underline{\pi}]$ which is compatible with Frobenius.

Let $\mathcal{O}_{\mathcal{E}}$ be the p-adic completion of $\mathfrak{S}[1/u]$. It is a discrete valuation ring with residue field $k((u))$. Put $\mathcal{E}=\mathrm{Frac}\mathcal{O}_{\mathcal{E}}$. The embedding $\mathfrak{S}\to W(R)$ extends to an embedding $\mathcal{E}\to W(\mathrm{Frac}\,R)$. Let $\mathcal{E}^{\mathrm{ur}}$ be the maximal unramified extension of $\mathcal{E}$ included in $W(\mathrm{Frac}\,R)[1/p]$ and $\mathcal{O}_{\mathcal{E}^{\mathrm{ur}}}$ its ring of integers. Since $\mathrm{Frac}\,R$ is algebraically closed (see [9], § A.3.1.6), the residue field $\mathcal{O}_{\mathcal{E}^{\mathrm{ur}}}/p$ is isomorphic to $k((u))^{\mathrm{sep}}$, a separable closure of $k((u))$. We will consider the tensor product

$\mathcal{O}_{\mathcal{E}^{\mathrm{ur}}} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p/\mathbb{Z}_p = \mathcal{E}^{\mathrm{ur}}/\mathcal{O}_{\mathcal{E}^{\mathrm{ur}}}$. It is an object of $\text{`Mod}^{r,\phi}_{/\mathfrak{S}}$ endowed with an action of G_∞.

Finally, the functor $\text{`}T_{\mathfrak{S}_\infty}$ is defined by the formula

$$\text{`}T_{\mathfrak{S}_\infty}(\mathfrak{M}) = \mathrm{Hom}_{\text{`Mod}^{r,\phi}_{/\mathfrak{S}}}(\mathfrak{M}, \mathcal{O}_{\mathcal{E}^{\mathrm{ur}}} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p/\mathbb{Z}_p)$$

for each $\mathfrak{M} \in \text{`Mod}^{r,\phi}_{/\mathfrak{S}_\infty}$. The restriction of $\text{`}T_{\mathfrak{S}_\infty}$ to the subcategory $\mathrm{Mod}^{r,\phi}_{/\mathfrak{S}_\infty}$ is denoted by $T_{\mathfrak{S}_\infty}$. If $\mathfrak{M} \in \mathrm{Mod}^{r,\phi}_{/\mathfrak{S}_1}$, the formula for $T_{\mathfrak{S}_\infty}(\mathfrak{M})$ can be simplified as follows:

$$T_{\mathfrak{S}_\infty}(\mathfrak{M}) = \mathrm{Hom}_{\text{`Mod}^{r,\phi}_{/\mathfrak{S}}}(\mathfrak{M}, \mathcal{O}_{\mathcal{E}^{\mathrm{ur}}}/p) = \mathrm{Hom}_{\text{`Mod}^{r,\phi}_{/\mathfrak{S}}}(\mathfrak{M}, k((u))^{\mathrm{sep}}).$$

Proposition 2.1.4. — *The composite $T_{\mathrm{qst}} \circ M_{\mathfrak{S}_\infty}$ is $T_{\mathfrak{S}_\infty}$ and it is an exact functor.*

If $\mathfrak{M} \in \mathrm{Mod}^{r,\phi}_{/\mathfrak{S}_1}$ is free of rank d over $\mathfrak{S}_1$, then $T_{\mathfrak{S}_\infty}(\mathfrak{M})$ is a vector space of dimension d over $\mathbb{F}_p$.

Proof. — It has been proved in § B.1.8.4 and § A.1.2 in [9]. □

Lemma 2.1.5. — *Let $\mathfrak{M} \in \mathrm{Mod}^{r,\phi}_{/\mathfrak{S}_\infty}$. Then $\bigcap_{f \in T_{\mathfrak{S}_\infty}(\mathfrak{M})} \ker f = 0$.*

Proof. — First, we show the lemma for $\mathfrak{M} \in \mathrm{Mod}^{r,\phi}_{/\mathfrak{S}_1}$. Put $\mathfrak{K} = \bigcap_{f \in T_{\mathfrak{S}_\infty}(\mathfrak{M})} \ker f$. Since u in invertible in $k((u))^{\mathrm{sep}}$, the quotient $\mathfrak{M}/\mathfrak{K}$ has no u-torsion and by proposition 2.1.1 (ii), it is an object of $\mathrm{Mod}^{r,\phi}_{/\mathfrak{S}_1}$. Furthermore, by definition of $\mathfrak{K}$, the map $\mathfrak{M} \to \mathfrak{M}/\mathfrak{K}$ induces a bijection $T_{\mathfrak{S}_\infty}(\mathfrak{M}/\mathfrak{K}) \to T_{\mathfrak{S}_\infty}(\mathfrak{M})$. By proposition 2.1.4, modules $\mathfrak{M}/\mathfrak{K}$ and $\mathfrak{M}$ have same rank and hence $\mathfrak{K} = 0$ as required.

It remains to prove that if $0 \to \mathfrak{M}' \to \mathfrak{M} \to \mathfrak{M}'' \to 0$ is an exact sequence in $\mathrm{Mod}^{r,\phi}_{/\mathfrak{S}_\infty}$ and if the conclusion is correct for $\mathfrak{M}'$ and $\mathfrak{M}''$, then it is also correct for $\mathfrak{M}$. Let $x \in \mathfrak{M}$ such that $f(x) = 0$ for all $f \in T_{\mathfrak{S}_\infty}(\mathfrak{M})$. If $y \in \mathfrak{M}''$ is the image of x, we have $g(y) = 0$ for all $g \in T_{\mathfrak{S}_\infty}(\mathfrak{M})$. Thus by assumption $y = 0$, and hence $x \in \mathfrak{M}'$. Let $g \in T_{\mathfrak{S}_\infty}(\mathfrak{M}')$. By exactness of $T_{\mathfrak{S}_\infty}$ (proposition 2.1.4), g can be extended to a map $f \in T_{\mathfrak{S}_\infty}(\mathfrak{M})$. We certainly have $g(x) = 0$. Then, using the assumption on $\mathfrak{M}'$, we finally get $x = 0$. □

Corollary 2.1.6. — *The functor $T_{\mathfrak{S}_\infty}$ is faithful.*

2.2. An equivalence of categories. — The aim of this subsection is to prove the following theorem.

THEOREM 2.2.1. — *Assume $r < p-1$. The functor $M_{\mathfrak{S}} : \mathrm{Mod}^{r,\phi}_{/\mathfrak{S}} \to \mathrm{Mod}^{r,\phi}_{/S}$ is an equivalence of categories.*

The full faithfulness was already seen (Proposition 2.1.2). Hence it remains to prove the essential surjectivity. Let $\mathcal{M} \in \mathrm{Mod}^{r,\phi}_{/S}$ and denote by d its rank over S. The heart of the proof is the following technical lemma.

LEMMA 2.2.2. — *With previous notations, there exists $\alpha_1, \dots, \alpha_d \in \mathrm{Fil}^r \mathcal{M}$ and a basis $e_1, \dots, e_d$ of $\mathcal{M}$ such that $e_i = \frac{1}{c^r}\phi_r(\alpha_i)$, $(\alpha_1, \dots, \alpha_d) = (e_1, \dots, e_d)B$ with B a $d \times d$ matrix with coefficients in $\mathfrak{S}$ and*

$$\mathrm{Fil}^r \mathcal{M} = \sum_{i=1}^{d} S\alpha_i + \mathrm{Fil}^p S\mathcal{M}. \tag{3}$$

Proof. — If R is a ring, we denote by $M_d(R)$ the algebra of $d \times d$ matrices with coefficients in R.

We first show that we can inductively construct $(\alpha_1^{(n)}, \dots, \alpha_d^{(n)}) \in \mathrm{Fil}^r \mathcal{M}$ such that

1. $(e_1^{(n)}, \dots, e_d^{(n)}) = c^{-r}\phi_r(\alpha_1^{(n)}, \dots, \alpha_d^{(n)})$ is a basis of $\mathcal{M}$;
2. there exist matrices $B^{(n)} \in M_d(\mathfrak{S})$ and $C^{(n)} \in M_d(p^n \mathrm{Fil}^{n+p} S)$ such that $(\alpha_1^{(n)}, \dots, \alpha_d^{(n)}) = (e_1^{(n)}, \dots, e_d^{(n)})(B^{(n)} + C^{(n)})$.

For $n = 0$, the result is a consequence of (the easy part of) Lemma 4.1.1 of [14]. Note also that Property (3) is satisfied with $\alpha_i^{(0)}$ instead of α_i. Now, assume that the $\alpha_i^{(n)}$'s are build. We put

$$(\alpha_1^{(n+1)}, \dots, \alpha_d^{(n+1)}) = (e_1^{(n)}, \dots, e_d^{(n)})B^{(n)}. \tag{4}$$

First remark that

$$\begin{aligned}(e_1^{(n+1)}, \dots, e_d^{(n+1)}) &= c^{-r}\phi_r(\alpha_1^{(n+1)}, \dots, \alpha_d^{(n+1)}) \\ &= c^{-r}\phi_r((\alpha_1^{(n)}, \dots, \alpha_d^{(n)}) - (e_1^{(n)}, \dots, e_d^{(n)})C^{(n)})) \\ &= (e_1^{(n)}, \dots, e_d^{(n)})(I - D^{(n)})\end{aligned}$$

where $c^{-r}\phi_r((e_1^{(n)}, \dots, e_d^{(n)})C^{(n)}) = (e_1^{(n)}, \dots, e_d^{(n)})D^{(n)}$. We claim that $p^{\lambda_n + n}$ divides $D^{(n)}$ with $\lambda_n = n+p-r-[\frac{n+p}{p-1}]$. Recall that for all $s \in \mathrm{Fil}^r S$ and $x \in \mathcal{M}$ we have $\phi_r(sx) = c^{-r}\phi_r(s)\phi_r(E(u)^r x)$. Moreover, by assumption, $C^{(n)} \in M_d(p^n \mathrm{Fil}^{n+p} S)$. So to prove the claim it suffices to show that $v_p(\phi_r(s)) \geqslant \lambda_n$ for all $s \in \mathrm{Fil}^{n+p} S$. Since s can be always represented by

$$s = \sum_{m=n+p}^{\infty} a_m(u)\frac{E(u)^m}{m!},\ a_m(u) \in W[u],\ a_m(u) \to 0 \quad p\text{-adically}$$

and $\phi(E(u)) = pc$, we reduce the proof to show that

$$m - v_p(m!) - r > n + p - r - \frac{n+p}{p-1} \quad \text{for any} \quad m \geqslant n+p$$

which is clear, using $v_p(m!) < \frac{m}{p-1}$.

It is easy to check $\lambda_n \geqslant 1$. Since $p^{\lambda_n+n} | D^{(n)}$, $(I - D^{(n)})$ is invertible and $(e_1^{(n+1)}, \dots, e_d^{(n+1)})$ is a basis of $\mathcal{M}$. Now by (4), we have

$$(\alpha_1^{(n+1)}, \dots, \alpha_d^{(n+1)}) = (e_1^{(n)}, \dots, e_d^{(n)})B^{(n)} = (e_1^{(n+1)}, \dots, e_d^{(n+1)})(I - D^{(n)})^{-1}B^{(n)}.$$

Put $A = (I - D^{(n)})^{-1}B^{(n)}$. To achieve the induction, it remains to write $A = B^{(n+1)} + C^{(n+1)}$ with $B^{(n+1)} \in M_d(\mathfrak{S})$ and $C^{(n+1)} \in M_d(p^{n+1}\mathrm{Fil}^{n+1+p}S)$. For that, write $D^{(n)} = p^{\lambda_n+n}E^{(n)}$ and

$$E^{(n)} = \sum_{i=0}^{n+p} b_i(u)\frac{E(u)^i}{i!} + \sum_{i=n+p+1}^{\infty} b_i(u)\frac{E(u)^i}{i!} = E_1^{(n)} + E_2^{(n)}$$

with $b_i(u) \in W[u]$. A simple computation on valuation gives $\frac{p^{\lambda_n+n}}{i!} \in \mathbb{Z}_p$ for all $i \leqslant n+p$. Thus $D_1^{(n)} = p^{\lambda_n+n}E_1^{(n)} \in M_d(\mathfrak{S})$. The conclusion then follows by expanding the series

$$A = \sum_{i=0}^{\infty} (D_1^{(n)} + D_2^{(n)})^i B^{(n)}$$

where $D_2^{(n)} = p^{\lambda_n+n}E_2^{(n)} \in M_d(p^{n+1}\mathrm{Fil}^{n+1+p}S)$.

To complete the proof of the lemma, remark that (4) implies

$$(\alpha_1^{(n+1)}, \dots, \alpha_d^{(n+1)}) - (\alpha_1^{(n)}, \dots, \alpha_d^{(n)}) = -(e_1^{(n)}, \dots, e_d^{(n)})C^{(n)}. \tag{5}$$

Hence all sequences $(\alpha_i^{(n)})_n$ converge (recall that p^n divides $C^{(n)}$). The convergence of all $e_i^{(n)}$ and then those of matrices $B^{(n)}$ follows. If α_i (resp. B) is the limit of $\alpha_i^{(n)}$ (resp. $B^{(n)}$), we have $\phi_r(\alpha_1, \dots, \alpha_d) = c^{-r}(e_1, \dots, e_d)$ and $(\alpha_1, \dots, \alpha_d) = (e_1, \dots, e_d)B$ with $B \in M_d(\mathfrak{S})$. It remains to check property (3). For that, we can reduce modulo p and noting $\alpha_i \equiv \alpha_i^{(0)} \pmod{p}$, we are done. □

Now, it is more or less easy to achieve the proof of theorem 2.2.1. First, we show that there exists $A \in M_d(\mathfrak{S})$ such that $BA = E(u)^r I$. Indeed, since $E(u)^r e_i \in \mathrm{Fil}^r \mathcal{M}$ for all i, Condition (3) implies that there exist matrices A', C' such that $BA' + C' = E(u)^r I$ and $C' \in M_d(\mathrm{Fil}^p S)$. Writing $A' = A'_0 + A'_1$ with $A'_0 \in M_d(W[u])$ and $A'_1 \in M_d(\mathrm{Fil}^p S)$, we may assume $A' \in M_d(W[u])$. Then $C' = E(u)^r I - BA'$ has coefficients in $\mathfrak{S} \cap \mathrm{Fil}^p S$. Therefore, $C' = E(u)^p C$ with $C \in M_d(\mathfrak{S})$. Now $BA' = E(u)^r(I - E(u)^{p-r}C)$ and $A = A'(I - E(u)^{p-r}C)^{-1} \in M_d(\mathfrak{S})$ is appropriate. Finally, we defined $\mathfrak{M} = \mathfrak{S}f_1 \oplus \cdots \oplus \mathfrak{S}f_d$ and we endow

it with ϕ given by $\phi(f_1,\dots,f_d)=(f_1,\dots,f_d)A$. It is then an easy exercise left to the reader to check that $M_{\mathfrak{S}}(\mathfrak{M})=\mathcal{M}$ as desired.

