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RADIATION FIELDS

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Abstract. — We study the "hyperboloidal Cauchy problem" for linear and semilinear wave equations on Minkowski space-time, with initial data in weighted Sobolev spaces allowing singular behavior at the boundary, or with polyhomogeneous initial data. Specifically, we consider nonlinear symmetric hyperbolic systems of a form which includes scalar fields with a $\lambda\phi^p$ nonlinearity, as well as wave maps, with initial data given on a hyperboloid; several of the results proved apply to general space-times admitting conformal completions at null infinity, as well to a large class of equations with a similar non-linearity structure. We prove existence of solutions with controlled asymptotic behavior, and asymptotic expansions for solutions when the initial data have such expansions. In particular we prove that polyhomogeneous initial data (satisfying compatibility conditions) lead to solutions which are polyhomogeneous at the conformal boundary $\mathcal{I}^+$ of the Minkowski space-time.

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RÉSUMÉ (*Champs rayonnants*). — Nous étudions le « problème de Cauchy hyperboloïdal » pour des équations d'ondes linéaires et semi-linéaires sur l'espace-temps de Minkowski, avec des données initiales, singulières au bord, dans des espaces de Sobolev à poids, où polyhomogènes. Plus précisement, nous considérons une classe de systèmes symétriques hyperboliques non-linéaires, compatibles avec l'équation d'onde scalaire $\lambda\phi^p$, ainsi qu'avec des applications d'onde, avec données initiales prescrites sur un hyperboloide. Plusieurs de nos résultats restent valables pour une classe générale d'espace-temps avec complétions conformes à l'infini isotrope, ainsi que pour une large classe d'équations avec une certaine structure des termes non-linéaires. Nous démontrons l'existence de solutions avec comportement asymptotique contrôlé, ainsi que des développements asymptotiques si les données initiales en possèdent. En particulier nous démontrons, sous une condition de compatibilité, que les données initiales polyhomogènes conduisent à des solutions polyhomogènes près du bord conforme $\mathcal{I}^+$ de l'espace-temps de Minkowski.

Contents

1. Introduction

Bondi *et al.* [6] together with Sachs [34] and Penrose [33], building upon the pioneering work of Trautman [36, 37], have proposed in the sixties a set of boundary conditions appropriate for the gravitational field in the radiation regime. A somewhat simplified way of introducing the Bondi-Penrose (BP) conditions is to assume existence of "asymptotically Minkowskian coordinates" $(x^\mu) = (t, x, y, z)$ in which the space-time metric $\mathfrak{g}$ takes the form

$$\mathfrak{g}_{\mu\nu} - \eta_{\mu\nu} = \frac{\overset{1}{h}_{\mu\nu}(t-r,\theta,\varphi)}{r} + \frac{\overset{2}{h}_{\mu\nu}(t-r,\theta,\varphi)}{r^2} + \cdots, \tag{1.1}$$

where $\eta_{\mu\nu}$ is the Minkowski metric $\mathrm{diag}(-1, 1, 1, 1)$, u stands for $t-r$, with r, θ, φ being the standard spherical coordinates on $\mathbb{R}^3$. The expansion above has to hold at, say, fixed u, with r tending to infinity. Existence of classes of solutions of the vacuum Einstein equations satisfying the asymptotic conditions (1.1) follows from the work in [20] together with [3, 4, 18, 19]. As of today it remains an open problem how general, within the class of radiating solutions of vacuum Einstein equations, are those solutions which display the behavior (1.1). Indeed, the results in [1–4, 17], [17] suggest strongly(1) that a more appropriate setup for such gravitational fields is that of *polyhomogeneous* asymptotic expansions:

$$\mathfrak{g}_{\mu\nu} - \eta_{\mu\nu} \in \mathcal{A}_{\rm phg}. \tag{1.2}$$

In the context of expansions in terms of a radial coordinate r tending to infinity, the space of polyhomogeneous functions is defined as the set of smooth functions which have an asymptotic expansion of the form

$$f \sim \sum_{i=0}^{\infty}\sum_{j=0}^{N_i} f_{ij}(u,\theta,\varphi)\frac{\ln^j r}{r^{n_i}}, \tag{1.3}$$

for some sequences n_i, N_i, with $n_i \nearrow \infty$. Here the symbol $\sim$ stands for "being asymptotic to": if the right-hand-side is truncated at some finite i, the remainder term falls off appropriately faster. Further, the functions f_{ij} are supposed to be smooth, and the asymptotic expansions should be preserved under differentiation.(2)

The suggestion, that the expansions (1.2) are better suited for describing the gravitational field in the radiation regime than (1.1), arises from the fact that *generic* – in a well defined sense – initial data constructed in [1–4, 17], are polyhomogeneous. This leads naturally to the question, whether polyhomogeneity of initial data is preserved under evolution dictated by wave equations.

In this paper we answer in the affirmative this question for semi-linear wave equations, and for the wave map equation, on Minkowski space-time. We develop a functional framework appropriate for the analysis of such questions. We prove local in time existence of solutions for classes of equations that include the semi-linear wave equations and the wave map equation on Minkowski space-time, with conormal and with polyhomogeneous initial data. We show that polyhomogeneity is preserved under evolution when appropriate (necessary) corner conditions are satisfied by the initial data. We note that the initial data considered here are more singular than allowed in the existing related results [7, 28, 31]. We are planning to analyse the corresponding problems for the vacuum Einstein equation in a forthcoming publication, see also [30].

(1) *Cf.* [29] and references therein for some further related results.

(2) The choice of the sequences n_i, N_i is not arbitrary, and is dictated by the equations at hand. For example, the analysis of $3+1$ dimensional Einstein equations in [17] suggests that consistent expansions can be obtained with $n_i = i$. On the other hand, Theorem 5.7 below gives actually $n_i = \frac{1}{2}i$ for wave-maps on $2+1$ dimensional Minkowski space-time. We note that the $2+1$ dimensional wave map equation is related to the vacuum Einstein equations with cylindrical symmetry (*cf., e.g.*, [5, 14, 15]).

Our main results are the existence and polyhomogeneity of solutions with appropriate polyhomogeneous initial data for the nonlinear scalar wave equation, and for the wave map equation. We achieve this in a few steps. First, we prove local existence of solutions of these equations in weighted Sobolev spaces, *cf.* Theorems 4.1 and 5.1. The next step is to obtain estimates on the time derivatives, *cf.* Theorems 4.4, 5.4 and 5.6. Those estimates are uniform in time in a neighborhood of the initial data surface if the initial data satisfy compatibility conditions. Somewhat surprisingly, we show that all initial data in weighted Sobolev spaces, not necessarily satisfying the compatibility conditions, evolve in such a way that the compatibility conditions will hold on all later time slices; see Corollary 4.5 and Theorems 5.4 and 5.6. Finally, in Theorems 4.10 and 5.7 we prove polyhomogeneity of the solutions with polyhomogeneous initial data; this requires a hierarchy of compatibility conditions. We hope to be able to show in a near future that polyhomogeneity of solutions can be established, for polyhomogeneous initial data, with a finite number of compatibility conditions.

The restriction to Minkowski space-time in Theorem 5.7 is not necessary, and is only made for simplicity of presentation of the results; the same remark applies to Theorem 4.1. Similarly the choice of the initial data hypersurface as the standard unit hyperboloid is not necessary.

This work is organised as follows: First, the reader is referred to Appendix A for definitions, notations, and the functional spaces involved; we also develop calculus in those spaces there. In Section 2 we briefly recall Penrose's conformal completions, as they provide the link between the asymptotic behavior of fields and the local analysis carried on in this work. In Section 3 we consider linear equations. There the key elements of our analysis are: a) Proposition 3.1 and its variations, which give *a priori* estimates in weighted Sobolev spaces; b) the mechanism for proving polyhomogeneity, provided in the proof of Theorem 3.4. The transition from the linear weighted Sobolev estimates to their nonlinear counterparts is done in Sections 4 and 5. This has already been outlined above, and requires a considerable amount of work. In Appendix B we prove several auxiliary results on ODE's, some of which are fairly straightforward; as those results are used in the body of the paper in various, sometimes involved, iterative arguments, it seemed convenient to have precise statements at hand.

Some of the results proved here have been announced in [16].

2. Conformal completions

The aim of this section is to set-up the framework necessary for our considerations; the results here are well known to relativists, but perhaps less so to the PDE community. In any case they are needed to establish notation. Consider, thus, an $n+1$ dimensional space-time $(\mathcal{M},\mathfrak{g})$ and let

$$\widetilde{\mathfrak{g}} = \Omega^2 \mathfrak{g}. \tag{2.1}$$

Let $\Box_h$ denote the wave operator associated with a Lorentzian metric h,

$$\Box_h f = \frac{1}{\sqrt{|\det h_{\rho\sigma}|}}\,\partial_\mu\big(\sqrt{|\det h_{\alpha\beta}|}\,h^{\mu\nu}\partial_\nu f\big).$$

We recall that the scalar curvature $R = R(\mathfrak{g})$ of $\mathfrak{g}$ is related to the corresponding scalar curvature $\widetilde{R} = \widetilde{R}(\widetilde{\mathfrak{g}})$ of $\widetilde{\mathfrak{g}}$ by the formula

$$\widetilde{R}\,\Omega^2 = R - 2n\Big\{\frac{1}{\Omega}\Box_{\mathfrak{g}}\Omega + \frac{n-3}{2}\frac{|\nabla\Omega|^2_{\mathfrak{g}}}{\Omega^2}\Big\}. \tag{2.2}$$

It then follows from (2.2) that we have the identity

$$\Box_{\widetilde{\mathfrak{g}}}\,(\Omega^{-(n-1)/2} f) = \Omega^{-(n+3)/2}\Big(\Box_{\mathfrak{g}}\, f + \frac{n-1}{4n}(\widetilde{R}\,\Omega^2 - R)f\Big). \tag{2.3}$$

It has been observed by Penrose [33] that the Minkowski space-time $(\mathcal{M},\eta)$ can be conformally completed to a space-time with boundary $(\widetilde{\mathcal{M}},\widetilde{\eta})$, $\widetilde{\eta} = \Omega^2\eta$ on $\mathcal{M}$, by adding to $\mathcal{M}$ two null hypersurfaces, usually denoted by $\mathcal{I}^+$ and $\mathcal{I}^-$, which can be thought of as end points ($\mathcal{I}^+$) and initial points ($\mathcal{I}^-$) of inextendible null geodesics [32,33,38]. We will only be interested in "the future null infinity" $\mathcal{I}^+$; an explicit construction (of a subset of $\mathcal{I}^+$) which is convenient for our purposes proceeds as follows: for $(x^0)^2 < \sum_i (x^i)^2$ we define

$$y^\mu = \frac{x^\mu}{x^\alpha x_\alpha}\cdot \tag{2.4}$$

In the coordinate system $\{y^\mu\}$ the Minkowski metric $\eta \equiv -(\mathrm{d}x^0)^2 + (\mathrm{d}x^1)^2 + (\mathrm{d}x^2)^2 + (\mathrm{d}x^3)^2 = \eta_{\alpha\beta}\,\mathrm{d}x^\alpha\mathrm{d}x^\beta$ takes the form

$$\eta = \frac{1}{\Omega^2}\,\eta_{\alpha\beta}\,\mathrm{d}y^\alpha\mathrm{d}y^\beta, \quad \Omega = \eta_{\alpha\beta}\,y^\alpha y^\beta. \tag{2.5}$$

We note that under (2.4) the exterior of the light cone $C_0^{x^\mu} \equiv \{\eta_{\alpha\beta}x^\alpha x^\beta = 0\}$ emanating from the origin of the x^μ-coordinates is mapped to the exterior of the light cone $C_0^{y^\mu} = \{\eta_{\alpha\beta}\,y^\alpha y^\beta = 0\}$ emanating from the origin of the y^μ-coordinates. The conformal completion is obtained by adding $C_0^{y^\mu}$ to $\mathcal{M}$,

$$\widetilde{\mathcal{M}} = \mathcal{M}\cup\big(C_0^{y^\mu}\setminus\{0\}\big),$$

with the obvious differential structure arising from the coordinate system y^μ. We shall use:

- the symbol $\mathcal{I}$ to denote $C_0^{y^\mu}\setminus\{0\}$, and
- $\mathcal{I}^+$ to denote $C_0^{y^\mu}\setminus\{0\}\cap\{y^0>0\}$.

As already mentioned, $\mathcal{I}$ so defined is actually a subset of the usual $\mathcal{I}$, but this will be irrelevant for our purposes.

We note that (2.4) is singular at the light cone $C_0^{x^\mu}$. This is again irrelevant from our point of view because we are only interested in the behavior of the solutions near $\mathcal{I}^+$, and finite speed of propagation allows us, for that purpose, to disregard what happens near $C_0^{x^\mu}$.

The above procedure can be adapted for several metrics of interest, such as the Schwarzschild, Kerr, or Robinson-Trautman metrics, to similarly yield conformal completions of space-time by the addition of null hypersurfaces $\mathcal{I}^+$. This observation was at the origin of Penrose's proposal to describe systems which are asymptotically flat in lightlike directions through the use of conformal completions.

It is noteworthy that the conformal technique allows one to reduce global-in-time existence problems to local ones; this has been exploited by various authors [8–13] for wave equations on a fixed background space-time. Further, Friedrich [22, 23, 26] has used this approach to obtain a global existence result for Einstein equations to the future of a "hyperboloidal" Cauchy surface, with "small" smoothly conformally compactifiable initial data, *cf.* also [21, 24], and [25].

On a more modest level, the identity (2.3) can be used as a starting point for the analysis of the asymptotic behavior of solutions of the scalar wave equation near $\mathcal{I}^+$, as it reduces the problem to a study of solutions near a null hypersurface. This is the approach used in this paper. There are associated identities for fields of any spin [33], which provide a convenient framework for similar questions for those fields.

3. A class of linear symmetric hyperbolic systems

In this section we define a class of linear symmetric hyperbolic first order systems on a set of the form $M_{x_0} \times I$, where M_{x_0} is defined at the beginning of Appendix A, and where I is an interval corresponding to the time variable, which will be denoted by τ, and we derive our key energy inequality in *weighted* Sobolev spaces. (We note that in some of our further applications the vector $\partial/\partial\tau$ will be lightlike, and not timelike as is usually the case. It should be pointed out that in our conventions the time variable is the last coordinate, allowing x to be the first variable, consistently with the conventions of the preceding sections.)

We start by introducing some notation for the sets within the "space-time" $M_{x_0} \times I$, which will be relevant in what follows:

(3.1a) $t \geq 0,\ \ 2(x_2 + t) < x_1 \leq x_0,\ \ \Sigma_{x_2,x_1,t} = \big\{\tau = t,\ x_2 < x < x_1 - 2t\big\},$

(3.1b) $T > 0,\ \ 2(x_2 + T) < x_1 \leq x_0,\ \ \Omega_{x_2,x_1,T} = \bigcup_{0<\tau<T} \Sigma_{x_2,x_1,\tau},$

(3.1c) $0 \leq 2t < x_1 \leq x_0,\ \ \Sigma_{x_1,t} = \big\{\tau = t,\, 0 < x < x_1 - 2t\big\},$

(3.1d) $0 < 2T < x_1,\quad \Omega_{x_1,T} = \bigcup_{0<t<T} \Sigma_{x_1,t}.$

There[3] is a natural identification between $\Sigma_{x_2,x_1,t}$ and M_{x_2,x_1-2t}, as defined at the beginning of Appendix A, similarly between $\Sigma_{x_1,t}$ and M_{x_1-2t}, and we shall freely make use of such identifications throughout. We shall write $\|f(t)\|_{\mathcal{H}_k^\alpha}$ for $\|f|_{\Sigma_{x_2,x_1,t}}\|_{\mathcal{H}_k^\alpha(\Sigma_{x_2,x_1,t})}$, or for $\|f|_{\Sigma_{x_1,t}}\|_{\mathcal{H}_k^\alpha(\Sigma_{x_1,t})}$, *etc.*; the distinction should be clear from the context.

We shall be interested in symmetric hyperbolic first order systems which in local coordinates take the form

$$\big[A^\mu(z^\nu)\partial_\mu + A(z)\big]f = F, \tag{3.2}$$

where $z^\nu = (y^i, \tau)$, with the following properties:

C1) f and F are sections of a bundle which is a direct sum of two N_1 and N_2 dimensional Riemannian bundles; we will write

$$f = \begin{pmatrix} \varphi \\ \psi \end{pmatrix}, \quad F = \begin{pmatrix} a \\ b \end{pmatrix}. \tag{3.3}$$

In local coordinates φ and a are thus $\mathbb{R}^{N_1}$ valued, while ψ and b are $\mathbb{R}^{N_2}$ valued. The respective scalar products will be denoted by $\langle\, .\, ,.\, \rangle_1$ and $\langle\, .\, ,.\, \rangle_2$. We shall use the generic symbol ∇ to denote[4] a covariant derivative compatible with those scalar products, *e.g.*, if X is a vector field on $\Omega_{x_0,T}$, then

$$X\big(\langle\phi,\psi\rangle_1\big) = \langle\nabla_X\phi,\psi\rangle_1 + \langle\phi,\nabla_X\psi\rangle_1, \tag{3.4}$$

similarly for $\langle\, .\, ,.\, \rangle_2$. The derivative ∇ will also be assumed to be compatible with every other structure at hand whenever useful in the context, *e.g.* a Riemannian metric on M, *etc.*

C2) The left hand side of (3.2) can be written as

$$\begin{pmatrix} E_-^\mu\nabla_\mu\varphi & +L\psi \\ -L^\dagger\varphi & +E_+^\mu\nabla_\mu\psi \end{pmatrix} + \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix}\begin{pmatrix} \varphi \\ \psi \end{pmatrix}, \tag{3.5}$$

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[3] The motivation for the factors of 2, and the general form of the sets considered, arises as follows: The set $\partial M \times I$ should be thought of as a smooth null hypersurface in space-time; *e.g.*, in Minkowski space-time with Minkowskian coordinates y^μ, it can be the intersection of the half-space $\{y^0 \geq \frac{1}{2}\}$ with the light cone emanating from the origin $y^\mu = 0$. Then τ is the Minkowski time, perhaps shifted by a constant, say $\tau = y^0 - \frac{1}{2}$. The coordinate x is a coordinate which vanishes on $\partial M \times I$, in the current example *e.g.*, $x = \sqrt{\sum (y^i)^2} - y^0$. Finally, in such a Minkowskian setup, the hypersurfaces $x = x_1 - 2\tau$, which determine one of the boundaries of the Σ's and Ω's defined in (3.1), correspond to the converging light cones $y^0 + \sqrt{\sum (y^i)^2} = \text{Const}$. The restrictions $2(x_2+t) < x_1 \leq x_0$ (in the definition of $\Sigma_{x_2,x_1,t}$) and $2(x_2+T) < x_1$ (in the definition of $\Omega_{x_2,x_1,T}$) are not necessary, and are only made for simplicity of discussion.

[4] In some situations (3.4) might fail to hold, and some undifferentiated supplementary terms will occur at the right-hand-side of (3.4). We note that our results will not be affected by the occurrence of such terms, provided those terms satisfy bounds as in (3.17).

where L is a first order differential operator. Here $L^\dagger$ denotes the formal adjoint of L, in the sense that if $\Omega = M$, or M_{x_1}, or M_{x_2,x_1}, and if φ, ψ are in $C_1(\overline{\Omega})$, then

$$\int_\Omega \langle \varphi, L\psi\rangle_1 \, \mathrm{d}\mu = \int_\Omega \langle L^\dagger \varphi, \psi\rangle_2 \, \mathrm{d}\mu, \tag{3.6}$$

where $\mathrm{d}\mu$ is a measure on M which will, we hope, be obvious from the context. By density Equation (3.6) will still hold with $\Omega = M_{x_2,x_1}$ for all $\alpha, \beta \in \mathbb{R}$, all $\varphi \in \mathcal{H}_1^\alpha(M_{x_2,x_1})$ and all $\psi \in \mathcal{H}_1^\beta(M_{x_2,x_1})$, at least for $x_1 > 0$. Equation (3.6) forces L not to contain any τ- or x-derivatives, where the letter x denotes a coordinate as defined in Section A, thus

$$L = \ell^A(x,v,\tau)\partial_A + \ell(x,v,\tau). \tag{3.7}$$

It follows that the principal part of the system (3.5) is of the form

$$\begin{pmatrix} E_-^\mu \partial_\mu & \ell^A \partial_A \\ (\ell^A)^t \partial_A & E_+^\mu \partial_\mu \end{pmatrix}, \tag{3.8}$$

where A^t denotes the transpose of a matrix A. Equation (3.8) explicitly shows that (3.5) is symmetric hyperbolic when the $E_\pm^\mu$'s are symmetric with $E_\pm^\tau$ positive definite; the notions of "symmetric hyperbolic" and "symmetrizable hyperbolic" are identified throughout this work.

The hypotheses above will be assumed throughout this section.

3.1. Estimates on the space derivatives of the solutions, $\alpha < -1/2$

Let us pass now to the description of the hypotheses needed to derive weighted energy estimates for space derivatives of f. To obtain such estimates, we shall require the existence of a constant C_1 such that the (matrix-valued) coefficients ℓ^A and ℓ satisfy, in the relevant range of τ's,

$$\|\ell(\tau)\|_{\mathcal{G}_k^0(M_{x_1-2\tau})} + \sum_A \|\ell^A(\tau)\|_{\mathcal{G}_k^0(M_{x_1-2\tau})} \le C_1. \tag{3.9}$$

Similarly writing

$$L^\dagger = \ell^{\dagger A}(x,v,\tau)\partial_A + \ell^\dagger(x,v,\tau), \tag{3.10}$$

we require

$$\|\ell^\dagger(\tau)\|_{\mathcal{G}_k^0(M_{x_1-2\tau})} + \sum_A \|\ell^{\dagger A}(\tau)\|_{\mathcal{G}_k^0(M_{x_1-2\tau})} \le C_1. \tag{3.11}$$

C3) The matrices $E_\pm^\mu$ are symmetric and satisfy

$$E_\pm^\mu n_\mu \ge \varepsilon \,\mathrm{Id}, \quad E_+^\mu \partial_\mu x \le -\varepsilon \,\mathrm{Id}, \quad |E_-^\mu \partial_\mu x| \le C_1 x, \tag{3.12}$$

for some $\varepsilon > 0$. Here n_μ denotes the field of future directed (*i.e.*, $b(d\tau, n) > 0$) b-unit normals to the surfaces $\{\tau = \mathrm{Const}\}$, where b is an auxiliary Riemannian metric h on M. (Later on we will mainly be interested in the case of E_+^μs of the form $E^\mu_\pm = e^\mu_\pm \otimes \mathrm{Id}$, for some vector fields $e^\mu_\pm$.) For simplicity we shall also assume

$$\partial_i E^\tau_\pm = 0; \tag{3.13}$$

this is by no means necessary, but is sufficient for the purposes of this paper. We will further assume[(5)] that the E^μ_-'s satisfy a bound of the form:

$$\begin{aligned}\big\|E^A_-(\tau)\big\|_{\mathcal{G}^0_k(M_{x_1-2\tau})} &+ \big\|E^x_-(\tau)\big\|_{\mathcal{G}^1_k(M_{x_1-2\tau})} \\ &+ \big\|\nabla_\mu E^\mu_-(\tau)\big\|_{L^\infty(M_{x_1-2\tau})} \le C_1.\end{aligned} \tag{3.14}$$

As far as the E^μ_+'s are concerned, we allow singular behavior which should, however, be somewhat less singular than $1/x$; to control that, we require existence of a function $\zeta : \mathbb{R}^+ \to \mathbb{R}^+$, satisfying $\lim_{x\to 0}\zeta(x) = 0$, such that for $0 < x \le x_1 - 2\tau$ we have

$$\big\|E^A_+(\tau)\big\|_{\mathcal{G}^{-1}_k(M_x)} + \big\|E^x_+(\tau)\big\|_{\mathcal{G}^0_k(M_x)} + \big\|x\nabla_\mu E^\mu_+(\tau)\big\|_{L^\infty(M_x)} \le \zeta(x). \tag{3.15}$$

When the operators $E^\mu_\pm\nabla_\mu$ are written out explicitly as

$$E^\mu_\pm\nabla_\mu = E^\mu_\pm\partial_\mu + B_\pm, \tag{3.16}$$

we require that for $0 < x < x_1 - 2\tau$,

$$\big\|B_-(\tau)\big\|_{\mathcal{G}^0_k(M_{x_1-2\tau})} \le C_1, \quad \big\|B_+(\tau)\big\|_{\mathcal{G}^{-1}_k(M_x)} \le \zeta(x). \tag{3.17}$$

C4) The matrices B_{ab}, where $a, b = 1, 2$, satisfy the bounds

$$\begin{cases}\big\|B_{12}(\tau)\big\|_{\mathcal{G}^{-1/2}_k(M_{x_1-2\tau})} + \big\|B_{21}(\tau)\big\|_{\mathcal{G}^{-1/2}_k(M_{x_1-2\tau})} \\ \qquad\qquad + \big\|B_{11}(\tau)\big\|_{\mathcal{G}^0_k(M_{x_1-2\tau})} \le C_1, \\ \big\|B_{22}(\tau)\big\|_{\mathcal{G}^{-1}_k(M_x)} \le \zeta(x),\end{cases} \tag{3.18}$$

this last equation holding again for $0 < x < x_1 - 2\tau$.

Our final hypothesis concerns the "acausal" nature of the boundary of $\Omega_{x_2,x_1,T}$:

C5) $\partial\Omega_{x_2,x_1,T}$ is "non-timelike", in the sense that for any covector n_μ, which is positive on outwards-pointing vectors and vanishes on vectors tangent to $\partial\Omega_{x_2,x_1,T}$ we have, on $\partial\Omega_{x_2,x_1,T} \cap \{\tau > 0\}$,

$$E^\mu_\pm n_\mu \ge 0. \tag{3.19}$$

(We note that (3.12) already guarantees that (3.19) holds on $\partial\Omega_{x_2,x_1,T} \cap \{\tau = T \text{ or } x = 0\}$.)

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(5) We use a convention in which the covariant derivatives $\nabla_\mu E^\mu_\pm$ include terms associated with the vector density character of X^μ defined by (3.21); in particular this should be taken into account when verifying that the estimates (3.14)–(3.15) hold.

The essential point of the above hypotheses is that the boundary $\{x=0\}$ is characteristic for Equation (3.2), with the dimension of the relevant kernel being constant over the boundary.[6] Weighted estimates, in the spirit of Proposition 3.1 below, near such characteristic boundaries hold for general symmetric hyperbolic systems, this will be discussed elsewhere.

Weighted energy inequalities in $\mathcal{H}^\alpha_k$ spaces with arbitrary values of k may be proved under various hypotheses on the coefficients which appear in (3.2). We note one such result for systems satisfying C1–C5, which lies in line with our remaining investigations. The restriction $\alpha \le -\frac12$ seems to be inherent to the problem at hand. We consider first the case $\alpha < -\frac12$; the case $\alpha = -\frac12$ is handled by the same methods, under somewhat more restrictive conditions on the coefficients, in Section 3.2.

Proposition 3.1. — *Suppose that $\alpha < -\frac12$, $k > \frac12 n+1$, $k\in\mathbb{N}$, and set either*

- $f(t) = f_{|\Sigma_{x_1,t}}$, $0 < x_1 \le x_0$, $0 \le t \le t_{\max} \equiv \frac12 x_1$, *or*
- $f(t) = f_{|\Sigma_{x_2,x_1,t}}$, $0 < 2x_2 < x_1 \le x_0$, $0 \le t < t_{\max} \equiv x_1 - 2x_2$.

Under the hypotheses C1–C5*, there exists a constant C_2 depending upon x_1, C_1, n, N, k and α, as well as upon the "error function" ζ and the boundary manifold ∂M, such that for all f satisfying* (3.2) *for which $f(0)\in H^{\mathrm{loc}}_k$ and for all $0 < t \le t_{\max}$ we have*

$$\|f(t)\|^2_{\mathcal{H}^\alpha_k(M_{x_1-2t})} \le C_2 \mathrm{e}^{C_2 t}\Big\{\|f(0)\|^2_{\mathcal{H}^\alpha_k(M_{x_1})} + \int_0^t \mathrm{e}^{C_2(t-s)}\big(\|a(s)\|^2_{\mathcal{H}^\alpha_k(M_{x_1-2s})} + \|b(s)\|^2_{\mathcal{H}^{\alpha-1/2}_k(M_{x_1-2s})}\big)\,\mathrm{d}s\Big\}, \tag{3.20}$$

with an identical estimate with M_{x_1-*} replaced by M_{x_2,x_1-*}.

Remark. — The condition $k > \frac12 n+1$ is needed to obtain C_1-weighted control of the solution; there are no restrictions on k if we have at our disposal an *a priori* C_1 weighted bound for f, and if the coefficients in the equation are suitably regular. In such a case, for $k \le \frac12 n+1$, the inequality (3.20) should be modified by adding a term $\|f(s)\|^2_{\mathcal{B}^\alpha_1(M_{x_1-2s})}$ under the integral appearing in (3.20).

Proof. — We start by proving (3.20) on sets M_{x_2,x_1-t}; in that case we are mainly interested to obtain uniform control for small values of x_2, with eventually x_2 tending to zero; without the uniformity in x_2 the estimate would of course be standard. Keeping this in mind, let X^μ be the "energy-momentum vector density",

$$X^\mu = \sum_{0\le|\beta|\le k} x^{-2\alpha-1+2\beta_1}\big\{\langle\mathcal{D}^\beta\varphi, E^\mu_-\mathcal{D}^\beta\varphi\rangle_1 + \langle\mathcal{D}^\beta\psi, E^\mu_+\mathcal{D}^\beta\psi\rangle_2\big\}. \tag{3.21}$$

[6] We are grateful to H. Friedrich for useful discussions concerning this point.

