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A model for Toepliz operators in the space of entire functions of exponential type

A MODEL FOR TOEPLIZ OPERATORS IN THE SPACE

OF ENTIRE FUNCTIONS OF EXPONENTIAL TYPE

BY

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RÉSUMÉ. — On construit, à l'aide de paquets d'ondes et d'un transport, un modèle pour les opérateurs de Toeplitz. Ce modèle est réalisé comme la composée d'un opérateur de multiplication par une fonction suivi d'un projecteur fixe. On définit ensuite une notion naturelle de symbole total pour les opérateurs de Toeplitz et l'on montre qu'une condition suffisante pour qu'un opérateur de Toeplitz soit positif est que son symbole total soit positif. L'inégalité fine de Gårding à un ordre arbitraire en découle.

ABSTRACT. — By use of wave packets and a push-forwards, we build a model for Toeplitz operators. This model is achived as the composition of the operator of multiplication by a function followed by a fixed projector. We define next a natural notion of total symbol for Toeplitz operators and we show that a sufficient condition to a Toeplitz operator to be positive is that its total symbol is nonnegative. Sharp Gårding inequality of any order follows from that.

0. Introduction

This paper summarizes an important part of my thesis. It describes a new modelization of Toeplitz operators. It establishes also a connection between the theory of pseudodifferential operators and the theory of spaces of entire functions. Moreover, we give a simple characterisation of Toeplitz operators as compositions of a multiplication by a function defined on $\mathbb{C}^n$ and a fixed projection.

We provide also a natural notion of « total symbol » for Toeplitz operators and show that nonnegativity of the total symbol is a sufficient condition for positivity of the corresponding Toeplitz operators. Sharp Gårding inequalities of any order for Toeplitz operators follow from it.

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1. Preliminaries

Let X be the boundary of a strictly convex bounded open set D of $\mathbb{C}^n$ that we assume to be smooth. We ask also D to contain the origin and can define its supporting function by

$$h(z) = \sup_{x \in D} \operatorname{Re} \bar{w} \cdot z.$$

$h(z)$ is a smooth convex and positively homogeneous function defined on $\mathbb{C}^n \setminus 0$. Since D is strictly convex, we can equip X with a differential form $\alpha \in \Omega^1(X)$ such that $\alpha \wedge (d\alpha)^{n-1}$ vanishes nowhere in X. This makes X into a contact manifold; equivalently, this provides a symplectic submanifold Σ of $T^*(X)$. Indeed, D is defined by an inequation $r < 0$ with $r \in C^\infty(\mathbb{C}^n)$, $dr \neq 0$ on X and set $\alpha = -i\partial r_{|X}$. Σ is then the halfline bundle spanned by α, i.e.

$$\Sigma = \big\{ (x, \xi) \, ; \; \xi = t\alpha_X, \; t > 0 \big\}.$$

Moreover, there is a positive measure $d\sigma$ on X with smooth positive density so that L^2 norms are well defined. $H^S(X)$ is the Sobolev space of generalized functions with s derivatives in L^2 and let $\mathcal{O}^s(X)$ be the intersection $H^s(X) \cap \operatorname{Ker} \bar{\partial}_b$ where $\bar{\partial}_b$ is the tangential Cauchy-Riemann system. $\mathcal{O}^\infty(X)$ and $\mathcal{O}^{-\infty}(X)$ stand respectively for $\bigcap \mathcal{O}^s(X)$ and $\bigcup \mathcal{O}^s(X)$. Let S_0 be the orthogonal projection : $H^s(X) \to \mathcal{O}^s(X)$. If Q is a pseudodifferential operator of order $m \in \mathbb{R}$ on X, the Toeplitz operator $T_Q : \mathcal{O}^s(X) \to \mathcal{O}^{s-m}(X)$ is defined by $T_Q f = S_o(Qf)$. We define also the principal symbol of the Toeplitz operator T_Q to be the principal symbol of Q restricted to the symplectic manifold Σ, that is :

$$\sigma(T_Q) = \sigma(Q)_{|\Sigma}.$$

Further informations about these notions and notations can be found in [1], [2] and [3].

