BULLETIN DE LA S. M. F.

H. ALEXANDER

Ends of varieties

Bulletin de la S. M. F., tome 120, nº 4 (1992), p. 523-547
<http://www.numdam.org/item?id=BSMF_1992__120_4_523_0>

Bull. Soc. math. France,
120, 1992, p. 523–547.

ENDS OF VARIETIES

BY

H. ALEXANDER (*)

RÉSUMÉ. — Nous étudions le comportement au bord d'une sous-variété de dimension complexe 1 dans un domaine pseudoconvexe de $\mathbb{C}^n$. Dans le cas d'une sous-variété avec un bord de mesure linéaire localement finie, nous obtenons des résultats sur les tangentes au bord, l'unicité de la sous-variété ayant un bord donné, l'accessibilité d'un point du bord et la mesure harmonique sur le bord.

ABSTRACT. — We study the boundary behavior of a one-dimensional subvariety of a strictly pseudoconvex domain in $\mathbb{C}^n$. When the boundary of the subvariety has locally finite linear measure, we obtain results on tangents to the boundary, uniqueness of the subvariety given the boundary, accessibility of boundary points and harmonic measure on the boundary.

Introduction

Let V be a subvariety of a domain D in $\mathbb{C}^n$. The end of V, denoted bV, is the set $\overline{V}\setminus V$ contained in the boundary bD of D. The terminology is due to GLOBEVNIK and STOUT, who studied the notion in a series of papers [12], [13], [14], [15]. Here we shall consider one-dimensional subvarieties, in strictly pseudoconvex domains, whose ends essentially have locally finite one-dimensional Hausdorff measure (which we shall refer to as "linear measure" and denote by $\mathcal{H}^1$).

Our first result concerns the general question of uniqueness. Given two irreducible subvarieties V_1 and V_2 of D, we want to conclude that $V_1 = V_2$ provided that $bV_1 \cap bV_2$ is, in some sense, sufficiently large. GLOBEVNIK and STOUT [13] showed that if D is the unit ball in $\mathbb{C}^2$ and each of the subvarieties is the image of the unit disc under a proper holomorphic map and each end is a rectifiable Jordan curve, then the two subvarieties

(*) Texte reçu le 6 septembre 1991, révisé le 25 novembre 1991.
H. ALEXANDER, University of Illinois at Chicago, Mathematics, Box 4348, Chicago, Illinois, USA.
Classification AMS : 32B15.

BULLETIN DE LA SOCIÉTÉ MATHÉMATIQUE DE FRANCE 0037–9484/1992/523/$ 5.00

coincide, provided that the two ends meet in a set of positive linear measure.

We shall not require that our varieties be parameterized by the unit disk or that their ends be anywhere arc-like. We do however need a topological assumption of local connectedness. Recall that a continuum is, by definition, a compact connected set of more than one point. To formulate our assumptions, we define the subset aV of bV as the set of points p in bV for which there exists a continuum X of finite linear measure contained in bV such that X is a (compact) neighborhood of p in bV. If X is a continuum of finite linear measure in R^n, then, by a theorem of Besicovitch [7], X is arcwise connected and is a disjoint union of a countable set of rectifiable Jordan arcs and a set of linear measure zero; moreover, X is locally connected, which can easily be proved directly or by using a result of Eilenberg and Harrold [6] to the effect that X is the continuous image of the unit interval. Consequently, we can alternatively describe aV as the set of points p of bV such that (i) bV has locally finite linear measure at p and (ii) bV is locally connected at p. In particular, aV is an open, sigma-compact subset of bV and is a countable union of continua of finite linear measure. We have the following uniqueness result :

Theorem 1. — *Let V_1 and V_2 be one-dimensional irreducible subvarieties of a strictly pseudoconvex domain D in $\mathbb{C}^n$. If $aV_1 \cap aV_2$ has positive linear measure, then $V_1 = V_2$.*

We should note that some sort of convexity condition is needed for this result; in our case, the subvarieties are both contained in a strictly pseudoconvex domain. Globevnik and Stout [13, Example 14] have indicated how the conclusion may otherwise fail, even if the ends are infinitely differentiable Jordan curves. Globevnik and Stout's proof of uniqueness is restricted to $\mathbb{C}^2$ because it is based on a result of Berntdsson [4] which does not hold in higher dimensions. The different method used here is to find "good" projections : to do this, we need to consider the tangents to bV at points of aV; these exist $\mathcal{H}^1$ a.e. on aV. In the case that bV is a $\mathcal{C}^2$ Jordan curve in the boundary of the unit ball, Forstnerič [10] has shown that the tangents to bV never lie in the complex tangential subspace of the tangent space to the unit sphere. It turns out that the existence of good projections is closely related to the existence of tangents which are not complex tangential to the boundary of the domain. Forstnerič's result, however, may fail if bV is only $\mathcal{C}^1$, as was observed by Rosay [18]. That is, in the $\mathcal{C}^1$ case, tangents to the curve may be complex tangential to bD. Nevertheless, the next result provides, in a quite general case, lots of tangents which are not complex tangential.

THEOREM 2. — *Let V be a one-dimensional irreducible subvariety of a strictly pseudoconvex domain D. Then $\mathcal{H}^1$ almost all points p of aV have the property that the tangent to bV at p exists and is not in the complex tangent space to bD at p.*

From this we obtain the following variant of the $\mathcal{C}^2$ case obtained by FORSTNERIČ [10] :

COROLLARY 1. — *Let Γ be a rectifiable Jordan curve in bD with D strictly pseudoconvex and $\overline{D}$ polynomially convex. Suppose that the tangents to Γ are complex tangential to bD at a set of points of Γ of positive linear measure. Then Γ is polynomially convex.*

For the proof we note that, by [1] and [2], $\widehat{\Gamma} \setminus \Gamma$ is either empty or is an irreducible one-dimensional subvariety of D whose end is exactly Γ. THEOREM 2 rules out the latter. The corollary is false if the Jordan curve is replaced by a continuum X of finite linear measure, even if X is a union of two real analytic curves; the reason being that the part of the polynomial hull of X inside D may be a non-empty subvariety whose end is a proper subset of X.

According to a classical result of F. and M. RIESZ [17], if J is a Jordan domain in the plane whose boundary bJ is a rectifiable Jordan curve, then harmonic measure on bJ (for some interior point of J) and arc length measure on bJ are mutually absolutely continuous measures. Part (a) of our next result can be viewed as a generalization to $\mathbb{C}^n$. A different formulation of the Riesz theorem, as, for example, given by GAMELIN [11, p. 45], involves annihilating ("orthogonal") measures. This relates to part (b) in our setting. The proof uses the "abstract" F. and M. RIESZ theorem [11].

THEOREM 3. — *Let V be an irreducible one-dimensional subvariety of a strictly pseudoconvex domain D in $\mathbb{C}^n$ such that $\mathcal{H}^1(bV \setminus aV) = 0$.*

(a) *Let μ_p be harmonic measure, with respect to V, on bV for some point p of V. Then μ_p and $\mathcal{H}^1{}_{|bV}$ (the restriction of linear measure to bV) are mutually absolutely continuous measures on bV.*

(b) *Let ν be a measure on bV which is orthogonal to $A(V)$. Then ν is absolutely continuous with respect to $\mathcal{H}^1$.*

(c) *If μ is a representing measure on bV for $p \in V$ for $A(V)$, then μ is absolutely continuous with respect to $\mathcal{H}^1$.*

Here $A(V)$ denotes the algebra of functions continuous on $\overline{V}$ and holomorphic on V. We briefly introduce harmonic measure for V below, in the usual way.

Our last results concern the accessibility of points on the boundary. Let V be an irreducible subvariety of a domain D. We say that a point p of bV is accessible from V it there is a real curve in V which approaches p asymptotically. Not every point of bV need be accessible : we shall give an example when D is the unit ball and V is a properly imbedded disk. However, we shall show that at every point of bV at which bV locally has finite linear measure, is accessible. The question of non-tangential accessibility is more subtle. We say that p is non-tangentially accessible from V if there exists a real curve in V which approaches p asymptotically and which approaches bD non-tangentially. Results on non-tangential accessibility for the case of holomorphic images of the unit disk were obtained by GLOBEVNIK and STOUT [13, Thm 9].

THEOREM 4. — *Let V be an irreducible one-dimensional subvariety of a strictly pseudoconvex domain D in $\mathbb{C}^n$. Every point of aV, except possibly for a set of linear measure zero, is non-tangentially accessible from V.*

Here is an outline of what follows : in section 1 we consider projections to a real line of a continuum in $\mathbb{R}^n$ of finite linear measure. We show that if the continuum is regular and has a tangent line at a point, then the projection of the continuum to a line not orthogonal to the tangent line is close to being one-one, in an appropriate measure theoretic sense. We apply this to a complexified projection in section 2 to prove THEOREM 2. THEOREM 1 is proved in section 3. We discuss harmonic measure and prove THEOREM 3 in section 4. We finish with a result on accessibility and a proof of THEOREM 4 in section 5.

1.1. — We begin with some preliminary results on rectifiable Jordan arcs and continua of finite linear measure in $\mathbb{R}^n$. As usual, $B(p,r)$ denotes the open ball of radius r about p and $\|p\|$, (p,q) denote the Euclidian norm and inner product. Let Γ be an open Jordan arc in $\mathbb{R}^n$, parameterized by arc length $\gamma : (a,b) \to \mathbb{R}^n$. Then $\|\gamma(t_1) - \gamma(t_2)\| \le |t_1 - t_2|$, γ is absolutely continuous, γ' exists a.e., $\|\gamma'(t)\| \le 1$ wherever γ' exists and $\|\gamma'(t)\| = 1$ a.e.

