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COMPARING HEAT OPERATORS THROUGH LOCAL ISOMETRIES OR FIBRATIONS

BY MANLIO BORDONI (*)

ABSTRACT. — Our aim is to generalize and improve the Kato's inequality, which compares the trace of the heat kernel of a compact Riemannian manifold with the one of a finite-sheeted covering of it. A comparison with the heat kernel of a suitable space-form gives, as a consequence, an analogous of Kato's inequality for non compact manifolds, which improves the classical inequality when the manifolds are compact. We get another generalization for local isometries, which are no more supposed to be covering maps (as a typical example, we apply this to the exponential map). Last, we consider Riemannian submersions with minimal fibers.

RÉSUMÉ. — COMPARAISON ENTRE OPÉRATEURS DE LA CHALEUR PAR ISOMÉTRIES LOCALES OU FIBRATIONS. — Notre but est de généraliser et d'améliorer l'inégalité de Kato, qui compare la trace du noyau de la chaleur d'une variété riemannienne compacte donnée à celle d'un revêtement riemannien fini de la variété. Une comparaison avec le noyau de la chaleur d'une variété simplement connexe de courbure constante convenablement choisie donne, comme conséquence, un analogue de l'inégalité de Kato qui améliore l'inégalité classique quand le revêtement n'est pas compact. On obtient une généralisation dans le cas où les variétés sont reliées par une isométrie locale (qui n'est pas obligatoirement un revêtement, un exemple typique étant donné par l'application exponentielle). Enfin, on traite le cas des submersions riemanniennes à fibres minimales.

1. Introduction

Let (M,g) be any connected Riemannian manifold of finite dimension n. Let us denote by $\Delta_M = \Delta_{(M,g)}$ the Laplace-Beltrami operator acting on functions and let us consider the heat equation:

$$\Big(\Delta_M + \frac{\partial}{\partial t}\Big)u = O, \tag{1.1}$$

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with Dirichlet or Neumann condition on the boundary if M has a nonempty boundary ∂M. The corresponding heat kernel will be denoted by $p_M(t,x,y)$ when the boundary is empty, $p_M^D(t,x,y)$ or $p_M^N(t,x,y)$ resp. when the boundary condition is Dirichlet's or Neumann's one. For a non compact manifold, we shall consider p_M to be the unique minimal heat kernel, *i.e.* the limit of the Dirichlet heat kernels of regular compact domains exhausting M; if M is complete and if its Ricci curvature is bounded from below, then p_M is the unique heat kernel on M (see [9, p. 189]). If M is compact, the spectrum of Δ_M is a discrete sequence $\{\lambda_i(M)\}_{i=0,1,2,\dots}$ (each eigenvalue is repeated according to its finite multiplicity); in this case, we shall also consider the trace $Z_M(t)$ of the heat operator $\mathrm{e}^{-\Delta_M t}$ (with positive t):

$$Z_M(t) = \sum_{i=0}^{+\infty} \mathrm{e}^{-\lambda_i(M)t}$$

and similar expressions for $Z_M^D(t)$ and $Z_M^N(t)$.

Given a mapping $f\colon (M',g') \to (M,g)$, our aim is to compare the heat kernels of the two manifolds, under suitable assumptions for f. It turns out that a good assumption is that f satisfies the following *Fubini's property* for every continuous function u on M':

$$\int_{M'} u(x')\,\mathrm{d}v_{g'}(x') = \int_M \Big\{ \int_{f^{-1}(x)} u_{|f^{-1}(x)}(y)\,\mathrm{d}v_{g'_x}(y) \Big\}\,\mathrm{d}v_g(x) \tag{1.2}$$

where $v_{g'}, v_g$ are the measures canonically associated to the metrics g', g, and where $v_{g'_x}$ denotes the measure associated to the metric g'_x induced on $F_x = f^{-1}(x)$ by g'.

Notice that, by Sard's theorem, if f is smooth on the outside U' of a closed subset of measure zero in M', and if $\dim M' > \dim M$, then the intersection of F_x with U' is a submanifold for almost every x, so that the integrals which occur in the formula (1.2) make sense. By the coarea formula (see [8, thm. 13.4.2]), this may be extended to the case where f is only a Lipschitz map. In this case, the differential $\mathrm{d}f_{x'}$ exists for a.e. x' and, considering its restriction to the orthogonal complement $H_{x'}$ of $T_{x'}(F_{f(x')})$ in $T_{x'}M'$, we may define its Jacobian as the determinant of this restriction. By Corollary 13.4.6 of [8], condition (1.2) is, in this case, equivalent to saying that this Jacobian is a.e. equal to ± 1. The property (1.2) is automatically satisfied for instance by Riemannian submersions and coverings, or by local isometries.

If f is a fibration of compact manifolds with typical fiber F, the so called *Kato's inequality* compares the trace of the heat operator on (M',g') with the one of the trivial fibration with the same typical fiber F. P. Bérard and S. Gallot [1] and in a different way G. Besson [4] show that, if f is a Riemannian submersion of compact boundaryless manifolds, whose fibers are totally geodesic submanifolds of M', then

$$Z_{M'}(t) \leq Z_{M\times F}(t) = Z_M(t)\cdot Z_F(t); \tag{1.3}$$

in particular, if f is a regular ℓ-sheeted Riemannian covering, one obtains:

$$Z_{M'}(t) \leq \ell \cdot Z_M(t) \tag{1.4}$$

(see also [22]); they also show that the inequality in (1.3) is an equality if and only if f is the trivial fibration. The inequality (1.4) was extended by J. Tysk [30] to a branched covering whose singularity set is a submanifold of M' of codimension at least 2.

In Section 2, we consider any mapping $f:(M',g') \to (M,g)$ which is locally isometric. In this case, Fubini's property is automatically satisfied. Most of the difficulties come from the fact that we don't assume that f is a covering map. A typical example is given by the exponential map, which is a local isometry on an open set in the tangent space (endowed with the pull-back metric), but not a covering. We show (Prop. 2.4) that the series $\sum_{y'\in f^{-1}(y)} p_{M'}(t,x',y')$ converges in the sense of distributions, and that its limit is not greater than $p_M(t,f(x'),y)$.

To obtain the equality case for boundaryless manifolds, we must assume that the manifolds are stochastically complete (Prop. 2.12); in this case, the sum of the series does not depend on $x' \in M'$ but only on $f(x') \in M$. Remember that a Riemannian manifold (X,g) is stochastically complete if and only if $\int_X p_X(t,x,y)\,\mathrm{d}v_g(y) = 1$ for any $x \in X$ and for any $t > 0$. A geometrical sufficient condition on a complete manifold (X,g) to be stochastically complete concernes the volume of geodesic balls (Grigor'yan theorem 2.9, see [20] and [21] for the proof).

In the case of manifolds with boundary, notice that, when M' has a nonempty boundary, the fact that M' is complete does not imply that M' is geodesically complete. In this case we show (Lemmas 2.2 and 2.3) a weak Hopf-Rinow theorem, and we prove that the restriction of f to the interior of M' is a covering map onto the interior of M. Proposition 2.4 also gives a sharp lower bound of the Dirichlet heat kernel p_M^D in terms of sums of $p_{M'}^D$. To obtain the equality case for manifolds with boundaries (Prop. 2.15), we must assume that f maps the boundary of M' onto the boundary of M, that M' is a complete metric space and that it satisfies the condition of Grigor'yan theorem 2.9.

When f is a ℓ-sheeted Riemannian covering of compact manifolds, we obtain (Cor. 2.18) a first improvement of Kato's inequality (1.4), in which appears explicitely the difference between $\ell \cdot Z_M(t)$ and $Z_{M'}(t)$. We obtain also a comparison between the heat kernels of M' and M in the case where f is not a covering map and, as a typical example, when f is the exponential map (Prop. 2.20, 2.22); this gives an estimate of the heat kernel of a manifold in terms of a computable euclidean one.

It is well known that the heat kernel $p_{M_K}(t, x_0, .)$ of the space-form (M_K, g_K) of constant curvature K only depends on t and on the distance from x_0. There are many works where a pointwise comparison between the heat kernel of a manifold (M, g) and the heat kernel of (M_K, g_K) is established, under suitable assumptions on the curvature of (M, g) (see for instance J. Cheeger and S.T. Yau [10], A. Debiard, B. Gaveau, E. Mazet [15], G. Courtois [11]). We give (Theorem 2.25) a unified proof of these results by clarifying the role played by the different singularities of the Laplacian of the functions which is obtained by transplantation of the heat kernel of M_K, extended by a constant outside a ball. By combining these results with the ones of Section 2, we obtain an effective improvement of Kato's inequality in the Corollaries 2.26, 2.27. The inequalities which appear in the corollaries are sharp (they are for example equalities in the case of the 2-sheeted covering of the real projective space by the standard sphere). These inequalities remain valid when the fibers have infinite cardinality.

In the case that $f:(M', g') \to (M, g)$ is a Riemannian submersion with minimal fibers (the manifolds are now assumed to be compact and boundaryless), we obtain that the resolvent and heat operators on M dominate the resolvent and heat operators on M' respectively (Prop. 3.6). To prove this result, we show that the mapping ϖ from $H_1(M')$ in $H_1(M)$ which sends u on ϖu, where $\varpi u(x)$ is the L^2-norm of u on F_x, is a symmetrization in the sense of G. Besson [5], which obeys a Kato-type inequality with respect to the Laplacians (Def. 3.2): then a generalized Beurling-Deny principle (3.3) gives the result.

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2. Kato's inequality for local (quasi) isometries and applications

a) Some topological remarks.

Let $f: X' \to X$ be any local homeomorphism from a Hausdorff topological space X' to a topological space X. *The unique lift lemma* is then valid, in the sense that *the continuous lift* passing through some point of X' of any continuous mapping $\gamma: Y \to X$, where Y is a connected topological space, *when it exists, is unique.* The proof is the classical one: let $c_1, c_2: Y \to X'$ be two continuous mappings satisfying $c_1(y_0) = c_2(y_0)$ for some $y_0 \in Y$ and $f \circ c_1 = \gamma = f \circ c_2$. The set of $y \in Y$ such that $c_1(y) = c_2(y)$ is closed and open because f is locally injective and X' is Hausdorff.

