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GALOIS-FIXED POINTS IN THE BRUHAT-TITS BUILDING OF A REDUCTIVE GROUP

by Gopal Prasad

Abstract. — We give a new proof of a useful result of Guy Rousseau on Galois-fixed points in the Bruhat-Tits building of a reductive group.

Résumé (*Points fixes de Galois dans l'immeuble de Bruhat-Tits d'un groupe réductif*)
Nous donnons une nouvelle preuve d'un résultat utile de Guy Rousseau sur les points fixes de Galois dans l'immeuble de Bruhat-Tits d'un groupe réductif.

Let k be a field with a nontrivial discrete valuation. We assume that k is complete and its residue field is perfect. Let p (≥ 0) be the characteristic of the residue field. Let G be an absolutely almost simple simply connected algebraic group defined over k. The Bruhat-Tits building $\mathcal{B}(G/\ell)$ of G/ℓ exists for any algebraic extension ℓ of k and it is functorial in ℓ (see [2, § 5] or [4]). If ℓ is a Galois extension of k, there is a natural action, by simplicial isometries, of the Galois group $\mathrm{Gal}(\ell/k)$ on the building $\mathcal{B}(G/\ell)$ (see [2, 4.2.12], or [4, Chap. II]). The convex subset consisting of points of $\mathcal{B}(G/\ell)$ fixed under $\mathrm{Gal}(\ell/k)$ will be denoted by $\mathcal{B}(G/\ell)^{\mathrm{Gal}(\ell/k)}$; $\mathcal{B}(G/\ell)^{\mathrm{Gal}(\ell/k)}$ contains $\mathcal{B}(G/k)$. It is known (and, in fact, this result is an important component of the Bruhat-Tits theory) that if ℓ is an unramified extension of k, then $\mathcal{B}(G/\ell)^{\mathrm{Gal}(\ell/k)}$ coincides with $\mathcal{B}(G/k)$, see [2, 5.1.25]. However, in general, the former is larger than $\mathcal{B}(G/k)$ (see [8, 2.6.1]). Guy Rousseau in his unpublished thesis [4] proved that if ℓ is a

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Gopal Prasad, Department of Mathematics, University of Michigan, Ann Arbor MI 48109-1109 (USA) • *E-mail :* gprasad@math.lsa.umich.edu

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tamely ramified finite Galois extension of k, then again $\mathcal{B}(G/\ell)^{\mathrm{Gal}(\ell/k)}$ coincides with $\mathcal{B}(G/k)$. This result has recently been used in the representation theory of, and harmonic analysis on, $G(k)$. The purpose of this note is to provide a short proof of the result.

Let $\mathfrak{K}$ be a field with a nontrivial discrete valuation and containing k as a valuated subfield. We assume that $\mathfrak{K}$ is *henselian* with respect to the given valuation and its residue field is perfect. Then G admits the Bruhat-Tits building $\mathcal{B}(G/\mathfrak{K})$ over $\mathfrak{K}$; see [2, § 5]. Let $\widehat{\mathfrak{K}}$ be the completion of $\mathfrak{K}$. Using the following version of Hensel's lemma: for any smooth variety V defined over $\mathfrak{K}$, $V(\mathfrak{K})$ is dense in $V(\widehat{\mathfrak{K}})$ in the topology on the latter induced by the topology on $\widehat{\mathfrak{K}}$, Bruhat, Tits and Rousseau have shown ([4, II, § 3]) that $\mathfrak{K}$-rank $G = \widehat{\mathfrak{K}}$-rank G, and the Bruhat-Tits building $\mathcal{B}(G/\widehat{\mathfrak{K}})$ of $G/\widehat{\mathfrak{K}}$ is equal to the building $\mathcal{B}(G/\mathfrak{K})$.

Let K be the completion of a fixed maximal unramified extension of k. Let L be a finite tamely ramified Galois extension of K and $\Gamma = \mathrm{Gal}(L/K)$. In view of the results of Bruhat and Tits, and of Bruhat, Tits and Rousseau mentioned above, to establish the theorem of Rousseau, it suffices to show that

$$\mathcal{B}(G/L)^{\Gamma} = \mathcal{B}(G/K).$$

This is what we will do below.