Suppose, first, that $f(0) \in H^{\mathrm{loc}}_{k+1}$; standard results [35, vol. III] show that $f(t)$ is an element of H^{loc}_{k+1}, and we then have[5]

$$\nabla_\mu X^\mu = N_1 + D_1 + D_2 + E_1 + E_2 + E_3, \tag{3.22}$$

where

$$\left\{\begin{aligned} N_1 &= \sum_{0\le|\beta|\le k} (2\beta_1 - 2\alpha - 1) x^{-2\alpha-2+2\beta_1} \big\langle \mathcal{D}^\beta\psi, (E_+^\mu\partial_\mu x)\mathcal{D}^\beta\psi\big\rangle_2,\\ D_1 &= 2\sum_{0\le|\beta|\le k} x^{-2\alpha-1+2\beta_1}\ \big\langle \mathcal{D}^\beta\varphi, E_-^\mu\nabla_\mu\mathcal{D}^\beta\varphi\big\rangle_1,\\ D_2 &= 2\sum_{0\le|\beta|\le k} x^{-2\alpha-1+2\beta_1}\big\langle \mathcal{D}^\beta\psi, E_+^\mu\nabla_\mu\mathcal{D}^\beta\psi\big\rangle_2,\\ E_1 &= \sum_{0\le|\beta|\le k} (2\beta_1 - 2\alpha - 1) x^{-2\alpha-1+2\beta_1}\Big\langle \mathcal{D}^\beta\varphi, \frac{(E_-^\mu\partial_\mu x)}{x}\mathcal{D}^\beta\varphi\Big\rangle_1,\\ E_2 &= \sum_{0\le|\beta|\le k} x^{-2\alpha-1+2\beta_1}\big\langle \mathcal{D}^\beta\varphi, (\nabla_\mu E_-^\mu)\mathcal{D}^\beta\varphi\big\rangle_1,\\ E_3 &= \sum_{0\le|\beta|\le k} x^{-2\alpha-1+2\beta_1}\big\langle \mathcal{D}^\beta\psi, (\nabla_\mu E_+^\mu)\mathcal{D}^\beta\psi\big\rangle_2. \end{aligned}\right. \tag{3.23}$$

Since $2\alpha+1<0$, from (3.12) one finds that

$$\int_{\Sigma_{x_2,x_1,s}} N_1 \,\mathrm{d}x\mathrm{d}\nu \le -|2\alpha+1|\varepsilon\cdot\|\psi\|^2_{\mathcal{H}_k^{\alpha+1/2}} \tag{3.24}$$

which is strictly negative except if ψ is identically zero, and can be used to control some of the error terms which occur at the right hand side of (3.22). (Here we have used the form (A.4) of $\|\psi\|^2_{\mathcal{H}_k^{\alpha+1/2}}$.) For example, to control E_3 we take any x_3 satisfying $2x_2 \le x_3 \le x_1 - 2s$ (we will make a more precise choice of x_3 later), and we write

$$\int_{\Sigma_{x_2,x_1,s}} E_3\,\mathrm{d}x\mathrm{d}\nu = E_{3,1} + E_{3,2}, \tag{3.25}$$

$$E_{3,1} \equiv \int_{\Sigma_{x_2,x_1,s}\bigcap\{x\ge x_3\}} E_3\,\mathrm{d}x\mathrm{d}\nu, \quad E_{3,2} \equiv \int_{\Sigma_{x_2,x_1,s}\bigcap\{x\le x_3\}} E_3\,\mathrm{d}x\mathrm{d}\nu.$$

By (3.15), $E_{3,2}$ can be estimated as follows:

$$|E_{3,2}| \le \sum_{0\le\beta\le k}\int_{\Sigma_{x_2,x_1,s}\cap\{x\le x_3\}} \zeta(x) x^{-2\alpha-2+2\beta_1}|\mathcal{D}^\beta\psi|^2\,\mathrm{d}x\mathrm{d}\nu \le \frac{(2\alpha+1)\varepsilon}{10}\|\psi\|^2_{\mathcal{H}_k^{\alpha+1/2}},$$

if x_3 is chosen small enough. Once this choice has been done, we can clearly estimate $E_{3,1}$ as

$$E_{3,1} \le C\|\psi\|^2_{\mathcal{H}_k^\alpha},$$

with some constant which is determined by x_3. The integrals of the error terms E_1 and E_2 are estimated in the obvious way, *cf.* (3.12) and (3.14):

$$\int_{\Sigma_{x_2,x_1,s}} (E_1+E_2)\,\mathrm{d}x\,\mathrm{d}\nu \le C\big\|\varphi(s)\big\|^2_{\mathcal{H}^\alpha_k}.$$

To control the terms D_1 and D_2 we use the evolution equations (3.5):

$$\begin{aligned}
(3.26)\quad E^\mu_-\nabla_\mu\mathcal{D}^\beta\varphi &= \mathcal{D}^\beta(E^\mu_-\nabla_\mu\varphi)+[E^\mu_-\nabla_\mu,\mathcal{D}^\beta]\varphi\\
&= -\mathcal{D}^\beta(L\psi+B_{11}\varphi+B_{12}\psi-a)+[E^\mu_-\nabla_\mu,\mathcal{D}^\beta]\varphi\\
&= -L\mathcal{D}^\beta\psi+\mathcal{D}^\beta a+E^\beta_4,\\
E^\beta_4 &= -[\mathcal{D}^\beta,L]\psi+[E^\mu_-\nabla_\mu,\mathcal{D}^\beta]\varphi-\mathcal{D}^\beta(B_{11}\varphi+B_{12}\psi),\\
(3.27)\quad E^\mu_+\nabla_\mu\mathcal{D}^\beta\psi &= L^\dagger\mathcal{D}^\beta\varphi+\mathcal{D}^\beta b+E^\beta_5,\\
E^\beta_5 &= [\mathcal{D}^\beta,L^\dagger]\varphi+[E^\mu_+\nabla_\mu,\mathcal{D}^\beta]\psi-\mathcal{D}^\beta(B_{21}\varphi+B_{22}\psi).
\end{aligned}$$

Integrating D_1+D_2 over $\Sigma_{x_2,x_1,s}$, one finds that the terms containing $L\mathcal{D}^\beta\psi$ and $-L^\dagger\mathcal{D}^\beta\varphi$ in (3.26) and (3.27) cancel out; the terms containing $\mathcal{D}^\beta a$ and $\mathcal{D}^\beta b$ are estimated as (here the somewhat arbitrarily chosen factor 10 can be replaced by any other larger number if desired)

$$\begin{aligned}
2\sum_{0\le|\beta|\le k}\int_{\Sigma_{x_2,x_1,s}} x^{-2\alpha-1+2\beta_1}\big(\langle\mathcal{D}^\beta\varphi,\mathcal{D}^\beta a\rangle_1+\langle\mathcal{D}^\beta\psi,\mathcal{D}^\beta b\rangle_2\big)\,\mathrm{d}x\,\mathrm{d}\nu&\\
\le \|\varphi\|^2_{\mathcal{H}^\alpha_k}+\|a\|^2_{\mathcal{H}^\alpha_k}+\frac{(2\alpha+1)\varepsilon}{10}\|\psi\|^2_{\mathcal{H}^{\alpha+1/2}_k}+\frac{10}{(2\alpha+1)\varepsilon}\|b\|^2_{\mathcal{H}^{\alpha-1/2}_k}&.
\end{aligned}$$

The terms containing the commutators $[\mathcal{D}^\beta,L]\psi$ and $[\mathcal{D}^\beta,L^\dagger]\varphi$, can be estimated[7] using the weighted commutator inequality (A.35), while the B_{11}, B_{12}, *etc.*, terms can be estimated using (A.34), by an expression of the form

$$(3.28)\qquad CC_1\Big(\|\psi\|^2_{\mathcal{H}^\alpha_k}+\|\varphi\|^2_{\mathcal{H}^\alpha_k}+\frac{(2\alpha+1)\varepsilon}{10}\|\psi\|^2_{\mathcal{H}^{\alpha+1/2}_k}\Big).$$

To estimate the commutator terms arising from $E^\mu_\pm$, we calculate, *e.g.*

$$\begin{aligned}
x^k[E^\mu_\pm\partial_\mu,\partial^k_x]\chi &= \sum_{i=1}^k\binom{i}{k}x^i(\partial^i_xE^\mu_\pm)x^{k-i}\partial^{k-i}_x\partial_\mu\chi=E_6+E_7,\\
E_6 &= \sum_{i=1,\,\mu\ne x}^k\binom{i}{k}x^i(\partial^i_xE^\mu_\pm)x^{k-i}\partial^{k-i}_x\partial_\mu\chi.
\end{aligned}$$

(7) This step requires weighted L^∞ control of ϕ and ψ, and weighted $W^{1,\infty}$ control of the coefficients in the equation. The hypothesis $k>\frac12 n+1$ is not needed if such *a priori* bounds are known.

The terms arising from E_6 are estimated in a straightforward way as in (3.28) using (A.36). The dangerous term E_7 can be written as

$$\begin{aligned} E_7 &\equiv \sum_{i=1}^{k} \binom{i}{k} x^i (\partial_x^i E_\pm^x) x^{k-i} \partial_x^{k-i+1} \chi \\ &= \sum_{i=1}^{k} \binom{i}{k} x^{i-1} (\partial_x^{i-1} \partial_x E_\pm^x) x^{k-i+1} \partial_x^{k-i+1} \chi, \end{aligned}$$

and can thus again be estimated as in (3.28) provided that $\partial_x E_-^x \in \mathcal{G}_{k-1}^0$, that $\partial_x E_+^x \in \mathcal{G}_{k-1}^{-1}$, and that (3.15) holds. Other terms in the $E_\pm^\mu$ commutators are handled in a similar way. Summarising, we have derived

$$\Big| \int_{\Sigma_{x_2,x_1,s}} \nabla_\mu X^\mu \mathrm{d}^n\mu \Big| \\ \leq CC_1 \big(\|a(s)\|^2_{\mathcal{H}_k^\alpha} + \|b(s)\|^2_{\mathcal{H}_k^{\alpha-1/2}} + \|\psi(s)\|^2_{\mathcal{H}_k^\alpha} + \|\varphi(s)\|^2_{\mathcal{H}_k^\alpha} \big),$$

where $\mathrm{d}^n\mu$ stands for $\mathrm{d}x\mathrm{d}\nu$, or for any measure uniformly equivalent to $\mathrm{d}x\mathrm{d}\nu$. Stokes theorem,

$$\int_{\Omega_{x_2,x_1,t}} \nabla_\mu X^\mu \mathrm{d}^n\mu \mathrm{d}\tau = \int_{\partial\Omega_{x_1,x_2,t}} X^\mu \mathrm{d}S_\mu,$$

and our hypotheses on the geometry of the problem lead to

$$\|f(t)\|^2_{\mathcal{H}_k^\alpha} \leq C\Big(\|f(0)\|^2_{\mathcal{H}_k^\alpha} + C_1 \int_0^t \big(\|a(s)\|^2_{\mathcal{H}_k^\alpha} + \|b(s)\|^2_{\mathcal{H}_k^{\alpha-1/2}} + \|f(s)\|^2_{\mathcal{H}_k^\alpha} \big) \mathrm{d}s \Big).$$

Gronwall's lemma establishes (3.20) on the family of hypersurfaces $\Sigma_{x_2,x_1,t}$ for $f(t) \in H^{\mathrm{loc}}_{k+1}$. If $f(t) \in H^{\mathrm{loc}}_k$, we approximate $f(0)$ by a sequence of functions $f_n(0)$, with $f_n(0) \in H^{\mathrm{loc}}_{k+1}$ converging to $f(0)$ in $\mathcal{H}_k^\alpha(\Sigma_{x_2,x_1,t})$, and we solve Equation (3.2) with initial data $f_n(0)$. The inequality (3.20) applied to the functions $f_n(t) - f_m(t)$ shows that $f_n(t)$ is Cauchy in $\mathcal{H}_k^\alpha$; passing to the limit $n \to \infty$ the desired result for f's such that $f(0) \in \mathcal{H}_k^\alpha(\Sigma_{x_2,x_1,t})$ easily follows.

Since all the constants above are x_2 independent, an elementary argument using the monotone convergence theorem shows that (3.20) for the $\Sigma_{x_1,t}$'s follows from the one for the $\Sigma_{x_2,x_1,t}$'s by passing to the limit $x_2 \to 0$. □

3.2. Estimates on the space derivatives of the solutions, $\alpha = -1/2$

When $\alpha = -\frac{1}{2}$ we do not have the $\beta_1 = 0$ negative terms in N_1 at our disposal in Equation (3.23), so that we cannot allow coefficients as singular as in the previous section. To handle that case we keep all the structure and regularity conditions already made, with the following supplementary restrictions: Equation (3.15) is replaced by

$$\|E_+^A(\tau)\|_{\mathcal{G}_k^0(M_x)} + \|E_+^x(\tau)\|_{\mathcal{G}_k^1(M_x)} + \|\nabla_\mu E_+^\mu(\tau)\|_{L^\infty(M_x)} \leq C_1. \tag{3.29}$$

Instead of (3.17) we require that

$$\big\|B_\pm(\tau)\big\|_{\mathcal{G}^0_k(M_{x_1-2\tau})} \le C_1, \tag{3.30}$$

while condition (3.18) becomes

$$\big\|B_{ab}(\tau)\big\|_{\mathcal{G}^0_k(M_{x_1-2\tau})} \le C_1. \tag{3.31}$$

We then obtain:

PROPOSITION 3.2. — *Suppose that* $k > \frac{1}{2}n + 1$, $k \in \mathbb{N}$, *and set either*

- $f(t) = f_{|\Sigma_{x_1,t}}$, $0 < x_1 \le x_0$, $0 \le t \le t_{\max} \equiv \frac{1}{2}x_1$, *or*
- $f(t) = f_{|\Sigma_{x_2,x_1,t}}$, $0 < 2x_2 < x_1 \le x_0$, $0 \le t < t_{\max} \equiv x_1 - 2x_2$.

Under the hypotheses C1–C5 *together with* (3.29)–(3.31) *there exists a constant* C_2 *depending upon* x_1, C_1, n, N, k *and the boundary manifold* ∂M, *such that for all* f *satisfying* $f(0) \in H^{\mathrm{loc}}_k$ *and for all* $0 < t \le t_{\max}$ *we have*

$$\begin{aligned}\big\|f(t)\big\|^2_{\mathcal{H}^{-1/2}_k(M_{x_1-2t})} \le C_2 \mathrm{e}^{C_2 t}\Big\{&\|f(0)\|^2_{\mathcal{H}^{-1/2}_k(M_{x_1})} \\ &+ \int_0^t \mathrm{e}^{C_2(t-s)}\big(\|a(s)\|^2_{\mathcal{H}^{-1/2}_k(M_{x_1-2s})} + \|b(s)\|^2_{\mathcal{H}^{-1/2}_k(M_{x_1-2s})}\big)\,\mathrm{d}s\Big\},\end{aligned} \tag{3.32}$$

with an identical estimate with M_{x_1-*} *replaced by* M_{x_2,x_1-*}.

Proof. — The proof is essentially identical, but simpler, to that of Proposition 3.1. We simply note that the key inequality (3.29) is replaced by

$$\begin{aligned}\Big|\int_{\Sigma_{x_2,x_1,s}} \nabla_\mu X^\mu \mathrm{d}^n\mu\Big| \le C C_1\big(&\|a(s)\|^2_{\mathcal{H}^{-1/2}_k} \\ &+ \|b(s)\|^2_{\mathcal{H}^{-1/2}_k} + \|\psi(s)\|^2_{\mathcal{H}^{-1/2}_k} + \|\varphi(s)\|^2_{\mathcal{H}^{-1/2}_k}\big).\end{aligned} \tag{3.33}$$

□

3.3. Estimates on the time derivatives of the solutions. — The hypotheses assumed in the previous section ensure that we can algebraically solve Equation (3.2) for $\partial_\tau f$, and then recursively obtain formulae for $\partial^i_\tau f$. Under the hypotheses of Proposition 3.1, it is then straightforward to obtain estimates on the norms

$$\big\|\big((x\partial_\tau)^i f\big)(\tau)\big\|_{\mathcal{H}^\alpha_{k-i}(\Sigma_{x_1-2\tau})}, \quad 0 \le i \le k,$$

provided suitable weighted conditions are imposed on the τ derivatives of the coefficients of Equation (3.2). However, we would like to obtain derivative estimates without the x factors, uniformly in τ. Clearly a necessary condition for the existence of such estimates is that

$$\big\|(\partial^i_\tau f)(0)\big\|_{\mathcal{H}^\alpha_{k-i}(\Sigma_{x_1})} < \infty, \quad 0 \le i \le k. \tag{3.34}$$

It turns out that (3.34) does not need to hold for arbitrary initial data $f(0) \in \mathcal{H}^\alpha_k$, and the requirement that it does lead to the *j-th order compatibility conditions*: by definition, these are the conditions on $f(0)$ which ensure that Equation (3.34) holds for $0 \leq i \leq j$. Since, for sufficiently differentiable solutions of Equation (3.2), all the derivatives $\partial_\tau^i f(0)$ can be explicitly written as an i-th order differential operator acting on $f(0)$, the compatibility conditions are conditions on the behavior of the initial data $f(0)$ near the "corner" $x = 0$; we shall therefore sometimes refer to them as *corner conditions*. We note that there can be corner conditions in weighted Sobolev spaces, or in weighted Hölder spaces; in this section we will be mainly interested in the latter, defined by Equation (3.39) below.

The following example is instructive in this context: For $0 \leq t < y$ let g be a solution of the $1+1$ dimensional wave equation

$$\Big(\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial y^2}\Big) g = 0, \tag{3.35}$$

with initial condition

$$g_{|t=0} = 2Cy^{\alpha+1}, \quad \frac{\partial g}{\partial t}\Big|_{t=0} = 2(\alpha+1)y^\alpha,$$

for some constants $C, \alpha \in \mathbb{R}$. From Equation (3.35) we can obtain a system of the form (3.5) by introducing $\tau = t$, $x = y - t$, $\varphi = (g, (\partial_\tau - 2\partial_x)g)$, $\psi = \partial_\tau g$, and setting $L = 0$, $E_-^\mu \partial_\mu = \partial_\tau \otimes \mathrm{id}$, $E_+^\mu \partial_\mu = (\partial_\tau - 2\partial_x)$, so that we have

$$\partial_\tau \begin{pmatrix} g \\ (\partial_\tau - 2\partial_x)g \end{pmatrix} - \begin{pmatrix} \psi \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad (\partial_\tau - 2\partial_x)\psi = 0.$$

The solution is

$$\begin{aligned} g &= (C+1)(y+t)^{\alpha+1} + (C-1)(y-t)^{\alpha+1} \\ &= (C+1)(2\tau + x)^{\alpha+1} + (C-1)x^{\alpha+1}. \end{aligned}$$

It follows that for each $0 \leq \tau \leq 1$, $k \in \mathbb{N}$, and $\beta < \min\{0, \alpha+1\}$, we have $g(\tau, .) \in \mathcal{H}_k^\beta((0,10])$, consistently with Proposition 3.1. Somewhat surprisingly, for $\tau > 0$ and for all $i \in \mathbb{N}$ the functions $\partial_\tau^i g(\tau, .)$ are smooth in x up to $x = 0$. However, the L^∞ bound for $\partial_\tau^i g(\tau, .)$ blows up as τ tends to zero except in the case

$$C = -1. \tag{3.36}$$

Condition (3.36) is precisely the corner condition needed for $\partial_\tau g(0, .)$ to be better behaved than $\partial_x g(0, .)$ at $x = 0$. In the example under consideration the fulfillment of the first order corner condition guarantees already that all the τ derivatives of g will be well behaved, but we do not expect this to be true in general.

Let us pass to a derivation of the desired estimates. We shall use a method which works directly in weighted Hölder spaces, avoiding the use of weighted Sobolev spaces; the price one pays is the need to consider systems somewhat less general than (3.5), but still general enough for our purposes. More precisely, in this section we restrict our attention to systems of the form

$$\partial_\tau\varphi + B_{11}\varphi + B_{12}\psi = L_{11}\varphi + L_{12}\psi + a, \tag{3.37a}$$

$$e_+\psi + B_{21}\varphi + B_{22}\psi = L_{21}\varphi + L_{22}\psi + b, \tag{3.37b}$$

with

$$e_+\psi \equiv (\partial_\tau - 2\partial_x)\psi.$$

We assume that the L_{ab}'s (where $a, b = 1, 2$) are first order differential operators of the form

$$L_{ab} = L^A_{ab}\partial_A + xL^\tau_{ab}\partial_\tau + xL^x_{ab}\partial_x, \tag{3.38}$$

with bounded coefficients L^μ_{ab}; no symmetry hypotheses are made. Clearly the intersection of systems of equations satisfying (3.37) with those of the form (3.5) is non-empty. (As we will see in Sections 4 and 5 below, non-linear wave equations on Minkowski space-time can be written in the form (3.37).) In particular Proposition 3.1 provides a large class of solutions of (3.37) such that for $\ell < k - \frac{1}{2}n$,

$$(\varphi, \psi)(\tau) \in \mathcal{H}^\alpha_k(M_{x_1-2\tau}) \subset \mathcal{C}^\alpha_\ell(M_{x_1-2\tau}).$$

We shall therefore assume that a solution $f = (\varphi, \psi)$ satisfying $f(\tau) \in \mathcal{C}^\alpha_\ell(M_{x_1-2\tau})$ is given, and study its τ-differentiability properties. For the purposes of the proof below it is convenient to introduce auxiliary spaces $\mathcal{C}^\alpha_{\ell|p}(\Omega)$ defined, for $p \le \ell$, as the space of functions f in $C_\ell(\Omega)$ such that the norm

$$\|f\|_{\mathcal{C}^\alpha_{\ell|p}(\Omega)} \equiv \sup_\Omega \sum_{\substack{0\le i+j+k+|\gamma|\le\ell \\ 0\le k\le p}} x^{-\alpha}\big|(x\partial_x)^i(x\partial_\tau)^j\mathcal{D}^\gamma_v\partial^k_\tau f\big|$$

is finite. Obviously, $\mathcal{C}^\alpha_{\ell|\ell} = \mathcal{C}^\alpha_\ell$. Similarly one defines $\mathcal{C}^{\alpha,\beta}_{\ell|p}(\Omega)$ using the norm

$$\|f\|_{\mathcal{C}^{\alpha,\beta}_{\ell|p}(\Omega)} \equiv \sup_\Omega \sum_{\substack{0\le i+j+k+|\gamma|\le\ell \\ 0\le k\le p}} \big(1 + |\ln x|\big)^{-\beta}x^{-\alpha}\cdot\big|(x\partial_x)^i(x\partial_\tau)^j\mathcal{D}^\gamma_v\partial^k_\tau f\big|.$$

Clearly $\mathcal{C}^\alpha_{\ell|p}(\Omega) = \mathcal{C}^{\alpha,0}_{\ell|p}(\Omega)$. We shall write $\mathcal{C}^{\alpha,\beta}_\ell$ for $\mathcal{C}^{\alpha,\beta}_{\ell|\ell}$.

Proposition 3.3. — *Let $\alpha \le 0$, $\ell \in \mathbb{N}$, write Ω for $\Omega_{x_1,T}$ (with $\Omega_{x_1,T}$ as in* (3.1d)*), and suppose that $L^\mu_{ab}, B_{ab} \in \mathcal{C}^0_\ell(\Omega)$, $a \in \mathcal{C}^\alpha_{\ell-1}(\Omega)$, $b \in \mathcal{C}^{\alpha-1}_{\ell-1}(\Omega)$. Consider $f \equiv (\varphi, \psi)$, a solution of* (3.37) *satisfying*

$$\forall\tau \in [0, T], \quad f(\tau) \in \mathcal{C}^\alpha_\ell(M_{x_1-2\tau}).$$

Then:

1) *For all $\epsilon > 0$ we have*

$$(\varphi, \psi) \in \mathcal{C}^{\alpha,\beta}_{\lfloor\ell/2\rfloor}\big(\Omega \cap \{x + 2\tau > \epsilon\}\big).$$

This implies, for any $\tau > 0$ *the compatibility conditions of order* $p = \lfloor \frac{1}{2}\ell \rfloor$ *(the integer part of* $\frac{1}{2}\ell$*) are satisfied by* $(\varphi(\tau), \psi(\tau))$*:*

$$\forall\, i,\ 1 \le i \le p, \quad \partial_\tau^i \varphi(\tau), \partial_\tau^i \psi(\tau) \in \mathcal{C}^{\alpha,\beta}_{\ell-i}(M_{x_1}), \tag{3.39}$$

Here $\beta = \lfloor \frac{1}{2}\ell \rfloor$ *if* $\alpha = 0$*, and* $\beta = 0$ *otherwise.*

2) *If there exists* $1 \le p \le \frac{1}{2}\ell$*,* $p \in \mathbb{N}$*, such that Equation* (3.39) *holds with* $\beta = 0$ *at* $\tau = 0$*, then*

$$(\varphi, \psi) \in \mathcal{C}^{\alpha,\beta}_{\ell-p|p}(\Omega) \subset \mathcal{C}^{\alpha,\beta}_{p}(\Omega), \tag{3.40}$$

with $\beta = p$ *if* $\alpha = 0$*, and* $\beta = 0$ *otherwise.*

Remark. — The method of proof here gives a number of well-controlled time derivatives smaller by a factor 2 than the number of space ones. This is, however, irrelevant, when $\ell = \infty$, which is the main point of interest in this work. We note that energy estimates as in the proof of Theorem 4.4 below provide an alternative, more complicated way of establishing a stronger statement, with more controlled time derivatives for large ℓ's. In the linear case considered here the function F occurring there vanishes, so that all the complications arising from the non-linearities disappear, and somewhat stronger results can be obtained using the methods there.

Proof. — By rearranging terms and redefining the L_{ab}'s, the B_{ab}'s, and the source functions a and b we may without loss of generality assume that

$$L^\tau_{ab} \equiv 0.$$

One can rewrite Equations (3.37) as $x\partial_\tau(\varphi, \psi)$ = a partial differential operator linear in $x\partial_x$ and ∂_v; by iteration this immediately yields $(\varphi, \psi) \in \mathcal{C}^\alpha_{\ell|0}$. Equation (3.37a) shows then that $\partial_\tau \varphi \in \mathcal{C}^\alpha_{\ell-1|0}$, hence $\varphi \in \mathcal{C}^\alpha_{\ell|1}$. On the other hand, Equation (3.37b) gives $e_+(\psi) \in \mathcal{C}^\alpha_{\ell-1|0} + \mathcal{C}^{\alpha-1}_{\ell-1}$, hence $\partial_\tau e_+(\psi) \in \mathcal{C}^{\alpha-1}_{\ell-2|0}$. Integrating Equation (3.37b) one finds

$$\psi(x, v^A, \tau) = \psi(x + 2\tau, v^A, 0) + \int_{\frac{1}{2}x}^{\tau+\frac{1}{2}x} e_+(\psi)\big(2v, v^A, \tau - v + \tfrac{1}{2}x\big)\,\mathrm{d}v. \tag{3.41}$$

(We note that for each $\epsilon > 0$ the first term above is uniformly C_ℓ on the set $\Omega \cap \{x + 2\tau \ge \epsilon\} \cap \{x \le x_0\}$.) Differentiating Equation (3.41) one obtains

$$\partial_\tau \psi(x, v^A, \tau) = \partial_\tau \psi(x + 2\tau, v^A, 0) + \int_{\frac{1}{2}x}^{\tau+\frac{1}{2}x} \partial_\tau e_+\big(\psi\big)(2v, v^A, \tau - v + \tfrac{1}{2}x\big)\,\mathrm{d}v;$$

since $\alpha \le 0$ and $\partial_\tau e_+(\psi) \in \mathcal{C}^{\alpha-1}_{\ell-2|0}$, straightforward estimations show that $\partial_\tau \psi \in \mathcal{C}^\alpha_{\ell-2|0}$, hence $\psi \in \mathcal{C}^\alpha_{\ell-1|1}$ if $\alpha \ne 0$, while $\psi \in \mathcal{C}^{0,1}_{\ell-1|1}$ when $\alpha = 0$.

Let $\beta_r = 0$ if $\alpha \neq 0$ and $\beta_r = r$ when $\alpha = 0$, and suppose that $\varphi \in \mathcal{C}^{\alpha,\beta_r}_{\ell+1-r|r}$ and $\psi \in \mathcal{C}^{\alpha,\beta_r}_{\ell-r|r}$ for some $1 \leq r \leq \frac{1}{2}(\ell-1)$; we have already shown this to hold for $r=1$. Equation (3.37a) gives

$$\partial_\tau \varphi \in \mathcal{C}^{\alpha,\beta_r}_{\ell-r-1|r} \implies \varphi \in \mathcal{C}^{\alpha,\beta_r}_{\ell-r|r+1}.$$

It then follows from Equation (3.37b) that

$$e_+(\psi) \in \mathcal{C}^{\alpha,\beta_r}_{\ell-r-1|r} \implies \partial^{r+1}_\tau e_+(\psi) \in \mathcal{C}^{\alpha-1,\beta_r}_{\ell-2r-2|0}.$$

Differentiating $r+1$ times Equation (3.41) with respect to τ we obtain

$$\begin{aligned}\partial^{r+1}_\tau \psi(x, v^A, \tau) &= \partial^{r+1}_\tau \psi(x+2\tau, v^A, 0)\\ &\quad + \int_{\frac{1}{2}x}^{\tau+\frac{1}{2}x} \partial^{r+1}_\tau e_+(\psi)\big(2v, v^A, \tau - v + \tfrac{1}{2}x\big)\,\mathrm{d}v,\end{aligned}$$

which gives $\partial^{r+1}_\tau \psi \in \mathcal{C}^{\alpha,\beta_r}_{\ell-2r-2|0}$, hence $\psi \in \mathcal{C}^{\alpha,\beta_r}_{\ell-r-1|r+1}$, and the induction is completed. ☐

3.4. Polyhomogeneous solutions. — We now wish to show that solutions with polyhomogeneous initial data will be polyhomogeneous. Let $\Omega_{x_0,T}$ be defined by Equation (3.1d); we shall denote by $\mathcal{A}^\delta_k(\Omega_{x_0,T})$ the space of functions f defined on $\Omega_{x_0,T}$ which can be written in the form

$$\sum_{i=0}^{k}\sum_{j=0}^{N_i} x^{i\delta} \ln^j x f_{ij} + f_{k\delta+\epsilon},$$

for some $\epsilon > 0$, some functions $f_{ij} \in C_\infty(\overline{\Omega_{x_0,T}})$, and some sequence (N_i) of non-negative integers. We also require that $f_{k\delta+\epsilon} \in \mathcal{C}^{k\delta+\epsilon}_\infty(\Omega_{x_0,T})$. We set

$$\mathcal{A}^\delta_\infty := \bigcap_{k\in\mathbb{N}} \mathcal{A}^\delta_k.$$

The following properties are useful in what follows:

- If $0 < x_1 < x_0 - \frac{1}{2}T$, then a function $f \in C_\infty(\Omega_{x_0,T})$ is in $\mathcal{A}^\delta_k(\Omega_{x_0T})$ if and only if for any coordinate patch $\mathcal{O}$ of ∂M we have $f \in \mathcal{A}^\delta_k(\mathcal{U}_{x_1})$, where $\mathcal{U}_{x_1} =]0, x_1[\times \mathcal{O} \times [0,T]$, and if $f \in C_\infty(\overline{\Omega_{\mathrm{int}}})$, where $\Omega_{\mathrm{int}} = \Omega_{x_0,T} \cap \{x \geq x_1\}$.
- For all $\epsilon > 0$ we have $\mathcal{C}^{\beta+p\delta+\epsilon}_\infty \subset x^\beta \mathcal{A}^\delta_p$; in particular $\mathcal{C}^\epsilon_\infty \subset \mathcal{A}^\delta_0$;
- It does not hold that $\mathcal{A}^\delta_k \subset \mathcal{C}^0_\infty$, however, for all $\epsilon > 0$ we have $\mathcal{A}^\delta_k \subset \mathcal{C}^{-\epsilon}_\infty$. More precisely, if $f \in \mathcal{A}^\delta_k$, then there exists $p \in \mathbb{N}$ such that $(1+|\ln x|^2)^{-p/2} f$ belongs to $\mathcal{C}^0_\infty$.
- As before we assume that $1/\delta \in \mathbb{N}$, which implies $x\mathcal{A}^\delta_k \subset \mathcal{A}^\delta_{k+1/\delta} \subset \mathcal{A}^\delta_{k+1}$.
- $\mathcal{A}^\delta_k$ is stable under multiplication: if $f, g \in \mathcal{A}^\delta_k$, then $fg \in \mathcal{A}^\delta_k$.

• $\mathcal{A}^\delta_k$ is stable under differentiation with respect to τ and to v, as well as under $x\partial_x$: if $f\in\mathcal{A}^\delta_k$, then $\partial_\tau f, X_i\cdot f$ $(i\ge 2)$, $x\partial_x f\in\mathcal{A}^\delta_k$, with the vector fields X_i defined in Appendix A, *cf.* Equation (A.7).

In this section we will consider systems of the form

$$\partial_\tau\varphi + B_{11}\varphi + B_{12}\psi = L_{11}\varphi + L_{12}\psi + a, \tag{3.42a}$$

$$\partial_x\psi + B_{21}\varphi + B_{22}\psi = L_{21}\varphi + L_{22}\psi + b, \tag{3.42b}$$

with the L_{ij}'s, $i,j=1,2$ of the form

$$L_{ij} = L^A_{ij}\partial_A + L^\tau_{ij}\partial_\tau + xL^x_{ij}\partial_x, \tag{3.43}$$

$$L^\mu_{11}\in x^\delta\mathcal{A}^\delta_{k-1} \quad\text{and}\quad L^\mu_{21}, L^\mu_{12}, L^\mu_{22}\in\mathcal{A}^\delta_k. \tag{3.44}$$

No symmetry hypotheses are made on the matrices L^μ_{ij}. Conditions (3.42a)–(3.44) are easily seen to be compatible with those made elsewhere in this paper, *cf., e.g.*, the proof of Corollary 3.5 below. The reader is warned, however, that the operators L_{ij} here do *not* coincide with those in (3.37): to bring (3.37) into the form (3.42) one needs to multiply Equation (3.37b) by $-\frac{1}{2}$, transfer the operator ∂_τ from the left- to the right-hand-side of (3.37), and appropriately redefine the L_{2j}'s.