2.3. Consequences. — The first consequence is the extension of the equivalence on torsion objects.

THEOREM 2.3.1. — *Assume $r<p-1$. The functor $M_{\mathfrak{S}_\infty}:\mathrm{Mod}^{r,\phi}_{/\mathfrak{S}_\infty}\to\mathrm{Mod}^{r,\phi}_{/S_\infty}$ is an equivalence of categories.*

Proof. — It remains to show the essential surjectivity. Let $\mathcal{M}$ be an object of $\mathrm{Mod}^{r,\phi}_{/S_\infty}$. By Theorem V.2.a of [6], there exist two objects $\hat{\mathcal{M}}$ and $\hat{\mathcal{M}}'$ in $\mathrm{Mod}^{r,\phi}_{/\mathfrak{S}_\infty}$, together with an exact sequence $0\to\hat{\mathcal{M}}'\to\hat{\mathcal{M}}\to\mathcal{M}\to 0$ in $\text{`}\mathrm{Mod}^{r,\phi}_{/S}$. Now, by Theorem 2.2.1, one can lift the canonical inclusion $\iota:\hat{\mathcal{M}}'\to\hat{\mathcal{M}}$ at $\mathrm{Mod}^{r,\phi}_{/\mathfrak{S}_\infty}$-level: there exist $\hat{\mathfrak{M}}$, $\hat{\mathfrak{M}}'$ in $\mathrm{Mod}^{r,\phi}_{/\mathfrak{S}_\infty}$ and $f:\hat{\mathfrak{M}}'\to\hat{\mathfrak{M}}$ a morphism in this category such that $M_{\mathfrak{S}}(f)=\iota$. The map $F=T_{\mathfrak{S}}(f)$ is an injective application between two free $\mathbb{Z}_p$-modules of same (finite) rank. Consequently, there exists $G:T_{\mathfrak{S}}(\hat{\mathfrak{M}}')\to T_{\mathfrak{S}}(\hat{\mathfrak{M}})$ (necessarily G_∞-equivariant) such that $F\circ G=G\circ F=p^n\mathrm{id}$ for an integer n. By full faithfulness of $T_{\mathfrak{S}}$, there exists a map $g:\hat{\mathfrak{M}}\to\hat{\mathfrak{M}}'$ satisfying $f\circ g=g\circ f=p^n\mathrm{id}$. It follows that $f\otimes_{\mathbb{Z}_p}\mathbb{Q}_p$ is bijective. Then we can apply Proposition 2.1.3: $\mathfrak{M}=\hat{\mathfrak{M}}/\hat{\mathfrak{M}}'$ is in $\mathrm{Mod}^{r,\phi}_{/\mathfrak{S}_\infty}$ and $M_{\mathfrak{S}_\infty}(\mathfrak{M})=\mathcal{M}$. □

PROPOSITION 2.3.2. — *Assume $r<p-1$ and choose M_{S_∞} a quasi-inverse of $M_{\mathfrak{S}_\infty}$. If $f:\mathcal{M}\to\mathcal{M}'$ is an injective (resp. surjective) morphism in $\mathrm{Mod}^{r,\phi}_{/S_\infty}$, then $M_{S_\infty}(f)$ is also. Moreover, the functor M_{S_∞} is exact.*

Proof. — Let $f:\mathcal{M}\to\mathcal{M}'$ be a morphism in $\mathrm{Mod}^{r,\phi}_{/S_\infty}$. Put $\mathfrak{M}=M_{S_\infty}(\mathcal{M})$, $\mathfrak{M}'=M_{S_\infty}(\mathcal{M}')$ and $g=M_{S_\infty}(f)$.

Assume f injective and denote by $\mathfrak{K}$ the kernel of g. By Proposition 2.1.1 (iii), we have $\mathfrak{K}\in\mathrm{Mod}^{r,\phi}_{/\mathfrak{S}_\infty}$. Put $\mathcal{K}=M_{\mathfrak{S}_\infty}(\mathfrak{K})$. Let $h:\mathcal{K}\to\mathcal{M}$ be the image under $M_{\mathfrak{S}_\infty}$ of the inclusion $\mathfrak{K}\to\mathfrak{M}$. The composite $f\circ h$ is zero and since f is injective, $h=0$. By faithfulness, the morphism $\mathfrak{K}\to\mathfrak{M}$ vanishes, and consequently $\mathfrak{K}=0$ and g is injective.

Now suppose f surjective and denote by $\mathfrak{C}$ the cokernel of g. We have $S\otimes_{(\phi),\mathfrak{S}}\mathfrak{C}=0$. Reducing modulo p, we get $S_1\otimes_{(\phi),\mathfrak{S}_1}\mathfrak{C}/p\mathfrak{C}=0$. Since $\mathfrak{C}/p\mathfrak{C}$ is a module of finite type over the principal ring $k[[u]]$, it is a direct sum of some $k[[u]]$ or $k[[u]]/u^n$ for suitable integers n. By computing the tensor product, it follows that the only solution is $\mathfrak{C}/p\mathfrak{C}=0$, *i.e* $\mathfrak{C}=p\mathfrak{C}$. Since $\mathfrak{C}$ is finitely generated, Nakayama's lemma gives $\mathfrak{C}=0$ as required.

For the exactness, take $0\to\mathcal{M}'\to\mathcal{M}\to\mathcal{M}''\to 0$ an exact sequence in $\mathrm{Mod}^{r,\phi}_{/S_\infty}$. We know that $M_{S_\infty}(\mathcal{M})\to M_{S_\infty}(\mathcal{M}'')$ is surjective. Call $\mathfrak{K}$ its

kernel: it is an object of $\mathrm{Mod}^{r,\phi}_{/\mathfrak{S}_\infty}$ and we have an exact sequence $0 \to \mathfrak{K} \to M_{S_\infty}(\mathcal{M}) \to M_{S_\infty}(\mathcal{M}'') \to 0$. Applying the exact functor $M_{\mathfrak{S}_\infty}$, we see that $M_{\mathfrak{S}_\infty}(\mathfrak{K})$ is the kernel of $\mathcal{M} \to \mathcal{M}''$. Hence, it is isomorphic to $\mathcal{M}'$ and we are done. □

Remark. — Although the functor $M_{\mathfrak{S}_\infty}$ is exact, the implication (f injective) $\Rightarrow$ ($M_{\mathfrak{S}_\infty}(f)$ injective) is not true if $er \geqslant p-1$. Here is a counter-example. Take $\mathfrak{M} = \mathfrak{S}_1$ with $\phi(1) = 1$, $\mathfrak{M}' = \mathfrak{S}_1$ with $\phi(1) = u^{p-1}$ and $f : \mathfrak{M}' \to \mathfrak{M}$, $1 \mapsto u$. It is injective. However, $\mathcal{M} = M_{\mathfrak{S}_\infty}$ is just S_1 endowed with $\mathrm{Fil}^r S_1$ and the canonical ϕ_r. On the other hand, $\mathcal{M}' = S_1$, $\mathrm{Fil}^r\mathcal{M}' = u^{er-p+1}\mathcal{M}'$ and $\phi_r(u^{er-p+1}) = (-1)^r$. The map $M_{\mathfrak{S}_\infty}(f)$ is the multiplication by u^p and sends $u^{(e-1)p}$ to 0; hence it is not injective.

Corollary 2.3.3. — *Assume $r < p-1$. Functors T_{qst} on $\mathrm{Mod}^{r,\phi}_{/S_\infty}$ and T_{st} on $\mathrm{Mod}^{r,\phi,N}_{/S_\infty}$ are faithful.*

Proof. — For T_{qst}, it is a direct consequence of Corollary 2.1.6 and Theorem 2.3.1. Let $f : \mathcal{M} \to \mathcal{M}'$ be a morphism in $\mathrm{Mod}^{r,\phi,N}_{/S_\infty}$. It can be seen as a morphism in $\mathrm{Mod}^{r,\phi}_{/S_\infty}$ and we have $T_{\mathrm{qst}}(f) = T_{\mathrm{st}}(f)$. If this morphism vanishes, then f have also to vanish thanks to the faithfulness of T_{qst}. This proves the corollary. □

Theorem 2.3.4. — *Assume $r < p-1$. Let $\mathcal{M}' \subset \mathcal{M}$ be two objects of $\mathrm{Mod}^{r,\phi}_{/S}$ such that $\mathcal{M}' \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \simeq \mathcal{M} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ and $\mathrm{Fil}^r\mathcal{M}' \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \simeq \mathrm{Fil}^r\mathcal{M} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$. Then the quotient $\mathcal{M}/\mathcal{M}'$ is an object of $\mathrm{Mod}^{r,\phi}_{/S_\infty}$. Furthermore every object of $\mathrm{Mod}^{r,\phi}_{/S_\infty}$ can be written in this way.*

Proof. — For the first part of the theorem, we use a similar argument as in the proof of Theorem 2.3.1. Let $\mathfrak{M}' \to \mathfrak{M}$ be an antecedent of the inclusion $\mathcal{M}' \to \mathcal{M}$. We first show that $\mathfrak{M}' \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \simeq \mathfrak{M} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$, and then by using proposition 2.1.3, we get $M_{\mathfrak{S}_\infty}(\mathfrak{M}/\mathfrak{M}') = \mathcal{M}/\mathcal{M}'$.

The second part is again Theorem V.2.a of [6]. □

Remark. — The condition $\mathrm{Fil}^r\mathcal{M}' \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \simeq \mathrm{Fil}^r\mathcal{M} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ is equivalent to $\mathrm{Fil}^r\mathcal{M}' = \mathcal{M}' \cap \mathrm{Fil}^r\mathcal{M}$. Indeed, if $x \in \mathcal{M}' \cap \mathrm{Fil}^r\mathcal{M}$ then $x \in \mathrm{Fil}^r\mathcal{M}' \otimes_{\mathbb{Z}_p} \mathbb{Q}_p = \mathrm{Fil}^r\mathcal{M} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ and $p^n x \in \mathrm{Fil}^r\mathcal{M}'$ for a suitable integer n. Since, by definition, $\mathcal{M}'/\mathrm{Fil}^r\mathcal{M}'$ has no p-torsion, we must have $x \in \mathrm{Fil}^r\mathcal{M}'$. The converse is easy.

2.4. Duality. — In [13], § 3.1, it is defined a duality on $\mathrm{Mod}^{r,\phi}_{/\mathfrak{S}_\infty}$ for all $r<\infty$. For $\mathfrak{M}\in\mathrm{Mod}^{r,\phi}_{/\mathfrak{S}_\infty}$, we put $\mathfrak{M}^\vee=\mathrm{Hom}_{\mathfrak{S}}(\mathfrak{M},\mathfrak{S}\otimes_{\mathbb{Z}_p}\mathbb{Q}_p/\mathbb{Z}_p)$. We then have a natural pairing:

$$\langle\cdot,\cdot\rangle:\mathfrak{M}\times\mathfrak{M}^\vee\to\mathfrak{S}\otimes_{\mathbb{Z}_p}\mathbb{Q}_p/\mathbb{Z}_p$$

which allows us to defined $\phi^\vee$ on $\mathfrak{M}^\vee$ by

$$\langle\phi(x),\phi^\vee(y)\rangle=c_0^{-r}E(u)^r\phi(\langle x,y\rangle)$$

(for all $x\in\mathfrak{M}$ and $y\in\mathfrak{M}^\vee$) where $c_0=\frac{E(0)}{p}\in W^\star$ and the latest ϕ is given by the usual operator on $\mathfrak{S}$. Here are main properties of the duality. We have a natural isomorphism $(\mathfrak{M}^\vee)^\vee\simeq\mathfrak{M}$, and a compatibility between duality and $T_{\mathfrak{S}_\infty}$ given by the following functorial isomorphism:

$$T_{\mathfrak{S}_\infty}(\mathfrak{M}^\vee)\simeq T_{\mathfrak{S}_\infty}(\mathfrak{M})^\vee(r). \tag{6}$$

where "(r)" is for the Tate twist.

If $r<p-1$, we can also find in the literature (see [6], Chapter V) a definition of a duality on $\mathrm{Mod}^{r,\phi}_{/S_\infty}$. In a few words, if $\mathcal{M}$ is an object of this category, we put $\mathcal{M}^\vee=\mathrm{Hom}_S(\mathcal{M},S\otimes_{\mathbb{Z}_p}\mathbb{Q}_p/\mathbb{Z}_p)$, $\mathrm{Fil}^r\mathcal{M}^\vee=\{f\in\mathcal{M}^\vee,\ f(\mathrm{Fil}^r\mathcal{M})\subset\mathrm{Fil}^rS\otimes_{\mathbb{Z}_p}\mathbb{Q}_p/\mathbb{Z}_p\}$ and if $f\in\mathrm{Fil}^r\mathcal{M}^\vee$, $\phi_r^\vee(f)$ is defined as the unique map making commutative the following diagram:

$$\begin{array}{ccc}\mathrm{Fil}^r\mathcal{M} & \xrightarrow{\phi_r} & \mathcal{M}\\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle \phi_r^\vee(f)}\\ \mathrm{Fil}^rS\otimes_{\mathbb{Z}_p}\mathbb{Q}_p/\mathbb{Z}_p & \xrightarrow{\phi_r} & S\otimes_{\mathbb{Z}_p}\mathbb{Q}_p/\mathbb{Z}_p\end{array}$$

We wish to compare these two constructions if $r<p-1$. For that, we put

$$\lambda=\prod_{n=1}^{\infty}\phi^n\left(\frac{E(u)}{pc_0}\right)\in S.$$

Now, strating from $\mathfrak{M}\in\mathrm{Mod}^{r,\phi}_{/\mathfrak{S}_\infty}$, we can define a natural map

$$M_{\mathfrak{S}_\infty}(\mathfrak{M}^\vee)\to M_{\mathfrak{S}_\infty}(\mathfrak{M})^\vee,\quad s\otimes f\mapsto\frac{1}{\lambda^r}\,sf.$$

A direct calculation gives $\phi(\lambda)=\frac{c}{\phi(c_0)}\lambda$, which implies that the previous isomorphism is compatible with ϕ, and hence a morphism in $\mathrm{Mod}^{r,\phi}_{/S_\infty}$. Hence, dualities on $\mathrm{Mod}^{r,\phi}_{/S_\infty}$ and $\mathrm{Mod}^{r,\phi}_{/\mathfrak{S}_\infty}$ are compatible under the equivalence $M_{\mathfrak{S}_\infty}$.

Corollary 2.4.1. — *Assume $r<p-1$. For any $\mathcal{M}\in\mathrm{Mod}^{r,\phi}_{/S_\infty}$, there exists a natural isomorphism $\mathcal{M}\to(\mathcal{M}^\vee)^\vee$ and a natural isomorphism:*

$$T_{\mathrm{qst}}(\mathcal{M}^\vee)\simeq T_{\mathrm{qst}}(\mathcal{M})^\vee(r).$$

REMARKS. — Corollary 2.4.1 is proved (with different methods) in [6] under the assumption $er < p-1$ or $r = 1$.

In *loc. cit.*, definition of duality is extended to $\mathrm{Mod}^{r,\phi,N}_{/S_\infty}$: the operator $N^\vee$ on $\mathcal{M}^\vee$ is defined by the formula $N^\vee(f) = N \circ f - f \circ N$ (where N is the given operator on $\mathcal{M}$). Using isomorphism (1), we directly obtain a version of Corollary 2.4.1 in this new situation.

3. A construction on $\mathrm{Mod}^{r,\phi}_{/\mathfrak{S}_\infty}$

In this section, we construct the functor Max^r discussed in the introduction and then prove Theorem 1.

The main ingredient of the proof is Theorem 2.3.1, together with a result of Fontaine (recalled in § 3.1) that claimed an equivalence between the category of torsion $\mathbb{Z}_p$-representations of G_∞ and a certain category of torsion ϕ-modules over $\mathcal{O}_\mathcal{E}$. Indeed, under these identifications, the functor T_{qst} just corresponds to the scalar extension from $\mathfrak{S}_\infty$ to $\mathcal{E}/\mathcal{O}_\mathcal{E}$. Our first step, which is achieved in § 3.2, is then a somehow careful study of lattices in some $\mathcal{O}_\mathcal{E}$-modules equipped with a Frobenius endomorphism: we essentially investigate their behaviour under sums and intersections, prove some finiteness properties and conclude to the existence of a maximal lattice among those corresponding to objects of $\mathrm{Mod}^{r,\phi}_{/\mathfrak{S}_\infty}$. This allows us finally to define Max^r by associating to each $\mathfrak{M} \in \mathrm{Mod}^{r,\phi}_{/\mathfrak{S}_\infty}$ the maximal lattice in $\mathfrak{M} \otimes_{\mathfrak{S}_\infty} \mathcal{E}/\mathcal{O}_\mathcal{E}$ (see Definition 3.3.1). In § 3.3, we enumerate several pleasant properties of Max^r and, in particular, we prove that its essential image is an abelian category on which the functor T_{qst} is fully faithful. In § 3.4, we develop the dual version of the theory by regarding minimal lattices (instead of maximal ones) and defining a functor Min^r. Again, we prove that its essential image is abelian and the restriction of T_{qst} to this subcategory is fully faithful. We then go back to the study of maximal objects: in § 3.5, we prove a certain reciprocity formula which computes $\mathrm{Max}^r(\mathcal{M})$ from $T_{\mathrm{qst}}(\mathcal{M})$. In § 3.6, we give a classification of simple objects of $\mathrm{Max}^r(\mathrm{Mod}^{r,\phi}_{/\mathfrak{S}_\infty})$ (recall that it is an abelian category) when the residue field k is algebraically closed. Finally in § 3.7, we summarize all our results and translate them in term of category $\mathrm{Mod}^{r,\phi}_{/S_\infty}$.