2. The wave packets transform

We first describe the Hilbert spaces of « entire functions of exponential type » we need and then define wave packets and wave packets transform.

DEFINITION. — *Let $s \in \mathbb{R}$, we denote by K^s the space of Lebesgue measurable functions u defined on $\mathbb{C}^n$ such that :*

$$\int_{\mathbb{C}^n} |u|^2 e^{-2h} q^{2s-1/2} \, d\lambda < +\infty.$$

$q(z)$ is the quantity $(1 + z \cdot \bar{z})^{1/2}$ and $d\lambda$ is the Lebesgue measure on $\mathbb{C}^n$. $\mathcal{K}^s$ is equiped with the inner product

$$(u, v)_s = \int_{\mathbb{C}^n} u \cdot \bar{v} \, e^{-2h} q^{2s-1/2} \, d\lambda.$$

Let E^s be the subclass of all functions $u \in \mathcal{K}^s$ that are analytic, that is :

$$\mathcal{E}^s = \mathcal{K}^s \cap A(\mathbb{C}^n).$$

At a point $x \in X$ we want to define a wave packet as the function $z \mapsto e^{\bar{x}z - h(z)}$, for every $z \in \mathbb{C}^n$. This leads to introduce the wave packets transform :

$$Wf(z) = \int_X e^{\bar{x} \cdot z} f(x) \, d\sigma(x).$$

The formal adjoint of W is thus defined by the integral :

$$W^* u(x) = \int_{\mathbb{C}^n} e^{x \cdot \bar{z} - 2h(z)} q(z)^{-1/2} u(z) \, d\lambda(z).$$

Obviously, W is a linear operator acting on $C^\infty(X)$ into the intersection $\mathcal{E}^\infty = \bigcap \mathcal{E}^s$. We show now continuity of the operators W and W^* as well as we work out their domain and range.

LEMMA 1.

(i) *W is a continuous linear operator : $C^\infty(X) \to \mathcal{E}^\infty$ and can be extended continuously into $C^{-\infty}(X) \to \mathcal{E}^{-\infty}$.*

(ii) *W^* is a continuous linear operator : $\mathcal{K}^\infty \to \mathcal{O}^\infty(X)$ and can be extended continuously into $\mathcal{K}^{-\infty} \to \mathcal{O}^{-\infty}(X)$.*

Proof. — There exists a vector field L such that $L(e^{\bar{x} \cdot z}) = e^{\bar{x} \cdot z}$; this L is of the form $|z|^{-1} L_1(z/|z|, x, \partial_x)$. L is defined as follow : there is a nonnegative function $g \in C^\infty(\mathbb{C}^n)$ homogeneous of degree 0 such that

$$\left(z \cdot d\bar{y} \mid z \cdot d\bar{y} \right) = |z|^2 g(z)$$

where (. | .) is the scalar product of differential forms on X. Then, we set :

$$L(f) = \frac{1}{|z|^2 g(z)} (z \cdot \mathrm{d}\bar{y} \mid \mathrm{d}f) = \frac{1}{|z|} L_1(f).$$

Thus, for every nonnegative integer N and $f \in C^\infty(X)$ we have

$$\int_X \mathrm{e}^{\bar{x}\cdot z} f(x)\, \mathrm{d}\sigma(x) = \int_X \mathrm{e}^{\bar{x}\cdot z} (^t L)^N f(x)\, \mathrm{d}\sigma(x)$$