If f is such that any continuous path $\gamma: [0, 1] \to X$ admits a continuous lift $c: [0, 1] \to X'$ beginning at any $x' \in f^{-1}(\gamma(0))$, and if X is arcwise connected, then all the fibers $f^{-1}(x)$ have the same cardinality: for $x_1, x_2 \in X$, let us *fix* a path γ_0 from x_1 to x_2. The mapping $f^{-1}(x_1) \to f^{-1}(x_2)$, which sends

$x_1' \in f^{-1}(x_1)$ to the endpoint of the unique lift of γ_0 starting from x_1', is bijective. Moreover, we can deduce the property of *lifting homotopies* of paths (with fixed endpoints) from the property of lifting paths. Suppose in fact that $\gamma_0 \simeq \gamma_1$ rel $\{0,1\}$ and let $H: R \to X$ be the homotopy, where R is the square $[0,1] \times [0,1]$. Let us fix a point $x' \in f^{-1}(\gamma_0(0))$. As each curve $\gamma_s(t) = H(s,t)$ admits a unique lift $c_s(t)$ starting from x', we can define a mapping $\widetilde{H}: R \to X'$ by setting $\widetilde{H}_s(t) = c_s(t)$. The mapping $\widetilde{H}$ is obviously continuous with respect to t. A covering of the image of $\widetilde{H}$ by open sets, each of which is homeomorphic (by f) to some open set in X, induces an open covering of the image of H. By compactness of R, we may consider a finite subdivision of R in closed rectangles $R_{ij} = [s_i, s_{i+1}] \times [t_j, t_{j+1}]$ such that the image by H of each of them lies in an open set $U_{ij} \subset X$ which is the homeomorphic image by f of some open set $V_{ij} \subset X'$ containing $\widetilde{H}(s_i, t_j)$. We are going to prove that $\widetilde{H}$ is continuous on each R_{ij} by an iterating process. In fact, $\widetilde{H}$ and $(f_{|V_{ij}})^{-1} \circ H$ coincide at (s_i, t_j), so they coincide on the interval $[s_i, s_{i+1}] \times \{t_j\}$, by uniqueness of the local continuous lift and because $\widetilde{H}$ is continuous on this interval (by the iteration assumption). For the same reason, as these two mappings coincide at (s, t_j), they coincide on $\{s\} \times [t_j, t_{j+1}]$ because $\widetilde{H}$ is continuous with respect to t. So $\widetilde{H}$ coincides with a continuous mapping on R_{ij} and the iteration assumption (continuity of $\widetilde{H}$ on $[s_i, s_{i+1}] \times \{t_{j+1}\}$) is satisfied. It follows that $\widetilde{H}$ is continuous on R and that $s \mapsto \widetilde{H}(s,1)$ is a continuous mapping with values in the discrete set $f^{-1}[\gamma_0(1)]$, thus it is constant and $\widetilde{H}$ has fixed endpoints.

Notice that f is not in general, under the previous assumptions, a covering map, even if we suppose that X' and X are manifolds. For a detailed discussion about this kind of topological problems, a good reference is the Chapter 2 of [19].

b) General facts about local isometries.

A mapping $f: (M', g') \to (M, g)$ is a *local isometry* if it is continuous and if, for any interior point x' in M', there exist two neighborhoods $V_{x'}$ of x' in M' and U_x of $x = f(x')$ in M such that $f_{|V_{x'}}$ is an isometry from $V_{x'}$ onto U_x. In the sequel, we will denote by $\mathrm{Int}(M)$ the interior of a manifold M, and we shall sometimes write $M = \mathrm{Int}(M) \cup \partial M$ to underline the existence of a boundary.

Let us denote by $i_f(x')$ the *isometry radius* of f at $x' \in \mathrm{Int}(M')$, *i.e.* the supremum of all positive r such that the restriction of f to the geodesic ball $B_{g'}(x', r)$ centered at x' and of radius r is an isometry, where $d_{g'}$ denotes the distance induced on M' by the metric g'; the isometry radius is stricly positive for any $x' \in \mathrm{Int}(M')$.

LEMMA 2.1. — *Let* $f:(M',g') \to (M,g)$ *be a local isometry. Then, for any interior point* x' *in* M', $f^{-1}[f(x')]$ *is a discrete closed set and any pair of points* $x'_1 \neq x'_2$ *in* $f^{-1}[f(x')]$ *satisfy*:

$$d_{g'}(x'_1,x'_2) \geq 2\max\big[i_f(x'_1),i_f(x'_2)\big].$$

Proof. — Assume (by commuting eventually x'_1 and x'_2) that $i_f(x'_1) \geq i_f(x'_2)$. Set $\delta = i_f(x'_1)$ and suppose that there exists a curve in M' from x'_1 to x'_2 of lenght less than 2δ. As $f \circ c$ is then a loop γ starting from $x = f(x')$ with the same length, its image lies entirely in $B_g(x,\delta)$, which is the isometric image by f of $B' = B_{g'}(x'_1,\delta)$. By ${f_{|B'}}^{-1}$, we may lift γ as a loop $\tilde{\gamma}$. By the uniqueness of the lift, we get $x'_2 =$ endpoint of $c =$ endpoint of $\tilde{\gamma} = x'_1$. □

We know that, in some cases, we can prove that $d_{g'}(x'_1,x'_2) \geq 2\,\mathrm{inj}_g(x)$, where $\mathrm{inj}_g(x)$ is the *injectivity radius* of (M,g) at the point x, *i.e.* the largest r such that exp_x is a diffeomorphism from the open ball of radius r in T_xM onto its image in M. This would be a major improvement of Lemma 2.1, because the lower bound would no more depend on f and on (M',g'). However, this is false in general: a counterexample is the mapping $f(z) = z^k$ from $(\mathbb{C} \setminus \{0\}, g')$ to $(\mathbb{C}, g)$, where g is the canonical metric and g' is the pull-back metric f^*g. Let $x = \varepsilon \in \mathbb{R}^+$, then the distance between two consecutive points in $f^{-1}(x)$ is smaller than $2\pi\varepsilon$, though the injectivity radius of $(\mathbb{C}, g)$ is infinite. Analogous counterexamples may be built from branched coverings.

Roughly speaking, there are two kinds of boundaries in a convenient compactification of (M',g'): the actual boundary $\partial M'$ and the so called "boundary at infinity". The problem comes from the fact that, when M' is not complete, this "boundary at infinity" may be at finite distance. We shall say that $M' = \mathrm{Int}(M') \cup \partial M'$ *is complete* if it is complete as a metric space $(M', d_{g'})$ endowed with the Riemannian distance. Notice that, when $\partial M'$ is not empty, Hopf-Rinow's theorem is no more valid in the usual formulation and the fact that M' is complete does not imply that M' is geodesically complete.

LEMMA 2.2. — *Let* $f:(M',g') \to (M,g)$ *be a surjective local isometry which maps* $\partial M'$ *onto* ∂M, *and let* $(M',d_{g'})$ *be complete. Then for any pair of distinct points* x'_1,x'_2 *in* $\mathrm{Int}(M')$, *lying in the same fiber* $f^{-1}(x)$, *one has*

$$d_{g'}(x'_1,x'_2) \geq 2\,\mathrm{inj}_g(x).$$

Proof. — We first prove *the existence of a (unique) lift for any curve* $\gamma: I_0 \to \mathrm{Int}(M)$. The set I of t's such that there exists a curve $c:[0,t] \to \mathrm{Int}(M')$ starting from any fixed point $x' \in f^{-1}(\gamma(0))$ and satisfying $f \circ c = \gamma_{|[0,t]}$, is open: in fact, if $t_0 \in I$, f is an isometry from a neighborhood V of $c(t_0)$ in $\mathrm{Int}(M')$ onto a neighborhood U of $\gamma(t_0)$ in $\mathrm{Int}(M)$ and $(f_{|V})^{-1} \circ \gamma$ gives an extension of c. The set I is also closed in I_0: in fact, considering a sequence $\{t_n\}$ in I which converges to some $t_0 \in I_0$, we get

$$d_{g'}\big[c(t_p), c(t_q)\big] \le L_{g'}\big(c_{|[t_p,t_q]}\big) = L_g\big(\gamma_{|[t_p,t_q]}\big)$$

(here $L_{g'}$ and L_g denote the lengths of curves with respect to the metrics g' and g resp.); so $\{c(t_n)\}_{n\in\mathbb{N}}$ is a Cauchy sequence and it converges to some point x'_0, which is an interior point of M' because f maps $\partial M'$ onto ∂M. Then f is an isometry from a neighborhood V of x'_0 in $\mathrm{Int}(M')$ onto a neighborhood U of $\gamma(t_0)$ in $\mathrm{Int}(M)$ and $\tilde{c} = ({f_{|V}}^{-1}) \circ \gamma$ is a local lift of γ, which provides an extension of γ on a neighborhood of t_0. So $t_0 \in I$ and, as I is both closed and open in I_0, I is equal to I_0.

In order to finish the proof, let us suppose the existence of a curve c in $\mathrm{Int}(M')$ from x'_1 to x'_2 whose length is smaller than $\delta = 2\,\mathrm{inj}_g(x)$. Then the loop $\gamma = f \circ c$ lies in the ball $B = B_g(x,\delta)$, which is diffeomorphic to an Euclidean ball. So γ is homotopic to a point in $B \subset \mathrm{Int}(M)$ by a homotopy H. By lifting this homotopy as in 2.a), we have that the lift $\widetilde{H}$ of H has fixed endpoints, so $x'_1 = \widetilde{H}(0,1) = \widetilde{H}(1,1) = x'_2$. □

REMARK. — Lemma 2.2 is classical when $\partial M'$ is empty, because f is then a Riemannian covering map. The analogous in the case "with boundary" is the following

LEMMA 2.3. — *Under the same assumptions as in Lemma* 2.2, *the restriction of f to* $\mathrm{Int}(M')$ *is a covering map onto* $\mathrm{Int}(M)$.

Proof. — We first notice that any geodesic c of (M', g'), which does not meet $\partial M'$, is defined on the whole of $\mathbb{R}$: an argument similar to the one of Lemma 2.2 (using moreover the fact that any geodesic adherent to x'_0 lifts by $\exp_{x'_0}^{-1}$ to a radius in $T_{x'_0}M'$ and thus admits an extension), shows that the domain of c is open and closed in $\mathbb{R}$.