We start with the following result, which assumes that the solutions have both space and time derivatives controlled, in the sense of weighted Sobolev spaces; recall that this hypothesis can be justified for equations satisfying moreover the hypotheses of the previous sections:

THEOREM 3.4. — *Let $\beta,\beta'\in\mathbb{R}$, $k\in\mathbb{N}\cup\{\infty\}$, and let (φ,ψ) be a solution of* (3.42) *in $\mathcal{C}^{\beta'}_\infty(\Omega_{x_0,T})$. Suppose that* (3.44) *holds, and that*

$$B_{11}\in\big(\mathcal{A}^\delta_k\cap L^\infty\big)(\Omega_{x_0,T}),\quad B_{12},B_{22},B_{21}\in\mathcal{A}^\delta_k(\Omega_{x_0,T}), \tag{3.45a}$$

$$a,b\in x^\beta\mathcal{A}^\delta_k(\Omega_{x_0,T}),\quad \varphi(0)\in x^\beta\mathcal{A}^\delta_k(M_{x_0}). \tag{3.45b}$$

Then

$$\varphi\in(x^\beta\mathcal{A}^\delta_k+\mathcal{A}^\delta_k)(\Omega_{x_0,T}),\quad \psi\in(x^{\beta+1}\mathcal{A}^\delta_k+x\mathcal{A}^\delta_k)(\Omega_{x_0,T})+C_\infty(\overline{\Omega_{x_0,T}}).$$

If one further assumes $L^\mu_{12},B_{12},a,\varphi(0)\in L^\infty(\Omega_{x_0,T})$, then it also holds that

$$\varphi\in(x^\beta\mathcal{A}^\delta_k+\mathcal{A}^\delta_k\cap L^\infty)(\Omega_{x_0,T}).$$

Proof. — It is convenient to decompose B_{11} in the obvious way as

$$B_{11}=B^0_{11}+B^\delta_{11},$$

with $B^\delta_{11}\in x^\delta\mathcal{A}^\delta_{k-1}$ and $B^0_{11}\in C_\infty$. We rewrite (3.42) as

$$\partial_\tau\varphi+B^0_{11}\varphi = c_1, \tag{3.46a}$$

$$\partial_x\psi = c_2, \tag{3.46b}$$

where

$$c_1 := L_{11}\varphi + L_{12}\psi + a - B_{12}\psi - B^{\delta}_{11}\varphi, \tag{3.47a}$$

$$c_2 := L_{21}\varphi + L_{22}\psi + b - B_{21}\varphi - B_{22}\psi, \tag{3.47b}$$

In what follows we let $\epsilon > 0$ be a positive constant, which can be made as small as desired, and which may change from line to line. We note that c_2 is in $\mathcal{C}^{\beta'-\epsilon}_{\infty} + x^{\beta}\mathcal{A}^{\delta}_{k}$, and integration in x of (3.46b), together with Propositions B.3 and B.6, gives $\psi = \psi_0 + \psi_{\beta'+1-\epsilon} + \psi_{\rm phg}$, where

$$\psi_0(.) = \begin{cases} \lim_{x\to 0}\psi(x,.) & \text{if } \beta' + 1 - \epsilon > 0, \\ 0 & \text{otherwise,} \end{cases}$$

with $\psi_0 \in C_{\infty}(\overline{\Omega_{x_0,T}})$, $\psi_{\beta'+1-\epsilon} \in \mathcal{C}^{\beta'+1-\epsilon}_{\infty}(\Omega_{x_0,T})$, $\psi_{\rm phg} \in x^{\beta+1}\mathcal{A}^{\delta}_{k}(\Omega_{x_0,T})$, hence

$$\psi \in C_{\infty} + \mathcal{C}^{\beta'+1-\epsilon}_{\infty} + x^{\beta+1}\mathcal{A}^{\delta}_{k}.$$

Since $L_{11}\varphi \in \mathcal{C}^{\beta'+\delta-\epsilon}_{\infty}$ (we have $\partial_x\varphi \in \mathcal{C}^{\beta'-1}_{\infty}$ and $xL^{x}_{11} \in x\mathcal{A}^{\delta}_{k} \cap \mathcal{C}^{\delta}_{0} \subset \mathcal{C}^{\delta}_{\infty}$; similarly for the other derivatives), we find that

$$c_1 \in \mathcal{A}^{\delta}_{k} + x^{\beta}\mathcal{A}^{\delta}_{k} + \mathcal{C}^{\beta'+\delta-\epsilon}_{\infty}.$$

We can then apply Proposition B.4 to (3.46a) to conclude that

$$\varphi \in \mathcal{A}^{\delta}_{k} + x^{\beta}\mathcal{A}^{\delta}_{k} + \mathcal{C}^{\beta'+p\delta-\epsilon}_{\infty}, \tag{3.48}$$

with $p = 1$. Coming back to c_2 we find now that $c_2 \in \mathcal{A}^{\delta}_{k} + x^{\beta}\mathcal{A}^{\delta}_{k} + \mathcal{C}^{\beta'+p\delta-\epsilon}_{\infty}$, and by Proposition B.6 we obtain

$$\psi \in C_{\infty} + x\mathcal{A}^{\delta}_{k} + x^{\beta+1}\mathcal{A}^{\delta}_{k} + \mathcal{C}^{\beta'+p\delta+1-\epsilon}_{\infty}, \tag{3.49}$$

still with $p = 1$. To conclude, we proceed by induction; let $\beta' + p\delta \leq \beta + k$ and suppose that Equations (3.48)–(3.49) hold; it follows that c_1 belong to $\mathcal{A}^{\delta}_{k} + x^{\beta}\mathcal{A}^{\delta}_{k} + \mathcal{C}^{\beta'+(p+1)\delta-\epsilon}_{\infty}$. Applying Proposition B.4 to (3.46a) gives (3.48) with p replaced by $p+1$. It follows that $c_2 \in \mathcal{A}^{\delta}_{k} + x^{\beta}\mathcal{A}^{\delta}_{k} + \mathcal{C}^{\beta'+(p+1)\delta-\epsilon}_{\infty}$; Proposition B.6 applied to (3.46b) gives (3.49) with p replaced by $p+1$, and the result is established. □

As a straightforward consequence of Theorem 3.4 we obtain:

Corollary 3.5. — *Let $\beta' \in \mathbb{R}$, let $(\varphi,\psi) \in \mathcal{C}^{\beta'}_{\infty}(\Omega_{x_0,T})$ be a solution of the system* (3.5)*, and suppose that*

$$B_{ij}, E^{\mu}_{\pm}, B_{\pm}, \ell, \ell^{\dagger}, \ell^{A}, (\ell^{A})^{\dagger} \in \mathcal{A}^{\delta}_{k}(\Omega_{x_0,T}), \tag{3.50a}$$

$$E^{\tau}_{-} \text{ and } E^{x}_{+} \text{ are invertible, with } (E^{\tau}_{-})^{-1}, (E^{x}_{+})^{-1} \in \mathcal{A}^{\delta}_{k}(\Omega_{x_0,T}), \tag{3.50b}$$

$$(E^{\tau}_{-})^{-1}E^{x}_{-} \in x\Big(\mathcal{A}^{\delta}_{k} \cap \mathcal{C}^{\delta}_{0}\Big)(\Omega_{x_0,T}), \quad (E^{\tau}_{-})^{-1}E^{A}_{-} \in x^{\delta}\mathcal{A}^{\delta}_{k-1}(\Omega_{x_0,T}), \tag{3.50c}$$

$$(E^{\tau}_{-})^{-1}(B_{11} + B_{-}) \in L^{\infty}(\Omega_{x_0,T}). \tag{3.50d}$$

If $a, b \in x^\beta \mathcal{A}^\delta_k(\Omega_{x_0,T})$ *and* $\varphi(0) \in x^\beta \mathcal{A}^\delta_k(M_{x_0})$*, with* $\beta \in \mathbb{R}$*, then*

$$\varphi \in (x^\beta \mathcal{A}^\delta_k + \mathcal{A}^\delta_k)(\Omega_{x_0,T}), \quad \psi \in (x^{\beta+1}\mathcal{A}^\delta_k + x\mathcal{A}^\delta_k)(\Omega_{x_0,T}) + C_\infty(\overline{\Omega_{x_0,T}}).$$

In particular, if $k = \infty$ *then the solution is polyhomogeneous.*

Proof. — We write Equation (3.5) as

$$\begin{cases} \partial_\tau \varphi + (E^\tau_-)^{-1}\big\{(B_{11} + B_-)\varphi + \ell\psi\big\} \\ \qquad\qquad = (E^\tau_-)^{-1}(E^i_- \partial_i \varphi - \ell^A \partial_A \psi + a), \\ \partial_x \psi - (E^x_+)^{-1}\big\{\ell^\dagger \varphi - (B_{22} + B_+)\psi\big\} \\ \qquad\qquad = (E^\tau_-)^{-1}\big((\ell^A)^\dagger \partial_A \varphi + E^\tau_+ \partial_\tau \psi + E^A_+ \partial_A \psi + b\big), \end{cases} \tag{3.51}$$

which is of the form (3.42), and we note that the hypotheses made on the coefficients of Equation (3.51) imply those of Theorem 3.4. □

An unsatisfactory feature of results such as Theorem 3.4 is that uniform estimates both on space and time derivatives of the solutions are assumed. Recall that uniform control of time derivatives can be obtained only if corner conditions are satisfied, and the hypotheses of Theorem 3.4 require an infinite number of those to be fulfilled. The same techniques can be used to obtain various expansions of solutions when a finite number of time derivatives are controlled only, but the statements turn to be out somewhat less elegant. We give an example of such results when $\delta = 1$:

Theorem 3.6. — *Let* $\beta \in \mathbb{R}$*,* $k \in \mathbb{N} \cup \{\infty\}$*, and let* (φ, ψ) *be a solution of* (3.42) *in* $\mathcal{C}^\beta_\ell(\Omega_{x_0,T})$ *for some* $\ell \geq 1$*. If Equations* (3.44)–(3.50) *hold with* $\delta = 1$*, then for any* $\lambda < 1$ *we have*

$$\begin{cases} \varphi \in \Big(x^\beta \mathcal{A}^1_k + \mathcal{A}^1_k + \displaystyle\bigcap_{\ell-2j-2\geq 0} \mathcal{C}^{\beta+j+\lambda}_{\ell-2j-2}\Big)(\Omega_{x_0,T}), \\ \psi \in \Big(x^{\beta+1}\mathcal{A}^1_k + x\mathcal{A}^1_k + \displaystyle\bigcap_{\ell-2j-1\geq 0} \mathcal{C}^{\beta+j+1+\lambda}_{\ell-2j-1}\Big)(\Omega_{x_0,T}) + C_\infty(\overline{\Omega_{x_0,T}}). \end{cases} \tag{3.52}$$

If one further assumes $L^\mu_{12}, B_{12}, a, \varphi(0) \in L^\infty(\Omega_{x_0,T})$*, then it also holds that*

$$\varphi \in \Big(x^\beta \mathcal{A}^1_k + \mathcal{A}^1_k \cap L^\infty + \bigcap_{\ell-2j-2\geq 0} \mathcal{C}^{\beta+j+\lambda}_{\ell-2j-2}\Big)(\Omega_{x_0,T}).$$

Proof. — The result is obtained through a repetition of the proof of Theorem 3.4, keeping track of the differentiability of the remainder terms. □

We are ready now to prove polyhomogeneity of solutions of the Cauchy problem for Equation (3.5). We consider only the simplest case of equations satisfying the conditions (3.53) below, considerably more general statements can be proved using similar methods. The differentiability hypotheses below are clearly satisfied by equations with smooth bounded coefficients; however, they also allow for a wide class of equations with polyhomogeneous coefficients.

We restrict ourselves to the case in which the corner conditions are satisfied to arbitrary order; if not, one obtains expansions as in (3.52), with a remainder in which a finite number only of time derivative are controlled; such results can be proved by identical arguments, compare the proof of Theorem 3.6. We hope to be able to show in a near future that the corner conditions are not needed, in which case one should obtain polyhomogeneous expansions in which uniformity is lost when the corner $\tau = x = 0$ is approached; this will be discussed elsewhere.

THEOREM 3.7. — *Consider a solution* $(\varphi, \psi) \in C_\infty \times C_\infty$ *of the system* (3.5)*, suppose that in addition to* (3.12)*,* (3.13)*,* (3.19)*, and* (3.50a) *we have*

$$B_{11}, B_-, E^\mu_\pm, \ell, \ell^\dagger \in L^\infty(\Omega_{x_0,T}), \tag{3.53a}$$

$$E^\mu_{-|x=0} = \partial_\tau \otimes \mathrm{id}, \quad E^\mu_{+|x=0} = (\partial_\tau - 2\partial_x) \otimes \mathrm{id}, \tag{3.53b}$$

$$E^x_\pm - E^x_{\pm|x=0}, E^\tau_\pm - E^\tau_{\pm|x=0} \in x^{1+\delta}\mathcal{A}^\delta_\infty(\Omega_{x_0,T}), \tag{3.53c}$$

$$E^A_- \in x\mathcal{A}^\delta_\infty(\Omega_{x_0,T}). \tag{3.53d}$$

If $a, b \in x^\beta \mathcal{A}^\delta_k(\Omega_{x_0,T})$ *and* $\varphi(0) \in x^\beta \mathcal{A}^\delta_k(M_{x_0})$*, with* $\beta \in \mathbb{R}$*, and if the initial data satisfy* corner conditions to arbitrary order*, in the sense that*

$$\forall i \in \mathbb{N}, \quad \partial^i_\tau \varphi(0), \partial^i_\tau \psi(0) \in \mathcal{C}^\lambda_\infty(M_{x_0}), \tag{3.54}$$

for some (i*-independent*) $\lambda \in \mathbb{R}$*, then*

$$\varphi \in (x^\beta \mathcal{A}^\delta_k + \mathcal{A}^\delta_k)(\Omega_{x_0,T}), \quad \psi \in (x^{\beta+1}\mathcal{A}^\delta_k + x\mathcal{A}^\delta_k)(\Omega_{x_0,T}) + C_\infty(\overline{\Omega_{x_0,T}}).$$

In particular, if $k = \infty$ *then the solution is polyhomogeneous.*

REMARK. — The class of initial data satisfying corner conditions to arbitrary order is rather large; for example, if an initial data set $(\varphi(0), \psi(0))$ satisfies them, and if f, g are arbitrary functions smooth up to boundary on the initial data hypersurface, then $(\varphi(0) + f, \psi(0) + g)$ will also satisfy those conditions. More generally, large classes of such initial data can be constructed using a polyhomogeneous generalisation of the Borel summation lemma.

Proof. — The hypothesis (3.54) with $i = 0$ and Proposition 3.1 show that for all $\tau \in [0, T]$ we have $\varphi(\tau), \psi(\tau) \in \mathcal{C}^\lambda_\infty(M_{x_0/2})$. Proposition 3.3 shows then that the hypotheses of Corollary 3.5 are satisfied, and the result follows. □

4. The semi-linear scalar wave equation

Let f be a solution of the semi-linear wave equation

$$\Box_{\mathfrak{g}} f = H(x^\mu, f), \tag{4.1}$$

here $\Box_{\mathfrak{g}}$ is the d'Alembertian associated with $\mathfrak{g}$. Set

$$\widetilde{f} = \Omega^{-(n-1)/2} f; \tag{4.2}$$

Letting $\widetilde{\mathfrak{g}} = \Omega^2 \mathfrak{g}$ as in (2.1), from (2.3) we obtain

$$\square_{\widetilde{\mathfrak{g}}} \widetilde{f} = \frac{n-1}{4n}\Big(\widetilde{R} - \frac{R}{\Omega^2}\Big)\widetilde{f} + \Omega^{-(n+3)/2} H(x^\mu, \Omega^{(n-1)/2}\widetilde{f}\,). \tag{4.3}$$

Let $\mathfrak{g} = \eta$ be the Minkowski metric; under the conformal transformation (2.4) one obtains from (2.5) that $\widetilde{\mathfrak{g}}$ is again the Minkowski metric, and (4.3) becomes

$$\square_\eta \widetilde{f} = \Omega^{-(n+3)/2} H(x^\mu, \Omega^{(n-1)/2}\widetilde{f}\,). \tag{4.4}$$

We shall assume that the initial data for f are given on a hypersurface $\Sigma \subset \mathcal{M}$, which, in a neighborhood $\mathcal{O}$ of $\mathcal{I}^+$ is given by the equation

$$\Sigma \cap \mathcal{O} = \big\{y^0 = \tfrac{1}{2}\big\}. \tag{4.5}$$

This corresponds to a hyperboloid in $\mathcal{M}$ given by the equation $x^0+1 = \sqrt{1+\vec{x}^{\,2}}$. It is convenient to introduce the following coordinate system (x, v, τ) in a $\widetilde{\mathcal{M}}$-neighborhood of $\mathcal{I}^+$:

$$\tau = y^0 - \tfrac{1}{2} \geq 0, \quad x = \Big(\sum (y^i)^2\Big)^{\frac{1}{2}} - y^0 \geq 0, \quad y^i = \Big(\sum (y^i)^2\Big)^{\frac{1}{2}} n^i(v), \tag{4.6}$$

$n^i(v) \in S^{n-1}$, with $v = (v^A)$ denoting spherical coordinates on S^{n-1}. Equation (2.5) gives

$$\Omega = x(2\tau + x + 1) \approx x. \tag{4.7}$$

If we let h denote the unit round metric on S^{n-1}, we then have

$$\eta = 2\mathrm{d}x\mathrm{d}\tau + \mathrm{d}x^2 + \big(x + \tau + \tfrac{1}{2}\big)^2 h, \tag{4.8}$$

$$\begin{aligned} \square_\eta \widetilde{f} &= \frac{1}{(x+\tau+\frac{1}{2})^{n-1}\sqrt{\det h}}\,\partial_\mu\big((x+\tau+\tfrac{1}{2})^{n-1}\sqrt{\det h}\,\eta^{\mu\nu}\partial_\nu \widetilde{f}\,\big) \\ &= \Big\{-\partial_\tau(\partial_\tau - 2\partial_x) + \frac{n-1}{x+\tau+\frac{1}{2}}\partial_x + \frac{\triangle_h}{(x+\tau+\frac{1}{2})^2}\Big\}\widetilde{f}, \end{aligned} \tag{4.9}$$

where $\triangle_h$ is the Laplace-Beltrami operator of the metric h. We set

$$e_- = \partial_\tau, \quad e_+ = \partial_\tau - 2\partial_x, \quad e_A = \frac{1}{(x+\tau+\frac{1}{2})} h_A, \tag{4.10}$$

$$\phi_- = e_-(\widetilde{f}\,), \quad \phi_+ = e_+(\widetilde{f}\,), \tag{4.11}$$

$$\phi_A = \psi_A = \frac{1}{(x+\tau+\frac{1}{2})} h_A(\widetilde{f}\,), \tag{4.12}$$

where h_A denotes an h-orthonormal frame on S^{n-1}. We use the symbol D to denote the covariant derivative operator associated to the metric h. (The usefulness of introducing two different objects for $h_A(\widetilde{f}\,)/(x+\tau+\frac{1}{2})$ will become clear shortly.) Equation (4.4) implies the following set of equations:

$$\left\{\begin{array}{l} e_-(\phi_+) - D_{e_A}\psi_A - \dfrac{n-1}{2(x+\tau+\frac12)}\phi_+ = -\dfrac{n-1}{2(x+\tau+\frac12)}\phi_- + a_+, \\ -e_A(\phi_+) + e_+(\psi_A) - \dfrac{1}{(x+\tau+\frac12)}\psi_A = b_A, \end{array}\right. \tag{4.13}$$

$$\left\{\begin{array}{l} e_-(\phi_A) - e_A(\phi_-) + \dfrac{1}{(x+\tau+\frac12)}\phi_A = a_A, \\ -D_{e_A}\phi_A + e_+(\phi_-) + \dfrac{n-1}{2(x+\tau+\frac12)}\phi_- = \dfrac{n-1}{2(x+\tau+\frac12)}\phi_+ + b_-, \end{array}\right. \tag{4.14}$$

$$e_-(\widetilde{f}\,) = \phi_-, \tag{4.15}$$

$$e_+(\widetilde{f}\,) = \phi_+, \tag{4.16}$$

with $a_A = b_A = 0$ and

$$a_+ \equiv b_- \equiv G \equiv H(x^\mu, \Omega^{(n-1)/2}\widetilde{f}\,). \tag{4.17}$$

4.1. Existence of solutions, space derivatives estimates. — We note that the partial differential operator standing on the left-hand-side of (4.13) is symmetric hyperbolic; the same holds true for (4.14), or for the joint system (4.13)–(4.16). Now, part of our technique consists in deriving weighted energy estimates for symmetric hyperbolic systems having the structure above, *cf.* Section 3. Each such system comes with his own estimates, so that for the systems (4.13) and (4.14) we can obtain estimates with different weights. This allows us to handle a reasonably wide range of non-linearities, giving existence and blow-up control for initial data in weighted Sobolev spaces (with conormal-type blow-up at $\mathcal{I}^+$):

THEOREM 4.1. — *Consider Equation* (4.1) *on* $\mathbb{R}^{n,1}$ *with initial data given on a hyperboloid* $\mathcal{S} \supset \Sigma_{x_0,0}$ *in Minkowski space-time, and satisfying*

$$\widetilde{f}_{|\Sigma_{x_0,0}} \equiv \Omega^{-(n-1)/2} f_{|\Sigma_{x_0,0}} \in \mathcal{H}^\alpha_{k+1}(\Sigma_{x_0,0}), \tag{4.18}$$

$$\partial_x(\Omega^{-(n-1)/2} f)_{|\Sigma_{x_0,0}} \in \mathcal{C}^\alpha_0(\Sigma_{x_0,0}) \cap \mathcal{H}^{\alpha-1/2}_k(\Sigma_{x_0,0}), \tag{4.19}$$

$$\partial_\tau(\Omega^{-(n-1)/2} f)_{|\Sigma_{x_0,0}} \in \mathcal{H}^\alpha_k(\Sigma_{x_0,0}), \tag{4.20}$$

with some $k > \frac12 n + 1$, $-1 < \alpha < -\frac12$. *Suppose further that* H *has a uniform zero of order* ℓ *at* $f = 0$, *in the sense of* (A.30), *with*

$$\ell \geq \left\{\begin{array}{ll} 4 & \text{if } n = 2, \\ 3 & \text{if } n = 3, \\ 2 & \text{if } n \geq 4. \end{array}\right. \tag{4.21}$$

Then:

1) *There exists* $0 < \tau_+ \leq T\ (< \frac{1}{2}x_0)$, *depending only upon* x_0 *and a bound on the norms of the initial data in the spaces appearing in Equations* (4.18)–(4.20), *and a solution* f *of Equation* (4.1), *defined on a set containing* Ω_{x_0,τ_+}, *satisfying the given initial conditions, and satisfying*

$$\|\widetilde{f}\|_{L^\infty(\Omega_{x_0,\tau_+})} < \infty.$$

2) *Further, if* τ_* *is such that* f *exists on* Ω_{x_0,τ_*} *and if* $\|\widetilde{f}\|_{L^\infty(\Omega_{x_0,\tau_*})} < \infty$, *then for* $0 \leq \tau < \tau_*$ *we have*

$$\widetilde{f}_{|\Sigma_{x_0,\tau}} \in L^\infty(\Sigma_{x_0,\tau}) \cap \mathcal{H}^\alpha_{k+1}(\Sigma_{x_0,\tau}),$$
$$\partial_\tau \widetilde{f}_{|\Sigma_{x_0,\tau}} \in \mathcal{H}^\alpha_k(\Sigma_{x_0,\tau}), \quad \partial_x \widetilde{f}_{|\Sigma_{x_0,\tau}} \in \mathcal{H}^{\alpha-\frac{1}{2}}_k(\Sigma_{x_0,\tau}) \cap \mathcal{C}^\alpha_0(\Sigma_{x_0,\tau}),$$

with a τ*-independent bound on all the norms.*

REMARKS

1) Integration in x of condition (4.19) implies that $\widetilde{f} \in L^\infty(\Sigma_{x_0,0})$.

2) Some further information can be found in Theorem 4.3 below.

3) If the inequality in (4.21) is not an equality for $n = 2, 3$ (no further restrictions for $n \geq 4$), then a proof similar, but simpler, basing on Proposition 3.2 instead of 3.1, leads to the same result with $\alpha = -\frac{1}{2}$.

Proof. — As before, we write $\|f(\tau)\|_{\mathcal{H}^\alpha_k}$ for $\|f_{|\Sigma_{x_0,\tau}}\|_{\mathcal{H}^\alpha_k(\Sigma_{x_0,\tau})}$, *etc.* Recall that the standard theory of hyperbolic systems (*cf., e.g.,* [35, chap. 16, vol. III]) shows that for any $0 < x_1 \leq x_0$ there exists $T(x_1) > 0$, satisfying $2x_1 + T \leq x_0$, and a solution $\widetilde{f}$ of (4.4), defined on $\Omega_{x_1,x_0,T}$, with initial data on Σ_{x_1,x_0} obtained from those on Σ_{x_0} by restriction. The idea of the proof is to derive x_1-independent, weighted *a priori* estimates for the solution. These estimates will guarantee that the existence time $T(x_1)$ does not shrink to zero as x_1 goes to zero; they will also guarantee that the weighted Sobolev regularity is preserved by evolution. We start with the following:

LEMMA 4.2. — *Under the hypotheses of Theorem* 4.1, *consider on* $\Omega_{x_1,x_0,T}$ *the system* (4.12)–(4.16), *set*

$$E_\alpha(t) = \big\|\widetilde{f}(t)\big\|^2_{\mathcal{H}^\alpha_k} + \big\|\phi_-(t)\big\|^2_{\mathcal{H}^\alpha_k} + \|\phi_+(t)\|^2_{\mathcal{H}^{\alpha-1/2}_k} + \sum_A \big\|\phi_A(t)\big\|^2_{\mathcal{H}^\alpha_k}. \tag{4.22}$$

Then there exists a x_1*-independent constant* C *such that*

$$E_\alpha(t) \leq C\Big\{E_\alpha(0)\mathrm{e}^{Ct} + \int_0^t \mathrm{e}^{C(t-s)} S(s)\mathrm{d}s\Big\}, \tag{4.23}$$

where

$$S(s) \equiv \sum_A \big\|a_A(s)\big\|^2_{\mathcal{H}^\alpha_k} + \big\|a_+(s)\big\|^2_{\mathcal{H}^{\alpha-1/2}_k} + \big\|b_-(s)\big\|^2_{\mathcal{H}^{\alpha-1/2}_k} + \sum_A \big\|b_A(s)\big\|^2_{\mathcal{H}^{\alpha-1}_k}. \tag{4.24}$$

Proof. — We wish, first, to apply Proposition 3.1 to the system consisting of Equation (4.14) together with $e_-(\widetilde{f}\,) = \phi_-$; in order to do this we set

$$\varphi = \begin{pmatrix} \widetilde{f} \\ \phi_A \end{pmatrix}, \quad \psi = \phi_-.$$

We choose $E^\mu_\pm \partial_\mu = e_\pm \otimes \mathrm{Id}$, we set

$$L\psi = \begin{pmatrix} 0 \\ -e_A(\psi) \end{pmatrix}, \tag{4.25}$$

and we define

$$\widetilde{E}_\alpha(t) = \big\|\widetilde{f}(t)\big\|^2_{\mathcal{H}^\alpha_k} + \big\|e_-(\widetilde{f}\,)(t)\big\|^2_{\mathcal{H}^\alpha_k} + \sum_A \big\|e_A(\widetilde{f}\,)(t)\big\|^2_{\mathcal{H}^\alpha_k}.$$

The hypotheses C1–C5 of Proposition 3.1 are readily verified, and for any $\alpha < -\frac{1}{2}$ the inequality (3.20) gives

$$\widetilde{E}_\alpha(t) \le C\Big\{\widetilde{E}_\alpha(0)\mathrm{e}^{Ct} + \int_0^t \mathrm{e}^{C(t-s)}\Big(\sum_A \big\|a_A(s)\big\|^2_{\mathcal{H}^\alpha} + \big\|\phi_+(s)\big\|^2_{\mathcal{H}^{\alpha-1/2}} + \big\|b_-(s)\big\|^2_{\mathcal{H}^{\alpha-1/2}}\Big)\mathrm{d}s\Big\}. \tag{4.26}$$

Next, we apply Proposition 3.1 directly to (4.13): setting

$$\widehat{E}_{\alpha'}(t) = \big\|e_+(\widetilde{f}\,)(t)\big\|^2_{\mathcal{H}^{\alpha'}_k} + \sum_A \big\|e_A(\widetilde{f}\,)(t)\big\|^2_{\mathcal{H}^{\alpha'}_k},$$

for any $\alpha' < -\frac{1}{2}$ it follows from (3.20) that

$$\widehat{E}_{\alpha'}(t) \le C\Big\{\widehat{E}_{\alpha'}(0)\mathrm{e}^{Ct} + \int_0^t \mathrm{e}^{C(t-s)}\big(\|a_+(s)\|^2_{\mathcal{H}^{\alpha'}} + \big\|\phi_-(s)\big\|^2_{\mathcal{H}^{\alpha'-1/2}} + \sum_A \big\|b_A(s)\big\|^2_{\mathcal{H}^{\alpha'-1/2}}\big)\mathrm{d}s\Big\}. \tag{4.27}$$

We set

$$E(t) = \widetilde{E}_\alpha(t) + \widehat{E}_{\alpha-1/2}(t).$$

It follows from (4.26) and (4.27) with $\alpha' = \alpha - \frac{1}{2}$ that we have

$$E(t) \le C\Big(E(0)\mathrm{e}^{Ct} + \int_0^t \mathrm{e}^{C(t-s)}\big(E(s) + S(s)\big)\mathrm{d}s\Big), \tag{4.28}$$

with $S(s)$ as in (4.24). Equation (4.23) with E_α replaced[8] by E follows now from Gronwall's Lemma. Since E_α is equivalent to E, our claims follow. □

Returning to the proof of Theorem 4.1, Lemma 4.2 applied to (4.13)–(4.15) gives (recall that G was defined in (4.17))

$$E_\alpha(t) \le C\Big(E_\alpha(0)\mathrm{e}^{Ct} + \int_0^t \mathrm{e}^{C(t-s)} \big\|G(s)\big\|^2_{\mathcal{H}_k^{\alpha-1/2}} \mathrm{d}s\Big). \tag{4.29}$$

By hypothesis the function H appearing in (4.1) has a uniform zero of order $\ell \ge 2$, in the sense of (A.30); we wish to use (A.31) to control the term containing $G(s)$ in (4.29). This requires an L^∞ bound on $\widetilde{f}$, which will be obtained next. As $k > \frac{1}{2}n + 1$, the Sobolev embedding (A.24) gives

$$\big\|e_-(\widetilde{f}\,)(s)\big\|^2_{\mathcal{C}_1^\alpha} + \big\|e_+(\widetilde{f}\,)(s)\big\|^2_{\mathcal{C}_1^{\alpha-1/2}} + \big\|e_A(\widetilde{f}\,)(s)\big\|^2_{\mathcal{C}_1^\alpha} \le CE_\alpha(s). \tag{4.30}$$

Now the conditions (4.21) on n and ℓ give

$$|G(\tau)| \le C\|\widetilde{f}(\tau)\|^\ell_{L^\infty} x^{\ell(n-1)/2-(n+3)/2} \le C\|\widetilde{f}(\tau)\|^\ell_{L^\infty} x^{-1/2},$$

so that (recall that $\alpha < -\frac{1}{2}$)

$$\big\|G(\tau)\big\|_{\mathcal{C}_0^\alpha} \le C\big\|\widetilde{f}(\tau)\big\|^\ell_{L^\infty}. \tag{4.31}$$

From (4.13) we have

$$\partial_\tau\phi_+ - \frac{n-1}{2(x+\tau+\frac{1}{2})}\phi_+ = D_{e_A}\psi_A - \frac{n-1}{2(x+\tau+\frac{1}{2})}\phi_- - G, \tag{4.32}$$

and (4.31) together with Proposition B.1 yield

$$\begin{aligned} \big\|\phi_+(t)\big\|_{\mathcal{C}_0^\alpha} &\le C\mathrm{e}^{Ct}\big\|\phi_+(0)\big\|_{\mathcal{C}_0^\alpha} \\ &\quad + C\int_0^t \mathrm{e}^{C(t-s)}\big(\|D_{e_A}\psi_A(s)\|_{\mathcal{C}_0^\alpha} + \|\phi_-(s)\|_{\mathcal{C}_0^\alpha} + \|G(s)\|_{\mathcal{C}_0^\alpha}\big)\mathrm{d}s \\ &\le C\mathrm{e}^{Ct}\big\|\phi_+(0)\big\|_{\mathcal{C}_0^\alpha} + \int_0^t \mathrm{e}^{C(t-s)} C\big(E_\alpha(s), \|\widetilde{f}(s)\|_{L^\infty}\big)\mathrm{d}s, \end{aligned} \tag{4.33}$$

for some continuous function $C(E_\alpha(\,.\,), \|f(\,.\,)\|_{L^\infty})$. Integration over $[x, x_0 - 2\tau]$ of $\partial_x\widetilde{f} = \frac{1}{2}(\phi_- - \phi_+)$ gives

$$\big|\widetilde{f}(\tau,x)\big| \le \big|\widetilde{f}(\tau, x_0-2\tau)\big| + \tfrac{1}{2}\big\|(\phi_- - \phi_+)(\tau)\big\|_{\mathcal{C}_0^\alpha}\int_x^{x_0-2\tau} s^\alpha \mathrm{d}s.$$

For any $0 \le \tau \le \tau_* < \frac{1}{2}x_0$ the $f(\tau, x_0 - 2\tau)$ term is estimated by a multiple of the initial energy in a standard way, which leads to the estimate (recall

[8] The constant C in Equation (4.23) does not necessarily coincide with that in (4.28).

that $\alpha > -1$)

$$\big\|\widetilde{f}(\tau)\big\|_{L^\infty} \le CE_\alpha(\tau) + Ce^{C\tau}\|\phi_+(0)\|_{\mathcal{C}_0^\alpha} + \int_0^\tau e^{C(\tau-s)}C\big(E_\alpha(s), \big\|\widetilde{f}(s)\big\|_{L^\infty}\big)\,ds. \tag{4.34}$$

Next, $\|G(s)\|_{\mathcal{H}_k^{\alpha-1/2}} \le C\|H(s,.,x^{(n-1)/2}\widetilde{f}\,)\|_{\mathcal{H}^{\alpha-1/2+(n+3)/2}}$, and our hypothesis that H has a uniform zero of order ℓ together with (A.31) gives

$$\big\|G(s)\big\|_{\mathcal{H}_k^{\alpha-1/2}} \le C\big(\|\widetilde{f}(s)\|_{L^\infty}\big)\|\widetilde{f}\|_{\mathcal{H}_k^{\alpha+(n+2)/2-\ell(n-1)/2}}.$$

In view of (4.34) this can be estimated by a function of $E_\alpha(s)$ and of $\|\widetilde{f}(s)\|_{L^\infty}$,

$$\big\|G(s)\big\|^2_{\mathcal{H}_k^{\alpha-1/2}} \le C\big(\|\widetilde{f}(s)\|_{L^\infty}\big)\big\|\widetilde{f}(s)\big\|^2_{\mathcal{H}_k^\alpha} \le C\big(\|\widetilde{f}(s)\|_{L^\infty}\big)E_\alpha(s), \tag{4.35}$$

provided that

$$\ell \ge \frac{n+2}{n-1} \tag{4.36}$$

(which coincides again with (4.21)). If (4.36) holds, from (4.29) and (4.34) we obtain

$$\big\|\widetilde{f}(\tau)\big\|_{L^\infty} + E_\alpha(\tau) \le Ce^{C\tau}\big(E_\alpha(0) + \big\|\partial_x\widetilde{f}(0)\big\|_{\mathcal{C}_0^\alpha} + \big\|\partial_\tau\widetilde{f}(0)\big\|_{\mathcal{H}_k^\alpha}\big) + \int_0^\tau \Phi\big(\tau, s, \|\widetilde{f}(s)\|_{L^\infty}, E_\alpha(s)\big)\,ds, \tag{4.37}$$

for some constant C, and for a function Φ which is bounded on bounded sets. It then easily follows that there exists a time τ_+ and a constant M, depending only upon x_0 and a bound on the norms of the initial data in the spaces appearing in Equations (4.18)–(4.20), such that $\|\widetilde{f}(\tau)\|_{L^\infty}$ and $E_\alpha(\tau)$ remain bounded by M for $0 \le \tau \le \tau_+$. Since all the objects above were x_1-independent, so is τ_+. By the usual continuation criterion (*cf., e.g.,* [35, Prop. 1.5, chap. 16, vol. III][(9)]) the solution exists on Ω_{x_1,x_0,τ_+} for all x_1; it thus follows that the maximally extended solution of the initial value problem considered here exists on a set which includes Ω_{x_0,τ_+}.

