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SOME EXCEPTIONAL COMPACT MATRIX PSEUDOGROUPS

BY

NICOLÁS ANDRUSKIEWITSCH (*)

RÉSUMÉ. — Nous construisons des groupes quantiques compacts de matrices de types E_6 et E_7.

ABSTRACT. — We construct compact matrix pseudogroups of types E_6 and E_7.

A mis hermanos Alejandro y Andrés.

0. Introduction

0.1. — Examples of non-commutative non-cocommutative Hopf algebras arise in the literature at least in three different ways : as quantized enveloping algebras ([KR], [Sk], [D1], [J]) ; as rings of functions on the formal (or algebraic) quantum group ([D2], [FRT], [T], [Ma]) ; or as compact matrix pseudogroups ([W1], [W2], [W3] and also [VS]). A natural question, already considered by several authors, is to relate these approaches. In this article, we shall construct compact matrix pseudogroups from representations of quantized enveloping algebras (QEA). For QEA of classical type, this was done in [R2], using an explicit construction of its "natural" representations from [Re].

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0.2. — A way to construct compact matrix pseudogroups, the Tannaka-Krein-Woronowicz theorem [W3], is to consider a concrete monoidal W^*-category and to complete the $*$-Hopf algebra built from the dual of spaces of morphisms with respecte to a suitable norm. In few words, the idea is to isolate the minimal number of properties of the category of unitary (finite dimensional) representations of a compact matrix pseudogroup and to show that a category satisfying those properties comes from a uniquely determined such object. The main observation in (the first part of) [R2] is that the category of finite dimensional representations of a QEA (with real positive parameter) is a concrete monoidal W^*-category, provided that each representation carries an "invariant" hermitian form. This last problem is solved once it is solved for a finite (one or two) number of representations; for these, the hermitian form is constructed explicitly using the formulas in [Re].

0.3. — Our approach, though very close in spirit to [W3], [R2], carries a technical simplification. Let us consider a finite dimensional representation ρ of a complex Hopf algebra $\mathcal{A}$ and let us assume for simplicity that it is isomorphic, as $\mathcal{A}$- module, to its the double dual. First, we consider its matrix coefficient algebra : it is a Hopf algebra contained in $\mathcal{A}^*$, spanned by the matrix coefficients of ρ and ρ^* (see 1.2). That is, we take as primary object the "universal enveloping algebra"; this point of view goes back to [Ko].

In the QEA-case, it can be considered as the ring of rational functions on the quantum algebraic group corresponding to the representation. It coincides with some previous constructions (see the papers cited above and also [L4, section 7], [APW]), but the existence of the antipode follows at once from our approach.

Next, we consider real algebras which, after extending scalars to $\mathbb{C}$, become isomorphic to $\mathcal{A}$. Equivalently, semilinear involutions of $\mathcal{A}$, preserving the multiplication. It turns out that the so found forms are *not* Hopf algebras; but they define a $*$-algebra structure on $\mathcal{A}^*$ and (provided that the space of the representation has a sesquilinear form invariant by the involution, cf. 1.3) a $*$-Hopf algebra structure in the coefficient algebra. It will be very interesting to understand what happens if we consider field extensions of degree higher than 2.

0.4. — The following step is to consider real forms of a QEA; remembering the well-known construction of real forms of simple Lie algebras, it is clear how to produce examples of such involutions. (It is also possible to present some of these forms, the "inner" ones, by generators and relations, see the Appendix.) This coincides with some of the proposed "quantum versions" of $\mathrm{su}(1,1)$.

0.5. — Next we remark that the realization in [Re] is also available for irreducible representations with minuscule highest weight, see 1.5; and they can be endowed with an inner product, invariant for the "Cartan involution", or equivalently, for the compact form of the QEA. (There is a small difference with [R2], where a different involution was considered.) In this way, we obtain twisted E_6 and E_7, as well as the classical twisted groups first constructed in [W3], [R2].

0.6. — The paper is organized as follows : in the first chapter, we present the construction outlined in 0.3 in a general form, with the hope that it will of some help to find more examples of compact matrix pseudogroups (see [AE]). This owes a great deal to [W3] and many proofs are inspired in *loc cit.* In the second chapter, we recall the definition of QEA and of its highest weight modules; we state our construction of real forms of QEA and prove that they give raise to $*$-Hopf algebras; finally, we obtain the minuscule compact matrix pseudogroups. In the Appendix, we give a presentation by generators and relations of some forms of QEA (over an arbitrary field of char 0) which splits after tensoring with a quadratic extension. It should be noted that this presentation depends on the choice of a maximal anisotropic Cartan subalgebra. (Namely, the span of $Z_1, \dots, Z_n$, cf. A.1.)

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1. Generalities

1.1 Dual Hopf algebras. — Let k be a field of characteristic 0, $\bar{k}$ its algebraic closure. A bialgebra over k is a data $(\mathcal{A}, m, 1, \Delta, \epsilon)$, where $(\mathcal{A}, m, 1)$ is an associative algebra with unit 1 over k, $(\mathcal{A}, \Delta, \epsilon)$ is a coassociative coalgebra with counit $\epsilon \in \mathcal{A}^*$ (i.e., it satisfies the dual axioms of an associative algebra with unit, cf. [D2]), and $\Delta : \mathcal{A} \to \mathcal{A} \otimes \mathcal{A}$

is an algebra morphism. In particular, $\mathcal{A}^*$ is an associative algebra with unit ϵ; but needs not to be a bialgebra, since in general the inclusion $\mathcal{A}^* \otimes \mathcal{A}^* \subset (\mathcal{A} \otimes \mathcal{A})^*$ is strict. A Hopf algebra over k is a data $(\mathcal{A}, m, 1, \Delta, \epsilon, S)$, where $(\mathcal{A}, m, 1, \Delta, \epsilon)$ is a bialgebra and $S : \mathcal{A} \to \mathcal{A}$ is an antimultiplicative isomorphism satisfying the axiom dual to the inversion in a group. S is called the *antipode.*

A representation of a Hopf algebra is a representation of its underlying algebra structure; equivalently, a module over a Hopf algebra is a module over the underlying algebra . For coalgebras, there is the dual notion of corepresentation : it is a pair (V, ρ), where V is a finite dimensional k-vector space and $\rho \in \operatorname{End}(V) \otimes \mathcal{A}$ satisfies $(\operatorname{id} \otimes \Delta)(\rho) = \rho \ominus \rho$. (We adopt the usual notation :

$$\ominus : \big(\operatorname{End}(V) \otimes \mathcal{A}\big) \otimes \big(\operatorname{End}(V) \otimes \mathcal{A}\big) \longrightarrow \operatorname{End}(V) \otimes \mathcal{A} \otimes \mathcal{A}$$

is given by

$$\ominus(\phi \otimes x \otimes \psi \otimes y) = \phi\psi \otimes x \otimes y).$$

There is also the notion of (right) comodule over a coalgebra $(\mathcal{A}, \Delta, \epsilon)$. It is a pair $(\mathcal{V}, \mu)$, where $\mathcal{V}$ is a k- vector space (allowed to be infinite dimensional) and μ is a linear application $\mathcal{V} \to \mathcal{V} \otimes \mathcal{A}$ satisfying

$$(\operatorname{id} \otimes \Delta)\mu = (\mu \otimes \operatorname{id})\mu \quad \text{and} \quad (\operatorname{id} \otimes \epsilon)\mu = \operatorname{id}.$$

The left comodules are defined in a similar way.

Let (V_1, ρ_1), (V_2, ρ_2) be two representations of the Hopf algebra $\mathcal{A}$. Then $(V_1 \otimes V_2, (\rho_1 \otimes \rho_2) \circ \Delta)$ and $(V_1^*, (\rho_1 \circ S)^t)$ are representations of $\mathcal{A}$. We shall denote $\rho^d = (\rho_1 \circ S)^t$ and $\rho_1 \otimes \rho_2$ instead of $(\rho_1 \otimes \rho_2) \circ \Delta$. The trivial representation of $\mathcal{A}$ is ϵ; we will say that $v \in V_1$ is invariant if $\rho_1(x)v = \epsilon(x)v$ for all $x \in \mathcal{A}$. Notice that $\operatorname{Hom}_{k-\mathrm{alg}}(\mathcal{A}, k)$ is a multiplicatively closed subset of $\mathcal{A}^*$. A representation of $\mathcal{A}$ on V is called *irreducible* if the only $\mathcal{A}$-endomorphisms of V are the multiples of the identity.

Let $\widehat{\mathcal{A}}$ be the set of isomorphy classes of irreducible representations of $\mathcal{A}$. Sometimes, we will write π for a representant of $\pi \in \widehat{\mathcal{A}}$. For any $\mathcal{A}$-module W and any $\pi \in \widehat{\mathcal{A}}$, let W_π the isotypic component of type π. Let us denote

$$\widehat{\mathcal{A}}[W] = \{\pi \in \widehat{\mathcal{A}} : W_\pi \neq 0\}.$$

Let V be a finite dimensional $\mathcal{A}$-module, $\{v_k\}$ a basis of V and $\{\mu_k\}$ its corresponding dual basis. Consider the module structure on $\operatorname{End}(V) \simeq$

$V^* \otimes V$ given by $(\rho^d \otimes \rho) \circ \Delta$. Let $\bar{t}_V : V^* \otimes V \to k$, $t_V : k \to V^* \otimes V$ given by

$$\bar{t}_V(\mu \otimes v) = \langle \mu, v\rangle, \quad t_V(1) = \sum_i \mu_i \otimes v_i.$$

We claim that $\bar{t}_V$, t_V are $\mathcal{A}$-morphisms, considering in k the trivial representation. This follows from the properties of the antipode and is trivial for $\bar{t}_V$. For t_V, if $\Delta(x) = \sum_j x'_i \otimes x''_j$, we have

$$\begin{aligned} x\sum_i \mu_i \otimes v_i &= \sum_{i,j} x'_j\mu_i \otimes x''_j v_i = \sum_{i,j,\ell}\langle x'_j\mu_i, v_\ell\rangle \mu_\ell \otimes x''_j v_i \\ &= \sum_{i,j,\ell} \mu_\ell \otimes x''_j \langle \mu_i, S(x'_j)v_\ell\rangle v_i \\ &= \sum_{j,\ell} \mu_\ell \otimes x''_j S(x'_j) v_\ell = \epsilon(x) t_V(1). \end{aligned}$$

Another trivial and well-known remark is that the canonical linear isomorphism $V \otimes k \simeq V$ intertwines the corresponding representations.

