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CANONICAL HOMOTOPY OPERATORS FOR THE $\bar\partial$ COMPLEX IN STRICTLY PSEUDOCONVEX DOMAINS

BY MATS ANDERSSON, JÖRGEN BOO AND JOAQUIM ORTEGA-CERDÀ (*)

ABSTRACT. — In a strictly pseudoconvex domain $D = \{\rho < 0\}$ in $\mathbb{C}^n$, we study the homotopy operators K_α for $\bar\partial$ that are canonical with respect to the metric $(-\rho)i\partial\bar\partial\log(1/-\rho)$ and weights $(-\rho)^\alpha$, $\alpha > 0$, and their relation to the canonical homotopy operator K_b for $\bar\partial_b$ on ∂D. We prove that the boundary values of the kernel of K_α in the ball are provided by well-known integral formulas due to Henkin, Skoda et *al.* We are able to compute the kernel for K_α in the interior of D, by using a technique for representing forms in D by complex tangential forms on the boundary of a higher dimensional domain. This is a generalization of a well-known technique for functions. In the ball we also prove the commutation rule $\partial/\partial z_\ell K_\alpha = K_{\alpha+1}\partial/\partial\zeta_\ell$, which generalizes a well-known fact about the weighted Bergman projections, and use it to construct homotopy formulas for $\partial\bar\partial$ in the ball.

RÉSUMÉ. — OPÉRATEURS D'HOMOTOPIE DANS LES DOMAINES STRICTEMENT PSEUDO-CONVEXES. — Dans un domaine strictement pseudoconvexe $D = \{\rho < 0\}$ dans $\mathbb{C}^n$, on étudie les opérateurs d'homotopie K_α pour les $\bar\partial$ canoniques pour la métrique $(-\rho)i\partial\bar\partial\log(1/-\rho)$ et les poids $(-\rho)^\alpha$, $\alpha > 0$ et leur relation avec l'opérateur canonique d'homotopie K_b pour le $\bar\partial_b$ dans ∂D. On démontre que les valeurs au bord du noyau de K_α pour la boule sont donnés par les formules intégrales de Henkin, Skoda et *al.* On parvient à calculer le noyau de K_α à l'intérieur du domaine D en utilisant une technique pour représenter les formes dans D par des formes tangentielle complexes au bord d'un domaine dans une dimension plus élevée. Il s'agit là d'une généralisation

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d'une technique bien connue dans les cas des fonctions. Dans la boule, on prouve aussi la loi de commutation $\partial/\partial z_\ell K_\alpha = K_{\alpha+1}\partial/\partial\zeta_\ell$, qui généralise un résultat déjà connu des projections de Bergman à poids. On utilise ce fait pour construire des formules d'homotopie pour le $\partial\bar\partial$ dans la boule.

1. Introduction

Estimates of growth and regularity of solutions to the $\bar\partial$-equation in domains in $\mathbb{C}^n$ were obtained by L^2-methods in the 1960's by Kohn, Hörmander et *al.* In the 1970's Henkin, Skoda, and others introduced formulas for representation of solutions that gave further information such as L^p-estimates, Hölder estimates, and so on. These two methods have been living side by side but the interplay between them is not fully understood. In this paper we offer a geometrical interpretation of the well-known Henkin-Skoda solution formula for $\bar\partial$ and its weighted analogues in strictly pseudoconvex domains. This leads to mutual exchange of information between the explicit formulas and L^2-theory. For instance, regularity properties of the abstractly defined operators can be derived from the explicit formulas, and in the other direction, by studying the adjoint of the $\bar\partial$-operator, we can prove a certain commutation property of the kernels and derivatives in the case of the ball, which makes it possible to construct explicit homotopy formulas for $\partial\bar\partial$.

The Henkin-Skoda solution formula provides the L^2-minimal solution for $\bar\partial$ in the ball but it does not coincide as an operator with the Kohn operator, which is the one that gives the minimal solution when applied to a $\bar\partial$ closed form and vanishes on forms that are orthogonal to the kernel of $\bar\partial$, with respect to the Euclidean L^2 norm. In [16], Harvey and Polking found an explicit expression for the Kohn operator, essentially expressed by rational functions, but anyway not as simple as the Henkin-Skoda formula. In order to interpret the Henkin-Skoda formula geometrically we have to introduce norms closely related to the Bergman norm.

Throughout this paper we let D be a smoothly bounded strictly pseudoconvex domain in $\mathbb{C}^n$, and ρ a strictly plurisubharmonic defining function. For $\alpha > 0$, let L^2_α be the space of locally square integrable forms in D such that

$$\|f\|^2_\alpha = \frac{\Gamma(n+\alpha)}{2^n\pi^n\Gamma(\alpha)}\int_D (-\rho)^\alpha |f|^2\,\mathrm{d}V \tag{1.1}$$

is finite, where $|f|$ is the norm of the form f with respect to the metric defined by the form

$$\Omega = (-\rho)\, i\,\partial\bar\partial \log\Big(\frac{1}{-\rho}\Big) \tag{1.2}$$

(in the case of the ball Ω is just distance to the boundary times the Bergman metric) and $\mathrm{d}V = \Omega^n/n!$. The volume form $\mathrm{d}V$ is equivalent to $(1/-\rho)$ times the Lebesgue measure, and therefore $A^2_\alpha = L^2_\alpha \cap \mathcal{O}(D)$ is the usual Bergman space with weight $(-\rho)^{\alpha-1}$. We let K_α be the operator on L^2_α defined so that $K_\alpha f$ is the minimal solution to $\bar\partial u = f$ if f is a $\bar\partial$-closed $(0, q+1)$-form in D (the existence of such a solution is well-known, see Theorem 2.2) and $K_\alpha f = 0$ if f is orthogonal to $\mathcal{K}_\alpha = L^2_\alpha \cap \operatorname{Ker} \bar\partial$. If P_α is the orthogonal projection of functions in L^2_α onto A^2_α, the Bergman projection, then the relation

$$K_\alpha \bar\partial f + \bar\partial K_\alpha f = f - P_\alpha f, \tag{1.3}$$

holds for f in $\operatorname{Dom} \bar\partial$. (It is enough to verify it separately for $f \in \mathcal{K}_\alpha$ and $f \in \mathcal{K}_\alpha^\perp \cap \operatorname{Dom} \bar\partial$, and both these cases follow immediately from the definition.)

These operators are natural to study for several reasons. To begin with, it is well-known that the $\bar\partial$-operator behaves like half a derivative in the complex tangential directions near the boundary of a strictly pseudoconvex domain. This is reflected in the standard estimates for $\bar\partial$. For instance, the well-known Henkin-Skoda estimate, [19] and [21], states that $\bar\partial u = f$ has a solution (f being a $\bar\partial$-closed $(0, q+1)$-form) such that

$$\int_{\partial D} |\bar\partial\rho \wedge u|_E \leq C \int_D (-\rho)^{-1/2} \big[\sqrt{-\rho}\,|f|_E + |\bar\partial\rho \wedge f|_E\big]. \tag{1.4}$$

Here, $|\ |_E$ denotes the Euclidean norm of a form, and since $-\rho$ is approximately the distance to the boundary, $\bar\partial\rho \wedge f$ determines the complex tangential part of f near the boundary. It is well-known that this boundary behaviour of $\bar\partial$ is reflected by the Bergman metric and therefore by Ω as well.

The estimate (1.4) was the first important success for weighted integral formulas; once they are constructed the estimate follows nicely, as the very feature of the formulas reflects this difference in normal and complex tangential part. This suggests that these operators better should be understood in terms of a metric like Ω that takes this difference into account.

One of our main results (Theorem 5.1) is that in the ball case, the boundary values of the operator due to Henkin and Skoda, and its weighted analogues, in fact coincide with (the boundary values of) the canonical operator K_α. Expressed in the inner product $(\ ,\ )_\alpha$ connected to $\|\ \|_\alpha$, the kernel for the boundary values of K_α has the simple expression

(1.5) $k_\alpha(\zeta, z)$

$$= \sum_{q=0}^{n-1} \frac{\Gamma(\alpha+n-q-1)}{\Gamma(n+\alpha)} \frac{\bar{z}\cdot \mathrm{d}\zeta \wedge (\mathrm{d}\bar{z}\cdot \mathrm{d}\zeta)^q}{(1-\bar{\zeta}\cdot z)^{\alpha-1+n-q}(1-\zeta\cdot\bar{z})^{q+1}}, \ z\in\partial\mathbb{B}.$$

In the general strictly pseudoconvex case one cannot hope for completely explicit formulas for K_α. However, it turns out that well-known solution operators approximately provide the boundary values of K_α, in much the same spirit that certain known explicit holomorphic projection operators approximate the Bergman and the Szegö projections in D, as was proved by Kerzman-Stein, [17], see [4] and [5].

The spaces L^2_α and operators K_α have connections to the boundary complex. In Section 2 we notice that $\|f\|_\alpha$ tends to $\|f\|_b$, the L^2-norm of the complex tangential part $f|_b$ of f over ∂D, when $\alpha \to 0$. It is also true that (the boundary values of) K_α tend to K_b, the canonical operator for $\bar\partial_b$. In the ball case this follows from Section 5, whereas the general case is treated in [4], [5].

There is another useful connection to the boundary complex that is exploited in Section 4. It is well-known and used by many authors the fact that if

$$\widetilde{D} = \{(z,w)\in\mathbb{C}^{n+1};\ \rho(z)+|w|^2<0\},$$

then $L^2(D)$ can be identified with the subspace $L^2(\partial\widetilde{D})$ consisting of functions that are rotation invariant in the last variable, and that furthermore, *via* this identification, the orthogonal projection onto the Bergman space A^2 in D, the Bergman projection, corresponds to the Szegö projection on $\partial\widetilde{D}$. We extend this representation to higher order forms, so that forms in D correspond to certain tangential forms on $\partial\widetilde{D}$ and so that the orthogonal projection of forms in L^2_1 onto $\mathcal{K}_1$ corresponds to the orthogonal projection of L^2_b onto $\mathcal{K}_b = L^2_b \cap \operatorname{Ker}\bar\partial_b$ on $\partial\widetilde{D}$. We also show that the canonical operator K_α can be represented by the complex tangential boundary values of the corresponding $\widetilde{K}_{\alpha-1}$ in $\widetilde{D}$. Since well-known formulas in the ball give the boundary values, we therefore get an effective procedure to compute the values of K_α in the interior. This is the done in Section 5. The resulting formulas have not previously occurred in the literature (as far as we know).