To establish point 2), suppose that a global *a priori* L^∞ bound on $\widetilde{f}$ is known. Then (4.29) and (4.35) give a linear integral inequality on E_α, and Gronwall's Lemma gives a global bound on E_α. Arguments of the last part of the proof of point 1) yield the result. □

For the purpose of estimating time derivatives of the solutions we will need a generalisation of Theorem 4.1, which covers the equations contained by time-differentiating Equations (4.13)–(4.16). There are lots of ways to relax those

(9) In that reference symmetric hyperbolic systems on a torus are considered; however simple domain of dependence considerations show that the results there apply to the setup here.

hypotheses; for simplicity we shall only make those generalisations which are strictly necessary for the arguments in the next section to go through. First, the fact that f is scalar valued plays no role in our considerations above; henceforth we assume that f has values in $\mathbb{R}^N$ for some $N \geq 1$. Next, the definitions (4.10) of $e_\pm$ and e_A will be kept. We will consider systems of the form

$$P\begin{pmatrix} \varphi \\ \psi \end{pmatrix} + \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix}\begin{pmatrix} \varphi \\ \psi \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix} + G, \tag{4.38a}$$

$$\varphi = \begin{pmatrix} \phi_+ \\ \phi_A \end{pmatrix}, \quad \psi = \begin{pmatrix} \phi_- \\ \psi_A \end{pmatrix} \tag{4.38b}$$

together with

$$\phi_A = \frac{1}{(x+\tau+\frac{1}{2})} h_A(\widetilde{f}\,) + B_{A,\phi}\tilde{f}, \tag{4.39a}$$

$$\psi_A = \frac{1}{(x+\tau+\frac{1}{2})} h_A(\widetilde{f}\,) + B_{A,\psi}\tilde{f}, \tag{4.39b}$$

$$e_-(\widetilde{f}\,) = B_0\phi_- + B_1\widetilde{f}, \tag{4.39c}$$

$$e_+(\widetilde{f}\,) = \phi_+, \tag{4.39d}$$

for some matrix valued functions $B_{A,\phi}$, $B_{A,\psi}$, B_0, B_1, with B_0 invertible. Here

$$P = \begin{pmatrix} e_- & \ell^A D_A \\ (\ell^A)^t D_A & e_+ \end{pmatrix} \tag{4.40}$$

is the (geometric) principal part of Equations (4.13)–(4.14). The nonlinear term $G = G(x^\mu, \widetilde{f}\,)$ will be labeled as

$$G = (G_{e_+(\phi_-)}, G_{e_+(\psi_A)}, G_{e_-(\phi_A)}, G_{e_+(\phi_-)}), \tag{4.41}$$

with the order of the components following that of Equations (4.13)–(4.14). The B_{ab}'s will be labeled as B_{ϕ_-,ϕ_+}, B_{ϕ_-,ϕ_A}, *etc.*; for example, in this notation, the second of Equations (4.14) takes the form

$$\begin{aligned} e_+(\phi_-) = {} & D_{e_A}\phi_A - B_{\phi_-,\phi_-}\phi_- - B_{\phi_-,\phi_+}\phi_+ \\ & - B_{\phi_-,\phi_A}\phi_A - B_{\phi_-,\psi_A}\psi_A + b_- + G_{e_+(\phi_-)}, \end{aligned} \tag{4.42}$$

with actually $B_{\phi_-,\phi_A} = B_{\phi_-,\psi_A} = 0$.

Some effort will be needed to prove the information of point 3) of the theorem that follows; this is needed to be able to iteratively apply that theorem in the next section:

Theorem 4.3. — *Consider the system* (4.38)–(4.39) *with*

$$\begin{aligned} \|a(\tau)\|_{\mathcal{H}^\alpha_k} + \|b(\tau)\|_{\mathcal{H}^\alpha_k} + \sup_{a,b=1,2}\|B_{ab}(\tau)\|_{\mathcal{C}^0_k} + \sup_{\substack{A=1,2 \\ \lambda=\phi,\psi}}\|B_{A,\lambda}(\tau)\|_{\mathcal{C}^0_k} \\ + \|B_0(\tau)\|_{\mathcal{C}^0_k} + \|B_0^{-1}(\tau)\|_{L^\infty} + \|B_1(\tau)\|_{\mathcal{C}^0_k} \leq \widetilde{C}, \end{aligned} \tag{4.43}$$

for some constant $\widetilde{C}$, and suppose that

$$G(x^\mu, \widetilde{f}\,) = \Omega^{-(n+3)/2} H(x^\mu, \Omega^{(n-1)/2}\widetilde{f}\,), \tag{4.44}$$

with $G_{e_-(\phi_A)} = 0$, with H having a uniform zero of order ℓ in the sense of (A.30)*, with ℓ satisfying* (4.21)*. If the initial data satisfy* (4.18)–(4.20) *with some $k > \frac{1}{2}n + 1$ and $-1 < \alpha < -\frac{1}{2}$, then:*

1) *The conclusions of point* 1) *of Theorem 4.1 hold with a time τ_+ depending only upon the constant $\widetilde{C}$ in* (4.43) *and a bound on the norms of the initial data in the spaces appearing in Equations* (4.18)–(4.20)*.*

2) *The conclusions of point* 2) *of Theorem 4.1 hold.*

3) *Under the hypotheses of point* 2) *of Theorem 4.1 we also have*

$$\big\|(x + 2\tau)\partial_\tau \widetilde{f}\big\|_{L^\infty(\Omega_{x_0,\tau_*})} < \infty. \tag{4.45}$$

Remarks

1) The condition $G_{e_-(\phi_A)} = 0$ can be weakened to

$$G_{e_-(\phi_A)}(x^\mu, \widetilde{f}\,) = \Omega^{-(n+2)/2} H_{e_-(\phi_A)}(x^\mu, \Omega^{(n-1)/2}\widetilde{f}\,), \tag{4.46}$$

for some function $H_{e_-(\phi_A)}$ with a uniform zero of order ℓ. Similarly it suffices to assume that

$$G_{e_+(\psi_A)}(x^\mu, \widetilde{f}\,) = \Omega^{-(n+4)/2} H_{e_+(\psi_A)}(x^\mu, \Omega^{(n-1)/2}\widetilde{f}\,), \tag{4.47}$$

for some function $H_{e_+(\psi_A)}$ with a uniform zero of order ℓ.

2) If the inequality in (4.21) is not an equality for $n = 2, 3$ (no further restrictions for $n \geq 4$), then the result remains true with $\alpha = -\frac{1}{2}$, see Remark 3) after Theorem 4.1.

Proof. — Let us start by remarking that, because $\psi_A = \phi_A$, in equations such as (4.42) we can replace B_{ϕ_-,ϕ_A} by $B_{\phi_-,\phi_A} + B_{\phi_-,\psi_A}$ obtaining a system in which $B_{\phi_-,\psi_A} = 0$. Proceeding similarly with the other equations we may thus without loss of generality assume that

$$B_{*,\psi_A} = 0. \tag{4.48}$$

The proof of points 1) and 2) is then identical to that of Theorem 4.1, with the following minor changes: Equation (4.32) is replaced by the equation

$$\begin{aligned} e_-(\phi_+) &+ B_{\phi_+,\phi_+}\phi_+ \\ &= D_{e_A}\phi_A - B_{\phi+-,\phi_-}\phi_- - B_{\phi_+,\phi_A}\phi_A + a_+ + G_{e_-(\phi_+)} \end{aligned} \tag{4.49}$$

to which Proposition B.1 still applies, recovering (4.33). Further, the equation $\partial_x \widetilde{f} = \frac{1}{2}(\phi_- - \phi_+)$ has to be replaced by

$$\partial_x \widetilde{f} + \frac{B_1}{2}\widetilde{f} = \frac{B_0\phi_- - \phi_+}{2},$$

and the desired conclusion is obtained by Proposition B.3. The remaining arguments do not require any modifications.

To prove point 3), from (4.42) we obtain

$$e_+\big[(x+2\tau)\phi_-\big] = (x+2\tau)\big(D_{e_A}\phi_A - B_{\phi_-,\phi_-}\phi_- - B_{\phi_-,\phi_A}\phi_A - B_{\phi_-,\phi_+}\phi_+ + b_- + G_{e_+(\phi_-)}\big). \tag{4.50}$$

From Equations (4.31), (4.33), and (4.39c) together with $\phi_-, \phi_A \in \mathcal{H}^\alpha_k \subset \mathcal{C}^\alpha_0$, $D_{e_A}\phi_A \in \mathcal{H}^\alpha_{k-1} \subset \mathcal{C}^\alpha_0$, we obtain

$$e_+\big[(x+2\tau)\phi_-\big] \le \widehat{C}x^{-\alpha},$$

for some constant $\widehat{C}$ depending only upon the initial data and $\|\widetilde{f}\|_{L^\infty(\Omega_{x_0,\tau_*})}$. Integrating as in the identity (3.41) we arrive at

$$\begin{aligned}
&\big|B_0^{-1}\{(x+2\tau)(\partial_\tau\widetilde{f} - B_1\widetilde{f})(x,v,\tau)\}\big| \\
&\quad\le \big|B_0^{-1}\Big\{(x+2\tau)\partial_\tau\widetilde{f}(x+2\tau,v,0)\Big\}\big| + C\big(\|\widetilde{f}\|_{L^\infty(\Omega_{x_0,\tau_*})} + \widehat{C}\big) \\
&\quad\le C\big(\|\partial_\tau\widetilde{f}\|_{\mathcal{C}^{-1}_0} + \|\widetilde{f}(0)\|_{L^\infty(\Omega_{x_0,\tau_*})} + \widehat{C}\big),
\end{aligned}$$

and Equation (4.45) follows. □

4.2. Estimates on the time derivatives of the solutions. — So far we have established existence of solutions with initial data in weighted Sobolev spaces, as well as weighted estimates on the space-derivatives of the solutions. The next step in proving polyhomogeneity is to establish estimates on time-derivatives. Similarly to the linear case, the question of corner conditions arises. In order to handle that, we introduce an index m, which corresponds to the number — perhaps zero — of corner conditions which are satisfied by the initial data. Next, the definition (A.30) of a uniform zero of order l has to be strengthened by adding conditions on time-derivatives: we shall require that for all $M \in \mathbb{R}$, for all $0 \le i \le \min(k,l)$ and for all $0 \le j \le m$ there exists a constant $\widehat{C} = \widehat{C}(M,m,k)$ such that for all $|p| \le M$ we have

$$\Big\|\frac{\partial^{i+j}F(\tau,.\,,p)}{\partial p^i\partial\tau^j}\Big\|_{\mathcal{C}^0_{k+m-i-j}} \le \widehat{C}|p|^{\ell-i}. \tag{4.51}$$

We start with the following:

Theorem 4.4. — *Let $\mathbb{N} \ni m \ge 0$; consider a solution $f : \Omega_{x_0,\tau_*} \to \mathbb{R}$ of* (4.1) *satisfying $\|\widetilde{f}\|_{L^\infty(\Omega_{x_0,\tau_*})} < \infty$, and suppose that*

$$0 \le i \le m+1, \quad \partial^i_\tau\widetilde{f}_{|\Sigma_{x_0,0}} \in \mathcal{H}^\alpha_{k+m+1-i}(\Sigma_{x_0,0}), \tag{4.52}$$

$$0 \le i \le m, \quad \partial_x\partial^i_\tau\widetilde{f}_{|\Sigma_{x_0,0}} \in \mathcal{C}^\alpha_0(\Sigma_{x_0,0}) \cap \mathcal{H}^{\alpha-1/2}_{k+m-i}(\Sigma_{x_0,0}), \tag{4.53}$$

with some $k > \frac{1}{2}n+1$ and $-1 < \alpha < -\frac{1}{2}$. Suppose, further, that H is smooth in f and has a uniform zero of order ℓ at $f = 0$, in the sense of (4.51), *with ℓ*

as in Equation (4.21). *Then for* $0 \le \tau < \tau_*$ *and for* $0 \le i \le m$, $0 \le j + i < k + m - \frac{1}{2}n$ *we have*

$$\text{(4.54a)}\quad \big[(\tau + 2x)\partial_\tau\big]^j \partial_\tau^i \widetilde{f}_{|\Sigma_{x_0,\tau}} \in L^\infty(\Sigma_{x_0,\tau}) \cap \mathcal{H}^\alpha_{k+m+1-i-j}(\Sigma_{x_0,\tau}),$$

$$\text{(4.54b)}\quad \partial_x\big[(\tau + 2x)\partial_\tau\big]^j \partial_\tau^i \widetilde{f}_{|\Sigma_{x_0,\tau}} \in \mathcal{H}^{\alpha-\frac{1}{2}}_{k+m-i-j}(\Sigma_{x_0,\tau}) \cap \mathcal{C}^\alpha_0(\Sigma_{x_0,\tau}),$$

and

$$\text{(4.55)}\qquad 0 \le p < k - \tfrac{1}{2}n, \quad \big[(\tau + 2x)\partial_\tau\big]^p \partial_\tau^{m+1} \widetilde{f}_{|\Sigma_{x_0,\tau}} \in \mathcal{H}^\alpha_{k-p}(\Sigma_{x_0,\tau}),$$

with τ*-independent bounds on the norms.*

Remark. — As before, in dimensions $n \ge 4$ the result remains valid for $\alpha = -\frac{1}{2}$; in dimensions $n = 2, 3$ the value $-\frac{1}{2}$ for α is allowed if the inequality in (4.21) is not an equality.

The proof below actually proves the analogous result for systems considered in Theorem 4.3, provided that obvious time-derivative conditions on the coefficients are added to (4.43), the simplest possibility being

$$\text{(4.56)}\qquad \big\|\partial_\tau^i a(\tau)\big\|_{\mathcal{H}^\alpha_{k+m-i}} + \big\|\partial_\tau^i b(\tau)\big\|_{\mathcal{H}^\alpha_{k+m-i}} + \sup_{a,b=1,2} \|\partial_\tau^i B_{ab}(\tau)\|_{\mathcal{C}^0_{k+m-i}} + \sup_{\substack{A=1,2\\ \lambda=\phi,\psi}} \big\|\partial_\tau^i B_{A,\lambda}(\tau)\big\|_{\mathcal{C}^0_{k+m-i}} + \big\|\partial_\tau^i B_0(\tau)\big\|_{\mathcal{C}^0_{k+m-i}} + \big\|\partial_\tau^i B_1(\tau)\big\|_{\mathcal{C}^0_{k+m-i}} \le \widetilde{C},$$

with $0 \le i \le m + k$; the same remark applies to Corollary 4.5 below. Before passing to that proof, we note that an important consequence of Theorem 4.4 is that corner conditions will hold at any time $\tau > 0$, regardless of whether or not they hold at $\tau = 0$:

Corollary 4.5. — *Under the conditions of point* 2) *of Theorem* 4.1, *for any* $0 < \tau < \tau_*$ *and for* $0 \le i < k - 1 - \frac{1}{2}n$ *we have*

$$\partial_\tau^i \widetilde{f}_{|\Sigma_{x_0,\tau}} \in L^\infty(\Sigma_{x_0,\tau}) \cap \mathcal{H}^\alpha_{k+1-i}(\Sigma_{x_0,\tau}), \quad \partial_\tau^{i+1} \widetilde{f}_{|\Sigma_{x_0,\tau}} \in \mathcal{H}^\alpha_{k-i}(\Sigma_{x_0,\tau}),$$
$$\partial_x \partial_\tau^i \widetilde{f}_{|\Sigma_{x_0,\tau}} \in \mathcal{H}^{\alpha-\frac{1}{2}}_{k-i}(\Sigma_{x_0,\tau}) \cap \mathcal{C}^\alpha_0(\Sigma_{x_0,\tau}).$$

We shall need the following simple Lemma:

Lemma 4.6. — *Let* $F(x^\mu, p)$ *be a function which is smooth in* p *at fixed* x^μ *and suppose that it has a uniform zero of order* $\ell \ge 1$ *in* p. *Then*

1) *For all* $i \in \mathbb{N}$ *the function* $\partial_\tau^i(F(x^\mu, u(x^\mu))$ *has a uniform zero of order* ℓ, *when viewed as a function of* $(u, \partial_\tau u, \dots, \partial_\tau^i u)$.
2) *Let* $H = \partial_p F$, *then* H *has a uniform zero of order* $\ell - 1$.

Proof. — Let $u = (u^i)$; smoothness of F in p allows us to write

$$F(\vec{x}, \tau, u) = A_{i_1,\dots,i_\ell} u^{i_1} \cdots u^{i_\ell}, \tag{4.57}$$

with some coefficients $A_{i_1,\dots,i_\ell} = A_{i_1,\dots,i_\ell}(\vec{x}, \tau, u)$ which are smooth in u, and totally symmetric in $i_1, \dots, i_\ell$; recall that the summation convention is used throughout. Point 2) immediately follows from (4.57). From that equation we also obtain

$$\begin{aligned}\partial_\tau F(\tau, \vec{x}, u) = \ & (\partial_\tau A_{i_1,\dots,i_\ell} + \partial_{u^i} A_{i_1,\dots,i_\ell} \partial_\tau u^i) u^{i_1} \cdots u^{i_\ell} \\ & + \ell A_{i_1,\dots,i_\ell} u^{i_1} \cdots u^{i_{\ell-1}} \partial_\tau u^{i_\ell},\end{aligned}$$

which proves point 1) for $i = 1$. The result then follows by straightforward induction. □

We can pass now to the proof of Theorem 4.4:

Proof. — We assume that Equations (4.38)–(4.39) are satisfied; Theorem 4.3 shows that (4.54)–(4.55) hold with $i = j = p = 0$. Consider the vector-valued function

$$\big(\widetilde{f}, (x+2\tau)\partial_\tau \widetilde{f}, \varphi, (x+2\tau)\partial_\tau \varphi, \psi, (x+2\tau)\partial_\tau \psi\big),$$

so that the new function $\widetilde{f}$ in (4.39) is $(\widetilde{f}, (x+2\tau)\partial_\tau \widetilde{f}\,)$, while the new functions φ, resp. ψ, in (4.38b) are $(\varphi, (x+2\tau)\partial_\tau \varphi)$, resp. $(\psi, (x+2\tau)\partial_\tau \psi)$. We claim that a set of equations of the form (4.38)–(4.39) holds for those new functions. Consider, for instance, Equation (4.39c); set

$$\widehat{f} := (x+2\tau)\partial_\tau \widetilde{f}, \quad \widehat{\phi}_- := (x+2\tau)\partial_\tau \phi_-,$$

etc., we have

$$\begin{aligned} e_-(\widehat{f}) &= \partial_\tau\big((x+2\tau)(B_0\phi_- + B_1 \widetilde{f}\,)\big) \\ &= B_0 \widehat{\phi}_- + (2B_0 + (x+2\tau)\partial_\tau B_0)\phi_- + B_1 \widehat{f} + (2B_1 + (x+2\tau)\partial_\tau B_1)\widetilde{f}, \end{aligned}$$

which is linear in $(\widetilde{f}, \widehat{f}, \phi_-, \widehat{\phi}_-)$. In fact

$$\begin{aligned} e_-\begin{pmatrix} \widetilde{f} \\ \widehat{f} \end{pmatrix} = & \begin{pmatrix} B_0 & 0 \\ 2B_0 + (x+2\tau)\partial_\tau B_0 & B_0 \end{pmatrix} \begin{pmatrix} \phi_- \\ \widehat{\phi}_- \end{pmatrix} \\ & + \begin{pmatrix} B_1 & 0 \\ 2B_1 + (x+2\tau)\partial_\tau B_1 & B_1 \end{pmatrix} \begin{pmatrix} \widetilde{f} \\ \widehat{f} \end{pmatrix}, \end{aligned}$$

and the new matrix B_0 is again invertible, as desired. Next,

$$\begin{aligned}
e_-(\widehat{\phi}_+) &= \partial_\tau\big((x+2\tau)\partial_\tau\phi_+\big)\\
&= \partial_\tau\big((x+2\tau)\big(-D_A\phi_A - B_{\phi_+\phi_-}\phi_-\\
&\qquad - B_{\phi_+\phi_A}\phi_A - B_{\phi_+\phi_+}\phi_+ + a_+ + G_{e_-(\phi_+)}\big)\big)\\
&= -D_A\widehat{\phi}_A - B_{\phi_+\phi_-}\widehat{\phi}_- - B_{\phi_+\phi_A}\widehat{\phi}_A - B_{\phi_+\phi_+}\widehat{\phi}_+\\
&\qquad + \mathrm{linear}(\varphi,\psi) + \widehat{a}_+ + G_{e_-(\widehat{\phi}_+)},\\
\widehat{a}_+ &= -2D_A\phi_A + \partial_\tau a_+ \ \in \mathcal{H}^\alpha_{k+m-1},\\
G_{e_-(\widehat{\phi}_+)} &= \partial_\tau(G_{e_-(\phi_+)})(x+2\tau),
\end{aligned}$$

where "linear" denotes terms which are linear in the relevant variables. The equation for $e_-(\widehat{\phi}_A)$ is handled in a similar way. The equations involving only e_+ or ∂_A are straightforward, since those operators commute with multiplication by $(x+2\tau)$. By Lemma 4.6 the new non-linearity has again a zero of order ℓ, when considered as a function of $(\widetilde{f},(x+2\tau)\partial_\tau\widetilde{f}\,)$. In order to apply Theorem 4.3 we need to check that the initial data are in the right spaces. Clearly

$$\big((x+2\tau)\partial_\tau\widetilde{f}\,\big)(0) = x\partial_\tau\widetilde{f}(0) \in \mathcal{H}^{\alpha+1}_{k+m} \subset \mathcal{H}^\alpha_{k+m}\cap L^\infty,$$

$$\big(\partial_x((x+2\tau)\partial_\tau\widetilde{f}\,)\big)(0) = \big(\partial_\tau\widetilde{f} + x\partial_x\partial_\tau\widetilde{f}\,\big)(0) \in \mathcal{H}^\alpha_{k+m-1} \subset \mathcal{C}^\alpha_0 \cap \mathcal{H}^{\alpha-1/2}_{k+m-1}.$$

Condition (4.20) requires some more work:

$$\begin{aligned}
\big(\partial_\tau\big((x+2\tau)\partial_\tau\widetilde{f}\,\big)\big)(0) &= \big(2\partial_\tau\widetilde{f} + x\partial_\tau^2\widetilde{f}\,\big)(0)\\
&= \big(2\partial_\tau\widetilde{f} + x(2\partial_x + e_+)\partial_\tau\widetilde{f}\,\big)(0)\\
&= \big(2\partial_\tau\widetilde{f} + 2x\partial_x\partial_\tau\widetilde{f} + xe_+(B_0\phi_- + B_1\widetilde{f}\,)\big)(0).
\end{aligned}$$

The first two terms are obviously in $\mathcal{H}^\alpha_{k+m-1}$, and so is $xe_+(B_1\widetilde{f}\,) = x(\partial_\tau - 2\partial_x)(B_-\widetilde{f}\,)$. Equation (4.42) gives

$$\begin{aligned}
\big(xe_+(\phi_-)\big)(0) &= x\big(D_{e_A}\phi_A - B_{\phi_-,\phi_-}\phi_- - B_{\phi_-,\phi_+}\phi_+\\
&\qquad -B_{\phi_-,\phi_A}\phi_A - B_{\phi_-,\psi_A}\psi_A + b_- + G_{e_+(\phi_-)}\big)(0).
\end{aligned}$$

The desired property $(xe_+(B_0\phi_-))(0) \in \mathcal{H}^\alpha_{k+m-1}$ follows immediately; the only non-trivial term is $xG_{e_+(\phi_-)}$, the $\mathcal{H}^\alpha_{k+m+1}$ norm of which can be estimated by a function of $\|\widetilde{f}(0)\|_{L^\infty}$ and $\|\widetilde{f}(0)\|_{\mathcal{H}^\alpha_{k+m+1}}$, *cf.* Equation (4.35). Now, $(x+2\tau)\partial_\tau\widetilde{f}$ is uniformly bounded on Ω_{x_0,τ_*} by point 3 of Theorem 4.3, so that we can apply point 2) of Theorem 4.3 to conclude that Equations (4.54)–(4.55) hold with $j=p=1$ and $m=0$; straightforward induction establishes Theorem 4.4 for the remaining j's and p's.

Consider, now, $m=1$; the result already established with $m=0$ shows that $\partial_\tau\widetilde{f}(\tau)$ exists and satisfies (4.54) with $i=1$ for any $\tau>0$; similarly (4.55) holds with $m=1$ for any $\tau>0$. Now, a calculation similar (but simpler) to the one done above shows that $(\widetilde{f}, \partial_\tau \widetilde{f}\,)$ satisfies a system of equations of the form (4.38)–(4.39) with initial data satisfying the conditions of Theorem 4.3 by hypothesis; the uniform bounds on some interval $[0, \tau_+)$ follow by point 1) of that theorem. We therefore have

$$\big\|(\widetilde{f}, \partial_\tau \widetilde{f}\,)\big\|_{L^\infty(\Omega_{x_0,\tau_*})} < \infty.$$

We can then apply the result already established for $m = 0$ to the system of equations satisfied by $(\widetilde{f}, \partial_\tau \widetilde{f}\,)$ to obtain the conclusion of Theorem 4.4 with $m = 1$. An induction upon m finishes the proof. □

4.3. Polyhomogeneous solutions. — The aim of this section is to establish polyhomogeneity of solutions of a large class of semi-linear systems of the form

$$\partial_\tau \varphi + B_{11}\varphi + B_{12}\psi \;=\; L_{11}\varphi + L_{12}\psi + a + G_\varphi, \tag{4.58a}$$

$$\partial_x \psi + B_{21}\varphi + B_{22}\psi \;=\; L_{21}\varphi + L_{22}\psi + b + G_\psi, \tag{4.58b}$$

with a nonlinearity $G = (G_\varphi, G_\psi)$ of the form

$$G = x^{-p\delta} H(x^\mu, x^{q\delta}\psi_1, x^{q\delta+1}\psi_2, x^{q\delta+1}\varphi). \tag{4.59}$$

Here we have decomposed ψ as

$$\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}; \tag{4.60}$$

this is motivated by different *a priori* estimates we have at our disposal for the appropriately defined components ψ_1 and ψ_2 of ψ in the applications we have in mind. Polyhomogeneity of solutions of (4.1) will follow as a special case, see Theorem 4.10 below. We will need to impose various restrictions on the function H, in order to do that some terminology will be needed. We shall say that a function $H(x^\mu, u)$ is *δ-polyhomogeneous in x with a uniform zero of order ℓ in u* if H is smooth in $u \in \mathbb{R}^N$ at fixed x^μ, if H satisfies (A.30) for any $0 \le i \le \min\{\ell, k\}$ and any $k \in \mathbb{N}$, if

$$\forall i \in \mathbb{N}, \quad \partial_u^i H(\,.\,, u) \in \mathcal{A}^\delta_\infty \tag{4.61}$$

at fixed constant u, and if we have the uniform estimate for constant u's

$$\forall \epsilon > 0,\ M \ge 0,\ i, k \in \mathbb{N},\ \exists\, C(\epsilon, M, i, k),\ \forall\, |u| \le M, \\ \big\|\partial_u^i H(\,.\,, u)\big\|_{\mathcal{C}_k^{-\epsilon}} \le C(\epsilon, M, i, k). \tag{4.62}$$

The qualification "in u" in "uniform zero of order ℓ in u" will often be omitted. The small parameter ϵ has been introduced above to take into account the possible logarithmic blow-up of functions in $\mathcal{A}^\delta_\infty$ at $x = 0$; for the applications to the nonlinear scalar wave equation or to the wave map equation on Minkowski space-time, the alternative simpler requirement would actually suffice:

$$\forall M \geq 0,\ i,k \in \mathbb{N},\ \exists C(M,i,k),\ \forall |u| \leq M, \quad \big\|\partial_u^i H(\,.\,,u)\big\|_{\mathcal{C}^0_k} \leq C(M,i,k), \tag{4.63}$$

again for constant u's. Clearly functions which are jointly smooth in u and in x^μ satisfy the above conditions; Lemma 4.7 below provides another class of such functions. The following simple facts about functions in the above class will be useful:

LEMMA 4.7. — *Let $m_1, m_2, k \in \mathbb{N}$, $m_1 \leq m_2$, and let $P(x^\mu, u)$ be a polynomial in $u = (u^1, \dots u^N)$ of the form*

$$P(x^\mu, u) = \sum_{m_1 \leq j \leq m_2} P_{i_1,\dots,i_j}(x^\mu) u^{i_1} \cdots u^{i_j},$$

with coefficients $P_{i_1,\dots,i_j}(x^\mu) \in \mathcal{A}^\delta_\infty$. Then:

1) *P is δ-polyhomogeneous in x with a uniform zero of order m_1.*
2) *If $f \in \mathcal{A}^\delta_k + \mathcal{C}^\lambda_\infty$ for some $\lambda > 0$, then for any $\epsilon > 0$ we have*

$$P(\,.\,, x^{q\delta} f) \in x^{m_1 q\delta}(\mathcal{A}^\delta_k + \mathcal{C}^{\lambda-\epsilon}_\infty).$$

The proof of Lemma 4.7 is elementary and will be left to the reader.

LEMMA 4.8. — *Let $k, q \in \mathbb{N}$ and let $H(x^\mu, u)$ be δ-polyhomogeneous with respect to x with a zero of order m in u. If*

$$f \in \begin{cases} \mathcal{A}^\delta_k \cap L^\infty + \mathcal{C}^\lambda_\infty & \text{if } q = 0, \\ \mathcal{A}^\delta_k + \mathcal{C}^\lambda_\infty & \text{otherwise,} \end{cases}$$

for some $\lambda > 0$, then for any $\epsilon > 0$

$$H(\,.\,, x^{q\delta} f) \in x^{mq\delta}(\mathcal{A}^\delta_k + \mathcal{C}^{\lambda-\epsilon}_\infty).$$

Proof. — We Taylor-expand H in u to order r, where r is any number satisfying $rq\delta > mq\delta + \lambda$. We then have

$$H(x^\mu, x^{q\delta} f) = P(x^\mu, x^{q\delta} f) + R,$$

where P is a polynomial and R is a remainder. We note that the coefficients of the expansion of P can be obtained by differentiating with respect to u and setting $u = 0$, and are therefore in $\mathcal{A}^\delta_\infty$ by (4.61). Further, the usual integral formula for the remainder in a Taylor expansion together with (4.62) shows that R has a uniform zero of order r, in the sense of Equation (A.30). The result follows from Lemma 4.7 and from Lemma A.5. □

We are ready now to pass to the proof of the non-linear analogue of Theorem 3.4:

THEOREM 4.9. — *Let* $p \in \mathbb{Z}, q, 1/\delta \in \mathbb{N}, -1 < \beta' \in \mathbb{R}$, $k \in \mathbb{N} \cup \{\infty\}$, *and let*

$$(\varphi, \psi) \in \mathcal{C}^{\beta'}_{\infty}(\Omega_{x_0,T}) \times \mathcal{C}^{\beta'}_{\infty}(\Omega_{x_0,T}), \quad \psi_1 \in L^{\infty}(\Omega_{x_0,T})$$

(ψ_1 *as in Equation* (4.60)), *be a solution of* (4.58) *with* G *of the form* (4.59), *where* H *is* δ*-polyhomogeneous in* x *with a uniform zero of order*

$$m > \frac{p - \delta^{-1}}{q}. \tag{4.64}$$

Suppose that Equations (3.43)–(3.44) *hold, and that*

(4.65a) $\quad B_{11} \in (\mathcal{A}^{\delta}_k \cap L^{\infty})(\Omega_{x_0,T}), \quad B_{12}, B_{22}, B_{21} \in \mathcal{A}^{\delta}_k(\Omega_{x_0,T}),$

(4.65b) $\quad a, b \in \mathcal{A}^{\delta}_k(\Omega_{x_0,T}), \quad \varphi(0) \in \mathcal{A}^{\delta}_k(M_{x_0}).$

Then

$$\varphi \in (x^{(mq-p)\delta}\mathcal{A}^{\delta}_k + \mathcal{A}^{\delta}_k)(\Omega_{x_0,T}) = x^{\min((mp-q)\delta,0)}\mathcal{A}^{\delta}_k(\Omega_{x_0,T}),$$

$$\psi \in x^{\min\{(mq-p)\delta+1,1\}}\mathcal{A}^{\delta}_k(\Omega_{x_0,T}) + C_{\infty}(\overline{\Omega_{x_0,T}}) \subset (\mathcal{A}^{\delta}_k \cap L^{\infty})(\Omega_{x_0,T}).$$

If one further assumes $L^{\mu}_{12}, B_{12}, a, \varphi(0), G_{\varphi}(\,.\,,0) \in L^{\infty}(\Omega_{x_0,T})$, *then it also holds that*

$$\varphi \in (x^{(mq-p)\delta}\mathcal{A}^{\delta}_k + \mathcal{A}^{\delta}_k \cap L^{\infty})(\Omega_{x_0,T}).$$

REMARK. — Obviously the theorem remains true if we replace G by a finite sum of nonlinearities satisfying the above hypotheses, with different p's and q's for each term of the sum.