LEMMA 1. — *Suppose that $\gamma'(t_0)$ exists. Then $\|\gamma'(t_0)\| = 1$ if and only if t_0 is a Lebesgue point of $\gamma'(t)$.*

Proof. — Suppose that t_0 is a Lebesgue point of γ'. By definition this means that

$$\frac{1}{|h|}\int_{t_0}^{t_0+h} \|\gamma'(t) - \gamma'(t_0)\|\,\mathrm{d}t \to 0 \quad \text{as} \quad h \to 0.$$

Then

$$1-\|\gamma'(t_0)\| = \left|\frac{1}{h}\int_{t_0}^{t_0+h} \|\gamma'(t)\|\,\mathrm{d}t\right| - \left|\frac{1}{h}\int_{t_0}^{t_0+h} \|\gamma'(t_0)\|\,\mathrm{d}t\right|$$
$$\leq \left|\frac{1}{h}\int_{t_0}^{t_0+h} \|\gamma'(t)-\gamma'(t_0)\|\,\mathrm{d}t\right| \longrightarrow 0.$$

Hence $1 \leq \|\gamma'(t_0)\|$. Hence $1 = \|\gamma'(t_0)\|$. Conversely, if $\|\gamma'(t_0)\| = 1$, then

$$\left\{\frac{1}{h}\int_{t_0}^{t_0+h} \|\gamma'(t)-\gamma'(t_0)\|\,\mathrm{d}t\right\}^2$$
$$\leq \left|\frac{1}{h}\int_{t_0}^{t_0+h} \|\gamma'(t)-\gamma'(t_0)\|^2\,\mathrm{d}t\right|$$
$$= 2\times\left|1-\Big(\frac{1}{h}\int_{t_0}^{t_0+h} \gamma'(t)\,\mathrm{d}t, \gamma'(t_0)\Big)\right|$$
$$= 2\times\left|1-\Big(\frac{\gamma(t_0+h)-\gamma(t_0)}{h}, \gamma'(t_0)\Big)\right| \to 2\big(1-\|\gamma'(t_0)\|^2\big) = 0.$$

We next show that a continuum X in $\mathbb{R}^n$ of finite linear measure has a tangent $\mathcal{H}^1$ a.e. in a strong sense. Namely, in the sense that the tangent cone reduces to a line. For $x \in \mathbb{R}^n$ and α a unit vector in $\mathbb{R}^n$ and $0 < \varepsilon < \frac{1}{2}\pi$ we define the cone (two-sided) at x in direction α and opening ε to be

$$S(x,\alpha,\varepsilon) = \Big\{y \in \mathbb{R}^n : \big|(y-x,\alpha)\big| \geq \cos\varepsilon \cdot \|y-x\|\Big\}.$$

We shall say that α is a *weak tangent* to X at x if

$$\lim_{r\to 0} \frac{1}{r}\,\mathcal{H}^1\big(X \cap \big(B(x,r)\setminus S(x,\alpha,\varepsilon)\big)\big) = 0$$

for all $\varepsilon > 0$. According to [7], a continuum X of finite linear measure has a weak tangent at $\mathcal{H}^1$ a.e. points of X. In fact, in [7], the word "weak" is omitted. We shall reserve the word tangent for a stronger notion. We shall say that α is a *tangent* of X at x if there exists a δ_0 such that $X \cap B(x,r) \subseteq S(x,\alpha,\varepsilon)$ if $r < \delta_0 = \delta_0(\varepsilon)$.

PROPOSITION 1. — *Let X be a continuum of finite linear measure in $\mathbb{R}^n$. Then X has a tangent $\mathcal{H}^1$ a.e. on X.*

Proof. — By a theorem of BESICOVITCH (see [7]), X is the disjoint union of a countable set of open rectifiable Jordan arcs $\{J_k\}$ and a set Z

with $\mathcal{H}^1(Z)=0$. It suffices to show that X has a tangent at $\mathcal{H}^1$ almost every point of J_k for each k. Fix k. Let $\gamma:(a,b)\to\mathbb{R}^n$ parameterize J_k by arc length. It suffices to show that X has a tangent at each point $p_0=\gamma(t_0)$ of J_k such that (i) X has a weak tangent at p_0 and (ii) t_0 is a Lebesgue point of γ'.

Put $\alpha=\gamma'(t_0)$. Then $\|\alpha\|=1$ and clearly α is the weak tangent to X at p_0, by uniqueness. Let $0<\varepsilon<\frac{1}{2}\pi$. We must show that $X\cap B(p_0,r)\subseteq S(p_0,\alpha,\varepsilon)$ if r is sufficiently small. Suppose not! Then there exists a sequence $\{x_n\}\subseteq X$, $x_n\to p_0$ and $x_n\notin S(p_0,\alpha,\varepsilon)$ for $n=1,2,\ldots$. Let $r_n=2\|x_n\|>0$. Then, since α is a weak tangent,

$$\frac{1}{r_n}\mathcal{H}^1\big(X\cap B(p_0,r_n)\setminus S\big(p_0,\alpha,\tfrac{1}{2}\varepsilon\big)\big)\longrightarrow 0. \tag{1}$$

Let Y_n be the connected component of $X\cap B(p_0,r_n)\setminus S\big(p_0,\alpha,\frac{1}{2}\varepsilon\big)$ which contains x_n. Since X is connected, $\overline{Y}_n$ has a non-empty intersection with $b\big[B(p_0,r_n)\setminus S(p_0,\alpha,\frac{1}{2}\varepsilon)\big]$. Because $\|x_n\|=\frac{1}{2}r_n$ and because $x_n\notin S(p_0,\alpha,\varepsilon)$ we conclude that distance of x_n to this boundary is at least $\|x_n\|\cdot\sin\big(\frac{1}{2}\varepsilon\big)\equiv\eta_n$. Hence diam $(Y_n)\geq\eta_n$. Therefore $\mathcal{H}^1(Y_n)\geq\eta_n$. Then (1) implies $\eta_n/r_n\to 0$. But $\eta_n/r_n=\frac{1}{2}\sin\big(\frac{1}{2}\varepsilon\big)>0$. Contradiction! This proves the proposition.

1.2. — Now suppose that γ is a closed rectifiable Jordan arc in $\mathbb{R}^n$ and that $\gamma:[a,b]\to\mathbb{R}^n$ parameterizes γ by arc length. Let $p_0\in\gamma$, $p_0=\gamma(t_0)$ and let e be a unit vector in $\mathbb{R}^n$. We shall consider the projection $\pi:\mathbb{R}^n\to\mathbb{R}$ given by $\pi(p)=(p-p_0,e)$ and the associated function $u:[a,b]\to\mathbb{R}$, $u(t)=\pi\big(\gamma(t)\big)=\big(\gamma(t)-p_0,e\big)$. Then $u(t_0)=0$, $u'(t)=\big(\gamma'(t),e\big)$ whenever $\gamma'(t)$ exists and $\big|u(t_1)-u(t_2)\big|\leq|t_1-t_2|$. Therefore u is absolutely continuous and $\mathcal{H}^1\big(u(A)\big)\leq\mathcal{H}^1(A)$ for all Borel sets $A\subseteq[a,b]$.

PROPOSITION 2. — *Let $t_0\in(a,b)$ be such that*

(a) *t_0 is a Lebesgue point of γ';*

(b) $\big(\gamma'(t_0),e\big)\neq 0$;

(c) $\pi^{-1}(0)\cap\gamma=\{p_0\}$.

Let $n(x)=\#\big\{t\in[a,b]:u(t)=x\big\}$ and $E=\big\{x\in\mathbb{R}:n(x)=1\big\}$. Let T be a Borel subset of $[a,b]$ such that t_0 is a point of density of T. Then 0 is a point of density of the two subsets of $\mathbb{R}$:

(i) *E, and*

(ii) *$u(T)$.*

Remarks :

(α) By LEMMA 1, from (a), $\gamma'(t_0)$ exists and is a unit vector. From (b), $u'(t_0) \neq 0$. Without loss of generality, we assume that $u'(t_0) :\equiv \eta > 0$.

(β) Hypothesis (a) implies that t_0 is a Lebesgue point for $|u'(t)|$. Indeed, for $h > 0$,

$$\begin{aligned}\frac{1}{2h}\int_{t_0-h}^{t_0+h} \big| |u'(t)| - |u'(t_0)| \big| \,\mathrm{d}t &\leq \frac{1}{2h}\int_{t_0-h}^{t_0+h} \big|u'(t) - u'(t_0)\big| \,\mathrm{d}t \\ &\leq \frac{1}{2h}\int_{t_0-h}^{t_0+h} \big|(\gamma'(t) - \gamma'(t_0), e)\big| \,\mathrm{d}t \\ &\leq \frac{1}{2h}\int_{t_0-h}^{t_0+h} \big| |\gamma'(t) - \gamma'(t_0)| \big| \,\mathrm{d}t \to 0.\end{aligned}$$

In particular, $\dfrac{1}{2h}\, I(h) :\equiv \dfrac{1}{2h}\displaystyle\int_{t_0-h}^{t_0+h} \big|u'(t)\big| \,\mathrm{d}t \to \big|u'(t_0)\big| = \eta.$

(γ) If $u^{-1}\big((x_1, x_2)\big) \subseteq (t_1, t_2) \subseteq [a, b]$ then

$$\int_{x_1}^{x_2} n(x)\,\mathrm{d}x \leq \int_{t_1}^{t_2} \big|u'(t)\big| \,\mathrm{d}t.$$

This is the usual Banach indicatrix; see [8, Thm 2.10.13].

Proof. — We may assume that $t_0 = 0$ and that $p_0 = 0 \in \mathbb{R}^n$. Let $\varepsilon > 0$. We claim

$$(2) \qquad (\exists\, \delta_0 > 0)\ (\forall\, x \neq 0)\ \big(\forall\, t \in [a, b]\big) \\ \Big\{ \big(|x| < \delta_0 \ \text{ and } \ u(t) = x\big) \Rightarrow |t| < |x|\,(1 + \varepsilon)/\eta \Big\}.$$

Suppose not! Then for each positive integer n there exists $x_n \neq 0$ and $t_n \in [a, b]$ such that $|x_n| < 1/n$, $u(t_n) = x_n$ and $|t_n| \geq (1 + \varepsilon)|x_n|/\eta$. If a subsequence $\{t_{n_j}\}$ converges to $t^* \in [a, b]$, $u(t^*) = \lim u(t_{n_j}) = \lim x_{n_j} = 0$. By (c), $\gamma(t^*) = p_0$ and so $t^* = 0$; i.e., we have $t_n \to 0$. Hence $u(t_n)/t_n \to u'(0) = \eta$. But $|u(t_n)/t_n| = |x_n/t_n| \leq \eta/(1 + \varepsilon)$. This is a contradiction; (2) follows.

Now if $0 < \delta < \delta_0$ then (2) implies

$$u^{-1}\big((-\delta, \delta)\big) \subseteq \big(-\delta(1 + \varepsilon)/\eta, \delta(1 + \varepsilon)/\eta\big).$$

By Remark (γ) we have

$$(3) \qquad \int_{-\delta}^{\delta} n(x)\,\mathrm{d}x \leq \int_{-\delta/\eta(1+\varepsilon)}^{\delta/\eta(1+\varepsilon)} \big|u'(t)\big| \,\mathrm{d}t \equiv I\big(\delta(1 + \varepsilon)/\eta\big).$$

Let E' be the complement of E in $\mathbb{R}$. If δ is sufficiently small, $n(x) \geq 1$ on $(-\delta, \delta)$ and so $n(x) \geq 1 + \chi_{E'}(x)$ on $(-\delta, \delta)$, where χ is the characteristic function. From (3) we get

$$2\delta + \mathcal{H}^1\big((-\delta, \delta) \setminus E\big) \leq I\big(\delta(1+\varepsilon)/\eta\big).$$

Hence

$$\frac{1}{2\delta}\,\mathcal{H}^1\big((-\delta, \delta) \setminus E\big) \leq -1 + \frac{1+\varepsilon}{\eta}\,\frac{I\big(\delta(1+\varepsilon)/\eta\big)}{2\delta(1+\varepsilon)/\eta}.$$

By Remark (β), $I(h)/(2h) \to \eta$ as $h \to 0$. We get

$$\limsup_{\delta \to 0} \frac{1}{2\delta}\,\mathcal{H}^1\big((-\delta, \delta) \setminus E\big) \leq -1 + (1+\varepsilon) = \varepsilon.$$

As ε is arbitrary, (i) follows.