A basic ingredient in our proofs is the formula for the formal adjoint $\bar\partial^*_\alpha$, that is computed in Section 3. It is a first order differential operator with coefficients that are smooth up to the boundary. It turns out that any smooth f belongs to the domain $\operatorname{Dom}\bar\partial^*_\alpha$ of the von Neumann adjoint of $\bar\partial$, and for $\alpha\geq 1$, $f\in\operatorname{Dom}\bar\partial^*_\alpha$ if and only if $f, \bar\partial^*_\alpha f\in L^2_\alpha$.

It is known since long ago that in the ball

$$\frac{\partial}{\partial z_j}P_\alpha f = P_{\alpha+1}\Big(\frac{\partial f}{\partial \zeta_j}\Big).$$

We prove (Theorem 6.1) that this formula extends to the operators K_α, acting on higher order forms. This formula is then used to construct homotopy formulas for the $\partial\bar\partial$ operator in the ball.

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2. Boundary behavior of the metric Ω

The pointwise inner product $\langle f,g\rangle$, defined by the form Ω, *cf.* (1.2), degenerates on the boundary of D, and in order to understand its asymptotic behavior, we shall express it in terms of $\beta = i\partial\bar\partial\rho$, which is equivalent to the Euclidean metric since ρ is strictly plurisubharmonic.

LEMMA 2.1. — *If $\langle\ ,\ \rangle_\beta$ denotes the inner product with respect to β, then for $(0,q)$-forms f and g we have*

$$\tag{2.1} \begin{cases} \langle f,g\rangle = \big((-\rho)\langle f,g\rangle_\beta + \langle\bar\partial\rho\wedge f, \bar\partial\rho\wedge g\rangle_\beta\big)/B,\\ (-\rho)\,\mathrm{d}V = B\beta^n/n\,!, \end{cases}$$

where the function $B = -\rho + |\bar\partial\rho|^2_\beta$ is smooth up to the boundary and nonvanishing.

Proof. — Let $\gamma = i\,\partial\rho\wedge\bar\partial\rho$ and $\omega = i\,\partial\bar\partial\log(1/-\rho)$. Choose an orthonormal frame $e_1,\dots,e_n$ with respect to β for the space of $(1,0)$-forms, such that $e_1 = \partial\rho/|\partial\rho|_\beta$. Then $\beta = i\sum_1^n e_j\wedge\bar e_j$, and

$$\tag{2.2} \begin{aligned} \omega = \frac{\beta}{(-\rho)} + \frac{\gamma}{(-\rho)^2} &= i\,\frac{\sum_{j=1}^n e_j\wedge\bar e_j}{(-\rho)} + i\,\frac{\partial\rho\wedge\bar\partial\rho}{(-\rho)^2}\\ &= a\,i\,e_1\wedge\bar e_1 + b\,i\sum_{j=2}^n e_j\wedge\bar e_j, \end{aligned}$$

where $a = B/(-\rho)^2$ and $b = 1/(-\rho)$. Therefore we have that

$$\begin{aligned} (-\rho)\,\mathrm{d}V &= (-\rho)\Omega^n/n\,! = (-\rho)^{n+1}\omega^n/n\,!\\ &= (-\rho)^{n+1}ab^{n-1}\beta^n/n\,! = B\beta^n/n\,!. \end{aligned}$$

The first equality in (2.1) is easily checked for $f = g = \bar e_{I_1}\wedge\dots\wedge\bar e_{I_q}$, and then the general case follows. ∎

Hence $\mathrm{d}V$ is equivalent to the Lebesgue measure divided by the distance to the boundary. If D is the ball and $\rho(z) = |z|^2-1$, then $B = 1$.

THEOREM 2.2. — *Suppose that* $\alpha > 0$. *For any* $\bar\partial$*-closed* $(0, q+1)$*-form* f *in* L^2_{loc} *there is a form* u *such that* $\bar\partial u = f$, *and*

$$\|u\|^2_\alpha \le C\|f\|^2_{\alpha+1}. \tag{2.3}$$

Of course, to define K_α, *cf.* Section 1, we only need the weaker statement that the $\bar\partial$-equation is solvable in L^2_α. Theorem 2.2 is readily proved by integral formulas; see *e.g.* [4], [5]. It is worth to notice that this theorem, however, does not immediately follow from the standard L^2-technique, but it requires an extra argument due to Donelly and Fefferman, [12].

REMARK 1. — From the proof of Lemma 2.1 it follows that if f is a (p, q)-form, then $|f|^2 dV = (-\rho)^{n-p-q}|f|^2_\omega \omega^n/n!$, which implies that

$$\|f\|^2_\alpha = \frac{\Gamma(n+\alpha)}{2^n\pi^n\Gamma(\alpha)}\|f\|^2_{n-p-q+\alpha,\omega}, \tag{2.4}$$

if

$$\|f\|^2_{\ell,\omega} = \int_D (-\rho)^\ell |f|^2_\omega \frac{\omega^n}{n!}.$$

Therefore, (2.3) can be rephrased as $\|u\|_{\ell,\omega} \le C_\ell\|f\|_{\ell,\omega}$ for all $\ell > n-q$.

Recall that a vector at a point $p \in \partial D$ is complex tangential if it is annihilated by both $\mathrm{d}\rho_{|p}$ and $\mathrm{d}^c\rho_{|p}$. If f is any form over ∂D, we denote its restriction to the complex tangential vectors by $f_{|b}$. This restriction is determined by $\mathrm{d}\rho \wedge \mathrm{d}^c\rho \wedge f$, and in particular if f is a $(0, q)$-form then $f_{|b}$ is determined simply by $\bar\partial\rho \wedge f$. In particular, $f_{|b}$ is smooth if and only if $\bar\partial\rho \wedge f$ is smooth *etc.* On the boundary, $\langle f, g\rangle$ degenerates to an inner product of the complex tangential parts $f_{|b}$ and $g_{|b}$ of f and g. When α tends to 0 we get the following inner product for complex tangential $(0, q)$-forms $f_{|b}$ and $g_{|b}$:

$$(f, g)_b = \frac{(n-1)!}{2^n\pi^n}\int_{\partial D}\langle f, g\rangle\,\mathrm{d}\sigma = \frac{(n-1)!}{2^n\pi^n}\int_{\partial D}\langle\bar\partial\rho\wedge f, \bar\partial\rho\wedge g\rangle_\beta\,\mathrm{d}\sigma/B,$$

where $\mathrm{d}\sigma = \mathrm{d}S/|d\rho|_\beta$ and $\mathrm{d}S$ is the surface measure induced by β.

Let L^2_b denote the corresponding L^2-space and let $\mathcal{K}_b$ be the kernel of $\bar\partial_b$ in L^2_b. It is well-known, see [13], that $\bar\partial_b$ has closed range, and therefore $\bar\partial_b u = f$ is solvable in L^2_b for a $(0, q)$-form f, $1 \le q \le n-1$, if and only if f is orthogonal to $\operatorname{Ker}\bar\partial^*_b$. For $q \le n-2$ this is equivalent to that $\bar\partial_b f = 0$. Therefore we can define the operator K_b on L^2_b such that $K_b f$ is the minimal solution to $\bar\partial_b u = f$ if f is orthogonal to $\operatorname{Ker}\bar\partial_b^*$ and $K_b f = 0$ if $f \in \operatorname{Ker}\bar\partial_b^*$. We then have the homotopy relation

$$\bar\partial_b K_b + K_b \bar\partial_b = I - P_b - S_b,$$

where S_b is the orthogonal projection of $(0, n-1)$-forms onto $\operatorname{Ker}\bar\partial_b^*$.

3. The adjoint operator $\bar\partial_\alpha^*$

Let $(\ ,\ )_\alpha$ denote the inner product connected to the norm $\|\ \|_\alpha$, *i.e.*,

$$(f,g)_\alpha = \frac{\Gamma(n+\alpha)}{2^n \pi^n \Gamma(\alpha)} \int_D (-\rho)^\alpha \langle f,g\rangle \, \mathrm{d}V,$$

and let $\bar\partial_\alpha^*$ be the formal adjoint of $\bar\partial$ with respect to $(\ ,\ )_\alpha$, *i.e.*,

$$(\bar\partial f, g)_\alpha = (f, \bar\partial_\alpha^* g)_\alpha$$

for all compactly supported smooth f and g. Our first objective is to find a formula for $\bar\partial_\alpha^*$ that reveals its behavior near the boundary. If θ is a form, we let $\theta\neg$ denote interior multiplication by θ, with respect to the metric β, *i.e.*,

$$\langle \theta\neg f, g\rangle_\beta = \langle f, \bar\theta \wedge g\rangle_\beta \quad \text{for all } g.$$

PROPOSITION 3.1. — *With the notation above, the formal adjoint is*

$$\bar\partial_\alpha^* = i\,[\partial, (\beta - (1/B)\gamma)\neg] + \frac{\alpha + n - p - q}{B}\,\partial\rho\neg$$

when acting on a (p,q)-form.

Since β is non-degenerate on $\overline{D}$, the operators involved have coefficients that are smooth up to the boundary and hence $\bar\partial_\alpha^*$ is a first order differential operator with smooth coefficients. Letting $\alpha = 0$, the proposition provides a formula for the $\bar\partial_b^*$ operator on ∂D.

Proof. — We use the notation from the proof of Lemma 2.1 and Remark 1. From (2.4) it follows that

$$(-\rho)\bar\partial_\alpha^* = \bar\partial_{n-p-q+\alpha,\omega}^*, \tag{3.1}$$

and since ω is a Kähler metric, see [8] or [14],

$$\bar\partial_{\alpha+n-p-q,\omega}^* = i\,[\partial, \omega\neg_\omega] + (\alpha + n - p - q)\frac{\partial\rho}{-\rho}\neg_\omega, \tag{3.2}$$

if $\neg_\omega$ denotes interior multiplication with respect to ω. Thus we just have to express $\omega\neg_\omega$ and $\partial\rho\neg_\omega$ in terms of $\neg$. From (2.2) it follows that $\bar{e}_1\neg_\omega e_j = \langle e_1, e_j\rangle_\omega = (1/a)\delta_{1j}$ and hence

$$\partial\rho\neg_\omega = \big((-\rho)^2/B\big)\partial\rho\neg. \tag{3.3}$$

Moreover, it is readily verified that

$$\begin{aligned}\omega\neg_\omega &= (1/a)\, i\, e_1\wedge\bar{e}_1\neg + (1/b)\, i\sum_{j=2}^{n}(-\rho)(e_j\wedge\bar{e}_j)\neg\\ &= (-\rho)(\beta-(1/B)\gamma)\neg.\end{aligned} \tag{3.4}$$

The desired formula now follows from the equations (3.1) to (3.4). ∎

The following simple consequence of Proposition 3.1 will be used in Section 6.