Proof. — The result is established by a repetition of the proof of Theorem 3.4, using Lemma A.5 and Lemma 4.8 to obtain the necessary estimates on the non-linear terms. We simply note that the condition on the order m of the non-linearity guarantees, using Lemma A.5, that

$$\partial_x \psi = c_2 \in \mathcal{C}^{\lambda-\epsilon}_{\infty},$$

with $\lambda = \min\{\beta', mq\delta - p\delta\} > -1$, hence $\psi \in L^{\infty}$ by integration. Decreasing β' if necessary we may without loss of generality assume that $\beta' = \lambda$. When applying Lemma 4.8 it is convenient to view the function H as a function of the variable $f := (\psi_1, x\psi_2, x\varphi) \in L^{\infty}$. The remaining details are left to the reader. □

As a straightforward corollary of Theorem 4.9 one obtains:

THEOREM 4.10. — *Let* $\delta = 1$ *in odd space dimensions, and let* $\delta = \frac{1}{2}$ *in even space dimensions. Consider Equation* (4.1) *on* $\mathbb{R}^{n,1}$, $n \geq 2$, *with initial data*

$$\widetilde{f}_{|\{\tau=0\}}, \quad \partial\widetilde{f}/\partial\tau_{|\{\tau=0\}} \in (\mathcal{A}^{\delta}_{\infty} \cap L^{\infty})(M_{x_0}).$$

Suppose further that $H(x^{\mu}, f)$ *is smooth in* f *at fixed* x^{μ}, *bounded and* δ*-polyhomogeneous in* x^{μ} *at constant* f, *and has a zero of order* ℓ *at* $f = 0$, *with* ℓ *as in* (4.21). *Then:*

1) *There exists* $\tau_+ > 0$ *such that* f *exists* Ω_{x_0,τ_+}, *with*

$$\|\widetilde{f}\|_{L^\infty(\Omega_{x_0,\tau_+})}. \tag{4.66}$$

2) *If the initial data are* compatible *polyhomogeneous in the sense that there exists* $\lambda < 1$ *such that*

$$\forall i \in \mathbb{N}, \quad \partial_x \partial_\tau^i \widetilde{f}(0) \in \mathcal{C}_\infty^{-\lambda}(M_{x_0}),$$

then the solution is polyhomogeneous on each neighborhood Ω_{x_0,τ_*} *of* $\mathcal{I}^+$ *on which* f *exists and satisfies* (4.66) *with* τ_+ *replaced by* τ_*.

Proof. — Point 1) is Theorem 4.1 specialised to polyhomogeneous initial data. To prove point 2) we set

$$\psi = \begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix}, \quad \text{where} \quad \psi_1 = \widetilde{f},\ \psi_2 = \begin{pmatrix}\phi_-\\ \phi_A\end{pmatrix}, \tag{4.67}$$

$$\varphi = \phi_+. \tag{4.68}$$

Then Equation (4.3) takes the form (4.58) with

$$G = -\Omega^{-(n+3)/2} H(x^\mu, \Omega^{(n-1)/2}\widetilde{f}) \equiv -\Omega^{-(n+3)/2} H(x^\mu, \Omega^{(n-1)/2}\psi_1), \tag{4.69}$$

$$G_\varphi = -G, \quad G_{\psi_1} = 0, \quad G_{\psi_2} = \begin{pmatrix}-G\\ 0\end{pmatrix}. \tag{4.70}$$

For n even we take $\delta = \frac{1}{2}$, $p = n+3$, $q = n-1$; the condition (4.64) then reads $m > (n+1)/(n-1)$, which coincides with (4.21). For n odd we take $\delta = 1$, $p = \frac{1}{2}(n+3)$, $q = \frac{1}{2}(n-1)$, and (4.21) guarantees again that (4.64) holds. □

5. Wave maps

Let $(\mathcal{N}, h)$ be a smooth Riemannian manifold, and let $f : (\mathcal{M}, \mathfrak{g}) \to (\mathcal{N}, h)$ solve the wave map equation. We will be interested in maps f which have the property that f approaches a constant map f_0 as r tends to infinity along light-like directions, $f_0(x) = p_0 \in \mathcal{N}$ for all $x \in \mathcal{M}$. Introducing normal coordinates around p_0 we can write $f = (f^a)$, $a = 1, \dots, N = \dim \mathcal{N}$, with the functions f^a satisfying the set of equations

$$\square_{\mathfrak{g}} f^a + \mathfrak{g}^{\mu\nu}\Gamma^a_{bc}(f)\,\frac{\partial f^b}{\partial x^\mu}\,\frac{\partial f^c}{\partial x^\nu} = 0, \tag{5.1}$$

where the Γ^a_{bc}'s are the Christoffel symbols of the metric h. Setting as before $\widetilde{f}^a = \Omega^{-(n-1)/2} f^a$, $\tilde{\mathfrak{g}} = \Omega^2 \mathfrak{g}$, we then have from (2.3),

$$\square_{\tilde{\mathfrak{g}}} \widetilde{f}^a = -\Omega^{-(n-1)/2}\tilde{\mathfrak{g}}^{\mu\nu}\Gamma^a_{bc}(\Omega^{(n-1)/2}\widetilde{f})\frac{\partial(\Omega^{(n-1)/2}\widetilde{f}^b)}{\partial x^\mu}\frac{\partial(\Omega^{(n-1)/2}\widetilde{f}^c)}{\partial x^\nu} + \frac{n-1}{4n}(\widetilde{R} - R\,\Omega^{-2})\widetilde{f}^a. \tag{5.2}$$

In particular if $(\mathcal{M},\mathfrak{g})$ is the Minkowski space-time (and if we use the same conformal transformation as in Section 2) we obtain a system of Equations (4.13)–(4.17) with $a_A = b_A = 0$, with the obvious replacements associated with $\widetilde{f}\mapsto \widetilde{f}^a$, and with G in (4.17) replaced by

$$\begin{aligned}(5.3)\ G^a := -\Gamma^a_{bc}(\Omega^{(n-1)/2}\widetilde{f})\Big\{&\Omega^{(n-1)/2}(-\phi^b_+\phi^c_- + \phi^b_A\phi^c_A)\\ &-(n-1)\Omega^{(n-3)/2}\widetilde{f}^c\big[\big(x\phi^b_+ - (1+x+2\tau)\phi^b_-\big) - (n-1)\widetilde{f}^b\big]\Big\}.\end{aligned}$$

5.1. Existence of solutions, space derivatives estimates. — As before, for even space-dimensions n the occurrence of non-integer powers of Ω above does not allow the use of the standard conformal method except for special target manifolds $(\mathcal{N},h)$, *cf.* [11]. This can be handled in our approach, and we show:

Theorem 5.1. — *Consider Equation* (5.1) *on* $\mathbb{R}^{n,1}$ *with initial data given on a hyperboloid* $\mathcal{S}\supset\Sigma_{x_0,0}$ *in Minkowski space-time, and satisfying*

$$(5.4)\quad \widetilde{f}^a{}_{|\Sigma_{x_0,0}} \equiv \Omega^{-(n-1)/2} f^a{}_{|\Sigma_{x_0,0}} \in \begin{cases}(\mathcal{H}^\alpha_{k+1}\cap L^\infty)(\Sigma_{x_0,0}) & \text{if } n\geq 3,\\ (\mathcal{H}^\alpha_{k+1}\cap \mathcal{C}^0_1)(\Sigma_{x_0,0}) & \text{if } n=2,\end{cases}$$

$$(5.5)\quad \partial_x(\Omega^{-(n-1)/2}f^a)_{|\Sigma_{x_0,0}} \in \mathcal{H}^\alpha_k(\Sigma_{x_0,0}),$$

$$(5.6)\quad \partial_\tau(\Omega^{-(n-1)/2}f^a)_{|\Sigma_{x_0,0}} \in \begin{cases}\mathcal{H}^\alpha_k(\Sigma_{x_0,0}) & \text{if } n\geq 3,\\ (\mathcal{H}^\alpha_k\cap L^\infty)(\Sigma_{x_0,0}) & \text{if } n=2.\end{cases}$$

for some $k>\frac12 n+1$, $-1<\alpha\leq -\frac12$. *Then:*

1) *There exists* $\tau_+>0$ *and a solution* f^a *of Equation* (5.1), *defined on a set containing* Ω_{x_0,τ_+}, *satisfying the given initial conditions, such that*

$$(5.7\text{a})\quad \|\widetilde{f}^a\|_{\mathcal{C}^0_1(\Omega_{x_0,\tau_+})}<\infty,\quad n=2,$$

$$\begin{aligned}(5.7\text{b})\quad \big\|xe_+(\widetilde{f}^a)\big\|_{L^\infty(\Omega_{x_0,\tau_+})} &+ \sum_{i=1}^r \|xX_i\widetilde{f}^a\|_{L^\infty(\Omega_{x_0,\tau_+})}\\ &+\|\widetilde{f}^a\|_{L^\infty(\Omega_{x_0,\tau_+})} + \|x\partial_\tau\widetilde{f}^a\|_{L^\infty(\Omega_{x_0,\tau_+})}<\infty,\quad n\geq 3.\end{aligned}$$

Here the X_i*'s are the vector fields defined in Section A, cf. Equation* (A.7).

2) *Further, if* τ_* *is such that* f^a *exists on* Ω_{x_0,τ_*} *with* (5.7) *holding with* $\tau_+=\tau_*$, *then for all* $0\leq\tau<\tau_*$ *we have uniformly in* τ

$$\widetilde{f}^a{}_{|\Sigma_{x_0,\tau}} \in L^\infty(\Sigma_{x_0,\tau})\cap\mathcal{H}^\alpha_{k+1}(\Sigma_{x_0,\tau}),$$

$$\partial_\tau\widetilde{f}^a{}_{|\Sigma_{x_0,\tau}}\in\mathcal{H}^\alpha_k(\Sigma_{x_0,\tau}),\quad \partial_x\widetilde{f}^a{}_{|\Sigma_{x_0,\tau}}\in\mathcal{H}^\alpha_k(\Sigma_{x_0,\tau}).$$

If $n=2$ *we also have uniform bounds in the following spaces*

$$\widetilde{f}^a{}_{|\Sigma_{x_0,\tau}}\in(\mathcal{C}^0_1\cap\mathcal{H}^\alpha_{k+1})(\Sigma_{x_0,\tau}),\quad \partial_\tau\widetilde{f}^a{}_{|\Sigma_{x_0,\tau}}\in(\mathcal{H}^\alpha_k\cap L^\infty)(\Sigma_{x_0,\tau}).$$

REMARK. — Integration of condition (5.5) implies of course that $\widetilde{f}$ belongs to $L^\infty(\Sigma_{x_0,0})$.

Proof. — The proof is similar to that of Theorem 4.1, but simpler, because we do not need to gain a $\frac{1}{2}$ in the decay rate, as done in Lemma 4.2. We write Equation (5.1) in the form (4.12)–(4.16), with $a_A = b_A = 0$ and with G in (4.17) replaced by G^a defined in (5.3). We write G^a as

$$G^a = A^a + B^a + C^a + D^a + E^a, \tag{5.8}$$

with the order of terms in (5.8) corresponding to that in (5.3). Since we are working in normal coordinates, Γ^a_{bc} has a uniform zero of order one in the sense of (A.30) at $f^a = 0$. We want to use Equation (3.20) to get an *a priori* estimate for the solutions of (5.1); for this we shall need to estimate the $\mathcal{H}^\alpha_k$ norms of all the terms which occur in (5.8). The simplest such term is E^a:

$$\begin{aligned}\|E^a\|_{\mathcal{H}^\alpha_k} &\equiv (n-1)^2\big\|\Gamma^a_{bc}(\Omega^{(n-1)/2}\widetilde{f})(\Omega^{(n-1)/2}\widetilde{f}^c)(\Omega^{(n-1)/2}\widetilde{f}^b)\Omega^{-1-(n-1)/2}\big\|_{\mathcal{H}^\alpha_k}\\ &\approx (n-1)^2\big\|\Gamma^a_{bc}(\Omega^{(n-1)/2}\widetilde{f})(\Omega^{(n-1)/2}\widetilde{f}^c)(\Omega^{(n-1)/2}\widetilde{f}^b)\big\|_{\mathcal{H}^{\alpha+(n+1)/2}_k},\end{aligned}$$

where we have used the fact that Ω/x is a smooth, and therefore bounded, function. The function $\Gamma^a_{bc}(\Omega^{(n-1)/2}\widetilde{f})(\Omega^{(n-1)/2}\widetilde{f}^c)(\Omega^{(n-1)/2}\widetilde{f}^b)$ can be viewed as a smooth function F of $x^{(n-1)/2}\widetilde{f}^a$ with a uniform zero of order three. We can thus apply (A.31) with $\ell = 3$ to obtain

$$\big\|E(s)\big\|_{\mathcal{H}^\alpha_k} \le C\big(\|\widetilde{f}(s)\|_{L^\infty}\big)\cdot\|\widetilde{f}\|_{\mathcal{H}^{\alpha+2-n}_k} \le C\big(\|\widetilde{f}(s)\|_{L^\infty}\big)\cdot\|\widetilde{f}\|_{\mathcal{H}^\alpha_k}, \tag{5.9}$$

since $n \ge 2$. We note that in dimensions larger than or equal to three we have at least one power of x "left unused" above, which will be made use of in estimating the remaining contributions to G^a. We proceed in a similar way with the other terms; in space dimension $n = 2$ we view

$$\Omega^{(n+1)/2}D^a \equiv \Omega^{(n+1)/2}(n-1)(1+x+2\tau)\Omega^{(n-3)/2}\Gamma^a_{bc}(\Omega^{(n-1)/2}\widetilde{f})\widetilde{f}^c\phi^b_-$$

as a smooth function F with a uniform zero of order three of $(x^{(n-1)/2}\widetilde{f}^a, x^{(n-1)/2}\phi^a_-)$, which leads to the estimate

$$\big\|D(s)\big\|_{\mathcal{H}^\alpha_k} \le C\big(\|\widetilde{f}(s)\|_{L^\infty}, \|\phi_-(s)\|_{L^\infty}\big)\big(\|\widetilde{f}\|_{\mathcal{H}^\alpha_k} + \|\phi_-(s)\|_{\mathcal{H}^\alpha_k}\big). \tag{5.10}$$

On the other hand, in dimension 3 or higher we can view $\Omega^{(n+1)/2}D^a$ as a function F with a uniform zero of order 3 of $(x^{(n-1)/2}\widetilde{f}^a, x^{(n-1)/2}x\phi^a_-)$, which implies

$$\big\|D(s)\big\|_{\mathcal{H}^\alpha_k} \le C\big(\|\widetilde{f}(s)\|_{L^\infty}, \|x\phi_-(s)\|_{L^\infty}\big)\big(\|\widetilde{f}\|_{\mathcal{H}^\alpha_k} + \|x\phi_-(s)\|_{\mathcal{H}^\alpha_k}\big). \tag{5.11}$$

Regardless of dimension we view

$$\Omega^{(n+1)/2}C^a \equiv \Omega^{(n+1)/2}(n-1)x\,\Omega^{(n-3)/2}\,\Gamma^a_{bc}(\Omega^{(n-1)/2}\widetilde{f})\widetilde{f}^c\phi^b_+$$

as a smooth function with a uniform zero of order 3 of $(x^{(n-1)/2}\widetilde{f}^a, x^{(n-1)/2}x\phi_+^a)$, obtaining thus

$$\big\|C(s)\big\|_{\mathcal{H}_k^\alpha} \le C\big(\|\widetilde{f}(s)\|_{L^\infty}, \|x\phi_+(s)\|_{L^\infty}\big)\big(\|\widetilde{f}\|_{\mathcal{H}_k^\alpha} + \|x\phi_+(s)\|_{\mathcal{H}_k^\alpha}\big). \tag{5.12}$$

Viewing B^a as a function of $(x^{(n-1)/2}\widetilde{f}^a, x^{(n-1)/2}x\phi_A^a)$, and viewing A^a as a function of $(x^{(n-1)/2}\widetilde{f}^a, x^{(n-1)/2}x\phi_-^a, x^{(n-1)/2}x\phi_+^a)$, one similarly obtains for $n \ge 3$

$$\begin{aligned}\big\|A(s)\big\|_{\mathcal{H}_k^\alpha} \le\ & C\big(\|\widetilde{f}(s)\|_{L^\infty}, \|x\phi_-(s)\|_{L^\infty}, \|x\phi_+(s)\|_{L^\infty}\big)\\ &\times\big(\|\widetilde{f}\|_{\mathcal{H}_k^\alpha} + \|x\phi_-(s)\|_{\mathcal{H}_k^\alpha} + \|x\phi_+(s)\|_{\mathcal{H}_k^\alpha}\big),\end{aligned} \tag{5.13}$$

$$\big\|B(s)\big\|_{\mathcal{H}_k^\alpha} \le C\big(\|\widetilde{f}(s)\|_{L^\infty}, \|x\phi_A(s)\|_{L^\infty}\big)\big(\|\widetilde{f}\|_{\mathcal{H}_k^\alpha} + \|x\phi_A(s)\|_{\mathcal{H}_k^\alpha}\big), \tag{5.14}$$

while in dimension 2 it holds that

$$\begin{aligned}\big\|A(s)\big\|_{\mathcal{H}_k^\alpha} \le\ & C\big(\|\widetilde{f}(s)\|_{L^\infty}, \|\phi_-(s)\|_{L^\infty}, \|x\phi_+(s)\|_{L^\infty}\big)\\ &\times\big(\|\widetilde{f}\|_{\mathcal{H}_k^\alpha} + \|\phi_-(s)\|_{\mathcal{H}_k^\alpha} + \|x\phi_+(s)\|_{\mathcal{H}_k^\alpha}\big),\end{aligned} \tag{5.15}$$

$$\big\|B(s)\big\|_{\mathcal{H}_k^\alpha} \le C\big(\|\widetilde{f}(s)\|_{L^\infty}, \|\phi_A(s)\|_{L^\infty}\big)\big(\|\widetilde{f}\|_{\mathcal{H}_k^\alpha} + \|\phi_A(s)\|_{\mathcal{H}_k^\alpha}\big). \tag{5.16}$$

Summarising, in space dimension 2 we have obtained

$$\begin{aligned}\big\|G(s)\big\|_{\mathcal{H}_k^\alpha} \le\ & C\big(\|\widetilde{f}(s)\|_{L^\infty}, \|\phi_-(s)\|_{L^\infty}, \|\phi_A(s)\|_{L^\infty}, \|x\phi_+(s)\|_{L^\infty}\big)\\ &\times\big(\|\widetilde{f}\|_{\mathcal{H}_k^\alpha} + \|\phi_-(s)\|_{\mathcal{H}_k^\alpha} + \|x\phi_+(s)\|_{\mathcal{H}_k^\alpha} + \|\phi_A(s)\|_{\mathcal{H}_k^\alpha}\big)\\ \le\ & C\big(\|\widetilde{f}(s)\|_{L^\infty}, \|\phi_-(s)\|_{L^\infty}, \|\phi_A(s)\|_{L^\infty}, \|x\phi_+(s)\|_{L^\infty}\big)\sqrt{E_\alpha(s)},\end{aligned} \tag{5.17}$$

where

$$E_\alpha(t) = \big\|\widetilde{f}(t)\big\|^2_{\mathcal{H}_k^\alpha} + \big\|\phi_-(t)\big\|^2_{\mathcal{H}_k^\alpha} + \big\|\phi_+(t)\big\|^2_{\mathcal{H}_k^\alpha} + \sum_A \big\|\phi_A(t)\big\|^2_{\mathcal{H}_k^\alpha}. \tag{5.18}$$

On the other hand in higher dimensions we can write

$$\begin{aligned}&\big\|G(s)\big\|_{\mathcal{H}_k^\alpha}\\ &\quad\le C\big(\|\widetilde{f}(s)\|_{L^\infty}, \|x\phi_A(s)\|_{L^\infty}, \|x\phi_-(s)\|_{L^\infty}, \|x\phi_+(s)\|_{L^\infty}\big)\sqrt{E_\alpha(s)}.\end{aligned} \tag{5.19}$$

To obtain a closed inequality from Equations (3.20) and (5.17) or (5.19), we need to control all the L_∞ norms occurring there. Since $k > \frac{1}{2}n + 1$, from Equation (5.17) and the weighted Sobolev embeddings we obtain

$$\big\|G(s)\big\|_{\mathcal{C}_1^\alpha} \le C\big(\|\widetilde{f}(s)\|_{L^\infty}, \|\phi_-(s)\|_{L^\infty}, \|\phi_A(s)\|_{L^\infty}, E_\alpha(s)\big), \tag{5.20}$$

if $n = 2$, or – from (5.19) –

$$\big\|G(s)\big\|_{\mathcal{C}_1^\alpha} \le C\big(\|\widetilde{f}(s)\|_{L^\infty}, E_\alpha(s)\big), \tag{5.21}$$

for $n \ge 3$. The identity

$$\widetilde{f}^a(\tau, x) = \widetilde{f}^a(\tau, x_0 - 2\tau) - \frac{1}{2}\int_x^{x_0-2\tau} (\phi_-^a - \phi_+^a)(\tau, s)\,\mathrm{d}s \tag{5.22}$$

yields

$$\big\|\widetilde f(s)\big\|_{L^\infty} \le C\big(\sqrt{E_\alpha(0)} + \|\phi_-(s)\|_{\mathcal{C}^\alpha_0} + \|\phi_+(s)\|_{\mathcal{C}^\alpha_0}\big) \le C\big(\sqrt{E_\alpha(0)} + \sqrt{E_\alpha(s)}\big) \tag{5.23}$$

for $n \ge 3$, while if $n = 2$ we use the estimate

$$\big\|\widetilde f(s)\big\|_{L^\infty} + \big\|\phi_A(s)\big\|_{L^\infty} \le C\big(\sqrt{E_\alpha(0)} + \|\phi_-(s)\|_{\mathcal{C}^\alpha_1} + \|\phi_+(s)\|_{\mathcal{C}^\alpha_1}\big) \le C\big(\sqrt{E_\alpha(0)} + \sqrt{E_\alpha(s)}\big). \tag{5.24}$$

In Equations (5.23)–(5.24), for notational simplicity we have estimated $\widetilde f^a(\tau, x_0 - 2\tau)$ and its angular derivatives by a multiple of the initial energy $E_\alpha(0)$; strictly speaking, this should be some functional of $(E_\alpha(0), \tau^*)$ for τ^* small enough; then such an estimate holds by standard methods for $0 \le \tau \le \tau_* < \frac12 x_0$. Further, such an inequality is correct if we already have a weighted L^∞ bound as assumed in point 2) of the theorem. If $n \ge 3$ Equations (3.20) for $\alpha < -\frac12$ or (3.32) if $\alpha = -\frac12$, (5.21) and (5.23) give

$$E_\alpha(\tau) \le C E_\alpha(0) + \int_0^\tau \Phi\big(E_\alpha(s)\big)\,\mathrm{d}s, \tag{5.25}$$

for some constant C, and for a function Φ which is bounded on bounded sets, and we conclude as in the proof of Theorem 4.1.

If $n = 2$, we note the identity

$$\phi_-(\tau, x) = \phi_-(0, x + 2\tau) + \int_0^\tau e_+(\phi_-)\big(\sigma, 2(\tau - \sigma) + x\big)\,\mathrm{d}\sigma. \tag{5.26}$$

From the second of Equations (4.14) we obtain

$$\big|e_+(\phi_-)(s,x)\big| \le C\big(\|\phi_-(s)\|_{\mathcal{C}^\alpha_0} + \|\phi_A(s)\|_{\mathcal{C}^\alpha_1} + \|\phi_+(s)\|_{\mathcal{C}^\alpha_0} + \|G(s)\|_{\mathcal{C}^\alpha_0}\big)x^\alpha,$$

so that

$$\big|\phi_-(\tau, x)\big| \le \big\|\phi_-(0)\big\|_{L^\infty} + C\int_0^\tau \big(\|\phi_-(\sigma)\|_{\mathcal{C}^\alpha_0} + \|\phi_A(\sigma)\|_{\mathcal{C}^\alpha_1} + \|\phi_+(\sigma)\|_{\mathcal{C}^\alpha_0} + \|G(\sigma)\|_{\mathcal{C}^\alpha_0}\big)\big(2(\tau - \sigma) + x\big)^\alpha\,\mathrm{d}\sigma. \tag{5.27}$$

It follows that

$$\big\|\phi_-(\tau)\big\|_{L^\infty} \le \big\|\phi_-(0)\big\|_{L^\infty} + C\int_0^\tau \big(\sqrt{E_\alpha(\sigma)} + \|G(\sigma)\|_{\mathcal{C}^\alpha_0}\big)(\tau - \sigma)^\alpha\,\mathrm{d}\sigma. \tag{5.28}$$

Let

$$F(s) \equiv \big\|\widetilde f(s)\big\|_{L^\infty} + \big\|\phi_-(s)\big\|_{L^\infty} + \big\|\phi_A(s)\big\|_{L^\infty} + \sqrt{E_\alpha(s)}\,. \tag{5.29}$$

It follows from (3.20), (5.24) and (5.28) that we have

$$F(\tau) \le C F(0) + \int_0^\tau \Phi\big(F(\sigma)\big)\big(1 + (\tau - \sigma)^\alpha\big)\,\mathrm{d}\sigma, \tag{5.30}$$

where Φ is a function bounded on bounded sets. We have the following:

LEMMA 5.2. — *There exists a time τ_*, depending only upon C, $F(0)$, and the function Φ, such that any positive continuous function $F : [0, \tau_+) \to \mathbb{R}$ satisfying the inequality* (5.30) *with $\alpha > -1$ is bounded from above by $CF(0)+1$ on $[0, \max(\tau_+, \tau_*))$.*

Proof. — Let

$$M = \sup_{0 \le x \le CF(0)+1} |\Phi(x)|;$$

if $M = 0$ the result is obviously true, so assume that $M \neq 0$. From Equation (5.30) we obtain that on any interval $[0, \tau)$ on which $F \le CF(0) + 1$ we have

$$F(\tau) \le CF(0) + \int_0^\tau M\big(1 + (\tau - \sigma)^\alpha\big)\,\mathrm{d}\sigma = CF(0) + M\Big(\tau + \frac{\tau^{\alpha+1}}{\alpha+1}\Big).$$

(Equation (5.30) with $\tau = 0$ shows that $CF(0) \ge F(0)$, and continuity of F implies that the set of such intervals is non-empty.) The result is established by choosing

$$\tau_* = \min\Big(\frac{1}{2M}, \Big[\frac{\alpha+1}{2M}\Big]^{1/(\alpha+1)}\Big). \qquad \square$$

Because the existence time τ_* in Theorem 5.1 does not depend upon x_1, Theorem 5.1 with $n = 2$ follows again by an argument identical to the one given at the end of Theorem 4.1. □

As in the case of the nonlinear wave equation (4.1), in order to obtain time derivative estimates we shall need a more general version of Theorem 5.1. Thus, we consider systems of the form (4.38)–(4.40) with a rather more general form of the non-linearity G appearing there. It should be clear from the proof of Theorem 5.1 that it is convenient to treat the case $n = 2$ separately, this will be considered in Section 5.3 below. We thus start with a result which holds in dimensions $n \ge 3$; the same proof gives similar results in dimension $n = 2$ for equations with a nonlinearity of higher order:

THEOREM 5.3. — *Let $n \ge 3$ and consider the system* (4.38)–(4.39) *with*

$$\begin{aligned}(5.31)\qquad \big\|a(\tau)\big\|_{\mathcal{H}^\alpha_k} + \big\|b(\tau)\big\|_{\mathcal{H}^\alpha_k} + \sup_{a,b=1,2}\big\|B_{ab}(\tau)\big\|_{\mathcal{C}^0_k}&\\ + \big\|B_0(\tau)\big\|_{\mathcal{C}^0_k} + \big\|B_0^{-1}(\tau)\big\|_{L^\infty} + \big\|B_1(\tau)\big\|_{\mathcal{C}^0_k} &\le \widetilde{C},\end{aligned}$$

for some constant $\widetilde{C}$, with the nonlinearity G in Equation (4.38a) *of the form*

$$(5.32)\quad G = x^{-(n+3)/2} H(x^\mu, x^{(n-1)/2}\widetilde{f}, x^{(n-1)/2} x\phi_A, x^{(n-1)/2} x\phi_+, x^{(n-1)/2} x\phi_-),$$

with $G_{e_-(\phi_A)} = 0$ (*cf. Equation* (4.41)), *and with* H *having a uniform zero of order* $\ell \geq 3$ *in the sense of* (A.30). *Suppose that the initial data satisfy*

$$\widetilde{f}^a{}_{|\Sigma_{x_0,0}} \equiv \Omega^{-(n-1)/2} f^a{}_{|\Sigma_{x_0,0}} \in (\mathcal{H}^\alpha_{k+1} \cap L^\infty)(\Sigma_{x_0,0}), \tag{5.33}$$

$$\partial_x \widetilde{f}^a{}_{|\Sigma_{x_0,0}} \in \mathcal{H}^\alpha_k(\Sigma_{x_0,0}), \tag{5.34}$$

$$\partial_\tau \widetilde{f}^a{}_{|\Sigma_{x_0,0}} \in \mathcal{H}^\alpha_k(\Sigma_{x_0,0}), \tag{5.35}$$

with some $k > \frac{1}{2}n + 1$, $-1 < \alpha \leq -\frac{1}{2}$, *then:*

1) *There exists* $\tau_+ > 0$, *depending only upon the constant* $\widetilde{C}$ *in* (5.31) *and a bound on the norms of the initial data in the spaces appearing in Equations* (5.33)–(5.35), *and a solution* f^a *of Equations* (4.38)–(4.39), *defined on a set containing* Ω_{x_0,τ_+}, *satisfying the given initial conditions, such that*

$$\begin{aligned} \big\| x e_+(\widetilde{f}^a) \big\|_{L^\infty(\Omega_{x_0,\tau_+})} + \sum_{i=1}^r \| x X_i \widetilde{f}^a \|_{L^\infty(\Omega_{x_0,\tau_+})} \\ + \| \widetilde{f}^a \|_{L^\infty(\Omega_{x_0,\tau_+})} + \| x \partial_\tau \widetilde{f}^a \|_{L^\infty(\Omega_{x_0,\tau_+})} < \infty. \end{aligned} \tag{5.36}$$

2) *Further, if* τ_* *is such that* f^a *exists on* Ω_{x_0,τ_*} *with* (5.36) *holding with* $\tau_+ = \tau_*$, *then for all* $0 \leq \tau < \tau_*$ *we have*

$$\widetilde{f}^a{}_{|\Sigma_{x_0,\tau}} \in L^\infty(\Sigma_{x_0,\tau}) \cap \mathcal{H}^\alpha_{k+1}(\Sigma_{x_0,\tau}), \tag{5.37a}$$

$$\partial_\tau \widetilde{f}^a{}_{|\Sigma_{x_0,\tau}} \in \mathcal{H}^\alpha_k(\Sigma_{x_0,\tau}), \tag{5.37b}$$

$$\partial_x \widetilde{f}^a{}_{|\Sigma_{x_0,\tau}} \in \mathcal{H}^\alpha_k(\Sigma_{x_0,\tau}), \tag{5.37c}$$

with uniform bounds in τ; *this implies*

$$\begin{aligned} \| x \partial_\tau \phi_+ \|_{L^\infty(\Omega_{x_0,\tau_*})} + \| x \partial_\tau \phi_A \|_{L^\infty(\Omega_{x_0,\tau_*})} \\ + \big\| (x + 2\tau) \partial_\tau \widetilde{f}^a \big\|_{L^\infty(\Omega_{x_0,\tau_*})} < \infty. \end{aligned} \tag{5.38}$$

If $k > \frac{1}{2}n + 2$ *then we also have*

$$\big\| x(x + 2\tau) \partial_\tau \phi_- \big\|_{L^\infty(\Omega_{x_0,\tau_*})} < \infty. \tag{5.39}$$

Proof. — The transition from Theorem 5.1 to Theorem 5.3 is rather similar to that from Theorem 4.1 to Theorem 4.3. We note that the estimates done in the course of the proof of Theorem 5.1, with $n \geq 3$ there, can be summed up in the inequality

$$\big\| x^{-(n+1)/2} H(x^\mu, x^{(n-1)/2} \widehat{f}) \big\|_{\mathcal{H}^\alpha_k} \leq C\big(\| \widehat{f} \|_{L^\infty}\big) \| \widehat{f} \|_{\mathcal{H}^\alpha_k}, \tag{5.40}$$

where

$$\widehat{f} := (\widetilde{f}, x\phi_A, x\phi_+, x\phi_-).$$

The minor modifications of the proof of Theorem 5.1 needed to obtain (5.37) and the estimate (5.38) on $(x+2\tau)\partial_\tau \widetilde{f}$ are identical to the ones described in the proof of Theorem 4.3. The estimate on $\| x \partial_\tau \phi_+ \|_{L^\infty(\Omega_{x_0,\tau_*})}$ is obtained directly from Equation (4.49) and from (5.40). The estimate on $\|x\partial_\tau\phi_A\|_{L^\infty(\Omega_{x_0,\tau_*})}$ is obtained from the (4.38a)–equivalent of the first of Equations (4.14). Next, for $k > \frac{1}{2}n+2$ Equations (4.42) and (5.40) give

$$e_+(\phi_-) \in \mathcal{H}^\alpha_{k-1} \subset \mathcal{C}^\alpha_1. \tag{5.41}$$

Differentiating Equation (5.26) with respect to x gives

$$\partial_x\phi_-(\tau,x) = \partial_x\phi_-(0,x+2\tau) + \int_0^\tau \big(\partial_x e_+(\phi_-)\big)\big(\sigma, 2(\tau-\sigma)+x\big)\,\mathrm{d}\sigma, \tag{5.42}$$

which together with (5.41) implies, by straightforward integration,

$$x(x+2\tau)\big|\partial_x\phi_-(\tau,x)\big| \le C. \tag{5.43}$$

This, (5.41), and the identity $\partial_\tau\phi_- = (\partial_\tau - 2\partial_x + 2\partial_x)\phi_- = e_+(\phi_-) + 2\partial_x\phi_-$ establish (5.39). □

5.2. Estimates on the time derivatives of the solutions, $n \ge 3$

To control the time derivatives of the solutions, as in Section 4.2 we introduce an index m which counts the number of corner conditions which are eventually satisfied by the initial data at the "corner" $\tau = x = 0$. As before we make a formal statement only for solutions of the wave-map equation (5.1), it should be clear from the proof that an analogous statement holds for solutions of (4.38)–(4.39) under appropriate conditions on the coefficients there.