Let $\mathcal{A}$ be an associative k-algebra with unit. Let $\mathcal{A}^{\text{opp}}$ be the opposite algebra. The left (resp. right) regular representation is the morphism $L : \mathcal{A} \to \operatorname{End}(\mathcal{A})$ (resp. $R : \mathcal{A}^{\text{opp}} \to \operatorname{End}(\mathcal{A})$) given by $L_x(y) = xy$ (resp. $R_x(y) = yx$). As L_x and R_y commute for every x, y, there is a representation $L \otimes R : \mathcal{A} \otimes \mathcal{A}^{\text{opp}} \to \operatorname{End}(\mathcal{A})$. Now assume that in addition $\mathcal{A}$ is a Hopf algebra. The adjoint representation of $\mathcal{A}$ is $\operatorname{ad} : \mathcal{A} \to \operatorname{End}(\mathcal{A})$

$$\operatorname{ad} = (L \otimes R) \circ (\operatorname{id} \otimes S) \circ \Delta.$$

Let $(\mathcal{A}, m_{\mathcal{A}}, 1_{\mathcal{A}}, \Delta_{\mathcal{A}}, \epsilon_{\mathcal{A}}, S_{\mathcal{A}})$, $(\mathcal{B}, m_{\mathcal{B}}, 1_{\mathcal{B}}, \Delta_{\mathcal{B}}, \epsilon_{\mathcal{B}}, S_{\mathcal{B}})$ be two Hopf algebras over k. A pairing between them is a bilinear form $\langle\, ,\, \rangle : \mathcal{A} \times \mathcal{B} \longrightarrow k$ such that for any $u, v \in \mathcal{A}$, $\alpha, \beta \in \mathcal{B}$

$$\begin{gathered} \big\langle u, m_{\mathcal{B}}(\alpha \otimes \beta)\big\rangle = \big\langle \Delta_{\mathcal{A}}(u), \alpha \otimes \beta\big\rangle, \quad \big\langle u \otimes v, \Delta_{\mathcal{B}}(\alpha)\big\rangle = \big\langle m_{\mathcal{A}}(u \otimes v), \alpha\big\rangle, \\ \big\langle 1_{\mathcal{A}}, \alpha\big\rangle = \epsilon_{\mathcal{B}}(\alpha), \quad \big\langle u, 1_{\mathcal{B}}\big\rangle = \epsilon_{\mathcal{A}}(u), \quad \big\langle u, S_{\mathcal{B}}(\alpha)\big\rangle = \big\langle S_{\mathcal{A}}(u), \alpha\big\rangle. \end{gathered}$$

We will say that $\mathcal{B}$ is *dual* to $\mathcal{A}$ if $\langle\, ,\, \rangle$ induces a monomorphism $\mathcal{B} \to \mathcal{A}^*$. It is clear what "$\mathcal{A}$ and $\mathcal{B}$ are dual" will mean.

It is well-known that for a given Hopf algebra the uniqueness of its dual Hopf algebra in general fails. For example, different dual Hopf algebras of the universal enveloping algebra of a finite dimensional real Lie algebra g appear in two different ways. First, we can consider a connected Lie group G whose Lie algebra is isomorphic to g; various kind of function algebras on G (rational, analytic, ...) give examples of different Hopf algebras dual to $U(g)$. But, secondly, there can exist many different connected Lie groups with Lie algebra isomorphic to g.

Let $\mathcal{B}$ be dual to $\mathcal{A}$. We shall identify $\mathcal{B}$ with its image in $\mathcal{A}^*$ under the map defined by the pairing. An $\mathcal{A}$-module-$\mathcal{B}$-comodule (an $\mathcal{A}$-$\mathcal{B}$-module, for shortness) is a pair (V, ρ), where V is a k-vector space and

$$\rho \in \operatorname{Hom}(\mathcal{A}, \operatorname{End} V) \simeq \mathcal{A}^* \otimes \operatorname{End} V \simeq \operatorname{End} V \otimes \mathcal{A}^*$$

satisfies : $\rho \in \operatorname{End} V \otimes \mathcal{B}$ and defines an $\mathcal{A}$-module and $\mathcal{B}$-comodule structures in V. It is not true that an $\mathcal{A}$-module defines a $\mathcal{B}$-comodule, there is no 2-dimensional representation of PSL(2).

1.2 The coefficient algebra. — Let $\mathcal{T}(V \otimes V^*)$ be the tensor algebra of the dual of the space of endomorphisms of a finite dimensional vector space V. Let $\{v_k\}$ be a basis of V and $\{\mu_k\}$ its corresponding dual basis. $\mathcal{T}(V \otimes V^*)$ carries a natural bialgebra structure, defined by

$$\Delta(v \otimes \mu) = \sum (v_k \otimes \mu) \otimes (v \otimes \mu_k), \qquad \epsilon(v \otimes \mu) = \langle \mu, v \rangle.$$

Let $\mathcal{A}$ be a Hopf algebra and $\rho : \mathcal{A} \to End(V)$ be a finite dimensional representation. There is an application $\phi_\rho = \phi : \mathcal{T}(V \otimes V^*) \to \mathcal{A}^*$ induced by $\langle \phi(v \otimes \mu), x \rangle = \langle \mu, x \cdot v \rangle$; $\phi(v \otimes \mu)$ are usually called the *matrix coefficients*. The image of ϕ is a bialgebra dual to $\mathcal{A}$ and ϕ is a morphism of bialgebras, because

$$\big\langle \phi(v \otimes \mu), xy \big\rangle = \big\langle (\phi \otimes \phi)\Delta(v \otimes \mu), x \otimes y \big\rangle.$$

(V, ρ) is a $\phi(\mathcal{T}(V \otimes V^*))$-comodule : we need to check that

$$\rho \in \phi(\mathcal{T}(V \otimes V^*)) \otimes \operatorname{End}(V).$$

But we have $\rho = \sum_{i,j} \phi(v_i \otimes \mu_j) \otimes (\mu_i \otimes v_j)$.

Let us consider $V \otimes V^*$ (resp. $\mathcal{A}^*$) with the representation

$$\rho_1(x)(v \otimes \mu) = \rho(x) v \otimes \mu \quad (\text{resp. } R^t).$$

Then $\phi : V \otimes V^* \to \mathcal{A}^*$ is a morphism of $\mathcal{A}$-modules. If we look at $\mathcal{T}(V \otimes V^*)$ as a graded $\mathcal{A}$-module, where the representation in $(V \otimes V^*)^{\otimes m}$ is $\rho_m = (\rho_i \otimes \rho_{m-i}) \circ \Delta$ (the coassociativity guarantees the arbitrariness of the i, $1 \leq i \leq m-1$, chosen), then $\phi^{\otimes m} : (V \otimes V^*)^{\otimes m} \to \mathcal{A}^*$ is still a morphism of $\mathcal{A}$-modules. For this, as $\phi^{\otimes m} = \Delta^t \circ (\phi_i \otimes \phi_{m-i})$, it suffices to prove $\Delta^t \circ ((R^t \otimes R^t) \circ \Delta(x))) = R^t(x) \circ \Delta^t$; i.e. that Δ^t is a morphism of $\mathcal{A}$-modules. But on one hand, we have

$$\begin{aligned}\langle \Delta^t \circ ((R^t \otimes R^t) \circ \Delta(x)))\alpha \otimes \beta, y\rangle &= \langle(((R^t \otimes R^t) \circ \Delta(x)))\alpha \otimes \beta, \Delta(y)\rangle \\ &= \langle \alpha \otimes \beta, \Delta(y)\Delta(x)\rangle,\end{aligned}$$

and on the other

$$\langle R^t(x) \circ \Delta^t \alpha \otimes \beta, y\rangle = \langle \alpha \otimes \beta, \Delta(yx)\rangle.$$

Let $\pi \in \widehat{\mathcal{A}}$ be finite dimensional. It is clear that $\phi_\pi(\pi \otimes \pi^*) \subseteq \mathcal{A}^*_\pi$ and the other inclusion also holds : let $f : \pi \to \mathcal{A}^*$ be a morphism of $\mathcal{A}$-modules, i.e. $\langle f(xv), y\rangle = \langle f(v), yx\rangle$ for any x, y in $\mathcal{A}$. If $\mu \in \pi^*$ is defined by $\langle \mu, v\rangle = \langle f(v), 1\rangle$, then the image of f and $\phi_\pi(\pi \otimes \mu)$ coincide.

The transposition $\tau : V \otimes V^* \to V^* \otimes V^{**} \simeq V^* \otimes V$ induces an isomorphism of algebras $\mathcal{T}(V \otimes V^*) \to \mathcal{T}(V^* \otimes V)$, still denoted τ. Clearly $\phi_{\rho^d}\tau = S^t\phi_\rho$; thus we have a morphism

$$S^t : \phi\big(\mathcal{T}(V \otimes V^*)\big) \to \phi\big(\mathcal{T}(V \otimes V^*)\big).$$

As $\mathcal{T}((V \otimes V^*) \oplus (V^* \otimes V))$ is the coproduct of $\mathcal{T}(V \otimes V^*)$ and $\mathcal{T}(V^* \otimes V)$ in the category of associative k-algebras, it inherits a bialgebra structure; we have a morphism of bialgebras (denoted $\phi_\rho \oplus \phi_{\rho^d}$) from $\mathcal{T}((V \otimes V^*) \oplus (V^* \otimes V))$ to $\mathcal{A}^*$.

PROPOSITION. — *Let us assume that*

(1) *there exists an isomorphism of vector spaces $M : V \to V$ such that $M(av) = S^2(a)M(v)$ for all $a \in \mathcal{A}$, $v \in V$.*

Then $\phi_\rho \oplus \phi_{\rho^d}(\mathcal{T}((V \otimes V^*) \oplus (V \otimes V^*)))$ is a Hopf algebra dual to $\mathcal{A}$ which will be denoted Coeff(ρ). *Moreover, (V, ρ) is an $\mathcal{A}$ –* Coeff(ρ)*-module.*

Proof. — Let $(V \otimes V^*) \oplus (V^* \otimes V) \to (V \otimes V^*) \oplus (V^* \otimes V)$ be the morphism

$$(v \otimes \mu, \eta \otimes w) \mapsto (M^{-1}w \otimes M^t(\eta), \mu \otimes v);$$

The corresponding algebra automorphism of $\mathcal{T}((V \otimes V^*) \oplus (V \otimes V^*))$ makes commutative the following diagram :

$$\begin{array}{ccc}\mathcal{T}\big((V \otimes V^*) \oplus (V \otimes V^*)\big) & \longrightarrow & \mathcal{T}\big((V \otimes V^*) \oplus (V \otimes V^*)\big) \\ {\scriptstyle \phi_\rho \oplus \phi_{\rho^d}}\Big\downarrow & & \Big\downarrow{\scriptstyle \phi_\rho \oplus \phi_{\rho^d}} \\ \mathcal{A}^* & \xrightarrow{\quad S^t \quad} & \mathcal{A}^*,\end{array}$$

as follows from

$$\big\langle M^t(\eta), xM^{-1}w\big\rangle = \big\langle M^t(\eta), M^{-1}(S^2(x)w)\big\rangle = \big\langle \eta, S^2(x)w\big\rangle.$$

Thus $S^t(\mathrm{Coeff}(\rho)) \subseteq \mathrm{Coeff}(\rho)$. $\square$

REMARK. — If the hypothesis of the Proposition is not fulfilled, the coefficient algebra can be constructed in the following way : let $\mathcal{V}_j = V \otimes V^*$, $\nu_j : \mathcal{V}_j \to \mathcal{V}_{j+1}$ the identity, $\phi_j : \mathcal{V}_j \to \mathcal{A}^*$ the application $\langle \phi_j(v \otimes \mu), x \rangle = \langle \mu, S^j(x) \cdot v \rangle$, for any $j \in \mathbb{Z}$. That is, $\phi_0 = \phi_\rho$, $\phi_1 = \phi_{\rho^d}$ (modulo the transposition), etc. By definition, the following diagram commutes :

$$\begin{array}{ccc} \mathcal{V}_j & \xrightarrow{\nu_j} & \mathcal{V}_{j+1} \\ \phi_j \Big\downarrow & & \Big\downarrow \phi_{j+1} \\ \mathcal{A}^* & \xrightarrow{S^t} & \mathcal{A}^*. \end{array}$$

Let $\Phi_\rho : \mathcal{T}(\oplus_{j \in \mathbb{Z}} \mathcal{V}_j) \to \mathcal{A}^*$, be defined by $\oplus \phi_j$, let $\nu = \oplus \nu_j$. Then $\Phi_\rho \circ \nu = S^t \circ \Phi_\rho$ and we can define a Hopf algebra structure in the image of Φ_ρ. (Different coalgebra structures are defined in $\mathcal{T}(\mathcal{V}_j)$, depending on the parity of j).