PROPOSITION 3.2. — *Suppose that D is the ball $\mathbb{B}$ and $\rho(z) = |z|^2 - 1$. If $\partial/\partial\zeta_j$ acts as a Lie derivative on forms, then*

$$\Big(\frac{\partial}{\partial\zeta_j}\Big)\bar\partial^*_\alpha = \bar\partial^*_{\alpha+1}\Big(\frac{\partial}{\partial\zeta_j}\Big).$$

Proof. — One readily verifies that $\partial/\partial\zeta_j$ commutes with $\beta\neg\partial$ and $\partial\rho\neg$. Since $\bar\partial\rho\neg\partial$ is the Lie derivative with respect to the vector field $\sum\zeta_j\frac{\partial}{\partial\zeta_j}$, we have that

$$\Big(\frac{\partial}{\partial\zeta_j}\Big)\bar\partial\rho\neg\partial = \bar\partial\rho\neg\partial\Big(\frac{\partial}{\partial\zeta_j}\Big) + \frac{\partial}{\partial\zeta_j}\cdot$$

The desired equality now follows from Proposition 3.1. ∎

An $f\in L^2_\alpha$ is in $\operatorname{Dom}\bar\partial^*_\alpha$ (the domain of the von Neumann adjoint) if there is a $g\in L^2_\alpha$ such that $(g,u)_\alpha = (f,\bar\partial u)_\alpha$ for all $u\in\operatorname{Dom}\bar\partial$. If this holds, then clearly $\bar\partial^*_\alpha f = g$ in the distribution sense, but in general the converse is not true, *i.e.*, there are $f\in L^2_\alpha$ with $\bar\partial^*_\alpha f\in L^2_\alpha$ such that yet f does not belong to $\operatorname{Dom}\bar\partial^*_\alpha$. Let $\mathcal{E}_q$ denote the space of $(0,q)$-forms in D that are smooth up to the boundary. In the Euclidean case and $\alpha = 1$ an $f\in\mathcal{E}_*$ is in the domain of $\bar\partial^*$ if and only if $\partial\rho\neg f = 0$ on the boundary. Our situation is much nicer.

PROPOSITION 3.3. — *If $\alpha > 0$ and $f, g\in\mathcal{E}_*$ then $(\bar\partial^*_\alpha f, g)_\alpha = (f,\bar\partial g)_\alpha$.*

Proof. — Since $\bar\partial^*_\alpha$ has smooth coefficients, the boundary integral that occurs when integrating by parts must vanish if $\alpha > 1$. Since the expression is analytic in α, the general case follows by analytic continuation. ∎

PROPOSITION 3.4. — *Suppose that* $\alpha \geq 1$. *If* $f, g \in L^2_\alpha$ *and* $\bar\partial^*_\alpha f$ *and* $\bar\partial g$ *are in* L^2_α *as well, then* $(\bar\partial^*_\alpha f, g)_\alpha = (f, \bar\partial g)_\alpha$. *That is,* $f \in L^2_\alpha$ *is in* $\operatorname{Dom}\bar\partial^*_\alpha$ *if and only if* $\bar\partial^*_\alpha f$ *is in* L^2_α.

Since the the image of $\bar\partial : L^2_{\alpha,q-1} \to L^2_{\alpha,q}$ is equal to $\mathcal{K}_\alpha$ for $q \geq 1$, we have in particular that $f \in L^2_{\alpha,q}$ is in $\mathcal{K}^\perp_{\alpha,q}$ if and only if $\bar\partial^*_\alpha f = 0$. Proposition 3.4 is an immediate consequence of Proposition 3.3 and the following approximation lemma.

LEMMA 3.5. — *Suppose that* $\mathcal{P}$ *is a first order linear differential operator with coefficients that are smooth up to the boundary, and suppose that* $\alpha \geq 1$. *If* f *and* $\mathcal{P}f$ *are in* L^2_α, *then there are* $f_j \in \mathcal{E}_*$ *such that* $f_j \to f$ *and* $\mathcal{P}f_j \to \mathcal{P}f$ *in* L^2_α.

To prove this lemma one first approximates f by a form defined in a neighborhood of $\overline{D}$ and then makes a standard regularization of this form. We omit the details. In general the lemma fails if $\alpha < 1$; *cf.* Remark 2 below.

PROPOSITION 3.6. — *For any* $\alpha > 0$, *we have that* $(\bar\partial^*_\alpha \phi, g)_\alpha = (\phi, \bar\partial g)_\alpha$ *if* $\phi \in \mathcal{E}_*$ *and* $g, \bar\partial g \in L^2_\alpha$. *That is, any* $\phi \in \mathcal{E}_*$ *belongs to* $\operatorname{Dom}\bar\partial^*_\alpha$.

Proof. — If $g, \bar\partial g \in L^2_\alpha$ then $g, \bar\partial g \in L^2_{\alpha'}$ for $\alpha' > \alpha$. Since $\bar\partial^*_{\alpha'}\phi$ is in $\mathcal{E}_*$, it follows from Proposition 3.4 that $(\bar\partial^*_{\alpha'}\phi, g)_{\alpha'} = 0$ for $\alpha' \geq 1$. The desired conclusion then follows by analytic continuation. ∎

The argument above breaks down if one only assumes that $\phi, \bar\partial^*_\alpha \phi \in L^2_\alpha$, since this does not imply that $\bar\partial^*_{\alpha'}\phi \in L^2_{\alpha'}$ for $\alpha' > \alpha$.

Let $K^*_\alpha : L^2_\alpha \to L^2_\alpha$ be the L^2_α-adjoint of K_α.

PROPOSITION 3.7. — *Let* $\alpha > 0$. *If* $f \in L^2_\alpha$, *then* $K_\alpha f$ *is in* $\operatorname{Dom}\bar\partial$, $K^*_\alpha f$ *is in* $\operatorname{Dom}\bar\partial^*_\alpha$ *and we have the orthogonal decomposition*

$$\bar\partial K_\alpha f + \bar\partial^*_\alpha K^*_\alpha f = f - P_\alpha f. \tag{3.5}$$

Proof. — If f is a function, then the equality is just (1.3). Therefore let us assume that f is a $(0, q)$ form, $q \geq 1$. By the very definition of K_α it follows that $\bar\partial K_\alpha f$ is equal to the orthogonal projection of f on $\mathcal{K}_\alpha$. Therefore, $K_\alpha f$ is in $\operatorname{Dom}\bar\partial$ and the operator $\bar\partial K_\alpha$ is self-adjoint. For any $g \in \operatorname{Dom}\bar\partial$ we have by (1.3) that

$$(K^*_\alpha f, \bar\partial g)_\alpha = (f, K_\alpha \bar\partial g)_\alpha = (f, g - \bar\partial K_\alpha g)_\alpha = (f - \bar\partial K_\alpha f, g)_\alpha,$$

since $\bar\partial K_\alpha$ is self-adjoint. This shows that $K^*_\alpha f \in \operatorname{Dom}\bar\partial^*_\alpha$ and $\bar\partial^*_\alpha K^*_\alpha f = f - \bar\partial K_\alpha f$. ∎

The preceding proof just depends on the solvability of $\bar\partial$, *i.e.* (1.3). If

$$\overline{\Box}_\alpha = \bar\partial\bar\partial^*_\alpha + \bar\partial^*_\alpha\bar\partial \quad\text{and}\quad E_\alpha = K_\alpha K^*_\alpha + K^*_\alpha K_\alpha,$$

then by similar arguments, one can prove that

$$\begin{aligned} \overline{\Box}_\alpha E_\alpha f &= f - P_\alpha f, \qquad f \in L^2_\alpha, \\ E_\alpha \overline{\Box}_\alpha f &= f - P_\alpha f, \qquad f \in \operatorname{Dom} \overline{\Box}_\alpha. \end{aligned} \tag{3.6}$$

There are similar statements for thè boundary complex. Since $(\ ,\ )_b$ is the limit of $(\ ,\ )_\alpha$ when $\alpha \to 0$, the analog of Proposition 3.3 folows by continuity, and therefore the analog of Proposition 3.4 holds as well, since the smooth forms are dense in the graph norm. If K^*_b, $\overline{\Box}_b$ and E^*_b are defined in the obvious way, then the analogues of Proposition 3.7 and (3.6) hold, if just $f - P_\alpha f$ is replaced by $f - P_b - S_b$.

REMARK 2. — Proposition 3.4 is not true for $0 < \alpha < 1$; at least not for $(0, n)$-forms. To see this, let D be the unit disk. Since $\bar\partial : L^2_{\alpha,0} \to L^2_{\alpha,1}$ is surjective, the statement would imply that $f \in L^2_{\alpha,1}$ vanishes if $\bar\partial^*_\alpha f = 0$. However, the latter equation means that $\partial(1-|z|^2)^\alpha f = 0$ and hence the kernel of $\bar\partial^*_\alpha : L^2_{\alpha,1} \to L^2_{\alpha,0}$ consists of all forms $f = (1-|z|^2)^{-\alpha}\bar h$, where h is holomorphic and $\int(1-|z|^2)^{-\alpha}|h|^2 < \infty$. For $q < n$ the corresponding result is true in the "limit case" when $\alpha \to 0$; therefore, one could guess that it is true even in the intermediate cases $0 < \alpha < 1$. In particular we would then have that

$$f \in L^2_\alpha \text{ and } \bar\partial^*_\alpha f = 0 \quad \text{implies} \quad f \in \mathcal{K}^\perp_\alpha \tag{3.7}$$

for $(0, q)$-forms, $1 \le q \le n-1$. Let us relate this statement to the norms $\|\ \|_{\ell,\omega}$. In view of (2.4) we have that $\bar\partial^*_\alpha f = 0$ if and only if $\bar\partial^*_{\ell,\omega} f = 0$, where $\ell = \alpha + n - q$. Since $|\partial \log(-1/\rho)|_\omega$ is bounded, ω is a complete metric, see [8], and therefore the compactly supported forms are dense in the graph norms with respect to the norms $\|\ \|_{\ell,\omega}$. In particular, this means that the formal adjoint $\bar\partial^*_{\ell,\omega}$ coincides with the corresponding von Neumann adjoint. Hence (3.7) holds if and only if the image of $\bar\partial : L^2_{\ell,\omega} \to L^2_{\ell,\omega}$ is dense in $\mathcal{K}_\alpha$ (as it follows that f is orthogonal to this image if $\bar\partial^*_{\ell,\omega} f = 0$). In view of Proposition 3.4, the image is dense if $\alpha \ge 1$. For $\alpha > 1$, it is in fact equal to $\mathcal{K}_\alpha$; this is the content of Theorem 2.2. However, $\bar\partial : L^2_{\ell,\omega} \to L^2_{\ell,\omega}$ is *not* surjective if $0 < \alpha \le 1$. To see this, let D be the ball and let $f = \bar\partial\bar h$, where h is some holomorphic function with $h(0) = 0$ that is C^1 up to the boundary. Then certainly $f \in \mathcal{K}_{\alpha,1}$, but it is not in the image of $\bar\partial$ unless $h = 0$. In fact, if $\bar\partial u = f$ and $\int(1-|\zeta|^2)^{\alpha-2}|u|^2 < \infty$, then since $g = u - \bar h$ is holomorphic, (and g and h are orthogonal with respect to radial measures) we would have that $\int(1-|\zeta|^2)^{\alpha-2}(|h|^2+|g|^2) < \infty$ which implies that $h = 0$.