Theorem 5.4. — *In dimension $n \ge 3$ let $\mathbb{N} \ni m \ge 0$. Consider a solution $f : \Omega_{x_0,\tau_*} \to \mathbb{R}$ of Equation (5.1) satisfying*

$$\big\|xe_+(\widetilde{f}^a)\big\|_{L^\infty(\Omega_{x_0,\tau_*})} + \sum_{i=1}^r \|xX_i\widetilde{f}^a\|_{L^\infty(\Omega_{x_0,\tau_*})} + \|\widetilde{f}^a\|_{L^\infty(\Omega_{x_0,\tau_*})} + \|x\partial_\tau\widetilde{f}^a\|_{L^\infty(\Omega_{x_0,\tau_*})} < \infty, \tag{5.44}$$

and suppose that

$$0 \le i \le m+1, \quad \partial^i_\tau\widetilde{f}^a{}_{|\Sigma_{x_0,0}} \in \mathcal{H}^\alpha_{k+m+1-i}(\Sigma_{x_0,0}), \tag{5.45}$$

$$0 \le i \le m, \quad \partial_x\partial^i_\tau\widetilde{f}^a{}_{|\Sigma_{x_0,0}} \in \mathcal{H}^\alpha_{k+m-i}(\Sigma_{x_0,0}), \tag{5.46}$$

with some $k > \frac{1}{2}n+2$, $-1 < \alpha \le -\frac{1}{2}$. Then for $0 \le \tau < \tau_*$ and for $0 \le i \le m$, we have

(5.47a) $0 \le j+i < k+m-\frac{1}{2}n$,

$$\big[(\tau+2x)\partial_\tau\big]^j\partial^i_\tau\widetilde{f}^a{}_{|\Sigma_{x_0,\tau}} \in L^\infty(\Sigma_{x_0,\tau}) \cap \mathcal{H}^\alpha_{k+m+1-i-j}(\Sigma_{x_0,\tau}),$$

(5.47b) $0 \le j+i < k+m-\frac{1}{2}n-1$

$$\partial_x[(\tau+2x)\partial_\tau]^j\partial^i_\tau\widetilde{f}^a{}_{|\Sigma_{x_0,\tau}} \in \mathcal{H}^\alpha_{k+m-i-j}(\Sigma_{x_0,\tau}),$$

and

$$0 \le p < k - \tfrac{1}{2}n, \quad \big[(\tau + 2x)\partial_\tau\big]^p \partial_\tau^{m+1} \widetilde{f}^a{}_{|\Sigma_{x_0,\tau}} \in \mathcal{H}^\alpha_{k-p}(\Sigma_{x_0,\tau}), \tag{5.48}$$

with τ-independent bounds on the norms.

Proof. — The proof is an inductive application of Theorem 5.3, as in the proof of Theorem 4.4, and will be omitted. □

5.3. Estimates on the time derivatives, $n = 2$.

— In space-dimension 2 the following equivalent of Theorem 5.3 holds:

THEOREM 5.5. — *Let $n = 2$, consider the system* (4.38)–(4.39)*, suppose that* (5.31) *holds for some constant $\widetilde{C}$, with the nonlinearity G in Equation* (4.38a) *of the form*

$$G = x^{-3/2} H(x^\mu, x^{1/2}\widetilde{f}, x^{1/2}\phi_A, x^{1/2}\phi_-, x^{3/2}\phi_+), \tag{5.49}$$

with $G_{e_-(\phi_A)} = 0$ (cf. Equation (4.41)*), and with H having a uniform zero of order $\ell \ge 3$ in the sense of* (A.30)*. Suppose that the initial data satisfy*

$$\widetilde{f}^a{}_{|\Sigma_{x_0,0}} \equiv \Omega^{-1/2} f^a{}_{|\Sigma_{x_0,0}} \in (\mathcal{H}^\alpha_{k+1} \cap \mathcal{C}^0_1)(\Sigma_{x_0,0}), \tag{5.50}$$

$$\partial_x(\Omega^{-1/2} f^a)_{|\Sigma_{x_0,0}} \in \mathcal{H}^\alpha_k(\Sigma_{x_0,0}), \tag{5.51}$$

$$\partial_\tau(\Omega^{-1/2} f^a)_{|\Sigma_{x_0,0}} \in (\mathcal{H}^\alpha_k \cap L^\infty)(\Sigma_{x_0,0}). \tag{5.52}$$

for some $k > 2$, $-1 < \alpha \le -\frac{1}{2}$. Then:

1) *There exists $\tau_+ > 0$, depending only upon the constant $\widetilde{C}$ in* (5.31) *and a bound on the norms of the initial data in the spaces appearing in Equations* (5.50)–(5.52)*, and a solution f^a of Equations* (4.38)–(4.39)*, defined on a set containing Ω_{x_0,τ_+}, satisfying the given initial conditions, such that*

$$\|\widetilde{f}^a\|_{\mathcal{C}^0_1(\Omega_{x_0,\tau_+})} < \infty. \tag{5.53}$$

2) *Further, for any τ_* such that f^a exists on Ω_{x_0,τ_*} with* (5.36) *holding with $\tau_+ = \tau_*$, we have for all $0 \le \tau < \tau_*$*

$$\widetilde{f}^a{}_{|\Sigma_{x_0,\tau}} \in (\mathcal{C}^0_1 \cap \mathcal{H}^\alpha_{k+1})(\Sigma_{x_0,\tau}), \quad \partial_\tau \widetilde{f}^a{}_{|\Sigma_{x_0,\tau}} \in (\mathcal{H}^\alpha_k \cap L^\infty)(\Sigma_{x_0,\tau}),$$

$$\partial_x \widetilde{f}^a{}_{|\Sigma_{x_0,\tau}} \in \mathcal{H}^\alpha_k(\Sigma_{x_0,\tau}),$$

with bounds uniform in τ. This implies

$$\|x\partial_\tau\phi_+\|_{L^\infty(\Omega_{x_0,\tau_*})} + \|\partial_\tau\widetilde{f}^a\|_{L^\infty(\Omega_{x_0,\tau_*})} < \infty. \tag{5.54}$$

If $k > 4$ then we also have

$$\big\|(x + 2\tau)\partial_\tau\phi_A\big\|_{L^\infty(\Omega_{x_0,\tau_*})} < \infty. \tag{5.55}$$

If $k > 4$ *and if* $\partial_\tau^2 \widetilde{f}_{|\Sigma_{x_0,\tau}} \in \mathcal{H}_{k-1}^{-1}$ *then it further holds that*

$$\left\|(x+2\tau)\partial_\tau\phi_-\right\|_{L^\infty(\Omega_{x_0,\tau_*})} < \infty. \tag{5.56}$$

Proof. — The proof of point 1) is essentially the same as that of Theorem 5.1, with the modifications discussed in the proof of Theorem 4.3. We note that the key estimates (5.17) and (5.20) hold in exactly the same form here, similarly for Equations (5.29)–(5.30). The estimate on $\partial_\tau \widetilde{f}$ in (5.54) follows from the definition of the norm in (5.53). The estimate on $\|x\partial_\tau\phi_+\|_{L^\infty(\Omega_{x_0,\tau_*})}$ is obtained directly from Equation (4.49) and from (5.17). To obtain (5.55) one needs to prove a bound on $\partial_A\phi_-$. This is obtained by differentiating (5.26) with respect to v^A and using the already known uniform bound for G in $\mathcal{H}_k^\alpha$, so that $\partial_A G \in \mathcal{H}_{k-1}^\alpha \subset \mathcal{C}_0^\alpha$. Finally,

$$e_+\big((x+2\tau)\partial_\tau\phi_-\big) = (x+2\tau)\partial_\tau\big(e_+(\phi_-)\big),$$

and integrating as in (5.26) one finds

$$\begin{aligned}(x+2\tau)\partial_\tau\phi_-(\tau,x) &= (x+2\tau)\partial_\tau\phi_-(0,x+2\tau)\\ &\quad + \int_0^\tau \big\{(2(\tau-\sigma)+x)\partial_\tau\big(e_+(\phi_-)\big)(\sigma,2(\tau-\sigma)+x)\big\}\,\mathrm{d}\sigma.\end{aligned} \tag{5.57}$$

The term at the right-hand-side of the first line of (5.57) is bounded because of the hypothesis on $\partial_\tau^2\widetilde{f}$. Expressing $e_+(\phi_-)$ by the right-hand-side of (4.42), one immediately finds that all the linear terms that arise after differentiation with respect to τ are in $\mathcal{H}_{k-1}^{\alpha-1}$ or better, and therefore give a finite contribution when integrated upon. The contribution from the non-linearity G can be rewritten as

$$x^{-1}\big\{H_{\widetilde{f}}\,\partial_\tau\widetilde{f} + H_{\phi^A}\partial_\tau\phi^A + H_{\phi_-}\partial_\tau\phi_- + H_{\phi_+}x\partial_\tau\phi_+\big\},$$

with appropriate functions H_* which, by Lemma 4.6, all have a uniform zero of order $\ell - 1 \geq 2$ in their arguments. This easily implies that the coefficients (including the x^{-1} factor) in front of the τ derivatives are in L^∞, and since each of the τ-derivative terms is in $\mathcal{H}_{k-1}^{\alpha-1}$ or better, the whole term is in $\mathcal{H}_{k-1}^{\alpha-1} \subset C_0^{\alpha-1}$. This is sufficient to lead to a finite contribution in (5.57), and (5.56) follows. □

We finally arrive at the two-dimensional equivalent of Theorem 5.4; comments identical to those made in Section 5.2 apply here. The main difference is that in dimension 2 we need the L^∞ bound on $\partial_\tau\widetilde{f}$ to obtain existence, which leads to the compatibility condition (5.62) on the second τ derivatives of $\widetilde{f}$ when one attempts to iteratively apply Theorem 5.5. The proof is again identical to that of Theorem 4.4 and will be omitted. Let us just mention that one easily checks that the conditions spelled out below guarantee that the initial data for the inductive system of equations are in the right spaces for the iterative application of Theorem 5.5. Further, Equations (5.54)–(5.56) provide the *a priori* bounds which guarantee that the existence time of the solution will not shrink at each iteration step.

THEOREM 5.6. — *In space-dimension* 2 *let* $\mathbb{N} \ni m \geq 0$. *Consider a solution* $f : \Omega_{x_0,\tau_*} \to \mathbb{R}$ *of Equation* (5.1) *satisfying*

$$\|\widetilde{f}^a\|_{\mathcal{C}_1^0(\Omega_{x_0,\tau_+})} < \infty. \tag{5.58}$$

and suppose that

$$0 \leq i \leq m, \quad \partial_\tau^i \widetilde{f}^a{}_{|\Sigma_{x_0,0}} \in (\mathcal{H}^\alpha_{k+m+1-i} \cap \mathcal{C}^0_1)(\Sigma_{x_0,0}), \tag{5.59}$$

$$0 \leq i \leq m, \quad \partial_x \partial_\tau^i \widetilde{f}^a{}_{|\Sigma_{x_0,0}} \in \mathcal{H}^\alpha_{k+m-i}(\Sigma_{x_0,0}), \tag{5.60}$$

$$\partial_\tau^{m+1} \widetilde{f}^a{}_{|\Sigma_{x_0,0}} \in (\mathcal{H}^\alpha_k \cap L^\infty)(\Sigma_{x_0,0}), \tag{5.61}$$

$$\partial_\tau^{m+2} \widetilde{f}^a{}_{|\Sigma_{x_0,0}} \in \mathcal{H}^{-1}_{k-1}(\Sigma_{x_0,0}), \tag{5.62}$$

with some $k > 4$, $-1 < \alpha \leq -\frac{1}{2}$. *Then for* $0 \leq \tau < \tau_*$ *and for* $0 \leq i \leq m$, *we have*

$$\begin{aligned} &0 \leq j + i < k + m - 3, \\ &\qquad \big[(\tau + 2x)\partial_\tau\big]^j \partial_\tau^i \widetilde{f}^a{}_{|\Sigma_{x_0,\tau}} \in (\mathcal{H}^\alpha_{k+m+1-i-j} \cap \mathcal{C}^0_1)(\Sigma_{x_0,\tau}), \end{aligned} \tag{5.63a}$$

$$\begin{aligned} &0 \leq j + i < k + m - 3 \\ &\qquad \partial_x \big[(\tau + 2x)\partial_\tau\big]^j \partial_\tau^i \widetilde{f}^a{}_{|\Sigma_{x_0,\tau}} \in \mathcal{H}^\alpha_{k+m-i-j}(\Sigma_{x_0,\tau}), \end{aligned} \tag{5.63b}$$

and

$$0 \leq p < k - 3, \quad \big[(\tau + 2x)\partial_\tau\big]^p \partial_\tau^{m+1} \widetilde{f}^a{}_{|\Sigma_{x_0,\tau}} \in \mathcal{H}^\alpha_{k-p}(\Sigma_{x_0,\tau}), \tag{5.64}$$

with τ*-independent bounds on the norms.* □

5.4. Polyhomogeneous solutions. — We are finally ready to prove polyhomogeneity at $\mathcal{I}$ of solutions of the wave map equation:

THEOREM 5.7. — *Let* $\delta = 1$ *in odd space dimensions, and let* $\delta = \frac{1}{2}$ *in even space dimensions. Consider Equation* (5.1) *on* $\mathbb{R}^{n,1}$, $n \geq 2$, *with initial data*

$$\partial_\tau^i \widetilde{f}^a{}_{|\{\tau=0\}} \in (\mathcal{A}^\delta_\infty \cap L^\infty)(M_{x_0}), \quad i = 0, 1, \;\; n = 2, \tag{5.65}$$

$$\widetilde{f}^a{}_{|\{\tau=0\}} \in (\mathcal{A}^\delta_\infty \cap L^\infty)(M_{x_0}), \quad \partial_\tau \widetilde{f}^a{}_{|\{\tau=0\}} \in \mathcal{A}^\delta_\infty(M_{x_0}), \quad n \geq 3. \tag{5.66}$$

Then:

1) *There exists* $\tau_+ > 0$ *such that* f^a *exists on* Ω_{x_0,τ_+}, *with*

$$\|\widetilde{f}^a\|_{\mathcal{C}^0_1(\Omega_{x_0,\tau_+})} < \infty, \quad n = 2, \tag{5.67a}$$

$$\begin{aligned} &\|xe_+(\widetilde{f}^a)\|_{L^\infty(\Omega_{x_0,\tau_+})} + \sum_{i=1}^r \|xX_i \widetilde{f}^a\|_{L^\infty(\Omega_{x_0,\tau_+})} \\ &\qquad + \|\widetilde{f}^a\|_{L^\infty(\Omega_{x_0,\tau_+})} + \|x\partial_\tau \widetilde{f}^a\|_{L^\infty(\Omega_{x_0,\tau_+})} < \infty, \quad n \geq 3. \end{aligned} \tag{5.67b}$$

2) *If the initial data are* compatible *polyhomogeneous in the sense that*

$$\forall i \in \mathbb{N}, \quad \partial_\tau^i \widetilde{f}^a(0) \in L^\infty(M_{x_0}),$$

then the solution is polyhomogeneous on each neighborhood Ω_{x_0,τ_*} *of* $\mathcal{I}^+$ *on which* f *exists and satisfies* (5.67) *with* τ_+ *replaced with* $\tau*$.

Proof. — Existence of solutions follows from Theorem 5.1. Theorems 5.4 and 5.6 give the time-derivative estimates which are necessary in Theorem 4.9. In order to apply that last theorem, we set

$$\varphi = \begin{pmatrix} \phi^c_+ \\ \phi^c_A \end{pmatrix}, \tag{5.68}$$

$$\psi_1 = (\widetilde{f}^c), \quad \psi_2 = (\phi^c_-). \tag{5.69}$$

Equation (5.2) takes then the form (4.58). As in Theorem 4.10, for $n \geq 4$ even we take $\delta = \frac{1}{2}$, $p = n+3$, $q = n-1$; while for $n \geq 3$ odd we take $\delta = 1$, $p = \frac{1}{2}(n+3)$, $q = \frac{1}{2}(n-1)$. For $n = 2$ we set $\delta = \frac{1}{2}$, $p = 3$, $q = 1$. The non-linearity here has a uniform zero of order 3, which is compatible with the hypotheses of Theorem 4.9, and the result follows by that last theorem. ☐

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Appendix A
Function spaces, embeddings, inequalities

Throughout this paper the letter C denotes a constant the exact value of which is irrelevant for the problem at hand, and which may vary from line to line.

Let M be a smooth manifold such that

$$\overline{M} \equiv M \cup \partial M$$

is a compact manifold with smooth boundary ∂M. Throughout this work the symbol x stands for a smooth defining function for ∂M, *i.e.*, a smooth function on $\overline{M}$ such that $\{x = 0\} = \partial M$, with dx nowhere vanishing on ∂M. It follows that there exists $x_0 > 0$ and a compact neighborhood K of ∂M on which x can be used as a coordinate, with K being diffeomorphic to $[0, x_0] \times \partial M$. For $0 \leq x_1 < x_2 \leq x_0$ we set

$$M_{x_1} = \big\{p \in M \mid 0 < x(p) < x_1\big\}, \tag{A.1a}$$

$$M_{x_1,x_2} = \big\{p \in M \mid x_1 < x(p) < x_2\big\}, \tag{A.1b}$$

$$\widetilde{\partial} M_{x_1} = \big\{p \in M \mid x(p) = x_1\big\} \approx \partial M. \tag{A.1c}$$

In what follows the symbol Ω will generally denote one of the sets M, M_{x_1}, or M_{x_1,x_2}. Any subset of $\overline{M}_{x_0}$ can be locally coordinatized by coordinates $y^i = (x, v^A)$, where the v^A's can be thought of as local coordinates on ∂M. We cover ∂M by a finite number of coordinate charts $\mathcal{O}_i$ so that the sets $\overline{\Omega}_i$, where

$$\Omega_i := (0, x_0) \times \mathcal{O}_i,$$

cover M_{x_0}. We use the usual multi-index notation for partial derivatives: for $\beta = (\beta_1, \dots, \beta_n) \in \mathbb{N}^n$ we set $\partial^\beta = \partial_1^{\beta_1} \cdots \partial_n^{\beta_n}$. We will write ∂_v^β for derivatives of the form $\partial_2^{\beta_2} \cdots \partial_n^{\beta_n}$, which do not involve the $x^1 \equiv x$ variable.

If $\mathcal{O}$ is an open set, for $k \in \mathbb{N} \cup \infty$ we let $C_k(\mathcal{O})$ denote the usual space of k-times differentiable functions on $\mathcal{O}$; the symbol $C_k(\overline{\mathcal{O}})$ is used to denote the set of those functions in $C_k(\mathcal{O})$ the derivatives of which, up to order k, extend by continuity to $\overline{\mathcal{O}}$. We emphasise that no uniformity is assumed in $C_k(\mathcal{O})$, so that functions there could grow without bound when approaching the boundary of $\mathcal{O}$. Nevertheless, the symbol $\|\cdot\|_{C_k}$ will denote the usual supremum norm of f and its derivatives up to order k. The symbol $C_{k+\lambda}(\mathcal{O})$ denotes the space of k-times continuously differentiable functions on $\mathcal{O}$, with λ-Hölder continuous k-th derivatives.

For $\alpha \in \mathbb{R}$, $k \in \mathbb{N}$ and $\lambda \in (0,1]$, we define $\mathcal{C}_0^\alpha(\Omega_i)$ (resp. $\mathcal{C}_{0+\lambda}^\alpha(\Omega_i)$, $\mathcal{C}_k^\alpha(\Omega_i)$, $\mathcal{C}_{k+\lambda}^\alpha(\Omega_i)$) as the spaces of appropriately differentiable functions such that the respective norms

$$\text{(A.2)} \quad \begin{cases} \|f\|_{\mathcal{C}_0^\alpha(\Omega_i)} \equiv \sup_{p\in\Omega_i} \big|x^{-\alpha} f(p)\big|, \\ \|f\|_{\mathcal{C}_{0+\lambda}^\alpha(\Omega_i)} \equiv \|f\|_{\mathcal{C}_0^\alpha(\Omega_i)} \\ \qquad + \sup\limits_{y\in\Omega_i} \ \sup\limits_{y\neq y'\in B(y,\frac{1}{2}x(y))\cap\Omega_i} \dfrac{x(y)^{-\alpha-\lambda}|f(y)-f(y')|}{|y-y'|^\lambda}, \\ \|f\|_{\mathcal{C}_k^\alpha(\Omega_i)} \equiv \sum\limits_{0\le|\beta|\le k} \|x^{\beta_1}\partial^\beta f\|_{\mathcal{C}_0^\alpha(\Omega_i)}, \\ \|f\|_{\mathcal{C}_{k+\lambda}^\alpha(\Omega_i)} \equiv \|f\|_{\mathcal{C}_{k-1}^\alpha(\Omega_i)} + \sum\limits_{|\beta|=k} \|x^{\beta_1}\partial^\beta f\|_{\mathcal{C}_{0+\lambda}^\alpha(\Omega_i)}, \end{cases}$$

are finite. Let $\mathcal{O}$ be an open subset of M, or a submanifold with boundary in M; for such sets we define:

$$\text{(A.3)} \quad \begin{cases} \|f\|_{\mathcal{C}_k^\alpha(\mathcal{O})} \equiv \sup_i \|f\|_{\mathcal{C}_k^\alpha(\Omega_i\cap\mathcal{O})} + \|f\|_{C_k(\complement M_{x_0/2}\cap\mathcal{O})}, \\ \|f\|_{\mathcal{C}_{k+\lambda}^\alpha(\mathcal{O})} \equiv \sup_i \|f\|_{\mathcal{C}_{k+\lambda}^\alpha(\Omega_i\cap\mathcal{O})} + \|f\|_{C_{k+\lambda}(\complement M_{x_0/2}\cap\mathcal{O})}. \end{cases}$$

We note that $f \in \mathcal{C}_{k+\lambda}^{\alpha+\sigma}(\Omega)$ if and only if $x^{-\sigma} f \in \mathcal{C}_{k+\lambda}^\alpha(\Omega)$.

We define the spaces $\mathcal{H}_k^\alpha(\Omega_i)$ as the spaces of those functions in $H_k^{\mathrm{loc}}(\Omega_i)$ for which the norms $\|.\|_{\mathcal{H}_k^\alpha(\Omega_i)}$ are finite, where

$$\text{(A.4)} \quad \|f\|^2_{\mathcal{H}_k^\alpha(\Omega_i)} = \sum_{0\le|\beta|\le k} \int_{\Omega_i} (x^{-\alpha+\beta_1}\partial^\beta f)^2 \frac{\mathrm{d}x}{x} \mathrm{d}\nu.$$

Here $d\nu$ is a measure on ∂M arising from some smooth Riemannian metric on ∂M. This is equivalent to

$$\text{(A.5)}\qquad \sum_{0\le\beta_1+|\beta|\le k}\int_{\Omega_i}(x^{-\alpha}(x\partial_x)^{\beta_1}\partial_v^\beta f)^2\,\frac{dx}{x}\,d\nu,$$

and it will sometimes be convenient to use (A.5) as the definition of $\|f\|^2_{\mathcal{H}^\alpha_k(\Omega_i)}$. For $\mathcal{O}$'s such that $\Omega_i\subset\mathcal{O}$ the spaces $\mathcal{H}^\alpha_k(\mathcal{O})$ are defined as the spaces of those functions in $H^{\mathrm{loc}}_k(\mathcal{O})$ for which the norm squared

$$\text{(A.6)}\qquad \|f\|^2_{\mathcal{H}^\alpha_k(\mathcal{O})}=\sum_i\|f\|^2_{\mathcal{H}^\alpha_k(\Omega_i)}+\|f\|^2_{H_k(\mathcal{O}\cap\complement M_{x_0/2})}$$

is finite. We note the equivalence of norms,

$$\|f\|_{H_0(\mathcal{O})}\approx\|f\|_{\mathcal{H}^{-1/2}_0(\mathcal{O})},$$

and that $\mathcal{H}^\alpha_k(M_{x_1,x_2})=H_k(M_{x_1,x_2})$ for all α and k whenever $x_1>0$, the norms being equivalent, with the constants involved depending upon x_1 and x_2, and degenerating in general when x_1 tends to 0.

It is often awkward to work with coordinate charts, in order to avoid that one can proceed as follows: Choose a fixed smooth complete Riemannian metric b on $\overline{M}$. Let x be any smooth defining function for ∂M, we let X_1 be the gradient of x with respect to the metric b; rescaling b by a smooth function if necessary we may without loss of generality assume that X_1 has length one in the metric b in a neighborhood of ∂M. As before we cover ∂M by a finite number of coordinate charts $\mathcal{O}_i$ with associated coordinates v^A; the v^A's are then propagated to a neighborhood of ∂M by requiring

$$X_1(v^A)=0.$$

This leads to a covering of M_{x_0} of the kind already used, and one easily checks that

$$X_1=\partial_x$$

in the resulting local coordinates. This gives then a globally defined vector ∂_x on M_{x_0}. For $i=2,\ldots,r$ we let X_i be any smooth vector fields on ∂M satisfying the condition that at any $p\in\partial M$ the linear combinations of the X_i exhaust the tangent space $T_p\partial M$. (If ∂M is a sphere S^{n-1}, a convenient choice is a basis of the collection of all Killing vectors of $(S^{n-1},\mathring{h})$, where $\mathring{h}$ is the unit round metric on S^{n-1}.) Over the domain of a chart (v^A) of ∂M, one thus has

$$\text{(A.7a)}\qquad \partial_A=\sum_{i=2}^r f^i_A(v^B)X_i,$$

$$\text{(A.7b)}\qquad X_i=\sum_{A=2}^n X_i^A(v^B)\partial_A,$$

for some locally defined smooth functions f^i_A, X^A_i; clearly things can be arranged so that those functions are bounded, together with all their partial derivatives. We propagate the X_i's to M_{x_0} by requiring $[X_1, X_i] = 0$, equivalently

$$\partial_x X^A_i = 0. \tag{A.8}$$

It follows that (A.7) still holds with x-independent functions. For any multi-index $\beta = (\beta_1, \beta_2, \ldots, \beta_r) \in \mathbb{N}^r$ we set, on M_{x_0},

$$\mathcal{D}^\beta f = X_1^{\beta_1} X_2^{\beta_2} \cdots X_r^{\beta_r} f = \partial_x^{\beta_1} X_2^{\beta_2} \cdots X_r^{\beta_r} f. \tag{A.9}$$

It follows that we have (here, $|\beta| = \beta_1 + \cdots + \beta_r$)

$$\|f\|_{\mathcal{C}^\alpha_k(M_{x_0})} \approx \sum_{0\le|\beta|\le k} \|x^{\beta_1}\mathcal{D}^\beta f\|_{\mathcal{C}^\alpha_0(M_{x_0})},$$

$$\|f\|^2_{\mathcal{H}^\alpha_k(M_{x_0})} \approx \sum_{0\le|\beta|\le k} \int_{M_{x_0}} (x^{-\alpha+\beta_1}\mathcal{D}^\beta f)^2 \frac{\mathrm{d}x}{x}\mathrm{d}\nu$$

(where $\approx$ denotes the fact that the norms are equivalent), *etc.*

There is a useful way of rewriting $\|.\|_{\mathcal{H}^\alpha_k(M_{x_0})}$ which proceeds as follows: for $f \in \mathcal{H}^\alpha_k(M_{x_0})$, $s \in (1,2)$, and $n \in \mathbb{N}$ we set

$$f_n(s,v) = f(x = x_0 s/2^n, v), \tag{A.10}$$

letting $\approx$ denote again equivalence of norms one then has, after a change of variables,

$$\begin{aligned} \|f\|^2_{\mathcal{H}^\alpha_k(M_{x_0})} &= \sum_{n\ge1}\sum_{0\le|\beta|\le k} \int_{[2^{-n}x_0, 2^{1-n}x_0]\times\partial M} \left|x^{-\alpha+\beta_1}\mathcal{D}^\beta f(x,v)\right|^2 \frac{\mathrm{d}x}{x}\mathrm{d}\nu \\ &\approx x_0^{-2\alpha}\sum_{n\ge1}\sum_{0\le|\beta|\le k} 2^{2n\alpha}\int_{[1,2]\times\partial M} \left|\mathcal{D}^\beta f_n(s,v)\right|^2 \mathrm{d}s\mathrm{d}\nu \\ &= x_0^{-2\alpha}\sum_{n\ge1} 2^{2n\alpha}\|f_n\|^2_{H_k([1,2]\times\partial M)}. \end{aligned} \tag{A.11}$$

More precisely, we write $A \approx B$ if there exist constants $C_1, C_2 > 0$ such that $C_1 A \le B \le C_2 A$. In (A.11) the relevant constants depend only upon α and k.

It turns out to be useful to have a formula similar to (A.11) for functions in $\mathcal{H}^\alpha_k(M_{x_2,x_1})$; this can be done for any x_1 and x_2, but in order to obtain uniform control of certain constants it is convenient to require $2x_2 \le x_1$. For such values of x_1 and x_2 we let $n_0(x_1,x_2) \in \mathbb{N}$ be such that $x_1/2^{n_0+1} \le x_2 \le x_1/2^{n_0}$. For $n \in \mathbb{N}$, $n \ge 1$, and for any $f : M_{x_2,x_1} \to \mathbb{R}^N$ we then define $f_n : (1,2)\times\partial M \to \mathbb{R}^N$ by

$$f_n(s,v) = \begin{cases} f(x_1 s/2^n, v) & \text{if } n \le n_0, \\ f(x_2 s, v) & \text{if } n = n_0 + 1, \\ 0 & \text{if } n > n_0 + 1. \end{cases} \tag{A.12}$$

(This coincides with the definition already given for M_{x_1}, when this set is thought of as being an "M_{x_2,x_1} with $x_2=0$", if we set $n_0=+\infty$.) A calculation as in (A.11) shows that for any $2x_2\le x_1\le x_0$, there exist constants C_1 and c_1, independent of x_0, x_1 and x_2, such that for all $f\in\mathcal{H}^\alpha_k(M_{x_2,x_1})$,

$$c_1x_1^{-2\alpha}\sum_n\big\{2^{n\alpha}\|f_n\|_{H_k([1,2]\times\partial M)}\big\}^2 \le \|f\|^2_{\mathcal{H}^\alpha_k(M_{x_2,x_1})}\le C_1x_1^{-2\alpha}\sum_n\big\{2^{n\alpha}\|f_n\|_{H_k([1,2]\times\partial M)}\big\}^2. \tag{A.13}$$

Equation (A.11) leads one to introduce[10] spaces $\mathcal{B}^\alpha_{k+\lambda}$, that arise naturally from weighted Sobolev embeddings, *cf.* Equation (A.25) below: we define

$$\|f\|^2_{\mathcal{B}^\alpha_{k+\lambda}(M_{x_0})} = x_0^{-2\alpha}\sum_{n\ge1}2^{2n\alpha}\|f_n\|^2_{C_{k+\lambda}([1,2]\times\partial M)}, \tag{A.14}$$

f_n as in (A.10), and we set

$$\mathcal{B}^\alpha_{k+\lambda}(M_{x_0})=\big\{f\in C_{k+\lambda}(M_{x_0})\ ;\ \|f\|_{\mathcal{B}^\alpha_{k+\lambda}(M_{x_0})}<\infty\big\}.$$

Clearly

$$\mathcal{B}^\alpha_{k+\lambda}(M_{x_0})\subset\mathcal{C}^\alpha_{k+\lambda}(M_{x_0}).$$

Since the general term f_N, as well as sums of the form $\Sigma_{n\ge N}f_n$, of a convergent series tend to zero as N tends to infinity, for $f\in\mathcal{B}^\alpha_{k+\lambda}(M_{x_0})$ we actually have

$$\lim_{x_1\to0}\|f\|_{\mathcal{C}^\alpha_{k+\lambda}(M_{x_1})}=0. \tag{A.15}$$

We have the trivial inclusion,

$$\alpha'>\alpha \implies \mathcal{C}^{\alpha'}_{k+\lambda}(M_{x_1})\subset\mathcal{H}^\alpha_k(M_{x_1}). \tag{A.16}$$

The fact that the inequality $\alpha'>\alpha$ in (A.16) is strict has various annoying consequences, which are best avoided by introducing yet another space – the space $\mathcal{G}^\alpha_k$ of functions in $H^k_{\mathrm{loc}}(M_{x_0})$ for which the norm squared

$$\|f\|^2_{\mathcal{G}^\alpha_k(M_{x_0})}=\sup_{n\ge1}\Big\{\sum_{0\le\beta\le k}\int_{[2^{-n}x_0,2^{1-n}x_0]\times\partial M}|x^{-\alpha+\beta_1}\mathcal{D}^\beta f(x,v)|^2\frac{\mathrm{d}x}{x}\mathrm{d}\nu\Big\} \tag{A.17}$$

is finite. We note that $\|f\|_{\mathcal{G}^\alpha_k(M_{x_0})}$ is equivalent to

$$x_0^{-\alpha}\sup_{n\ge1}\big\{2^{n\alpha}\|f_n\|_{H_k([1,2]\times\partial M)}\big\}, \tag{A.18}$$

[10] The symbol $\mathcal{B}$ might suggest to the reader that we specifically have Besov spaces in mind; this is not the case, and we hope that the notation will not lead to confusion.

with $f_n(s,v) = f(x_0 s/2^n, v)$, as in (A.10). To define the $\mathcal{G}_k^\alpha(M_{x_2,x_1})$'s, assuming again that $x_2 \le x_{1/2}$, we let $I_n(x_1,x_2)$ be defined as

$$I_n = \begin{cases} (2^{-n}x_1, 2^{1-n}x_1) & \text{if } n \le n_0, \\ (x_2, 2x_2) & \text{if } n = n_0+1, \\ \varnothing & \text{if } n > n_0+1, \end{cases} \tag{A.19}$$

where n_0 is as in (A.12). For all $f \in H_k^{\mathrm{loc}}(M_{x_2,x_1})$ we set

$$\|f\|^2_{\mathcal{G}_k^\alpha(M_{x_2,x_1})} = \sup_n \Big\{ \sum_i \sum_{0\le|\beta|\le k} \int_{\Omega_i \cap \{I_n \times \partial M\}} (x^{-\alpha+\beta_1}\mathcal{D}^\beta f)^2 \frac{\mathrm{d}x}{x}\,\mathrm{d}\nu \Big\} \tag{A.20}$$

(we identify $(a,b)\times\partial M$ and $M_{a,b}$). Similarly to (A.13), there exist constants c_2 and C_2, which do *not* depend upon x_0, x_1, and x_2, such that for all $2x_2 \le x_1 \le x_0$,

$$\begin{aligned} c_2 x_1^{-\alpha} \sup_n \|f_n\|_{H_k([1,2]\times\partial M)} &\le \|f\|_{\mathcal{G}_k^\alpha(M_{x_2,x_1})} \\ &\le C_2 x_1^{-\alpha} \sup_n \|f_n\|_{H_k([1,2]\times\partial M)}. \end{aligned} \tag{A.21}$$

We have the obvious inequality

$$\|f\|_{\mathcal{G}_k^\alpha(\Omega)} \le \|f\|_{\mathcal{H}_k^\alpha(\Omega)}, \tag{A.22}$$

together with the modified version of (A.16),

$$\alpha' \ge \alpha \implies \mathcal{C}_{k+\lambda}^{\alpha'} \subset \mathcal{G}_k^\alpha; \tag{A.23}$$

in particular the function $(x,v) \mapsto x^\alpha$ is in $\mathcal{G}_k^\alpha(M_{x_0})$.