1.3 $*$-Hopf algebras. — Let us assume that in addition the field k is provided with an involution γ. A k $*$-algebra is a pair $(\mathcal{A}, *)$, where $\mathcal{A}$ is an associative algebra and $*$ is an application $\mathcal{A} \to \mathcal{A}$, $v \mapsto v^*$ satisfying

$$(v^*)^* = v, \quad (v+w)^* = v^* + w^*, \quad (vw)^* = w^* v^*, \quad (\lambda v)^* = \gamma(\lambda) v^*.$$

A $*$-Hopf algebra is a Hopf algebra provided with a star operation for the algebra structure, such that

$$\Delta(v^*) = \Delta(v)^*, \quad S(S(v^*)^*) = v.$$

Let k^γ be the field of γ-fixed points. Let $\mathcal{A}$ be a Hopf algebra and $\mathcal{A}_0$ a k^γ-form of the algebra structure of $\mathcal{A}$. That is, $\mathcal{A}_0 \otimes_{k^\gamma} k \simeq \mathcal{A}$ is an isomorphism of algebras. Let $T : \mathcal{A} \to \mathcal{A}$,

$$T(a \otimes \lambda) = a \otimes \gamma(\lambda)$$

($a \in \mathcal{A}_0$, $\lambda \in k$). Let $\alpha \mapsto \alpha^*$ be the automorphism of $\mathcal{A}^*$ given by

$$\langle \alpha^*, x \rangle = \gamma\big(\langle \alpha, T(x) \rangle\big).$$

PROPOSITION. — *Let us assume that*

$$\Delta \circ T = (T \otimes T) \circ \tau \circ \Delta. \qquad (2)$$

Then $(\mathcal{A}^*, *)$ is a $*$-algebra. Moreover, let $\mathcal{B} \subset \mathcal{A}^*$ be a $*$-stable Hopf algebra dual to $\mathcal{A}$ and suppose that

$$S_{\mathcal{A}} T S_{\mathcal{A}} = T. \tag{3}$$

Then $\mathcal{B}$ is a $*$-Hopf algebra.

Proof. — Both assertions are proved in a straightforward way. For example,

$$\begin{aligned}\big\langle S_{\mathcal{B}}(S_{\mathcal{B}}(\alpha^*)^*), v\big\rangle &= \big\langle (S_{\mathcal{B}}(\alpha^*)^*), S_{\mathcal{A}}(v)\big\rangle = \gamma\big(\big\langle S_{\mathcal{B}}(\alpha^*), T S_{\mathcal{A}}(v)\big\rangle\big)\\ &= \big\langle \alpha, T S_{\mathcal{A}} T S_{\mathcal{A}}(v)\big\rangle = \langle \alpha, v\rangle\end{aligned}$$

since T^2 is the identity. Let us observe that ϵ is a $*$-morphism. Let $x \in \mathcal{A}$. Then

$$\begin{aligned}\epsilon^*(x) &= \gamma\big(\epsilon(Tx)\big) = \gamma\big(m(\mathrm{id}\otimes S)\Delta T(x)\big)\\ &= \gamma\big(m(T\otimes T)(\mathrm{id}\otimes S^{-1})\tau\Delta(x)\big)\\ &= \gamma\big(Tm(S^{-1}\otimes S^{-1})(S\otimes \mathrm{id})\tau\Delta(x)\big)\\ &= \gamma\big(TS^{-1}m(S\otimes \mathrm{id})\Delta(x)\big) = \gamma\big(TS^{-1}\epsilon(x)\big) = \gamma\big(\epsilon(x)\big). \quad \square\end{aligned}$$

1.4 Compact matrix pseudogroups. — Let k, k^γ, etc. as in 1.3. A $*$-matrix Hopf algebra is a pair $(\mathcal{B}, u)$, where $\mathcal{B}$ is a $*$-Hopf algebra and u is a matrix with coefficients in $\mathcal{B}$ such that :

(4) the entries of u and $(u)^*$ generate $\mathcal{B}$;

(5) $\Delta_{\mathcal{B}}(u_{j\ell}) = \sum_i u_{ji}\, u_{i\ell}$;

(6) $\sum_i S_{\mathcal{B}}(u_{ji})u_{i\ell} = \delta_{j\ell}\, 1_{\mathcal{B}}, \quad \sum_i u_{ji} S_{\mathcal{B}}(u_{i\ell}) = \delta_{j\ell}\, 1_{\mathcal{B}}$.

If $J : V \to W$ is an antilinear morphism of k-vector spaces (i.e. $J(tv) = \gamma(t)J(v)$), then $J^* : W^* \to V^*$ denotes the antilinear morphism given by $\langle J^*(\mu), v\rangle = \gamma(\langle \mu, J(v)\rangle)$.

LEMMA. — *Let $\rho : \mathcal{A} \to \mathrm{End}(V)$ be a finite dimensional representation of a Hopf algebra $\mathcal{A}$ satisfying* (1); *let $T : \mathcal{A} \to \mathcal{A}$ be an antilinear involution satisfying* (2), (3). *Let us assume that* :

$$\begin{cases}\textit{there exists an antilinear isomorphism } J : V \to V^* \\ \textit{such that } J(xv) = T(x)J(v),\ x \in \mathcal{A},\ v \in V.\end{cases} \tag{7}$$

Let $(u_{ij}) = (\phi_\rho(v_j \otimes \mu_i)) \in \mathrm{Coeff}(\rho)^{\dim V \times \dim V}$. Then $(\mathrm{Coeff}(\rho), u)$ is a $*$-matrix Hopf algebra.

Proof. — (5) holds by definition of Coeff(ρ) and (6) by the axioms of the antipode. Coeff(ρ) is generated by u_{ij} and $S(u_{ij})$. Thanks to PROPOSITION 1.3, it suffices to prove

$$\phi_\rho(v \otimes \mu)^* = \phi_{\rho^d}(J(v) \otimes (J^{-1})^*(\mu)).$$

But

$$\begin{aligned}\big\langle \phi_{\rho^d}(J(v) \otimes \big(J^{-1})^*(\mu)\big), x\big\rangle &= \big\langle (J^{-1})^*(\mu), xJ(v)\big\rangle \\ &= \gamma\big(\langle \mu, T(x)v\rangle\big) = \big\langle \phi_\rho(v \otimes \mu)^*, x\big\rangle. \quad \square\end{aligned}$$

Let us remark that $\alpha \in \text{Coeff}(\rho)_\pi$ implies $\alpha^* \in \text{Coeff}(\rho)_{\pi^d}$, as follows from the proof of the Proposition.

(7) is equivalent to the existence of a sesquilinear form $(\,|\,) : V \times V \to k$ satisfying $(xv \mid w) = (v \mid TS^{-1}(x)w)$. The equivalence is given by the formula $(v \mid w) = J(w)(v)$. The sesquilinear form $(\,|\,) : V^* \times V^* \to k$ given by $(\mu \mid \eta) = (MJ^{-1}(\eta) \mid J^{-1}(\mu))$ also satisfies that recipe.

We want to extend the sesquilinear form to Coeff(ρ). First, we define a sesquilinear form on $V \otimes V^*$ (resp. on $V^* \otimes V$) by

$$\big(v_1 \otimes \mu_1 \mid v_2 \otimes \mu_2\big) = \big(v_1 \mid v_2\big)\big(J^{-1}(\mu_1) \mid J^{-1}(\mu_2)\big)$$

(resp. by

$$(\eta_1 \otimes w_1 \mid \eta_2 \otimes w_2) = \big(MJ^{-1}\eta_1 \mid J^{-1}\eta_2\big)(w_1 \mid w_2)\big).$$

Let us consider the representation of $\mathcal{A}$ in the orthogonal direct sum $V \otimes V^* \oplus V^* \otimes V$ given by

$$x(v \otimes \mu, \eta \otimes w) = (xv \otimes \mu, x\eta \otimes w).$$

Then for any α, $\beta \in V \otimes V^* \oplus V^* \otimes V$ we have

$$\big(x\alpha \mid \beta\big) = \big(\alpha \mid TS^{-1}(x)\beta\big). \tag{8}$$

Extend the sesquilinear form to $\mathcal{T}(V \otimes V^* \oplus V^* \otimes V)$ in the canonical way. Then (8) holds for any α, $\beta \in \mathcal{T}(V \otimes V^* \oplus V^* \otimes V)$. Indeed, it is easy to see that if V_i $(i = 1, 2)$ are $\mathcal{A}$-modules provided with sesquilinear forms satisfying (8), then the sesquilinear form on $V_1 \otimes V_2$ given by $(v_1 \otimes v_2 \mid w_1 \otimes w_2) = (v_1 \mid w_1)(v_2 \mid w_2)$ still satisfies (8). (To see this, it is necessary to use (2) and that S also satisfies (2).)

For the rest of the section, we shall assume that $k = \mathbb{C}$, $k^\gamma = \mathbb{R}$. A compact matrix pseudogroup [W2] is a $*$-matrix Hopf algebra $\mathcal{A}$ endowed with a norm $\| \ \|$ such that $\|xx^*\| = \|x\| \cdot \|x^*\|$. (This is equivalent to the definition in [W2], as remarked in the arguments after [W3, Prop. 3.5].)

We need to "normalize" M (cf. (1)). Let us assume that $J = J^*$; recall that then for any $\xi : V^* \to V^*$, $(J^{-1})^t \xi J^t = J^{-1}\xi J$. On the other hand, $J^{-1}M^tJ$ is a linear endomorphism of V satisfying $J^{-1}M^tJ(xv) = S^2(x)J^{-1}M^tJ(v)$, thanks to (3). If in addition, ρ is irreducible, there exists $c \in \mathbb{C}^\times$ such that $J^{-1}M^tJ = cM$. Moreover

$$M = \bar{c}(J^{-1})^t M^t J^t = \bar{c}J^{-1}M^tJ = \bar{c}cM,$$

i.e., $|c| = 1$. Hence we can choose $\lambda \in \mathbb{C}^\times$ such that $J^{-1}\lambda M^tJ = \lambda M$. In what follows, we will replace M by λM, in order to have

$$J^{-1}M^tJ = M.$$

PROPOSITION. — *Let $\rho : \mathcal{A} \to \mathrm{End}(V)$ be an irreducible finite dimensional representation of a Hopf algebra $\mathcal{A}$ satisfying* (1), *$T : \mathcal{A} \to \mathcal{A}$ an antilinear involution satisfying* (2), (3), *$J : V \to V^*$ an antilinear isomorphism satisfying* (7). *Let us assume*

(9) *$J = J^*$ and $(\mid)$ is a positive defined hermitian form.*

(10) *M is positive defined.*

Then $\mathrm{Coeff}(\rho)$ *is a compact matrix pseudogroup.*

Proof. — (9) and (10) implies that the extension of $(\mid)$ to

$$\mathcal{T}(V \otimes V^* \oplus V^* \otimes V)$$

constructed below is positive defined. As in the classical case, it is immediate that $\mathcal{T}(V \otimes V^* \oplus V^* \otimes V)$ decomposes in orthogonal direct sum of irreducible submodules. Hence $\mathrm{Coeff}(\rho)$ inherits an inner product satisfying (8) (perhaps not in a unique way).