For (ii), we consider $S = [a, b] \setminus T$. If $\delta > 0$ is small,

$$(-\delta, \delta) \subseteq u\big([a, b]\big) \subseteq u(T) \cup u(S).$$

Hence

$$\mathcal{H}^1\big(u(T) \cap (-\delta, \delta)\big) \geq 2\delta - \mathcal{H}^1\big(u(S) \cap (-\delta, \delta)\big). \tag{4}$$

We have $u(S) \cap (-\delta, \delta) \subseteq u\big(S \cap u^{-1}\big((-\delta, \delta)\big)\big)$. Let $\varepsilon > 0$. By (2) we get a δ_0 such that $0 < \delta < \delta_0$ implies $u^{-1}(-\delta, \delta) \subseteq \big(-\delta(1+\varepsilon)/\eta, \delta(1+\varepsilon)/\eta\big)$. Therefore $u(S) \cap (-\delta, \delta) \subseteq u\big(S \cap \big(-\delta(1+\varepsilon)/\eta, \delta(1+\varepsilon)/\eta\big)\big)$. Hence

$$\mathcal{H}^1\big(u(S) \cap (-\delta, \delta)\big) \leq \mathcal{H}^1\Big(S \cap \big(-\delta(1+\varepsilon)/\eta, \delta(1+\varepsilon)/\eta\big)\Big).$$

Since $\mathcal{H}^1\big(S \cap (-r, r)\big)/(2r) \to 0$ as $r \to 0$ we conclude that

$$\frac{1}{2\delta}\,\mathcal{H}^1\big(u(S) \cap (-\delta, \delta)\big) \to 0$$

as $\delta \to 0$. This, with (4), gives (ii).

We next extend the previous proposition to a continuum of finite linear measure. Suppose that X is a continuum of finite linear measure in $\mathbb{R}^n$ and suppose that Γ is a (necessarily rectifiable) Jordan arc contained in X, parameterized by $\gamma : [a, b] \to \mathbb{R}^n$ in arc length. Let $p_0 \in \Gamma$, $p_0 = \gamma(t_0)$ with $t_0 \in (a, b)$ and define $\pi : \mathbb{R}^n \to \mathbb{R}$ and $u : [a, b] \to \mathbb{R}$ as above, for a fixed unit vector e.

PROPOSITION 3. — *Suppose that*

(a) t_0 *is a Lebesgue point of* γ' ;

(b) X *has a tangent at* p_0 ;

(c) X *is regular at* p_0 ;

(d) $(\gamma'(t_0), e) \neq 0$; *and*

(e) $\pi^{-1}(0) \cap X = \{p_0\}$.

Let $N(x) = \#\{p \in X : \pi(p) = x\}$ *and let* $A = \{x \in \mathbb{R} : N(x) = 1\}$. *Then* 0 *is a point of density of* A.

Remark. — X is regular at p_0, means, by definition, that

$$\lim_{r \downarrow 0} \frac{1}{2r} \mathcal{H}^1\big(X \cap B(p_0, r)\big) = 1.$$

Since X is a continuum of finite linear measure, it is regular at $\mathcal{H}^1$ almost all of its points [7]. Also it follows easily from (a) and Lemma 1 that Γ is regular at p_0.

Proof. — We define E, as in PROPOSITION 2, as the set where $\pi_{|\Gamma}$ has multiplicity one. By PROPOSITION 2, E has 0 as a point of density and therefore its complement E' has density 0 at 0. Let $C = \pi(X \setminus \Gamma)$. If δ is sufficiently small, $\pi(X) \supseteq (-\delta, \delta)$, and then $A' \cap (-\delta, \delta) \subseteq (E' \cup C) \cap (-\delta, \delta)$, for if $x \in A' \cap \pi(X) \setminus C$ then $x \in \pi(\Gamma) \setminus E \subseteq E'$. We need to show that A' has density 0 at 0. It suffices to show that C has density 0 at 0.

Let $\eta = \big(\gamma'(t_0), e\big)$; $\eta \neq 0$ by (d). We may assume $\eta > 0$. Let $\varepsilon > 0$. Then we claim that there exists $\delta_0 > 0$ such that if $0 < \delta < \delta_0$, then

$$C \cap (-\delta, \delta) \subseteq \pi\big((X \setminus \Gamma) \cap B\big(p_0, (1+\varepsilon)\delta/\eta\big)\big). \tag{5}$$

Suppose not! Then there exist $\delta_n \downarrow 0$ and $x_n \in \pi(X \setminus \Gamma) \cap (-\delta_n, \delta_n)$ but

$$x_n \notin \pi\big((X \setminus \Gamma) \cap B\big(p_0, (1+\varepsilon)\delta_n/\eta\big)\big).$$

Hence $x_n = \pi(p_n)$, $p_n \in X \setminus \Gamma$, $||p_n - p_0|| \geq (1+\varepsilon)\delta_n/\eta$. Then $p_n \to p_0$. For if a subsequence $p_{n_j} \to p^* \in X$, then $\pi(p^*) = \lim \pi(p_{n_j}) = \lim x_{n_j} = 0$. By (e), $p^* = p_0$. Now, by (a) and (b), $\gamma'(t_0)$ is the tangent to X at p_0. Since $p_n \to p_0$, some subsequence of $(p_n - p_0)/||p_n - p_0|| \to \pm\gamma'(t_0)$. We may assume this is the case for the original sequence. Hence $\alpha_n :\equiv \big((p_n - p_0)/||p_n - p_0||, e\big) \to \pm\eta$. But

$$|\alpha_n| = \frac{|x_n|}{||p_n - p_0||} \leq \frac{\delta_n}{(1+\varepsilon)\delta_n/\eta} = \frac{\eta}{(1+\varepsilon)}.$$

This is a contradiction and we conclude that (5) holds. From (5) we get

$$\mathcal{H}^1(C\cap(-\delta,\delta)) \le \mathcal{H}^1\big[X\cap B(p_0,(1+\varepsilon)\delta/\eta)\big] - \mathcal{H}^1\big[\Gamma\cap B(p_0,(1+\varepsilon)\delta/\eta)\big]$$

and therefore

$$\frac{1}{2\delta}\mathcal{H}^1\big(C\cap(-\delta,\delta)\big) \le \frac{1+\varepsilon}{\eta}\Big\{\Big[\frac{\mathcal{H}^1\big(X\cap B(p_0,(1+\varepsilon)\delta/\eta)\big)}{2(1+\varepsilon)\delta/\eta}\Big] - \Big[\frac{\mathcal{H}^1\big(\Gamma\cap B(p_0,(1+\varepsilon)\delta/\eta)\big)}{2(1+\varepsilon)\delta/\eta}\Big]\Big\}.$$

Since X and Γ are both regular at p_0, each of the two quotients in square brackets converges to 1 as $\delta \to 0$. Hence C has density 0 at 0.

2.1. — For the proof of THEOREM 2 we shall need two lemmas. Let D be a strictly pseudoconvex domain in $\mathbb{C}^n$. Recall that $A(D)$ denotes the algebra of complex-valued functions which are continuous on $\overline{D}$ and holomorphic on D.

LEMMA 2. — *Let Γ be a rectifiable Jordan arc in bD. Let E be a compact subset of Γ such that at each point of E the tangent to Γ exists and is complex tangential to bD. Then E is a peak interpolation set for $A(D)$.*

LEMMA 3. — *Let Ω be a bounded, open, simply connected subset of the complex plane with $\mathcal{H}^1(b\Omega) < \infty$. Suppose that $\omega \subseteq b\Omega$ is a Borel set with $\mathcal{H}^1(\omega) > 0$, that $F \in H^\infty(\Omega)$ and that for all $z \in \omega$ there is a path σ_z in Ω which approaches z asymptotically such that the limit of $F(\zeta)$ as ζ approaches z along σ_z exists and equals 0. Then $F \equiv 0$.*

We shall prove these lemmas after proving THEOREM 2.

2.2. — We prove THEOREM 2 by contradiction and suppose that the tangent to aV, which exists $\mathcal{H}^1$ a.e. on aV, is complex tangential to bD on a set of positive $\mathcal{H}^1$ measure. We claim then that there exists $p_0 \in aV$, a continuum $X \subseteq aV$ which is a neighborhood of p_0 in aV and with $\mathcal{H}^1(X) < \infty$, a Jordan arc Γ, $p_0 \in \Gamma \subseteq X$ such that $\gamma : (a,b) \to \mathbb{C}^n$ parametrizes Γ by arclength, $\gamma(t_0) = p_0$, and a compact set $E \subseteq \Gamma$ such that the tangent to a aV exists at each point of E and is complex tangential to bD, $\mathcal{H}^1(E) > 0$, and that (a)–(e) of PROPOSITION 3 hold with $e = \gamma'(t_0)$ and such that t_0 is a point of density of $T = \gamma^{-1}(E)$.

To see this, we use the fact that aV has a tangent and is regular $\mathcal{H}^1$ a.e. and the fact that aV is a countable union of Jordan arcs and a null set to get a p_0 and a Γ satisfying (a), (b) and (c) with aV in place of X, (d) and the statement on T. Then choose X as a sufficiently small neighborhood of p_0 in aV, using (b) to achieve (e). Here we view $\mathbb{C}^n$ as being $\mathbb{R}^{2n}$, the real inner product $(\cdot\,,\cdot)$ being the real part of the Hermitian inner product $\langle\cdot\,,\cdot\rangle$. In particular, we have the complex projection $g:\mathbb{C}^n\to\mathbb{C}$ given by $g(p)=\langle p-p_0,e\rangle$ and satisfying $\pi(p)=\operatorname{Re}\big(g(p)\big)$. Finally we can replace Γ by a subarc to have $\Gamma\subseteq X$. As before we have $u:(a,b)\to\mathbb{R}$ given by $u(t)=\pi\big(\gamma(t)\big)=\operatorname{Re}\big(g\big(\gamma(t)\big)\big)$.