so that we get the inequality

$$\left| Wf(z) \right| \leq C_N(f) |z|^{-2N} \mathrm{e}^{h(z)}$$

with

$$C_N(f) = \sup_{|z|=1} \int_X \left| (^t L)^N f \right| \mathrm{d}\sigma.$$

This proves that $Wf \in \mathcal{E}^\infty$. Continuity follows from the fact that $f \mapsto C_N(f)$ is a seminorm on $C^N(X)$. Moreover, W is well defined on distribution densities space $C^{-\infty}(X)$ since X is compact and the function $x \mapsto \mathrm{e}^{\bar{x}\cdot z}$ is smooth. Indeed, let f be a distribution of $C^{-\infty}(X)$, the value $Wf(z)$ is equal to $\langle f, e^{\bar{x}\cdot z} \rangle$ and moreover, a distribution with compact support is of finite order. The inequality that results from the last fact shows that $Wf \in \mathcal{E}^{-\infty}$ and the continuity of $W : C^{-\infty}(X) \to \mathcal{E}^{-\infty}$ when $C^{-\infty}(X)$ is equiped with the weak topology. The first statement is then proved and the second one can be obtained by duality. $\quad\square$

LEMMA 2.

(i) *The range of the operator* $W : C^\infty(X) \to \mathcal{E}^\infty$ *is dense.*

(ii) *The range of the operator* $W^* : \mathcal{K}^\infty \to \mathcal{O}^\infty(X)$ *is dense.*

Proof. — Let $\mathcal{E}^-$ be the space of entire functions $u \in A(\mathbb{C}^n)$ such that $|u(z)| \leq C\mathrm{e}^{h(z)-r|z|}$ for some $C > 0$ and $r > 0$. Such a function is the Laplace transform of a measure μ defined on X. We get thus $u(z) = \langle \mu, \mathrm{e}^{\bar{x}\cdot z} \rangle$. Let $f(y) = \int S(y, y')\, \mathrm{d}\mu(y')$ be the projection onto $C^\infty(X)$ thanks to the Szegö projection. This integral is definite since the Szegö kernel is definite for y, y' belonging to $\overline{D}$ and is smooth in the complement of the diagonal of $X \times X$ ([4]). We obtain $u(z) = \langle \mu, \mathrm{e}^{\bar{x}\cdot z} \rangle = Wf(z)$. This shows the inclusion $\mathcal{E}^- \subseteq W(C^\infty(X))$. Since the inclusion $\mathcal{E}^- \subseteq \mathcal{E}^s$ is continuous with dense range, we conclude to the first statement.

The image of $C_0^\infty(\mathbb{C}^n)$ by W^* contains all exponential functions of type $x \mapsto C\mathrm{e}^{a\cdot x}$ with $C > 0$ and $a \in \mathbb{C}^n$. The set generated by these

exponentials is dense in $\mathcal{O}^0(X)$ and then in $\mathcal{O}^\infty(X)$ since the range of W^* is obviously analytic. Moreover, we get these inclusions :

$$\{Ce^{a \cdot x}\} \subseteq W^*\big(C_0^\infty(\mathbb{C}^n)\big) \subseteq W^*(\mathcal{K}^\infty) \subseteq \mathcal{O}^\infty(X),$$

which implies the second statement after continuity of W^* and density of $\{Ce^{a \cdot x}\}$ in $\mathcal{O}^\infty(X)$. $\square$

The composition W^*W is defined on $C^\infty(X)$ and ranges into $\mathcal{O}^\infty(X)$. The composition formula reads :

$$W^*W f(x) = \iint_{X \times \mathbb{C}^n} \mathrm{e}^{x \cdot \bar{z} + \bar{y} \cdot z - 2h(z)} q(z)^{-\frac{1}{2}} f(y) \, \mathrm{d}\sigma(y) \, \mathrm{d}\lambda(z).$$

LEMMA 3. — W^*W *is a Fourier integral operator with complex-valued phase function, of order* 0, *elliptic, « adapted to* id_Σ *». Therefore,* W^*W *is a continuous operator on* $L^2(X)$ *and induces on* X *a « quantized contact structure ».* (see [2], [3]).