Let $\psi : \mathrm{Coeff}(\rho) \to \mathbb{C}$ be defined by $\psi(\alpha) = (\alpha \mid \epsilon)$. (Recall that ϵ is the counit of $\mathcal{A}$ and the unit of $\mathrm{Coeff}(\rho)$.) We shall prove

$$\psi(\alpha\alpha^*) > 0 \quad \text{for any} \quad \alpha \neq 0. \tag{11}$$

Let π, $\pi' \in \widehat{\mathcal{A}}[\mathrm{Coeff}(\rho)]$, $\alpha \in \pi \otimes \pi^*$, $\alpha' \in \pi' \otimes \pi'^*$. Let $\pi \otimes \pi' = \bigoplus_j \pi_j$ be the decomposition as a direct sum of irreducible components. In other words, if $x \in \mathcal{A}$, then the image of x in $(\pi \otimes \pi')^* \otimes (\pi \otimes \pi')$ is $\sum x_j$, where x_j is the image of x in $\pi_j^* \otimes \pi_j$. It follows that

$$\phi_\pi(\alpha)\phi_{\pi'}(\alpha') = \sum_j \phi_{\pi_j}\big(\Upsilon(\alpha \otimes \alpha')_j\big).$$

Here, $\Upsilon : \pi \otimes \pi^* \otimes \pi' \otimes \pi'^* \simeq (\operatorname{End} \pi)^* \otimes (\operatorname{End} \pi')^* \to (\operatorname{End}(\pi \otimes \pi'))^* \simeq \pi \otimes \pi' \otimes \pi^* \otimes \pi'^*$ is the canonical application, i.e.

$$v \otimes \mu \otimes v' \otimes \mu' \stackrel{\Upsilon}{\longmapsto} v \otimes v' \otimes \mu \otimes \mu';$$

and, if i_j (resp. p_j) is the inclusion of π_j on $\pi \otimes \pi'$ (resp. the orthogonal projection from $\pi \otimes \pi'$ onto π_j), then $\langle T_j, f\rangle = \langle T, i_j f p_j\rangle$ ($T \in \operatorname{End}\big(\pi \otimes \pi'\big)^*$, $f \in \operatorname{End}(\pi_j)$).

By construction, this decomposition is orthogonal. It follows from the Schur Lemma (see below) that $(\phi_\pi(\alpha)\phi_{\pi'}(\alpha') \mid \epsilon) = 0$ if π^* and π' are not isomorphic. Hence we are reduced to prove (11) for $\alpha \in \operatorname{Coeff}(\rho)_\pi$ or even for $\alpha = \phi_\rho(v \otimes \mu) \in \operatorname{Coeff}(\rho)_\rho$. As $i_\epsilon = t_\rho$, we have

$$\begin{aligned}(\alpha\alpha^* \mid \epsilon) &= \big\langle v \otimes J(v) \otimes \mu \otimes J^{-1}(\mu), \sum_{i,j} J(v_i) \otimes v_i \otimes v_j \otimes J(v_j)\big\rangle \\ &= (\mu \mid \mu)\,(v \mid v)\end{aligned}$$

and (11) follows. The proposition can be now proved in a standard way. □

LEMMA (SCHUR). — *Let V, $W \in \widehat{\mathcal{A}}[\operatorname{Coeff}(\rho)]$. Assume that* (9), (10) *hold.*

a) $\dim \operatorname{Hom}_{\mathcal{A}}(V, W)$ *is* 1 (*resp.* 0) *if V and W are* (*resp. are not*) *isomorphic.*

b) $\dim \operatorname{Hom}_{\mathcal{A}}(k_\epsilon, V \otimes W)$ *is* 1 (*resp.* 0) *if V^* and W are* (*resp. are not*) *isomorphic.*

Proof. — (See [W3, Prop. 2.5].)

(a) is immediate (we asume (9) and (10) to be sure that the irreducibility of V implies that it has no proper submodules).

(b) follows from (a) and the isomorphism

$$\operatorname{Hom}_{\mathcal{A}}(V, W) \simeq \operatorname{Hom}_{\mathcal{A}}(k_\epsilon, V^* \otimes W)$$

given by $G \mapsto (\operatorname{id} \otimes G) \circ t_{V^*}$, whose inverse is $F \mapsto (\bar{t}_V \otimes \operatorname{id}) \circ (\operatorname{id} \otimes F)$. (The verification is straightforward.) □

2. Examples

2.1 Quantized enveloping algebras. — Let q be an indeterminate. Given $N, M, d \in \mathbb{N}_0$, we denote (as usual)

$$[N]^!_d = \prod_{1\le h\le N} \frac{q^{dh}-q^{-dh}}{q^d-q^{-d}} \in k[q,q^{-1}],$$

$$\begin{bmatrix} M+N \\ N \end{bmatrix}_d = \frac{[M+N]^!_d}{[M]^!_d[N]^!_d} \in k[q,q^{-1}].$$

Let g be a simple finite dimensional split Lie algebra; let A be the corresponding Cartan matrix. There exists a diagonal matrix such that $D=(d_1,\dots,d_n)\in \mathbb{Z}^{n\times n}$, $DA = AD$ and $\det D \neq 0$.

It is well-known that the universal enveloping algebra of g is the associative k-algebra given by generators e_i, f_i, h_i and relations

$$h_ih_j = h_jh_i, \tag{12}$$

$$h_ie_j - e_jh_i = a_{ij}e_j, \quad h_if_j - f_jh_i = -a_{ij}f_j, \tag{13}$$

$$e_if_j - f_je_i = \delta_{ij}h_j, \tag{14}$$

and if $i \neq j$

$$\begin{cases} \displaystyle\sum_{h+\ell=1-a_{ij}} \binom{1-a_{ij}}{h}(-1)^h e_i^\ell e_j e_i^h = 0, \\ \displaystyle\sum_{h+\ell=1-a_{ij}} (-1)^h \binom{1-a_{ij}}{h} f_i^\ell f_j f_i^h = 0. \end{cases} \tag{15}$$

The quantized enveloping algebra $U_q(g) = U_{k\,q}(g)$ is defined as the associative $k(q)$-algebra given by generators E_i, F_i, K_i, K_i^{-1} and relations

$$K_iK_i^{-1} = K_i^{-1}K_i = 1, \quad K_iK_j = K_jK_i, \tag{16}$$

$$K_iE_j = q^{d_ia_{ij}}E_jK_i, \quad K_iF_j = q^{-d_ia_{ij}}F_jK_i, \tag{17}$$

$$E_iF_j - F_jE_i = \delta_{ij}\frac{K_i - K_i^{-1}}{q^{d_i}-q^{-d_i}}, \tag{18}$$

and if $i \neq j$

$$\sum_{h+\ell=1-a_{ij}} (-1)^h E_i^{(\ell)}E_jE_i^{(h)} = 0, \qquad \sum_{h+\ell=1-a_{ij}} (-1)^h F_i^{(\ell)}F_jF_i^{(h)} = 0. \tag{19}$$

Here $E_i^{(N)}$ denotes E_i^N divided by $[N]_{d_i}^!$ (idem for $F_i^{(N)}$). (We follow the presentation in [L2].)

Let us introduce the following notation, for any elements x, y in an associative $k(q)$-algebra :

$$\mathcal{B}_d^N(x,y) = \sum_{h=0}^{N} (-1)^h \begin{bmatrix} N \\ h \end{bmatrix}_d x^{N-h} y x^h.$$

Thus (19) can be rewritten as

$$\mathcal{B}_{d_i}^{1-a_{ij}}(E_i, E_j) = 0, \quad \mathcal{B}_{d_i}^{1-a_{ij}}(F_i, F_j) = 0. \tag{20}$$

$U(A)$ is a Hopf algebra with comultiplication Δ, antipode S and counit ϵ defined by

$$\begin{cases} \Delta(E_i) = E_i \otimes 1 + K_i \otimes E_i, \\ \Delta(F_i) = F_i \otimes K_i^{-1} + 1 \otimes F_i, \\ \Delta K_i = K_i \otimes K_i, \end{cases} \tag{21}$$

$$S(E_i) = -K_i^{-1} E_i, \quad S(F_i) = -F_i K_i, \quad S(K_i) = K_i^{-1}, \tag{22}$$

$$\epsilon(E_i) = 0, \quad \epsilon(F_i) = 0, \quad \epsilon(K_i) = 1. \tag{23}$$

Note that

$$S^{-1}(E_i) = -E_i K_i^{-1}, \quad S^{-1}(F_i) = -K_i F_i, \quad S^{-1}(K_i) = K_i^{-1}. \tag{24}$$

It follows from [L3], [L4] that we can specialize q to any non-zero element of k.

2.2 Minuscule highest weight modules. — The notion of "integrable" representation was introduced in [Sk2], [J] and in full generality in [L1], [R1].

Let us fix a quantized enveloping algebra $U = U_k(A)$. Let U^0 be the subalgebra of U generated by $K_i^{\pm 1}$ for all i. U^0 is a commutative algebra, isomorphic to the algebra of Laurent polynomials in K_i. $\mathrm{Hom}_{k(q)-\mathrm{alg}}(U_0, k(q))$ is isomorphic to $(k(q)^\times)^n$ via

$$\phi \longmapsto \big(\phi(K_1), \dots, \phi(K_n)\big).$$

Let P (resp. $Q^\vee$) be the free abelian group with basis ω_i (resp. $\alpha_i^\vee$), $1 \le i \le n$. Let $\langle\ ,\ \rangle : P \times Q^\vee \to \mathbb{Z}$ be the bilinear pairing defined by $\langle \omega_i, \alpha_j^\vee \rangle = \delta_{ij}$. Let $\alpha_j \in P$ be defined by $\langle \alpha_j, \alpha_i^\vee \rangle = a_{ij}$ and let Q (resp. Q^+) be the subgroup (resp. the subsemigroup) of P generated by $\alpha_1, \dots, \alpha_n$. Let $(\mid) : Q^\vee \times Q^\vee \to \mathbb{Z}$ be the symmetric bilinear non-degenerate form defined by $(\alpha_i^\vee \mid \alpha_j^\vee) = d_j^{-1} a_{ij} = d_i^{-1} a_{ji}$. $(\mid)$ defines a morphism $\nu : Q^\vee \to P$ by $\langle \nu(\alpha^\vee), \beta^\vee \rangle = (\alpha^\vee \mid \beta^\vee)$; we have $Q \subset \nu(Q^\vee)$ because $\alpha_i = d_i \nu(\alpha_i^\vee)$. In particular, we obtain a symmetric bilinear non-degenerate form, still denoted $(\mid)$, on Q; we have $(\alpha_i \mid \alpha_j) = d_i a_{ij} = d_j a_{ji}$. We even have $(\mid) : P \times Q \to \mathbb{Z}$.