Since $e=\gamma'(t_0)$ is the tangent to X at p_0 by (a) and (b), if $\delta_0>0$ is sufficiently small,

$$aV\cap B(p_0,\delta_0)=X\cap B(p_0,\delta_0)\subseteq S\big(p_0,e,\tfrac{1}{6}\pi\big).$$

Since

$$g\Big(S\big(p_0,e,\tfrac{1}{6}\pi\big)\Big)\subseteq\big\{\lambda\in\mathbb{C}:|\operatorname{Im}\lambda|\le\frac{1}{\sqrt{3}}\,|\operatorname{Re}\lambda|\big\}$$

we get

$$g\big(X\cap B(p_0,\delta_0)\big)\subseteq\Big\{\lambda\in\mathbb{C}:|\operatorname{Im}\lambda|\le\frac{1}{\sqrt{3}}\,|\operatorname{Re}\lambda|\Big\}. \tag{6}$$

For $0<\delta<\delta_0$, consider $W=V\cap B(p_0,\delta)$, a subvariety of $B(p_0,\delta)\cap D$. Then

$$bW\subseteq\big(X\cap\overline{B}(p_0,\delta)\big)\cup\big(bB(p_0,\delta)\cap V\big),\ \text{ and}$$
$$g(bW)\subseteq g\big(X\cap B(p_0,\delta_0)\big)\cup g\big(bB(p_0,\delta)\cap V\big).$$

Since $\{p\in V:g(p)=0\}$ is discrete and countable (otherwise $g\equiv 0$ on V and so $g\equiv 0$ on $\Gamma\subseteq bV$ and so $u\equiv 0$, contradicting

$$u'(t_0)=\big(\gamma'(t_0),e\big)\ne 0$$

by (d)) we may choose δ so that $0\notin g\big(bB(p_0,\delta)\cap V\big)$. From (6) and (e) we conclude that $g^{-1}(0)\cap bW=\{p_0\}$ and that $0\notin g\big(bB(p_0,\delta)\cap\overline{V}\big)$. Hence for $\rho>0$ sufficiently small

$$Q_\rho\cap g\big(bB(p_0,\delta)\cap\overline{V}\big)=\emptyset$$

where

$$Q_\rho\equiv\Big\{\lambda\in\mathbb{C}:|\operatorname{Re}\lambda|\le\rho\ \text{ and }\ |\operatorname{Im}\lambda|\le\rho\Big\}.$$

Again from (6) we have

$$g\big(X\cap\overline{B}(p_0,\delta)\big)\cap\big\{\lambda\in\mathbb{C}:|\operatorname{Re}\lambda|<\rho\big\}\subseteq Q_\rho.$$

By PROPOSITION 2, $u(T)$ has 0 as a point of density and by PROPOSITION 3, the set A defined there has 0 as a point of density. Set

$$A_1 = A \cap u(T) \cap (-\rho, \rho).$$

Then 0 is a point of density of A_1. For $x \in \mathbb{R}$ we denote the vertical line $\{\lambda \in \mathbb{C} : \operatorname{Re}\lambda = x\}$ by ℓ_x. Then by the definition of A, for each $x \in A_1$, $\ell_x \cap Q_\rho \cap g(X)$ contains exactly one point : $\lambda_x = x + iv_x$ with $-\rho < v_x < \rho$. Moreover $g^{-1}(\lambda_x) \cap X$ consists of a single point $p_x \in \Gamma$. In particular the segment $[-\rho, \rho] \times \{\rho\}$ is disjoint from $g(bW)$ and so is contained in a component Ω_1 of $\mathbb{C} \setminus g(bW)$. Likewise $[-\rho, \rho] \times \{-\rho\}$ is contained in a component Ω_2. Then for $j = 1$ and 2, $g : g^{-1}(\Omega_j) \cap W \to \Omega_j$ is a proper holomorphic map and hence a branched cover of multiplicity m_j, with $m_j \geq 0$.

Since the set of singular values of $g_{|V}$ is countable, by removing a set of linear measure zero from A_1 we can assume that for every $x \in A_1$ (i) ℓ_x contains no singular values of $g \mid V$ and (ii) every point of A_1 is a point of density of A_1.

Now fix $x \in A_1$. Let $m = m(x) = \#\big(g^{-1}(\lambda_x) \cap W\big)$. Each neighborhood of each point in $g^{-1}(\lambda_x) \cap W$ is mapped by g to a neighborhood of $\lambda_x \in b\Omega_1 \cap b\Omega_2$. We conclude that $m \leq m_1$ and $m \leq m_2$.

We claim that it is not true that $m_1 = m = m_2$. Suppose it were true! Then $g^{-1}(\lambda_x) \cap W = \{p_1, p_2, \ldots, p_m\}$. Choose $m+1$ disjoint neighborhoods $N ; N_1, N_2, \ldots, N_m$ of $p_x ; p_1, p_2 \ldots, p_n$ respectively in $\mathbb{C}^n$. Then $g(W \cap N_j)$ is a neighborhood of λ_x for $j = 1, 2, \ldots, m$. Since $g_{|g^{-1}(\Omega_j) \cap W}$ has multiplicity $m = m_j$ over Ω_j, we conclude that $g(W \cap N)$ is disjoint from $\Omega_1 \cup \Omega_2$. Since $\Omega_1 \cup \Omega_2$ contains, for each $x' \in A_1$, $\big(\ell_{x'} \setminus \{\lambda_{x'}\}\big) \cap Q_\rho$, it follows that the open set $g(W \cap N)$ is disjoint from $\ell_{x'} \cap Q_\rho$ for $x' \in A_1$. We can take N of the form $B(p_x, \varepsilon) \subseteq B(p_0, \delta)$. Arguing, as above, that $\{p \in V : g(p) = \lambda_x\}$ is discrete, we get that ε can be chosen so that

$$\lambda_x \notin g\big(bB(p_x, \varepsilon) \cap \overline{V}\big). \tag{7}$$

(We use the fact that $g^{-1}(\lambda_x) \cap X = \{p_x\}$, since $x \in A_1$.) By (7) we can choose a small open rectangle R about λ_x with $\overline{R}$ disjoint from $g\big(bB(p_x, \varepsilon) \cap \overline{V}\big)$. We can take $R \subseteq Q_\sigma$ of the form $(x_1, x_2) \times (y_1, y_2)$ where

$$-\sigma < x_1 < x < x_2 < \sigma, \qquad x_1, x_2 \in A_1,$$
$$-\sigma < y_1 < v_x < y_2 < \sigma.$$

Since $bR \subseteq \Omega_1 \cup \Omega_2 \cup \{\lambda_{x_j} : j = 1, 2\}$ if $(x_2 - x_1)$ is sufficiently small, it follows that every component of $g(W \cap N)$ which meets R is contained in R. Since $p_x \in bV$, there exists $q \in V \cap N = W \cap N$ such that $g(q) \in R$. Let W_1 be the connected component of $W \cap N$ which contains q. Then $g(W_1)$ meets R and so is contained in R. Now we claim that $bW_1 \subseteq X$. Indeed, $bW_1 \subseteq b(W \cap N) \subseteq X \cup S$ where $S = bB(p_x, \varepsilon) \cap \overline{V}$. But by choice of R, $g(S) \cap \overline{R}$ is empty. The claim thus follows. It implies that W_1 is a subvariety of D which is contained in V, which is irreducible. A contradiction!

Thus for all $x \in A_1$, either $m(x) < m_1$ or $m(x) < m_2$. Thus we may assume that the first condition holds on a Borel subset B of A_1 of positive measure. In particular, we can now say that $m_1 > 0$. Set

$$\omega = \{\lambda_x : x \in B\} = \big(B \times [-\sigma, \sigma]\big) \cap g(X),$$

a Borel subset of $b\Omega_1$ with $\mathcal{H}^1(\omega) \geq \mathcal{H}^1(B) > 0$.

For $x \in B$, set $\gamma_x = \big\{x + it : v_x < t < \sigma\big\} \subseteq \ell_x \cap Q_\sigma$. Then $\gamma_x \subseteq \Omega_1$. Since γ_x contains no critical values of $g_{|V}$, $g^{-1}(\gamma_x) \cap W$ is a disjoint union of m_1 Jordan arcs $\gamma_1^x, \ldots, \gamma_{m_1}^x$. Fix j, $1 \leq j \leq m_1$. The cluster set of $\{p \in \gamma_j^x\}$ in $\mathbb{C}^n$ as $g(p) \to \lambda_x$ is connected and is contained in $g^{-1}(\lambda_x) \cap \overline{W}$ which consists of $m + 1$ points, m of them in $g^{-1}(\lambda_x) \cap W$ and one point in bW, namely, p_x. Therefore the cluster set reduces to a single point and so γ_j^x approaches one of the $m + 1$ points asymptotically. Since g maps neighborhoods in W of each of the m points of $g^{-1}(\lambda_x) \cap W$ homeomorphically to a neighborhood of λ_x, it follows, since $m < m_1$ that one of the γ_j^x approaches p_x. Recall that $p_x \in E \subseteq \Gamma$, since $A_1 \subseteq u(T)$.

Now we apply Lemma 2 to $E \subseteq \Gamma$ and obtain a peak function $f_1 \in A(D)$. Set $f = 1 - f_1$. Then $f \neq 0$ on $\overline{D} \setminus E$; in particular, $f \neq 0$ on V and $f = 0$ on E. For $\lambda \in \Omega_1$, $g^{-1}(\lambda) \cap W = \{w^1, w^2, \ldots, w^{m_1}\}$, counting multiplicity. Define a bounded function F on Ω_1 by

$$F(\lambda) = \prod_{\substack{g^{-1}(\lambda) \cap W = \\ \{w^1, w^2, \ldots, w^{m_1}\}}} f(w^j).$$

It is standard that F is a well-defined bounded holomorphic function on Ω_1. Let Ω equal to the component of $\Omega_1 \cap Q_\sigma^\circ$ which contains $[-\sigma, \sigma] \times \{\sigma\}$ in its closure. Then $b\Omega \subseteq bQ_\sigma \cup g(X)$ and so $\mathcal{H}^1(b\Omega) < \infty$. Also $\omega \subseteq b\Omega$ and $\gamma_x \subseteq \Omega$ for $x \in A_1$. Fix $\lambda_x \in \omega$. We have seen that some $\gamma_j^x \to p_x \in E$. Hence $f(w) \to 0$ as $w \in \gamma_j^x \to p_x$, since $f(p_x) = 0$. It follows that $F(\lambda) \to 0$ as $\lambda \in \gamma_x \to \lambda_x$. By LEMMA 3, $F \equiv 0$. But $f \neq 0$ on V implies $F \neq 0$ on Ω. This is a contradiction. THEOREM 2 follows.

2.3. — We shall deduce LEMMA 2 from a theorem of DAVIE and ØKSENDAL [5]; the case when Γ is smooth and E a subarc was already noted in [5]. Following the notation of [5], we write $T(\zeta)$ for the complex tangent space to bD at $\zeta \in bD$ and $L(\zeta)$ for its orthogonal complement in the (real) tangent space to bD at ζ. Also if S is a real linear subspace of $\mathbb{C}^n$, and Y is a subset, $d_S(Y)$ denotes the diameter of the orthogonal projection of Y to S. Clearly $d_S(Y) \leq \mathrm{diam(Y)}$.