4. Up and down in dimension

Given our domain D and defining function ρ in $\mathbb{C}^n$, let

$$\tilde\rho(z,w) = \rho(z) + |w|^2 \quad \text{for} \quad (z,w) \in \mathbb{C}^{n+1}.$$

Then $\tilde\rho$ is a strictly plurisubharmonic defining function for the strictly pseudoconvex domain $\widetilde D = \{\tilde\rho < 0\}$ in $\mathbb{C}^{n+1}$. Hence, anything done so far applies equally well to $\widetilde D$. For a $(0,q)$-form f in D, we put

$$\tilde f(z,w) = f(z).$$

We notice that f is determined by the complex tangential values $\tilde f_{|b}$ of $\tilde f$. In fact, $\tilde f_{|b} = 0$ means that $(\bar\partial\rho(z) + w\,\mathrm{d}\bar w)\wedge f(z) = 0$ on $\partial\widetilde D$, in particular at points $(z, \sqrt{-\rho(z)})$, and this implies that $f(z) = 0$.

In what follows $(\ ,\ )_\alpha$ means $(\ ,\ )_b$ when $\alpha = 0$.

PROPOSITION 4.1. — *Let D and $\widetilde D$ be as above. Then for $(0,q)$-forms we have*

(i) $(f,g)_\alpha = (\tilde f,\tilde g)_{\alpha-1}$ *for* $\alpha \ge 1$. *In particular,* $f \in L^2_\alpha(D)$ *if and only if* $\tilde f \in L^2_{\alpha-1}(\widetilde D)$ *for* $\alpha > 1$ *and* $f \in L^2_1(D)$ *if and only if* $\tilde f_{|b} \in L^2_b(\partial\widetilde D)$.

(ii) $\bar\partial f = g$ *in* D *if and only if* $\bar\partial\tilde f = \tilde g$ *in* $\widetilde D$ *if and only if* $\bar\partial_b\tilde f_{|b} = \tilde g_{|b}$.

(iii) $(\bar\partial^*_\alpha f)^\sim = \bar\partial^*_{\alpha-1}\tilde f$ *for* $f \in \mathcal{E}_*(\overline D)$ *for* $\alpha > 1$, *and* $(\bar\partial^*_1 f)^\sim{}_{|b} = \bar\partial^*_b\tilde f_{|b}$.

(iv) $f \in \mathcal{K}^\perp_\alpha$ *if and only if* $\tilde f \in \mathcal{K}^\perp_{\alpha-1}$, *and* $f \in \mathcal{K}^\perp_1$ *if and only if* $\tilde f_{|b} \in \mathcal{K}^\perp_b$.

(v) *If* $f \in L^2_\alpha$, $\alpha > 1$, *then* $f = f_1 + f_2$ *is the decomposition in* $\mathcal{K}_\alpha$ *and* $\mathcal{K}^\perp_\alpha$ *if and only if* $\tilde f = \tilde f_1 + \tilde f_2$ *is the decomposition in* $\mathcal{K}_{\alpha-1}$ *and* $\mathcal{K}^\perp_{\alpha-1}$, *and if* $\alpha = 1$, *then* $f = f_1 + f_2$ *is the decomposition in* $\mathcal{K}_1$ *and* $\mathcal{K}^\perp_1$ *if and only if* $\tilde f_{|b} = \tilde f_{1|b} + \tilde f_{2|b}$ *is the decomposition in* $\mathcal{K}_b$ *and* $\mathcal{K}^\perp_b$.

Proof. — Since $\widetilde\beta = i\,\partial\bar\partial\rho + i\,\mathrm{d}w\wedge\mathrm{d}\bar w$ we find that, at each point, dw is orthogonal to all $\mathrm{d}z_j$ with respect to $\widetilde\beta$, and moreover $\mathrm{d}w$ has norm one. In other words, if a, a', b, b' contain no differentials of w, then

$$\big\langle a + b\wedge\bar\partial|w|^2, a' + b'\wedge\bar\partial|w|^2\big\rangle_{\widetilde\beta} = \langle a,a'\rangle_\beta + |w|^2\langle b,b'\rangle_\beta.$$

It follows that $|\bar\partial\tilde\rho|^2_{\tilde\beta} = |\bar\partial\rho|^2_\beta + |w|^2$ and therefore $\widetilde{B} = B$. In view of (2.1) we also get

$$\langle \tilde f, \tilde g\rangle^{\sim} = \langle f, g\rangle. \tag{4.1}$$

Moreover, *cf.* (2.1),

$$(-\tilde\rho)\,\mathrm{d}\widetilde{V} = \widetilde{B}\widetilde{\beta}_{n+1} = B\beta_n \wedge i\,\mathrm{d}w \wedge \mathrm{d}\bar w = (-\rho)\,\mathrm{d}V \wedge i\,\mathrm{d}w\wedge \mathrm{d}\bar w,$$

and therefore,

$$\begin{aligned}\int_{\widetilde D}(-\tilde\rho)^{\alpha-1}\langle\tilde f,\tilde g\rangle^{\sim}\,\mathrm{d}\widetilde V &= \int_{\widetilde D}(-\tilde\rho)^{\alpha-2}\langle\tilde f,\tilde g\rangle^{\sim}(-\tilde\rho)\,\mathrm{d}\widetilde V\\ &= \int_D\int_{|w|^2\le-\rho(z)}\big(-\rho-|w|^2\big)^{\alpha-2} i\,\mathrm{d}w\wedge\mathrm{d}\bar w\langle f,g\rangle(-\rho)\,\mathrm{d}V\\ &= \frac{2\pi}{\alpha-1}\int_D(-\rho)^\alpha\langle f,g\rangle\,\mathrm{d}V.\end{aligned}$$

This proves part (i) for $\alpha > 1$. The case $\alpha = 1$ follows by continuity.

Part (ii) is obvious, just noting that $\bar\partial_b\tilde f_{|b} = (\bar\partial\tilde f)_{|b} = (\bar\partial f)^{\sim}{}_{|b}$.

To see part (iii), first notice that if ϕ is a form that only contains differentials of w, then $\phi\neg\tilde f = 0$. Moreover, $(\partial f)^{\sim} = \partial\tilde f$. Therefore, (iii) follows from Proposition 3.1 .

Let us now consider part (iv). The nontrivial direction is that $\tilde f \in \mathcal{K}^\perp_{\alpha-1}$ if $f\in\mathcal{K}^\perp_\alpha$ (we assume $\alpha > 1$, the case $\alpha = 1$ is similar). It follows from Theorem 4.1 that the operator $^\sim : L^2_\alpha(D) \to L^2_{\alpha-1}(\widetilde D)$ is bounded. Let

$$M : L^2_{\alpha-1}(\widetilde D) \longrightarrow L^2_\alpha(D)$$

be its adjoint. We claim that $\bar\partial Mg = 0$ if $\bar\partial g = 0$. Clearly, $\bar\partial Mg = 0$ in the distribution sense means that $(\bar\partial^*_\alpha\phi, Mg)_\alpha = 0$ for all compactly supported smooth forms ϕ. However, for any such ϕ we have

$$(\bar\partial^*_\alpha\phi, Mg)_\alpha = \big((\bar\partial^*_\alpha\phi)^{\sim}, g\big)_{\alpha-1} = (\bar\partial^*_{\alpha-1}\tilde\phi, g)_{\alpha-1},$$

and the last term vanishes by Proposition 3.6. Now take $f\in\mathcal{K}^\perp_\alpha$ and $g\in\mathcal{K}_{\alpha-1}$. Then $(\tilde f, g)_{\alpha-1} = (f, Mg)_\alpha = 0$ by the assumption on f since $\bar\partial Mg = 0$. Thus part (iv) is proved.

The last statement is an immediate consequence of (ii) and (iv). ∎

With the same argument as above it follows that

$$\bar\partial Mg = M\bar\partial g \quad \text{if} \quad \bar\partial g \in L^2_{\alpha-1}(\widetilde{D}).$$

It is possible to verify this equality directly by means of the formula for Mg in Proposition 5.4 below.

Let ϕ be a complex tangential form on $\partial\widetilde{D}$. We say that ϕ is invariant if $\tau^*\phi = \phi$ for all $\tau(z,w) = (z, \mathrm{e}^{i\theta}w)$. Notice that $\tau^*\phi$ is well defined since $\tau^*\,\mathrm{d}\tilde\rho = \mathrm{d}\tilde\rho$ and $\tau^*\,\mathrm{d}^c\tilde\rho = \mathrm{d}^c\tilde\rho$.

PROPOSITION 4.2. — *There is a one-to-one correspondence between forms $f \in L^2_1(D)$ and invariant complex tangential forms $\phi \in L^2_b(\partial\widetilde{D})$. Moreover, f is smooth on $\overline{D}$ if and only if ϕ is smooth.*

Let $\mathcal{E}^b_*(\partial\widetilde{D})$ denote the space of smooth tangential $(0,*)$ forms. If $f \in \mathcal{E}_*(D)$ then clearly $f_{|b} \in \mathcal{E}^b_*(\partial\widetilde{D})$. The proposition states that the converse holds as well. This is less trivial, and it will be clear from the proof that, in general, one loses one half unit of regularity on the "complex normal" component of f. For instance, $f_{|b}$ is bounded if and only if f admits a representation $f = u_1 + \bar\partial\rho(z) \wedge u_2/\sqrt{-\rho(z)}$, where u_j are bounded.

Proof of Proposition 4.2. — Suppose that ϕ is an invariant form and let

$$\Phi(z,w) = w\,\mathrm{d}\bar w \wedge a(z,w) + b(z,w)$$

be a representing $(0,q)$-form, where a and b contain no differentials $\mathrm{d}\bar w$. After possibly taking a mean value of all rotations in w, we may assume that Φ itself is rotation invariant in w, and then it follows that actually a and b only depends on z. Therefore $\phi = \tilde f_{|b}$ if $f(z) = -\bar\partial\rho(z)\wedge a(z) + b(z)$. The uniqueness of f is clear by the remark before Proposition 4.1, and in view of part (i) of this proposition, $f \in L^2_1(D)$ if and only if $f_{|b} \in L^2_b(\partial\widetilde{D})$.