For any fixed $x \in \mathrm{Int}(M)$, let $f^{-1}(x) = \{x'_j\}_{j\in J}$. The injectivity radius of M' at x'_j is greater than the injectivity radius of M at x, because every minimizing geodesic of M which does not meet the boundary lifts to minimizing geodesics of M' (f does not increase the distances). Thus, for any r smaller than $\mathrm{inj}_g(x)$, the map $f_j = \exp_x \circ df_{|x'_j} \circ \exp_{x'_j}^{-1}$ is a (diffeomorphic) isometry from $V'_j = B_{g'}(x'_j, r)$ onto $U = B_g(x,r)$ whose differential at x'_j coincides with that of f, so f_j coincides with f on V'_j. In order to prove that $f^{-1}(U) = \bigcup_{j\in J} V'_j$, let $x' \in f^{-1}(U)$ and γ be the minimizing geodesic from $f(x')$ to x, which entirely lies in U. The existence of a lift c defined on the same interval proves the existence of a geodesic c (whose length is less than r) from x' to some point x'_j in $f^{-1}(x)$. It implies that $x' \in B_{g'}(x'_j, r)$. □

c) Comparison between heat kernels.

PROPOSITION 2.4. — *Let $f:(M',g') \to (M,g)$ be a local isometry (as in Lemma 2.1). Then for any positive t, for any $x' \in M'$ and for any $y \in M$, if $p_{M'}$ and p_M are the (minimal) heat kernerls of M' and M, one has, in the sense of distributions:*

$$\sum_{y'\in f^{-1}(y)} p_{M'}(t,x',y') \le p_M(t,f(x'),y); \tag{2.5}$$

the same result is true for Dirichlet heat kernels if M' and M have nonempty boundaries.

Proof. — Let us consider a compact exhaustion $\{\Omega'_j\}_{j\in\mathbb{N}}$ of $\mathrm{Int}(M')$ by regular domains: by definition, $p_{M'}(t,x',y') = \lim_{j\to\infty} p^D_{\Omega'_j}(t,x',y')$.

Let $C_0^\infty(M)$ be the space of the functions on M with compact support included in the *interior* of M. The coarea formula (see [8, thm. 13.4.2]) gives, for any $\varphi \in C_0^\infty(M)$:

$$\int_M \Big[\sum_{y'\in f^{-1}(y)} p_{M'}(t,x',y')\Big]\cdot\varphi(y)\,\mathrm{d}v_g(y) = \int_{M'} p_{M'}(t,x',y')\cdot(\varphi\circ f)(y')\,\mathrm{d}v_{g'}(y'). \tag{2.6}$$

The main difficulty is that $\varphi\circ f$ is generally *not* in $C_0^\infty(M')$, nevertheless it is bounded on M'. As $\int_{\Omega'_j} p^D_{\Omega'_j}(t,x',y')\,\mathrm{d}v_{g'}(y') \le 1$, Lebesgue monotone convergence theorem implies that $\int_{M'} p_{M'}(t,x',y')\,\mathrm{d}v_{g'}(y') \le 1$. Applying (2.6) to any positive test function, we get that

$$\Big|\int_M \Big[\sum_{y'\in f^{-1}(y)} p_{M'}(t,x',y')\Big]\cdot\varphi(y)\,\mathrm{d}v_g(y)\Big| \le \|\varphi\|_{C^0}$$

and the series converges (in the sense of distributions) for any fixed x' to a Radon measure T_t on M (notice that $T_t \equiv 0$ on the complementary of the image by f of the interior of M').

For every j, the (finite, *cf.* Lemma 2.1) sum

$$T_t^j = \sum_{y'\in f^{-1}(y)\cap\Omega'_j} p^D_{\Omega'_j}(t,x',y'),$$

for any fixed $x' \in \Omega'_j$, is a function on M, whose limit as $j\to\infty$ (in the sense of distributions) is, by the same arguments as in (2.6), equal to T_t by Lebesgue monotone convergence theorem. Let us denote by ν the inward unit normal vector field on $\partial\Omega'_j$ and by $a_{g'}$ the $(n-1)$-dimensional measure on $\partial\Omega'_j$ induced by the restriction of g' on $\partial\Omega'_j$. Replacing φ by $\Delta_M\varphi$ in the formula (2.6) and noticing that $(\Delta_M\varphi)\circ f = \Delta_{M'}(\varphi\circ f)$ at any interior point of M', we get, when φ is a nonnegative function in $C_0^\infty(M)$:

$$
\begin{aligned}
(2.7)\quad \langle \Delta_M T_t^j, \varphi\rangle &= \int_{\Omega'_j} p^D_{\Omega'_j}(t,x',y')\cdot \Delta_{M'}(\varphi\circ f)(y')\,\mathrm{d}v_{g'}(y')\\
&= \int_{\Omega'_j} \Delta_{M'} p^D_{\Omega'_j}(t,x',y')\cdot(\varphi\circ f)(y')\,\mathrm{d}v_{g'}(y')\\
&\qquad + \int_{\partial\Omega'_j} p^D_{\Omega'_j}(t,x',y')\cdot \frac{\partial(\varphi\circ f)}{\partial\nu}(y')\,\mathrm{d}a_{g'}(y')\\
&\qquad\qquad - \int_{\partial\Omega'_j} \frac{\partial p^D_{\Omega'_j}}{\partial\nu}(t,x',y')\cdot(\varphi\circ f)(y')\,\mathrm{d}a_{g'}(y')\\
&\le -\Big\langle \frac{\partial}{\partial t} T_t^j, \varphi\Big\rangle
\end{aligned}
$$

because the first boundary term vanishes by the Dirichlet boundary condition and the second one, which does not vanish in general, is negative ($\partial p^D_{\Omega'_j}/\partial\nu$ is nonnegative). It comes that $(\Delta_M + \partial/\partial t)T_t^j \le 0$ in the sense of distibutions.

Let us denote by T_0^j the limit of T_t^j when t goes to 0, we have, by (2.6):

$$
\langle T_0^j, \varphi\rangle = \int_{\Omega'_j} p^D_{\Omega'_j}(0,x',y')\cdot(\varphi\circ f)(y')\,\mathrm{d}v_{g'}(y') = \varphi\big[f(x')\big]
$$

so T_0^j is the Dirac measure $\delta_{f(x')}$. Moreover, the support of T_t^j is included in the compact subset $f(\Omega'_j)$ of $\mathrm{Int}(M)$. The maximum principle then gives

$$
T_t^j(\cdot) \le p_M\big(t, f(x'), \cdot\big)
$$

and hence the claim when $j\to\infty$. As each term of the series which defines T_t is nonnegative, it implies that $T_t \in L^1(M)$ because the coarea formula implies that $\int_M \sum_{y'\in f^{-1}(y)} p_{M'}(t,x',y')\,\mathrm{d}v_g(y) \le 1$. Generally T_t is not C^1.

For the Dirichlet problem, as f maps $\mathrm{Int}(M')$ into $\mathrm{Int}(M)$, no point in ∂M has a counterimage by f in $\mathrm{Int}(M')$. By the minimality, $p_{M'}$ goes to 0 on $\partial M'$ and it follows that T_t verifies the Dirichlet condition on ∂M. □

Notice that the assumptions of Proposition 2.4 *do not exclude* that f may map points belonging to $\partial M'$ to points belonging to $\mathrm{Int}(M)$ (a trivial example is $M' = \Omega$ =regular domain with boundary in M). This is the case of the exponential mapping, see the following application 2.g).

The inequality (2.5) is not in general an equality, even if we assume $f(\partial M') = \partial M$. For example, let us consider the local isometry $f: M' = \mathbb{R}^+ \to M = S^1$ defined by $f(x') = \mathrm{e}^{ix'}$, where $\mathbb{R}^+ = (0, +\infty)$ is considered as a non complete boundaryless manifold. The minimal heat kernel of M' automatically satisfies Dirichlet conditions at 0 ($\{0\}$ is the part of the "boundary at infinity" which is at finite distance) and at $+\infty$; it writes

$$p_{M'}(t, x', y') = k(t, x', y') - k(t, -x', y')$$

where $k(t, x', y') = (4\pi t)^{-1/2} \mathrm{e}^{-|x'-y'|^2/4t}$ is the usual heat kernel of the real line. When y'_0 lies in $(0, 2\pi)$ and when $x_0 = \mathrm{e}^{2i\pi}$, the invariance of k by the diagonal action of translations gives:

$$\begin{aligned} \sum_{y' \in f^{-1}(\mathrm{e}^{iy'_0})} p_{M'}(t, 2\pi, y') &= \sum_{p \in \mathbb{Z}+} \big(k(t, 2\pi, y'_0 + 2p\pi) - k(t, -2\pi, y'_0 + 2p\pi)\big) \\ &= k(t, 2\pi, y'_0) + k(t, 0, y'_0) \\ &< \sum_{p \in \mathbb{Z}} k(t, 2p\pi, y'_0) = p_M\big(t, f(2\pi), \mathrm{e}^{iy'_0}\big). \end{aligned}$$

This example generalizes to any covering $f: M'' \to M$, when we consider the excision $M' = M'' \setminus D$ of a closed domain D with non trivial capacity from M'' and the local isometry $f: M' \to M$. In fact, $p_{M'}(t, x'_0, y') < p_{M''}(t, x'_0, y')$ implies that

$$\sum_{y' \in f^{-1}(y) \setminus D} p_{M'}(t, x'_0, y') < \sum_{y' \in f^{-1}(y)} p_{M''}(t, x'_0, y') = p_M\big(t, f(x'_0), y\big),$$

the last equality being proved in the following Proposition 2.12.

d) The equality case.