Let $\zeta \in E$. Then, for all $\eta > 0$, there exists a subarc J of Γ containing ζ such that $\mathcal{H}^1(J) < \eta$ and $d_{L(\zeta)}(J) \leq \eta\ \mathrm{diam(J)}$. This follows easily from the fact that the tangent to Γ at ζ exists and is complex tangential to bD at ζ. Let W be an open subset of Γ containing E and such that $\mathcal{H}^1(W) < 2\mathcal{H}^1(E)$.

Let $\varepsilon > 0$ and set $\eta = \min\big(\varepsilon, \varepsilon/\big(4\mathcal{H}^1(E)\big)\big)$. For each $\zeta \in E$, choose an interval J as above such that $d_{L(S)}(J) \leq \eta \cdot \mathrm{diam}(J)$ and also such that $\mathrm{diam(J)} < \eta$ and $J \subseteq W$. Let $J_1, J_2, \ldots, J_N$ be a finite subcover of E with $\zeta_1, \zeta_2, \ldots, \zeta_N$ the corresponding points. Write $d_L(k)$ for $d_{L(\zeta_k)}(J_k)$ and similarly $d_T(k)$. By discarding some J_i's, without changing their union, we may assume that no point belongs to more than two J_i's; hence

$$\sum \mathcal{H}^1(J_k) \leq 2\mathcal{H}^1(W) < 4\mathcal{H}^1(E).$$

Therefore $\sum \mathrm{diam(J_k)} < 4\mathcal{H}^1\mathrm{(E)}$.

Finally we get

(i) $\sum d_T(k)^2 \leq \eta \sum d_T(k) \leq \eta \sum \mathrm{diam}(J_k) < 4\eta\mathcal{H}^1(E) \leq \varepsilon$, and

(ii) $\sum d_L(k) \leq \eta \sum \mathrm{diam} J_k < 4\eta\mathcal{H}^1(E) \leq \varepsilon$.

The lemma now follows from Theorem 1 of [5].

2.4 Proof of lemma 3. — Let $\psi : U \to \Omega$ be a Riemann map. Recall [3] that ψ extends to be a continuous map of bU onto $b\Omega$. Set $\omega_1 = \psi^{-1}(\omega) \subseteq bU$ and $F_1 = F \circ \psi$ in $H^\infty(U)$. Let F_1^* denote the a.e. defined radial limit of F_1 on bU. Set

$$N = \Big\{ \mathrm{e}^{i\theta} : F_1^*\big(\mathrm{e}^{i\theta}\big) \text{ exists and equals } 0 \Big\}.$$

We claim that $\psi(N \cap \omega_1) = \omega$. Let $z \in \omega$ and set $\widetilde{\sigma}_z = \psi^{-1} \circ \sigma_z$. Then $\widetilde{\sigma}_z$ is a path in U. Its cluster set on bU is connected, hence is a subarc. But ψ maps this subarc to z. We conclude that the subarc reduces to a single point λ with $\psi(\lambda) = z \in \omega$. Hence $\lambda \in \omega_1$. Also $F_1(\zeta) \to 0$ as $\zeta \to \lambda$ along $\widetilde{\sigma}_z$. By a classical result of Lindelöf, $F_1^*(\lambda) = 0$; i.e. $\lambda \in N$. Thus $\lambda \in N \cap \Omega_1$ and this gives the claim.

Next, we claim that $\mathcal{H}^1(N) > 0$. This implies that $F_1 \equiv 0$ by Fatou's lemma and hence that $F \equiv 0$, as desired.

To verify the claim, we use the fact that $\psi' \in H^1$ (Hardy space) and so $\operatorname{Re}\psi$ and $\operatorname{Im}\psi$ are absolutely continuous on bU; cf. [3]. We have $\mathcal{H}^1\big(\psi(N)\big) \geq \mathcal{H}^1\big(\psi(N \cap \omega_1)\big) = \mathcal{H}^1(\omega) > 0$. By the projection lemma of [3] since $\psi(N) \subseteq b\Omega$, a continuum with $\mathcal{H}^1(b\Omega) < \infty$, either $\mathcal{H}^1\big(\operatorname{Re}\psi(N)\big) > 0$ or $\mathcal{H}^1\big(\operatorname{Im}\psi(N)\big) > 0$. Suppose the former, without loss of generality. Then, as $\operatorname{Re}\psi$ is absolutely continuous, it follows that $\mathcal{H}^1(N) > 0$; cf. [3, Lemma 1]. This gives the lemma.

3.1. — We can now use PROPOSITION 3 with THEOREM 2 to prove THEOREM 1. It will be convenient to assume henceforth that D is strictly convex. This is justified by the imbedding theorem of FORNAESS [9] and HENKIN [16]. Let $N(p)$ be the outward unit normal vector to bD at $p \in bD$. Fix $p_0 \in bD$, we set $e = iN(p_0)$ and define $g : \mathbb{C}^n \to \mathbb{C}$ by $g(p) = \langle p - p_0, e\rangle$ and $\pi : \mathbb{C}^n \to \mathbb{R}$ by $\pi = \operatorname{Re} \circ g$. For $p \in \overline{D}$, by strict convexity, $\operatorname{Im} g(p) \geq 0$ and $\operatorname{Im} g(p) > 0$ for $p \in \overline{D} \setminus \{p_0\}$.

LEMMA 4. — *Let $X \subseteq bD$ be a continuum and suppose that the tangent to X exists at $p_0 \in X$ and that this tangent is not complex tangential to bD. Then the tangent to $g(X) \subseteq \mathbb{C}$ at $\lambda = 0$ is the real axis* $\{\lambda \in \mathbb{C} : \operatorname{Im}\lambda = 0\}$.

Proof. — Let $\lambda_n (\neq 0) \in g(X)$ be such that $\lambda_n \to 0$. Say $\lambda_n = g(p_n)$, $p_n \in X$. Then the fact $\pi^{-1}(0) \cap X = \{p_0\}$ yields that $p_n \to p_0$. Passing to a subsequence, $(p_n - p_0)/\|p_n - p_0\| \to \tau$, the tangent to X at p_0. Since $X \subseteq bD$, $\big(\tau, N(p_0)\big) = 0$. Hence $\operatorname{Im}\langle \tau, e\rangle = 0$. We have

$$\frac{\lambda_n}{\|p_n - p_0\|} = \Big\langle \frac{p_n - p_0}{\|p_n - p_0\|}, e \Big\rangle.$$

Hence $\operatorname{Im}\big(\lambda_n/\|p_n - p_0\|\big) \to \operatorname{Im}\langle \tau, e\rangle = 0$. As τ is not a complex tangent, $\operatorname{Re}\langle \tau, e\rangle :\equiv b \neq 0$. Hence

$$\operatorname{Re}\big(\lambda_n/\|p_n - p_0\|\big) \to \operatorname{Re}\langle \tau, e\rangle = b.$$

It follows that $\operatorname{Im}\lambda_n/\operatorname{Re}\lambda_n \to 0/b = 0$. This gives the lemma.

The next lemma describes the nice projections which will be used to prove the remaining theorems. As above, we write

$$Q_\sigma = \{\lambda \in \mathbb{C} : |\operatorname{Re}\lambda| < \sigma,\ |\operatorname{Im}\lambda| < \sigma\}.$$

LEMMA 5. — *Let V be an irreducible subvariety of D, $p_0 \in aV$. Let X be a continuum in bD with $\mathcal{H}^1(X) < \infty$. Suppose that X contains a neighborhood of p_0 in bV and that the tangent to X at p_0 exists and is not complex tangential to bD. Let $A \subseteq \mathbb{R}$ be defined, as in Proposition 3, as $\{x \in \mathbb{R} : \#(\pi^{-1}(x) \cap X) = 1\}$. Let E be a compact subset of $aV \cap X$. Suppose that 0 is a point of density of $A \cap \pi(E)$. Then for all $\sigma > 0$ sufficiently small,*

(a) $g(bV \setminus X) \subseteq \{\lambda : \operatorname{Im}\lambda > 2\sigma\}$.

(b) $g(X) \cap Q_\sigma \subseteq \{\lambda : \operatorname{Im}\lambda < \frac{1}{2}\sigma\}$.

(c) *$L := g(X) \cup bQ_\sigma$ is connected and if Ω is the component of $\mathbb{C} \setminus L$ containing $(-\sigma,\sigma) \times (\frac{1}{2}\sigma,\sigma)$, and $W := g^{-1}(\Omega) \cap V$, then $g : W \to \Omega$ is a homeomorphism, Ω is simply connected and $\mathcal{H}^1(b\Omega) < \infty$.*

(d) *Let $B = A \cap \pi(E) \cap (-\sigma,\sigma)$. For $x \in B$, $g(X) \cap \ell_x \cap Q_\sigma$ is a single point $\lambda_x = x + iv_x \in b\Omega$ and $g^{-1}(\lambda_x) \cap \overline{V}$ is a single point $p_x \in E \subseteq aV$. Let γ_x be the segment $\{x + iy : v_x < y < \sigma\}$. Then $g^{-1}(\gamma_x) \cap W$ is a Jordan curve in W which approahces p_x asymptotically. The set*

$$\omega = \{\lambda_x : x \in B\} = \big(B \times (-\sigma,\sigma)\big) \cap g(X)$$

is a Borel subset of $b\Omega$ and $\mathcal{H}^1(\omega) > 0$.

Proof. — Since $g^{-1}(0) \cap bD = \{p_0\}$, and since X is a neighborhood of p_0 in bV, (a) holds for small σ. LEMMA 4 gives (b) for small σ. For (c) we note first that L is connected if $g(X)$ and bQ_σ are not disjoint, which is true for small σ. Then by (a) and (b) L is disjoint from $(-\sigma,\sigma) \times (\frac{1}{2}\sigma,\sigma)$ for small σ. Clearly then Ω is a simply connected domain and $b\Omega \subseteq L$ and so $\mathcal{H}^1(b\Omega) \le \mathcal{H}^1(L) < \infty$. Then $g : W \to \Omega$ is a proper map and so is a branched cover of multiplicity $m \ge 0$. We must show that $m = 1$.

Suppose that $m = 0$. Fix $x_1, x_2 \in B$ with $-\sigma < x_1 < 0 < x_2 < \sigma$. Let $W' = g^{-1}(Q^0_\sigma) \cap V$. Since $p_0 \in bV$, there exists $q \in W'$ such that $g(q) \in Q^0_\sigma$ and $x_1 < \operatorname{Re} g(q) < x_2$. Let W'' be the component of W' which contains q. Since $m = 0$, $g(W'')$ does not meet Ω and therefore $g(W'')$ does not meet $\ell_x \cap Q_\sigma$ for $x \in B$. Set $R = (x_1, x_2) \times (-\sigma,\sigma)$. Then $g(W'')$ does not meet bR but does meet R. Hence, by connectness, $g(W'') \subseteq R$. Since $bW'' \subseteq X \cup g^{-1}(bQ_\sigma)$, it follows that $bW'' \subseteq X$. Hence W'' is a subvariety of D and $W'' \subseteq V$. This implies $W'' = V$, a contradiction if σ is small.