If f is smooth on $\overline{D}$, then it is clear that $\tilde f \in \mathcal{E}^b_*(\partial\widetilde{D})$. Conversely, if ϕ is smooth then there is a smooth invariant representing form Φ over $\partial\widetilde{D}$ as above, which means that $b(z)$ and $wa(z)$ are smooth on $\partial\widetilde{D}$. Then clearly $b(z)$ is smooth on $\overline{D}$ so we have to prove that $a(z)$ is smooth on $\overline{D}$ as well. For simplicity we assume that $n = 1$ and that D is the unit disk. The possible problem is when w is close to 0. Here

$$(w,t) \mapsto \big(\mathrm{e}^{it}\sqrt{1-|w|^2}, w\big)$$

are coordinates on $\partial\widetilde{D}$ and so $(w,t) \mapsto wa(\mathrm{e}^{it}\sqrt{1-|w|^2})$ is a C^∞-function. In particular, for real x, $(x,t) \mapsto xa(\mathrm{e}^{it}\sqrt{1-x^2})$ is smooth and odd in x. Therefore there is a smooth function u such that $xa(e^{it}\sqrt{1-x^2}) = xu(x^2,t)$. Thus $a(re^{it}) = u(1-r^2,t)$ and hence it is smooth up to the boundary. $\square$

Notice that any invariant $(0,n)$-form $\phi \in L^2_b(\partial\widetilde{D})$ is orthogonal to $\operatorname{Ker}\bar\partial_b^*$; in fact, this is equivalent to the solvability of $\bar\partial_b u = \phi$ in $L^2_b(\partial\widetilde{D})$, which in turn follows from Propositions 4.1 and 4.2. This can also be verified directly, see [6].

Let $\widetilde{K}_\alpha$ denote the canonical homotopy operator in $\widetilde{D}$ with respect to $\tilde\rho$, and let $\widetilde{K}_b$ be the canonical operator on $\partial\widetilde{D}$ (corresponding to $\alpha = 0$), and similarily with $\widetilde{P}_\alpha$ and $\widetilde{P}_b$. From Propositions 4.1 and 4.2 we get the following basic result.

THEOREM 4.3. — *With the notation above we have that*

$$(K_\alpha f)\tilde{} = \widetilde{K}_{\alpha-1}\tilde f, \qquad (P_\alpha f)\tilde{} = \widetilde{P}_{\alpha-1}\tilde f$$

if $\alpha > 1$ *and*

$$(K_1 f)\tilde{}_{|b} = \widetilde{K}_b \tilde f_{|b}, \quad (P_1 f)\tilde{}_{|b} = \widetilde{P}_b \tilde f_{|b}.$$

Thus $K_\alpha f$ (and $P_\alpha f$) can be reconstructed from the complex tangential boundary values $\widetilde{K}_{\alpha-1}\tilde f_{|b}$ (and $\widetilde{P}_{\alpha-1}\tilde f_{|b}$). In particular, $K_\alpha f$ is smooth if (and only if) $\widetilde{K}_{\alpha-1}\tilde f_{|b}$ is.

Proof. — First suppose that $f \in \mathcal{K}_\alpha$. Then $u = K_\alpha f$ solves $\bar\partial u = f$ and $u \in \mathcal{K}_\alpha^\perp$. By Proposition 4.1, therefore, $\bar\partial\tilde u = \tilde f$ and $\tilde u \in \mathcal{K}_{\alpha-1}^\perp$, so that $\tilde u = \widetilde{K}_{\alpha-1}\tilde f$. On the other hand, if $f \in \mathcal{K}_\alpha^\perp$, then $\tilde f \in \mathcal{K}_{\alpha-1}^\perp$ and therefore, $K_\alpha f$ as well as $K_{\alpha-1}\tilde f$ vanish. The other statements follow in the same way. $\square$

For the Bergman projections P_α, this theorem is wellknown and has been used by many authors, see e.*g*. [1], [7] and [20].

5. Integral representation in the ball

In this section we consider the unit ball $\mathbb{B}$ in $\mathbb{C}^n$ and $\rho(z) = |z|^2 - 1$. There are well-known explicit formulas for the canonical boundary operators K_b, P_b (and S_b) on $\partial\mathbb{B}$. By repeated use of (4.2) one can therefore compute K_α for all positive integers α. Since the boundary operators are known to preserve regularity we obtain regularity for K_α, in view of Proposition 4.2. However, using (4.3) we can obtain the same results for all α.

It is well-known that the Bergman projection P_α is given by

$$Pf(z) = \big(f, \overline{p_\alpha(\,\cdot\,, z)}\,\big)_\alpha,$$

where

$$p_\alpha(\zeta, z) = \frac{1}{(1 - \bar\zeta \cdot z)^{n+\alpha}}\cdot \tag{5.1}$$

There are similar formulas for P_b and S_b on $\partial\mathbb{B}$, with kernels

$$p_b(\zeta, z) = (1 - \bar\zeta \cdot z)^{-n} \quad \text{and} \quad s_b(\zeta, z) = \delta(z) \wedge \delta(\zeta)\big((1 - \zeta \cdot \bar z)\big)^{-n},$$

where δ is the $(0, n-1)$-form

$$\delta = \sum_{j=1}^{n} (-1)^{j+1} \zeta_j \widehat{\mathrm{d}\zeta_j},$$

in the sense that $P_b f(z)$ and $S_b f(z)$ are the the boundary values of the holomorphic function $(f, \overline{p_b(\cdot, z)}\,)_b$ and the anti-holomorphic form $(f, \overline{s_b(\cdot, z)}\,)_b$, respectively. Moreover it is wellknown and not hard to verify that P_α as well as P_b and S_b preserve regularity.

REMARK 3. — We thus have adopt the convention that the kernel $t_\alpha(\zeta, z)$ corresponds to the operator $T_\alpha f(z) = (f, \overline{t(\,\cdot\,, z)}\,)_\alpha$. For $(0,q)$ forms f and g, $\langle f, g\rangle \mathrm{d}V = c_q f \wedge \bar g \wedge \Omega_{n-q}$, where $c_q = 1$ if q is even and $c_q = -i$ if q is odd. Therefore the corresponding operator on $(0, q+1)$-forms can be written in the form $\int f \wedge T$ if $T(\zeta, z) = t_\alpha(\zeta, z) \wedge \Omega_{n-q}(\zeta)$.

THEOREM 5.1. — *Let $\mathbb{B}$ be the ball in $\mathbb{C}^n$ and let $\alpha \geq 0$. The boundary values of the kernel $k_\alpha(\zeta, z)$ for the canonical operator K_α are given by*

$$k_\alpha(\zeta, z) = \sum_{q=0}^{n-1} \frac{\Gamma(\alpha + n - q - 1)}{\Gamma(n + \alpha)} \, \frac{\bar z \cdot \mathrm{d}\zeta \wedge (\mathrm{d}\bar z \cdot \mathrm{d}\zeta)^q}{(1 - \bar\zeta \cdot z)^{\alpha - 1 + n - q}(1 - \zeta \cdot \bar z)^{q+1}}, \tag{5.2}$$

for $z \in \partial\mathbb{B}$. Moreover, the values in the interior are given by

$$\begin{aligned} k_\alpha(\zeta, z) = \sum_{q=0}^{n-1} c_{n,\alpha,q} \frac{1}{(1 - \bar\zeta \cdot z)^{n+\alpha-q}(1 - \zeta \cdot \bar z)^{q+1}(1 - |a|^2)^n} \\ \times \Big[\big[(1 - \bar\zeta \cdot z) P_{n-q-1}^{\alpha-1, -n} \bar z \cdot \mathrm{d}\zeta - (1 - |z|^2) P_{n-q-1}^{\alpha, -n} \bar\zeta \cdot \mathrm{d}\zeta\big] \wedge (\mathrm{d}\bar z \cdot \mathrm{d}\zeta)^q \\ + q P_{n-q-1}^{\alpha, -n} \bar z \cdot \mathrm{d}\zeta \wedge (\mathrm{d}\bar z \cdot \mathrm{d}\zeta)^{q-1} \wedge \bar\partial |z|^2 \wedge \bar\zeta \cdot \mathrm{d}\zeta\Big], \end{aligned} \tag{5.3}$$

where

$$c_{n,\alpha,q} = \frac{\Gamma(n+\alpha-q-1)}{\Gamma(n+\alpha)} \frac{\Gamma(\alpha)\Gamma(n-q)}{\Gamma(n+\alpha-q-1)},$$

$$1-|a|^2 = 1 - \frac{(1-|\zeta|^2)(1-|z|^2)}{|1-\bar{\zeta}\cdot z|^2},$$

and $P^{\alpha,-n}_{n-q-1}$ *and* $P^{\alpha-1,-n}_{n-q-1}$ *are polynomials in* $|a|^2$ *of degree* $n-1-q$. *More precisely,* $P^{\alpha,\beta}_m = P^{\alpha,\beta}_m(1-2|a|^2)$, *where* $P^{\alpha,\beta}_m(x)$ *are the Jacobi polynomials*

$$P^{\alpha,\beta}_m(x) = \frac{(-1)^m}{2^m m!}(1-x)^{-\alpha}(1+x)^{-\beta}\frac{\mathrm{d}^m}{\mathrm{d}x^m}\{(1-x)^{m+\alpha}(1+x)^{m+\beta}\}.$$

Let K^b_α be the operator, defined by the kernel (5.2) that maps forms in $\mathbb{B}$ to complex tangential forms. In view of Remark 3 one can confirm that this is precisely the solution operator for $\bar\partial_b$ found by Skoda and Henkin for $\alpha = 1$. The general case follows for instance from Example 1 (iii) in [9]; for integer values of α, see also [7]. It is wellknown that K^b maps $\mathcal{E}_*$ into $\mathcal{E}^b_*$ (*i.e.* smooth tangential forms) and that the relation

$$\bar\partial_b K^b_\alpha + K^b_\alpha \bar\partial = I - P_\alpha, \tag{5.5}$$

holds. When $\alpha \to 0$ then this relation tends to $\bar\partial_b K_b + K_b\bar\partial = I - P_b - S_b$.