**3.1. The category $\mathrm{`Mod}^{\phi}_{/\mathcal{O}_\mathcal{E}}$.** — Let's recall classical results about the classification of $\mathbb{Z}_p$-representations of $G_\infty$. Denote by $\mathrm{`Mod}^{\phi}_{/\mathcal{O}_\mathcal{E}}$ the category of torsion étale ϕ-modules over $\mathcal{O}_\mathcal{E}$. By definition, an object of $\mathrm{`Mod}^{\phi}_{/\mathcal{O}_\mathcal{E}}$ is an $\mathcal{O}_\mathcal{E}$-module M killed by a power of p and equipped with a Frobenius $\phi : M \to M$ that induces a bijection $\mathrm{id} \otimes \phi : \phi^\star M \to M$ (where $\phi^\star M = \mathcal{O}_\mathcal{E} \otimes_{(\phi),\mathcal{O}_\mathcal{E}} \mathfrak{M}$).

REMARK. — Since we are only interested in p-torsion modules, the definition does not change if we substitute the ring $\mathfrak{S}[1/u]$ to $\mathcal{O}_{\mathcal{E}}$ (in other words, we do not need to complete p-adically). In the sequel, we will just work with $\mathfrak{S}[1/u]$.

We have a functor $`T_{\mathcal{O}_{\mathcal{E}}} : `\mathrm{Mod}^{\phi}_{/\mathcal{O}_{\mathcal{E}}} \to \mathrm{Rep}_{\mathbb{Z}_p}(G_\infty)$ defined by

$$`T_{\mathcal{O}_{\mathcal{E}}}(M) = \mathrm{Hom}_{`\mathrm{Mod}^{\phi}_{/\mathcal{O}_{\mathcal{E}}}}(M, \mathcal{O}_{\mathcal{E}^{\mathrm{ur}}} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p/\mathbb{Z}_p).$$

THEOREM 3.1.1. — *The functor $`T_{\mathcal{O}_{\mathcal{E}}}$ is exact and fully faithful.*

Proof. — See [9], § A.1.2. □

Furthermore $`T_{\mathfrak{S}_\infty}$ factors through $`T_{\mathcal{O}_{\mathcal{E}}}$ as follows: if $`M_{\mathcal{O}_{\mathcal{E}}} : \mathrm{Mod}^{r,\phi}_{/\mathfrak{S}_\infty} \to `\mathrm{Mod}^{\phi}_{/\mathcal{O}_{\mathcal{E}}}$ is defined by $`M_{\mathcal{O}_{\mathcal{E}}}(\mathfrak{M}) = \mathfrak{M} \otimes_{\mathfrak{S}} \mathcal{O}_{\mathcal{E}} = \mathfrak{M} \otimes_{\mathfrak{S}} \mathfrak{S}[1/u]$ (since $E(u)$ is invertible in $\mathcal{O}_{\mathcal{E}}$, the map $\mathrm{id} \otimes \phi : \phi^\star[`M_{\mathcal{O}_{\mathcal{E}}}(\mathfrak{M})] \to `M_{\mathcal{O}_{\mathcal{E}}}(\mathfrak{M})$ is bijective), the equality $`T_{\mathfrak{S}_\infty} = `T_{\mathcal{O}_{\mathcal{E}}} \circ `M_{\mathcal{O}_{\mathcal{E}}}$ holds. In a slightly different situation, $`M_{\mathcal{O}_{\mathcal{E}}}$ is the functor $j^\star$ of [9]. From now on, we will use the notation $\mathfrak{M}[1/u]$ for $`M_{\mathcal{O}_{\mathcal{E}}}(\mathfrak{M})$. In [9], Fontaine defines an adjoint $j_\star$ to his functor $j^\star$. In the sequel, we will adapt his construction to our settings.

3.2. The ordered set $F^r_{\mathfrak{S}}(M)$. — In this subsection, we fix $M \in `\mathrm{Mod}^{\phi}_{/\mathcal{O}_{\mathcal{E}}}$. Our aim is to study the structure of the "set" of previous images of $M$ under $`M_{\mathcal{O}_{\mathcal{E}}}$. We begin by the following definition:

DEFINITION 3.2.1. — Let $\mathcal{F}^r_{\mathfrak{S}}(M)$ be the category whose objects are couples $(\mathfrak{M}, f)$ where $\mathfrak{M}$ is an object of $\mathrm{Mod}^{r,\phi}_{/\mathfrak{S}_\infty}$ and $f : \mathfrak{M}[1/u] \to M$ is an isomorphism. Morphisms in $\mathcal{F}^r_{\mathfrak{S}}(M)$ are morphisms in $\mathrm{Mod}^{r,\phi}_{/\mathfrak{S}_\infty}$ that are compatible with f.

Let $F^r_{\mathfrak{S}}(M)$ be the (partially) ordered set (by inclusion) of $\mathfrak{M} \in \mathrm{Mod}^{r,\phi}_{/\mathfrak{S}_\infty}$ contained in M such that $\mathfrak{M}[1/u] = M$.

The following lemma is easy:

LEMMA 3.2.2. — *The category $\mathcal{F}^r_{\mathfrak{S}}(M)$ is equivalent to (the category associated to) the ordered set $F^r_{\mathfrak{S}}(M)$.*

Supremum and infimum.

PROPOSITION 3.2.3. — *The ordered set $F^r_{\mathfrak{S}}(M)$ has finite supremum and finite infimum.*

Proof. — Obviously, it suffices to prove that for any $\mathfrak{M}'$ and $\mathfrak{M}''$ in $F^r_{\mathfrak{S}}(M)$, $\sup(\mathfrak{M}', \mathfrak{M}'')$ and $\inf(\mathfrak{M}', \mathfrak{M}'')$ exist.

For the supremum, it is enough to show that $\mathfrak{M} = \mathfrak{M}' + \mathfrak{M}''$ (where the sum is computed in M) is an object of $\mathrm{Mod}^{r,\phi}_{/\mathfrak{S}_\infty}$ (it is obvious that $\mathfrak{M}[1/u] = M$). For this, remark that $\mathfrak{M}$ satisfies Condition (2) defined page 192 (since $\mathfrak{M}'$ and $\mathfrak{M}''$ satisfy it). The conclusion then follows from Proposition 2.1.1 (ii).

In the same way, for the infimum, we want to prove that $\mathfrak{M} = \mathfrak{M}' \cap \mathfrak{M}''$ satisfies $\mathfrak{M}[1/u] = M$ and is in $\mathrm{Mod}^{r,\phi}_{/\mathfrak{S}_\infty}$. Since $\mathfrak{M}'$ is finitely generated, there exists an integer s such that $u^s\mathfrak{M}' \subset \mathfrak{M}''$ and the first point is clear. Now, let $x \in \mathfrak{M}$. Because $\mathfrak{M}'$ and $\mathfrak{M}''$ are in $\mathrm{Mod}^{r,\phi}_{/\mathfrak{S}_\infty}$, there exists $x' \in \phi^\star\mathfrak{M}'$ and $x'' \in \phi^\star\mathfrak{M}''$ such that $E(u)^r x = \mathrm{id} \otimes \phi(x') = \mathrm{id} \otimes \phi(x'')$ (if $r = \infty$, it must be replaced by a sufficiently large integer). By definition, $\mathrm{id} \otimes \phi$ is injective on $\phi^\star M$. It follows that $x' = x'' \in \phi^\star\mathfrak{M}$. Consequently, Condition (2) holds for $\mathfrak{M}$. Moreover, since $\mathfrak{S}$ in noetherian, $\mathfrak{M} \subset \mathfrak{M}'$ is finitely generated over $\mathfrak{S}$. Finally, it is obviously killed by a power of p, and without u-torsion. Proposition 2.1.1 ends the proof. □

Some finiteness property.

LEMMA 3.2.4. — *Fix $\mathfrak{M} \in F^r_{\mathfrak{S}}(M)$. There exists an integer ℓ (depending only on $\mathfrak{M}$) such that $\lg_{\mathfrak{S}}(\mathfrak{M}'/\mathfrak{M}) \leqslant \ell$ for any $\mathfrak{M}' \in F^r_{\mathfrak{S}}(M)$ with $\mathfrak{M} \subset \mathfrak{M}'$.*

Proof. — First, we prove by *dévissage* that it is sufficient to consider the case where M is killed by p. Denote by $\mathfrak{M}(p)$ (resp. $\mathfrak{M}'(p)$) the kernel of the multiplication by p on $\mathfrak{M}$ (resp. $\mathfrak{M}'$). We have the following commutative diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathfrak{M}(p) & \longrightarrow & \mathfrak{M} & \longrightarrow & \mathfrak{M}/\mathfrak{M}(p) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathfrak{M}'(p) & \longrightarrow & \mathfrak{M}' & \longrightarrow & \mathfrak{M}'/\mathfrak{M}'(p) & \longrightarrow & 0 \end{array}$$

where both horizontal sequences are exact, and all vertical arrows are injective. Snake lemma then shows that the sequence $0 \to \frac{\mathfrak{M}'(p)}{\mathfrak{M}(p)} \to \frac{\mathfrak{M}'}{\mathfrak{M}} \to \frac{\mathfrak{M}'/\mathfrak{M}(p)}{\mathfrak{M}'/\mathfrak{M}(p)} \to 0$ remains exact, and we are done.

If M is killed by p, the argument is the following. Since $\mathrm{id}\otimes\phi : \phi^\star\mathfrak{M} \to \mathfrak{M}$ is injective (Proposition 2.1.1 (i)), the map $\mathfrak{M}/u\mathfrak{M} \to \langle \mathrm{im}\, \phi\rangle / u \langle \mathrm{im}\, \phi\rangle$ induced by ϕ is also injective. By definition, there exists an integer s such that $E(u)^s\mathfrak{M} \subset \langle \mathrm{im}\, \phi\rangle$. (If r is finite, one can choose $s = r$.) We then have the implication

$$(x \notin u\mathfrak{M}) \implies (\phi(x) \notin u^{es+1}\mathfrak{M}). \tag{7}$$

Furthermore, there exists an integer n such that $u^n\mathfrak{M}' \subset \mathfrak{M}$. Choose n minimal (not necessary positive). Then we can find $x \in \mathfrak{M}'$ such that $u^{n-1}x \notin \mathfrak{M}$. Therefore $u^n x \in \mathfrak{M}$ but $u^n x \notin u\mathfrak{M}$. Applying Implication (7), we get $\phi(u^n x) \notin u^{es+1}\mathfrak{M}$, that is to say $u^n\phi(x) \notin u^{1+es-(p-1)n}\mathfrak{M}$. On the other hand, $u^n\phi(x) \in u^n\mathfrak{M}' \subset \mathfrak{M}$. It follows $1 + es - (p-1)n \geqslant 0$. Thus $n \leqslant t = E(\frac{es+1}{p-1})$ (here E denotes the integer part). From $u^n\mathfrak{M}' \subset \mathfrak{M}$, we get $u^t\mathfrak{M}' \subset \mathfrak{M}$ and Lemma is proved (with $\ell = t \dim_{k((u))} M$). □

LEMMA 3.2.5. — *Assume $r < \infty$. There exists an integer ℓ (depending only on M) such that $\lg_{\mathfrak{S}}(\mathfrak{M}'/\mathfrak{M}) \leqslant \ell$ for any $\mathfrak{M}$ and $\mathfrak{M}'$ in $F^r_{\mathfrak{S}}(M)$ with $\mathfrak{M} \subset \mathfrak{M}'$.*

Proof. — The proof of Lemma 3.2.4 shows that ℓ can be chosen equal to $\lg_{\mathcal{O}_\mathcal{E}}(M) \times E(\frac{er+1}{p-1})$, which depends only on M. □

COROLLARY 3.2.6. — *The ordered set $F^r_{\mathfrak{S}}(M)$ always has a greatest element. Furthermore, if $r < \infty$, $F^r_{\mathfrak{S}}(M)$ has a smallest element.*

REMARK. — The proof of Lemma 3.2.4 gives an upper bound for the length of any chain in $F^r_{\mathfrak{S}}(M)$, that is:

$$1 + \lg_{\mathcal{O}_\mathcal{E}}(M) \times E\left(\frac{er+1}{p-1}\right).$$

In particular, if $er < p-1$, the set $F^r_{\mathfrak{S}}(M)$ contains at most one element. This latest assertion will be used several times in the sequel.

Functoriality. — In view of possible generalizations, we would like to rephrase quickly previous properties in a more categorical and functorial way.

PROPOSITION 3.2.7. — *The category $\mathcal{F}_{\mathfrak{S}}(M)$ has finite (direct) sums and finite products.*

PROPOSITION 3.2.8. — *The category $\mathcal{F}_{\mathfrak{S}}(M)$ is noetherian in the following sense: if*

$$\mathfrak{M}_1 \xrightarrow{f_1} \mathfrak{M}_2 \xrightarrow{f_2} \cdots \xrightarrow{f_{n-1}} \mathfrak{M}_n \xrightarrow{f_n} \cdots$$

is an infinite sequence of morphisms, all f_n are isomorphisms for n big enough.

If r is finite, the category $\mathcal{F}_{\mathfrak{S}}(M)$ is artinian in the following sense: if $\mathfrak{M}_1 \xleftarrow{f_1} \mathfrak{M}_2 \xleftarrow{f_2} \cdots$ is an infinite sequence of morphisms, all f_n are isomorphisms for n big enough.

PROPOSITION 3.2.9. — *Let $\mathfrak{M}_1,\dots,\mathfrak{M}_n$ (resp. $\mathfrak{M}'_1,\dots,\mathfrak{M}'_n$) be objects of $\mathcal{F}_{\mathfrak{S}}(M)$ (resp. $\mathcal{F}_{\mathfrak{S}}(M')$). Let $f_i : \mathfrak{M}_i \to \mathfrak{M}'_i$ be morphisms in $\mathrm{Mod}^{r,\phi}_{/\mathfrak{S}_\infty}$. Put $\mathfrak{M} = \sup(\mathfrak{M}_1,\dots,\mathfrak{M}_n)$ and $\mathfrak{M}' = \sup(\mathfrak{M}'_1,\dots,\mathfrak{M}'_n)$. Then there exists a unique map $f : \mathfrak{M} \to \mathfrak{M}'$ making commutative all diagrams*

$$\begin{array}{ccc} \mathfrak{M}_i & \xrightarrow{f_i} & \mathfrak{M}'_i \\ \downarrow & & \downarrow \\ \mathfrak{M} & \xrightarrow{f} & \mathfrak{M}' \end{array}$$

We put $f = \sup(f_1,\dots,f_n)$.

Furthermore, the association $(f_1,\dots,f_n) \mapsto \sup(f_1,\dots,f_n)$ is functorial in an obvious sense.

REMARK. — Of course, the analogous statement with inf is also true.

IMPORTANT REMARK. — Since $\text{`}T_{\mathcal{O}_\mathcal{E}}$ is fully faithful, the functor $\text{`}M_{\mathcal{O}_\mathcal{E}}$ can be replaced by $T_{\mathfrak{S}}$ in definition 3.2.1. Hence, it is possible to define supremum and infimum without reference to the auxiliary category $\text{`Mod}^{\phi}_{/\mathcal{O}_\mathcal{E}}$.

3.3. Maximal objects. — In this subsection, we give (and prove) some pleasant properties of objects arising as the greatest element of one set $F_{\mathfrak{S}}(M)$.

The functor Max^r

DEFINITION 3.3.1. — Let $\mathfrak{M} \in \mathrm{Mod}^{r,\phi}_{/\mathfrak{S}_\infty}$. We define $\mathrm{Max}^r(\mathfrak{M})$ to be the greatest element of $F^r_{\mathfrak{S}}(\mathfrak{M}[1/u])$. It is endowed with an homomorphism $\iota^{\mathfrak{M}}_{\max} : \mathfrak{M} \to \mathrm{Max}^r(\mathfrak{M})$ in the category $\mathrm{Mod}^{r,\phi}_{/\mathfrak{S}_\infty}$.

An object $\mathfrak{M}$ of $\mathrm{Mod}^{r,\phi}_{/\mathfrak{S}_\infty}$ is said *maximal* (in $\mathrm{Mod}^{r,\phi}_{/\mathfrak{S}_\infty}$) [(4)] if the map $\iota^{\mathfrak{M}}_{\max}$ is an isomorphism.