If S_k denotes a space of functions, where $k \in \mathbb{N}$ is a differentiability index, we set $S_\infty \equiv \bigcap_{k\in\mathbb{N}} S_k$, *e.g.*, $\mathcal{G}_\infty^\alpha \equiv \bigcap_{k\in\mathbb{N}} \mathcal{G}_k^\alpha$, *etc.*

We note the following:

Proposition A.1. — *Let* $\Omega = M$, *or* $\Omega = M_{x_1}$, $0 < x_1 \le x_0$, *or* $\Omega = M_{x_2,x_1}$, $2x_2 < x_1 \le x_0$, *and let* $\mathcal{H}_k^\alpha = \mathcal{H}_k^\alpha(\Omega)$, *etc. For* $k' \in \mathbb{N}$, $\lambda \in [0,1]$, $0 \le k'+\lambda \le k - \frac{1}{2}n \notin \mathbb{N}$ *or* $0 \le k'+\lambda < k-\frac12 n \in \mathbb{N}$ *we have the continuous embeddings*

$$\mathcal{H}_k^\alpha \subset \mathcal{B}_{k'+\lambda}^\alpha \subset \mathcal{C}_{k'+\lambda}^\alpha, \quad \mathcal{H}_k^\alpha \subset \mathcal{G}_k^\alpha \subset \mathcal{C}_{k'+\lambda}^\alpha, \tag{A.24}$$

and there exists an x_2*-independent constant* C *such that we have*

$$\forall f \in \mathcal{H}_k^\alpha, \quad \|f\|_{\mathcal{B}_{k'+\lambda}^\alpha(\Omega)} \le C\|f\|_{\mathcal{H}_k^\alpha(\Omega)}, \tag{A.25}$$

$$\forall f \in \mathcal{G}_k^\alpha, \quad \|f\|_{\mathcal{C}_{k'+\lambda}^\alpha(\Omega)} \le C\|f\|_{\mathcal{G}_k^\alpha(\Omega)}. \tag{A.26}$$

Proof. — (A.25)–(A.26) follow immediately from (A.11) and (A.13), together with the standard Sobolev embedding; the remaining inclusions in (A.24) are trivial. □

All other inequalities involving Sobolev spaces have their counterpart in the weighted setting; we shall in particular need various weighted versions of the Moser inequalities. The reader should note the different weights for the members of Equation (A.31) below – this shift of weights in this inequality is the key to our handling of nonlinear equations.

Proposition A.2. — *Let $\Omega = M$, or $\Omega = M_{x_1}$, $0 < x_1 \le x_0$, or $\Omega = M_{x_2,x_1}$, $2x_2 < x_1 \le x_0$, and let $\mathcal{H}_k^\alpha = \mathcal{H}_k^\alpha(\Omega)$, etc.*

1) *There exists a constant $C = C(\alpha,\alpha',\beta,k,x_1)$ such that, for all $f \in \mathcal{H}_k^{\alpha'} \cap \mathcal{C}_0^\alpha$ and $g \in \mathcal{H}_k^\beta \cap \mathcal{C}_0^{\alpha+\beta-\alpha'}$, we have*

$$\|fg\|_{\mathcal{H}_k^{\alpha+\beta}} \le C\big(\|f\|_{\mathcal{C}_0^\alpha}\|g\|_{\mathcal{H}_k^\beta} + \|f\|_{\mathcal{H}_k^{\alpha'}}\|g\|_{\mathcal{C}_0^{\alpha+\beta-\alpha'}}\big). \tag{A.27}$$

Further, for all $|\gamma| \le k$,

$$\begin{aligned} &\big\|x^{\gamma_1}\mathcal{D}^\gamma(fg) - (x^{\gamma_1}\mathcal{D}^\gamma f)g\big\|_{\mathcal{H}_0^{\alpha+\beta}} \\ &\qquad \le C\Big\{\|f\|_{\mathcal{C}_0^\alpha}\|g\|_{\mathcal{H}_k^\beta} + \|f\|_{\mathcal{H}_{k-1}^{\alpha'}}\big(\|x\partial_x g\|_{\mathcal{C}_0^{\alpha+\beta-\alpha'}} + \sum_{i=2}^r \|X_i g\|_{\mathcal{C}_0^{\alpha+\beta-\alpha'}}\big)\Big\}, \end{aligned} \tag{A.28}$$

where the vector fields X are defined in Equation (A.7).

2) *Let $F \in C_k(M \times \mathbb{R}^N)$ be a function such that for all $B \in \mathbb{R}^+$ there exists a constant $C_1 = C_1(B)$ so that, for all $p \in \mathbb{R}^N$, $|p| \le B$, we have*

$$\big\|F(\,.\,,p)\big\|_{\mathcal{C}_k^0(M_{x_0})} \le C_1.$$

Then for all $\alpha < 0$, $\beta \in \mathbb{R}$, and $B \in \mathbb{R}^+$ there exists a constant $C_2(B,k,\alpha,\beta,x_1)$ such that for all $\mathbb{R}^N$-valued functions $f \in \mathcal{H}_k^{\alpha-\beta}(\Omega)$ with $\|x^\beta f\|_{L^\infty(\Omega)} \le B$ we have

$$\big\|F(\,.\,,x^\beta f)\big\|_{\mathcal{H}_k^\alpha} \le C_2\big(1 + \|f\|_{\mathcal{H}_k^{\alpha-\beta}}\big). \tag{A.29}$$

Further, if F has a uniform *zero of order $\ell > 0$ at $p = 0$, in the sense that for all $B \in \mathbb{R}$ there exists a constant $\widehat{C}(B)$ such that for all $|p| \le B$ and $0 \le i \le \min(k,\ell)$,*

$$\Big\|\frac{\partial^i F(\,.\,,p)}{\partial p^i}\Big\|_{\mathcal{C}_{k-i}^0(M_{x_0})} \le \widehat{C}(B)|p|^{\ell-i}, \tag{A.30}$$

then for all $\alpha \in \mathbb{R}$, $\beta \ge 0$, there exists a constant $C_3(\widehat{C},\ell,k,\alpha,\beta,B)$ such that, for all $f \in \mathcal{H}_k^{\alpha-\ell\beta}(\Omega)$ with $\|f\|_{L^\infty(\Omega)} \le B$, we have

$$\big\|F(\,.\,,x^\beta f)\big\|_{\mathcal{H}_k^\alpha} \le C_3\|f\|_{\mathcal{H}_k^{\alpha-\ell\beta}}. \tag{A.31}$$

Remark. — Hypothesis (A.30) will hold if F is *e.g.* a polynomial in p with coefficients of p^j vanishing for $j < \ell$, and being functions belonging to $\mathcal{C}_k^0$ for $j \ge \ell$.

Proof. — We shall give a detailed proof of (A.29) and (A.31), the inequalities (A.27)–(A.28) follow by an analogous argument using [35, Vol. III, p. 10, Equ. (3.21)–(3.22)], *cf.* the calculation of Proposition A.3 below. Let, similarly to (A.10),

$$F_n(s,v) = F\Big(\Big(x = \frac{x_0 s}{2^n}, v\Big);\ \Big(\frac{x_0 s}{2^n}\Big)^{\beta} f\Big(x = \frac{x_0 s}{2^n}, v\Big)\Big);$$

from Equation (A.11) we have

$$\big\|F(\,.\,,x^\beta f)\big\|^2_{\mathcal{H}^\alpha_k(M_{x_0})} \approx x_0^{2\alpha} \sum_{n\ge 1} 2^{2n\alpha} \|F_n\|^2_{H_k([1,2]\times\partial M)}. \tag{A.32}$$

We have the obvious bound

$$\sup_{[1,2]\times\partial M} \Big|\Big(\frac{x_0 s}{2^n}\Big)^{\beta} f\Big(\frac{x_0 s}{2^n}, v\Big)\Big| \le \|x^\beta f\|_{L^\infty(M_{x_0})} \le M.$$

Further the partial derivatives of $(s,v) \mapsto F_n(s,v,p)$ with respect to s and v at $p\in\mathbb{R}^N$ fixed, $|p|\le M$, can be bounded by a constant depending only upon

$$\sup_{|p|\le M} \big\|F(\,.\,,p)\big\|_{C^0_k(M_{x_0})}.$$

The usual Moser inequalities [35, Vol. III, p. 11, Equ. (3.30)] give

$$\|F_n\|^2_{H_k([1,2]\times\partial M)} \le C\big(1 + 2^{-2n\beta}\|f_n\|^2_{H_k([1,2]\times\partial M)}\big),$$

with f_n as in (A.10), and with a constant C depending upon k and M. Inserting this in (A.32) one obtains (recall that $\alpha < 0$)

$$\begin{aligned} \big\|F(\,.\,,x^\beta f)\big\|^2_{\mathcal{H}^\alpha_k(M_{x_0})} &\le C \sum_{n\ge 1} 2^{2n\alpha}\big(1 + 2^{-2n\beta}\|f_n\|^2_{H_k([1,2]\times\partial M)}\big) \\ &\le C\big(1 + \|f\|_{\mathcal{H}^{\alpha-\beta}_k(M_{x_0})}\big). \end{aligned} \tag{A.33}$$

This establishes (A.29) for $\Omega = M_{x_0}$, and (A.29) with $\Omega = M$ readily follows. The remaining Ω's are handled in a similar way.

To establish (A.31), we note the inequality

$$\Big|\frac{\partial^{|\gamma|+i} F_n(\,.\,,p)}{\partial y^\gamma \partial p^i}\Big| \le C|p|^{\max(\ell-i,0)},$$

which follows from (A.30) when $|\gamma|+i \le k$. Letting y stand for $(s,v)\in[1,2]\times\partial M$, it then follows that for $|\sigma|\le k$ we have

$$\begin{aligned} |\partial^\sigma F_n| &= \Big| \sum_{|\gamma|+|\sigma_1|+\cdots+|\sigma_i| = |\sigma|} C(\sigma_1,\ldots,\sigma_i,\beta)\Big(\frac{x_0}{2^n}\Big)^{\beta(|\sigma_1|+\cdots+|\sigma_i|)} \\ &\qquad\qquad \times \frac{\partial^{|\gamma|+i} F_n}{\partial y^\gamma \partial p^i}\partial^{\sigma_1}(s^\beta f_n)\cdots\partial^{\sigma_i}(s^\beta f_n)\Big| \\ &\le 2^{-\ell\beta n} C \sum_{|\sigma_1|+\cdots+|\sigma_i|\le|\sigma|} \big|\partial^{\sigma_1}(s^\beta f_n)\big|\cdots\big|\partial^{\sigma_i}(s^\beta f_n)\big|. \end{aligned}$$

The usual inequalities [35, Vol. III, Chap. 13, Sect. 3] give

$$\|F_n\|_{H_k([1,2]\times\partial M)} \le C(k,M)2^{-\ell\beta n}\|f_n\|_{H_k([1,2]\times\partial M)},$$

for some constant $C(k,M)$, and one concludes from (A.32), as in (A.33). □

We have the following sharper version of (A.27)–(A.28):

PROPOSITION A.3. — *Let $\Omega = M$, or $\Omega = M_{x_1}$, $0 < x_1 \le x_0$, or $\Omega = M_{x_2,x_1}$, $2x_2 \le x_1 \le x_0$, and let $\mathcal{H}^\alpha_k = \mathcal{H}^\alpha_k(\Omega)$, etc. There exists a constant $C_s = C_s(\alpha,\beta,k)$ such that, for all $f \in \mathcal{H}^\alpha_k \cap \mathcal{B}^\alpha_0$ and $g \in \mathcal{G}^\beta_k \cap \mathcal{C}^\beta_0$ we have*

$$\text{(A.34)}\qquad \|fg\|_{\mathcal{H}^{\alpha+\beta}_k} \le C_s\big(\|f\|_{\mathcal{B}^\alpha_0}\|g\|_{\mathcal{G}^\beta_k} + \|f\|_{\mathcal{H}^\alpha_k}\|g\|_{\mathcal{C}^\beta_0}\big).$$

Moreover it also holds that

$$\text{(A.35)}\quad \forall|\gamma|\le k,\quad \big\|x^{\gamma_1}\mathcal{D}^\gamma(fg) - (x^{\gamma_1}\mathcal{D}^\gamma f)g\big\|_{\mathcal{H}^{\alpha+\beta}_0}$$
$$\le C\Big\{\|f\|_{\mathcal{B}^\alpha_0}\|g\|_{\mathcal{G}^\beta_k} + \|f\|_{\mathcal{H}^\alpha_{k-1}}\big(\|x\partial_x g\|_{\mathcal{C}^\beta_0} + \sum_{i=2}^r \|X_i g\|_{\mathcal{C}^\beta_0}\big)\Big\},$$

where the vector fields X are defined in Equation (A.7)*.*

REMARK. — A useful, though less elegant, inequality related to (A.34) is

$$\text{(A.36)}\quad \forall\,|\gamma+\sigma|\le k,$$
$$\big\|x^{\gamma_1}(\mathcal{D}^\gamma f)x^{\sigma_1}(\mathcal{D}^\sigma g)\big\|_{\mathcal{H}^{\alpha+\beta}_0} \le C_s\big(\|f\|_{\mathcal{B}^\alpha_0}\|g\|_{\mathcal{G}^\beta_k} + \|f\|_{\mathcal{H}^\alpha_k}\|g\|_{\mathcal{C}^\beta_0}\big).$$

Proof. — We will prove (A.35), the proof of (A.34) is essentially identical. When $\Omega = M_{x_0}$ we do the rescaling $f_n(s,v) = f(x_0 s/2^n, v)$, $g_n(s,v) = g(x_0 s/2^n, v)$, we then have, for all $|\gamma|\le k$,

$$\text{(A.37)}\ \big\|x^{\gamma_1}\mathcal{D}^\gamma(fg) - (x^{\gamma_1}\mathcal{D}^\gamma f)g\big\|^2_{\mathcal{H}^{\alpha+\beta}_0}$$
$$\approx x_0^{-2(\alpha+\beta)}\sum_n 2^{2n(\alpha+\beta)}\big\|\mathcal{D}^\gamma(f_n g_n) - (\mathcal{D}^\gamma f_n)g_n\big\|^2_{H_0([1,2]\times\partial M)}$$
$$\le Cx_0^{-2(\alpha+\beta)}\sum_n 2^{2n(\alpha+\beta)}\big(\|f_n\|^2_{L^\infty}\|g_n\|^2_{H_k} + \|f_n\|^2_{H_{k-1}}\|\mathcal{D}g_n\|^2_{L^\infty}\big)$$
$$\le Cx_0^{-2(\alpha+\beta)}\Big\{\Big(\sum_n 2^{2n\alpha}\|f_n\|^2_{L^\infty}\Big)\sup_n\big(2^{2n\beta}\|g_n\|^2_{H_k}\big)$$
$$+\Big(\sum_n 2^{2n\alpha}\|f_n\|^2_{H_{k-1}}\Big)\sup_n\big(2^{2n\beta}\|\mathcal{D}g_n\|^2_{L^\infty}\big)\Big\}$$
$$\approx C\big(\|f\|^2_{\mathcal{B}^\alpha_0}\|g\|^2_{\mathcal{G}^\beta_k} + \|f\|^2_{\mathcal{H}^\alpha_{k-1}}\|g\|^2_{\mathcal{C}^\beta_1}\big)$$
$$\le C_s\big(\|f\|_{\mathcal{B}^\alpha_0}\|g\|_{\mathcal{G}^\beta_k} + \|f\|_{\mathcal{H}^\alpha_{k-1}}\|g\|_{\mathcal{C}^\beta_1}\big)^2.$$

(In the third line above we have used the inequality [35, Vol. III, p. 10, Equ. (3.22)].) The case $\Omega = M$ follows immediately from the above; the case $\Omega = M_{x_2x_1}$ is treated similarly using (A.12)–(A.13) and (A.19)-(A.21). □

Similar results can be proved in weighted Hölder spaces:

LEMMA A.4. — *Let* $\Omega = M$*, or* $\Omega = M_{x_1}$*,* $0 < x_1 \le x_0$*, or* $\Omega = M_{x_2,x_1}$*,* $2x_2 \le x_1 \le x_0$*, and let* $\mathcal{C}_k^\alpha = \mathcal{C}_k^\alpha(\Omega)$*. Let* $f \in \mathcal{C}_k^\alpha \cap \mathcal{C}_0^\beta$ *and* $g \in \mathcal{C}_k^\gamma \cap \mathcal{C}_0^\delta$ *with* $\alpha + \delta = \gamma + \beta = \sigma$*. Then we have* $fg \in \mathcal{C}_k^\sigma$ *and*

$$\|fg\|_{\mathcal{C}_k^\sigma} \le C_i\big(\|f\|_{\mathcal{C}_0^\beta}\|g\|_{\mathcal{C}_k^\gamma} + \|g\|_{\mathcal{C}_0^\delta}\|f\|_{\mathcal{C}_k^\alpha}\big), \tag{A.38}$$

Proof. — The proof is very similar to that of Propositions A.2 and A.3. We use the same conventions as in (A.12), (A.19). We have $\|fg\|_{\mathcal{C}_k^\sigma} \approx \sup_n 2^{n\sigma}\|f_n g_n\|_{C_k(\omega)}$, where

$$\omega \equiv [1,2] \times \partial M, \tag{A.39}$$

similarly for f and g. The interpolation inequality [27, App. A]

$$\|f_n g_n\|_{C_k(\omega)} \le C(\|f_n\|_\infty \|g_n\|_{C_k(\omega)} + \|g_n\|_\infty \|f_n\|_{C_k(\omega)})$$

leads to the conclusion. □

We have the following $\mathcal{C}_k^\beta$ equivalent of the second part of Proposition A.2, with a similar proof, based on Lemma A.4:

LEMMA A.5. — *Let* F *be a function satisfying the hypotheses of point* 2) *of Proposition* A.2*, with a uniform zero of order* ℓ *in* p *in the sense of Equation* (A.30)*. Then, for any* $\epsilon > 0$*,* $\beta \in \mathbb{R}$ *and* $f \in \mathcal{C}_k^\beta \cap L^\infty$ *we have* $F(\,.\,, x^\epsilon f) \in \mathcal{C}_k^{\beta+\ell\epsilon}$*, and there exists a constant* C *depending upon* $\|f\|_{L^\infty}$ *such that*

$$\big\|F(\,.\,, x^\epsilon f)\big\|_{\mathcal{C}_k^{\beta+\ell\epsilon}} \le C\big(\|f\|_\infty\big) \cdot \|f\|_{\mathcal{C}_k^\beta}. \tag{A.40}$$

The space of polyhomogeneous functions $\mathcal{A}_{\rm phg} = \mathcal{A}_{\rm phg}(M)$ is defined as the set of smooth functions on $\overline{M}$ which have an asymptotic expansion of the form

$$f \sim \sum_{i=0}^{\infty}\sum_{j=0}^{N_i} f_{ij} x^{n_i} \ln^j x, \tag{A.41}$$

for some sequences n_i, N_i, with $n_i \nearrow \infty$. The polyhomogeneous expansions of the introduction are of this form if r there is replaced by $1/x$; this corresponds to the conformal transformation of Section 2, which brings "null infinity" to a finite distance. We emphasise that we allow non-integer values of the n_i's; however, we shall mostly be interested in rational ones, as those arise naturally in the problem at hand. Here the symbol $\sim$ stands for "being asymptotic to": if the right-hand-side is truncated at some finite i, the remainder term falls off appropriately faster. Further, the functions f_{ij} are supposed to be smooth on $\overline{M}$, and the asymptotic expansions should be preserved under differentiation. It is easily checked that the space $\mathcal{A}_{\rm phg}$ is independent of the choice of the function x, within the class of defining functions for ∂M.

Appendix B

ODE's in weighted spaces

In our handling of PDE's we will need ODE estimates to obtain information about solutions, we thus begin with some *a priori* estimates in weighted spaces for ODE's. While the results are well-known in principle, and easy to prove, we present them in detail here because their precise form is necessary for our arguments elsewhere in this work. For a vector w we denote by $\|w\|$ or by $|w|$ the usual Euclidean norm, while for a matrix b the symbol $\|b\|$ denotes its matrix norm.

B.1. Solutions of $\partial_\tau\varphi + b\varphi = c$ in weighted spaces. — Let $\mathcal{O}$ be an open subset of ∂M, which might be the whole of ∂M, or a coordinate patch of ∂M with coordinates v^A, whichever appropriate in the context; we set

(B.1) $$\mathcal{U}_{x_2,x_1} \equiv (x_2,x_1)\times\mathcal{O}\times[0,T],$$

(B.2) $$\mathcal{S}_{x_2,x_1} \equiv (x_2,x_1)\times\mathcal{O},$$

with $0 \le x_2 < x_1$. The time variable τ will usually be the last variable, so τ will run from $[0,T]$ whenever $\mathcal{U}_{x_2,x_1}$ is involved. Strictly speaking, $\mathcal{U}_{x_2,x_1}$ should carry an extra T index, but we have not done that in order not to overburden notation. To avoid ambiguities we emphasize that the spaces $\mathcal{C}^0_k(\mathcal{U}_{x_2,x_1})$ in the Proposition below are defined as in the previous section, with the v^A variables there corresponding here to some local coordinates on $\mathcal{O}$ *together with* the time variable τ; the time derivative ∂_τ should be understood as a one-sided one at $\tau=0$ and at $\tau=T$.

Proposition B.1. — *Let $\alpha\in\mathbb{R}$, $b\in\mathcal{C}^0_k(\mathcal{U}_{x_2,x_1},\mathrm{End}(\mathbb{R}^N))$, $c\in\mathcal{C}^\alpha_k(\mathcal{U}_{x_2,x_1},\mathbb{R}^N)$, then the unique solution φ of the equation*

(B.3) $$\partial_\tau\varphi + b\varphi = c,$$

with initial data $\widetilde{\varphi}\equiv\varphi_{|\tau=0}\in\mathcal{C}^\alpha_k(\mathcal{S}_{x_2,x_1},\mathbb{R}^N)$ is in $\mathcal{C}^\alpha_k(\mathcal{U}_{x_2,x_1},\mathbb{R}^N)$ with

(B.4) $$\|\varphi\|_{\mathcal{C}^\alpha_k(\mathcal{U}_{x_2,x_1})} \le C\big(n,N,k,T,x_1,\|b\|_{\mathcal{C}^0_k(\mathcal{U}_{x_2,x_1})}\big)\big(\|\widetilde{\varphi}\|_{\mathcal{C}^\alpha_k(\mathcal{S}_{x_2,x_1})}+\|c\|_{\mathcal{C}^\alpha_k(\mathcal{U}_{x_2,x_1})}\big).$$

We also have the estimates

$$\text{(B.5)}\quad \|\varphi(\tau)\|_{\mathcal{C}_0^\alpha(\mathcal{S}_{x_2,x_1})} \le C\mathrm{e}^{\|b\|_\infty \tau}\Big\{\|\varphi(0)\|_{\mathcal{C}_0^\alpha(\mathcal{S}_{x_2,x_1})} + \int_0^\tau \mathrm{e}^{-\|b\|_\infty s}\|c(s)\|_{\mathcal{C}_0^\alpha(\mathcal{S}_{x_2,x_1})}\mathrm{d}s\Big\},$$

$$\text{(B.6)}\quad \|\varphi(\tau)\|_{\mathcal{C}_k^\alpha(\mathcal{S}_{x_2,x_1})} \le C\mathrm{e}^{C\|b\|_\infty \tau}\Big\{\|\varphi(0)\|_{\mathcal{C}_k^\alpha(\mathcal{S}_{x_2,x_1})} + \int_0^\tau \mathrm{e}^{-C|b|_\infty s}\|c(s)\|_{\mathcal{C}_k^\alpha(\mathcal{S}_{x_2,x_1})}\mathrm{d}s + \int_0^\tau \mathrm{e}^{(1-C)|b|_\infty s}\|b(s)\|_{\mathcal{C}_k^\alpha(\mathcal{S}_{x_2,x_1})}\Big(\|\varphi(0)\|_{\mathcal{C}_0^\alpha(\mathcal{S}_{x_2,x_1})} + \int_0^s \mathrm{e}^{-|b|_\infty t}\|c(t)\|_{\mathcal{C}_0^\alpha(\mathcal{S}_{x_2,x_1})}\mathrm{d}t\Big)\mathrm{d}s\Big\}.$$

REMARKS

1) Analogous results in $\mathcal{B}_k^\alpha$ spaces can be proved by similar arguments.

2) An *a priori* estimate in weighted Sobolev spaces for (B.3) follows from Proposition 3.1 by setting $E_-^\mu\partial_\mu = \partial_\tau \otimes \mathrm{id}$ and $L \equiv \psi \equiv b \equiv 0$ there.

Proof. — Let $k \in \mathbb{N}^*$, and let $\beta = (\beta_1, \beta_2, \dots \beta_n)$ be a multi-index with $|\beta| \le k$; $\partial^\beta\varphi$ verifies the equation

$$\text{(B.7)}\qquad \partial_\tau\partial^\beta\varphi = -\partial^\beta(b\varphi) + \partial^\beta c.$$

Let $\epsilon > 0$ and set

$$e(.\,,t,\epsilon) = \Big(\epsilon + \sum_{|\beta|\le k} x^{2(\beta_1-\alpha)}\langle\partial^\beta\varphi, \partial^\beta\varphi\rangle\Big)^{1/2},$$

$$E(t,\epsilon) = \|e(.,t,\epsilon)\|_{L^\infty(\mathcal{S}_{x_2,x_1})}.$$

When $k = 0$ one easily finds $\partial_\tau e \le \|b\|e + x^{-\alpha}|c|$, and (B.5) readily follows. For $k > 0$ we have

$$\begin{aligned}\partial_\tau e &= \frac{1}{e}\sum_{|\beta|\le k} x^{2(\beta_1-\alpha)}\langle\partial_\tau\partial^\beta\varphi, \partial^\beta\varphi\rangle\\ &\le \frac{1}{e}\sum_{|\beta|\le k} x^{2(\beta_1-\alpha)}|\partial^\beta(-b\varphi + c)|\cdot|\partial^\beta\varphi|\\ &\le \frac{C(k,n)}{e}\big(\|b\varphi\|_{\mathcal{C}_k^\alpha(\mathcal{S}_{x_2,x_1})} + \|c\|_{\mathcal{C}_k^\alpha(\mathcal{S}_{x_2,x_1})}\big)e\\ &\le C(k,n)\big(\|b\varphi\|_{\mathcal{C}_k^\alpha(\mathcal{S}_{x_2,x_1})} + \|c\|_{\mathcal{C}_k^\alpha(\mathcal{S}_{x_2,x_1})}\big),\end{aligned}$$

where $C(k,n)$ is a constant depending upon k and the space dimension n, and which arises from the inequality $\sum_{i=1}^p |a_i| \le \sqrt{p}\sqrt{\sum_i |a_i|^2}$ for any real sequence (a_i). The weighted interpolation inequalities, Lemma A.4, imply

$$\|b\varphi\|_{\mathcal{C}_k^\alpha(\mathcal{S}_{x_2,x_1})} \le C\big(\|b\|_{L^\infty(\mathcal{S}_{x_2,x_1})}\|\varphi\|_{\mathcal{C}_k^\alpha(\mathcal{S}_{x_2,x_1})} + \|b\|_{\mathcal{C}_k^0(\mathcal{S}_{x_2,x_1})}\|\varphi\|_{\mathcal{C}_0^\alpha(\mathcal{S}_{x_2,x_1})}\big),$$

where C is a constant which depends upon k, N and n. It follows that

$$\begin{aligned}\partial_\tau e \le\ & C\big(\|b\|_{L^\infty(\mathcal{S}_{x_2,x_1})}\|\varphi\|_{\mathcal{C}_k^\alpha(\mathcal{S}_{x_2,x_1})} + \|b\|_{\mathcal{C}_k^0(\mathcal{S}_{x_2,x_1})}\|\varphi\|_{\mathcal{C}_0^\alpha(\mathcal{S}_{x_2,x_1})} \\ &\qquad + \|c\|_{\mathcal{C}_k^\alpha(\mathcal{S}_{x_2,x_1})}\big) \\ \le\ & C\big(\|b\|_\infty E(\epsilon,t) + \|b\|_{\mathcal{C}_k^0(\mathcal{S}_{x_2,x_1})}\|\varphi\|_{\mathcal{C}_0^\alpha(\mathcal{S}_{x_2,x_1})} + \|c\|_{\mathcal{C}_k^\alpha(\mathcal{S}_{x_2,x_1})}\big),\end{aligned}$$

with perhaps a different constant C. By integration we obtain

$$\begin{aligned}e(t) \le\ & e(0) + C\int_0^t \big(\|b\|_\infty E(s,\epsilon) + \|b(s)\|_{\mathcal{C}_k^0(\mathcal{S}_{x_2,x_1})}\|\varphi(s)\|_{\mathcal{C}_0^\alpha(\mathcal{S}_{x_2,x_1})} \\ &\qquad + \|c(s)\|_{\mathcal{C}_k^\alpha(\mathcal{S}_{x_2,x_1})}\big)\,\mathrm{d}s,\end{aligned}$$

from which we deduce

$$\begin{aligned}E(t,\epsilon) \le\ & E(0,\epsilon) + C\int_0^t \big(\|b\|_\infty E(s,\epsilon) + \|b(s)\|_{\mathcal{C}_k^0(\mathcal{S}_{x_2,x_1})}\|\varphi(s)\|_{\mathcal{C}_0^\alpha(\mathcal{S}_{x_2,x_1})} \\ &\qquad + \|c(s)\|_{\mathcal{C}_k^\alpha(\mathcal{S}_{x_2,x_1})}\big)\,\mathrm{d}s.\end{aligned}$$

Using Gronwall's Lemma and letting $\epsilon \to 0$ one obtains

$$\begin{aligned}E(t,0) \le\ & \mathrm{e}^{C\|b\|_\infty t}E(0,0) + C\int_0^t \mathrm{e}^{C\|b\|_\infty(t-s)}\Big(\|b(s)\|_{\mathcal{C}_k^0(\mathcal{S}_{x_2,x_1})}\cdot\|\varphi(s)\|_{\mathcal{C}_0^\alpha(\mathcal{S}_{x_2,x_1})} \\ &\qquad + \|c(s)\|_{\mathcal{C}_k^\alpha(\mathcal{S}_{x_2,x_1})}\Big)\,\mathrm{d}s.\end{aligned}$$

The estimate (B.5) for $\|\varphi\|_{\mathcal{C}_0^\alpha(\mathcal{S}_{x_2,x_1})}$ inserted in the last inequality leads to Equation (B.6). The time-derivative estimates follow immediately from the above and from the equation satisfied by φ. □

B.2. Solutions of $\partial_x\phi + b\phi = c$ in weighted spaces. — All the results in this section, as well as in Section B.4 below, remain valid if we replace the set $\mathcal{U}_{x_2,x_1}$ defined in Equation (B.1) with $\mathcal{S}_{x_2,x_1}$ defined in (B.2) – the time dimension does not play a preferred role in the current problem. We start with the following elementary result; the point is to ensure that the relevant constants are x_2 independent:

LEMMA B.2. — *Let $g \in \mathcal{C}_k^\alpha(\mathcal{U}_{x_2,x_1},\mathbb{R}^N)$, $0 \le x_2 < x_1$. Then f defined for $\alpha > -1$ by*

$$f(x,v^A,\tau) = \int_{x_2}^x g(s,v^A,\tau)\,\mathrm{d}s$$

is in $\mathcal{C}_k^{\alpha+1}(\mathcal{U}_{x_2,x_1},\mathbb{R}^N)$, with

$$\|f\|_{\mathcal{C}_k^{\alpha+1}(\mathcal{U}_{x_2,x_1})} \le \max\Big\{1,\frac{1}{\alpha+1}\Big\}\|g\|_{\mathcal{C}_k^\alpha(\mathcal{U}_{x_2,x_1})}.$$

Similarly f_2 defined by

$$f_2(x,v,\tau) = -\int_x^{x_1} g(s,v,\tau)\mathrm{d}s$$

satisfies

$$\begin{aligned} &\big(1+(\ln x)^2\big)^{-1/2} f_2 \in \mathcal{C}_k^0(\mathcal{U}_{x_2,x_1}) && \textit{for } \alpha=-1,\\ &f_2 \in \mathcal{C}_k^{\min\{\alpha+1,0\}}(\mathcal{U}_{x_2,x_1}) && \textit{for } \alpha<0 \textit{ and } \alpha\neq -1,\end{aligned}$$

with

$$\|f_2\|_{\mathcal{C}_k^{\min\{\alpha+1,0\}}(\mathcal{U}_{x_2,x_1})} \le \max\Big\{1, \Big|\frac{1}{1+\alpha}\Big|, \Big|\frac{x_1^{\alpha+1}}{1+\alpha}\Big|\Big\}\|g\|_{\mathcal{C}_k^\alpha(\mathcal{U}_{x_2,x_1})}.$$

Proof. — We have the trivial relations

$$\int_{x_2}^x s^\alpha \mathrm{d}s \le \frac{1}{\alpha+1}\, x^{\alpha+1} \quad \text{for } \alpha>-1, \qquad \int_x^{x_1} s^{-1}\mathrm{d}s = \ln x_1 - \ln x,$$

as well as the commutation rules:

$$\partial_x\int_a^x g\mathrm{d}x = g(x), \quad \partial_{v^A}\int_a^x g\mathrm{d}x = \int_a^x \partial_{v^A} g\mathrm{d}x, \quad \partial_\tau\int_a^x g\mathrm{d}x = \int_a^x \partial_\tau g\mathrm{d}x.$$

Note that

$$\|f\|_{\mathcal{C}_k^{\alpha+1}(\mathcal{U}_{x_2,x_1})} = \|\partial_x f\|_{\mathcal{C}_{k-1}^\alpha(\mathcal{U}_{x_2,x_1})} + \sum_{0\le i+|\delta|\le k} \|\partial_\tau^i\partial_{v^A}^\delta f\|_{\mathcal{C}_0^{\alpha+1}(\mathcal{U}_{x_2,x_1})}, \tag{B.8}$$

with $\|\partial_x f\|_{\mathcal{C}_{k-1}^\alpha(\mathcal{U}_{x_2,x_1})} = \|g\|_{\mathcal{C}_{k-1}^\alpha(\mathcal{U}_{x_2,x_1})}$. To estimate $\partial_\tau^i\partial_{v^A}^\delta f$ one writes

$$|\partial_\tau^i\partial_v^\delta f| \le \int_{x_2}^x |\partial_\tau^i\partial_v^\delta g|\mathrm{d}s \le \int_{x_2}^x \|\partial_\tau^i\partial_v^\delta g\|_{\mathcal{C}_0^\alpha} s^\alpha \mathrm{d}s \le \frac{1}{\alpha+1}\, x^{\alpha+1}\|\partial_\tau\partial_v^\delta g\|_{\mathcal{C}_0^\alpha}.$$

The results for f_2 are established in a similar way. □

We shall use the following notation

$$\mathcal{I}_{x_2} = \{x = x_2\}, \tag{B.9}$$

with the range of the other variables being in principle clear from the context; this is the equivalent of the set $\tilde{\partial} M_{x_2}$ of Equation (A.1) when the set-up described there is assumed.