Proof. — Note that the symplectic cone Σ is the set of covectors (x, ξ) of the form $x = 2\partial h/\partial \bar{z}(z)$ and $\xi = -i\bar{z} \cdot \mathrm{d}x_{|X}$. Now, the phase function is :

$$\phi = -i\big(x \cdot \bar{z} + \bar{y} \cdot z - 2h(z)\big).$$

It can easily be checked that ϕ is a good nondegenerated complex-valued phase function [8]. The canonical relation $\mathcal{C}$ associated to ϕ is the range of the cotangent map

$$x = 2\frac{\partial h}{\partial \bar{z}}(z), \qquad \xi = -i\bar{z},$$

$$\bar{y} = 2\frac{\partial h}{\partial z}(z), \qquad \bar{\eta} = iz.$$

The real points of this relation form the graph of id_Σ. It follows that the Schwartz kernel of W^*W is a lagrangien distribution of $I^0(X \times X \,;\, \mathcal{C})$ (see [7], [8]). It can be shown that the set of critical points

$$C = \Big\{(x, y, z) \in X \times X \times \mathbb{C}^n \setminus 0 \,;\, x = y = 2\frac{\partial h}{\partial \bar{z}}(z)\Big\}$$

is a conic submanifold along which $\mathrm{Im}\,\phi$ is transversally elliptic. A detailed proof of this can be found in my thesis; it is similar to that one of corollary (1.3) in [4]. $\square$

If $f \in H^s(X)$ with $s \geq 0$, it is clear that $Wf \in \mathcal{E}^s$. Indeed, same arguments as in the last lemma show that $W^* q^{2s} W$ is a Fourier integral operator of order $2s$. We get :

$$\left(W^* q^{2s} W f, f\right) = \left(q^{2s} W f, W f\right)_0 = \int_{\mathbb{C}^n} q^{2s} |Wf|^2 \mathrm{e}^{-2h} q^{-\frac{1}{2}} \, \mathrm{d}\lambda$$

which is finite. It follows that $Wf \in \mathcal{E}^s$. When $s < 0$, we can deduce the same statement thanks to a duality argument.

LEMMA 4. — *For every* $s \in \mathbb{R}$:
  (i) $W(H^s(X)) = W(\mathcal{O}^s(X))$.
  (ii) $W^*(\mathcal{K}^s) = W^*(\mathcal{E}^s)$.

Proof. — The first statement is obvious since $W S_0 = W$. Let $P_0 : \mathcal{K}^0 \to \mathcal{E}^0$ be the orthogonal projection (recall that $\mathcal{E}^s = \mathcal{K}^s \cap \operatorname{Ker} \bar{\partial}$). As for the Szegö projection, P_0 can be restricted or extented to : $\mathcal{K}^s \to \mathcal{E}^s$. Thus the second statement follows from the equality $W^* P_0 = W^*$.   □

We can state now the main theorem of this section.

THEOREM 1. — *For every* $s \in \mathbb{R}$:
  (i) $W : \mathcal{O}^s(X) \to \mathcal{E}^s$ *is bijective and continuous.*
  (ii) $W^* : \mathcal{E}^s \to \mathcal{O}^s(X)$ *is bijective and continuous.*

Proof. — From LEMMA 4, we have :

$$W\left(C^\infty(X)\right) \subseteq W\left(H^s(X)\right) = W\left(\mathcal{O}^s(X)\right) \subseteq \mathcal{E}^s.$$

LEMMA 2 shows in fact that $W(C^\infty(X))$ is dense in $\mathcal{E}^s$ and since $W^* W$ is elliptic, it follows that the range is closed; therefore, the last two inclusions become equalities and the map $W : \mathcal{O}^s(X) \to \mathcal{E}^s$ is onto. LEMMA 2 and 4 show that $W^*(\mathcal{E}^s)$ is dense in $\mathcal{O}^s(X)$ which implies that the orthogonal set of $W^*(\mathcal{E}^s)$ is reduced to $\{0\}$. But this orthogonal set contains $\operatorname{Ker} W$ which shows that W is one-to-one. Continuity of $W : \mathcal{O}^s(X) \to \mathcal{E}^s$ follows from LEMMA 3. The first statement is thus proved and the second one is similarly proved.   □

This theorem expresses the fact that the space $\mathcal{E}^s$ is a model for the initial waves $\mathcal{O}^s(X)$. In other words, we may say that we made a microlocal analysis of these waves and have obtained their asymptotic behaviour in terms of entire functions of exponentiel type.