REMARKS. — By § B.1.5.3 of [9], a ϕ-module over $\mathfrak{S}$ *killed by a power of* p satisfies Condition (2) with $r = \infty$, if and only if $\mathrm{id} \otimes \phi : \phi^\star\mathfrak{M}[1/u] \to \mathfrak{M}[1/u]$ is bijective. It follows that for any $\mathfrak{M} \in \mathrm{Mod}^{\infty,\phi}_{/\mathfrak{S}_\infty}$, $\mathrm{Max}^\infty(\mathfrak{M}) = j_\star(\mathfrak{M}[1/u])$ where $j_\star$ is the functor defined in § B.1.4 of *loc. cit.*

In general, $\mathrm{Max}^r(\mathfrak{M})$ and $\mathrm{Max}^{r+1}(\mathfrak{M})$ does not coincide. For instance, take r such that $er \geqslant p$ and consider $\mathfrak{M} = \mathfrak{S}e_1 \oplus \mathfrak{S}e_2$ with $\phi(e_1) = ue_1 + u^{er}e_2$ and $\phi(e_2) = u^p e_1$. Then $\mathfrak{M}$ is maximal in $\mathrm{Mod}^{r,\phi}_{/\mathfrak{S}_\infty}$ but *not* in $\mathrm{Mod}^{r+1,\phi}_{/\mathfrak{S}_\infty}$ since the submodule of $\mathfrak{M}[1/u]$ generated by e_1 and $\frac{e_2}{u}$ is in $F^{r+1}_{\mathfrak{S}}(\mathfrak{M}[1/u])$.

PROPOSITION 3.3.2. — *The previous definition gives rise to a functor* $\mathrm{Max}^r : \mathrm{Mod}^{r,\phi}_{/\mathfrak{S}_\infty} \to \mathrm{Mod}^{r,\phi}_{/\mathfrak{S}_\infty}$.

(4) When the value of r is clear by the context, we just say *maximal*.

Proof. — We have to prove that any map $f : \mathfrak{M} \to \mathfrak{M}'$ induces a map $\mathrm{Max}^r(\mathfrak{M}) \to \mathrm{Max}^r(\mathfrak{M}')$. Let $g = f \otimes_{\mathfrak{S}} \mathfrak{S}[1/u]$. By proposition 2.1.1 (iii), $g(\mathrm{Max}^r(\mathfrak{M}))$ is in $\mathrm{Mod}^{r,\phi}_{/\mathfrak{S}_\infty}$. Hence $g(\mathrm{Max}^r(\mathfrak{M})) \subset \mathrm{Max}^r(\mathfrak{M}')$ and we are done. □

REMARK. — The collection of homomorphisms $(\iota^{\mathfrak{M}}_{\max})$ defines a natural transformation between the identity functor and Max^r.

We now show several properties of Max^r.

PROPOSITION 3.3.3. — *The functor* Max^r *is a projection, that is* $\mathrm{Max}^r \circ \mathrm{Max}^r = \mathrm{Max}^r$. *In particular, for any* $\mathfrak{M} \in \mathrm{Mod}^{r,\phi}_{/\mathfrak{S}_\infty}$, *the object* $\mathrm{Max}^r(\mathfrak{M})$ *is maximal.*

Proof. — Just remark that $\mathrm{Max}^r(\mathfrak{M})[1/u] = \mathfrak{M}[1/u]$. □

PROPOSITION 3.3.4. — *The functor* Max^r *is left exact.*

Proof. — Let $0 \to \mathfrak{M}' \to \mathfrak{M} \to \mathfrak{M}'' \to 0$ be an exact sequence in $\mathrm{Mod}^{r,\phi}_{/\mathfrak{S}_\infty}$. We have the following commutative diagram:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathfrak{M}' & \longrightarrow & \mathfrak{M} & \longrightarrow & \mathfrak{M}'' & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \iota^{\mathfrak{M}'}_{\max}} & & \downarrow{\scriptstyle \iota^{\mathfrak{M}}_{\max}} & & \downarrow{\scriptstyle \iota^{\mathfrak{M}''}_{\max}} & & \\
0 & \longrightarrow & \mathrm{Max}^r(\mathfrak{M}') & \longrightarrow & \mathrm{Max}^r(\mathfrak{M}) & \longrightarrow & \mathrm{Max}^r(\mathfrak{M}'') & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathfrak{M}'[1/u] & \longrightarrow & \mathfrak{M}[1/u] & \longrightarrow & \mathfrak{M}''[1/u] & \longrightarrow & 0
\end{array}$$

where the first line is exact by assumption and the last one is also exact because of the flatness of $\mathfrak{S}[1/u]$ over $\mathfrak{S}$. We have to show that the middle line is exact. Injectivity is obvious.

Let's prove $\mathrm{Max}^r(\mathfrak{M}') = \mathrm{Max}^r(\mathfrak{M}) \cap \mathfrak{M}'[1/u]$. The inclusion $\subset$ is clear. Now, remark that $\mathfrak{M}'_{\max} = \mathrm{Max}^r(\mathfrak{M}) \cap \mathfrak{M}'[1/u]$ is a $\mathfrak{S}$-submodule of $\mathfrak{M}'[1/u]$ of finite type, which is stable under ϕ. Let $x \in \mathfrak{M}'_{\max}$. Then there exists $y \in \phi^\star \mathrm{Max}^r(\mathfrak{M})$ and $z \in \phi^\star \mathfrak{M}'[1/u]$ such that $E(u)^r x = \mathrm{id} \otimes \phi(y) = \mathrm{id} \otimes \phi(z)$ (if $r = \infty$, it must be replaced by a sufficiently large integer). Since $\mathrm{id} \otimes \phi : \phi^\star \mathfrak{M}[1/u] \to \mathfrak{M}[1/u]$ is injective, we have $y = z \in \phi^\star \mathfrak{M}'_{\max}$. Hence $\mathfrak{M}'_{\max}$ is an object of $`\mathrm{Mod}^{r,\phi}_{/\mathfrak{S}}$ and the claimed equality is indeed true. This gives directly the exactness at middle. □

REMARK. — Unfortunately, Max^r is not right exact (even on $\mathrm{Mod}^{r,\phi}_{/\mathfrak{S}_1}$) if $er \geqslant p-1$. For instance, consider $\mathfrak{M} = \mathfrak{S}_1 e_1 \oplus \mathfrak{S}_1 e_2$ equipped with ϕ defined by $\phi(e_1) = e_1$ and $\phi(e_2) = ue_1 + u^{p-1}e_2$. Denote by $\mathfrak{M}'$ the submodule of $\mathfrak{M}$ generated by e_1. We can easily see that $\mathfrak{M}$ and $\mathfrak{M}'$ are both *maximal* objects of $\mathrm{Mod}^{r,\phi}_{/\mathfrak{S}_1}$. However, $\mathfrak{M}/\mathfrak{M}'$ is isomorphic to $\mathfrak{S}_1$ with $\phi(1) = u^{p-1}$. It is not maximal since $\frac{1}{u}\mathfrak{S}_1$ is finitely generated and stable under ϕ.

PROPOSITION 3.3.5. — *Let $\mathfrak{M} \in \mathrm{Mod}^{r,\phi}_{/\mathfrak{S}_\infty}$. The couple $(\mathrm{Max}^r(\mathfrak{M}), \iota^{\mathfrak{M}}_{\max})$ is characterized by the following universal property:*

- *the morphism $T_{\mathfrak{S}_\infty}(\iota^{\mathfrak{M}}_{\max})$ is an isomorphism;*
- *for each couple $(\mathfrak{M}', f)$ where $\mathfrak{M}' \in \mathrm{Mod}^{r,\phi}_{/\mathfrak{S}_\infty}$ and $f : \mathfrak{M} \to \mathfrak{M}'$ becomes an isomorphism under $T_{\mathfrak{S}_\infty}$, there exists a unique map $g : \mathfrak{M}' \to \mathrm{Max}^r(\mathfrak{M})$ such that $g \circ f = \iota^{\mathfrak{M}}_{\max}$.*

Proof. — The first point is clear.

Take $(\mathfrak{M}', f)$ as in the proposition. Since the quotient $\mathfrak{M}/\mathrm{Max}^r(\mathfrak{M})$ is killed by a power of u, the map g is uniquely determined. On the other hand, by full faithfulness of ${}^{\backprime}T_{\mathcal{O}_\mathcal{E}}$, f induces an isomorphism $\tilde{f} : \mathfrak{M}[1/u] \to \mathfrak{M}'[1/u]$. Denote by g the restriction of $\tilde{f}^{-1}$ to $\mathfrak{M}'$. Since $\mathfrak{M}'$ is finitely generated over $\mathfrak{S}$, $g(\mathfrak{M}')$ is also and hence $g(\mathfrak{M}') \subset \mathrm{Max}^r(\mathfrak{M})$ (by definition of Max^r). In other words, g induces a map $\mathfrak{M}' \to \mathrm{Max}^r(\mathfrak{M})$ and it is easy to check that $g \circ f = \iota^{\mathfrak{M}}_{\max}$.

It remains to prove that the universal property characterizes $\mathrm{Max}^r(\mathfrak{M})$. But if $\mathfrak{M}'$ satisfies also the universal property, we get two maps $\mathfrak{M}' \to \mathrm{Max}^r(\mathfrak{M})$ and $\mathrm{Max}^r(\mathfrak{M}) \to \mathfrak{M}'$ whose composites in both direction must be identity. □

The category $\mathrm{Max}^{r,\phi}_{/\mathfrak{S}_\infty}$

DEFINITION 3.3.6. — We put $\mathrm{Max}^{r,\phi}_{/\mathfrak{S}_\infty} = \mathrm{Max}^r(\mathrm{Mod}^{r,\phi}_{/\mathfrak{S}_\infty})$. It is a full subcategory of $\mathrm{Mod}^{r,\phi}_{/\mathfrak{S}_\infty}$.

We now show several pleasant properties of this category.

PROPOSITION 3.3.7. — *The functor $\mathrm{Max}^r : \mathrm{Mod}^{r,\phi}_{/\mathfrak{S}_\infty} \to \mathrm{Max}^{r,\phi}_{/\mathfrak{S}_\infty}$ is a left adjoint to the inclusion functor $\mathrm{Max}^{r,\phi}_{/\mathfrak{S}_\infty} \to \mathrm{Mod}^{r,\phi}_{/\mathfrak{S}_\infty}$.*

Proof. — Let $f : \mathfrak{M} \to \mathfrak{M}'$ be a morphism in $\mathrm{Mod}^{r,\phi}_{/\mathfrak{S}_\infty}$ and assume that $\mathfrak{M}'$ is maximal. We have to prove that there exists a unique map $\tilde{f} : \mathrm{Max}^r(\mathfrak{M}) \to \mathfrak{M}'$ such that $\tilde{f} \circ \iota^{\mathfrak{M}}_{\max} = f$. The uniqueness is implied by the following observation: $\mathfrak{M}'$ has no u-torsion, and $\mathrm{Max}^r(\mathfrak{M})/\mathfrak{M}$ is canceled by a power of u. For the existence, just remark that $\tilde{f} = \mathrm{Max}^r(f)$ is appropriate. □

THEOREM 3.3.8. — *The category* $\mathrm{Max}^{r,\phi}_{/\mathfrak{S}_\infty}$ *is abelian. More precisely, if* $f : \mathfrak{M} \to \mathfrak{M}'$ *is a morphism in* $\mathrm{Max}^{r,\phi}_{/\mathfrak{S}_\infty}$

- *the kernel of* f *in the usual sense is an object of* $\mathrm{Max}^{r,\phi}_{/\mathfrak{S}_\infty}$ *and is the kernel of* f *in the abelian category* $\mathrm{Max}^{r,\phi}_{/\mathfrak{S}_\infty}$;
- *the cokernel of* f *in the usual sense,* $\mathrm{coker}\, f$*, is an object of* $\mathrm{Mod}^{r,\phi}_{/\mathfrak{S}_\infty}$ *and* $\mathrm{Max}^r(\frac{\mathrm{coker}\, f}{u\text{-torsion}})$ *is the cokernel of* f *in the abelian category* $\mathrm{Max}^{r,\phi}_{/\mathfrak{S}_\infty}$*; moreover if* f *is injective, then* $\mathrm{coker}\, f$ *has no* u*-torsion ;*
- *the image (resp. coimage) of* f *in the usual sense is an object of* $\mathrm{Mod}^{r,\phi}_{/\mathfrak{S}_\infty}$ *and its image under the functor* Max^r *is the image (resp. coimage) of* f *in the abelian category* $\mathrm{Max}^{r,\phi}_{/\mathfrak{S}_\infty}$.

Proof. — Let $f : \mathfrak{M} \to \mathfrak{M}'$ be a morphism in $\mathrm{Max}^{r,\phi}_{/\mathfrak{S}_\infty}$. By Proposition 2.1.1 (iii), $\mathfrak{K} = \ker f$ is an object of $\mathrm{Mod}^{r,\phi}_{/\mathfrak{S}_\infty}$. It remains to prove that it is maximal. Denote by $\mathfrak{M}_{\max}$ the $\mathfrak{S}$-submodule of $\mathfrak{M}[1/u]$ generated by $\mathrm{Max}^r(\mathfrak{K})$ and $\mathfrak{M}$. It satisfies Condition (2) (because $\mathrm{Max}^r(\mathfrak{K})$ and $\mathfrak{M}$ satisfy it) and hence, by Proposition 2.1.1 (ii), it is an object of $\mathrm{Mod}^{r,\phi}_{/\mathfrak{S}_\infty}$ included in $\mathfrak{M}[1/u]$. Since $\mathfrak{M}$ is assumed to be maximal, we get $\mathfrak{M}_{\max} \subset \mathfrak{M}$ and then $\mathrm{Max}^r(\mathfrak{K}) \subset \mathfrak{M}$. It follows $\mathrm{Max}^r(\mathfrak{K}) \subset \mathfrak{M} \cap \mathfrak{K}[1/u] \subset \mathfrak{K}$ (for the last inclusion, use $\mathfrak{K}[1/u] = \ker(f \otimes_{\mathfrak{S}} \mathfrak{S}[1/u])$), and $\mathrm{Max}^r(\mathfrak{K}) = \mathfrak{K}$.

With Proposition 3.3.7, it is easy to prove that $\mathrm{Max}^r(\frac{\mathrm{coker}\, f}{u\text{-torsion}})$ is the cokernel of f in $\mathrm{Max}^{r,\phi}_{/\mathfrak{S}_\infty}$. The implication ($f$ injective) $\Rightarrow$ ($\mathrm{coker} f \in \mathrm{Mod}^{r,\phi}_{/\mathfrak{S}_\infty}$) is showed as in Proposition 3.3.4.

We have already seen that the usual image of f, say $\mathrm{im}\, f$, is an object of $\mathrm{Mod}^{r,\phi}_{/\mathfrak{S}_\infty}$ (Proposition 2.1.1 (iii)). Let $g : \mathrm{im}\, f \to \mathfrak{M}'$ be the natural inclusion. We have $\mathrm{coker}\, g = \mathrm{coker}\, f$. On the other hand, since $\mathrm{Max}^r(g)$ is an injective morphism between two maximal objects, its cokernel has no u-torsion. Together with $g \otimes_{\mathfrak{S}} \mathfrak{S}[1/u] = \mathrm{Max}^r(g) \otimes_{\mathfrak{S}} \mathfrak{S}[1/u]$, it implies $\mathrm{coker}\, \mathrm{Max}^r(g) = \frac{\mathrm{coker}\, f}{u\text{-torsion}}$. Now, applying the left-exact functor Max^r (see Proposition 3.3.4) to the exact sequence $0 \to \mathrm{Max}^r(\mathrm{im}\, f) \to \mathfrak{M}' \to \frac{\mathrm{coker}\, f}{u\text{-torsion}} \to 0$, we get $\mathrm{Max}^r(\mathrm{im}\, f) = \ker(\mathfrak{M}' \to \mathfrak{C})$ where $\mathfrak{C} = \mathrm{Max}^r(\frac{\mathrm{coker}\, f}{u\text{-torsion}})$. Statement about image is then proved.