Proposition B.3. — *Let $0\le x_2 < x_1$, suppose that $b \in \mathcal{C}_k^{-\epsilon}(\mathcal{U}_{x_2,x_1}, \mathrm{End}(\mathbb{R}^N))$, $0\le\epsilon<1$, $c\in\mathcal{C}_k^\alpha(\mathcal{U}_{x_2,x_1},\mathbb{R}^N)$, and let ϕ be a solution in $C_k^{\mathrm{loc}}(\mathcal{U}_{x_2,x_1})$ of the equation*

$$\partial_x\phi + b\phi = c. \tag{B.10}$$

Then the following hold:

1) *If* $\alpha < -1$ *, then* $\phi \in \mathcal{C}_k^{\alpha+1}(\mathcal{U}_{x_2,x_1})$ *and we have, for* $\alpha + 2 - \epsilon \neq 0$ *and for* $x_2 \leq x_3 \leq x_1$ *small enough so that* $C(\|b\|_{\mathcal{C}_0^{-\epsilon}}, x_3) < 1$,

$$\|\phi\|_{\mathcal{C}_0^{\alpha+1}(\mathcal{U}_{x_2,x_3})} \leq \frac{1}{1 - C(\|b\|_{\mathcal{C}_0^{-\epsilon}}, x_3)} \Big(x_3^{-\alpha-1} \|\phi\|_{C_0(\mathcal{I}_{x_3})} + \frac{1}{|1+\alpha|} \|c\|_{\mathcal{C}_0^{\alpha}(\mathcal{U}_{x_2,x_3})} \Big), \tag{B.11}$$

where

$$C(\|b\|_{\mathcal{C}_0^{-\epsilon}}, x_3) = \frac{x_3^{1-\epsilon}}{|2+\alpha-\epsilon|} \|b\|_{\mathcal{C}_0^{-\epsilon}(\mathcal{U}_{x_2,x_3})}. \tag{B.12}$$

Moreover, if $x_2 \leq x_3 \leq x_1$ *is small enough so that* $C_i C(\|b\|_{\mathcal{C}_0^{-\epsilon}}, x_3) < 1$, *where* C_i *is the constant in the interpolation inequality* (*A*.38)*, then*

$$\begin{aligned} \|\phi\|_{\mathcal{C}_k^{\alpha+1}(\mathcal{U}_{x_2,x_3})} \leq C_\alpha\big(\|b\|_{\mathcal{C}_0^{-\epsilon}}, C_i, x_3, k\big) \big\{ \|\phi(x_3)\|_{C_k(\mathcal{I}_{x_3})} + \|c\|_{\mathcal{C}_k^{\alpha}(\mathcal{U}_{x_2,x_3})} \\ + \|b\|_{\mathcal{C}_k^{-\epsilon}(\mathcal{U}_{x_2,x_3})} (\|\phi(x_3)\|_{C_0(\mathcal{I}_{x_3})} + \|c\|_{\mathcal{C}_0^{\alpha}(\mathcal{U}_{x_2,x_3})}) \big\}, \end{aligned} \tag{B.13}$$

with $C_\alpha(\|b\|_{\mathcal{C}_0^{-\epsilon}}, C_i, x_3)$ *an increasing function in the first and third variable.*

2) *If* $\alpha = -1$*, then* $(1 + (\ln x)^2)^{-1/2} \phi \in \mathcal{C}_k^0(\mathcal{U}_{x_2,x_1})$.

3) *If* $\alpha > -1$*, then* $\phi_{x_2} \equiv \lim_{x \to x_2} \phi$ *is in* $C_k(\mathcal{I}_{x_2})$*, with*

$$\phi - \phi_{x_2} \in \mathcal{C}_k^{1-\epsilon}(\mathcal{U}_{x_2,x_1}) + \mathcal{C}_k^{\alpha+1}(\mathcal{U}_{x_2,x_1}), \tag{B.14}$$

$\phi \in \mathcal{C}_k^{\alpha+1}(\mathcal{U}_{x_2,x_1})$ *if* $\phi_{x_2} = 0$*, and*

$$\|\phi\|_{L^\infty(\mathcal{U}_{x_2,x_3})} \leq \frac{1}{1 - C'(\|b\|_{\mathcal{C}_0^{-\epsilon}}, x_3)} \Big(\|\phi\|_{L^\infty(\mathcal{I}_{x_3})} + \frac{x_3^{1+\alpha}}{1+\alpha} \|c\|_{\mathcal{C}_0^{\alpha}(\mathcal{U}_{x_2,x_3})} \Big) \tag{B.15}$$

for $x_2 \leq x_3 \leq x_1$ *small enough so that*

$$C'\big(\|b\|_{\mathcal{C}_0^{-\epsilon}}, x_3\big) := \frac{x_3^{1-\epsilon}}{1-\epsilon} \|b\|_{\mathcal{C}_0^{-\epsilon}(\mathcal{U}_{x_2,x_3})} < 1.$$

Moreover for x_3 *small enough so that* $C_i C'(\|b\|_{\mathcal{C}_0^{-\epsilon}}, x_3) < 1$ *we also have*

$$\begin{aligned} \|\phi\|_{\mathcal{C}_k^{0}(\mathcal{U}_{x_2,x_3})} \leq C'_\alpha\big(\|b\|_{\mathcal{C}_0^{-\epsilon}}, C_i, x_3, k\big) \big\{ \|\phi(x_3)\|_{C_k(\mathcal{I}_{x_3})} + \|c\|_{\mathcal{C}_k^{\alpha}(\mathcal{U}_{x_2,x_3})} \\ + \|b\|_{\mathcal{C}_k^{-\epsilon}(\mathcal{U}_{x_2,x_3})} \big(\|\phi(x_3)\|_{C_0(\mathcal{I}_{x_3})} + \|c\|_{\mathcal{C}_0^{\alpha}(\mathcal{U}_{x_2,x_3})}\big) \big\}, \end{aligned} \tag{B.16}$$

with C'_α *an increasing function in its first and third argument.*

Remarks

1) The inequalities above are standard when $x_2 > 0$ and when the constants are allowed to depend upon x_2, regardless of whether or not x_3 can be made small. As already mentioned, the point here is to make sure that the constants do not blow up as x_2 gets small.

2) Log-weighted estimates are easily derived in case 2); they will, however, not be needed in what follows.

Proof. — 1) For simplicity, we will write $\mathcal{C}_k^\delta$ for $\mathcal{C}_k^\delta(\mathcal{U}_{x_3,x_2})$. Let ϕ be a (local) solution of (B.10), corresponding to initial data at $\{x = x_1\}$ in $C_k(\mathcal{I}_{x_1})$. For $a > 0$ set

$$e_a(x, v^A, \tau) := \Big(a + \sum_{|\beta|\le k} x^{2\beta_1}\langle \partial^\beta\phi \mid \partial^\beta\phi\rangle\Big)^{1/2},$$

and $e := e_0$. Let $x_3 \in (x_2, x_1)\cap(0,1]$ be such that $(x_3^{1-\epsilon}/|2+\alpha-\epsilon|)\cdot\|b\|_{\mathcal{C}_0^{-\epsilon}} < 1$. We have for all $x_2 < x \le x_3$,

$$\begin{aligned}(\text{B.17})\qquad -\partial_x e_a \;&=\; -\frac{1}{e_a}\sum \beta_1 x^{2\beta_1-1}\langle\partial^\beta\phi|\partial^\beta\phi\rangle \quad \text{I}\\ &\quad -\frac{1}{e_a}\sum_{|\beta|\le k} x^{2\beta_1}\langle\partial^\beta\partial_x\phi|\partial^\beta\phi\rangle \quad \text{II}\ ,\end{aligned}$$

Since β_1 is non-negative we have $-\partial_x e_a(x, v^A, \tau) \le$ II ; further

$$\begin{aligned}(\text{B.18})\qquad \text{II} &= \frac{1}{e_a}\sum_{|\beta|\le k} x^{2\beta_1}\big\langle\partial^\beta(b\phi - c) \mid \partial^\beta\phi\big\rangle\\ &\le \frac{1}{e_a}\sum_{|\beta|\le k}\big(|x^{\beta_1}\partial^\beta c| + |x^{\beta_1}\partial^\beta(b\phi)|\big)\cdot|x^{\beta_1}\partial^\beta\phi|\\ &\le \sum_{|\beta|\le k}|x^{\beta_1}\partial^\beta c| + \big|x^{\beta_1}\partial^\beta(b\phi)\big|.\end{aligned}$$

Clearly

$$\begin{aligned}\sum|x^{\beta_1}\partial^\beta c| \;&=\; x^\alpha\sum|x^{-\alpha+\beta_1}\partial^\beta c| \le x^\alpha\|c\|_{\mathcal{C}_k^\alpha},\\ \sum_{|\beta|\le k}|x^{\beta_1}\partial^\beta(b\phi)| \;&=\; x^{\alpha+1-\epsilon}\sum_{|\beta|\le k}\big|x^{-\alpha-1+\epsilon+\beta_1}\partial^\beta(b\phi)\big| \le x^{\alpha+1-\epsilon}\|b\phi\|_{\mathcal{C}_k^{\alpha+1-\epsilon}},\end{aligned}$$

which gives

$$(\text{B.19})\qquad -\partial_x e_a \;\le\; x^\alpha\|c\|_{\mathcal{C}_k^\alpha} + x^{\alpha+1-\epsilon}\|b\phi\|_{\mathcal{C}_k^{\alpha+1-\epsilon}}.$$

Consider, first, the case $k = 0$; in this case (B.19) reads

$$-\partial_x e_a \le x^\alpha\|c\|_{\mathcal{C}_0^\alpha} + x^{\alpha+1-\epsilon}\|b\|_{\mathcal{C}_0^{-\epsilon}}\|\phi\|_{\mathcal{C}_0^{\alpha+1}},$$

which, after integrating over $[x_3, x]$ and passing to the limit $a \to 0$, gives (recall that $\alpha < -1$)

$$\begin{aligned} e(x, v^A, \tau) &\le e(x_3, v^A, \tau) + \Big(-\frac{x^{\alpha+1}}{(1+\alpha)} + \frac{x_3^{\alpha+1}}{(1+\alpha)} \Big) \|c\|_{\mathcal{C}_0^\alpha} \\ &\quad + \Big(\frac{x_3^{\alpha+2-\epsilon}}{(2+\alpha-\epsilon)} - \frac{x^{\alpha+2-\epsilon}}{(2+\alpha-\epsilon)} \Big) \|b\|_{\mathcal{C}_0^{-\epsilon}} \cdot \|\phi\|_{\mathcal{C}_0^{\alpha+1}} \\ &\le \|\phi\|_{C_0(\mathcal{I}_{x_3})} + \frac{x^{\alpha+1}}{|1+\alpha|} \|c\|_{\mathcal{C}_0^\alpha} \\ &\quad + \Big(\frac{x_3^{\alpha+2-\epsilon}}{(2+\alpha-\epsilon)} - \frac{x^{\alpha+2-\epsilon}}{(2+\alpha-\epsilon)} \Big) \|b\|_{\mathcal{C}_0^{-\epsilon}} \cdot \|\phi\|_{\mathcal{C}_0^{\alpha+1}}. \end{aligned} \tag{B.20}$$

Suppose for the moment that $\alpha + 2 - \epsilon < 0$; Equation (B.20) yields

$$e(x, v^A, \tau) \le \|\phi\|_{C_0(\mathcal{I}_{x_3})} + \frac{x^{\alpha+1}}{|1+\alpha|} \|c\|_{\mathcal{C}_0^\alpha} + \frac{x^{\alpha+2-\epsilon}}{|2+\alpha-\epsilon|} \|b\|_{\mathcal{C}_0^{-\epsilon}} \|\phi\|_{\mathcal{C}_0^{\alpha+1}}, \tag{B.21}$$

and since $x^{-1-\alpha} \le x_3^{-1-\alpha} \le 1$ we obtain

$$\begin{aligned} x^{-\alpha-1} e(x, v^A, \tau) &\le x_3^{-1-\alpha} \|\phi\|_{C_0(\mathcal{I}_{x_3})} + \frac{1}{|1+\alpha|} \|c\|_{\mathcal{C}_0^\alpha} \\ &\quad + \frac{x_3^{1-\epsilon}}{|2+\alpha-\epsilon|} \|b\|_{\mathcal{C}_0^{-\epsilon}} \cdot \|\phi\|_{\mathcal{C}_0^{\alpha+1}}. \end{aligned}$$

On the other hand, if $\alpha + 2 - \epsilon > 0$ then

$$e(x, v^A, \tau) \le \|\phi\|_{C_0(\mathcal{I}_{x_3})} + \frac{x^{\alpha+1}}{|1+\alpha|} \|c\|_{\mathcal{C}_0^\alpha} + \frac{x_3^{\alpha+2-\epsilon}}{(2+\alpha-\epsilon)} \|b\|_{\mathcal{C}_0^{-\epsilon}} \cdot \|\phi\|_{\mathcal{C}_0^{\alpha+1}},$$

which gives

$$\begin{aligned} x^{-\alpha-1} e(x, v^A, \tau) &\le x_3^{-1-\alpha} \|\phi\|_{C_0(\mathcal{I}_{x_3})} + \frac{1}{|1+\alpha|} \|c\|_{\mathcal{C}_0^\alpha} \\ &\quad + \frac{x_3^{1-\epsilon}}{|2+\alpha-\epsilon|} \|b\|_{\mathcal{C}_0^{-\epsilon}} \|\phi\|_{\mathcal{C}_0^{\alpha+1}}. \end{aligned}$$

The inequality $\|\phi\|_{\mathcal{C}_0^{\alpha+1}(\mathcal{U}_{x_2,x_3})} \le \sup_{[x_2,x_3]} x^{-1-\alpha} e$ shows that in all cases we have

$$\|\phi\|_{\mathcal{C}_0^{\alpha+1}(\mathcal{U}_{x_2,x_3})} \le \frac{1}{1 - C(\|b\|_{\mathcal{C}_0^{-\epsilon}}, x_3)} \Big(x_3^{-1-\alpha} \|\phi\|_{C_0(\mathcal{I}_{x_3})} + \frac{1}{|1+\alpha|} \|c\|_{\mathcal{C}_0^\alpha} \Big),$$

with the constant as in Equation (B.12). Consider, now, any $0 < k \in \mathbb{N}$; Equation (B.19) and the interpolation inequality (A.38) give

$$-\partial_x e_a \le x^\alpha \|c\|_{\mathcal{C}_k^\alpha} + x^{\alpha+1-\epsilon} C_i \big(\|b\|_{\mathcal{C}_0^{-\epsilon}} \cdot \|\phi\|_{\mathcal{C}_k^{\alpha+1}} + \|b\|_{\mathcal{C}_k^{-\epsilon}} \cdot \|\phi\|_{\mathcal{C}_0^{\alpha+1}} \big).$$

An argument identical to the one before, considering separately the cases $\alpha + 2 - \epsilon > 0$ or < 0, leads to

$$\begin{aligned}\|\phi\|_{\mathcal{C}_k^{\alpha+1}} \le\ & \frac{C(k)}{1 - C_i C(\|b\|_{\mathcal{C}_0^{-\epsilon}}, x_3)} \Big(x_3^{-1-\alpha} \|\phi\|_{C_k(\mathcal{I}_{x_3})} + \frac{1}{|1+\alpha|} \|c\|_{\mathcal{C}_k^{\alpha}} \Big) \\ & + \frac{C(k) C_i}{1 - C_i C(\|b\|_{\mathcal{C}_0^{-\epsilon}}, x_3)} \cdot \frac{x_3^{1-\epsilon}}{|2+\alpha-\epsilon|} \|b\|_{\mathcal{C}_k^{-\epsilon}} \cdot \|\phi\|_{\mathcal{C}_0^{\alpha+1}} \\ \le\ & \frac{C(k, x_3)}{1 - C_i C(\|b\|_{\mathcal{C}_0^{-\epsilon}}, x_3)} \Big(\|\phi\|_{C_k(\mathcal{I}_{x_3})} + \|c\|_{\mathcal{C}_k^{\alpha}} \\ & \quad + \frac{C(k) C_i}{1 - C_i C(\|b\|_{\mathcal{C}_0^{-\epsilon}}, x_3)} \|b\|_{\mathcal{C}_k^{-\epsilon}} \big(x_3^{-\alpha-1} \|\phi\|_{C_0(\mathcal{I}_{x_3})} + \frac{1}{|1+\alpha|} \|c\|_{\mathcal{C}_0^{\alpha}} \big) \Big),\end{aligned}$$

which gives (B.13). We have thus shown that $\phi \in \mathcal{C}_k^{\alpha+1}(\mathcal{U}_{x_2,x_3})$; the property that $\phi \in \mathcal{C}_k^{\alpha+1}(\mathcal{U}_{x_2,x_1})$ immediately follows.

2) The proof is identical, except for a few obvious modifications in the calculations.

3) To obtain the L^∞ estimate, we start from (B.17)–(B.18) with $k = 0$, which upon integration and passing to the limit $a \to 0$ gives

$$e(x, v^A, \tau) \le e(x_3, v^A, \tau) + \frac{x_3^{\alpha+1}}{1+\alpha} \|c\|_{\mathcal{C}_0^{\alpha}} + \frac{x_3^{1-\epsilon}}{1-\epsilon} \|b\|_{\mathcal{C}_0^{-\epsilon}} \|\phi\|_{\mathcal{C}_0^{0}},$$

from which we deduce

$$\|\phi\|_{L^\infty(\mathcal{U}_{x_2,x_3})} \le \|\phi\|_{L^\infty(\mathcal{I}_{x_3})} + \frac{x_3^{\alpha+1}}{\alpha+1} \|c\|_{\mathcal{C}_0^{\alpha}} + \frac{x_3^{1-\epsilon}}{1-\epsilon} \|b\|_{\mathcal{C}_0^{-\epsilon}} \|\phi\|_{L^\infty(\mathcal{U}_{x_2,x_3})},$$

and (B.15) follows. The proof of (B.16) is similar to that of the analogous statement in point 1. From what has been said it can be seen that $\phi_{x_2} \equiv \lim_{x \to x_2} \phi$ exists and is in $C_k(\mathcal{I}_{x_2})$. It remains to show that $\phi - \phi_{x_2}$ satisfies (B.14). When b is a multiple of the identity, we can integrate (B.10) to obtain

$$\phi(x, .) = \phi_{x_2}(.) \mathrm{e}^{-\int_{x_2}^{x} b(s,.) \mathrm{d}s} + \int_{x_2}^{x} \mathrm{e}^{\int_x^y b(s,.) \mathrm{d}s} c(y, .) \mathrm{d}y, \tag{B.22}$$

from which the result easily follows. The general case can be established by manipulations similar to the previous ones. □

B.3. Polyhomogeneous solutions of $\partial_\tau \varphi + b\varphi = c$. — We pass now to an analysis of ODE's with polyhomogeneous sources. The results here have an auxiliary character, and several of them are rather elementary; they will be needed to handle the real problem at hand, with partial differential operators. Let $\mathcal{O}$ be an open subset of ∂M, we set

$$\mathcal{U}_{x_1} = (0, x_1] \times \mathcal{O} \times [0, T]. \tag{B.23}$$

It will be seen in Sections 4 and 5 that[11] integer space-dimensions force us to consider polyhomogeneous expansions with half-integer power of x; in order to account for that, we introduce an index

$$\delta = \frac{1}{d},$$

where d is a non-zero integer, $d \in \mathbb{N}^*$. We will mostly be interested in the case $\delta = \frac{1}{2}$ or $\delta = 1$, however other values are also possible in the formalism here. Results analogous to the ones below hold for the general polyhomogeneous expansions of Equation (A.41), which can be established by similar methods. We find it of interest that a consistent framework can be obtained in the setting considered below:

Proposition B.4. — *Let $\beta \in \mathbb{R}$ and consider the system*

$$\text{(B.24a)} \qquad \partial_\tau \varphi + b\varphi = c,$$

$$\text{(B.24b)} \qquad \varphi|_{\{\tau=0\}}(x,v) \equiv \widetilde{\varphi}(x,v) = x^\beta \sum_{i=0}^{p} \sum_{j=0}^{N_i} x^{i\delta} \ln^j x \widetilde{\varphi}_{ij}(x,v) + \widetilde{\varphi}_{p\delta+\beta+\epsilon}(x,v),$$

$$\text{(B.24c)} \qquad \widetilde{\varphi}_{ij} \in C_\infty(\overline{\{\tau = 0\}}), \quad \widetilde{\varphi}_{p\delta+\beta+\epsilon} \in \mathcal{C}_\infty^{p\delta+\beta+\epsilon}(\{\tau=0\}),$$

with

$$\text{(B.25a)} \qquad b(x,v,\tau) = \sum_{i=0}^{p} \sum_{j=0}^{N'_i} x^{i\delta} \ln^j x b_{ij}(x,v,\tau) + b_{p\delta+\epsilon}(x,v,\tau),$$

$$\text{(B.25b)} \qquad b_{p\delta+\epsilon} \in \mathcal{C}_\infty^{p\delta+\epsilon}(\mathcal{U}_{x_1}), \quad b_{ij} \in C_\infty(\overline{\mathcal{U}}_{x_1}),$$

$$\text{(B.25c)} \qquad c(x,v,\tau) = x^\beta \sum_{i=0}^{p} \sum_{j=0}^{N''_i} x^{i\delta} \ln^j x c_{ij}(x,v,\tau) + c_{p\delta+\beta+\epsilon}(x,v,\tau),$$

$$\text{(B.25d)} \qquad c_{p\delta+\beta+\epsilon} \in \mathcal{C}_\infty^{p\delta+\beta+\epsilon}(\mathcal{U}_{x_1}), \quad c_{ij} \in C_\infty(\overline{\mathcal{U}}_{x_1}),$$

where $0 < \epsilon < \delta$, and $(N_i), (N'_i), (N''_i)$ are sequences with integer values, and with

$$b \in L^\infty(\mathcal{U}_{x_1}).$$

Then the solution φ takes the form

$$\text{(B.26)} \qquad \varphi(x,v,\tau) = x^\beta \sum_{i=0}^{p} \sum_{j=0}^{M_i} x^{i\delta} \ln^j x \varphi_{ij}(x,v,\tau) + \varphi_{p\delta+\beta+\epsilon}(x,v,\tau),$$

with $\varphi_{ij} \in C_\infty(\overline{\mathcal{U}}_{x_1})$, M_k is an integer sequence and $\varphi_{p\delta+\beta+\epsilon} \in \mathcal{C}_\infty^{p\delta+\beta+\epsilon}(\mathcal{U}_{x_1})$.

To prove the proposition we shall need the following lemma:

[11] This is due to occurrence of the factor $\Omega^{(n-1)/2}$ in equations such as (4.2).

LEMMA B.5. — *Under the hypotheses of Proposition B.4, suppose that in addition we have*

$$\widetilde{\varphi}_{p\delta+\beta+\epsilon} = b_{p\delta+\epsilon} = c_{p\delta+\beta+\epsilon} = 0.$$

Then for any $\epsilon \in (0,\delta)$ *we have*

$$\varphi = x^\beta \sum_{i=0}^{p} \sum_{j=0}^{M_i} x^{i\delta} \ln^j x \varphi_{ij} + \varphi_{p\delta+\beta+\epsilon}, \tag{B.27}$$

with $\varphi_{ij} \in C_\infty(\overline{\mathcal{U}}_{x_1})$, $\varphi_{p\delta+\beta+\epsilon} \in \mathcal{C}_\infty^{p\delta+\beta+\epsilon}(\mathcal{U}_{x_1})$, *for some integer-valued sequence* M_k.

Proof. — Inserting (B.27) in the equation (B.24a) and tracking the coefficients in front of $x^{i\delta}\ln^j x$ one finds the following set of equations:

$$M_0 = \max\{N_0, N_0''\}, \quad M_{i+1} = \max\big\{\max_{0\le k\le i} M_k + N'_{i-k}, N''_{i+1}, N_{i+1}\big\},$$

$$i \in [\![0,p]\!],\ j \in [\![0,M_i]\!], \quad \partial_\tau \varphi_{ij} + \sum_{k=0}^{i} \sum_{\ell=0}^{\min\{N'_k,j\}} b_{k\ell}\varphi_{i-kj-\ell} = c_{ij},$$

$$\partial_\tau \varphi_{p\delta+\beta+\epsilon} + b\varphi_{p\delta+\beta+\epsilon} = -\sum_{i=p+1}^{2p} x^\beta \sum_{j=0}^{M_i} x^{i\delta}\ln^j x \Big\{ \sum_{k=0}^{i} \sum_{\ell=0}^{\min\{N'_k,j\}} b_{k\ell}\varphi_{i-kj-\ell}\Big\}.$$

Here $[\![a,b]\!] := [a,b]\cap\mathbb{N}$. This system is easily solved: one begins with $i=0$ and solves the equations for j running from 0 to M_0. This can then be repeated for $i=1$, *etc.*, until $i=p$ is reached. This provides the functions φ_{ij}. Finally, one solves the last equation for the remainder term $\varphi_{p\delta+\beta+\epsilon}$, with initial value zero, noting that the right hand side of the resulting equation is in $\mathcal{C}_\infty^{p\delta+\beta+\epsilon}(\mathcal{U}_{x_1})$, and one concludes using Proposition B.1. □

Proof of Proposition B.4. — With the notation of the proposition, we set $b_{\mathrm{phg}} = b - b_{p\delta+\epsilon}$, $c_{\mathrm{phg}} = c - c_{p\delta+\beta+\epsilon}$, $\widetilde{\varphi}_{\mathrm{phg}} = \tilde{\varphi} - \widetilde{\varphi}_{p\delta+\beta+\epsilon}$. We use the Lemma above to obtain a solution φ_{phg} of the problem

$$\partial_\tau\varphi + b_{\mathrm{phg}}\varphi = c_{\mathrm{phg}}, \tag{B.28}$$

$$\varphi_{|\Sigma} = \widetilde{\varphi} = x^\beta \sum_{i=0}^{p}\sum_{j=0}^{N_i} x^{i\delta}\ln^j x \widetilde{\varphi}_{ij}(x,v). \tag{B.29}$$

Then we solve

$$\partial_\tau\varphi' + b\varphi' = c_{p\delta+\beta+\epsilon} - b_{p\delta+\epsilon}\varphi_{\mathrm{phg}}$$

with $\varphi'_{|\tau=0} = \widetilde{\varphi}_{p\delta+\beta+\epsilon}$. We have $\varphi' \in \mathcal{C}_\infty^{p\delta+\beta+\epsilon}(\mathcal{U}_{x_1})$ according to Proposition B.1. To conclude we set $\varphi = \varphi_{\mathrm{phg}} + \varphi'$ which is of the required form, and solves (B.24a). □

B.4. Polyhomogeneous solutions of $\partial_x\varphi + b\varphi = c$

PROPOSITION B.6. — *Let φ be a solution in $C^{\mathrm{loc}}_\infty(\mathcal{U}_{x_1})$ of*

$$\partial_x\varphi + \frac{b}{x}\varphi = c, \tag{B.30}$$

and suppose that (B.25) *holds with some $\epsilon \in]0,\delta[$, $\beta \in \mathbb{R}$, and with some integer-valued sequences (N_i'), (N_i''). If $b = o(x)$ (equivalently, $b_{0j}(0,v,\tau) = 0$), then*

$$\varphi = \sum_{i=0}^{p}\sum_{j=0}^{M_i} x^{i\delta}\ln^j x\widehat{\varphi}_{ij} + x^{\beta+1}\sum_{i=0}^{p}\sum_{j=0}^{M_i} x^{i\delta}\ln^j x\varphi_{ij} + \varphi_{p\delta+1+\beta+\epsilon}, \tag{B.31}$$

with $\widehat{\varphi}_{ij}, \varphi_{ij} \in C_\infty(\overline{\mathcal{U}}_{x_1})$ and $\varphi_{p\delta+1+\beta+\epsilon} \in \mathcal{C}^{p\delta+1+\beta+\epsilon}_\infty(\mathcal{U}_{x_1})$, for some integer sequence (M_i).

Proof. — Proposition B.3 shows that for $\beta > -1$ the limit

$$\varphi_0(.) := \lim_{x\to 0}\varphi(x,.)$$

exists and is a smooth function on $\mathcal{O}\times[0,T]$. When b is a multiple of the identity matrix the result is obtained by a straightforward analysis of the formula

$$\varphi(x,.) = \varphi_0(.)\mathrm{e}^{-\int_0^x b(s,.)\mathrm{d}s} + \int_0^x \mathrm{e}^{\int_x^y b(s,.)\mathrm{d}s}c(y,.)\mathrm{d}y, \tag{B.32}$$

using the estimates of Lemma B.2. For $\beta < -1$, and again for b – a multiple of the identity matrix – we use instead

$$\varphi(x,.) = \varphi\big(\tfrac{1}{2}x_1,.\big)\mathrm{e}^{-\int_{x_1/2}^x b(s,.)\mathrm{d}s} + \int_{\frac{1}{2}x_1}^x \mathrm{e}^{\int_x^y b(s,.)\mathrm{d}s}c(y,.)\mathrm{d}y. \tag{B.33}$$

In the general case, we first note that it follows from Proposition B.3 that there exists $\lambda \in \mathbb{R}$ such that $\psi \in \mathcal{C}^\lambda_\infty$. We then write

$$\partial_x\psi - c = -\frac{b}{x}\psi \in \mathcal{C}^{\lambda+\delta-1}_\infty; \tag{B.34}$$

integrating gives $\psi - \int_0^x c \in \mathcal{C}^{\lambda+\delta}_\infty$. Inserting this equation in the right-hand-side of (B.34) and integrating again one obtains a similar equation with a remainder term falling-off one power of δ faster. The result is proved by repeating this procedure a finite number of times. □

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