3. The transport

In the section we will transport Toeplitz operators into operators that will be called « multiplicators ». We define also these operators and show how they arise from the transport.

The set of bounded linear operators of order $m \in \mathbb{R}$ defined on $\mathcal{O}^s(X)$ is denoted by $L^m(\mathcal{O})$. The intersection $L^{-\infty}(\mathcal{O}) = \bigcap L^m(\mathcal{O})$ is the space of regularizing operators; it is in fact equal to $\mathcal{T}^{-\infty}(X)$, the set of Toeplitz operators associated with a regularizing pseudodifferential operator on X.

We want to make the wave packets transform into a unitary operator. Thus, since W^*W is elliptic, we may set :

$$U = W(W^*W)^{-1/2}.$$

Clearly, we get
$$UU^* = P_0 \quad \text{and} \quad U^*U = S_0,$$

where P_0 is the orthogonal projection : $\mathcal{K}^s \to \mathcal{E}^s$ and S_0 is the Szegö projection : $H^s(X) \to \mathcal{O}^s(X)$.

In the proof of LEMMA 3 of § 2, we see that the symplectic cone is the image of $\mathbb{C}^n$ by the map :

$$\chi : z \mapsto \left(2\frac{\partial h}{\partial \bar{z}}(z), -i\bar{z}\right).$$

χ is a canonical map (symplectomorphism) if $\mathbb{C}^n$ is equiped with the symplectic form $2i\partial\bar{\partial}h$. Indeed, the Liouville form is

$$\lambda = \operatorname{Re}\xi \cdot \mathrm{d}x = \operatorname{Re}\big(\mathrm{d}(x \cdot \xi) - x \cdot \mathrm{d}\xi\big)$$

since $x = 2\dfrac{\partial h}{\partial \bar{z}}$ and $\xi = -i\bar{z}$ we get $x \cdot \mathrm{d}\xi = -2i\bar{\partial}h$ and the symplectic form is
$$\omega = \mathrm{d}\lambda = -2i\,\mathrm{d}\left(\frac{\partial h}{\partial \bar{z}} \cdot \mathrm{d}\bar{z}\right) = 2i\partial\bar{\partial}h.$$

Obviously, W^*W and U^*U are Fourier integral operators associated with the same complex canonical relation $\mathcal{C}$ which is also the one of any Toeplitz operators when they are viewed as Fourier integral operators.

We define now our « multiplicators ».

From now on, we take $m \in \mathbb{R}$ and $\delta \in [0, \frac{1}{2})$. We denote by $\mathcal{T}_\delta^m(X)$ the set of Toeplitz operators of order m and type $(1-\delta, \delta)$ and by Σ_δ^m the space of all functions of the form $a \circ \chi$ where $a \in S_{1-\delta}^m(\Sigma)$ is a pseudodifferential symbol of order m and type $(1 - \delta, \delta)$.

DEFINITION. — *A multiplicator of order* m *and type* $(1 - \delta, \delta)$ *is an operator defined by*

$$u \mapsto P_0(a \cdot u)$$

for every $u \in \mathcal{E}^s$ *and some* $a \in \Sigma_\delta^m$.