Finally, by definition, the usual coimage (resp. coimage in $\mathrm{Max}^{r,\phi}_{/\mathfrak{S}_\infty}$) of f is the usual cokernel (resp. cokernel in $\mathrm{Max}^{r,\phi}_{/\mathfrak{S}_\infty}$) of the inclusion $\ker f \to \mathfrak{M}$. The announced property about coimages follows and then also the identification between image and coimage. □

LEMMA 3.3.9. — *Let* $\alpha : \mathfrak{M}' \to \mathfrak{M}$ *and* $\beta : \mathfrak{M} \to \mathfrak{M}''$ *be morphisms in* $\mathrm{Max}^{r,\phi}_{/\mathfrak{S}_\infty}$ *such that* $\beta \circ \alpha = 0$. *The sequence* $0 \to \mathfrak{M}' \to \mathfrak{M} \to \mathfrak{M}'' \to 0$

is exact in (the abelian category) $\mathrm{Max}^{r,\phi}_{/\mathfrak{S}_\infty}$ *if and only if the sequence* $0 \to \mathfrak{M}'[1/u] \to \mathfrak{M}[1/u] \to \mathfrak{M}''[1/u] \to 0$ *is exact.*

Moreover, the functor $\backslash\!M_{\mathcal{O}_\mathcal{E}} : \mathrm{Max}^{r,\phi}_{/\mathfrak{S}_\infty} \to \mathrm{Mod}^{r,\phi}_{/\mathcal{O}_\mathcal{E}}$ *is fully faithful.*

REMARK. — The reader should be very careful with the following point. There are two *different* notions of exact sequences in $\mathrm{Max}^{r,\phi}_{/\mathfrak{S}_\infty}$. The first one is given by the structure of abelian category whereas the second one is just the "restriction" of the notion of exact sequence in $\mathrm{Mod}^{r,\phi}_{/\mathfrak{S}_\infty}$. From now on, we will only consider the first one. This is for instance the reason why Corollary 3.3.11 is not in contradiction with the counter-example given after Proposition 3.3.4.

Proof. — By description of kernels and cokernels given in Theorem 3.3.8, we have the following: the sequence $0 \to \mathfrak{M}' \to \mathfrak{M} \to \mathfrak{M}'' \to 0$ is exact in $\mathrm{Max}^{r,\phi}_{/\mathfrak{S}_\infty}$ if and only if $0 \to \mathfrak{M}' \to \mathfrak{M} \to \mathfrak{M}''$ is exact (as a sequence of $\mathfrak{S}$-modules) and $\mathrm{coker}(\mathfrak{M} \to \mathfrak{M}'')$ is killed by a power of u. The first part of lemma then follows.

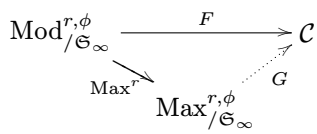
Since for all $\mathfrak{M} \in \mathrm{Max}^{r,\phi}_{/\mathfrak{S}_\infty}$, we have $\mathfrak{M} \subset \mathfrak{M}[1/u]$, the functor $\backslash\!M_{\mathcal{O}_\mathcal{E}}$ is clearly faithful. Let $\mathfrak{M}$ and $\mathfrak{M}'$ be two objects of $\mathrm{Max}^{r,\phi}_{/\mathfrak{S}_\infty}$ and $f : \mathfrak{M}[1/u] \to \mathfrak{M}'[1/u]$. We have to show that f sends $\mathfrak{M}$ to $\mathfrak{M}'$. Using Proposition 2.1.1 (iii), we have $f(\mathfrak{M}) \in \mathrm{Mod}^{r,\phi}_{/\mathfrak{S}_\infty}$ and by the proof of Proposition 3.2.3, $f(\mathfrak{M}) + \mathfrak{M}'$ (computed in $\mathfrak{M}'[1/u]$) is in $\mathrm{Mod}^{r,\phi}_{/\mathfrak{S}_\infty}$. Hence, by definition of maximal objects $f(\mathfrak{M}) + \mathfrak{M}' \subset \mathfrak{M}'$, and then $f(\mathfrak{M}) \subset \mathfrak{M}'$ as required. □

COROLLARY 3.3.10. — *The functor* $T_{\mathfrak{S}_\infty}$ *defined on* $\mathrm{Max}^{r,\phi}_{/\mathfrak{S}_\infty}$ *is exact and fully faithful.*

COROLLARY 3.3.11. — *The functor* $\mathrm{Max}^r : \mathrm{Mod}^{r,\phi}_{/\mathfrak{S}_\infty} \to \mathrm{Max}^{r,\phi}_{/\mathfrak{S}_\infty}$ *is exact.*

THEOREM 3.3.12. — *The functor* $\mathrm{Max}^r : \mathrm{Mod}^{r,\phi}_{/\mathfrak{S}_\infty} \to \mathrm{Max}^{r,\phi}_{/\mathfrak{S}_\infty}$ *realizes the localization of* $\mathrm{Mod}^{r,\phi}_{/\mathfrak{S}_\infty}$ *with respect to morphisms* f *such that* $T_{\mathfrak{S}_\infty}(f)$ *is an isomorphism.*

Proof. — Take $\mathcal{C}$ a category and $F : \mathrm{Mod}^{r,\phi}_{/\mathfrak{S}_\infty} \to \mathcal{C}$ a functor that satisfies the following implication: if $T_{\mathfrak{S}_\infty}(f)$ is an isomorphism, then $F(f)$ too. We have to show that there exists a unique functor G making the following diagram commutative:



If $\mathfrak{M}$ is in $\mathrm{Max}^{r,\phi}_{/\mathfrak{S}_\infty}$, we must have $G(\mathfrak{M}) = F \circ \mathrm{Max}^r(\mathfrak{M}) = F(\mathfrak{M})$. This proves the uniqueness and gives a candidate for G. Finally, we only have to check that

for all $\mathfrak{M} \in \mathrm{Mod}^{r,\phi}_{/\mathfrak{S}_\infty}$, there exists a canonical isomorphism between $F(\mathfrak{M})$ and $G(\mathrm{Max}^r(\mathfrak{M})) = F(\mathrm{Max}^r(\mathfrak{M}))$. It is given by $F(\iota^{\mathfrak{M}}_{\max})$. □

How to recognize maximal objects?— It seems to be difficult to find a criteria to recognize maximal objects among objects of $\mathrm{Mod}^{r,\phi}_{/\mathfrak{S}_\infty}$. Nevertheless, we have the following stability property.

PROPOSITION 3.3.13. — *The category* $\mathrm{Max}^{r,\phi}_{/\mathfrak{S}_\infty}$ *is stable under extensions in* $\mathrm{Mod}^{r,\phi}_{/\mathfrak{S}_\infty}$.

REMARK. — The proposition means that if $0 \to \mathfrak{M}' \to \mathfrak{M} \to \mathfrak{M}'' \to 0$ is an exact sequence in $\mathrm{Mod}^{r,\phi}_{/\mathfrak{S}_\infty}$ (and not in $\mathrm{Max}^{r,\phi}_{/\mathfrak{S}_\infty}$ — that does not make sense) and if $\mathfrak{M}'$ and $\mathfrak{M}''$ are maximal, then $\mathfrak{M}$ is also. Hence, the proposition does *not* imply that $\mathrm{Max}^{r,\phi}_{/\mathfrak{S}_\infty}$ is the smallest full subcategory of $\mathrm{Mod}^{r,\phi}_{/\mathfrak{S}_\infty}$ containing simple objects described in § 3.6.

Proof. — Assume that $0 \to \mathfrak{M}' \to \mathfrak{M} \to \mathfrak{M}'' \to 0$ is an exact sequence in $\mathrm{Mod}^{r,\phi}_{/\mathfrak{S}_\infty}$ and $\mathfrak{M}'$ and $\mathfrak{M}''$ are maximal. We have the following diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathfrak{M}' & \longrightarrow & \mathfrak{M} & \longrightarrow & \mathfrak{M}'' & \longrightarrow & 0 \\ & & \| & & \downarrow & & \downarrow f & & \\ 0 & \longrightarrow & \mathfrak{M}' & \longrightarrow & \mathrm{Max}^r(\mathfrak{M}) & \longrightarrow & \mathfrak{C} & \longrightarrow & 0 \end{array}$$

where $\mathfrak{C}$ is defined as the cokernel of $\mathfrak{M}' \to \mathrm{Max}^r(\mathfrak{M})$. A diagram chase shows that f is injective. Moreover by Theorem 3.3.8, $\mathfrak{C} \in \mathrm{Mod}^{r,\phi}_{/\mathfrak{S}_\infty}$ and it is easy to check that $\mathfrak{M}''[1/u] = \mathfrak{C}[1/u]$. Since $\mathfrak{M}''$ is maximal, we must have $\mathfrak{M}'' = \mathfrak{C}$, *i.e.* f bijective. It follows that $\mathfrak{M} = \mathrm{Max}^r(\mathfrak{M})$ as required. □

Then we have a sufficient condition to be maximal.

LEMMA 3.3.14. — *Let* $\mathfrak{M} \in \mathrm{Mod}^{r,\phi}_{/\mathfrak{S}_1}$. *If* $\mathrm{coker}\,(\mathrm{id} \otimes \phi)$ *is killed by* u^{p-2} *then* $\mathfrak{M}$ *is maximal.*

Proof. — It follows from the proof of Lemma 3.2.4. □

COROLLARY 3.3.15. — *If* $er < p-1$, *then* $\mathrm{Max}^{r,\phi}_{/\mathfrak{S}_\infty} = \mathrm{Mod}^{r,\phi}_{/\mathfrak{S}_\infty}$.

3.4. Minimal objects. — In this section, we develop a dual notion of maximal objects (called *minimal objects*), that satisfies analogous properties. According to Corollary 3.2.6, we need to assume $r < \infty$.

The functor Min^r.

DEFINITION 3.4.1. — Let $\mathfrak{M} \in \mathrm{Mod}^{r,\phi}_{/\mathfrak{S}_\infty}$. The object $\mathrm{Min}^r(\mathfrak{M})$ is defined as the smallest element of $F^r_{\mathfrak{S}}(\mathfrak{M}[1/u])$. It is endowed with an homomorphism $\iota^{\mathfrak{M}}_{\min} : \mathrm{Min}^r(\mathfrak{M}) \to \mathfrak{M}$ in the category $\mathrm{Mod}^{r,\phi}_{/\mathfrak{S}_\infty}$.

An object $\mathfrak{M}$ of $\mathrm{Mod}^{r,\phi}_{/\mathfrak{S}_\infty}$ is said *minimal* (in $\mathrm{Mod}^{r,\phi}_{/\mathfrak{S}_\infty}$) if $\iota^{\mathfrak{M}}_{\min}$ is an isomorphism.

PROPOSITION 3.4.2. — *The previous definition gives rise to a functor* $\mathrm{Min}^r : \mathrm{Mod}^{r,\phi}_{/\mathfrak{S}_\infty} \to \mathrm{Mod}^{r,\phi}_{/\mathfrak{S}_\infty}$. *Moreover, the collection of map* $(\iota^{\mathcal{M}}_{\min})$ *defines a natural transformation between* Min^r *and the identity functor.*

Proof. — Consider $f : \mathfrak{M}_1 \to \mathfrak{M}_2$ a map in $\mathrm{Mod}^{r,\phi}_{/\mathfrak{S}_\infty}$. In order to prove that Min^r is a functor, we have to show that $f(\mathrm{Min}^r(\mathfrak{M}_1)) \subset \mathrm{Min}^r(\mathfrak{M}_2)$. Since $\mathrm{Mod}^{r,\phi}_{/\mathfrak{S}_\infty}$ is stable under images (Proposition 2.1.1 (iii)), we can assume successively that f is surjective, then injective.

Assume first f is surjective. Put $F = f \otimes_{\mathfrak{S}} \mathfrak{S}[1/u]$ and $\mathfrak{M}'_1 = F^{-1}(\mathrm{Min}^r(\mathfrak{M}_2))$. From the surjectivity of f and $(\mathrm{Min}^r\mathfrak{M}_2)[1/u] = \mathfrak{M}_2[1/u]$, we deduce $\mathfrak{M}'_1[1/u] = \mathfrak{M}_1[1/u]$. Moreover, if $\mathfrak{K} = \ker f$, we have the following commutative diagram:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \phi^\star\mathfrak{K}[1/u] & \longrightarrow & \phi^\star\mathfrak{M}'_1 & \longrightarrow & \phi^\star\mathrm{Min}^r(\mathfrak{M}_2) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \mathrm{id}\otimes\phi}\ \sim & & \downarrow{\scriptstyle \mathrm{id}\otimes\phi'_1} & & \downarrow{\scriptstyle \mathrm{id}\otimes\phi_2} & & \\
0 & \longrightarrow & \mathfrak{K}[1/u] & \longrightarrow & \mathfrak{M}'_1 & \longrightarrow & \mathrm{Min}^r(\mathfrak{M}_2) & \longrightarrow & 0
\end{array}$$

Hence $\mathrm{coker}\,(\mathrm{id} \otimes \phi'_1)$ can be seen as a submodule of $\mathrm{coker}\,(\mathrm{id} \otimes \phi'_2)$ and so it is killed by $E(u)^r$ (if $r = \infty$, it must be replaced by a sufficiently large integer). Therefore, by Proposition 2.1.1 (ii), $\mathfrak{M}'_1 \in F^r_{\mathfrak{S}}(\mathfrak{M}_1[1/u])$ and $\mathrm{Min}^r(\mathfrak{M}_1) \subset \mathfrak{M}'_1$. The conclusion follows.

Now, assume f is injective and consider $\mathfrak{M}_1$ as a subobject of $\mathfrak{M}_2$. Put $\mathfrak{M}'_1 = \mathfrak{M}_1[1/u] \cap \mathrm{Min}^r(\mathfrak{M}_2)$. Since $(\mathrm{Min}^r\mathfrak{M}_2)[1/u] = \mathfrak{M}_2[1/u]$, we have $\mathfrak{M}'_1[1/u] = \mathfrak{M}_1[1/u]$. Let $x \in \mathfrak{M}'_1$. There exists $y \in \phi^\star\mathfrak{M}_1[1/u]$ and $z \in \phi^\star\mathrm{Min}^r(\mathfrak{M}_2)$ such that $x = \mathrm{id} \otimes \phi(y) = \mathrm{id} \otimes \phi(z)$. Since $\mathrm{id} \otimes \phi$ is injective on $\mathfrak{M}_2[1/u]$, we must have $y = z \in \mathfrak{M}'_1$. So, by Proposition 2.1.1 (ii), $\mathfrak{M}'_1 \in F^r_{\mathfrak{S}}(\mathfrak{M}_1[1/u])$. Hence $\mathrm{Min}^r(\mathfrak{M}_1) \subset \mathfrak{M}'_1$, and we are done.

The last statement of the proposition is then obvious. □

PROPOSITION 3.4.3. — *The functor* Min^r *is a projection, that is* $\mathrm{Min}^r \circ \mathrm{Min}^r = \mathrm{Min}^r$.

Proof. — Just use $\mathrm{Min}^r(\mathfrak{M})[1/u] = \mathfrak{M}[1/u]$. □

LEMMA 3.4.4. — *Let* $f : \mathfrak{M} \to \mathfrak{M}'$ *be a morphism in* $\mathrm{Mod}^{r,\phi}_{/\mathfrak{S}_\infty}$. *Then* $f(\mathrm{Min}^r(\mathfrak{M})) = \mathrm{Min}^r(f(\mathfrak{M}))$.

Proof. — First note that $f(\mathfrak{M})$ is an object of $\mathrm{Mod}^{r,\phi}_{/\mathfrak{S}_\infty}$ (Proposition 2.1.1 (iii)) and consequently the formula $\mathrm{Min}^r(f(\mathfrak{M}))$ makes sense.

The inclusion $\subset$ has been proved in Proposition 3.4.2. Put $\mathfrak{M}'' = f(\mathrm{Min}(\mathfrak{M}))$. By Proposition 2.1.1 (iii), it is an object of $\mathrm{Mod}^{r,\phi}_{/\mathfrak{S}_\infty}$ such that $\mathfrak{M}''[1/u] = f(\mathfrak{M})[1/u]$. Hence $\mathrm{Min}(f(\mathfrak{M})) \subset \mathfrak{M}''$ as required. □

COROLLARY 3.4.5. — *Let* $f : \mathfrak{M} \to \mathfrak{M}'$ *be a morphism in* $\mathrm{Mod}^{r,\phi}_{/\mathfrak{S}_\infty}$. *If* f *is injective (resp. surjective), then* $\mathrm{Min}^r(f)$ *is also.*

REMARK. — Dualizing the example given after Proposition 3.3.4, we see that Min^r is not "middle-exact".