In order to characterize the cases where the inequality of Proposition 2.4 is an equality, let us study the very important notion of

DEFINITION 2.8 (stochastic completeness). — *A Riemannian manifold (X, g) will be called stochastically complete if and only if $\int_X p_X(t, x, y)\, \mathrm{d}v_g(y) = 1$ for any $x \in X$ and for any $t > 0$.*

A compact Riemannian manifold is always stochastically complete. Many authors have given sufficient geometric conditions for *complete* non compact Riemannian manifolds to be stochastically complete. For instance, M.P. Gaffney [16] proved that a sufficient condition is that the volume of the geodesic balls has subexponential growth. S.T. Yau proved that a sufficient condition is that the Ricci curvature is bounded from below (*cf.* [31]; see also [9, p. 191, thm. 5]); notice that this condition is automatically satisfied by any geodesically complete Riemannian manifold (M', g') which admits a local isometry onto a compact manifold (M, g). K. Ichihara [24] and P. Hsu [23] extended this result to the case where the Ricci curvature is allowed to go to $-\infty$ at infinity, provided that this growth is controlled in terms of the distance function. L. Karp and P. Li [25] proved that any geodesically complete manifold, whose geodesic balls $B(x_0, R)$ satisfy $\mathrm{Vol}(B(x_0, R)) \leq \mathrm{e}^{cR^2}$ when $R \to +\infty$ (for some point x_0 and some constant c), is stochastically complete — see also E.B. Davies [14] and M. Takeda [29] for different proofs. A. Grigor'yan (*cf.* [20], see [21, thm 9.1 and Section 6 for more detailed proofs]) gave the following sufficient condition which contains the above ones:

THEOREM 2.9. — *Any Riemannian manifold (X,g), which is a complete metric space with respect to d_g and whose geodesic balls $B(x_0,R)$ centered at same point x_0 satisfy the condition*

$$\int^{\infty} \frac{R\,\mathrm{d}R}{\log\big[\mathrm{Vol}(B(x_0,R))\big]} = +\infty \tag{2.10}$$

is stochastically complete.

Let us denote by $r_{\min}(x)$ the infimum of the Ricci curvature at the point $x \in X$, *i.e.*

$$r_{\min}(x) = \inf\Big\{\frac{\mathrm{Ric}(u,u)}{g(u,u)} : u \in T_xX \setminus \{0\}\Big\},$$

and let $(r_{\min})_-(x) = \sup(0, -r_{\min}(x))$.

PROPOSITION 2.11. — *Let (X,g) be any geodesically complete Riemannian manifold. Let us suppose that there exists same point $x_0 \in X$ and some $p > \dim(X)$ such that*

$$\limsup_{R\to+\infty}\Big[\frac{1}{R^2}\log\Big(1 + \int_{B(x_0,R)} (r_{\min})_-^{\frac{1}{2}p}(x)\,\mathrm{d}v_g(x)\Big)\Big] < +\infty,$$

then (X,g) is stochastically complete.

Proof. — For a fixed positive ε less than the injectivity radius of X at x_0, let us consider the geodesic ball $B(x_0, \varepsilon)$ and the sphere $\partial B(x_0, \varepsilon)$ centered at x_0

and of radius ε. We shall call "normal coordinates system" the mapping Ψ from $(-\infty,+\infty)\times\partial B(x_0,\varepsilon)$ onto X defined by $\Psi(t,x)=\exp_x(tN_x)$, where N_x is the outward unit normal vector at $x\in\partial B(x_0,\varepsilon)$ (Ψ is a diffeomorphism from some open subset in $(-\infty,+\infty)\times\partial B(x_0,\varepsilon)$ onto an open subset in X whose complementary is of measure zero). The density of the Riemannian measure v_g with respect to the product measure in normal coordinates is defined by $\Psi^*\,\mathrm{d}v_g = J(t,x)^{n-1}\,\mathrm{d}t\,\mathrm{d}x$. For any $x\in\partial B(x_0,\varepsilon)$, let $(t_-(x),t_+(x))$ be the greatest interval on which the geodesic $t\mapsto\Psi(t,x)$ minimizes the distance from $\Psi(t,x)$ to $\partial B(x_0,\varepsilon)$, and let us define $J_+(t,x)$ equal to $J(t,x)$ if $t\in(t_-(x),t_+(x))$ and equal to zero elsewhere. Then, the $(n-1)$-dimensional volume of the sphere $\partial B(x_0,R)$ is given by

$$L(R)=\mathrm{Vol}_{n-1}\big(\partial B(x_0,R)\big)=\int_{\partial B(x_0,\varepsilon)}J_+(R,x)^{n-1}\,\mathrm{d}x.$$

Let us denote by $L'(R)$ the supremum limit, when $h>0$ goes to zero, of $(L(R+h)-L(R))/h$. It is proved in [18, p. 198] that, for $R>\varepsilon$,

$$L'(R)\le L(R)^{\frac{p-2}{p-1}}\Big[(n-1)^{p-1}\int_{\partial B(x_0,\varepsilon)}\eta_+(x)^{p-1}\,\mathrm{d}x + C(p)\int_{B(x_0,R+\varepsilon)\setminus B(x_0,\varepsilon)}(r_{\min})_-^{\frac{p}{2}}(x)\,\mathrm{d}v_g(x)\Big]^{\frac{1}{p-1}},$$

where η is the mean curvature of the geodesic sphere $\partial B(x_0,\varepsilon)$, where $\eta_+(x)=\sup(0,\eta(x))$, and where $C(p)$ is an explicit constant only depending on p (and on $n=\dim(X)$).

By assumption, there exist R_0 and a constant K such that

$$1+\int_{B(x_0,R)}(r_{\min})_-^{\frac{p}{2}}(x)\,\mathrm{d}v_g(x)\le \mathrm{e}^{KR^2}$$

for $R\ge R_0$, and the above inequality becomes:

$$\frac{L'(R)}{L(R)^{\frac{p-2}{p-1}}}\le C'\cdot\mathrm{e}^{\frac{KR^2}{p-1}}.$$

Integrating two times, we get, for R great enough and with suitable constants (see [18, p. 198]):

$$L(R)^{\frac{1}{p-1}}\le a+b\cdot\mathrm{e}^{\frac{KR^2}{p-1}}\le b\cdot\mathrm{e}^{\frac{K'R^2}{p-1}},$$

$$\mathrm{Vol}\big(B(x_0,R)\big)\le a'+b'\cdot\mathrm{e}^{K'R^2}\le b'\cdot\mathrm{e}^{K''R^2}.$$

Thus $R^{-1}\log[\mathrm{Vol}(B(x_0,R))]\le a''/R+K''R$ and

$$\int_{R_0}^{\infty}\frac{R\,\mathrm{d}R}{\log\big[\mathrm{Vol}(B(x_0,R))\big]}=+\infty.$$

By Grigor'yan theorem 2.9, this implies that (X,g) is stochastically complete. ∎

PROPOSITION 2.12. — *Let $f\colon(M',g')\to(M,g)$ be a local isometry between boundaryless manifolds.*

(i) *If (M',g') is stochastically complete, then (M,g) is also stochastically complete when f is surjective.*

(ii) *If (M,g) is assumed to be stochastically complete, then the equality*

$$\sum_{y'\in f^{-1}(y)} p_{M'}(t,x',y')=p_M\big(t,f(x'),y\big) \tag{2.13}$$

holds (for any positive t, for any $x'\in M'$ and for almost every $y\in M$) if and only if (M',g') is stochastically complete.

Proof. — By Proposition 2.4 and the coarea formula, we have

$$\begin{aligned} 0&\le\int_M\Big[p_M\big(t,f(x'),y\big)-\sum_{y'\in f^{-1}(y)}p_{M'}(t,x',y')\Big]\,\mathrm{d}v_g(y)\\ &=\int_M p_M\big(t,f(x'),y\big)\,\mathrm{d}v_g(y)-\int_{M'}p_{M'}(t,x',y')\,\mathrm{d}v_{g'}(y'). \end{aligned} \tag{2.14}$$

Stochastic completeness of (M',g') implies $\int_{M'}p_{M'}(t,x',y')\,\mathrm{d}v_{g'}(y')=1$ and thus $\int_M p_M(t,f(x'),y)\,\mathrm{d}v_g(y)\ge 1$. But property $\int_M p_M(t,f(x'),y)\,\mathrm{d}v_g(y)\le 1$ is always valid, hence $\int_M p_M(t,f(x'),y)\,\mathrm{d}v_g(y)=1$ and (M,g) is stochastically complete.

If (M',g') and (M,g) are stochastically complete, the above inequality (2.14) is an equality and this implies the equality (2.13). On the other hand, if (M,g) is stochastically complete and if the equality (2.13) holds, then the above inequality (2.14) is an equality; this implies that

$$1=\int_M p_M\big(t,f(x'),y\big)\,\mathrm{d}v_g(y)=\int_{M'}p_{M'}(t,x',y')\,\mathrm{d}v_{g'}(y')$$

and (M',g') is stochastically complete. ∎

e) The case with boundary.

Let us consider the case where $(M',\partial M',g')$ and $(M,\partial M,g)$ are Riemannian manifolds with boundaries. Let us endow these manifolds with the Riemannian distances $d_{g'}$ and d_g (*i.e.* $d_{g'}(x',y')$ is the infimum of the g'-length of curves joining x' and y'). We want to compare the heat kernels of M' and M (with Dirichlet boundary condition), denoted $p^D_{M'}$ and p^D_M.