This operator is denoted by $\mathrm{Op}(a)$ and the set of such operators is denoted by

$$\mathrm{OP}\,\Sigma_\delta^m = \left\{ \mathrm{Op}(a)\,;\; a \in \Sigma_\delta^m \right\}.$$

If $L^m(\mathcal{E})$ is the set of bounded linear operators defined on $\mathcal{E}^s$ of order m, then $\mathrm{OP}\,\Sigma_\delta^m$ is included in $L^m(\mathcal{E})$.

Now any operator $T \in L^m(\mathcal{O})$ can be transported into an operator of $L^m(\mathcal{E})$ by the transport map :

$$T \mapsto UTU^*.$$

LEMMA 1. — *For every* $T \in \mathcal{T}_\delta^m(X)$, *we have*

$$UTU^* - \mathrm{Op}\big(\sigma(T) \circ \chi\big) \in L^{m+2\delta-1}(\mathcal{E}).$$

Recall that $\sigma(T)$ is the principal symbol of T.

Proof. — Let P be a pseudodifferential operator such that $T = S_0 P S_0$. The assertion is equivalent to

$$W^*W\big(PW^*W - W^*\sigma(T) \circ \chi W\big) \in L^{m+2\delta-1}(\mathcal{O}).$$

For $f \in \mathcal{O}^\infty(X)$, write $u = Wf$. In order to estimate PW^*u, we will use local coordinates, a partition of unity and the formula of stationnary phase [8, formula 2.28]. To that purpose, take local coordinates in a neighbourhood of a point $x \in X$ and let $p(x, \xi)$ be the local expression of the symbol of P. If $\tilde{p}$ is an almost-analytic extension of p in the sense of A. MELIN and J. SJÖSTRAND [8] we can use the stationnary phase method to estimate the local expression of PW^*u at the point x. After somme calculations, we get

$$P\big(e^{\bar{x}\cdot z - 2h(z)}\big) = e^{\bar{x}\cdot z - 2h(z)}\big(p(x, -i\bar{z}) + s(x, z)\big)$$

with $s(x, z) = \mathrm{O}(|z|^{m+2\delta-1})$ when $|z| \to +\infty$.

It remains to show that the operator $f \mapsto PW^*Wf - W^*p \circ \chi Wf$ is of order $m+2\delta-1$. Indeed, this operator is a Fourier integral operator with complex phase $\phi = -i(x \cdot \bar{z} + \bar{y} \cdot z - 2h(z))$ whose critical points (x, ξ) are such that $x = 2\partial h/\partial \bar{z}(z)$ and $\xi = -i\bar{z}$. Together with the last estimation, the proof is complete. $\square$

LEMMA 2. — *For every $a \in \Sigma_\delta^m$, we have $U^* a U \in \mathcal{T}_\delta^m(X)$.*

Proof. — As was said before, any Fourier integral operator associated with the canonical relation $\mathcal{C}$ is a Toeplitz operator : So is $U^* a U$ after the LEMMA 3 of §2.

This leads to simple characterization of Toeplitz operators :

THEOREM 2. — *For every $m \in \mathbb{R}$ and $\delta \in [0, \frac{1}{2})$, we have :*
  (i) $\mathcal{T}_\delta^m(X) = U^* \operatorname{OP} \Sigma_\delta^m U + L^{-\infty}(\mathcal{O})$.
  (ii) $U \mathcal{T}_\delta^m(X) U^* = \operatorname{OP} \Sigma_\delta^m + L^{-\infty}(\mathcal{E})$.

Proof. — Let $T \in \mathcal{T}_\delta^m(X)$ according to LEMMAS 1 and 2, we have

$$T - U \operatorname{Op}(\sigma(T) \circ \chi) U^* \in \mathcal{T}_\delta^{m+2\delta-1}(X).$$

Set $a_0 = \sigma(T) \circ \chi$. By an iterative procedure, we get two sequences $a_j \in \Sigma_\delta^{m+j(2\delta-1)}$ and $p_j \in S_{1-\delta}^{m+j(2\delta-1)}(\Sigma)$ such that $a_j = p_j \circ \chi$.