PROPOSITION 3.4.6. — *Let* $\mathfrak{M} \in \mathrm{Mod}^{r,\phi}_{/\mathfrak{S}_\infty}$. *The couple* $(\mathrm{Min}(\mathfrak{M}), \iota^{\mathfrak{M}}_{\min})$ *is characterized by the following universal property:*

- *the morphism* $T_{\mathfrak{S}_\infty}(\iota^{\mathfrak{M}}_{\min})$ *is an isomorphism;*
- *for each couple* $(\mathfrak{M}', f)$ *where* $\mathfrak{M}' \in \mathrm{Mod}^{r,\phi}_{/\mathfrak{S}_\infty}$ *and* $f : \mathfrak{M}' \to \mathfrak{M}$ *becomes an isomorphism under* $T_{\mathfrak{S}_\infty}$, *there exists a unique map* $g : \mathrm{Min}(\mathfrak{M}) \to \mathfrak{M}'$ *such that* $f \circ g = \iota^{\mathfrak{M}}_{\max}$.

Proof. — The first point is clear. Take $(\mathfrak{M}', f)$ as in the proposition. Since $T_{\mathfrak{S}_\infty}(f)$ is an isomorphism, f induces an isomorphism $\mathfrak{M}'[1/u] \to \mathfrak{M}[1/u]$ (by full faithfulness of ${}^\backprime T_{\mathcal{O}_\mathcal{E}}$). Hence, f is injective, and we can consider $\mathfrak{M}'$ as a subobject of $\mathfrak{M}$. It is then sufficient to prove that $\mathrm{Min}^r(\mathfrak{M}) \subset \mathfrak{M}'$ but this follows from the definition of Min^r. □

The category $\mathrm{Min}^{r,\phi}_{/\mathfrak{S}_\infty}$.

DEFINITION 3.4.7. — We put $\mathrm{Min}^{r,\phi}_{/\mathfrak{S}_\infty} = \mathrm{Min}^r(\mathrm{Mod}^{r,\phi}_{/\mathfrak{S}_\infty})$. It is a full subcategory of $\mathrm{Mod}^{r,\phi}_{/\mathfrak{S}_\infty}$.

PROPOSITION 3.4.8. — *The functor* $\mathrm{Min}^r : \mathrm{Mod}^{r,\phi}_{/\mathfrak{S}_\infty} \to \mathrm{Min}^{r,\phi}_{/\mathfrak{S}_\infty}$ *is a right adjoint of the inclusion functor* $\mathrm{Min}^{r,\phi}_{/\mathfrak{S}_\infty} \to \mathrm{Mod}^{r,\phi}_{/\mathfrak{S}_\infty}$.

Proof. — We have to prove that if $f : \mathfrak{M} \to \mathfrak{M}'$ is any morphism in $\mathrm{Mod}^{r,\phi}_{/\mathfrak{S}_\infty}$ with $\mathfrak{M}$ minimal, then f factors through $\iota^{\mathfrak{M}'}_{\min}$. This is a direct consequence of Proposition 3.4.2. □

THEOREM 3.4.9. — *The category* $\mathrm{Min}^{r,\phi}_{/\mathfrak{S}_\infty}$ *is abelian. More precisely, if* $f : \mathfrak{M} \to \mathfrak{M}'$ *is a morphism in* $\mathrm{Min}^{r,\phi}_{/\mathfrak{S}_\infty}$

- *the kernel of f in the usual sense is an object of* $\mathrm{Mod}^{r,\phi}_{/\mathfrak{S}_\infty}$ *whose image under* Min^r *is a kernel of f in the abelian category* $\mathrm{Min}^{r,\phi}_{/\mathfrak{S}_\infty}$
- *the cokernel of f in the usual sense,* $\mathrm{coker}\, f$*, may have u-torsion; however* $\frac{\mathrm{coker}\, f}{u\text{-torsion}}$ *is an object of* $\mathrm{Min}^{r,\phi}_{/\mathfrak{S}_\infty}$ *which is a cokernel of f in the abelian category* $\mathrm{Min}^{r,\phi}_{/\mathfrak{S}_\infty}$
- *the image (resp. coimage) of f in the usual sense is an object of* $\mathrm{Min}^{r,\phi}_{/\mathfrak{S}_\infty}$ *and is the image (resp. coimage) of f in the abelian category* $\mathrm{Min}^{r,\phi}_{/\mathfrak{S}_\infty}$.

Proof. — During the proof, we will denote by $\ker f$, $\mathrm{coker}\, f$, $\mathrm{im}\, f$ and $\mathrm{coim}\, f$ the objects computed in the usual sense.

The assertion about kernels results from Propositions 2.1.1 (iii) and 3.4.8. Let's prove the assertion about cokernels. Denote by $\mathfrak{C}$ the quotient of $\mathrm{coker}\, f$ by its u-torsion. Obviously $\mathfrak{C}$ has no u-torsion. Moreover, it satisfies Condition (2), it is finitely generated and it is killed by a power of p (since it is a quotient of $\mathfrak{M}'$). Hence, by Proposition 2.1.1 (ii), $\mathfrak{C} \in \mathrm{Mod}^{r,\phi}_{/\mathfrak{S}_\infty}$. Lemma 3.4.4 applied to the surjective morphism $\mathfrak{M}' \to \mathfrak{C}$ then shows that $\mathfrak{C}$ is minimal.

By definition, the image (in $\mathrm{Min}^{r,\phi}_{/\mathfrak{S}_\infty}$) of f, called $\mathfrak{I}$, is the kernel (in $\mathrm{Min}^{r,\phi}_{/\mathfrak{S}_\infty}$) of $\mathfrak{M}' \to \mathfrak{C}$. Hence $\mathrm{im}\, f \subset \mathfrak{I}$ and the quotient $\mathfrak{I}/\mathrm{im}\, f$ is killed by a power of u. It follows that $\mathrm{Min}^r(\mathrm{im}\, f) = \mathrm{Min}^r(\mathfrak{I}) = \mathfrak{I}$. But, by Lemma 3.4.4, $\mathrm{im}\, f$ is already minimal. Thus $\mathfrak{I} = \mathrm{im}\, f$ as required. The argument is quite similar for coimage (remark that since $\mathrm{coim}\, f$ is isomorphic to $\mathrm{im}\, f$, it is also minimal). □

Lemma 3.4.10. — *Let $\alpha : \mathfrak{M}' \to \mathfrak{M}$ and $\beta : \mathfrak{M}' \to \mathfrak{M}''$ be morphisms in* $\mathrm{Min}^{r,\phi}_{/\mathfrak{S}_\infty}$ *such that $\beta \circ \alpha = 0$. The sequence $0 \to \mathfrak{M}' \to \mathfrak{M} \to \mathfrak{M}'' \to 0$ is exact (in the abelian category)* $\mathrm{Min}^{r,\phi}_{/\mathfrak{S}_\infty}$ *if and only if the sequence $0 \to \mathfrak{M}'[1/u] \to \mathfrak{M}[1/u] \to \mathfrak{M}''[1/u] \to 0$ is exact.*

Moreover, the functor $\grave{M}_{\mathcal{O}_\mathcal{E}} : \mathrm{Min}^{r,\phi}_{/\mathfrak{S}_\infty} \to \mathrm{Mod}^{r,\phi}_{/\mathfrak{S}_\infty}$ *is fully faithful.*

Proof. — The first part of lemma follows from the description of kernels and cokernels given above.

Since for all $\mathfrak{M} \in \mathrm{Min}^{r,\phi}_{/\mathfrak{S}_\infty}$, we have $\mathfrak{M} \subset \mathfrak{M}[1/u]$, the functor is clearly faithful. Let $\mathfrak{M}$ and $\mathfrak{M}'$ be two objects of $\mathrm{Min}^{r,\phi}_{/\mathfrak{S}_\infty}$ and $f : \mathfrak{M}[1/u] \to \mathfrak{M}'[1/u]$. We have to show that f sends $\mathfrak{M}$ to $\mathfrak{M}'$. The proof is the same as in Proposition 3.4.2. □

Corollary 3.4.11. — *The functor $T_{\mathfrak{S}_\infty}$ defined on* $\mathrm{Min}^{r,\phi}_{/\mathfrak{S}_\infty}$ *is exact and fully faithful.*

Corollary 3.4.12. — *The functor* $\mathrm{Min}^r : \mathrm{Mod}^{r,\phi}_{/\mathfrak{S}_\infty} \to \mathrm{Min}^{r,\phi}_{/\mathfrak{S}_\infty}$ *is exact.*

Link with duality.

PROPOSITION 3.4.13. — *Assume r finite. For all $\mathfrak{M} \in \mathrm{Mod}^{r,\phi}_{/\mathfrak{S}_\infty}$, we have natural isomorphisms*

$$\mathrm{Min}^r(\mathfrak{M}^\vee) \simeq \mathrm{Max}^r(\mathfrak{M})^\vee \quad \textit{and} \quad \mathrm{Max}^r(\mathfrak{M}^\vee) \simeq \mathrm{Min}^r(\mathfrak{M})^\vee.$$

In particular, duality permutes subcategories $\mathrm{Min}^{r,\phi}_{/\mathfrak{S}_\infty}$ and $\mathrm{Max}^{r,\phi}_{/\mathfrak{S}_\infty}$.

Proof. — Formula (6) implies that, given a morphism f in the category $\mathrm{Mod}^{r,\phi}_{/\mathfrak{S}_\infty}$, $T_{\mathfrak{S}_\infty}(f)$ is an isomorphism if and only if $T_{\mathfrak{S}_\infty}(f^\vee)$ is. Then the proposition is a formal (and easy) consequence of the universal properties defining Max^r (Proposition 3.3.5) and Min^r (Proposition 3.4.6) on the one hand, and the full faithfulness of $T_{\mathfrak{S}_\infty}$ on $\mathrm{Max}^{r,\phi}_{/\mathfrak{S}_\infty}$ (Corollary 3.3.10) and $\mathrm{Min}^{r,\phi}_{/\mathfrak{S}_\infty}$ (Corollary 3.4.11) on the other hand. □

3.5. A reciprocity formula. — In this subsection, we will use the functor $j_\star$ of Fontaine defined in § B.1.4 of [9]. For $M \in {}'\mathrm{Mod}^{\phi}_{/\mathcal{O}_\mathcal{E}}$, define the ordered set $G_\mathfrak{S}(M)$ as the set of $\mathfrak{S}$-submodules $\mathfrak{M} \subset M$ such that $\mathfrak{M}$ is of finite type over $\mathfrak{S}$, stable under ϕ and $\mathrm{id} \otimes \phi : \phi^\star\mathfrak{M}[1/u] \to \mathfrak{M}[1/u]$ is bijective. Recall that, by definition:

$$j_\star M = \bigcup_{\mathfrak{M} \in G_\mathfrak{S}(M)} \mathfrak{M}.$$

In the same way, we put for any $r \in \{0, 1, \dots, \infty\}$:

$$j^r_\star M = \bigcup_{\mathfrak{M} \in G^r_\mathfrak{S}(M)} \mathfrak{M}$$

where $G^r_\mathfrak{S}(M)$ is the ordered set of all $\mathfrak{M} \in \mathrm{Mod}^{r,\phi}_{/\mathfrak{S}_\infty}$ with $\mathfrak{M} \subset M$ (we do not ask $\mathfrak{M}[1/u]$ to be equal to M). By § B.1.5.3 of [9], the equality $G_\mathfrak{S}(M) = G^\infty_\mathfrak{S}(M)$ holds. Moreover, if $\mathfrak{M}$ is an object of $\mathrm{Mod}^{r,\phi}_{/\mathfrak{S}_\infty}$, (the proof of) Proposition 3.2.3 shows that greatest elements of $F^r_\mathfrak{S}(M)$ and $G^r_\mathfrak{S}(M)$ coincide. Hence $\mathrm{Max}^r(\mathfrak{M}) = j^r_\star(\mathfrak{M}[1/u])$.

Following [13], we define for $r \in \{0, 1, \dots, \infty\}$:

$$\mathfrak{S}^{f,r}_n = j^r_\star(\mathcal{O}_{\mathcal{E}^{\mathrm{ur}}}/p^n\mathcal{O}_{\mathcal{E}^{\mathrm{ur}}}) \subset \mathcal{O}_{\mathcal{E}^{\mathrm{ur}}}/p^n\mathcal{O}_{\mathcal{E}^{\mathrm{ur}}} \quad \text{and} \quad \mathfrak{S}^{f,r} = \varprojlim_n \mathfrak{S}^{f,r}_n \subset \mathcal{O}_{\mathcal{E}^{\mathrm{ur}}}.$$

For all integer n, $\mathfrak{S}^{f,r}_n$ is an object of ${}'\mathrm{Mod}^{r,\phi}_{/\mathfrak{S}_\infty}$, and obviously $\mathfrak{S}^{f,\infty}_n = \bigcup_{r\in\mathbb{N}} \mathfrak{S}^{f,r}_n$. By Proposition 2.5.1 of *loc. cit.*, they are stable under ϕ and the action of G_∞. Furthermore, this proposition implies that $\mathfrak{S}^{f,\infty}$ is the period ring $\mathfrak{S}^{\mathrm{ur}}$ traditionally used in this context (for instance in [11], [14], [13]).

Finally, if $\mathfrak{M} \in \mathrm{Mod}^{r,\phi}_{/\mathfrak{S}_\infty}$ is canceled by p^n, the formula for $T_{\mathfrak{S}_\infty}(\mathfrak{M})$ can be "simplified" as follows:

$$T_{\mathfrak{S}_\infty}(\mathfrak{M}) = \mathrm{Hom}_{\backslash \mathrm{Mod}^{r,\phi}_{/\mathfrak{S}}}(\mathfrak{M}, \mathfrak{S}^{f,r}_n).$$

(To prove this, it is enough to show that the image of any $f \in T_{\mathfrak{S}_\infty}(\mathfrak{M})$ is an object of $\mathrm{Mod}^{r,\phi}_{/\mathfrak{S}_\infty}$, which follows from Proposition 2.1.1.)

Here is the main theorem of this subsection:

THEOREM 3.5.1. — *Let* $\mathfrak{M} \in \mathrm{Mod}^{r,\phi}_{/\mathfrak{S}_\infty}$ *killed by* p^n. *Then* $\mathrm{Max}^r(\mathfrak{M}) = \mathrm{Hom}_{\mathbb{Z}_p[G_\infty]}(T_{\mathfrak{S}_\infty}(\mathfrak{M}), \mathfrak{S}^{f,r}_n)$.

REMARK. — It seems that such a formula does not exist with Min^r (instead of Max^r). Indeed, it would probably imply the left-exactness of Min^r, which is known to be false (see remark after Corollary 3.4.5).