Since K^b_α maps $\mathcal{E}_*$ into $\mathcal{E}^b_*$, we can use (4.2) and Proposition 4.2 to conclude that K_α preserves regularity, at least for $\alpha \geq 1$. However, by looking directly at the formula for the interior values one can check that this is true for all α. Another possibility is to confirm, following [7], that K^b_α actually has an analytic extension to $\operatorname{Re}\alpha > -n$, that maps $\mathcal{E}_* \to \mathcal{E}^b_*$. From the proof of Theorem 5.1 it follows that one can obtain K^*_α from the boundary values of the operator in $\widetilde{B}$ obtained by taking the formal adjoint of the kernel for $K_{\alpha-1}$. It is then easy to check that this operator preserves regularity. However, again one can just as well consider the kernel for K^*_α directly. Anyway, we have

THEOREM 5.2. — *For any* $\alpha \geq 0$, *the projections* P_α *and the canonical operators* K_α *and their adjoints* K^*_α *preserve regularity. In particular, we have regularity for the orthogonal decomposition* (3.5). *Moreover, since* E_α *preserves regularity we have regularity for the* $\overline{\Box}_\alpha$*-equation.*

REMARK 4. — Since

$$P^{\alpha-1,-n}_{n-q-1}(1) = \Gamma(n+\alpha-q-1)/\Gamma(\alpha)\Gamma(n-q),$$

$k_\alpha(\zeta, z)$ coincides with $k^b_\alpha(\zeta, z)$ as tangential forms when z is on the boundary. Notice that $1-|a|^2 = \mathcal{O}(|\zeta - z|^2)$ for (ζ, z) on compact subsets of $\mathbb{B}\times\mathbb{B}$. Since

$$P^{\alpha-1,-n}_{n-q-1}(-1) = P^{\alpha,-n}_{n-q-1}(-1) \quad \text{and} \quad \bar z\cdot \mathrm{d}\zeta \wedge \bar\zeta \cdot \mathrm{d}\zeta = \mathcal{O}(|\zeta - z|)$$

it follows that $k^q_\alpha(\zeta, z) = \mathcal{O}(|\zeta - z|^{-2n+1})$ as expected. If $n = 1$ (and $q = 0$) then

$$1 - |a|^2 = \frac{|\zeta - z|^2}{|1 - \bar\zeta z|^2}$$

and therefore the kernel reduces to

$$k_\alpha(\zeta, z) = \frac{1}{\alpha}\,\frac{\bar z \mathrm{d}\zeta}{(1-\bar\zeta z)^\alpha(\zeta - z)},$$

which corresponds to the weighted Cauchy integrals in the unit disk. For further discussion about the nature of the singularity of this kernel, see [4], [5]. For an alternative way of writing the kernel, see [6].

Remark 5. — Previously there have appeared several constructions of explicit operators B_α that gives the L^2_α minimal solution when applied to a $\bar\partial$-closed $(0,1)$-form. One such formula was found by Charpentier in [11]. Other possibilities are provided by Example 1 in [9], by making various choices of the section S, *e.g.* $S = (1 - \bar\zeta \cdot z)\bar z - (1 - |z|^2)\bar\zeta$ or $S = -(1 - \bar z \cdot \zeta)\bar\zeta + (1 - |\zeta|^2)\bar z$. The first choice give back the Charpentier kernel. However one can verify, see [6], that none of these kernels coincide with the kernel for the canonical operator K_α. From the geometrical interpretation it is clear that this kernel neither is equal to the kernel for the Kohn operator found by Harvey and Polking, [16].

Proof of Theorem 5.1. — Let K^b_α denote the operator that is defined by the kernels (5.2). Our starting point is the knowledge that this operator map smooth forms onto smooth tangential forms and that (5.5) holds. Now let

$$h^b_\alpha = \sum_{q=0}^{n-1} \frac{\Gamma(\alpha + n - q - 1)}{\Gamma(n+\alpha)q}\,\frac{(\mathrm{d}\bar z \cdot \mathrm{d}\zeta)^q}{(1 - \bar\zeta\cdot z)^{\alpha-1+n-q}(1 - \zeta\cdot\bar z)^q}, \quad z \in \partial\mathbb{B},$$

where $1/x^0$ means $\log x$, and let H^b_α denote the corresponding operator. Then even H^b_α maps smooth forms onto smooth tangential forms and since $\partial_\zeta h^b_\alpha = k^b_\alpha$ it follows from Proposition 3.3 that

$$K^b_\alpha f = H^b_\alpha \bar\partial^*_\alpha f \tag{5.6}$$

for smooth forms f.

Now let $\alpha > 1$ and consider the operator $\widetilde{K}^b_{\alpha-1}$ and its corresponding kernel $\widetilde{k}^b_{\alpha-1}$ on $\widetilde{\mathbb{B}}$ in $\mathbb{C}^{n+1}$. It is readily verified that the tangential forms $\widetilde{K}^b_{\alpha-1}\tilde{f}$ and $\widetilde{H}^b_{\alpha-1}\tilde{f}$ are invariant if f is a form in $\mathbb{B}$. Therefore it follows from Propositions 4.2 and 4.1 and (5.6) that these operators define operators K'_α and H'_α in $\mathbb{B}$, which preserve regularity, and such that

$$K'_\alpha = H'_\alpha \bar{\partial}^*_\alpha. \tag{5.7}$$

We are to show that in fact K'_α is equal to K_α. We already know that the explicit operator P_α represents the orthogonal projection P_α, and that $\widetilde{P_\alpha f} = P_{\alpha-1}\tilde{f}$, *cf.* (4.2). Hence it follows that

$$\bar{\partial} K'_\alpha f + K'_\alpha \bar{\partial} f = f - P_\alpha f. \tag{5.8}$$

LEMMA 5.3. — *The operator H'_α preserves regularity and $(H'_\alpha f, \phi)_\alpha = (f, H'_\alpha \phi)$ for $\phi, f \in \mathcal{E}_*$.*

Taking this for granted, we can conclude the proof. Let $\Pi = \bar{\partial} K'_\alpha$. Then it follows from (5.8) that Π is a projection $\mathcal{E}_* \to \mathcal{K}_\alpha \cap \mathcal{E}_*$, and from (5.7) and the lemma it follows that $(\Pi f, \phi)_\alpha = (f, \Pi\phi)_\alpha$ for smooth ϕ, f. Therefore,

$$\|\Pi f\|_\alpha \le \|f\|_\alpha$$

(in fact, $\|\Pi f\|^2 = (\Pi f, \Pi f) = (\Pi^2 f, f) = (\Pi f, f) \le \|f\|\|\Pi f\|$) and since $\mathcal{E}_*$ is dense in L^2_α we find that Π is (the restriction to $\mathcal{E}_*$ of) the orthogonal projection $L^2_\alpha \to \mathcal{K}_\alpha$. In particular, we find that the orthogonal decomposition $L^2_\alpha = \mathcal{K}_\alpha \oplus \mathcal{K}^\perp_\alpha$ preserves regularity. Now, let f be a smooth $\bar{\partial}$-closed form. Then it follows from (5.8) that $K'_\alpha f$ is a smooth solution to $\bar{\partial} u = f$, and since the projection Π preserves regularity, the minimal solution u is smooth as well. Another application of (5.7) and (5.8) yields that $u = K'_\alpha f$, since $K'_\alpha u = H'_\alpha \bar{\partial}^*_\alpha u = 0$. Thus we have proved that actually $K'_\alpha f$ is equal to the canonical $K_\alpha f$ if f is smooth and $\alpha > 1$ (or even $\alpha \ge 1$).

To handle the case $\alpha > 0$ we have to assume that the entire theorem is already proved for $\alpha > 1$. We notice that the explicit operator $K'_\alpha f$, defined for $f \in \mathcal{E}_*$ and $\alpha > 1$ by the kernel (5.3), has an analytic continuation to $\alpha > 0$ and by the remark preceding Theorem 5.2 it is a mapping $\mathcal{E}_* \to \mathcal{E}_*$. Again by analytic continuation it follows that $\bar{\partial} K'_\alpha \colon \mathcal{E}_* \to \mathcal{E}_*$ is self-adjoint and that (5.8) holds for all $\alpha > 0$. As before we can then conclude that ∂K_α is the orthogonal projection onto $\mathcal{K}'_\alpha$, and that $K'_\alpha f = K_\alpha f$ for all $\alpha > 0$.

To obtain (5.3) and for the proof of Lemma 5.3 we need the following proposition.

PROPOSITION 5.4. — *Let* $M\colon L^2_{\alpha-1}(\widetilde D)\to L^2_\alpha(D)$ *be the adjoint of* $\tilde{}\colon L^2_\alpha(D)\to L^2_{\alpha-1}(\widetilde D)$. *Suppose that* $g(\zeta,\xi)=a(\zeta,\xi)+\mathrm{d}\bar\xi\wedge b(\zeta,\xi)$. *Then*

$$Mg(\zeta)=\frac{\alpha-1}{\pi}\int_{|\tau|<1}\big(1-|\tau|^2\big)^{\alpha-2}\times\Big(a(\zeta,\sqrt{-\rho}\,\tau)-\frac{\bar\tau}{\sqrt{-\rho}}\bar\partial\rho\wedge b\big(\zeta,\sqrt{-\rho}\,\tau\big)\Big)\,\mathrm{d}\lambda(\tau).$$

From (5.2) we have that

$$\tilde k^{b,q}_{\alpha-1}(\zeta,w;z,z_{n+1})=A\bar z\cdot\mathrm{d}\zeta\wedge(\mathrm{d}\bar z\cdot\mathrm{d}\zeta)^q+\big(\bar z_{n+1}\wedge(\mathrm{d}\bar z\cdot\mathrm{d}\zeta)^q+q(\mathrm{d}\bar z\cdot\mathrm{d}\zeta)^{q-1}\wedge\mathrm{d}\bar z_{n+1}\big)\wedge\mathrm{d}w,$$

$$A=\frac{\Gamma(\alpha+n-q-1)}{\Gamma(n+\alpha)}\,\frac{1}{(1-\bar\zeta\cdot z-\bar w z_{n+1})^{\alpha+n-q-1}(1-\zeta\cdot\bar z-w\bar z_{n+1})^{q+1}}.$$

