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ON NONIMBEDDABILITY OF HARTOGS FIGURES INTO COMPLEX MANIFOLDS

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Abstract. — We prove the impossibility of imbeddings of Hartogs figures into general complex manifolds which are close to an imbedding of an analytic disc attached to a totally real collar. Analogously we provide examples of the so called thin Hartogs figures in complex manifolds having no neighborhood biholomorphic to an open set in a Stein manifold.

Résumé (*Sur la non-prolongabilité des figures de Hartogs dans les variétés complexes*)
Nous prouvons qu'il est impossible en général de plonger une figure de Hartogs dans une variété complexe proche d'un plongement du disque analytique attaché à une bande totallement réelle. De manière analogue, nous construisons un exemple d'une marmite de Hartogs dans une variété complexe qui n'admet pas un voisignage plongeable dans une variété de Stein.

1. Introduction

We discuss the possibility of imbeddings of Hartogs figures into general complex manifolds.

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Let Δ denote the unit disk in $\mathbb{C}$, $\Delta(r)$ a disk of radius r, Δ^2 a unit bidisk in $\mathbb{C}^2$ and $A_{r_1}^{r_2}$ an annulus $\Delta(r_2) \setminus \bar{\Delta}(r_1), r_1 < r_2$. Recall that a "thick" *Hartogs figure* (or simply Hartogs figure) is a set of the form

$$\begin{aligned} H(\varepsilon) &:= \big\{(z,w) \in \mathbb{C}^2 : |z| < \varepsilon,\ |w| < 1+\varepsilon \text{ or } |z| < 1+\varepsilon,\ 1-\varepsilon < |w| < 1+\varepsilon\big\} \\ &= \Delta(\varepsilon) \times \Delta(1+\varepsilon) \cup \Delta(1+\varepsilon) \times A_{1-\varepsilon}^{1+\varepsilon}, \end{aligned}$$

for some ε, $0 < \varepsilon < 1$.

Let X be a complex manifold of dimension 2 which is foliated by complex curves over the unit disk. More precisely, there is a holomorphic submersion $\pi : X \to \Delta$ with connected fibers $X_z := \pi^{-1}(z)$. Hartogs figures $H(\varepsilon)$ are naturally foliated over the disk in the first factor $\mathbb{C}_z$ of $\mathbb{C}^2_{z,w}$ and we denote the corresponding projection by $\pi_1 : H(\varepsilon) \to \Delta(1+\varepsilon)$. A holomorphic mapping $f : (H(\varepsilon), \pi_1) \to (X, \pi)$ is called foliated if there exists a holomorphic map $\zeta : \Delta(1+\varepsilon) \to \Delta$ such that $\pi(f(z,w)) = \zeta(z)$.

Let $\mathbb{S}^1 := \{w \in \mathbb{C} : |w| = 1\}$ denote the unit circle. Suppose further we are given a smooth family $\Gamma = \{\gamma_z : z \in \Delta\}$ of diffeomorphic images of $\mathbb{S}^1$ with $\gamma_z \subset X_z$ such that:

(i) there are $z \in \Delta$ arbitrarily close to 0 for which γ_z bound a disk in X_z;

(ii) but γ_0 is not supposed to bound a disk in X_0.

Denote by $\mathbb{S}^1_a := \{a\} \times \mathbb{S}^1 \subset \mathbb{C}^2$ circles in corresponding fibers of $H(\varepsilon)$. Set also $\Delta_a := \{a\} \times \Delta \subset \mathbb{C}^2$.

Question. — *Does there exist $\varepsilon > 0$ such that a "thick" Hartogs figure $H(\varepsilon)$ can be holomorphically imbedded into X in the following way:*

1) *imbedding $f : H(\varepsilon) \to X$ is foliated;*

2) *$f(\Delta_0) \subset X_a$ for some $a \in \Delta$ and $f(\mathbb{S}^1_0)$ is homologous to γ_a (this means, in particular, that for this a the curve γ_a is homologous to zero in X_a);*

3) *the curve $f(\mathbb{S}^1_1)$ is contained in X_0 and is homologous to γ_0 in X_0?*

In [2, p. 124] and [1, p. 146] the existence of such imbedding is used as an obvious fact. The main goal of this note is to provide an example giving the negative answer to this question. We shall construct the following:

Example. — There exists a complex surface X with a holomorphic submersion π onto the unit disk Δ such that:

1) all fibers $X_z := \pi^{-1}(z)$ are disks with possible punctures;

2) the fiber X_0 over the origin is a punctured disk; the subset $U \subset \Delta$ consisting of such z that the fiber X_z is a disk, is nonempty, open and $\partial U \ni 0$;

3) for any circle γ_0 around the puncture in X_0 and for any circle γ_a in any of $X_a, a \in U$, there does not exist a foliated holomorphic map f from any "thick" Hartogs figure $H(\varepsilon)$ to X such that $f(\Delta_0) \subset X_a$ and $f(\mathbb{S}^1_1) \subset X_0$ is homologous to γ_0 in the fiber X_0.