Proof. — Put $\tilde{\mathfrak{M}} = \mathrm{Hom}_{\mathbb{Z}_p[G_\infty]}(T_{\mathfrak{S}_\infty}(\mathfrak{M}), \mathfrak{S}^{f,r}_n)$. It is endowed with a Frobenius ϕ (given by the Frobenius on $\mathfrak{S}^{f,r}_n$). Moreover, biduality gives a natural map compatible with Frobenius:

$$\iota : \mathrm{Max}^r(\mathfrak{M}) \to \mathrm{Hom}_{\mathbb{Z}_p[G_\infty]}(T_{\mathfrak{S}_\infty}(\mathrm{Max}^r(\mathfrak{M})), \mathfrak{S}^{f,r}_n) \simeq \tilde{\mathfrak{M}}.$$

By Remark A.1.2.7.(a) of [9], the composite

$$\begin{aligned}\mathfrak{M}[1/u] \xrightarrow{\iota \otimes_{\mathfrak{S}} \mathfrak{S}[1/u]} \tilde{\mathfrak{M}}[1/u] \hookrightarrow & \mathrm{Hom}_{\mathbb{Z}_p[G_\infty]}(T_{\mathfrak{S}_\infty}(\mathrm{Max}^r(\mathfrak{M})), \mathcal{O}_{\mathcal{E}^{\mathrm{ur}}}/p^n\mathcal{O}_{\mathcal{E}^{\mathrm{ur}}}) \\ = & \mathrm{Hom}_{\mathbb{Z}_p[G_\infty]}(\backslash T_{\mathcal{O}_\mathcal{E}}(\mathfrak{M}[1/u]), \mathcal{O}_{\mathcal{E}^{\mathrm{ur}}}/p^n\mathcal{O}_{\mathcal{E}^{\mathrm{ur}}})\end{aligned}$$

is bijective. Hence, $\iota \otimes_{\mathfrak{S}} \mathfrak{S}[1/u]$ is also a bijection. We want to prove that ι itself is an isomorphism. Injectivity is clear since $\mathrm{Max}^r(\mathfrak{M})$ has no u-torsion. Since $\mathrm{Max}^r(\mathfrak{M}) = j^r_\star(\mathfrak{M}[1/u])$, surjectivity will follow from the statement "every $f \in \tilde{\mathfrak{M}}$ is contained in an object $\mathfrak{N} \in G^r_{\mathfrak{S}}(\mathfrak{M}[1/u])$". Let us prove the claim. Consider $e_1, \ldots, e_d$ a generating family of $\mathfrak{M}$ and put $x_i = f(e_i)$. By definition of $\mathfrak{S}^{f,r}_n$, there exists $\mathfrak{N}_i \subset \mathcal{O}_{\mathcal{E}^{\mathrm{ur}}}/p^n\mathcal{O}_{\mathcal{E}^{\mathrm{ur}}}$ with $\mathfrak{N}_i \in \mathrm{Mod}^{r,\phi}_{/\mathfrak{S}_\infty}$ and $x_i \in \mathfrak{N}_i$. Then, as usual using Proposition 2.1.1, we can check that $\mathfrak{N} = \mathrm{Hom}_{\mathbb{Z}_p[G_\infty]}(T_{\mathfrak{S}_\infty}(\mathrm{Max}^r(\mathfrak{M})), \sum_{i=1}^d \mathfrak{N}_i)$ answers the question. □

3.6. Simple objects. — For simplicity, we assume in this subsection $c_0 = 1$ (recall that $c_0 = \frac{E(0)}{p}$). Of course, it is not crucial but assuming this will allow us to simplify several formulas and several definitions of objects.

We fix an element $r \in \{0, 1, 2, \ldots, \infty\}$.

3.6.1. *Definitions and basic properties.*

DEFINITION 3.6.1. — Let $\mathcal{S}'$ be the set of sequences of integers between 0 and er that are periodic (from the start). To a sequence $(n_i) \in \mathcal{S}'$, we associate several numeric invariants:

- its dimension d: it is the smallest period of (n_i);
- for $i \in \mathbb{Z}/d\mathbb{Z}$, the integer $s_i = n_i p^{d-1} + n_{i+1}p^{d-2} + \cdots + n_{i+d-1}$;
- for $i \in \mathbb{Z}/d\mathbb{Z}$, $t_i = \frac{s_i}{p^d-1} \in \mathbb{Q}/\mathbb{Z}$ and $t = t_0$.

We also associate an object $\mathfrak{M}(n_i) \in \mathrm{Mod}^{r,\phi}_{/\mathfrak{S}_1}$ defined as follows:

- as a $\mathfrak{S}_1$-module, $\mathfrak{M}(n_i) = \bigoplus_{i\in\mathbb{Z}/d\mathbb{Z}} e_i\,\mathfrak{S}_1$;
- for all $i \in \mathbb{Z}/d\mathbb{Z}$, $\phi(e_i) = u^{n_i} e_{i+1}$.

Let $\mathcal{S}$ be the subset of $\mathcal{S}'$ consisting of all sequences (n_i) for which the elements $t_0, \ldots, t_{d-1}$ are pairwise distinct (in $\mathbb{Q}/\mathbb{Z}$).

PROPOSITION 3.6.2. — *Assume $r < \infty$. Let (n_i) and (m_i) be two sequences in $\mathcal{S}'$. If $n_i + m_i = er$ for all i, then duality permutes objects $\mathfrak{M}(n_i)$ and $\mathfrak{M}(m_i)$.*

Proof. — Easy computation. □

LEMMA 3.6.3. — *Let $(n_i) \in \mathcal{S}$ and s be a non negative integer. Let (E) be the equation $\phi^d(x) = u^s x$ in variable $x \in \mathfrak{M}(n_i)$ (resp. $x \in \mathfrak{M}(n_i)[1/u]$). Then (E) has a non zero solution if and only if there exists $i \in \mathbb{Z}/d\mathbb{Z}$ (necessarily unique) and v a non negative integer (resp. an integer) such that $s - s_i = v(p^d - 1)$. In this case, the set of solutions is $\{\alpha u^v e_i,\ \alpha \in k \cap \mathbb{F}_{p^d}\}$.*

Proof. — First, remark that if $p^d - 1$ divides $s - s_i$ and $s - s_j$, we get $s_i \equiv s_j \pmod{p^d - 1}$ and then $t_i \equiv t_j \pmod{\mathbb{Z}}$. Hence, by assumption, $i = j$ (in $\mathbb{Z}/d\mathbb{Z}$). This justifies the uniqueness of i.

An easy computation gives $\phi^d(e_i) = u^{s_i} e_i$ for all i. Write $x = x_0 e_0 + \cdots + x_{d-1}e_{d-1}$ with $x_i \in \mathfrak{S}_1 = k[[u]]$ (resp $x_i \in \mathfrak{S}[1/u]$). Then (E) becomes the system $u^{s_i} x_i^{p^d} = u^s x_i$, and the lemma follows. □

PROPOSITION 3.6.4. — *Let (n_i) and (n'_i) be in $\mathcal{S}$. The objects $\mathfrak{M}(n_i)$ and $\mathfrak{M}(n'_i)$ are isomorphic if and only if there exists an integer b such that $n'_{i+b} = n_i$ for all i.*

Proof. — The condition is obviously sufficient. Now, take (n_i), d and s_i, *etc.* as in Definition 3.6.1. We have to show that knowing $\mathfrak{M} = \mathfrak{M}(n_i)$, we can recover the sequence (n_i) up to a shift. Since d is the dimension of $\mathfrak{M}$, it is clearly determined. By Lemma 3.6.3, integers s_i are exactly integers s such that there exists $x \in \mathfrak{M}$, $x \notin u\mathfrak{M}$ such that $\phi^d(x) = u^s x$. So, their set is also determined. Moreover if x_i is a non zero solution of $\phi^d(x_i) = u^{s_i} x_i$, we can

write $x_i = \alpha_i e_i$ with $\alpha_i \in k$. It follows that ϕ maps $x_i\mathfrak{S}_1$ to $x_{i+1}\mathfrak{S}_1$ and then that the sequence (s_i) is determined up to circular permutation. It remains to prove that the knowledge of (s_i) determines the sequence (n_i). But we have an equality

$$\begin{pmatrix} s_0 \\ s_1 \\ \vdots \\ s_{d-1} \end{pmatrix} = M \begin{pmatrix} n_{d-1} \\ n_0 \\ \vdots \\ n_{d-2} \end{pmatrix}$$

where M is a matrix with integer coefficients whose reduction modulo p is identity. Hence it is invertible and proposition follows. □

3.6.2. *Maximum and minimum objects.* — Here, we compute functors Min^r and Max^r on objects $\mathfrak{M}(n_i)$. We first define several subsets of $\mathcal{S}'$.

DEFINITION 3.6.5. — Put $m = \min\{er, p-1\}$.

Let $\mathcal{S}_{\max} \subset \mathcal{S}'$ be the set of sequences of integers between 0 and m that are periodic except that the constant sequence with value $p-1$ is removed from $\mathcal{S}_{\max}$ (if necessary).

If $r < \infty$, define $\mathcal{S}_{\min} \subset \mathcal{S}'$ as the set of sequences of integers between $er - m$ and er that are periodic except that the constant sequence with value $er - (p-1)$ is removed from $\mathcal{S}_{\min}$ (if necessary).

LEMMA 3.6.6. — *We have $\mathcal{S}_{\max} \subset \mathcal{S}$ and $\mathcal{S}_{\min} \subset \mathcal{S}$ (if r is finite).*

Proof. — Exercise. (For Max, one may consider expansion of t_i's in p-basis.) □

Until the end of this subsection, the assumption $r < \infty$ will always be implicit when dealing with minimal objects.

PROPOSITION 3.6.7. — *Let $(n_i) \in \mathcal{S}_{\max}$ (resp $(n_i) \in \mathcal{S}_{\min}$). Then $\mathfrak{M}(n_i)$ is maximal (resp. minimal).*

Proof. — By duality, we only have to prove the statement with Max. By examining the proof of Lemma 3.2.4, we see that $\mathrm{Max}(\mathfrak{M}(n_i)) \subset \frac{1}{u}\mathfrak{M}(n_i)$. Assume by contradiction, that there exists an element $x \in \mathrm{Max}(\mathfrak{M}(n_i))$, $x \notin \mathfrak{M}(n_i)$ and write $ux = x_0e_0 + \cdots + x_{d-1}e_{d-1}$ with $x_i \in \mathfrak{S}_1$ and $x_j \notin u\mathfrak{S}_1$ for one index j. A computation gives:

$$\phi(x) = \frac{\phi(x_0)}{u^{p-n_0}}e_1 + \cdots + \frac{\phi(x_{d-2})}{u^{p-n_{d-2}}}e_{d-1} + \frac{\phi(x_{d-1})}{u^{p-n_{d-1}}}e_0.$$

This element have to lie in $\mathrm{Max}(\mathfrak{M}(n_i))$. Therefore $p - n_j \leqslant 1$, *i.e.* $n_j \geqslant p-1$. So $n_j = p-1$. Repeating the argument with $\phi(x)$ instead of x, we obtain $n_{j+1} = p-1$, and so on. Finally, $n_i = p-1$ for all i and $(n_i) \notin \mathcal{S}_{\max}$. □

PROPOSITION 3.6.8. — *For any $(n_i) \in \mathcal{S}$, there exists a sequence $(m_i) \in \mathcal{S}_{\max}$ (resp. $(m_i) \in \mathcal{S}_{\min}$) such that $\mathrm{Max}(\mathfrak{M}(n_i)) = \mathfrak{M}(m_i)$ (resp. $\mathrm{Min}(\mathfrak{M}(n_i)) = \mathfrak{M}(m_i)$).*

Proof. — By duality, we only have to prove the statement with Max. Denote by s_i' the unique integer in $[0, p^d - 1[$ congruent to s_i modulo $p^d - 1$, and define m_i to be the quotient in the Euclidean division of s_i' by p. It is easy to see that the m_i's $(0 \leqslant i \leqslant d-1)$ are digits in p-basis of s_0', and that this property implies $(m_i) \in \mathcal{S}_{\max}$. Now, put $q_i = \frac{s_i - s_i'}{p^d - 1}$: it is the quotient in the Euclidean division of s_i by p. These numbers are non negative integers and they satisfy the relation $pq_i + m_i = q_{i+1} + n_i$ for all $i \in \mathbb{Z}/d\mathbb{Z}$.

Denote by $\mathfrak{M}'$ the submodule of $\mathfrak{M}[1/u]$ generated by $e_i' = \frac{1}{u^{q_i}} e_i$. A direct computation gives $\phi(e_i') = u^{m_i} e_{i+1}'$, and then $\mathfrak{M}' \simeq \mathfrak{M}(m_i)$. Moreover Proposition 3.6.7 shows that $\mathfrak{M}'$ is maximal. The conclusion follows. □

REMARK. — If (n_i) is in $\mathcal{S}'$ but not in $\mathcal{S}$, almost all arguments of the proof are still correct. The only problem is that the sequence (m_i) obtained is periodic with period less than d.

COROLLARY 3.6.9. — *Let $(n_i) \in \mathcal{S}$. If $\mathfrak{M}(n_i)$ is maximal (resp. minimal) then (n_i) is in $\mathcal{S}_{\max}$ (resp. $\mathcal{S}_{\min}$).*

Proof. — By Proposition 3.6.8, we can find a sequence $(m_i) \in \mathcal{S}_{\max}$ such that $\mathfrak{M}(n_i) = \mathrm{Max}(\mathfrak{M}(n_i)) \simeq \mathfrak{M}(m_i)$. By Proposition 3.6.4, there exists an integer b such that $n_i = m_{i+b}$ for all i, and then $(n_i) \in \mathcal{S}_{\max}$. □

COROLLARY 3.6.10. — *Let (n_i) and (n_i') be in $\mathcal{S}$. Objects $\mathrm{Max}(\mathfrak{M}(n_i))$ and $\mathrm{Max}(\mathfrak{M}(n_i'))$ (resp. $\mathrm{Min}(\mathfrak{M}(n_i))$) and $\mathrm{Min}(\mathfrak{M}(n_i'))$) are isomorphic if and only if there exists an integer b such that $t \equiv p^b t' \pmod{\mathbb{Z}}$ (with obvious notations).*

Proof. — Easy after Proposition 3.6.4 and proof of Proposition 3.6.8. □

3.6.2.1. *Classification.* — With notations of [16], § 1, an easy computation gives the following theorem.

THEOREM 3.6.11. — *Let $(n_i) \in \mathcal{S}_{\max}$. Then $T_{\mathfrak{S}_\infty}(\mathfrak{M}(n_i))$ is an irreducible representation of G_∞ whose tame inertia weights are exactly the n_i's.*

REMARK. — For $(n_i) \in \mathcal{S}_{\min}$, tame inertia weights of $T_{\mathfrak{S}_\infty}(\mathfrak{M}(n_i))$ are not simply linked with the n_i's. Precisely, to make the computation, the method is to write the rational number t_i in p-basis and then to read its digits.

PROPOSITION 3.6.12. — *We assume k to be algebraically closed. Let $(n_i) \in \mathcal{S}$. The object $\mathrm{Max}(\mathfrak{M}(n_i))$ (resp. $\mathrm{Min}(\mathfrak{M}(n_i))$) is simple in $\mathrm{Max}^{r,\phi}_{/\mathfrak{S}_\infty}$ (resp. $\mathrm{Min}^{r,\phi}_{/\mathfrak{S}_\infty}$). All simple objects can be written in this form.*

Proof. — If $er < p-1$, the proposition was already proved in § 4 of [7]. From now on, we assume $er \geqslant p-1$. Moreover, it suffices, using duality, to show the proposition with Max.

By the exactness and the full faithfulness of $T_{\mathfrak{S}_\infty}$ on $\mathrm{Max}^{r,\phi}_{/\mathfrak{S}_\infty}$ (Corollary 3.3.10), in order to show that $\mathrm{Max}(\mathfrak{M}(n_i))$ is simple, it is enough to justify that $T_{\mathfrak{S}_\infty}(\mathrm{Max}(\mathfrak{M}(n_i)))$ is an irreducible representation, which is a direct consequence of the previous theorem. Now, consider $\mathfrak{M} \in \mathrm{Max}^{r,\phi}_{/\mathfrak{S}_\infty}$ a simple object. By the previous theorem and the classification of irreducible representations given in § 1.5 and § 1.6 of [16] [(5)], there exists a quotient of $T_{\mathfrak{S}_\infty}(\mathfrak{M})$ isomorphic to $T_{\mathfrak{S}_\infty}(\mathfrak{M}(n_i))$ for some sequence $(n_i) \in \mathcal{S}_{\max}$. Since $er \geqslant p-1$, we have $\mathfrak{M}(n_i) \in \mathrm{Mod}^{r,\phi}_{/\mathfrak{S}_\infty}$ and $\mathfrak{M}(n_i) = \mathrm{Max}^r(\mathfrak{M}(n_i))$ (since (n_i) is in $\mathcal{S}_{\max}$). Finally, full faithfulness of $T_{\mathfrak{S}_\infty}$ on $\mathrm{Max}^{r,\phi}_{/\mathfrak{S}_\infty}$ gives a non-vanishing morphism $\mathfrak{M}(n_i) \to \mathfrak{M}$, and the proposition follows. □

Corollary 3.6.13. — *Assume that k is algebraically closed. Then, for any r, the essential closure of the category $T_{\mathfrak{S}_\infty}(\mathrm{Mod}^{r,\phi}_{/\mathfrak{S}_\infty})$ is stable under quotients and subobjects.*

Proof. — Noting that $T_{\mathfrak{S}_\infty}(\mathrm{Mod}^{r,\phi}_{/\mathfrak{S}_\infty}) = T_{\mathfrak{S}_\infty}(\mathrm{Max}^{r,\phi}_{/\mathfrak{S}_\infty})$, the corollary is a direct consequence of Theorem 3.6.11, Proposition 3.6.12 and Property 6.4.2 of [7]. □

3.7. Reformulation with $\mathrm{Mod}^{r,\phi}_{/S_\infty}$. — Under the equivalence of Theorem 2.3.1, almost all results proved in previous subsections can be translated into statements about $\mathrm{Mod}^{r,\phi}_{/S_\infty}$. Here is a quite long theorem that summarizes what one obtains.