In view of Proposition 5.4 and (the proof of) Proposition 4.2 we then get that

$$\begin{aligned}k^q_\alpha(\zeta,z)=&\frac{\Gamma(\alpha+n-q-1)}{\Gamma(n+\alpha)}\,\frac{1}{(1-\bar\zeta\cdot z)^{\alpha+n-q-1}(1-\zeta\cdot\bar z)^{q+1}}\\&\times\Big\{\bar z\wedge(\mathrm{d}\bar z\cdot\mathrm{d}\zeta)^q m_{\alpha-2,\alpha+n-q-1,q+1}\\&-\frac{1}{\sqrt{(1-|\zeta|^2)(1-|z|^2)}}\Big((1-|z|^2)(\mathrm{d}\bar z\cdot\mathrm{d}\zeta)^q-\bar z\cdot\mathrm{d}\zeta\wedge q(\mathrm{d}\bar z\cdot\mathrm{d}\zeta)^{q-1}\wedge\bar\partial|z|^2\Big)\\&\wedge\partial|\zeta|^2\wedge m'_{\alpha-2,\alpha+n-q-1,q+1}\Big\},\end{aligned}$$

where

$$m_{\alpha-2,j,k}=\frac{\alpha-1}{\pi}\int_{|\tau|<1}\frac{(1-|\tau|^2)^{\alpha-2}\,\mathrm{d}m(\tau)}{(1-a\bar\tau)^j(1-\bar a\tau)^k},$$

$$m'_{\alpha-2,j,k}=\frac{\alpha-1}{\pi}\int_{|\tau|<1}\frac{(1-|\tau|^2)^{\alpha-2}\tau\,\mathrm{d}m(\tau)}{(1-a\bar\tau)^j(1-\bar a\tau)^k}$$

and $a=\sqrt{1-|\zeta|^2}\sqrt{1-|z|^2}/(1-\bar\zeta\cdot z)$. An integration by parts in the expression for m' reveals that $m'_{\alpha-2,j,k}=(aj/\alpha)m_{\alpha,j+1,k}$ and hence

$$\begin{aligned}k^q_\alpha(\zeta,z)=&\frac{\Gamma(\alpha+n-q-1)}{\Gamma(n+\alpha)}\\&\Big[\frac{m_{\alpha-2,\alpha+n-q-1,q+1}}{(1-\bar\zeta\cdot z)^{\alpha+n-q-1}(1-\zeta\cdot\bar z)^{q+1}}\bar z\mathrm{d}\zeta\wedge(\mathrm{d}\bar z\cdot\mathrm{d}\zeta)^q\\&-\frac{\alpha+n-q-1}{\alpha}\,\frac{m_{\alpha-1,\alpha+n-q,q+1}}{(1-\bar\zeta\cdot z)^{\alpha+n-q}(1-\zeta\cdot\bar z)^{q+1}}\\&\big((1-|z|^2)(\mathrm{d}\bar z\cdot\mathrm{d}\zeta)^q-\bar z\cdot\mathrm{d}\zeta\wedge q(\mathrm{d}\bar z\cdot\mathrm{d}\zeta)^{q-1}\wedge\bar\partial|z|^2\big)\wedge\partial|\zeta|^2\Big].\end{aligned}$$

Observe that $m_{\alpha,j,k} = 1$ when $a = 0$. It is not hard to see that $m_{\gamma,j,k}$ only depends on $|a|^2$. More precisely, see [3], $m_{\gamma,j,k} = F(j,k,\gamma+2,|a|^2)$, where F is the hypergeometric function

$$\begin{aligned} F(m,b,c,z) &= \frac{(c-1)!}{(b-1)!(c-b-1)!}\int_0^1 t^{b-1}(1-t)^{c-b-1}\frac{\mathrm{d}t}{(1-tz)^m} \\ &= 1 + \frac{mb}{c\cdot 1}z + \frac{m(m+1)b(b+1)}{c(c+1)\cdot 1\cdot 2}z^2 + \cdots. \end{aligned}$$

The hypergeometric functions that appear in the kernel are of the forms

$$F\big(\alpha+n-q-1, q+1, \alpha, |a|^2\big) \quad \text{and} \quad F\big(\alpha+n-q, q+1, \alpha+1, |a|^2\big)$$

and therefore, if we apply the well-known formula (see for instance [15]),

$$F(m,b,c,z) = (1-z)^{c-m-b}F(c-m, c-b, c, z),$$

the functions that appear are

$$\frac{F(-n+q+1, \alpha-q-1, \alpha, |a|^2)}{(1-|a|^2)^n} \quad \text{and} \quad \frac{F(-n+q+1, \alpha-q, \alpha+1, |a|^2)}{(1-|a|^2)^n}.$$

If m is a non-positive integer then actually $F(m,b,c,z)$ is polynomial in z of degree $-m$. More precisely,

$$F\big(-m, \alpha+1+\beta+m, \alpha+1, |a|^2\big) = \frac{\Gamma(\alpha+1)\Gamma(m+1)}{\Gamma(\alpha+1+m)}P_m^{\alpha,\beta}(1-2|a|^2),$$

where $P_m^{\alpha,\beta}(x)$ are the Jacobi polynomials (5.1). Thus the hypergeometric functions that appear in our expression for the kernel are in fact rational functions. If we replace F by its rational expression in (5.4) and plug it into (5.3) we obtain the stated formula for $k_\alpha^q(\zeta,z)$. Thus the proof of Theorem 5.1 is complete. ∎

Proof of Lemma 5.3. — The kernel for the operator $H_\alpha^q f(z)$ is obtained in the same manner from $H_{\alpha-1}^{b,q}(z,z_{n+1})$, and a similar computation as for k_α^q yields that

$$\begin{aligned} h_\alpha^q(\zeta,z) = c_{n,q,\alpha}\Big(&\frac{(\mathrm{d}\bar z\cdot\mathrm{d}\zeta)^q}{(1-\bar\zeta\cdot z)^{\alpha+n-q-1}(1-\zeta\cdot\bar z)^q}m_{\alpha-2,\alpha+n-q-1,q} \\ &- c\,\frac{q(\mathrm{d}\bar z\cdot\mathrm{d}\zeta)^{q-1}\wedge\bar\partial|z|^2\wedge\partial|\zeta|^2}{(1-\bar\zeta\cdot z)^{\alpha+n-q}(1-\zeta\cdot\bar z)^q}m_{\alpha-1,\alpha+n-q,q}\Big) \end{aligned}$$

for $q > 0$ and

$$h_\alpha(\zeta, z) = \frac{c_{n,0,\alpha}}{(1-\bar\zeta\cdot z)^{\alpha+n-1}}\int_{|\tau|<1}\frac{(1-|\tau|^2)^{\alpha-2}\big[\log(1-\zeta\cdot\bar z)+\log(1-\bar a\tau)\big]\mathrm{d}m(\tau)}{(1-a\bar\tau)^{\alpha+n-1}}$$

for some real constants $c_{n,q,\alpha}$ and c. Therefore these kernels satisfy

$$h_\alpha^q(z,\zeta) = (-1)^q\,\overline{h_\alpha^q(\zeta,z)}$$

and hence they correspond to self-adjoint operators. ∎

REMARK 6. — For $q > 0$, h_α^q can be expressed in terms of Jacobi polynomials in the same way as k_α^q. In the expression above for the function h_α^0, the first term is independent of a (because of rotation invariance) and hence is equal to $\log(1-\zeta\cdot\bar z)$. The function

$$\frac{\alpha-1}{\pi}\int_{|\tau|<1}\frac{(1-|\tau|^2)^{\alpha-2}\log(1-\bar a\tau)\,\mathrm{d}m(\tau)}{(1-a\bar\tau)^{\alpha+n-1}}$$

is equal to $\Phi(|a|^2)$, where $\Phi(0) = 0$ and (by a simple computation)

$$\Phi'(|a|^2) = \frac{\alpha+n}{\pi}\int_{|\tau|<1}\frac{(1-|\tau|^2)^{\alpha-1}\,\mathrm{d}m(\tau)}{(1-\bar\tau a)^{\alpha+n}(1-\tau\bar a)}\cdot$$

The last term is a rational function in $|a|^2$ as before and Φ will involve a logarithm. An alternative way to compute k_α^q is to first compute h_α^q and then use that $k_\alpha^q = \partial_\zeta h_\alpha^q$. In that computation it is worth to notice that

$$\partial m_{\alpha,j,k} = \frac{jk}{\alpha}m_{\alpha+1,j+1,k+1}\partial|a|^2.$$

Proof of Proposition 5.4. — We need the following slightly more general version of (4.1),

$$\langle\tilde f, a + \mathrm{d}\bar\xi\wedge b\rangle^{\tilde{}} = \langle f, a\rangle - \frac{\xi}{-\rho}\langle f, \bar\partial\rho\wedge b\rangle, \tag{5.9}$$

which is obtained in the same way. If $g = a + \mathrm{d}\bar\xi\wedge b$,

$$Mg(\zeta) = \frac{\alpha-1}{\pi}\int_{|\xi|<\sqrt{-\rho}}(-\rho)^{-(\alpha-1)}\big(-\rho-|\xi|^2\big)^{\alpha-2}\Big(a(\zeta,\xi) - \frac{\bar\xi}{-\rho}\bar\partial\rho\wedge b(\zeta,\xi)\Big)\,\mathrm{d}\lambda(\xi),$$

and after the change of variables $\xi = \sqrt{-\rho}\,\tau$ we get the desired expression. □

Notice, by the way, that the right-hand side of (5.9) is equal to

$$\langle \tilde{f}, a - (-\rho)^{-1}\bar{\xi}\bar{\partial}\rho \wedge b\rangle^{\sim}$$

which means that the mapping $a + \mathrm{d}\bar{\xi} \wedge b \mapsto a - (-\rho)^{-1}\bar{\xi}\bar{\partial}\rho \wedge b$ is the orthogonal projection onto the image of $\sim$.

6. Homotopy formulas for $\partial\bar{\partial}$

THEOREM 6.1. — *Let D be the ball and $\alpha > 0$. Then*

$$\frac{\partial P_\alpha f}{\partial z_j} = P_{\alpha+1}\Big(\frac{\partial f}{\partial \zeta_j}\Big), \tag{6.1}$$

if f is a smooth function, and

$$\frac{\partial K_\alpha f}{\partial z_j} = K_{\alpha+1}\Big(\frac{\partial f}{\partial \zeta_j}\Big), \tag{6.2}$$

if f is a smooth form.

The formula (6.1) is well-known and has been used by several authors, see *e.g.* [10]. It follows from the simple equality $(\partial f/\partial z_j, h)_{\alpha+1} = (f, z_j h)_\alpha$ for holomorphic h. However, (6.2) seems to be new.

Proof. — We first consider (6.2). In view of Theorem 5.2 we may assume that either $f \in \mathcal{K}_\alpha \cap \mathcal{E}_*$ or $f \in \mathcal{K}_\alpha^\perp \cap \mathcal{E}_*$. If $f \in \mathcal{K}_\alpha^\perp$, then $\bar{\partial}_\alpha^* f = 0$ and therefore $\bar{\partial}_{\alpha+1}^*(\partial f/\partial\zeta_j) = 0$ according to Proposition 3.2. In view of Proposition 3.4 therefore $\partial f/\partial\zeta_j \in \mathcal{K}_{\alpha+1}^\perp$. Hence by definition, $(\partial/\partial z_j)K_\alpha f = 0 = K_{\alpha+1}(\partial f/\partial\zeta_j)$. If u is the L_α^2 minimal solution to $\bar{\partial}u = f$, then by the same argument, $\partial u/\partial z_j$ is the $L_{\alpha+1}^2$ minimal solution to $\bar{\partial}v = \partial f/\partial z_j$. This proves (6.2) in case $\bar{\partial}f = 0$. The equality (6.1) follows from (6.2) and (1.3). □

In [2], [3] were found operators M_α acting on d-closed (1,1)-forms such that

$$M_\alpha \partial\bar{\partial} u = u - \Pi_\alpha u,$$

where Π_α is the orthogonal projection in L_α^2 onto the pluriharmonic functions. Explicitly,

$$\Pi_\alpha = P_\alpha + \overline{P}_\alpha - \Pi_\alpha^0, \tag{6.3}$$

where, as before, P_α is the holomorphic projection and $\Pi^0_\alpha = P_\alpha \overline{P}_\alpha$ is the (orthogonal) projection onto the constants. In particular, $M_\alpha\theta$ provides the L^2_α minimal solution to $\partial\bar\partial u = \theta$ if $\mathrm{d}\theta = 0$. Sharp estimates of the kernels were given.