This example will be constructed in §2. In §3 we shall discuss imbeddings of the so called "thin" Hartogs figure into Stein manifolds and answer the question asked us by Evgeny Poletsky.

Acknowledgments. — We would like to thank E. Poletsky, who was the first who asked us the question about possibility of imbeddings of Hartogs figures into general complex manifolds. We would like also to acknowledge M. Brunella for sending us the preprint [3] where his erroneous argument with Hartogs figures is replaced by another approach using a sort of "nonparametric" Levi-type extension theorem.

We would like to give our thanks to the Referee of this note for his valuable suggestions which we had use in §3.

At any rate the question about possibility of imbeddings of Hartogs figures into a general complex manifold seems to be of some interest.

2. Construction of the example

Our example is based on the violation of the argument principle.

Let J_{st} denote the usual complex structure in $\mathbb{C}^2_{z,w}$.

Take a function $\lambda(t) \in C^\infty(\mathbb{R}), 0 \le \lambda \le 1$, which satisfies

$$\lambda(t) = \begin{cases} 0 & \text{for } t < \frac{1}{9}, \\ 1 & \text{for } t > \frac{4}{9}. \end{cases}$$

For $k = 0, 1, 2, \dots$ consider the following domain M^k in $\mathbb{C} \times \Delta \subset \mathbb{C}^2_{z,w}$:

$$M^k := (\mathbb{C} \times \Delta) \setminus \big\{(z,w) : \tfrac{1}{3} \le |z| \le \tfrac{2}{3},\ w^2 = z^k \lambda(|z|^2) \text{ or } |z| \ge \tfrac{1}{3},\ w = 0\big\}.$$

Let J_k be the (almost) complex structure on M^k with the basis of (1,0)-forms constituted by $\mathrm{d}z$ and $\mathrm{d}w + a_k \mathrm{d}\bar z$, where

$$a_k(z,w) = \begin{cases} \dfrac{w z^{k+1} \lambda'(|z|^2)}{w^2 - z^k \lambda(|z|^2)} & \text{for } \frac{1}{3} < |z| < \frac{2}{3}, \\ 0 & \text{otherwise.} \end{cases}$$

We shall denote by $\Lambda^{p,q}_{J_k}(M^k)$ the subspace in $\Lambda^{p+q}(M^k)$ consisting of (p,q)-forms relative to J_k.

Lemma 1. — *J_k is well defined on the whole of M^k, is (formally) integrable, hence (M^k, J_k) is a complex manifold. Moreover:*

(i) $J_k = J_{\mathrm{st}}$ *on* $M^k \setminus \big(\overline{A^{2/3}_{1/3}} \times \Delta\big)$;

(ii) *the functions* $f_k(z,w) = w + (z^k/w)\lambda(|z|^2)$ *and* $g(z,w) = z$ *are* J_k*-holomorphic on* M^k;

(iii) $\mathrm{ind}_{|w|=1-\varepsilon} f_k(z,w) = -1$ *for* $|z| \ge 1$ *and* $0 < \varepsilon < \frac{1}{6}$.

Proof. — (i) Integrability condition on an almost complex structure J reads as $\mathrm{d}\Lambda_J^{1,0} \subset \Lambda_J^{2,0} + \Lambda_J^{1,1}$ where $\Lambda_J^{2,0}$ is the linear span of $\Lambda_J^{1,0} \wedge \Lambda_J^{1,0}$ and $\Lambda_J^{1,1}$ is the same for $\Lambda_J^{1,0} \wedge \Lambda_J^{0,1}$. Any form $\alpha \in \Lambda_{J_k}^{1,0}$ is represented as $\alpha_1 \mathrm{d}z + \alpha_2(\mathrm{d}w + a_k \mathrm{d}\bar{z})$ with smooth α_1, α_2, hence, $\mathrm{d}\alpha \equiv \alpha_2 \mathrm{d}a_k \wedge \mathrm{d}\bar{z} \bmod (\Lambda_{J_k}^{2,0} + \Lambda_{J_k}^{1,1})$. Now,