Theorem 3.7.1. — *Assume $r < p-1$.*

– (cf. Proposition 3.3.5) *For each $\mathcal{M} \in \mathrm{Mod}^{r,\phi}_{/S_\infty}$, there exists a unique couple $(\mathrm{Max}^r(\mathcal{M}), \iota^{\mathcal{M}}_{\max} : \mathcal{M} \to \mathrm{Max}^r(\mathcal{M}))$ (where $\mathrm{Max}^r(\mathcal{M})$ is an objects of $\mathrm{Mod}^{r,\phi}_{/S_\infty}$ and $\iota^{\mathcal{M}}_{\max}$ a morphism in this category) such that:*
  - *the morphism $T_{\mathrm{qst}}(\iota^{\mathcal{M}}_{\max})$ is an isomorphism;*
  - *for any $\mathcal{M}' \in \mathrm{Mod}^{r,\phi}_{/S_\infty}$ endowed with a morphism $f : \mathcal{M} \to \mathcal{M}'$ such that $T_{\mathrm{qst}}(f)$ is an isomorphism, there exists a unique $g : \mathcal{M}' \to \mathrm{Max}^r(\mathcal{M})$ such that $g \circ f = \iota^{\mathcal{M}}_{\max}$.*

  This construction gives rise to a functor $\mathrm{Max}^r : \mathrm{Mod}^{r,\phi}_{/S_\infty} \to \mathrm{Mod}^{r,\phi}_{/S_\infty}$ together with a natural transformation $\iota_{\max} : \mathrm{id}_{\mathcal{M}} \to \mathrm{Max}^r$.

(5) In this reference, the classification is made for G_K-representations, but it is easily seen that the same arguments works with G_∞-representations.

- (cf. Proposition 3.3.3) *The function* Max^r *is a projection, that is* $\mathrm{Max}^r \circ \mathrm{Max}^r = \mathrm{Max}^r$.
- (cf. Proposition 3.3.7 and Theorem 3.3.12) *The functor* $\mathrm{Max}^r : \mathrm{Mod}^{r,\phi}_{/S_\infty} \to \mathrm{Max}^{r,\phi}_{/S_\infty}$ *is a left adjoint to the inclusion* $\mathrm{Max}^{r,\phi}_{/S_\infty} \to \mathrm{Mod}^{r,\phi}_{/S_\infty}$ *and realizes the localization of* $\mathrm{Mod}^{r,\phi}_{/S_\infty}$ *with respect to morphisms* f *such that* $T_{\mathrm{qst}}(f)$ *is an isomorphism.*
- (cf. Theorem 3.3.8) *The essential image of* Max^r, *called* $\mathrm{Max}^{r,\phi}_{/S_\infty}$, *is an abelian category.*
- (cf. Corollary 3.3.10) *The restriction of* T_{qst} *on* $\mathrm{Max}^{r,\phi}_{/S_\infty}$ *is exact and fully faithful.*

Of course, we also have a dual version of this theorem with minimal objects. It gives rise to a function $\mathrm{Min}^r : \mathrm{Mod}^{r,\phi}_{/S_\infty} \to \mathrm{Mod}^{r,\phi}_{/S_\infty}$ whose essential image is denoted by $\mathrm{Min}^{r,\phi}_{/S_\infty}$. It is clear that duality on $\mathrm{Mod}^{r,\phi}_{/S_\infty}$ discussed in § 2.4 permutes functors Max^r and Min^r and categories $\mathrm{Max}^{r,\phi}_{/S_\infty}$ and $\mathrm{Min}^{r,\phi}_{/S_\infty}$.

Furthermore, if k is algebraically close, we can give a classification of simple objects of $\mathrm{Max}^{r,\phi}_{/S_\infty}$ and $\mathrm{Min}^{r,\phi}_{/S_\infty}$. For any sequence $(n_i) \in \mathcal{S}$ (see Definition 3.6.1) put $\mathcal{M}(n_i) = M_{\mathfrak{S}_\infty}(\mathfrak{M}(n_i))$. It is described as follows:

- $\mathcal{M}(n_i) = \bigoplus_{i \in \mathbb{Z}/d\mathbb{Z}} f_i\, S_1$;
- $\mathrm{Fil}^r \mathcal{M}(n_i) = \sum_{i \in \mathbb{Z}/d\mathbb{Z}} u^{er-n_i} f_i\, S_1$;
- for all $i \in \mathbb{Z}/d\mathbb{Z}$, $\phi_r(u^{er-n_i} f_i) = (-1)^r f_{i+1}$.

THEOREM 3.7.2. — *Assume the residue field* k *algebraically closed, and* $r < p-1$.

For all sequence $(n_i) \in \mathcal{S}_{\max}$ *(resp.* $(n_i) \in \mathcal{S}_{\min}$*), the object* $\mathcal{M}(n_i)$ *is simple in* $\mathrm{Max}^{r,\phi}_{/S_\infty}$ *(resp. in* $\mathrm{Min}^{r,\phi}_{/S_\infty}$*). Every simple object of* $\mathrm{Max}^{r,\phi}_{/S_\infty}$ *(resp. of* $\mathrm{Min}^{r,\phi}_{/S_\infty}$*) is isomorphic to* $\mathcal{M}(n_i)$ *for a certain sequence* $(n_i) \in \mathcal{S}_{\max}$ *(resp.* $(n_i) \in \mathcal{S}_{\min}$*). Moreover, two objects* $\mathcal{M}(n_i)$ *and* $\mathcal{M}(m_i)$ *are isomorphic if and only if there exists an integer* b *such that* $n_i = m_{i+b}$ *for all* i.

The G_∞*-representation* $T_{\mathrm{qst}}(\mathcal{M}(n_i))$ *is irreducible and its tame inertia weights are exactly the* n_i*'s.*

4. The case $r = 1$

We assume $r = 1$. The forgetful functor $\mathrm{Mod}^{1,\phi,(N)}_{/S_\infty} \to \mathrm{Mod}^{1,\phi}_{/S_\infty}$ is an equivalence of categories (see Lemma 5.1.2 of [5]), and therefore, quasi-semi-stable representations are exactly restrictions to G_∞ of quotients of two lattices in a

crystalline representation with Hodge-Tate weights in $\{0,1\}$. Moreover, they are also (restrictions to G_∞ of) representations of the form $\mathcal{G}(\bar{K})$ where $\mathcal{G}$ is a finite flat group scheme over $\mathcal{O}_K$ killed by a power of p. Let denote by $\mathrm{Rep}_\infty^{[0,1]}(G_K)$ (resp. $\mathrm{Rep}_\infty^{[0,1]}(G_\infty)$) their category. We have the following commutative diagram

$$\begin{array}{ccccc} \mathrm{Mod}^{1,\phi,(N)}_{/S_\infty} & & \xrightarrow{\quad T_{\mathrm{st}} \quad} & & \mathrm{Rep}_\infty^{[0,1]}(G_K) \\ \sim\downarrow & & & & \downarrow \\ \mathrm{Mod}^{1,\phi}_{/S_\infty} & \xrightarrow{\mathrm{Max}^1} & \mathrm{Max}^{1,\phi}_{/S_\infty} & \xrightarrow{T_{\mathrm{qst}}} & \mathrm{Rep}_\infty^{[0,1]}(G_\infty) \end{array}$$

where vertical arrows represent forgetful functors.

PROPOSITION 4.0.1. — *The functor T_{st} factors through $\mathrm{Max}^{1,\phi}_{/S_\infty}$.*

Proof. — By the last statement of Theorem 3.7.1, it is sufficient to prove that if $T_{\mathrm{qst}}(f)$ is an isomorphism, then $T_{\mathrm{st}}(f)$ is also (where f in any map in $\mathrm{Max}^{1,\phi}_{/S_\infty}$). But it is obvious since $T_{\mathrm{qst}}(f) = T_{\mathrm{st}}(f)$. □

COROLLARY 4.0.2. — *The functor $\mathrm{Rep}_\infty^{[0,1]}(G_K) \to \mathrm{Rep}_\infty^{[0,1]}(G_\infty)$ is fully faithful. In other words, if $F : T \to T'$ is a G_∞-equivariant map between two objects of $\mathrm{Rep}_\infty^{[0,1]}(G_K)$, then it is G_K-equivariant.*

Moreover, $T_{\mathrm{st}} : \mathrm{Max}^{1,\phi}_{/S_\infty} \to \mathrm{Rep}_\infty^{[0,1]}(G_K)$ is fully faithful.

Proof. — If $\mathcal{M}$ and $\mathcal{M}'$ are objects of $\mathrm{Max}^{1,\phi}_{/S_\infty}$, the composite

$$\begin{aligned} \mathrm{Hom}_{\mathrm{Max}^{1,\phi}_{/S_\infty}}(\mathcal{M},\mathcal{M}') \to \mathrm{Hom}_{\mathrm{Rep}_\infty^{[0,1]}(G_K)}(T_{\mathrm{st}}(\mathcal{M}'),T_{\mathrm{st}}(\mathcal{M})) \\ \to \mathrm{Hom}_{\mathrm{Rep}_\infty^{[0,1]}(G_\infty)}(T_{\mathrm{qst}}(\mathcal{M}'),T_{\mathrm{qst}}(\mathcal{M})) \end{aligned}$$

is bijective (by full faithfulness of T_{qst}) whereas the second map is obviously injective. This implies that both maps are bijective. Since $T_{\mathrm{st}} : \mathrm{Max}^{1,\phi}_{/S_\infty} \to \mathrm{Rep}_\infty^{[0,1]}(G_K)$ is essentially surjective (by definition of $\mathrm{Rep}_\infty^{[0,1]}(G_K)$), the corollary follows. □

REMARK. — The first part of corollary was already known (Theorem 3.4.3 of [4]). However, the proof given here is slightly different.

5. Perspectives and questions

The semi-stable and crystalline case. — Of course, one may ask if the previous theory can be extended to the semi-stable case. Precisely:

QUESTION 1. — Can we find a simple criteria to recognize an object of $\mathrm{Mod}^{r,\phi,N}_{/S_\infty}$ that can be written as a quotient of two strongly divisible modules?

QUESTION 2. — Are Theorems 3.7.1 and 3.7.2 (with $N(f_i)=0$) still true if we replace $\mathrm{Mod}^{r,\phi}_{/S_\infty}$ by $\mathrm{Mod}^{r,\phi,N}_{/S_\infty}$ ($\mathrm{Max}^{r,\phi}_{/S_\infty}$ by $\mathrm{Max}^{r,\phi,N}_{/S_\infty}$, and T_{qst} by T_{st})?

It seems quite difficult to find a satisfying answer to question 1. For the moment, the authors do not know if any object can be written such as a quotient, although they conjecture it is false. On the other hand, Question 2 seems more accessible and is actually partially answered in [8].

Finally note that links between crystalline and semi-stable torsion theory seem to be more complicated than it looks. Denote by $\mathrm{Mod}^{r,\phi,(N)}_{/S_\infty}$ the full subcategory of $\mathrm{Mod}^{r,\phi,N}_{/S_\infty}$ gathering objects $\mathcal{M}$ satisfying $N(\mathcal{M})\subset(uS+\mathrm{Fil}^1S)\mathcal{M}$. If $r=1$, we saw that the forgetful functor $\mathrm{Mod}^{r,\phi,(N)}_{/S_\infty}\to\mathrm{Mod}^{r,\phi}_{/S_\infty}$ is an equivalence and then one can identify $\mathrm{Mod}^{r,\phi,(N)}_{/S_\infty}$ and $\mathrm{Mod}^{r,\phi}_{/S_\infty}$. However, if $r>1$, this functor is not anymore fully faithful and consequently one can *not* identify $\mathrm{Mod}^{r,\phi,(N)}_{/S_\infty}$ as a subcategory of $\mathrm{Mod}^{r,\phi}_{/S_\infty}$.

Here is a counter-example. Assume $e\geqslant\frac{p-1}{r-1}$. Assume also that there exists $\lambda\in S_1$ such that $\lambda^{p-1}\equiv c \pmod p$. Put $\mathcal{M}=e_1S_1\oplus e_2S_1$, and let $\mathrm{Fil}^r\mathcal{M}$ be the submodule of $\mathcal{M}$ generated by e_1, $u^{e+p-1}e_2$ and $\mathrm{Fil}^pS_1\mathcal{M}$. Equip $\mathcal{M}$ with a Frobenius by putting $\phi_r(e_1)=e_1$ and $\phi_r(u^{e+p-1}e_2)=e_2$. Then it is possible to define on $\mathcal{M}$ two monodromy operators N_1 and N_2 by the formulas $N_1(e_1)=N_2(e_1)=0$, $N_2(e_1)=\lambda u^pe_2$, $N_2(e_2)=0$. These operators give rise to two objects $\mathcal{M}_1$ and $\mathcal{M}_2$ of $\mathrm{Mod}^{r,\phi,(N)}_{/S_\infty}$. They are not isomorphic since $N\circ\phi_r$ vanishes on $\mathrm{Fil}^r\mathcal{M}_1$ but not on $\mathrm{Fil}^r\mathcal{M}_2$. Moreover, one can prove that associated Galois representations (*via* the functor T_{st}) are not isomorphic.

Going further, we can evaluate what should be $\mathrm{Min}(\mathcal{M}_1)$ and $\mathrm{Min}(\mathcal{M}_2)$. For simplicity, assume $e<p-1$. Define $\mathcal{M}'=e'_1S_1\oplus e'_2S_1$ endowed with $\mathrm{Fil}^r\mathcal{M}'$ generated by e'_1, $u^ee'_2$ and $\mathrm{Fil}^pS\mathcal{M}'$. Put $\phi_r(e'_1)=e'_1$ and $\phi_r(u^ee'_2)=e'_2$. Again, we can equip $\mathcal{M}'$ with two monodromy operators N_1 and N_2 defined by $N_1(e'_1)=N_1(e'_2)=0$, $N_2(e'_1)=\lambda e'_2$ and $N_2(e'_2)=0$. Call $\mathcal{M}'_1$ and $\mathcal{M}'_2$ the corresponding objects of $\mathrm{Mod}^{r,\phi,N}_{/S_\infty}$. For $i\in\{1,2\}$, we have a morphism $\mathcal{M}'_i\to\mathcal{M}_i$ (in $\mathrm{Mod}^{r,\phi,N}_{/S_\infty}$) and we can check that it induces an isomorphism *via* T_{st}. Moreover, since $e\leqslant p-2$, $\mathcal{M}'_1$ and $\mathcal{M}'_2$ should be minimal. Therefore

$\mathrm{Min}^r(\mathcal{M}_i)$ should be equal to $\mathcal{M}'_i$ and the implication $(\mathcal{M} \in \mathrm{Mod}^{r,\phi,(N)}_{/S_\infty}) \Rightarrow (\mathrm{Min}(\mathcal{M}) \in \mathrm{Mod}^{r,\phi,(N)}_{/S_\infty})$ should (surprisingly) be false.

A point of view with sheaves. — Proposition 3.3.7 and Theorem 3.3.8 show that the situation is quite similar to what happens with presheaves and sheaves. More concretely we may ask the following question:

QUESTION 3. — Is it possible to see objects of $\mathrm{Mod}^{r,\phi}_{/\mathfrak{S}_\infty}$ (resp. $\mathrm{Max}^{r,\phi}_{/\mathfrak{S}_\infty}$) as global sections of some presheaves (resp. sheaves) on a certain site, in such a way that the functor Max corresponds to the functor "associated sheaf"?

Is it possible to find such presheaves and sheaves in certain cohomology groups of certain varieties?

In order to precise the latest question, assume $r = 1$. Consider $\mathcal{G}$ a finite flat group scheme killed by a power of p over $\mathcal{O}_K$. In [3], Breuil manages to associate to $\mathcal{G}$ an object $\mathcal{M} \in \mathrm{Mod}^{r,\phi}_{/S_\infty}$ using geometric construction. We can ask the following:

QUESTION 4. — Is it possible to find an only geometric recipe that associates to $\mathcal{G}$ the object $\mathrm{Max}(\mathcal{M})$? For instance, can we obtain this recipe by sheafifying (in a certain way) the construction of Breuil?

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