Since $2\delta - 1 < 0$, there exists a symbol $p \in S_{1-\delta}^m(\Sigma)$ such that $p \sim \Sigma p_j$ and $T - U \operatorname{Op}(p \circ \chi) U^* \in L^{-\infty}(\mathcal{O})$. This gives :

$$\mathcal{T}_\delta^m(X) \subseteq U^* \operatorname{OP} \Sigma_\delta^m U + L^{-\infty}(\mathcal{O}).$$

The opposite inclusion follows from LEMMA 2 and from the fact that a regularizing operator of $L^{-\infty}(\mathcal{O})$ is a Toeplitz operator of any order. This shows the first statement and the second one can be deduced from the first one thanks to the transport.

It is important to say that $L^{-\infty}(\mathcal{E})$ is not completely included in $\operatorname{OP} \Sigma^{-\infty}$, that is : regularizing operators on the $\mathcal{E}^s$-side may have no symbol in $\Sigma^{-\infty}$. See for instance the operator that for every $u \in \mathcal{E}^s$ gives the constant function $u(0)$.

Note also that an analytic differential operator on $\mathbb{C}^n$ with constant complex coefficients and of any order belongs to $L^0(\mathcal{E})$. On the other hand, multiplication by a polynomial of order m in the z and $\bar{z}$ variables is an element of $L^m(\mathcal{E})$. This is a situation opposite to the Toeplitz operators'one.

4. Symbols

We define in this section the notions of principal and total symbols of Toeplitz operators. We describe the symbol calculus which is similar to the pseudodifferential's one and give sharp Gårding inequalities.

As for pseudodifferential operators, the principal symbol of a multiplicator is defined by the isomorphism

$$\operatorname{OP} \Sigma_\delta^m / \operatorname{OP} \Sigma_\delta^{m-1} \simeq \Sigma_\delta^m / \Sigma_\delta^{m-1}.$$

Principal symbol of a multiplicator A is written $\sigma_{pr}(A)$; formally, we just wrote $\sigma(A)$, but because of the introduction of the notion of «total» symbol, we need to subscribe «pr» or «tot» below σ. According to THEOREM 2, a multiplicator $A \in \mathrm{OP}\,\Sigma_\delta^m$ can be written in the form :

$$A = UTU^*$$

with $T \in \mathcal{T}_\delta^m(X)$ and LEMMA 1 of § 3 shows that

$$\sigma_{\mathrm{pr}}(T) \circ \chi = \sigma_{\mathrm{pr}}(A).$$

The principal symbol calculus of Toeplitz operators is completely transported (see [2], [3]).

THEOREM 3. Principal symbol calculus. — *Let* $m, m' \in \mathbb{R}$ *and let* $\delta \in [0, 1/2)$; $A \in \mathrm{OP}\,\Sigma_\delta^m$ *and* $B \in \mathrm{OP}\,\Sigma_\delta^{m'}$, *we have* :

   (i) $A^* \in \mathrm{OP}\,\Sigma_\delta^m$ *and* $\sigma_{\mathrm{pr}}(A^*) = \overline{\sigma_{\mathrm{pr}}(A)}$.
   (ii) $AB \in \mathrm{OP}\,\Sigma_\delta^{m+m'} + L^{-\infty}(E)$ *and* $\sigma_{\mathrm{pr}}(AB) = \sigma_{\mathrm{pr}}(A) \cdot \sigma_{\mathrm{pr}}(B)$.
   (iii) $[A, B] \in \mathrm{OP}\,\Sigma_\delta^{m+m'+2\delta-1} + L^{-\infty}(E)$ *and*

$$\sigma_{\mathrm{pr}}([A, B]) = -i\{\sigma_{\mathrm{pr}}(A), \sigma_{\mathrm{pr}}(B)\}.$$

Here $\{\,.\,,\,.\,\}$ is the Poisson brackets on $\mathbb{C}^n$ equiped with the symplectic form $2i\partial\bar\partial h$. We will not reproduce here the proof since it is only an easy exercice of translation.