We can identify U^0 with the group algebra (over $k(q)$) of Q via $K_i \to \alpha_i$; the image of $\alpha = \sum c_i \alpha_i$ will be denoted K_α; i.e. $K_\alpha = \prod K_i^{c_i}$. Every $\omega \in P$ defines a group homorphism $Q^\vee \to k(q)^\times$ by $\alpha^\vee \mapsto q^{\langle \omega, \alpha^\vee \rangle}$; and hence a homomorphism $Q \to k(q)^\times$ by $\alpha \mapsto q^{\langle \omega, \nu^{-1}(\alpha) \rangle}$. We shall denote by e^ω the corresponding $k(q)$-algebra homomorphism $U^0 \to k(q)$. For example, $e^{\alpha_j}(K_i) = q^{\langle \alpha_j, d_i \alpha_i^\vee \rangle} = q^{d_i a_{ij}}$. We have a monomorphism of groups $P \to \operatorname{Hom}_{k(q)-\mathrm{alg}}(U_0, k(q))$; we shall identify P, Q, Q^+ with their images in what follows.

Let M be a $U(A)$-module. For $w \in (k(q)^\times)^n$, let

$$M_w = \{ x \in M : K_i x = w_i x \ \forall i \}.$$

There is a partial order in $(k(q)^\times)^n$ given by $w \le w'$ if and only if $w^{-1} w' \in Q^+$. M is called a highest weight module if it is generated by some $x \in M_w$ ($w \in k(q)$ is then called the highest weight of M) which is annihilated by E_i for every i. It is known ([L1, Prop. 2.6]) that for any $w \in (k(q)^*)^n$ there exists a simple highest weight module of highest weight w, unique up to isomorphism, denoted $L(w)$.

On the other hand, M is integrable if $M = \sum_w M_w$ and E_i, F_i are locally nilpotent endomorphisms of M. Then (cf. [L1, Prop. 3.2]) $L(w)$ is integrable if $w \in P$. This exhausts the integrable ireducible highest weight modules, modulo tensoring with one-dimensional representations.

Now we shall present a realization of minuscule highest weight modules. It is inspired in the formulas in [Re].

Let us recall [B] that a dominant weight $w \in P - 0$ is called *minuscule* if

$$\langle w, \alpha^\vee \rangle = 0,\ 1 \text{ or } -1, \quad \text{for any root } \alpha^\vee.$$

The fact about minuscule weights that will be useful for us is the following: if τ and $\tau + \alpha_i$ are weights of a minuscule highest weight module $\mathcal{V}$, then $\langle \tau, \alpha_i^\vee \rangle = -1$. In particular, $\tau - \alpha_i$ and $\tau + 2\alpha_i$ are not weights of $\mathcal{V}$.

The minuscule weights form a system of representants of $P/Q - 0$. Hence, there are minuscule weights for irreducible root systems of types A, B, C, D, E_6, E_7, cf. [B, Tables]. Let $\mathcal{V}$ be a minuscule highest weight module, $\Pi(\mathcal{V})$ the set of its weights; it is the orbit under the Weyl group of the highest weight.

PROPOSITION. — *Let $\mathcal{V}$ be a minuscule highest weight module of a complex Lie algebra of type X, corresponding to a representation ρ. Let $q \in \mathbb{C} - 0$, and let $t \in \mathbb{C}$ such that* $\exp(t) = q$. *The assignement*

$$E_i \mapsto \rho(e_i), \quad F_i \mapsto \rho(f_i), \quad K_i \mapsto \exp t\, d_i \rho(h_i).$$

gives rise to a $U_q(X)$-module structure on $\mathcal{V}$.

Proof. — We need to check relations (16),...,(19). Relation (16) is trivial. Relation (17) follows from (13) and the following well-known formula which holds for any pair of elements x, y in any associative algebra :

$$x^j y = \sum_{i=o}^{j} \binom{j}{i} (\operatorname{ad} x)^i y\, x^{j-i}.$$

For (18) and (19) we will use that the highest weight is minuscule. (18) means in our case

$$\rho(h_i) = \frac{\exp(t\, d_i h_i) - \exp(-t\, d_i h_i)}{\exp(t\, d_i) - \exp(-t\, d_i)}.$$

But $\rho(h_i)$ is diagonalizable, with eigenvalues $\langle w, \alpha_i^\vee \rangle$, $w \in \Pi(\mathcal{V})$, which are 0 or ± 1 by hypothesis. Thus we are reduced to prove

$$s = \frac{\exp(us) - \exp(-us)}{\exp(u) - \exp(-u)}, \quad \text{for any } u \in \mathbb{C}, \quad s = 0, 1\,;$$

and this is clear. Finally, let us check (19). It is identical to (15) if $a_{ij} = 0$; but if $a_{ij} < 0$, $w \in \Pi(\mathcal{V})$ and $x \in V$ is a weight vector of weight w, then $e_i^h e_j e_i^{1-a_{ij}-h} x$ is a weight vector of weight $w + \alpha_j + (1 - a_{ij})\alpha_i$. Looking at the α_i-string trough w, we see that $e_i^h e_j e_i^{1-a_{ij}-h} x = 0$ and hence (19) holds. ▯

2.3 Algebraic quantum groups. — Let $L(w)$ be an integrable irreducible highest weight module over the quantum algebra $U(A)$, $w \in P$, let ρ be the corresponding representation. It follows from [L1, Th. 4.12] that $L(w)$ is a finite dimensional $k(q)$-module, of dimension, say, m. Let G be the algebraic subgroup of $\mathrm{GL}(m)$ corresponding to the image of g by its irreducible representation of highest weight w. Then $\mathrm{Coeff}(\rho)$ is a Hopf algebra dual to $U(A)$ which will be denoted $k_q[G]$. Indeed, (1) is true : let $v \in L(w)$ be a highest weight vector and consider the representation $\widetilde{\rho} = \rho \circ S^2$. As $S^2(E_i) = q^{-2d_i} E_i$ and $S^2(K_i) = K_i$, v is a highest weight vector for this new representation. It follows the existence of an isomorphism $M \in \mathrm{End}(L(w))$ such that $M(x) \circ \rho = S^2(x) \circ M$ for any x : we can apply PROPOSITION 1.2.

2.4 Real forms of quantized enveloping algebras. — Let g be a complex simple (finite dimensional) Lie algebra, whose enveloping algebra is presented by (12), ..., (15). The antilinear involution given by

$$e_i \mapsto -f_i, \quad f_i \mapsto -e_i, \quad h_i \mapsto -h_i$$

is called the *Cartan involution* and will be denoted by ω. Let θ be a diagram automorphism and define t_θ by

$$t_\theta(e_i) = e_{\theta(i)}, \quad t_\theta(f_i) = f_{\theta(i)}, \quad t_\theta(h_i) = h_{\theta(i)}.$$

It is well-known (cf. for example [K, exercice 8.9]) that, modulo conjugation, all the linear involutions of g are contained in the following list :

(25) t_j, $1 \leq j \leq n$, defined by

$$t_j(e_i) = e_i, \quad t_j(f_i) = f_i, \quad t_j(h_i) = h_i, \quad \text{if} \quad \mathrm{i} \neq \mathrm{j}$$
$$t_j(e_j) = -e_j, \quad t_j(f_j) = -f_j, \quad t_j(h_j) = h_j.$$

(26) t_θ, if θ is a diagram automorphism of order two.

(27) $t_{\theta,j} = t_\theta t_j$, if θ is a diagram automorphism of order two and j is fixed by θ.

Now let q be an indeterminate. We shall consider the following two involutions of $\mathbb{C}(q)$:

$$\gamma_1(f(q)) = \bar{f}(q), \quad \gamma_2(f(q)) = \bar{f}(q^{-1}).$$

γ_1 corresponds to the field $\mathbb{R}(q)$ and γ_2 to the field of real rational functions on S^1. In other words, when considering γ_1 (resp. γ_2), we can specialize to q real (resp. $q \in S^1$).

We shall consider now involutions with respect to γ_1. The γ_2-involutions are constructed by composing with respect to the $\gamma_1\gamma_2$-involution Ξ given by

$$E_i \mapsto F_i K_i, \quad F_i \mapsto K_i^{-1} E_i, \quad K_i \mapsto K_i, \quad q \mapsto q^{-1}.$$

Note that Ξ is multiplicative, comultiplicative and $\Xi S = S \Xi$.

The Cartan involution is the γ_1-antilinear involution denoted by Ω and given by

$$E_i \mapsto -F_i, \quad F_i \mapsto -E_i, \quad K_i^{\pm 1} \mapsto K_i^{\mp 1}.$$

Now we define the following linear automorphisms of $U_q(g)$:

(28) $\widetilde{T}_j$, $1 \le j \le n$, defined by

$$\widetilde{T}_j(E_i) = (-1)^{\delta_{ij}} E_i, \quad \widetilde{T}_j(F_i) = (-1)^{\delta_{ij}} F_i, \quad \widetilde{T}_j(K_i) = K_i.$$

(29) $\widetilde{T}_\theta$, where θ is a diagram automorphism, given by

$$\widetilde{T}_\theta(E_i) = E_{\theta(i)}, \quad \widetilde{T}_\theta(F_i) = F_{\theta(i)}, \quad \widetilde{T}_\theta(K_i) = K_{\theta(i)}.$$

(30) $\widetilde{T}_{\theta,j} = T_\theta T_j$, if θ is of order two and j is fixed by θ.

Now it is easy to see that any $\widetilde{T}$ in the list (28), ...,(30) commutes with Ω; we define $T = \widetilde{T}\Omega$.

PROPOSITION.

i) Ω *satisfies* (2), (3).

ii) *If* $\widetilde{T}$ *in the list* (28),...,(30) *is of order* 2 *then* T *satisfies* (2), (3).

iii) *Any* $\widetilde{T}$ *in the list* (28),...,(30) *is a morphism of Hopf algebras.*

iv) *Let* $L(w)$ *an irreducible highest weight module,* $w \in P$. *Let* T *be* Ω *or an involution from* (28). *Then there exists* $J : L(w) \to L(w)^*$ *satisfying* (7).

v) *For* T *from* (29), (30), iv) *holds if* $w \in P$ *is* θ*-stable.*

Proof.

i) to iii) are easily verified by definition.

iv) and v) can be proved as suggested in [K, Lemma 11.5] : take the real QEA $U_{\mathbb{R}}$ and its highest weight module $L(w)_{\mathbb{R}}$, proceed as in [K, Prop. 9.4] (this is posible because $T(U_{\mathbb{R}}) \subseteq U_{\mathbb{R}}$) and extend the scalars to $\mathbb{C}$.