$$\mathrm{d}a_k \wedge \mathrm{d}\bar{z} = \frac{\partial a_k}{\partial z}\mathrm{d}z \wedge \mathrm{d}\bar{z} + \frac{\partial a_k}{\partial w}\mathrm{d}w \wedge \mathrm{d}\bar{z} + \frac{\partial a_k}{\partial \bar{w}}\mathrm{d}\bar{w} \wedge \mathrm{d}\bar{z} \equiv \frac{\partial a_k}{\partial w}\mathrm{d}w \wedge \mathrm{d}\bar{z}$$

mod $\Lambda_{J_k}^{1,1}$ since $\partial a_k / \partial \bar{w} = 0$. Finally,

$$\mathrm{d}w \wedge \mathrm{d}\bar{z} = (\mathrm{d}w + a_k \mathrm{d}\bar{z}) \wedge \mathrm{d}\bar{z} \in \Lambda_{J_k}^{1,1}.$$

(ii) Really,

$$\begin{aligned}\mathrm{d}f_k(z,w) &= \Big(k\frac{z^{k-1}}{w}\lambda + \lambda' \frac{z^k}{w}\bar{z}\Big)\mathrm{d}z + \lambda' \frac{z^{k+1}}{w}\mathrm{d}\bar{z} + \Big(1 - \frac{z^k}{w^2}\lambda\Big)\mathrm{d}w \\ &= \Big(k\frac{z^{k-1}}{w}\lambda + \lambda' \frac{z^k}{w}\bar{z}\Big)\mathrm{d}z + \Big(1 - \frac{z^k}{w^2}\lambda\Big)(\mathrm{d}w + a_k \mathrm{d}\bar{z}) \in \Lambda_{J_k}^{1,0},\end{aligned}$$

i.e. $\bar{\partial} f_k = 0$. The case of $g(z,w) = z$ is obvious since $\mathrm{d}z \in \Lambda_{J_k}^{1,0}(M^k)$.

(iii) is obvious. □

The following lemma tells that regions $R_1 = \{|z| < \frac{1}{3}\}$ and $R_2 = \{|z| > 1\}$ on M^k are separated by a sort of a "barrier" $\overline{A}_{1/3}^{2/3} \times \Delta$ in the sense that a foliated holomorphic map

$$f : (\zeta, \eta) \longmapsto \big(z(\zeta), w(\zeta, \eta)\big)$$

from $H(\varepsilon)$ to (M^k, J_k) such that $f(\mathbb{S}^1_\zeta) \sim \mathbb{S}^1_{z(\zeta)}$ which starts at R_1 (i.e. $|z(0)| < \frac{1}{3}$), cannot reach R_2 (i.e., $|z(1)|$ cannot be greater then 1).

Lemma 2. — *Let $f : (\zeta, \eta) \mapsto (z(\zeta), w(\zeta, \eta))$ be any foliated holomorphic map $(H(\varepsilon), J_{\mathrm{st}}) \to (M^k, J_k)$ such that*

(i) $|z(0)| < \frac{1}{3}$,

(ii) $f(\mathbb{S}^1_\zeta) \sim \mathbb{S}^1_{z(\zeta)}$ *in* $M^k_{z(\zeta)}$ *for all* $\zeta \in \Delta(1+\varepsilon)$.

Then $z(\Delta) \subset \Delta$.

Proof. — Suppose not. Set $U = z^{-1}(\Delta)$. Then $U \neq \Delta$ and there exist a point $\zeta_0 \in \Delta \cap \partial U$ and a curve $\gamma(t)$ from $\gamma(0) = 0$ to $\gamma(1) = \zeta_0$ such that $\gamma(t) \in U$ for $0 \leq t < 1$. The function $F_k(\zeta, \eta) = f_k(z(\zeta), w(\zeta, \eta))$ is holomorphic in $H(\varepsilon)$ and therefore holomorphically extends onto the bidisk $\Delta^2_{1+\varepsilon}$. Since

$$F_k(\zeta, \eta) = w(\zeta, \eta) + \frac{z(\zeta)^k}{w(\zeta, \eta)}\lambda\big(|z(\eta)|^2\big),$$