Now suppose $m > 1$. By removing a countable set from B we can assume that γ_x contains only regular values of $g_{|V}$ for each $x \in B$ and so $g^{-1}(\gamma_x) \cap W$ is a disjoint union of m Jordan arcs $\gamma_1^x, \gamma_2^x, \dots, \gamma_m^x$. For each $x \in B$, $g(V)$ is disjoint from the set $\ell_x \cap \{\lambda : \operatorname{Im}\lambda < v_x\}$; this is because $g(X)$ and $g(bV \setminus X)$ and therefore $g(bV)$ are disjoint from the set. As $g_{|V}$ is an open map, $\lambda_x \notin g(V)$. Hence $g^{-1}(\lambda_x) \cap \overline{V}$ is the unique point p_x of $g^{-1}(\lambda_x) \cap aV$. We conclude that each γ_j^x approaches p_x asymptotically. Suppose that $\lambda_0 \in \Omega$ is such that $g^{-1}(\lambda_0) \cap W$ contains m (> 1) distinct points $w_1^0, w_2^0, \dots, w_m^0$. Choose a polynomial f in $\mathbb{C}^n$ which separates these m points. Define a function F on Ω by

$$F(\lambda) = \prod_{\substack{W \cap g^{-1}(\lambda) = \{p_1, p_2, \dots p_m\} \\ i<j}} \big(f(p_i) - f(p_j)\big)^2.$$

Then F is a well-defined bounded holomorphic function on Ω and $F(\lambda_0) \neq 0$. For $x \in B$, $f(\zeta) \to f(p_x)$ as $\zeta \in \gamma_j^x \to p_x$, for $1 \leq j \leq m$. It follows that $F(\lambda) \to 0$ as $\lambda \in \gamma_x \to \lambda_x \in \omega$. $\mathcal{H}^1(\omega) \geq \mathcal{H}^1(B) > 0$. By LEMMA 3, $F \equiv 0$. Contradiction! We conclude that $m = 1$ and therefore $g : W \to \Omega$ is a homeomorphism. Our arguments also give (d).

3.2. — We now prove THEOREM 1. By hypothesis, $\mathcal{H}^1(aV_1 \cap aV_2) > 0$. For $j = 1$ and 2, aV_j is a disjoint union of a countable set of rectifiable Jordan arcs and an $\mathcal{H}^1$-null set. It follows that there exist rectifiable Jordan arcs $\Gamma_j \subseteq aV_j$ such that $\mathcal{H}^1(\Gamma_1 \cap \Gamma_2) > 0$. Let $\gamma_j : (a_j, b_j) \to \mathbb{C}^n$ parameterize Γ_j by arc length and set $T_j = \gamma_j^{-1}(\Gamma_1 \cap \Gamma_2)$.

LEMMA 6. — *Every point $p \in \Gamma_1 \cap \Gamma_2$, except for a set of $\mathcal{H}^1$ measure zero, has the following properties :*

(a) *$\gamma_j^{-1}(p)$ is a Lebesgue point of γ_j' and is a point of density of T_j, for $j = 1$ and 2.*

(b) *$aV_1 \cup aV_2$ has a tangent at p.*

(c) *$aV_1 \cup aV_2$ is regular at p.*

(d) *the tangent to $aV_1 \cup aV_2$ at p is not complex tangential to bD.*

Proof. — It suffices to show that each of these conditions hold $\mathcal{H}^1$ a.e. on $\Gamma_1 \cap \Gamma_2$. Part (a) follows from the fact that almost every point of (a_j, b_j) is a Lebesgue point of γ_j' and almost every point of T_j is a point of density of T_j.

At each point of $aV_1 \cap aV_2$, the set $aV_1 \cup aV_2$ is locally connected and has a neighborhood of finite linear measure. Hence PROPOSITION 1 implies that (b) holds a.e. on $\Gamma_1 \cap \Gamma_2$. Likewise for (c) because continua of finite linear measure are regular $\mathcal{H}^1$ a.e. Finally, THEOREM 2 gives (d).

Now fix $p_0 \in \Gamma_1 \cap \Gamma_2$ such that (a)–(d) of LEMMA 6 hold at p_0. Let X be a compact connected neighborhood of p_0 in $aV_1 \cup aV_2$ with $\mathcal{H}^1(X) < \infty$. Set $t_j = \gamma_j^{-1}(p_0)$, $j = 1, 2$. Then $aV_1, aV_2, \Gamma_1, \Gamma_2$ and X all have the same tangent at p_0 and this tangent is not complex tangential to bD at p_0; the tangent is $\gamma_1'(t_1) = \gamma_2'(t_2)$, with a possible change of orientation of γ_2. We set $e = iN(p_0)$ as usual. We can now apply PROPOSITION 3 to X and Γ_1 and to the set A defined there to conclude that 0 is a point of density of A. Similarly, we apply PROPOSITION 2 to Γ_1 and T_1 to obtain that 0 is a point of density of $\pi(\Gamma_1 \cap \Gamma_2)$; we note that in PROPOSITION 2, for $u :\equiv \pi \circ \gamma_1$, $u(T_1) = \pi(\Gamma_1 \cap \Gamma_2)$. Hence 0 is a point of density of $A \cap \pi(E)$ where we have set $E :\equiv \Gamma_1 \cap \Gamma_2$.

We can now apply LEMMA 5 twice. First to V_1 and X and then to V_2 and X. We conclude that if $\sigma > 0$ is sufficiently small, then $g : W_j \to \Omega$ is a homeomorphism for $j = 1$ and 2, with $W_j = g^{-1}(\Omega) \cap V_j$. We claim that $W_1 = W_2$. This implies that $V_1 = V_2$ and completes the proof of THEOREM 1. Suppose not! Then there exists $\lambda_0 \in \Omega$ such that $w_1^0 \neq w_2^0$ where $w_j^0 = g^{-1}(\lambda_0) \cap W_j$. Choose a polynomial f such that $f(w_1^0) \neq f(w_2^0)$. Define a function F on Ω by

$$F(\lambda) = f \circ \left(g_{|W_1}\right)^{-1} - f \circ \left(g_{|W_2}\right)^{-1}.$$

F is a bounded holomorphic function on Ω and $F(\lambda_0) \neq 0$. Let $x \in B$ and set $\sigma_j^x = \left(g_{|W_j}\right)^{-1}(\gamma_x)$. Then $\sigma_j^x \to p_x$ and so $f(\zeta) \to f(p_x)$ as $\zeta \in \sigma_j^x \to p_x$. Hence $F(\lambda) \to 0$ as $\lambda \in \gamma_x \to \lambda_x$. Hence $F \equiv 0$ by LEMMA 3. This is the desired contradiction.

4.1. — We shall briefly recall the definition of harmonic measure in our setting. Let V be an irreducible subvariety of complex dimension one of a strictly pseudoconvex domain D in $\mathbb{C}^n$. Let $\tau : \widetilde{V} \to V$ be the usual normalization. We shall say that a continuous real valued function ϕ on V is subharmonic if $\tau^*(\phi) = \phi \circ \tau$ is subharmonic on the Riemann surface $\widetilde{V}$. If ϕ is subharmonic on V and continuous on $\overline{V} = V \cup bV$, then the usual maximum principle holds. For a real-valued continuous function u on bV we apply the usual Perron process to get a continuous function $\tilde{u}$ on V such that $\tau^*(\tilde{u})$ is harmonic on $\widetilde{V}$. Barrier functions exist at each point of bV; in fact the real parts of peaking functions in $A(D)$ can be employed. Consequently $\tilde{u}$ attains the boundary values u and so extends to be continuous on $\overline{V}$. For $p \in V$, the functional $u \mapsto \tilde{u}(p)$ is positive and linear and therefore there is a unique positive measure μ_p on bV such that $\tilde{u}(p) = \int_{bV} u \, \mathrm{d}\mu_p$; μ_p is harmonic measure for p on bV (relative to V). It follows from Harnack's equality that for p_1 and p_2 in V there exists $C > 0$ such that $\mu_{p_2} \leq C\mu_{p_1}$.

LEMMA 7. — *Let F be a compact subset of bV such that $\mu_p(F) = 0$ for some $p \in V$. Then there exists a real continuous function h on V such that $\tau^*(h)$ is harmonic on $\widetilde{V}$ and such that, for each $\zeta \in F$,*

$$\lim_{\substack{z \to \zeta, \\ z \in V}} h(z) = \infty.$$

Moreover, $h \geq 1$ on V.

Proof. — First, since $\mu_p(F) = 0$, there exists a pointwise increasing sequence of continuous real-valued functions $\{u_n\}$ on bV such that $u_n \geq n$ on F, $1 \leq u_n$ on bV, and $\int u_n d\mu_p < 2$ for all n. Let h be the limit of the increasing sequence $\tilde{u}_n$. Since $\tilde{u}_n(p) < 2$, $h(p) \leq 2$ and therefore, by Harnack, h is continuous and finite on V and $\tau^*(h)$ is harmonic on $\widetilde{V}$. The limit statement follows from the fact that, as $z \in V \to \zeta \in F$, $\liminf h(z) \geq \lim h_n(z) \geq n$ for all n.

Also $1 \leq u_n$ on bV implies $1 \leq h$ on V.

4.2 Proof of theorem 3a. — We first show $\mathcal{H}^1|_{bV} \ll \mu_p$. Suppose not! Then there exists a compact subset F of bV such that $\mu_p(F) = 0$ and $\mathcal{H}^1(F) > 0$. Since $\mathcal{H}^1(bV \setminus aV) = 0$, there exists a rectifiable Jordan arc $\Gamma \subseteq aV$ such that $\mathcal{H}^1(E) > 0$, where $E = \Gamma \cap F$. Let $\gamma : (a, b) \to \mathbb{C}^n$ parametrize Γ by arclength and set $T = \gamma^{-1}(E)$. Choose $t_0 \in T$ such that all of the following conditions hold : t_0 is a point of density of T, t_0 is a Lebesgue point of γ', $p_0 \equiv \gamma(t_0)$ is a regular point of aV, $\gamma'(t_0)$ is the tangent to aV at p_0 and is not complex tangential to bD. In fact, by LEMMA 1, PROPOSITION 1 and THEOREM 2, almost all points of T will do. Set $e = iN(p_0)$, $g(p) = \langle p - p_0, iN(p_0)\rangle$ and $\pi = \mathrm{Re} \circ g$, as usual. By PROPOSITIONS 2 and 3, 0 is a point of density of $\pi(E) \cap A$, where A is defined in PROPOSITION 3. Arguing as in § 3.2, we can choose a continuum X which is a neighborhood of p_0 in aV such that $\mathcal{H}^1(X) < \infty$. The hypotheses of LEMMA 5 are valid and we get a $\sigma > 0$ such that (a)–(d) of LEMMA 5 hold ; in particular we have Q_σ, the homeomorphism $g : W \to \Omega$ and $\omega \subseteq b\Omega$.