2.5 Minuscule compact matrix pseudogroups.

PROPOSITION. — *Let* $\mathcal{V}$ *be a minuscule highest weight module of a quantized enveloping algebra* U_q, *where* q *is a real positive parameter. Then* $\mathcal{V}$ *has an* Ω*-invariant inner product* $(\ |\ )$. *That is,*

$$(xv \mid w) = \big(v \mid \Omega S^{-1}(x) w\big) \quad \textit{for all} \quad x \in U_q,\ v, w \in \mathcal{V}.$$

Proof. — Combining PROPOSITION 2.2 and the classical theory (see for example [K, Th. 11.7]) we observe that $\mathcal{V}$ has an ω-invariant inner product $(\ |\ )_0$; i.e. we have

$$(e_i v \mid w)_0 = (v \mid f_i w)_0, \quad (f_i v \mid w)_0 = (v \mid e_i w)_0,$$
$$(h_i v \mid w)_0 = (v \mid h_i w)_0,$$

for all $v, w \in \mathcal{V}$; clearly, the decomposition of $\mathcal{V}$ in weight spaces is orthogonal.

Let τ be a weight. Let $\varrho \in P$ be defined by $\langle \varrho, \alpha_i^\vee \rangle = 1$, for all i. Then we define

$$(v \mid w) = q^{(\varrho|\tau)}(v \mid w)_0 \quad \text{for all} \quad v, w \in \mathcal{V}_\tau.$$

Clearly, this defines an inner product on $\mathcal{V}$. Let us proceed with the invariance. We need to prove

$$(E_i v \mid w) = (v \mid F_i K_i w), \quad (F_i v \mid w) = (v \mid K_i^{-1} E_i w),$$
$$(K_i v \mid w) = (v \mid K_i w),$$

for all $v, w \in \mathcal{V}$. The third equality is clear. For the first, we can assume $v \in \mathcal{V}_\tau - 0$, $w \in \mathcal{V}_{\tau+\alpha_i} - 0$. Then

$$\begin{aligned}(E_i v \mid w) &= q^{(\varrho|\tau+\alpha_i)}(e_i v \mid w)_0 = q^{(\varrho|\tau+\alpha_i)}(v \mid f_i w)_0 = q^{(\varrho|\alpha_i)}(v \mid F_i w)\\ &= q^{d_i-(\tau|\alpha_i)-2d_i}(v \mid F_i K_i w) = (v \mid F_i K_i w).\end{aligned}$$

The proof of the second is similar. $\square$

Combining the preceding with PROPOSITION 1.4, we obtain twisted deformations of any compact connected simple Lie group of type A, B, C, D, E_6, E_7. (Note that instead of (10), we can use in the proof of PROPOSITION 1.4 an invariant form on $\mathcal{V}^*$ satisfying (9).)

Appendix : generators and relations

A.0. — In [A] we presented a construction by generators and relations of k-forms of (symmetrizable) "derived" Kac-Moody algebras over $\bar{k}$. This construction can be roughly described as "*glueing together*" suitably choosen three dimensional simple Lie algebras (TDS for short) over k. (The TDS over k are in one-to-one correspondence with the quaternion algebras $sq(a, b)$ over k, cf. A.1 below). In particular, we constructed there k-forms of simple finite dimensional Lie algebras over $\bar{k}$.

In this paper we propose a definition of quantized enveloping algebras of those k-forms of g constructed in [A]. We begin by the quantized enveloping algebra of the quaternion algebra $sq(a,b)$. The observation is the following : $k[x,x^{-1}]$ is the coordinate ring of the hyperbola

$$\{(x,y) : xy = 1\}$$

which is isomorphic to $\{(r,s) : r^2 - s^2 = 1\}$. But the split form of $s\ell(2,\bar{k})$ corresponds to $a = 1$, $b = -1$. Thus we replace, in the general case, $k[x,x^{-1}]$ by $k[r,s]$ where r, s are two commuting variables satisfying $r^2 + ab\,s^2 = 1$.

A.1 Presentation of certain simple Lie algebras. — Let X, Y, Z be a basis of a 3-dimensional k-vector space V. For fixed $a,b \in k^* = k - 0$ a Lie algebra structure on V, which we shall denote $sq(a,b)$ is defined by the rule :

$$[X,Y] = 2Z, \quad [Y,Z] = -2bX, \quad [Z,X] = -2aY.$$

$sq(a,b)$ is simple and any simple 3-dimensional Lie algebra (TDS, for short) over k arises in this way. $sq(a,b)$ can be realized as Lie algebra of the traceless elements of a suitable quaternion algebra. $s\ell(2,k)$ is isomorphic to $sq(1,-1)$ and if $k = \mathbb{R}$, $sq(-1,-1)$ is $su(2,\mathbb{R})$. $sq(1,1)$ is also isomorphic to $s\ell(2,\mathbb{R})$; but whereas in $sq(-1,1)$ Z spans a split Cartan subalgebra, in $sq(1,1)$ Z is compact.

Now we recall the construction from [A], which is a generalization (and a consequence) of Serre's theorem. If g is a Lie algebra, V a g-module, $X \in g$, $v \in V$, $\alpha \in k$ and $n = 0,1,2,3$, we put

$$\mathcal{P}_n^\alpha(X,v) = \begin{cases} 0 & \text{if} \quad n = 0, \\ \alpha v & \text{if} \quad n = 1, \\ 4\alpha X v & \text{if} \quad n = 2, \\ 10\alpha X^2 v - 9\alpha^2 v & \text{if} \quad n = 3. \end{cases}$$

Let us fix elements of k^*, a_i, b_i, s_{ij} $(1 \le i,\ j \le n)$, satisfying the relations $s_{ij} = s_{ih}s_{hj} \quad \forall\, i,h,j \quad b_j/a_j = (b_i/a_i)s_{ij}$.

We define $g_k(A, a_i, s_{ij}, b_i)$, as the Lie algebra over k given by $3n$ generators $\{X_i, Y_i, Z_i : 1 \le i \le n\}$ and relations

(31) $$[Z_i, Z_j] = 0,$$

(32) $$[X_i, Y_i] = 2Z_i,$$

(33) $$[Z_i, X_j] = -a_i s_{ij}^{-1} a_{ij} Y_j, \quad [Y_j, Z_i] = -b_i s_{ij} a_{ij} X_j\ ;$$

and if $i \neq j$

$$[X_i, Y_j] = s_{ij}[Y_i, X_j], \quad [X_i, X_j] = -a_i\, b_i^{-1} s_{ij}^{-1} [Y_i, Y_j], \tag{34}$$

$$\begin{cases} (\operatorname{ad} X_i)^{1-a_{ij}} X_j = \mathcal{P}^{a_i}_{a_{ij}}(\operatorname{ad} X_i, X_j), \\ (\operatorname{ad} X_i)^{1-a_{ij}} Y_j = \mathcal{P}^{a_i}_{a_{ij}}(\operatorname{ad} X_i, Y_j). \end{cases} \tag{35}$$

Let $a, b \in k^*$ and consider the data $a_i = a$, $s_{ij} = 1$, $b_j = b$, for all i, j. We will denote $g_k(A, a, b)$ instead of $g_k(A, a_i, s_{ij}, b_i)$ in this case; $g_k(A, 1, -1)$ is isomorphic to the split form of the simple Lie algebra of Cartan matrix A, cf. [A].

A.2 Quantized enveloping algebras of TDS. — Let $a, b \in k^*$, q an indeterminate. Let

$$C_1(q) = \tfrac{1}{2}(q^2 + q^{-2}), \quad C_2(q) = \tfrac{1}{2}(q^2 - q^{-2}).$$

We define $U(2, a, b) = U_k(2, a, b)$, the quantized enveloping algebra of the TDS $sq(a, b)$, as the associative $k(q)$-algebra given by generators X, Y, R, S and relations

$$R^2 + abS^2 = 1, \quad RS = SR, \tag{36}$$

$$RX = C_1 XR - C_2 aYS, \quad SX = C_1 XS + C_2 b^{-1} YR, \tag{37}$$

$$RY = C_2 bXS + C_1 YR, \quad SY = -C_2 a^{-1} XR + C_1 YS, \tag{38}$$

$$XY - YX = -\frac{4abS}{q - q^{-1}} \cdot \tag{39}$$

LEMMA.

i) *If k' is an extension of k, there is a natural isomorphism $U_k(a, b) \otimes k' \simeq U_{k'}(a, b)$.*

ii) *Let $\lambda, \mu \in k^*$. Then there is an isomorphism between $U_k(a\lambda^2, b\mu^2)$ (with generators X', Y', R', S') and $U_k(a, b)$, given by*

$$X' \mapsto \lambda X, \quad Y' \mapsto \mu Y, \quad R' \mapsto R, \quad S' \mapsto (\lambda\mu)^{-1} S.$$

In particular, the order two automorphism of $U(a, b)$ given by $\lambda = -1$, $\mu = 1$ is called the Cartan involution.

iii) *There is an isomorphism from* $U(2) = U$ *(the quantized enveloping algebra of* $s\ell(2)$*, with generators* E*,* F*,* K*,* K^{-1}*) onto* $U(1,-1)$*, given by*

$$E \mapsto \tfrac{1}{2}(X-Y), \quad F \mapsto \tfrac{1}{2}(X+Y), \quad K \mapsto (R+S), \quad K^{-1} \mapsto (R-S)$$

with inverse

$$X \mapsto (E+F), \quad Y \mapsto (-E+F), \quad R \mapsto \tfrac{1}{2}(K+K^{-1}), \quad S \mapsto \tfrac{1}{2}(K-K^{-1}).$$

iv) *Let us assume that* $-ab$ *is not a square in* k *and choose a quadratic extension* k' *containing a square root* $\sqrt{-ab}$ *which we shall fix once and for all. Then there is an isomorphism from* $U_k(a,b)\otimes k'$ *onto* $U_{k'}$ *given by*

$$R \mapsto \tfrac{1}{2}(K+K^{-1}), \qquad S \mapsto \tfrac{1}{2}(K-K^{-1})/\sqrt{-ab},$$
$$X \mapsto E + aF, \qquad Y \mapsto \sqrt{-ab}\,\big(-E/a + F\big),$$

whose inverse is given by

$$K \mapsto R + \sqrt{-ab}\,S, \qquad K^{-1} \mapsto R - \sqrt{-ab}\,S,$$
$$E \mapsto \tfrac{1}{2}\big(X - (a/\sqrt{-ab})\,Y\big), \qquad F \mapsto \tfrac{1}{2}\big(X/a + (1/\sqrt{-ab})\,Y\big).$$

Let γ *be the non-trivial element of* $Gal(k'/k)$ *; i.e.* $\gamma(\sqrt{-ab}) = -\sqrt{-ab}$ *;* γ *extends to an automorphism in* $Gal(k'(q)/k(q))$*, letting* q *invariant, which we shall still denote* γ*. Let* $T : U_{k'} \to U_{k'}$ *be defined by* $T(v\otimes\lambda) = v\otimes\gamma(\lambda)$*, for* $v\in U_k(a,b)$*,* $\lambda\in k'$*, with the above identification.* T *is an antilinear isomorphism of algebras. Then* $(U_{k'}^*, *)$ *is a* $*$*-algebra, with the star operation* $* : U_{k'}^* \to U_{k'}^*$ *given by* $\langle\alpha^*, x\rangle = \gamma(\langle\alpha, T(x)\rangle)$*.*

v) *Let* $k'_q[G]$ *be an algebraic quantum group corresponding to some finite dimensional representation* $U_{k'} \to \mathrm{End}(V)$*. Then* $k'_q[G]$ *is a* $*$*-subalgebra of* $U_{k'}^*$ *and a* $*$*-Hopf algebra.*

A.3 Quantized enveloping algebras of $g_k(A, a_i, s_{ij}, b_i)$.