Proposition 2.4 already provides a sharp lower bound of p^D_M in terms of sums of $p^D_{M'}$. For the equality case, we get the following sufficient condition:

PROPOSITION 2.15. — *Let $f:(M',g') \to (M,g)$ be a surjective local isometry which maps $\partial M'$ onto ∂M (and $\mathrm{Int}(M')$ onto $\mathrm{Int}(M)$ by assumption). Let us suppose that $(M',d_{g'})$ is a complete metric space, and that the volume of balls in $(M',d_{g'})$ satisfies*

$$\int^{\infty} \frac{R\,\mathrm{d}R}{\log\big[\mathrm{Vol}(B(x'_0,R))\big]} = +\infty$$

for some point $x'_0 \in M'$. Then, for any positive t, for any $x' \in M'$ and for almost every $y \in M$, one has

$$\sum_{y'\in f^{-1}(y)} p^D_{M'}(t,x',y') = p^D_M\big(t,f(x'),y\big).$$

Proof. — As $\partial M'$ is regular, the *double* $M' \# M'$ of the manifold $M' = \mathrm{Int}(M') \cup \partial M'$ is the boundaryless manifold which is obtained by quotienting the disjoint union of two samples M'_1 and M'_2 of M' by the relation

$$\psi'_1(x') \sim \psi'_2(x') \text{ if and only if } x' \in \partial M',$$

where ψ'_i denotes the canonical mapping from M' onto M'_i, $i = 1,2$. Then $M' \# M'$ has a structure of C^∞-manifold and the metrics g'_i (images of g' by ψ'_i) give a $C^{0,1}$ Riemannian metric h' on $M' \# M'$ (which is C^∞ on the interior of each M'_i). Then $(M'\#M', d_{h'})$ is complete, where $d_{h'}$ is the Riemannian distance induced by h'. To any curve from $\psi'_1(x')$ to $\psi'_2(y')$ one associates a curve from $\psi'_1(x')$ to $\psi'_1(y')$ lying in M'_1 and with the same length (just map any portion of the curve which lies in M'_2 into M'_1 by the isometry $\psi'_1 \circ (\psi'_2)^{-1}$). This implies that

$$d_{h'}\big(\psi'_1(x'), \psi'_2(y')\big) \geq d_{h'}\big(\psi'_1(x'), \psi'_1(y')\big) = d_{g'}(x',y').$$

Thus, if we denote by $B_{h'}(R)$ and $B_{g'}(R)$ the balls of $(M' \# M', d_{h'})$ and $(M', d_{g'})$ of radius R and centered at $\psi'_1(x'_0)$ and x'_0 respectively, we deduce that $B_{h'}(R) \subset \psi'_1(B_{g'}(R)) \cup \psi'_2(B_{g'}(R))$ and that $\mathrm{Vol}(B_{h'}(R)) \leq 2\,\mathrm{Vol}(B_{g'}(R))$. We thus get

$$\int^{\infty} \frac{R\,\mathrm{d}R}{\log\big[\mathrm{Vol}(B_{h'}(R))\big]} = +\infty.$$

By Grigor'yan theorem 2.9, this implies that $(M' \# M', h')$ is stochastically complete.

In fact, let us notice that Grigor'yan theorem is still valid for C^∞ manifolds endowed with a $C^{0,1}$ metric whose singular set is a smooth $(n-1)$-dimensional submanifold. Let us underline the reason why the proof, given in [20] and [21] section 9, still works in the case $(M' \# M', h')$: in this proof, the only argument which involves the smoothness of the metric is the Green's formula which writes, for any smooth functions u, φ on $M' \# M'$ such that u is compactly supported:

$$(2.16)\quad \int_{M'\#M'} \varphi \cdot \Delta_{h'} u = \int_{M'\#M'} h'(\nabla u, \nabla\varphi) = \int_{M'_1} \varphi \cdot \Delta_{h'} u + \int_{M'_2} \varphi \cdot \Delta_{h'} u$$
$$= \int_{M'} (\varphi \circ \psi'_1) \cdot \Delta_{M'}(u \circ \psi'_1) + \int_{M'} (\varphi \circ \psi'_2) \cdot \Delta_{M'}(u \circ \psi'_2)$$

where $\Delta_{h'}$ and $\Delta_{M'}$ are the Laplacians of $(M'\#M', h')$ and (M', g') respectively. In fact, the two first equalities come from the fact that the integrals on $\partial M'_1$ and on $\partial M'_2$ of $(\varphi \cdot \partial u/\partial\nu)$ give zero because $\partial M'_1 = \partial M'_2$ and because the exterior normal $\nu^1_{\rm ext}$ to $\partial M'_1$ is identified with the inner normal $\nu^2_{\rm int}$ to $\partial M'_2$, and thus $du(\nu^1_{\rm ext}) - du(\nu^2_{int}) = 0$ on $\partial M'_1 = \partial M'_2$. The last equality comes from the fact that ψ'_1 and ψ'_2 are isometries.

Let us now denote by σ the involutive isometry of $(M' \# M', h')$ which coincides with $\psi'_2 \circ (\psi'_1)^{-1}$ on M'_1 and with $\psi'_1 \circ (\psi'_2)^{-1}$ on M'_2. As every eigenspace of $\Delta_{h'}$ splits in two subspaces: the spaces of odd (resp. even) functions with respect to σ, and as $(\Delta_{h'} u) \circ \psi'_i = \Delta_{M'}(u \circ \psi'_i)$, we get the following simple expression for $p^D_{M'}$ (*cf.* [17, p. 48]):

$$(2.17)\quad p^D_{M'}(t, x', y') = p_{M'\#M'}\big(t, \psi'_1(x'), \psi'_1(y')\big) - p_{M'\#M'}\big(t, \psi'_1(x'), \psi'_2(y')\big).$$

We may now buid the double $(M \# M, h)$ of (M, g) and a local isometry $F\colon (M' \# M', h') \to (M \# M, h)$ by setting $F[\psi'_i(x')] = \psi_i[f(x')]$, where ψ_i is the identification of M with one of the samples M_i of M in $M \# M$. As f maps $\partial M'$ onto ∂M, F is a well defined local isometry; we may then apply Proposition 2.12 (notice that its proof, as the one of Proposition 2.4, is still valid in this case), which gives, for $i = 1, 2$:

$$\sum_{y' \in f^{-1}(y)} p_{M'\#M'}\big(t, \psi'_1(x'), \psi'_i(y')\big) = p_{M\#M}\big(t, \psi_1(f(x')), \psi_i(y)\big).$$

Plugging this equality in (2.17), we get

$$\sum_{y' \in f^{-1}(y)} p^D_{M'}(t, x', y') = p_{M\#M}\big(t, \psi_1(f(x')), \psi_1(y)\big) - p_{M\#M}\big(t, \psi_1(f(x')), \psi_2(y)\big) = p^D_M\big(t, (f(x')), y\big),$$

the last equality being obtained by applying (2.17) to $M \# M$. □

f) An application to regular or singular coverings with finite fiber.

Propositions 2.12 and 2.15 gives a first improvement of the Kato's inequality (1.4) for any Riemannian covering with finite fibers. Namely, we have:

COROLLARY 2.18. — *Let $f: M' \to M$ be a regular ℓ-sheeted Riemannian covering between compact boundaryless manifolds. Then we have for any positive t:*

$$Z_{M'}(t) + \int_{M'} \sum_{\substack{y' \in f^{-1}(f(x')) \\ y' \neq x'}} p_{M'}(t, x', y')\, \mathrm{d}v_{g'}(x') = \ell \cdot Z_M(t) \tag{2.19}$$

which implies the classical Kato's inequality (1.4). *Equality case in* (1.4) *occurs if and only if the covering is trivial, i.e. if and only if M' is the disjoint union of ℓ copies of M.*

The result is also valid, mutatis mutandis, for the Dirichlet heat kernels of compact manifolds with boundary, if we assume that f maps $\partial M'$ onto ∂M.

Proof. — Assume that the manifolds are boundaryless: by setting $f(x') = y$ in formula (2.13), an integration on M' gives (2.19), which implies Kato's inequality by the positivity of the heat kernel. If Kato's inequality is an equality for at least one positive t, then (2.19) implies $p_{M'}(t, x', y') \equiv 0$ for any $x' \neq y'$ in $F_{f(x')}$, thus x' and y' lie in two different connected components of M'. The same argument works in the boundary case. ∎

When the cardinality of the fiber is infinite (for instance when M' is the universal covering of a compact manifold), we cannot consider the trace of the heat operator on M'. Nevertheless the inequality (2.5) gives, by integration, for any regular compact domain $\Omega' \subset M'$:

$$Z^D_{\Omega'}(t) \leq \sup_{x \in f(\Omega')} \big(\sharp(f^{-1}(x) \cap \Omega')\big) Z^D_{f(\Omega')}(t) \leq \sup_{x \in f(\Omega')} \big(\sharp(f^{-1}(x) \cap \Omega')\big) Z_M(t).$$

An easy application of the minmax principle allows to extend the result of Corollary 2.18 to *local almost-isometries*, where we call a C^1-mapping $f: (M', g') \to (M, g)$ a local almost-isometry if and only if there exist two positive constants a and b such that

$$a^2 g' \leq f^* g \leq b^2 g'.$$

PROPOSITION 2.20. — *Let $f: (M', g') \to (M, g)$ be a local almost-isometry between compact manifolds. Then*

$$Z_{(M', g')}(t) \leq \sup_{x \in M} \big(\sharp f^{-1}(x)\big) \cdot Z_{(M,g)}\Big(\frac{a^{n+2}}{b^n} t\Big).$$

Proof. — Integrating inequality (2.5) provides a comparison between the heat kernels of the local isometric manifolds (M', f^*g) and (M, g):

$$\sup_{x\in M}\big(\sharp f^{-1}(x)\big)\cdot Z_{(M,g)}(t)\geq Z_{(M',f^*g)}(t).$$

The minmax principle implies (see for instance [19, cor. 4.63]) that for any i:

$$\frac{a^n}{b^{n+2}}\lambda_i(M',g')\leq\lambda_i(M',f^*g)\leq\frac{b^n}{a^{n+2}}\lambda_i(M',g')$$

and hence that $Z_{(M',f^*g)}(t)\geq Z_{(M',g')}\big((b^n/a^{n+2})\,t\big)$. □

REMARK 2.21. — If M' is connected (and f still C^1), the results 2.18 (resp. 2.20) can be settled under the following *a priori* weaker assumption: there exists some subset A of mesaure zero in M such that the restriction of f to $M'\setminus f^{-1}(A)$ is a regular Riemannian covering (resp. a local almost-isometry). In fact, f is then automatically a Riemannian covering (resp. a local almost-isometry) on the whole of M' (if not, the complement of the union of the subsets $\{y\in M'\mid a^2g'_y\leq(f^*g)_y\leq b^2g'_y\}$ and $\{y\in M'\mid(\operatorname{Jac}f)_y=0\}$ is, by connexity, a non empty open subset in $f^{-1}(A)$ which maps onto an open subset in A, a contradiction).

g) Application to the exponential map.