By abstract nonsense it follows that there is an operator R_α, acting on 3-forms, such that $\partial\bar\partial M_\alpha\theta = \theta - R_\alpha\,\mathrm{d}\theta$ for any (reasonable) $(1,1)$-form θ. We will now construct (semi-explicit) operators M_α and R_α with these properties and moreover we will extend to the case of (q,q)-forms. It should be emphasized that the operator M_α constructed below, when acting on $(1,1)$-forms (probably) is not the same as the operator in [2], [3], but it anyway gives the minimal solutions, and admits the same estimates.

So far, the operator K_α is just defined on $(0,q)$-forms. We can extend to an operator $K_\alpha : \mathcal{E}_{p,*} \to \mathcal{E}_{p,*-1}$ by the formula

$$K_\alpha\big(a_{IJ}(\zeta)\,\mathrm{d}\bar\zeta^J \wedge \mathrm{d}\zeta^I\big)(z) = K_\alpha\big(a_{IJ}(\zeta)\,\mathrm{d}\bar\zeta^J\big) \wedge \mathrm{d}z^I.$$

The operator P_α is extended in the same way. It is then clear that the formula

$$\bar\partial K_\alpha + K_\alpha\bar\partial = I - P_\alpha$$

still holds. The main observation now is

Proposisition 6.2. — *Let K_α be the canonical operator with respect to α, and P_α the corresponding orthogonal holomorphic projection. Then*

$$\partial K_\alpha = -K_{\alpha+1}\partial \quad\text{and}\quad \partial P_\alpha = P_{\alpha+1}\partial.$$

Proof. — By the definition and Theorem 6.1 we have

$$\begin{aligned}
\partial K_\alpha\big(a(\zeta)\,\mathrm{d}\bar\zeta^J \wedge \mathrm{d}\zeta^I\big) &= \partial K_\alpha\big(a(\zeta)\,\mathrm{d}\bar\zeta^J\big) \wedge \mathrm{d}z^I \\
&= \sum (-1)^{|J|-1} \frac{\partial}{\partial z_k} K_\alpha(a\,\mathrm{d}\bar\zeta^J) \wedge \mathrm{d}z_k \wedge \mathrm{d}z^I \\
&= \sum (-1)^{|J|-1} K_{\alpha+1}\Big(\frac{\partial a}{\partial \zeta_k}\,\mathrm{d}\bar\zeta^J\Big) \wedge \mathrm{d}z_k \wedge \mathrm{d}z^I \\
&= K_{\alpha+1}\Big(\sum (-1)^{|J|-1} \frac{\partial a}{\partial \zeta_k}\,\mathrm{d}\bar\zeta^J \wedge \mathrm{d}\zeta_k \wedge \mathrm{d}\zeta^I\Big) \\
&= -K\big(\partial a(\zeta) \wedge \mathrm{d}\bar\zeta^J \wedge \mathrm{d}\zeta^I\big).
\end{aligned}$$

The statement about P_α now follows by the computation

$$\partial P_\alpha u = \partial(u - K_\alpha\bar\partial u) = \partial u - K_{\alpha+1}\bar\partial\partial u = P_{\alpha+1}\partial u. \quad \square$$

We define $\overline{K}_\alpha : \mathcal{E}_{*,q} \to \mathcal{E}_{*-1,q}$ by the formula

$$\overline{K}_\alpha f = \overline{K_\alpha \bar{f}}.$$

For (q,q)-forms we then let

$$M_\alpha = \tfrac{1}{2} i(\overline{K}_\alpha K_{\alpha+1} - K_\alpha \overline{K}_{\alpha+1}),$$

$$D_\alpha = \tfrac{1}{2}(\partial \overline{K}_\alpha \bar\partial K_\alpha + \bar\partial K_\alpha \partial \overline{K}_\alpha)$$

if $q \geq 1$. For $q = 0$ we let

$$D_\alpha = P_\alpha \overline{P}_\alpha.$$

Finally,

$$\Pi_\alpha = \bar\partial K_\alpha + \partial \overline{K}_\alpha - D_\alpha.$$

Certainly M_α, D_α and Π_α, map real forms onto real ones. Moreover we have

THEOREM 6.3.

(a) *The operator* $D_\alpha : \mathcal{E}_{q,q} \to \mathcal{E}_{q,q}$ *is a projection onto the* d-*closed forms.*

(b) *The operator* $\Pi_\alpha : \mathcal{E}_{q,q} \to \mathcal{E}_{q,q}$ *is a projection onto the* $\partial\bar\partial$-*closed forms.*

(c) *The operator* M_α *is a homotopy operator for* $i\,\partial\bar\partial$ *in the following sense,*

$$M_\alpha(i\,\partial\bar\partial u) = u - \Pi_\alpha u, \tag{6.4}$$

$$i\,\partial\bar\partial M_\alpha \theta = D_{\alpha+1}\theta = \theta - R_{\alpha+1}\,\mathrm{d}\theta, \tag{6.5}$$

where

$$R_{\alpha+1}\,\mathrm{d} = \overline{K}_{\alpha+1}\partial + K_{\alpha+1}\bar\partial - \tfrac{1}{2}\big(\overline{K}_{\alpha+1}\partial K_{\alpha+1}\bar\partial + K_{\alpha+1}\bar\partial \overline{K}_{\alpha+1}\partial\big).$$

In particular, $M_\alpha\theta$ is a solution to $i\,\partial\bar\partial u = \theta$ if $\mathrm{d}\theta = 0$. Actually any operator $\overline{K}_\alpha K_\beta$ solves the $\partial\bar\partial$-equation. This is obvious from Proposition 7.1, since if $\mathrm{d}\theta = 0$ then $K_\beta\theta$ is a ∂-closed solution to $\bar\partial f = \theta$ and thus $\overline{K}_\alpha f$ solves $\partial u = f$ which means that $\partial\bar\partial u = \theta$.

Recall that $\bar\partial K_\alpha$ is a projection of (q,q)-forms onto the $\bar\partial$-closed ones, and analogously for its conjugate, and therefore (b) says that the projection Π of (q,q)-forms onto $\operatorname{Ker}\partial\bar\partial$ is the sum of one projection onto

$\operatorname{Ker}\bar\partial$ and one onto $\operatorname{Ker}\partial$ minus one projection onto $\operatorname{Ker} d$. When $q=0$ this is just the formula (6.3) above.

Proof. — It is enough to prove part (c) since (a) and (b) are immediate consequences of (6.4) and (6.5). To see (6.4), note that by Proposition 6.2,

$$\begin{aligned}-\partial\bar\partial \overline{K}_\alpha K_{\alpha+1} &= \partial\overline{K}_{\alpha+1}\bar\partial K_{\alpha+1} = (I-\overline{K}_{\alpha+1}\partial)(I-K_{\alpha+1}\bar\partial)\\ &= I-\overline{K}_{\alpha+1}\partial - K_{\alpha+1}\bar\partial + \overline{K}_{\alpha+1}\partial K_{\alpha+1}\bar\partial\end{aligned}$$

if $q\geq 2$. From this, (6.4) follows. The case $q=1$ is completely analogous, just replace the projections $\bar\partial K_\alpha$ and $\partial\overline{K}_\alpha$ by P_α and $\overline{P}_\alpha$. By Proposition 6.2 again we have

$$\begin{aligned}\overline{K}_\alpha K_{\alpha+1}\partial\bar\partial &= -\overline{K}_\alpha\partial K_\alpha\bar\partial = (I-\partial\overline{K}_\alpha)(I-\bar\partial K_\alpha)\\ &= I-\big(\partial\overline{K}_\alpha+\bar\partial K_\alpha+\partial\overline{K}_\alpha\bar\partial K_\alpha\big),\end{aligned}$$

which implies (6.5). ∎

Instead of trying to give a complete description of all the operators involved, we concentrate on M_α. It is possible to give a semi-explicit expression as some real-analytic function of the quantities $1-|\zeta|^2$, $1-\bar\zeta\cdot z$ and some simple forms but we restrict our ambition to indicate that it admits some expected L^1-estimates. A straight forward estimation, see [4], [5], gives the estimates

$$\int_{\mathbb{B}}(1-|z|^2)^{\ell}|K_\alpha f|\,dV \leq C\int_{\mathbb{B}}(1-|z|^2)^{\ell-1/2}|f|\,dV, \tag{6.6}$$

for any $\ell>0$ and

$$\int_{\partial\mathbb{B}}|K_\alpha f|\,dV \leq C\int_{\mathbb{B}}(1-|z|^2)^{-1/2}|f|\,d\sigma, \tag{6.7}$$

if α is sufficiently large (depending on ℓ). For the operators M_α we have the following expected result.

PROPOSITION 6.4. — *For (q,q)-forms θ and any $\ell>0$ we have the estimates*

$$\int_{\mathbb{B}}(1-|z|^2)^{\ell}|M\theta|\,dV \leq C\int_{\mathbb{B}}(1-|z|^2)^{\ell+1}|\theta|\,dV, \tag{6.8}$$

and

$$\int_{\partial\mathbb{B}} |M\theta|\,\mathrm{d}\sigma \le C \int_{\mathbb{B}} |\theta|\,\mathrm{d}V. \tag{6.9}$$

if $M = M_\alpha$ *and* α *is large enough (depending on* ℓ*).*

Note that for a (p,q) form θ,

$$|\theta| \sim (-\rho)|\theta|_\beta + \sqrt{-\rho}\,\big(|\partial\rho\wedge\theta|_\beta + |\bar\partial\rho\wedge\theta|_\beta\big) + |\partial\rho\wedge\bar\partial\rho\wedge\theta|_\beta$$

For $q = 1$, eq. (6.9) gives the Henkin-Skoda estimate of solutions to the $\partial\bar\partial$-equation, and therefore the statement may be thought of as a generalized version. It can certainly be proved by the usual method as well, but our purpose is to point out that actually our operator M works. Proposition 6.5 follows from (6.6) and (6.7), observing that the kernel for the commutator

$$f \longmapsto \partial|z|^2 \wedge K_\alpha f - K_\alpha\big(\partial|\zeta|^2 \wedge f\big)$$

is $\mathcal{O}(|\zeta - z|)$ times the kernel for K_α. For the details we refer to [6].

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