A Toeplitz operator, modulo regularizing operators, is not canonically represented by a total symbol. Anyway, recall the definition of a multiplicator :

$$\mathrm{Op}(a)u = P_0(a \cdot u).$$

It is natural to call this $a \in \Sigma_\delta^m$ the total symbol of the multiplicator $\mathrm{Op}(a)$. According to THEOREM 2, any Toeplitz operator $T \in \mathcal{T}_\delta^m(X)$ can be written :

$$T = U\,\mathrm{Op}(a)U^* + R$$

with $a \in \Sigma_\delta^m$ and $R \in L^{-\infty}(\mathcal{O})$.

DEFINITION. — *The class modulo* $\Sigma^{-\infty}$ *of the symbol* $a \in \Sigma_\delta^m$ *that satisfies the equality*

$$T = U\,\mathrm{Op}(a)U^* + R$$

will be called the «total symbol of T *» and written* $\sigma_{\mathrm{tot}}(T)$.

Symbol calculus for total symbols is much more complicated. The only obvious result we have is

$$\sigma_{\text{tot}}(A^*) = \overline{\sigma_{\text{tot}}(A)}$$

for any $A \in \text{OP} \, \Sigma_\delta^m$. This follows from the equalities :

$$\big(\text{Op}(a)u, v\big)_0 = (au, v)_0 = (u, \bar{a}v)_0 = \big(u, \text{Op}(\bar{a})v\big)_0.$$

The most important feature of multiplicators deals with positivity.

THEOREM 4. — *A sufficient condition for a Toeplitz operator to be positive is that its total symbol is nonnegative.*

Proof. — If $a \in \Sigma_\delta^m$ is nonnegative, it is clear that the multiplicator $\text{Op}(a)$ is positive since

$$\big(\text{Op}(a)u, u\big)_s = \int_{\mathbb{C}^n} a \cdot |u|^2 e^{-2h} q^{2s - \frac{1}{2}} \, d\lambda.$$

Moreover, positivity of a Toeplitz operator is equivalent to the positivity of the transported operator modulo $L^{-\infty}(\mathcal{E})$ which is a multiplicator; this follows from THEOREM 1. □

This gives an easy sharp Gårding inequality.

COROLLARY. — *Let $T \in \mathcal{T}_\delta^1(X)$ such that $\sigma_{\text{pr}}(T) \geq 0$, there exists a constant $C > 0$ such that*

$$\text{Re}(Tf, f) \geq -C\|f\|^2$$

for any $f \in \mathcal{O}^1(X)$.

The proof is obvious.

We can state also the sharp Gårding inequality of any order for Toeplitz operators.

THEOREM 5. — *Let $T \in \mathcal{T}_\delta^m(X)$ be a Toeplitz operator. Assume that its total symbol $a \in \Sigma^m$ can be split into $a = a_1 + a_2$ such that a_1 is nonnegative and a_2 is of order zero. Then there exists $C > 0$ such that*

$$\text{Re}(Tf, f) \geq -C\|f\|^2$$

for every $f \in \mathcal{O}^{m-1}(X)$.

The proof is obvious.

This result can be applied to regular total symbols, that is symbols that can be written as an asymptotic series

$$a \sim \Sigma a_{m-j}$$

such that $a_{m-j} \in \Sigma^{m-j}$ is positively homogeneous of order $(m - j)$, m being the order of a. In that case, we can ask that the part $(a_m + \cdots + a_1)$ be nonnegative in order to get the sharp Gårding inequality.

Notes

From the corollary we get the sharp Gårding inequality of order 1 for pseudodifferential operators : this was shown by L. HÖRMANDER in 1966. To my knowledge, the statement is new for Toeplitz operators. As for the sharp Gårding inequality of any order, it is new for Toeplitz operators.

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