Since $\mu_p(F) = 0$, LEMMA 7 gives a function h with $\tau^*(h)$ harmonic on $\widetilde{V}$. Set $u = h \circ (g_{|W})^{-1}$ on Ω. Then u is continuous and harmonic on Ω, since possible isolated singular points for u are removable. Let v be a harmonic conjugate for u on the simply connected domain Ω. Consider $\lambda_x \in \omega$. By LEMMA 7, $h(z) \to \infty$ as $z \in g^{-1}(\gamma_x) \to p_x \in F$, in the notation of LEMMA 5. Hence $u(\lambda) \to \infty$ as $\lambda \in \gamma_x \to \lambda_x \in \omega$. Set $f(\lambda) = \mathrm{e}^{-(u(\lambda)+iv(\lambda))}$ for $\lambda \in \Omega$. Then f is a bounded holomorphic function on Ω and $f \neq 0$ in Ω. By LEMMA 3, $f \equiv 0$ in Ω. Contradiction ! We conclude that $\mathcal{H}^1|_{bV} \ll \mu_p$.

Next we show that $\mu_p \ll \mathcal{H}^1{}_{|bV}$. Let $E \subseteq bV$ be a Borel set with $\mathcal{H}^1(E) = 0$. Let K be any compact subset of E. Then $\mathcal{H}^1(K) = 0$ and by the corollary to Theorem 2 of DAVIE and ØKSENDAL [5], K is a peak interpolation set for $A(D)$. Hence there exists a function $f \in A(D)$ such that $f = 1$ on K and $|f| < 1$ on $\overline{D} \setminus K$. We have

$$0 = \lim f^n(p) = \lim \int f^n \,\mathrm{d}\mu_p = \int_K 1 \,\mathrm{d}\mu_p = \mu_p(K)$$

By the regularity of μ_p, we get $\mu_p(E) = 0$; i.e., $\mu_p \ll \mathcal{H}^1{}_{|bV}$.

Remark. — The last paragraph is equally valid for any representing measure for evaluation at p in place of μ_p. This gives part (c) of THEOREM 3. Combining this with part (a) we see that every representing measure for p is absolutely continuous with respect to μ_p. In the terminology of the abstract F. and M. RIESZ theorem [11], μ_p is a "dominant" representing measure for evaluation at p.

4.3 Proof of theorem 3b. — Fix $p \in V$. Then μ_p is a representing measure for evaluation at p for the algebra $A(V)$. Let $\nu = \nu_a + \nu_s$ be the Lebesgue decomposition of ν with respect to μ_p; $\nu_a \ll \mu_p$ and $\nu_s \perp \mu_p$. By the abstract F. and M. RIESZ theorem [11, p. 44], ν_s is orthogonal to $A(V)$. By part (a), $\nu_a \ll \mathcal{H}^1{}_{|bV}$ and $\nu_s \perp \mathcal{H}^1{}_{|bV}$. Thus it suffices to show that $\nu_s = 0$. There exists a Borel set $E \subseteq bV$ such that ν_s is concentrated on E and $\mathcal{H}^1(E) = 0$. Let K be any compact subset of E. Then $\mathcal{H}^1(K) = 0$ and, as noted above, DAVIE and ØKSENDAL proved that K is a peak interpolation set for $A(D)$. This implies that $|\nu_s|(K) = 0$. Indeed, let $g \in A(D)$ be such that $g \equiv 1$ on K and $|g| < 1$ on $\overline{D} \setminus K$ and let u be any continuous function on K; extend u to a function on $\overline{D}$ in $A(D)$ and note that $\int_K u d\nu_s = \lim_{n\to\infty} \int u g^n \,\mathrm{d}\nu_s = 0$, since $ug^n \in A(D)$ and ν_s is orthogonal to $A(D)$. By the regularity of ν_s we conclude that $\nu_s = 0$.

5.1. — We next consider accessibility of points of bV.

PROPOSITION 4. — *Let V be an irreducible one-dimensional subvariety of a strictly pseudoconvex domain D in $\mathbb{C}^n$. Let $p_0 \in bV$ be such that there exists a neighborhood N of p_0 in bV with $\mathcal{H}^1(N) < \infty$. Then p_0 is accessible from V.*

Proof. — As above, we may assume that D is strictly convex. Also as above we have the projection $g : \mathbb{C}^n \to \mathbb{C}$, $g(p) = \langle p - p_0, iN(p_0)\rangle$ with $\operatorname{Im} g(p) > 0$ for $p \in \overline{D} \setminus \{p_0\}$. Fix $q_0 \in V$ and choose $c > 0$ such that $g(bV \setminus N) \subseteq \{\lambda \in \mathbb{C} : \operatorname{Im}\lambda > c\}$ and $c < \operatorname{Im} g(q_0)$.

Let k_t denote the horizontal line $\{\lambda \in \mathbb{C} : \operatorname{Im}\lambda = t\}$ for $t \in \mathbb{R}$. Set $n(t) = \#\{p \in bV : g(p) \in k_t\}$. Then, since $\mathcal{H}^1(N) < \infty$, $\int_0^c n(t)\,\mathrm{d}t < \infty$

and, in particular, $n(t)$ is finite a.e. for $0 < t < c$. Choose $\delta_n \downarrow 0$, $\delta_n < c$ such that $n(\delta_n)$ is finite and such that k_{δ_n} contains no singular values of $g_{|V}$ for all n. Set $V_n = V \cap g^{-1}\{\lambda \in \mathbb{C} : \operatorname{Im}\lambda < \delta_n\}$. Then V_n is non-empty for each n, since $p_0 \in bV$.

We claim that V_n, a subvariety of $\{p \in \mathbb{C}^n : \operatorname{Im} g(p) < \delta_n\} \cap D$, has a finite number of components. Indeed the finite set $F_n = g(bV) \cap k_{\delta_n} \subseteq k_{\delta_n}$ divides k_{δ_n} into a finite set of line segments $\{\gamma\}$ such that $g^{-1}(\gamma) \cap V$ is again a finite union of disjoint arcs $\{\sigma\}$, each mapped homeomorphically to γ by g. This is because $g : V \cap g^{-1}(\Omega) \to \Omega$ is a branced cover for each component Ω of $\mathbb{C} \setminus g(bV)$ and there is no branching over k_{δ_n}. Each σ is contained in the closure of one and only one component of V_n. Also each component of V_n is such that any of its points can be joined to q_0 by a path α in V. By connectedness, the path α, which can be chosen to avoid the finite set $V \cap g^{-1}(F_n)$, must meet one of the σ's. Consequently the closure of the component meets and therefore must contain one of the σ's. Thus the number of components of V_n is at most the number of σ's.

Next we shall choose inductively a sequence $\{W_n\}$ such that W_n is a component of V_n, $W_{n+1} \subseteq W_n$ and $p_0 \in \overline{W}_n$, as follows. Since $p_0 \in \overline{V}_1$ and V_1 contains a finite number of components, choose W_1 as any component of V_1 with $\overline{W}_1$ containing p_0. Given $W_1, W_2, \ldots, W_n$ as above, note that $V_{n+1} \cap W_n$ is non-empty since $p_0 \in \overline{W}_n$. Clearly the set of components of $V_{n+1} \cap W_n$ is a subset of the finite set of components of V_{n+1} and consequently, one of these components contains p_0 in its closure. Choose this component to be W_{n+1}.

Now choose any sequence $\{q_n\}$ such that $q_n \in W_n$. Then clearly $q_n \to p_0$. We can join q_1 to q_2 by a path in W_1 and then join q_2 to q_3 by a path in W_2, etc. The sum of these paths gives a path in V which approaches p_0 asymptotically.

5.2. Example. — Set $\rho(\theta) = \theta/(1 + 2\theta)$ for $0 \leq \theta$. Let S be the spiral $\{\rho(\theta)\mathrm{e}^{i\theta} : 0 \leq \theta < \infty\}$ and let C be the circle $\left\{\lambda \in \mathbb{C} : |\lambda| = \frac{1}{2}\right\}$. Then $\overline{S} = S \cup C$. Let Ω be the bounded component of $\mathbb{C} \setminus \overline{S}$. Then C is (the underlying set of) a prime end of the simply connected domain Ω. There is a Riemann map $f : U \to \Omega$, where U is the open unit disk, such that f extends to be a continuous map $\overline{U} \setminus \{1\} \to \overline{\Omega} \setminus C$ and $|f|$ extends continuously to $\overline{U}$ such that $|f|(1) = \frac{1}{2}$ and $|f| \leq \frac{1}{2}$ on $\overline{U}$. Hence there exists a continuous function g on $\overline{U}$, holomorphic on U, such that $|g| = \sqrt{1 - |f|^2}$ on bU. In particular, $(\frac{1}{2})^2 + |\beta|^2 = 1$, where $\beta = g(1)$. Define $\Phi : U \to \mathbb{B}_2 =$ the open unit ball in $\mathbb{C}^2$ by $\Phi(\lambda) = \big(f(\lambda), g(\lambda)\big)$. Then Φ is a proper map and so its image is a complex submanifold V

of $\mathbb{B}_2$ with $bV = \Phi\big(bU \setminus \{1\}\big) \cup C \times \{\beta\}$. The points of $C \times \{\beta\} \subseteq bV$ are not accessible from V. Indeed, if γ were a curve in V approaching some point of $C \times \{\beta\}$, then $z_1 \circ \gamma$ would be a curve in Ω approaching a point of C. But no such curve in Ω exists.

The construction also shows that THEOREM 1 cannot have its hypotheses greatly weakened. Namely, if V_1 and V_2 have connected ends of finite linear measure, then Theorem 1 can be applied to see that $V_1 = V_2$ provided that $\mathcal{H}^1(bV_1 \cap bV_2) > 0$.

This would not be true if we only knew that the ends had σ-finite $\mathcal{H}^1$ measure. Namely, take $V_1 = V$ as above and take

$$V_2 = \big\{(\lambda, \beta) \in \mathbb{C}^2 : |\lambda|^2 + |\beta|^2 < 1\big\},$$

a subvariety of the unit ball $\mathbb{B}_2$. Then $bV_2 = C \times \{\beta\}$ and $bV_1 \cap bV_2 = C \times \{\beta\}$ has positive measure, $\mathcal{H}^1(bV_2) < \infty$, bV_1 is $\mathcal{H}^1$ σ-finite and is the union of two real analytic curves, but $V_1 \neq V_2$. One can also show that

$$\big\{ \text{ the polynomial hull of } bV_1 \big\} \setminus bV_1 = V_1 \cup V_2.$$

5.3. — For the proof of THEOREM 4 we shall assume that D is strictly convex. Choose a point p_0 such that there exists an open rectifiable Jordan arc Γ in aV continuing p_0, such that aV is regular at p_0, such that the tangent to aV exists and is not complex tangential to bD. By our previous arguments, $\mathcal{H}^1$ almost all points $p_0 \in aV$ will suffice. Then, taking E as a compact neighborhood of p_0 in aV, the argument of THEOREM 3 shows that the conclusion of LEMMA 5 holds for σ sufficiently small, where $g(p) \equiv \langle p - p_0, e\rangle$ and $e = iN(p_0)$.