Let $\mathcal{X}$, $\mathcal{Y}$ be two indeterminates and let

$$M_s = \{\mathcal{X}^s, \mathcal{X}^{s-1}\mathcal{Y}, \mathcal{X}^{s-2}\mathcal{Y}\mathcal{X}, \dots\}$$

be the set of words in $\mathcal{X}$, $\mathcal{Y}$ of lenght s. For $w \in M_s$, we set $o_{\mathcal{Y}}(w)$ (resp. $o_{\mathcal{X}}(w)$) for the number of times that $\mathcal{Y}$ (resp., $\mathcal{X}$) appears in w.

Obviously, $o_{\mathcal{X}}(w)+o_{\mathcal{Y}}(w)=s$. In the free associative k-algebra generated by $\mathcal{X}$, $\mathcal{Y}$ we have

$$(\mathcal{X}+\mathcal{Y})^s=\sum_{w\in M_s} w,\quad (\mathcal{X}-\mathcal{Y})^s=\sum_{w\in M_s}(-1)^{o_{\mathcal{Y}}(w)}w.$$

Now if x, y are elements of an associative algebra and $S\subseteq M_s$, then $S(x,y)$ will denote the image of S under the map corresponding to $\mathcal{X}\mapsto x$, $\mathcal{Y}\mapsto y$.

Let us introduce, for a fixed Cartan matrix A as in 2.1, the polynomials on q :

$$C_1^{ij}(q)=\tfrac{1}{2}(q^{d_i a_{ij}}+q^{-d_i a_{ij}}),\quad C_2^{ij}(q)=\tfrac{1}{2}(q^{d_i a_{ij}}-q^{-d_i a_{ij}}).$$

Let also introduce the sets I_j^s, for $j=1,2$ and $0\le s\le 1-a_{ij}$; and for $(m,n)\in I_j^s$, the integer $q_{m,n}$ by

$$I_1^s=\Big\{(m,n)\in M_s(\mathcal{X},\mathcal{Y})\times M_{1-a_{ij}-s}(\mathcal{X},\mathcal{Y}):\\ -o_{\mathcal{Y}}(mn)=2q_{m,n}\in 2\mathbb{Z}\Big\},$$

$$I_2^s=\Big\{(m,n)\in M_s(\mathcal{X},\mathcal{Y})\times M_{1-a_{ij}-s}(\mathcal{X},\mathcal{Y}):\\ -o_{\mathcal{Y}}(mn)=2q_{m,n}+1\in 2\mathbb{Z}+1\Big\}.$$

(Note that I_1^s, I_2^s depend also on a_{ij} ; this is not reflected in the notation.)

We define $U_k(A,a_i,s_{ij},b_i)$, the proposed quantized enveloping algebra of $g_k(A,a_i,s_{ij},b_i)$, as the associative $k(q)$-algebra given by generators X_i, Y_i, R_i, S_i and relations

$$1=R_i^2+a_ib_iS_i^2, \tag{40}$$

$$R_iS_j=S_jR_i,\quad R_iR_j=R_jR_i,\quad S_iS_j=S_jS_i, \tag{41}$$

$$R_iX_j=C_1^{ij}(q)X_jR_i-C_2^{ij}(q)a_is_{ij}^{-1}Y_jS_i, \tag{42}$$

$$S_iX_j=C_1^{ij}(q)X_jS_i+C_2^{ij}(q)b_i^{-1}s_{ij}^{-1}Y_jR_i, \tag{43}$$

$$R_iY_j=C_2^{ij}(q)b_is_{ij}X_jS_i+C_1^{ij}(q)Y_jR_i, \tag{44}$$

$$S_iY_j=-C_2^{ij}(q)a_i^{-1}s_{ij}X_jR_i+C_1^{ij}(q)Y_jS_i, \tag{45}$$

$$X_iY_i-Y_iX_i=\frac{-4a_ib_iS_i}{q^{d_i}-q^{-d_i}}, \tag{46}$$

and if $i \neq j$

$$X_iY_j - Y_jX_i = s_{ij}(Y_iX_j - X_jY_i), \tag{47}$$

$$X_iX_j - X_jX_i = -a_ib_i^{-1}s_{ij}^{-1}(Y_iY_j - Y_jY_i) \tag{48}$$

$$\sum_{s=0}^{1-a_{ij}} (-1)^s \begin{bmatrix} 1-a_{ij} \\ s \end{bmatrix}_{d_i} \Big\{ \sum_{(m,n)\in I_1^s(X_i,Y_i)} (-a_i^{-1}b_i)^{q_{m,n}} mX_jn + s_{ij}^{-1}\Big(\sum_{(m,n)\in I_2^s(X_i,Y_i)} (-a_i^{-1}b_i)^{q_{m,n}} mY_jn \Big) \Big\} = 0 \tag{49}$$

$$\sum_{s=0}^{1-a_{ij}} (-1)^s \begin{bmatrix} 1-a_{ij} \\ s \end{bmatrix}_{d_i} \Big\{ \sum_{(m,n)\in I_1^s(X_i,Y_i)} (-a_i^{-1}b_i)^{q_{m,n}} mY_jn - a_i^{-1}b_is_{ij}\Big(\sum_{(m,n)\in I_2^s(X_i,Y_i)} (-a_i^{-1}b_i)^{q_{m,n}} mX_jn \Big) \Big\} = 0. \tag{50}$$

For example, if $a_{ij} = 0$, (49) and (50) take the form

$$X_iX_j - X_jX_i = a_ib_i^{-1}s_{ij}^{-1}(Y_iY_j - Y_jY_i), \tag{49}$$

$$X_iY_j - Y_jX_i = -s_{ij}(Y_iX_j - X_jY_i), \tag{50}$$

which, combined with (47), (48) gives that X_i (resp. Y_i) commutes with X_j, Y_j. If $a_{ij} = -1$, the relations are

$$b_i\mathcal{B}^2(X_i,X_j) - a_i\mathcal{B}^2(Y_i,X_j) = a_is_{ij}\Big\{X_iY_iY_j + Y_iX_iY_j - (q^{d_i}+q^{-d_i})(X_iY_jY_i + Y_iY_jX_i) + Y_jX_iY_i + Y_jY_iX_i\Big\}, \tag{49}$$

$$b_i\mathcal{B}^2(X_i,Y_j) - a_i\mathcal{B}^2(Y_i,Y_j) = -s_{ij}b_i\Big\{X_iY_iX_j + Y_iX_iX_j - (q^{d_i}+q^{-d_i})(X_iX_jY_i + Y_iX_jX_i) + X_jX_iY_i + X_jY_iX_i\Big\}. \tag{50}$$

Let a, $b \in k^*$ and consider the data $a_i = a$, $s_{ij} = 1$, $b_j = b$, for all i, j. We will denote $U_k(A,a,b)$ instead of $U_k(A,a_i,s_{ij},b_i)$ in this case; it is the quantized enveloping algebra of $g_k(A,a,b)$, cf. A.1.

LEMMA.

i) *If k' is an extension of k, there is a natural isomorphism* $U_k(A,a_i,s_{ij},b_i) \otimes_k k' \simeq U_{k'}(A,a_i,s_{ij},b_i)$.

ii) *Let* $\lambda_i, \gamma_i \in k^*$. *Then we have an isomorphism from* $U(A, a_i, t_{ij}, b_i)$ *to* $U(A, a_i\lambda_i^2, s_{ij}\mu_{ij}, b_i\gamma_i^2)$.

iii) *Let* $c, d \in k^*$ *and let us assume that there exists* $\lambda_i, \gamma \in k^*$, *satisfying* : $\lambda_i^2 = ca_i^{-1}$, $\gamma^2 = db_1^{-1}$. *Then* $U(A, a_i, s_{ij}, b_i)$ *is isomorphic to* $U(A, c, d)$.

iv) $U(A, 1, -1)$ *is isomorphic to* $U(A)$ (*cf.* 2.1).

v) $U_k(A, a_i, s_{ij}, b_i) \otimes_k \bar{k}$ *is isomorphic to* $U_{\bar{k}}(A)$.

Proof.

i) is obvious.

ii) Let $\mu_{ij} = (\gamma_j\lambda_i)/(\gamma_i\lambda_j)$. We have $\mu_{ij} = \mu_{ih}\mu_{hj}$, for all i, h, j. Let X_i', Y_i', R_i', S_i' be the generators of $U(A, a_i\lambda_i^2, s_{ij}\mu_{ij}, b_i\gamma_i^2)$, (40′), …, (50′) the defining relations. The assignement

$$X_j' \mapsto \lambda_j X_j \quad Y_j' \mapsto \gamma_j Y_j \quad R_j' \mapsto R_j \quad S_j' \mapsto (\lambda_j\gamma_j)^{-1}S_j$$

gives the claimed isomorphism. We leave the checking of the well-definiteness to the reader; it is a straightforward computation.

iii) follows from ii), putting $\gamma_j = \gamma\lambda_j\lambda_1^{-1}s_{j1}$.

For iv), the isomorphism from $U_k(A)$ onto $U_k(A, 1, -1)$, is given by

$$E_i \mapsto \tfrac{1}{2}(X_i - Y_i), \quad F_i \mapsto \tfrac{1}{2}(X_i + Y_i), \quad K_i^{\pm} \mapsto (R_i \pm S_i)$$

with inverse given by

$$\begin{aligned} X_i &\mapsto E_i + F_i, & Y_i &\mapsto -E_i + F_i,\\ R_i &\mapsto \tfrac{1}{2}(K_i + K_i^{-1}), & S_i &\mapsto \tfrac{1}{2}(K_i - K_i^{-1}). \end{aligned}$$

It is necessary to check that relations (16),…,(19) are equivalent to relations (40),…,(50). It is clear that (16) is equivalent to (40), (41). Let us assume that (16),…,(19) holds. Let us deduce (42) :

$$\begin{aligned} 2(C_1^{ij}X_jR_i - C_2^{ij}Y_jS_i) &= C_1^{ij}(E_j + F_j)(K_i + K_i^{-1})\\ &\qquad - C_2^{ij}(-E_j + F_j)(K_i - K_i^{-1})\\ &= (C_1^{ij} + C_2^{ij})(E_jK_i + F_jK_i^{-1})\\ &\qquad + (C_1^{ij} - C_2^{ij})(E_jK_i^{-1} + F_jK_i)\\ &= K_iE_j + K_i^{-1}F_j + K_i^{-1}E_j + K_iF_j = 2R_iX_j. \end{aligned}$$