A typical example of local almost-isometry is the exponential map

$$\exp_{x_0}: T_{x_0}M\longrightarrow M,$$

defined on an open subset V of $T_{x_0}M$ onto the complementary of the conjugate locus of M (for instance, if (M,g) has negative curvature, the conjugate locus is empty and V coincides with the whole of $T_{x_0}M$). When V is endowed with the pull-back metric, the exponential map satisfies the conditions of Lemma 2.1 and of Proposition 2.4 and *not* the ones of Lemma 2.2 and of Proposition 2.12, thus we get only the inequality case for the comparison between the heat kernels of M and V, and not the equality one. We then obtain the following improvement of Proposition 5.9 of [7]:

PROPOSITION 2.22. — *Let (M,g) be any complete Riemannian manifold without boundary whose sectional curvature lies in the interval $[-K_{\min}^2,K_{\max}^2]$, and let us consider the exponential mapping $\exp_{x_0}: B_R\to M$ where B_R is the euclidean ball of radius $R<\pi/K_{\max}$ ($R<+\infty$ if $K_{\max}=0$). Then we have*

$$\sup_{x\in M}\big(\sharp(\exp_{x_0}^{-1}(x)\cap B_R)\big)\cdot Z_M(t)\geq Z^D_{B_R}\Big(\frac{K_{\max}^{n+2}}{K_{\min}^n}\cdot\frac{\operatorname{sh}^n(K_{\min}R)}{\sin^{n+2}(K_{\max}R)}\cdot R^2t\Big).$$

Proof. — If we assume that the sectional curvature of M lies in the interval $[-K^2_{\min}, K^2_{\max}]$, then Rauch's comparison theorem gives, at points $v \in V$:

$$\frac{\sin^2(K_{\max}\|v\|)}{K^2_{\max}\cdot\|v\|^2}g_0 \le \exp^*_{x_0} g \le \frac{\operatorname{sh}^2(K_{\min}\|v\|)}{K^2_{\min}\cdot\|v\|^2}g_0$$

where g_0 is the euclidean metric g_{x_0} of $T_{x_0}M$. This proves that $\exp_{x_0}$, restricted to B_R, is a local almost-isometry as defined before Proposition 2.20. Applying Proposition 2.20, we finish the proof. □

h) An improvement of Kato's inequality between heat kernels.

Let (M,g) be any Riemannian manifold of dimension n, and let (M_K, g_K) be the n-dimensional simply connected Riemannian manifold of constant sectional curvature K (briefly the K space-form). It is well known that, for any $x_0 \in M_K$, the heat kernel $p_{M_K}(t, x_0, \cdot)$ is invariant by the isotropy group of x_0 and thus it only depends on t and on the distance from x_0:

$$p_{M_K}(t, x_0, \cdot) = \varphi\big(t; d_K(x_0, \cdot)\big) = \varphi_t \circ \rho_K(\cdot) \tag{2.23}$$

where d_K denotes the spherical, euclidean or hyperbolic distance according that K is positive, null or negative respectively (by homotheties, we may normalize the metric g_K on M_K in order that $K = +1, 0, -1$), and where $\rho_K(\cdot) = d_K(x_0, \cdot)$.

Let us denote by $p^N_{\rho_0}(t, x_0, \cdot)$ the heat kernel on the geodesic ball $B_K(x_0, \rho_0)$ of (M_K, g_K) under Neumann boundary conditions. As before, there exists a function $\psi: (0, +\infty) \times [0, \rho_0) \to \mathbb{R}^+$ such that $p^N_{\rho_0}(t, x_0, \cdot) = \psi_t \circ \rho_K(\cdot)$. We define Φ_{ρ_0} by

$$\Phi_{\rho_0}(t, \rho) = \begin{cases} \psi_t(\rho) & \text{if } \rho < \rho_0, \\ \psi_t(\rho_0) & \text{if } \rho \ge \rho_0. \end{cases} \tag{2.24}$$

We shall give below a unified proof of the following theorem, clarifying by the way some distributional tools already present in [12]:

THEOREM 2.25 (see [10], [15], [11], [12]). — *Let (M,g) be any n-dimensional Riemannian boundaryless manifold.*

(i) *If the Ricci curvature of (M,g) is bounded from below by $(n-1)Kg$, then*

$$p_M(t,x,y) \ge \varphi\big(t, d_g(x,y)\big)$$

where φ is the function defined in (2.23);

(ii) *if all sectional curvatures of (M,g) are bounded from above by K, then*

$$p_M(t,x,y) \le \Phi_{\rho_0}\big(t, d_g(x,y)\big)$$

where Φ_{ρ_0} is the function defined in (2.24) *and where $\rho_0 = \operatorname{inj}_g(x)$ if $K \le 0$ and $\rho_0 = \min(\operatorname{inj}_g(x), \pi/\sqrt{K})$ if $K > 0$.*

If the inequalities in (i) *and* (ii) *are equalities, then* (M,g) *is isometric to* (M_K,g_K).

Proof of (i). — We shall show that $(\Delta_M + \partial/\partial t)\varphi \leq 0$ in the sense of distibutions, and that, as t goes to 0^+, $\varphi(t, d_g(x,\cdot))$ goes to the Dirac measure $\delta_x(\cdot)$: since p_M is a solution of the heat equation on M, the maximum principle gives (i) and the characterization of the equality case.

To show the inequality above, let x be any fixed point in M, and let us set $\rho = d_g(x,\cdot)$ and $\varphi_t = \varphi(t,\cdot)$, so that $\varphi_t'(\cdot) = \partial\varphi/\partial r\,(t,\cdot)$. A simple calculation gives

$$\begin{aligned}\Delta_M(\varphi_t \circ \rho) &= -\varphi_t'' \circ \rho + (\varphi_t' \circ \rho)\Delta_M \rho \\ &= -\varphi_t'' \circ \rho + (\varphi_t' \circ \rho)(\Delta_M \rho)_{\mathrm{reg}} + (\varphi_t' \circ \rho)(\Delta_M \rho)_{\mathrm{sing}}.\end{aligned}$$

$(\Delta_M\rho)_{\mathrm{reg}}$ is the regular part of $\Delta_M\rho$, which is defined on the complementary of $\mathrm{cut}(x)$, the cut locus of x (remember that $\mathrm{cut}(x)$ is a closed set of measure zero). It is classical (see for instance [17, p. 40]) that

$$(\Delta_M\rho)_{\mathrm{reg}} = -\frac{1}{a}\frac{\partial a}{\partial r},$$

where a is the density of the Riemannian measure v_g with respect to the product measure in polar coordinates around x, *i.e.* (*cf.* the proof of Proposition 2.11) $\Psi^* \mathrm{d}v_g = a(r,X)\,\mathrm{d}r\,\mathrm{d}X$, where $\Psi(r,X) = \exp_x(rX)$ is a diffeomorphism from an open subset U in $(0,+\infty) \times S_x$ onto $M \setminus (\{x\} \cup \mathrm{cut}(x))$ (S_x is the unit sphere of T_xM). In the case of the space-form (M_K, g_K), let us define the open subset U_K, the diffeomorphism Ψ_K from U_K onto $M_K \setminus (\{x_0\} \cup \mathrm{cut}(x_0))$ and the function a_K (instead of a) in the same way; notice the existence of a function $b_K : \mathbb{R}^+ \to \mathbb{R}^+$ such that $a_K = b_K \circ \rho_K$. As $a = 0$ at focal points of x, $(\Delta_M\rho)_{\mathrm{reg}}$ is not bounded, but it lies in $L^1_{\mathrm{loc}}(M)$ (see [26]). The singular part $(\Delta_M\rho)_{\mathrm{sing}}$ is a positive distribution which is supported on $\mathrm{cut}(x)$ (see for instance [26]). The derivative $\varphi_t' \circ \rho$ has no singular part because it is C^1-piecewise and hence in $L^1_{\mathrm{loc}}(M)$ (φ_t is smooth and ρ is C^1-piecewise). As $\varphi_t' < 0$ (see [9, p. 192]), $(\Delta_M + \partial/\partial t)(\varphi_t \circ \rho)$ is not greater than its regular part, which is

$$\Big[\Big(\Delta_M + \frac{\partial}{\partial t}\Big)(\varphi_t \circ \rho)\Big]_{\mathrm{reg}} = -\varphi_t'' \circ \rho - \frac{1}{a}\frac{\partial a}{\partial r}\varphi_t' \circ \rho + \frac{\partial \varphi_t}{\partial t} \circ \rho.$$

Since $(a^{-1}\partial a/\partial r)(r,X) \leq b_K'/b_K\,(r)$ by Bishop's comparison theorem (see [6, pp. 256–257]), denoting

$$L(\varphi_t) = -\varphi_t'' - \frac{b_K'}{b_K}\cdot\varphi_t' + \frac{\partial\varphi_t}{\partial t},$$

we get $(\Delta_M + \partial/\partial t)(\varphi_t \circ \rho) \leq L(\varphi_t) \circ \rho$. On the other side, we have $0 = (\Delta_{M_K} + \partial/\partial t)(\varphi_t \circ \rho_K) = L(\varphi_t) \circ \rho_K$. This ends the proof when $K \leq 0$, for ρ_K is then defined on $[0, +\infty)$. When $K > 0$, ρ_K is defined on $[0, \pi/\sqrt{K})$, which, by Myers' theorem, contains the interval of definition of ρ. Thus we get $(\Delta_M + \partial/\partial t)(\varphi_t \circ \rho) \leq 0$ in all cases.

It remains to show that $\lim_{t\to 0^+} \varphi_t \circ \rho(\cdot) = \delta_x(\cdot)$ in the sense of distributions. For any test-function h we have indeed, identifying isometrically $T_x M$ with $T_{x_0} M_K$,

$$\begin{aligned}\langle \varphi_t \circ \rho, h\rangle &= \int_{\Psi(U)} \varphi_t \circ \rho(y) h(y)\, \mathrm{d}v_g(y) \\ &= \int_{\Psi_K(U)} (\varphi_t \circ \rho \circ \Psi \circ \Psi_K^{-1})(z)(h \circ \Psi \circ \Psi_K^{-1})(z) \frac{a}{a_K} \big(\Psi_K^{-1}(z)\big)\, \mathrm{d}v_{g_K}(z) \\ &= \int_{\Psi_K(U)} \varphi_t \circ \rho_K(z)(h \circ \Psi \circ \Psi_K^{-1})(z) \frac{a}{a_K} \big(\Psi_K^{-1}(z)\big)\, \mathrm{d}v_{g_K}(z)\end{aligned}$$

which goes to $h \circ \Psi \circ \Psi_K^{-1}(x_0) = h(x)$ as t goes to 0^+ because $a(r, X) \sim a_K(r, X) \sim r^{n-1}$ when r goes to zero. □

Proof of (ii). — This proof is similar to the one of (i). For the sake of simplicity, let us denote $\Phi = \Phi_{\rho_0}$. We shall show that $(\Delta_M + \partial/\partial t)\Phi$ is a non-negative distribution. As $\lim_{t\to 0^+} \Phi(t, d_g(x, \cdot)) = \delta_x(\cdot)$ (same proof as in (i)), the claim follows by maximum principle.