Thus we have the homeomorphism $g : W = V \cap g^{-1}(\Omega) \to \Omega$. We identify $e^{\perp}$ in $\mathbb{C}^n$ with $\mathbb{C}^{n-1}$ and define a $\mathbb{C}^{n-1}$-valued holomorphic $\psi(\lambda)$ for $\lambda \in \Omega$ as follows. For $\lambda \in \Omega$, write $g^{-1}(\lambda) = w = p_0 + \lambda \mathrm{e} + w' \in W$, with $\langle w', e\rangle = 0$, the orthogonal decomposition. Now define $\psi(\lambda)$ for $\lambda \in \Omega$ by $\psi(\lambda) = w'/\lambda \in \mathrm{e}^{\perp} = \mathbb{C}^{n-1}$; more explicitly,

$$\psi(\lambda) = \frac{1}{\lambda}\big[g^{-1}(\lambda) - p_0 - \lambda e\big].$$

LEMMA 8. — *ψ is a bounded $\mathbb{C}^{n-1}$-valued holomorphic function on Ω.*

Assuming the lemma, say $\|\psi(\lambda)\| \le M$ for $\lambda \in \Omega$, we shall complete the proof of THEOREM 4. We know that $it \in \Omega$ for $0 < t < \sigma$. Define the curve γ in V by $\gamma(t) = g^{-1}(it) \in V$ for $0 < t < \sigma$. Then $\gamma \to p_0$ as $t \downarrow 0$. To see that γ approaches p_0 non-tangentially, write

$$\gamma(t) = g^{-1}(it) = p_0 + ite + w' = p_0 + t\big(-N(p_0)\big) + it\psi(it),$$

using $\psi(\lambda) = w'/\lambda$. Since $-N(p_0)$ is the inward unit normal to bD at p_0, we must show, for non-tangential approach, that $\|w'\|/t$ is bounded as $t \downarrow 0$. But this quotient equals $\|\psi(it)\|$, which is bounded by M. The theorem follows.

5.4 Proof of lemma 8. — We first show that ψ is in the Hardy space $H^2(\Omega, \mathbb{C}^{N-1})$. For this, we need to show that $\|\psi(\lambda)\|^2$ has a harmonic majorant on Ω. Since D is strictly convex, there is a large ball containing D whose boundary contains p_0 and such that this boundary is tangent to bD at p_0. As $-N(p_0)$ is the inward normal to D at p_0 this ball is of the form $B\big(p_0 - RN(p_0), R\big)$ for some $R > 0$. Hence

$$W = g^{-1}(\Omega) \cap V \subseteq D \subseteq B\big(p_0 - RN(p_0), R\big).$$

Writing $w = p_0 + \lambda e + w'$, $\langle w', e\rangle = 0$, we get

$$\begin{aligned} R^2 &> \|w - \big(p_0 - RN(p_0)\big)\|^2 \\ &= \|(\lambda - iR)e + w'\|^2 \\ &= |\lambda - iR|^2 + \|w'\|^2, \end{aligned}$$

since $N(p_0) = -ie$. Thus $\|w'\|^2 < R^2(1 - |\lambda/R - i|^2)$. Set

$$h(\zeta) = \frac{1 - |\zeta - i|^2}{|\zeta|^2}$$

for $\zeta \neq 0$ in $\mathbb{C}$. We have $\|\psi(\lambda)\|^2 = \|w'\|^2/|\lambda|^2 \leq h(\lambda/R)$. A computation shows that $h(\zeta)$ is harmonic on the upper half plane and therefore $\lambda \mapsto h(\lambda/R)$ is harmonic on Ω. Thus $\psi \in H^2$.

5.5. — Next we show that ψ is bounded on $b\Omega \setminus \{0\}$.

LEMMA 9. — *There exists an $M > 0$ such that for all $\zeta \in b\Omega \setminus \{0\}$*

$$\limsup_{\substack{\lambda \to \zeta \\ \lambda \in \Omega}} \|\psi(\lambda)\| \leq M.$$

Proof. — We have $\|\psi(\lambda)\|^2 \leq u(\lambda) \equiv h(\lambda/R)$ for all $\lambda \in \Omega$, with $h(\zeta) = (1-|\zeta-i|^2)/|\zeta|^2$. It suffices thus to show that u, which satisfies $u \geq 0$ on Ω, is bounded above on $b\Omega \setminus \{0\}$. Suppose not! Then there exist $\lambda_n \in b\Omega$, $\lambda_n \neq 0$ such that $u(\lambda_n) \to \infty$ and $\lambda_n \to 0$. Since $g(bV)$ contains a neighborhood of 0 in $b\Omega$, there exist $w_n \in bV$ such that $g(w_n) = \lambda_n$ and then $w_n \to p_0$, $w_n \neq p_0$.

Since bD is smooth, there exists an internally tangent sphere at $p_0 \in bD$; i.e., there exists a $\delta > 0$ such that $B(p_0 - \delta N(p_0), \delta) \subseteq D$. Then bD is exterior to this ball and therefore $\|w_n - (p_0 - \delta N(p_0))\|^2 \geq \delta^2$. Writing $w_n = p_0 + \lambda_n e + w_n'$, $\langle e, w_n' \rangle = 0$, we get $\delta^2 \leq |\lambda_n - i\delta|^2 + \|w_n'\|^2$. Setting $\lambda_n = s_n + it_n$ we get

$$t_n \leq \frac{1}{2\delta}\big(|\lambda_n|^2 + \|w_n'\|^2\big). \tag{8}$$

Since the unit tangent τ to bV at p_0 is not complex tangential to bD, we have $\tau = ae + w_0'$ where $0 < a \leq 1$. Passing to a subsequence if necessary, we can assume that $(w_n - p_0)/\|w_n - p_0\| \to ae + w_0'$. Taking the inner product with e yields

$$\frac{\lambda_n}{\sqrt{|\lambda_n|^2 + \|w_n'\|^2}} \to a.$$

Taking real parts gives $s_n/\sqrt{|\lambda_n|^2 + \|w_n'\|^2} \to a$. By (8) we get

$$t_n \leq \frac{1}{2\delta}\left[\frac{|\lambda_n|^2 + \|w_n'\|^2}{s_n^2}\right] s_n^2.$$

Since the quotient in brackets converges to $1/a^2$, we get a $C > 0$ such that $t_n \leq Cs_n^2$ for all n. Now we have

$$u(\lambda_n) = \frac{R^2 - |\lambda_n - iR|^2}{|\lambda_n|^2} = \frac{2t_nR}{s_n^2 + t_n^2} - 1 \leq 2CR - 1.$$

This contradicts the fact that $u(\lambda_n) \to \infty$.

5.6. — To complete the proof of LEMMA 8 we consider a Riemann map $\varphi : U \to \Omega$ and the pull-back $\Psi = \varphi^*(\psi) = \psi \circ \varphi$ of ψ. The harmonic majorant of $\|\psi\|^2$ pulls back to one for $\|\Psi\|^2$ and so $\Psi \in H^2(U, \mathbb{C}^{n-1})$, the usual Hardy space. By LEMMA 9, the boundary function Ψ^* of Ψ satisfies $\|\Psi^*\|^2 \leq M$ a.e. on bU and therefore Ψ is a bounded holomorphic function on U. Hence ψ is bounded on Ω.

BIBLIOGRAPHY

[1] ALEXANDER (H.). — *Polynomial approximation and hulls in sets of finite linear measure in* $\mathbb{C}^n$, Amer. J. Math., t. **93**, 1971, p. 65–74.
[2] ALEXANDER (H.). — *The polynomial hull of a rectifiable curve in* $\mathbb{C}^n$, Amer. J. Math., t. **110**, 1988, p. 629–640.

[3] ALEXANDER (H.). — *Linear measure on plane continua of finite linear measure*, Ark. Mat., t. **27**, 1989, p. 169–177.
[4] BERNDTSSON (B.). — *Integral formulae for the $\partial\overline{\partial}$-equation and zeros of bounded holomorphic functions on the unit ball*, Math. Ann., t. **249**, 1980, p. 163–176.
[5] DAVIE (A.) and ØKSENDAL (B.). — *Peak interpolation sets for some algebras of analytic functions*, Pacific J. Math., t. **41**, 1972, p. 81–87.
[6] EILENBERG (S.) and HARROLD (O.). — *Continua of finite linear measure I.*, Amer. J. Math., t. **65**, 1943, p. 137–146.
[7] FALCONER (K.). — *The Geometry of Fractal Sets, Cambridge.* — Cambridge Univ. Press, 1985.
[8] FEDERER (H.). — *Geometric Measure Theory.* — Springer-Verlag, 1969.
[9] FORNAESS (J.-E.). — *Embedding strictly pseudoconvex domains in convex domains*, Amer. J. Math., t. **98**, 1976, p. 529–569.
[10] FORSTNERIČ (F.). — *Regularity of varieties in strictly pseudoconvex domains*, Publ. Math., t. **32**, 1988, p. 145–150.
[11] GAMELIN (T.). — *Uniform Algebras.* — Prentice Hall, 1969.
[12] GLOBEVNIK (J.) and STOUT (E.L.). — *The ends of varieties*, Amer. J. Math., t. **108**, 1986, p. 1355–1410.
[13] GLOBEVNIK (J.) and STOUT (E.L.). — *The ends of discs*, Bull. Soc. Math. France, t. **114**, 1986, p. 175–195.
[14] GLOBEVNIK (J.) and STOUT (E.L.). — *Boundary regularity for holomorphic maps from the disc to the ball*, Math. Scand., t. **60**, 1987, p. 31–38.
[15] GLOBEVNIK (J.) and STOUT (E.L.). — *Analytic discs with rectifiable simple closed curves as ends*, Ann. Math., t. **127**, 1988, p. 389–401.
[16] HENKIN and CIRKA (E.). — *Boundary properties of holomorphic functions of several complex variables*, J. Sov. Math., t. **5**, 1976, p. 612–687.
[17] RIESZ (F.) and (M.). — *Uber die Randwerte einer analytischen Funktion*, 4e Congres des Math. Scand., Stockholm, 1916, p. 27–44.
[18] ROSAY (J.-P.). — *A remark on a theorem of Forstnerič.* — Preprint, 1989.
[19] RUDIN (W.). — *Function Theory in the Unit Ball in $\mathbb{C}^n$.* — Springer-Verlag, 1980.