(43), (44), (45) can be proved in a similar way. For (46), we have :

$$X_iY_i - Y_iX_i = (E_i + F_i)(-E_i + F_i) - (-E_i + F_i)(E_i + F_i)$$
$$= 2(E_iF_i - F_iE_i) = 2\,\frac{K_i - K_i^{-1}}{q^{d_i} - q^{-d_i}} = \frac{-4S_i}{q^{d_i} - q^{-d_i}},$$

and similarly for (47), (48). Now we deduce (49), (50). We shall use (20). Let us add the left-hand sides of (49) and (50) :

$$\sum_{s=0}^{1-a_{ij}} (-1)^s \begin{bmatrix} 1-a_{ij} \\ s \end{bmatrix}_{d_i} \Big\{ \sum_{(m,n)\in I_1^s(X_i,Y_i)} mX_jn + \sum_{(m,n)\in I_2^s(X_i,Y_i)} mY_jn \Big\}$$
$$+ \sum_{s=0}^{1-a_{ij}} (-1)^s \begin{bmatrix} 1-a_{ij} \\ s \end{bmatrix}_{d_i} \Big\{ \sum_{(m,n)\in I_1^s(X_i,Y_i)} mY_jn + \sum_{(m,n)\in I_2^s(X_i,Y_i)} mX_jn \Big\}$$
$$= \sum_{s=0}^{1-a_{ij}} (-1)^s \begin{bmatrix} 1-a_{ij} \\ s \end{bmatrix}_{d_i} \sum_{(m,n)\in M_s\times M_{1-a_{ij}-s}} (mY_jn + mX_jn)$$
$$= \sum_{s=0}^{1-a_{ij}} (-1)^s \begin{bmatrix} 1-a_{ij} \\ s \end{bmatrix}_{d_i} \sum_{m\in M_s(X_i,Y_i)} \sum_{n\in M_{1-a_{ij}-s}(X_i,Y_i)} m(Y_j + X_j)n$$
$$= \sum_{s=0}^{1-a_{ij}} (-1)^s \begin{bmatrix} 1-a_{ij} \\ s \end{bmatrix}_{d_i} (X_i + Y_i)^s (X_j + Y_j)(X_i + Y_i)^{1-a_{ij}-s}$$
$$= \sum_{s=0}^{1-a_{ij}} (-1)^s \begin{bmatrix} 1-a_{ij} \\ s \end{bmatrix}_{d_i} F_i^s F_j F_i^{1-a_{ij}-s}.$$

Substracting the left-hand sides of (49) and (50), we obtain :

$$\sum_{s=0}^{1-a_{ij}} (-1)^s \begin{bmatrix} 1-a_{ij} \\ s \end{bmatrix}_{d_i}$$
$$\sum_{(m,n)\in M_s\times M_{1-a_{ij}-s}} \Big((-1)^{o_{\mathcal{Y}}(mn)} mX_jn - (-1)^{o_{\mathcal{Y}}(mn)} mY_jn\Big)$$
$$= \sum_{s=0}^{1-a_{ij}} (-1)^s \begin{bmatrix} 1-a_{ij} \\ s \end{bmatrix}_{d_i} \sum_{m\in M_s} (-1)^{o_{\mathcal{Y}}(m)}$$
$$\sum_{n\in M_{1-a_{ij}-s}} (-1)^{o_{\mathcal{Y}}(n)} m(X_j - Y_j)n$$

$$= \sum_{s=0}^{1-a_{ij}} (-1)^s \begin{bmatrix} 1-a_{ij} \\ s \end{bmatrix}_{d_i} (X_i - Y_i)^s (X_j - Y_j)(X_i - Y_i)^{1-a_{ij}-s}$$

$$= \sum_{s=0}^{1-a_{ij}} (-1)^s \begin{bmatrix} 1-a_{ij} \\ s \end{bmatrix}_{d_i} E_i^s E_j E_i^{1-a_{ij}-s}.$$

Now we turn to the implication (40),..., (50) $\Rightarrow$ (16),..., (19). The last computation shows how to obtain (19) from (49) and (50); (17) and (18) will be left to the reader.

v) follows from the preceding items.

Let us assume that $-a_\ell b_\ell$ is not a square in k, for some $\ell : 1 \le \ell \le n$, and choose a quadratic extension k' containing a square root $\sqrt{-a_\ell b_\ell}$ which we shall fix in the rest of the section. As $-a_j b_j = -a_\ell b_\ell (a_j a_\ell^{-1} s_{\ell j})^2$, $\sqrt{-a_\ell b_\ell} a_j a_\ell^{-1} s_{\ell j}$ is a square root of $-a_j b_j$ which will be denoted $\sqrt{-a_j b_j}$. Clearly

$$\sqrt{-a_i b_i} = \sqrt{-a_j b_j} a_i a_j^{-1} s_{ji} \quad \forall\, i, j.$$

Then there is an isomorphism from $U(A, a_i, s_{ij}, b_i)$ onto $U_{k'}(A)$ given by

$$R_i \mapsto \tfrac{1}{2}(K_i + K_i^{-1}), \quad S_i \mapsto \tfrac{1}{2}(K_i - K_i^{-1})/\sqrt{-a_i b_i}$$

$$X \mapsto E_i + a_i F_i, \qquad Y_i \mapsto \sqrt{-a_i b_i}\,(-E_i/a_i + F_i),$$

whose inverse is given by

$$K_i \mapsto R_i + \sqrt{-a_i b_i}\, S_i, \qquad K_i^{-1} \mapsto R_i - \sqrt{-a_i b_i}\, S_i$$

$$E_i \mapsto \tfrac{1}{2}\big(X_i - (a_i/\sqrt{-a_i b_i}) Y_i\big), \quad F_i \mapsto \tfrac{1}{2}\big(X_i/a_i + (1/\sqrt{-a_i b_i}) Y_i\big).$$

Let γ be the non-trivial element of $\mathrm{Gal}(\mathrm{k}'/\mathrm{k})$; one has

$$\gamma(\sqrt{-a_i b_i}) = -\sqrt{-a_i b_i}\,;$$

γ extends to an automorphism in $\mathrm{Gal}(\mathrm{k}'(\mathrm{q})/\mathrm{k}(\mathrm{q}))$, letting q invariant, which we shall still denote γ. Let $T : U_{k'}(A) \to U_{k'}(A)$ be defined by $T(v \otimes \lambda) = v \otimes \gamma(\lambda)$, for $v \in U(A, a_i, s_{ij}, b_i)\, \lambda \in k'$, with the above identification.

PROPOSITION.

i) $(U_{k'}(A)^*, *)$ *is a* $*$*-algebra, with the application* $* : U_{k'}(A)^* \to U_{k'}(A)^*$ *given by* $\langle \alpha^*, x\rangle = \gamma(\langle \alpha, T(x)\rangle)$.

ii) *Let* $k'_q[G]$ *be an algebraic quantum group corresponding to some finite dimensional representation* $U_{k'}(A) \to \mathrm{End}(V)$. *Then* $k'_q[G]$ *is a* $*$*-subalgebra of* $U_{k'}(A)^*$ *and a* $*$*-Hopf algebra.*

BIBLIOGRAPHY

[A] ANDRUSKIEWITSCH (N.). — *Some forms of Kac Moody algebras*, J. Algebra, (to appear).

[AE] ANDRUSKIEWITSCH (N.) and ENRIQUEZ (B.). — *Examples of Compact Matrix Pseudogroups arising from the twisting operation*, Preprint.

[APW] ANDERSON (H.H.), POLO (P.) and WEN (K.). — Representations of quantum algebras, *Invent. Math.*, t. **104**, 1991, p. 1–60.

[B] BOURBAKI (N.). — *Groupes et algèbres de Lie, Chap. 4, 5 et 6.* — Hermann, Paris, 1968.

[D1] DRINFELD (V.G.). — Hopf algebras and the Yang-Baxter equation, *Soviet Math. Dokl.*, t. **32**, 1985, p. 254–258.

[D2] DRINFELD (V.G.). — *Quantum groups*, Proc. of the ICM, Berkeley, 1986, pp. 798–820.

[FRT] FADEEV (L.D.), RESHETIKHIN (N. Yu) and TAKHTAJAN (L.A.). — *Quantization of Lie Groups and Lie algebras*, in Algebraic Analysis, Papers dedicated to Professor Mikio Saito on the Occasion of His Sixtieth Birthday, Academic Press, 1988, pp. 129–139.

[J] JIMBO (M.). — A q-difference analogue of $U(g)$ and the Yang-Baxter equation, *Lett. Math. Phys.*, t. **10**, 1985, p. 63–69.

[L1] LUSZTIG (G.). — Quantum deformations of certain simple modules over enveloping algebras, *Adv. in Math.*, t. **70**, 1988, p. 237–249.

[L2] LUSZTIG (G.). — Modular representantions and quantum groups, *Contemp. Math.*, t. **82**, 1989, p. 59–77.

[L3] LUSZTIG (G.). — Finite dimensional Hopf algebras arising from quantized universal enveloping algebras, *J. of Amer. Math. Soc.*, t. **3** (1), p. 257–296.

[L4] LUSZTIG (G.). — *Quantum groups at roots of* 1, Geom. Dedicata, 1990.

[K] KAC (V.). — *Infinite dimensional Lie algebras*, Prog. in Math., vol. 44, Birkhauser, Boston, 1983.

[Ko] KOSTANT (B.). — Groups over $\mathbb{Z}$, *Proc. Symp. Pur Math., Amer. Math. Soc.*, t. **6**, 1966, p. 90–98.

[KR] KULISH (P.P.) and RESHETIKHIN (N. Yu). — The quantum linear problem for the sine Gordon equation and higher representations, *J. Soviet Math.*, t. **23** (4), 1983, p. 2435–2440.

[Ma] MANIN (Yu.). — *Quantum Groups.* — Montreal Lectures, 1988.

[Re] RESHETIKHIN (N.). — *Quantized universal enveloping algebras, the Yang-Baxter equation and invariants of links*, I–II, Preprint, L.O.M.I..

[R1] ROSSO (M.). — Finite dimensional representations of the quantum analog of the enveloping algebra of a complex simple Lie algebra, *Commun. Math. Phys.*, t. **117**, 1988, p. 581–593.

[R2] ROSSO (M.). — Algèbres envelopantes quantifiées, groupes quantiques compacts de matrices et calcul différentiel non commutatif, *Duke Math. J*, t. **61**, 1990, p. 11–39.

[Sk] SKLYANIN (E.K.). — On an algebra generated by quadratic relations, *Uspekhi Mat. Nauk*, t. **40**, **2**, 1985, p. 214.

[Sk2] SKLYANIN (E.K.). — Some algebraic structures connected with the Yang-Baxter equation. Representations of the quantum algebra, *Funct. An. Appl.*, t. **17**, 1983, p. 34–48.

[T] TAKEUCHI (M.). — Quantum Orthogonal and Symmetric Groups and their Embedding into Quantum GL_n, *Proc. Japan Acad. Ser. A*, t. **65**, **2**, 1989, p. 55–58.

[VS] VAKSMAN (L.L.) and SOIBELMAN (Ya.S.). — Algebra of functions on the quantum group SU(2), *Funct. An. Appl.*, t. **22**, 1988, p. 170–181.

[W1] WORONOWICZ (S.L.). — Twisted SU(2) Group. An example of Non-Commutative Differential Calculus, *Publ. RIMS, Kyoto Univ.*, t. **23**, 1987, p. 117–181.

[W2] WORONOWICZ (S.L.). — Compact matrix pseudogroups, *Comm. in Math. Phys.*, t. **111**, 1987, p. 613–665.

[W3] WORONOWICZ (S.L.). — Tannaka-Krein duality for compact matrix pseudogroups. Twisted $SU(N)$ groups, *Invent. Math.*, t. **93**, 1988, p. 35–76.