Let S be a maximal K-split torus of G. It is a well known consequence of a theorem of Steinberg (see [6], [1, 8.6]) that G is quasi-split over K, *i.e.* it contains a Borel subgroup defined over K. Hence, the centralizer $\mathcal{T}$ of S in G is a maximal K-torus. The maximal L-split subtorus T of $\mathcal{T}$ is defined over K since $\mathcal{T}$ is. If $\mathcal{T}$ does not split over L, then in fact, $T = S$, and $T(L)$ $(= S(L))$ is Γ-equivariantly isomorphic to $(L^\times)^r$; where $r = L$-rank G $(= K$-rank $G)$. On the other hand, if $\mathcal{T}$ splits over L, then $T = \mathcal{T}$. In this case, let a (≥ 0) be the number of Galois-orbits in the Tits index (*cf.* [7]) of G/K containing more than one vertex and b be the number of vertices (in the Tits index) fixed under the Galois group, and $\mathfrak{L}(\subset L)$ be the splitting field of $\mathcal{T}$ if G is not a triality form of type ${}^6D_4$, and let it be a fixed cubic extension of K contained in the splitting field of $\mathcal{T}$ if G is a triality form of type ${}^6D_4$. Then as G is simply connected, $T(L) = \mathcal{T}(L)$ is Γ-equivariantly isomorphic to $((\mathfrak{L}\otimes_K L)^\times)^a \cdot (L^\times)^b$, with Γ acting trivially on $\mathfrak{L}$ and acting in the natural way on L.

Since the centralizer of S in G is a torus containing the torus T, the restriction to S of any root of G with respect to T is nontrivial. This implies that the apartment A corresponding to the maximal K-split torus S in the building $\mathcal{B}(G/K)$, which is contained in the apartment, in the building $\mathcal{B}(G/L)$, corresponding to the maximal L-split torus T, is not contained in a wall of the latter. Let C be a chamber (*i.e.* a simplex of maximal dimension) lying in the apartment A, and $\mathcal{C}$ be a chamber in the apartment corresponding to the maximal L-split torus T, in the building $\mathcal{B}(G/L)$, containing a point x of C in its interior. As the point x is fixed under the Galois group Γ, $\mathcal{C}$ is Γ-stable.

Hence the Iwahori subgroup I of $G(L)$ determined by the chamber $\mathcal{C}$ is also Γ-stable.

Let y be a point of the convex subset $\mathcal{B}(G/L)^{\Gamma}$. Then the geodesic $[x,y]$ is contained in $\mathcal{B}(G/L)^{\Gamma}$. Since x is an interior point of the chamber $\mathcal{C}$, the geodesic $[x,y]$ can't be contained in a wall of any apartment of the building $\mathcal{B}(G/L)$. Therefore, the points of $[x,y]$ sufficiently close to y, but possibly not the point y itself, lie in the interior of a chamber $\mathcal{C}'$ of the building $\mathcal{B}(G/L)$. This chamber is necessarily Γ-stable. We shall show that there is a maximal L-split torus T', T' defined over K and containing a maximal K-split torus S', such that $\mathcal{C}'$ lies in the apartment A' determined by T' in the building $\mathcal{B}(G/L)$.

Let I' be the Iwahori subgroup of $G(L)$ determined by $\mathcal{C}'$. This Iwahori subgroup is also stable under Γ. Let $g\in G(L)$ be such that $I'=gIg^{-1}$. Then for $\gamma\in\Gamma$, as $\gamma(I')=I'$,

$$c(\gamma):=g^{-1}\gamma(g)$$

normalizes I and hence it belongs to it. $\gamma\mapsto c(\gamma)$ is a I-valued 1-cocycle on Γ. The maximal L-split tori of G associated with $I'=gIg^{-1}$ (*i.e.* the tori such that the associated apartments contain the chamber $\mathcal{C}'$) are of the form $ghTh^{-1}g^{-1}$, $h\in I$. We will now show that there exists an $u\in I$ such that for any $\gamma\in\Gamma$, the element

$$(gu)^{-1}\gamma(gu)\ \bigl(=u^{-1}c(\gamma)\gamma(u)\bigr)$$

belongs to $I\cap T(L)$.