The radial function $\Phi_t \circ \rho$ is regular inside $B_g(x, \rho_0)$ and constant outside $B_g(x, \rho_0)$, thus its differential is bounded and the singular Laplacian is a measure supported by $\partial B_g(x, \rho_0)$, whose sign is given by the gap in the radial derivative. But this gap does not exist, indeed $\lim_{\varepsilon\to 0} \Phi_t'(\rho_0 + \varepsilon) = 0$ because $\Phi_t(\rho)$ is constant for $\rho \geq \rho_0$, and $\lim_{\varepsilon\to 0} \Phi_t'(\rho_0 - \varepsilon) = 0$ because $p_{\rho_0}^N$ verifies the Neumann boundary condition. Therefore $[\Delta_M(\Phi_t \circ \rho)]_{\mathrm{sing}} = 0$, *i.e.* $\Delta_M(\Phi_t \circ \rho) \in L^1_{\mathrm{loc}}(M)$. The same computations of the regular Laplacian done in the proof of (i) then gives:

$$\Delta_M(\Phi_t \circ \rho) = \big[\Delta_M(\Phi_t \circ \rho)\big]_{\mathrm{reg}} = \begin{cases} -\Phi_t'' \circ \rho - \dfrac{1}{a}\dfrac{\partial a}{\partial r}(\Phi_t' \circ \rho) & \text{if } \rho < \rho_0, \\ 0 & \text{if } \rho \geq \rho_0. \end{cases}$$

Use now the Rauch's comparison theorem instead of the Bishop's one: from $(a^{-1}\partial a/\partial r)(r, X) \geq b_K'/b_K(r)$ we obtain $(\Delta_M + \partial/\partial t)(\Phi_t \circ \rho) \geq 0$ if $\rho < \rho_0$.

It remains to show that

$$\Big(\Delta_M + \frac{\partial}{\partial t})(\Phi_t \circ \rho\Big) = \frac{\partial \Phi_t}{\partial t} \circ \rho = \frac{\partial \psi_t}{\partial t}(\rho_0) \geq 0$$

for $\rho \geq \rho_0$. The fact that $\psi_t \circ \rho$ satisfies the heat equation on $[0,\rho_0]$ implies

$$\frac{\partial \psi_t}{\partial t}(\rho_0) = \psi_t''(\rho_0) + \frac{b_K'}{b_K} \cdot \psi_t'(\rho_0) = \psi_t''(\rho_0)$$

because the Neumann condition writes $\psi_t'(\rho_0) = 0$. As $\psi_t'(\rho) \leq 0$ on $[0,\rho_0]$, it comes that $\partial\psi_t/\partial t\,(\rho_0) \geq 0$ (*i.e.* the point where the temperature $\psi_t(r)$ attains its minimum has a not decreasing temperature with respect to the time). ▯

Let us now consider any local isometry $f\colon(M',g') \to (M,g)$. Let ρ_0 be the injectivity radius $\mathrm{inj}(M')$ of (M',g') when $K \leq 0$ and $\rho_0 = \min(\mathrm{inj}(M'), \pi/\sqrt{K})$ when $K > 0$. We just apply the results of Lemma 2.2, of Propositions 2.4 or 2.12 and 2.15 and of Theorem 2.25 written for (M',g') and the properties of φ and Φ to obtain

COROLLARY 2.26. — *Let $f\colon(M',g') \to (M,g)$ be a local isometry of n-dimensional manifolds. For any $x \in M$ and for any fixed $x' \in F_x = f^{-1}(x)$:*

(i) *if* $\mathrm{Ricci}(M,g) \geq (n-1)Kg$, *then*

$$p_{M'}(t,x',x') + \sum_{y' \in F_x \setminus \{x'\}} \varphi\big(t, d_{g'}(x',y')\big) \leq p_M(t,x,x);$$

moreover, if the volume of M is finite, we have

$$\int_{U'} p_{M'}(t,x',x')\,\mathrm{d}v_{g'}(x') + \varphi\big(t, \sup_{x\in M} d_{\min}(x)\big)\cdot \mathrm{Vol}(M) \leq \int_M p_M(t,x,x)\,\mathrm{d}v_g(x)$$

where $U' \subset M'$ is a fundamental domain of f, and where $d_{\min}(x)$ is the infimum of $d_{g'}(x',y')$ for $x', y' \in F_x$, $x' \neq y'$;

(ii) *if f is surjective and maps $\partial M'$ onto ∂M, if $(M',d_{g'})$ is complete, and if the sectional curvatures of (M,g) are bounded from above by K, then, for any $x' \in f^{-1}(x)$,*

$$p_M(t,x,x) \leq p_{M'}(t,x',x') + \sum_{y' \in F_x \setminus \{x'\}} \Phi_{\rho_0}\big(t, d_{g'}(x',y')\big);$$

moreover, if the volume of M is finite, we have

$$\int_M p_M(t,x,x)\,dv_g(x) \leq \int_{U'} p_{M'}(t,x',x')\,\mathrm{d}v_{g'}(x') + \int_M (\sharp F_x - 1)\cdot \Phi_{\rho_0}\big(t, d_{\min}(x)\big)\,\mathrm{d}v_g(x).$$

These results give an analogous to Kato's inequality in the case where M' is not compact (and even not complete!). Moreover, also in the compact case, this Corollary improves the Kato's inequality in the following way.

COROLLARY 2.27. — *Let $f:(M',g') \to (M,g)$ be a ℓ-sheeted regular Riemannian covering of a n-dimensional compact manifold M:*

(i) *if* $\mathrm{Ricci}(M,g) \geq (n-1)Kg$, *then*

$$Z_{M'}(t) + \ell(\ell-1)\cdot\varphi\big(t,\mathrm{diam}(M')\big)\cdot\mathrm{Vol}(M) \leq \ell\cdot Z_M(t)$$

where $\mathrm{diam}(M')$ *denotes the diameter of* (M',g');

(ii) *if the sectional curvatures of* (M,g) *are bounded from above by* K, *then*

$$\ell\cdot Z_M(t) \leq Z_{M'}(t) + \ell(\ell-1)\cdot\Phi_{\rho_0}\big(t,2\,\mathrm{inj}(M)\big)\cdot\mathrm{Vol}(M)$$

where $\mathrm{inj}(M)$ *is the injectivity radius of* (M,g).

REMARK 2.28. — The sharpness of the above inequalities is proved by considering the 2-sheeted Riemannian covering $f:(S^n,\mathrm{can}) \to (P^n(\mathbb{R}),\mathrm{can})$. The fiber consists at any $x \in P^n(\mathbb{R})$ of two points x' and $-x'$, so in this case $d_{\min}(x) = d_{\max}(x) = \rho_0 = \pi = 2\,\mathrm{inj}(P^n(\mathbb{R}))$ for any x. As $p_{S^n}(t,x',x') + p_{S^n}(t,x',-x') = p_{P^n(\mathbb{R})}(t,x,x)$ and $p_{S^n}(t,x',-x') = \varphi(t,d(x',-x')) = \Phi_{\rho_0}(t,2\,\mathrm{inj}(P^n(\mathbb{R}))) = \varphi(t,\pi)$, we have

$$\begin{aligned} Z_{S^n}(t) - 2Z_{P^n(\mathbb{R})}(t) &= 2\,\mathrm{Vol}(P^n(\mathbb{R}))\cdot\varphi\big(t,\mathrm{diam}(S^n)\big) \\ &= 2\,\mathrm{Vol}\big(P^n(\mathbb{R})\big)\cdot\Phi_{\rho_0}\Big(t,2\,\mathrm{inj}\big(P^n(\mathbb{R})\big)\Big) \end{aligned}$$

and the inequalities of Corollaries 2.26,2.27 are equalities in this case.

3. Riemannian submersions with minimal fibers

We are going now to consider the case in which $f:(M',g') \to (M,g)$ is a Riemannian submersion of compact boundaryless manifolds. A simple calculation (*cf.* [4, p. 277]) gives, for any $u \in C^\infty(M')$:

$$\Delta_M\Big(\int_{F_x} u\Big) = \int_{F_x}\Delta_{M'}u - \int_{F_x}\mathrm{div}_{M'}(uH), \tag{3.1}$$

where H is the mean curvature vector of the fiber $F_x = f^{-1}(x)$. For any fixed positive t and $x' \in M'$, $T_t = \int_{F_y} p_{M'}(t,x',y')\,\mathrm{d}v_{g'_y}(y')$ is a Radon measure on M which verifies, in the sense of distributions:

$$\Big(\Delta_M + \frac{\partial}{\partial t}\Big)T_t = -\int_{F_y}\mathrm{div}_{M'}\big(p_{M'}(t,x',y')H(y')\big)\,\mathrm{d}v_{g'_y}(y') \text{ and } \lim_{t\to0^+} T_t = \delta_{f(x')}.$$

If we assume that all the fibers are minimal submanifolds of M', *i.e.* $H \equiv 0$, (3.1) implies that the Laplacians commute with the integration on the fibers (*cf.* [2, Prop. 3.13]). In this case, for any $x' \in M'$, the integral of $p_{M'}(t,x',\cdot)$ on a fiber F_y does not depend on x' but only on $x = f(x')$ and

$$\int_{F_y} p_{M'}(t,x',y')dv_{g'_y}(y') = p_M\big(t,f(x'),y\big)$$

(see also [4]). It is also easy to verify that $\Delta_{M'}(\varphi \circ f) = (\Delta_M \varphi) \circ f$ and that the functions of the type $\varphi \circ f$ are stable under $\Delta_{M'}$. We deduce that $\mathrm{Spec}(\Delta_M) \subset \mathrm{Spec}(\Delta_{M'})$ and hence that $Z_{M'}(t) \geq Z_M(t)$.

We shall give a domination theorem in the opposite sense for the heat and resolvent operators by the method of symmetrization in the sense of G. Besson [5]. Let us recall this definition.