we see that $\mathrm{ind}_{|\eta|=1} F_k(0,\eta) = \mathrm{ind}_{|\eta|=1} w(0,\eta) \geq 0$ due to J_{st}-holomorphicity of $w(0,\eta)$ on $\{0\} \times \Delta(1+\varepsilon)$. But $|z(\zeta_0)| = 1$, so $\lambda(|z(\zeta_0)|^2) = 1$ and therefore

$$\Big|\frac{z(\zeta_0)^k}{w(\zeta_0,\eta)}\Big| \cdot \lambda\big(|z(\zeta_0)|^2\big) > 1 > \big|w(\zeta_0,\eta)\big| \quad \text{for } |\eta| = 1.$$

As $w(\zeta_0,\eta)$ is holomorphic on $\{\zeta_0\} \times \Delta$, one has

$$\mathrm{ind}_{|\eta|=1} F_k(\zeta_0,\eta) = \mathrm{ind}_{|\eta|=1} \frac{1}{w(\zeta_0,\eta)} = -1$$

due to the condition (ii) of the lemma. This contradicts to the holomorphicity of F_k on $\Delta^2_{1+\varepsilon}$. □

Construction of the counterexample. — Let now

$$K_j : |z - c_j| < r_j$$

be a family of mutually disjoint discs in Δ converging to 0 and Σ_j be the intersection of $\bar{K}_j \times \Delta$ with

$$\Big\{\frac{1}{3} r_j \leq |z - c_j| \leq \frac{2}{3} r_j,\ w^2 = \Big(\frac{z - c_j}{r_j}\Big)^{k_j} \lambda\Big(\frac{|z-c_j|^2}{r_j^2}\Big)\Big\} \cup \Big\{|z - c_j| \geq \frac{1}{3} r_j,\ w = 0\Big\}.$$

Let X be the domain $\Delta^2 \setminus \big(\bigcup_j \Sigma_j \cup \big\{z \notin \bigcup_j K_j,\ w = 0\big\}\big)$ and J be the complex structure (integrable!) in X with the basis of (1,0)-forms constituted by $\mathrm{d}z$ and $\mathrm{d}w + b\mathrm{d}\bar{z}$ where

$$b(z,w) = a_{k_j}\Big(\frac{z - a_j}{r_j}, w\Big) \quad \text{for } z \in K_j,\ j = 1, 2, \ldots$$

and $b = 0$ otherwise. Then $J \in C^\infty(X)$ if $k_j \uparrow \infty$ sufficiently fast. $\pi : X \to \Delta$ denotes the natural projection which X inherits as a domain in $\Delta \times \Delta$.

Checking of the following lemma is straightforward (due to Lemma 2) and is left to the reader.

LEMMA 3. — *Projection π is holomorphic and therefore (X, π) is a holomorphic fibration. Moreover:*

(i) *X_z are disks with punctures; X_0 is a punctured disk; the leaf X_z is a disk for $z \in \bigcup_j \{|z - c_j| < \frac{1}{3} r_j\}$;*

(ii) *there exists no foliated holomorphic map $f = (z, w) : H(\varepsilon) \to (X, J)$ such that $|z(0) - c_j| < \frac{1}{3} r_j$ for some j, $z(1) = 0$ and $f(\mathbb{S}^1_\zeta) \sim \mathbb{S}^1_{z(\zeta)}$ for all $\zeta \in \Delta(1+\varepsilon)$.*

REMARK. — In Lemmas 2 and 3, condition $f(\mathbb{S}^1_\zeta) \sim \mathbb{S}^1_{z(\zeta)}$ can be weakened to $f(\mathbb{S}^1_\zeta) \sim d \cdot \mathbb{S}^1_{z(\zeta)}$ for some $d \in \mathbb{N}$.

3. Imbeddings of the "thin" Hartogs figure

The following question was asked us by Evgeny Poletsky. The "thin" Hartogs figure is the following set in $\mathbb{C}^2$

$$\begin{aligned} H &= \big\{(z,w)\in\mathbb{C}^2 : z=0,\ |w|\le 1 \text{ or } |\mathrm{Re}\, z|\le 1,\ \mathrm{Im}\, z=0,\ |w|=1\big\} \\ &= \big(\{0\}\times\bar{\Delta}\big)\cup\big([-1,1]\times\mathbb{S}^1\big). \end{aligned}$$

Let X be some complex manifold of dimension two and suppose we are given a continuous map $f : H\to X$ such that $f(0,w)$ is holomorphic on the disk $\{0\}\times\Delta$.