Let I^{+} be the maximal normal pro-unipotent subgroup of I. Let F be the residue field of K (F is also the residue field of L). From our assumption that the residue field of k is perfect, it follows that F is algebraically closed. Now if F and K are of same characteristic, then the ring of integers of K contains a subfield which projects isomorphically onto the residue field F, and if the fields F and K are of unequal characteristics, then the group of units of K contains a canonical subgroup which projects isomorphically onto $F^{\times}$ (see [5, II, Prop. 6 and 8]). From this and the explicit description of $T(L)$ given above, it is obvious that the maximal bounded subgroup $I\cap T(L)$ of $T(L)$ contains a subgroup Δ stable under the natural action of the Galois group Γ on $T(L)$ such that I is a semi-direct product $I^{+}\rtimes\Delta$ of the normal subgroup I^{+} and Δ. For $\gamma\in\Gamma$, let

$$c(\gamma)=g^{-1}\gamma(g)=i(\gamma)\delta(\gamma),$$

with $i(\gamma)\in I^{+}$, and $\delta(\gamma)\in\Delta$. Then for $\gamma,\gamma'\in\Gamma$,

$$\begin{aligned}c(\gamma\gamma')&=c(\gamma)\cdot\gamma\bigl(c(\gamma')\bigr)\\&=i(\gamma)\delta(\gamma)\cdot\gamma\bigl(i(\gamma')\delta(\gamma')\bigr)\\&=i(\gamma)\cdot\delta(\gamma)\gamma\bigl(i(\gamma')\bigr)\delta(\gamma)^{-1}\cdot\delta(\gamma)\gamma\bigl(\delta(\gamma')\bigr).\end{aligned}$$

Hence,

$$(*)\qquad i(\gamma\gamma') = i(\gamma)\cdot\delta(\gamma)\gamma\big(i(\gamma')\big)\delta(\gamma)^{-1} \quad\text{and}\quad \delta(\gamma\gamma') = \delta(\gamma)\gamma\big(\delta(\gamma')\big).$$

We define a new action of Γ on I^+: For $\gamma \in \Gamma$ and $u \in I^+$, let

$$\gamma \circ u = \delta(\gamma)\gamma(u)\delta(\gamma)^{-1}.$$

According to (*), $\gamma \mapsto i(\gamma)$ is a I^+-valued 1-cocycle on Γ with respect to this action. The Iwahori subgroup I admits a decreasing filtration by Γ-stable normal subgroups I_n, $n \geq 1$, converging to the trivial subgroup $\{1\}$, such that $I_1 = I^+$ and for all n, I_n/I_{n+1} is a finite dimensional F-vector space (*cf.* [3, § 2]). Now as L is a tamely ramified finite Galois extension of K, the Galois group Γ is a finite group of order prime to p, and hence the cohomology groups $H^1(\Gamma, I_n/I_{n+1})$ are trivial, so the cohomology set $H^1(\Gamma, I^+)$ is also trivial. From this we conclude that there exists an element $u \in I^+$ such that

$$i(\gamma) = u(\gamma\circ u)^{-1} = u\delta(\gamma)\gamma(u)^{-1}\delta(\gamma)^{-1}.$$

Then $u^{-1}i(\gamma)\delta(\gamma)\gamma(u) = \delta(\gamma)$. Now,

$$\begin{aligned}(gu)^{-1}\gamma(gu) &= u^{-1}c(\gamma)\gamma(u) = u^{-1}i(\gamma)\delta(\gamma)\gamma(u)\\ &= \delta(\gamma) \quad \big(\in \Delta \subset T(L)\big).\end{aligned}$$