Definition 3.2. — *Let K^+ be a cone in a real Hilbert space K; the cone is supposed to be self-dual, i.e. every w such that $\langle w, K^+\rangle \geq 0$ lies in K^+. A mapping S from a Hilbert space H to K^+ is a symmetrization if*

(a) $\big|\langle u_1,u_2\rangle\big| \leq \langle Su_1,Su_2\rangle$ *for any $u_1,u_2 \in H$, with equality when $u_1 = u_2$;*

(b) *for any $w \in K^+$ and for any $u_1 \in H$, there exists $u_2 \in H$ such that u_1,u_2 are w-paired, i.e. $Su_2 = w$ and*

$$\langle u_1,u_2\rangle = \langle Su_1,Su_2\rangle = \langle Su_1,w\rangle.$$

As a symmetrization is a Lipschitz map, one can define S on a dense subset of H only, namely, in the present application, S will be defined on $C^\infty(M')$ which is dense in $H = L^2(M')$.

A symmetrization S is said to obey Kato's inequality with respect to two operators A and B acting on H and K respectively if S does not increase energy, i.e. for any $u_1,u_2 \in H$ which are Su_2-paired

$$q_B(Su_1,Su_2) \leq q_A(u_1,u_2)$$

where q_A and q_B denote the quadratic forms associated to A and B respectively.

Under the conditions (a) and (b) above, G. Besson gives in [5] a consequence which can be seen as a generalized Beurling-Deny principle:

Generalized Beurling-Deny Principle 3.3. — *If S is a symmetrization which obeys Kato's inequality, then the operators $(B + \lambda I)^{-1}$ (for any $\lambda > -\lambda_0(B)$) and e^{-tB} dominate the operators $(A + \lambda I)^{-1}$ and e^{-tA} respectively, where we say that an operator L acting on K dominates an operator T acting on H if $L(S(u)) \geq S(T(u))$ for any $u \in H$.*

Let now $f\colon (M',g') \to (M,g)$ be any surjective mapping, which verifies Fubini's property (1.2) (the manifolds are assumed to be connected, compact and without boundary). For any $u\colon M' \to \mathbb{R}$ for which it makes sense, we define $\varpi u\colon M \to \mathbb{R}$ to be the L^2-norm of the restriction of u to the fibers:

$$(\varpi u)(x) = \Big(\int_{F_x} \big(u_{|F_x}(y)\big)^2 \, \mathrm{d}v_{g'_x}(y)\Big)^{\frac{1}{2}} \tag{3.4}$$

(this map was introduced, in a more general context, in [7]). It is easy to see that if u is in $L^2(M')$, then ϖu is in $L^2(M)$ and that ϖ preserves the L^2-norms:

$$\|\varpi u\|_{L^2(M)} = \|u\|_{L^2(M')}. \tag{3.5}$$

PROPOSITION 3.6. — *Let $f\colon (M',g') \to (M,g)$ be a Riemannian submersion with minimal fibers, then the resolvent operator $(\Delta_M+\lambda)^{-1}$ and the heat operator $\mathrm{e}^{-t\Delta_M}$ dominate $(\Delta_{M'}+\lambda)^{-1}$ and $\mathrm{e}^{-t\Delta_{M'}}$ respectively for any positive t and λ.*

A remark similar to 2.21 could be done for Proposition 3.6. The proof of this proposition is a direct consequence of the generalized Beurling-Deny criterion (3.3) and of the following lemmas.

LEMMA 3.7. — *Let $f\colon (M',g') \to (M,g)$ be a surjective mapping of compact boundaryless manifolds, which verifies Fubini's property (1.2). Then the mapping $\varpi\colon L^2(M') \to L^2(M)$ defined by (3.4) is a symmetrization.*

Proof. — K^+ is the self-dual cone of non-negative functions in $L^2(M)$. Condition (a) is a consequence of Cauchy-Schwarz inequality:

$$\begin{aligned}\big|\langle\!\langle u_1,u_2\rangle\!\rangle_{L^2(M')}\big| &= \Big|\int_{M'} u_1(x')u_2(x')\,\mathrm{d}v_{g'}(x')\Big| \\ &\le \int_M \Big|\int_{F_x} u_1(y)u_2(y)dv_{g'_x}(y)\Big|\,\mathrm{d}v_g(x) \le \langle\!\langle \varpi u_1,\varpi u_2\rangle\!\rangle_{L^2(M)}.\end{aligned}$$

To show (b), let $u_1 \in C^\infty(M')$ and $w \in K^+$. We define, for $x' \in M'$ and $x = f(x') \in M$:

$$u_2(x') = \begin{cases} \dfrac{(w\circ f)(x')}{(\varpi u_1 \circ f)(x')}\, u_1(x') & \text{if } \varpi u_1(x) \neq 0, \\ (w\circ f)(x')h(x') & \text{if } \varpi u_1(x) = 0 \end{cases} \tag{3.8}$$

where h is any measurable function on M' such that $\big(\int_{F_x}(h(y))^2\,\mathrm{d}v_{g'_x}(y)\big)^{\frac{1}{2}} = 1$ (for instance $h(x') = (\mathrm{Vol}(F_{f(x')}))^{-\frac{1}{2}}$). The function u_2 verifies $\varpi u_2 = w$ and lies in $L^2(M')$ by (3.4). A direct calculation shows that u_1,u_2 are w-paired. □

The good assumption to ensure that ϖ obeys Kato's inequality is that f is a Riemannian submersion with minimal fibers, namely we have:

LEMMA 3.9. — *Let $f:(M',g') \to (M,g)$ be a Riemannian submersion with minimal fibers. Then the symmetrization ϖ defined by* (3.4) *obeys Kato's inequality with respect to the Laplacians $\Delta_{M'}, \Delta_M$.*

Proof. — Let us consider an orthonormal basis $\{e_i\}$ of differential tangent vector fields on a neighborhood of x in M. Their horizontal lifts $\{e_i^*\}$ on M' are horizontal orthonormal vector fields at each point. As the fibers are minimal submanifolds, the holonomy induces a diffeomorphism from F_x to F_y which preserves the measure and so the horizontal derivatives commute with the integrals on the fibers (see [2, Lemma 3.14]). Let us define, for $u \in C^\infty(M)$:

$$(\varpi u)_\varepsilon(x) = \big(\varpi u(x)^2 + \varepsilon^2\big)^{\frac{1}{2}}.$$

We have for the i-th component of the differential of $(\varpi u)_\varepsilon$, $i = 1, \dots, \dim(M)$:

$$\begin{aligned}\big(d(\varpi u)_\varepsilon(x)\big)^i = e_i(\varpi u)_\varepsilon(x) &= \frac{1}{(\varpi u)_\varepsilon(x)} \int_{F_x} u(y)(e_i^* u)(y)\, \mathrm{d}v_{g'_x}(y) \\ &= \frac{1}{(\varpi u)_\varepsilon(x)} \int_{F_x} u(y)(du(y))^i\, \mathrm{d}v_{g'_x}(y),\end{aligned}$$

and by the Cauchy-Schwarz inequality:

$$\big|d(\varpi u)_\varepsilon(x)\big|^2 \le \Big(\frac{\varpi u(x)}{(\varpi u)_\varepsilon(x)}\Big)^2 \int_{F_x} \big|du(y)\big|^2\, \mathrm{d}v_{g'_x}(y)$$

since in $|du(y)|^2$ one must take into account the derivatives related to vertical vector fields (*i.e.* tangent to the fiber). We obtain, as ε goes to 0, if ϖu is considered as the limit of $(\varpi u)_\varepsilon$ in the sense of distributions:

$$(3.10) \qquad \begin{cases} \big|d(\varpi u)(x)\big|^2 = 0 & \text{if } \varpi u(x) = 0, \\ \big|d(\varpi u)(x)\big|^2 \le \displaystyle\int_{F_x} \big|du(y)\big|^2\, \mathrm{d}v_{g'_x}(y) & \text{if } \varpi u(x) \ne 0 \end{cases}$$

hence ϖu belongs to $H^1(M)$ and satisfies $\|\varpi u\|_{H^1(M)} \le \|u\|_{H^1(M')}$.

Recall that, as the fibers are minimal, the Laplacians commute with the integrals on the fibers, then

$$\begin{aligned}\Delta_M(\varpi u)^2_\varepsilon(x) = \Delta_M \int_{F_x} u(y)^2\, \mathrm{d}v_{g'_x}(y) &= \int_{F_x} \Delta_{M'} u^2(y)\, \mathrm{d}v_{g'_x}(y) \\ &= \int_{F_x} \big(2u(y)\Delta_{M'}u(y) - 2|du(y)|^2\big)\, \mathrm{d}v_{g'_x}(y)\end{aligned}$$

which gives, by comparison with

$$\Delta_M(\varpi u)^2_\varepsilon(x) = 2(\varpi u)_\varepsilon(x)\Delta_M(\varpi u)_\varepsilon(x) - 2\big|d(\varpi u)_\varepsilon(x)\big|^2$$

and by the previous estimate (3.10) of $|d(\varpi u)(x)|^2$:

$$(\varpi u)_\varepsilon(x)\Delta_M(\varpi u)_\varepsilon(x) \leq \int_{F_x} u(y)\Delta_{M'}u(y)\,\mathrm{d}v_{g'_x}(y).$$

Let us now consider the mapping u_2 defined by (3.8) with $u_1 = u$. Then

$$\langle\!\langle \Delta_M(\varpi u)_\varepsilon, \varpi u_2\rangle\!\rangle_{L^2(M)} \leq \int_M \Big(\int_{F_x} u(y)\Delta_{M'}u(y)\,\frac{\varpi u_2 \circ f(y)}{(\varpi u)_\varepsilon \circ f(y)}\,\mathrm{d}v_{g'_x}(y)\Big)\,\mathrm{d}v_g(x)$$

which gives, as ε goes to 0:

$$\langle\!\langle d(\varpi u), d(\varpi u_2)\rangle\!\rangle_{L^2(M)} \leq \langle\!\langle \Delta_{M'}u, u_2\rangle\!\rangle_{L^2(M)} = \langle\!\langle du, du_2\rangle\!\rangle_{L^2(M')}$$

i.e. ϖ obeys Kato's inequality with respect to the Laplacians (use in the computations the fact that, when $\varpi u(x) = 0$, it implies $\int_{F_x} u(y)\,\mathrm{d}v_{g'_x}(y) = 0$ and $\int_{F_x} \Delta_{M'}u(y)\,\mathrm{d}v_{g'_x}(y) = \Delta_M \int_{F_x} u(y)\,\mathrm{d}v_{g'_x}(y)$). □

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