Question. — *Assume in addition that $f : H\to X$ is an imbedding. Can one always find a neighborhood $V\supset f(H)$ and an imbedding of V into some Stein manifold?*

The answer to this question turns out to be negative too. Define $f : H\to (M_0,J_0)$ as follows:

- $z(0,\eta)=0$,
- $w(0,\eta)=(1-\varepsilon)\eta$ on $\{0\}\times\bar{\Delta}$ and
- $z(\zeta,\mathrm{e}^{i\theta})=\zeta, w(\zeta,\mathrm{e}^{i\theta})=(1-\varepsilon)\mathrm{e}^{i\theta}$ with $0<\varepsilon<\frac{1}{10}$ on $[-1,1]\times\mathbb{S}^1$,

i.e. f is a scaled tautological imbedding.

Lemma 4. — *There is no neighborhood of $f(H)$ which can be holomorphically imbedded into some Stein manifold.*

Proof. — Suppose that such a neighborhood $V\supset f(H)$ exists and $p : V\to Y$ is a holomorphic imbedding of V into a Stein manifold Y. Let π be the projection $(z,w)\to z$ of M_0 to $\mathbb{C}$. After shrinking V, if necessary, we can assume about the projection $\pi_{|V} : V\to\mathbb{C}$ the following:

(i) for z in a neighborhood W_1 of the origin in $\mathbb{C}_z$ $\pi^{-1}(z)$ is a disk;

(ii) there is a neighborhood W_2 of $[-1,1]$ on $\mathbb{C}_z$ such that $\pi^{-1}(z)$ is an annulus for all $z\in W_2\setminus\bar{W}_1$;

Remark that for any $z\in[-1,1]\setminus\{0\}$ the set $\pi^{-1}(z)\cap f(H)$ is a circle $\{z\}\times\{|w|=1-\varepsilon\}$ denoted by γ_z. It is not difficult to see that for every $z\in W_2$ the curve $\Gamma_z := p(\gamma_z)$ bounds a disk D_z in Y. This follows from the moment conditions. The family $\{D_z\}_{z\in W_2}$ holomorphically depends on z and any function holomorphic on V, and thus on $p(V)$, holomorphically extend onto every D_z.

Therefore, f_0 from the construction of the example in §2 holomorphically extends onto each D_z and therefore should have $\mathrm{ind}_{\gamma_z}(f_0\circ p)=\mathrm{ind}_{\Gamma_z} f_0\ge 0$, which contradicts Lemma 1. ☐

Remark. — Using techniques from [5] and [6] one can show that $f(H)$ has no neighborhood which can be holomorphically imbedded into a holomorphically convex Kähler manifold.

The rest of this paragraph is inspired by the remarks of the Referee. An imbedding $f : H \to X$ of a "thin" Hartogs figure into a complex surface X has a natural invariant. Namely the Maslov index of the curve $\Gamma_0 = f(\gamma_0)$ on the totally real (in the neighborhood of Γ_0) manifold $L := f([-1,1] \times \mathbb{S}^1)$. See [4, p. 156] on the for the definition of Maslov index. In our example Maslov index is zero.

The following simple example with Maslov index 1 is due to the Referee. Consider natural inclusions $H \subset \mathbb{C}^2 \subset \mathbb{CP}^2$. Blow-up points $(0, \frac{1}{2})$ and $(0, -\frac{1}{2})$. Then blow-down the strict transform of the line $\{w = 0\}$. Let

$$\sigma : \mathbb{CP}^2 \longrightarrow \mathbb{CP}^1 \times \mathbb{CP}^1$$

be the birational map obtained. Set $f := \sigma_{|H}$ and $\widetilde{H} = f(H)$. It is clear that there are no nonconstant holomorphic functions in any neighborhood of $\widetilde{H}$ and therefore no neighborhood of $\widetilde{H}$ can be imbedded into a Stein manifold.

The Maslov index of $\{z = 0\} \cap H$ is 0 and therefore, after contraction of the line $\{w = 0\}$ it becomes 1.

In [8] an example in $\mathbb{CP}^2$ is constructed with Maslov index 2.

It would be interesting to construct examples of imbeddings of the thin Hartogs figure into complex surfaces having no neighborhoods imbeddable into Stein manifolds and having negative Maslov index.

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