Hence the maximal L-split torus $T' := guT(gu)^{-1}$ and the subtorus $S' := guS(gu)^{-1}$ are defined over K. Also, the restriction to T of the conjugation by gu is defined over K and so S' ($\subset T'$) is a maximal K-split torus of G. Therefore, the apartment A' corresponding to T', in the building $\mathcal{B}(G/L)$, is stable under the action of the Galois group Γ and A'^{Γ} is the apartment corresponding to the maximal K-split torus S' in the building $\mathcal{B}(G/K)$. As $u \in I$, the apartment A' contains the chamber $\mathcal{C}'$ and so also the point y. Now since $y \in A'^{\Gamma}$, we conclude that $y \in \mathcal{B}(G/K)$, which implies that $\mathcal{B}(G/L)^{\Gamma} = \mathcal{B}(G/K)$.

Remark 1. — If a k-group G is centrally k-isogenous to the direct product of a k torus C and simply connected almost k-simple groups G_i, $1 \leq i \leq n$, and ℓ is a Galois extension of k, then the (enlarged) Bruhat-Tits building of G/ℓ is the product of the Bruhat-Tits buildings of C/ℓ and of G_i/ℓ, $1 \leq i \leq n$.

The building of C/ℓ is $X_\ell(C) \otimes_{\mathbb{Z}} \mathbb{R}$, where $X_\ell(C)$ is the free abelian group of one-parameter subgroups of C defined over ℓ. This implies at once that $\mathcal{B}(C/\ell)^{\mathrm{Gal}(\ell/k)} = \mathcal{B}(C/k)$.

For a semi-simple group $\mathcal{G}$ defined over a finite separable extension k' of k, the Bruhat-Tits building of $R_{k'/k}(\mathcal{G})/\ell$ is of course the building of $\mathcal{G}(k' \otimes_k \ell)$.

Using the above observations, it is easy to deduce from the result proved above that $\mathcal{B}(G/\ell)^{\mathrm{Gal}(\ell/k)} = \mathcal{B}(G/k)$ for an arbitrary connected reductive k-group G and any finite tamely ramified Galois extension ℓ of k.

Remark 2 (due to Ching-Li Chai). — Let k be a field with a nontrivial discrete valuation. We assume that the field is henselian with respect to the given valuation and its residue field is perfect. For a finite extension ℓ of k, let $\hat{\ell}$ denote the completion of ℓ. Let G be a connected reductive group defined over k. Then for any finite extension ℓ of k, G admits the Bruhat-Tits building $\mathcal{B}(G/\ell)$ ([2, § 5]), and the Bruhat-Tits building $\mathcal{B}(G/\hat{\ell})$ of $G/\hat{\ell}$ is equal to $\mathcal{B}(G/\ell)$, [4, II, § 3]. Now if ℓ is a tamely ramified finite Galois extension of k with Galois group Γ, then $\hat{\ell}/\hat{k}$ is also a tamely ramified Galois extension whose Galois group is canonically isomorphic to Γ. As it follows from the above that $\mathcal{B}(G/\hat{\ell})^{\Gamma} = \mathcal{B}(G/\hat{k})$, we conclude that $\mathcal{B}(G/\ell)^{\Gamma} = \mathcal{B}(G/k)$. We should note here that in Rousseau's thesis, this result has been proven also when the residue field of k is not perfect, and under some additional hypothesis on the reductive group G, if the valuation on k is real but not discrete.

Remark 3. — Let G be a connected reductive group defined over a discretely valuated henselian field k. Let T be a torus of G defined and anisotropic over k. Let ℓ be the splitting field of T; ℓ is a finite Galois extension of k. We assume that ℓ is tamely ramified over k and T is a maximal ℓ-split torus of G.

Using Rousseau's theorem established above, one can associate to T a canonical point of the Bruhat-Tits building $\mathcal{B}(G/k)$ fixed under $T(k)$ as follows. Let A be the apartment of the building $\mathcal{B}(G/\ell)$ corresponding to T. Then as T is anisotropic over k, the Galois group Γ of ℓ/k has a unique fixed point in A and by Rousseau's theorem, this point actually lies in $\mathcal{B}(G/k)$.

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