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SUBELLIPTIC VARIATIONAL PROBLEMS

BY

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RÉSUMÉ. — En utilisant la méthode directe et l'itération de MOSER, nous démontrons l'existence et la C^{μ}-régularité du point stationnaire pour le problème variationnel elliptique dégénéré $I(\mu) = \int_{\Omega} F(x, u, Xu)\, dx$ où $X = (X_1, \dots, X_m)$ est un système de champs de vecteurs C^{∞} réels qui satisfait à la condition de Hörmander. Les hypothèses sur $F(x, u, \xi)$ sont analogues à celles faites pour les problèmes elliptiques.

ABSTRACT. — Using the direct method and the MOSER's process, we prove the existence and C^{μ} regularity of stationary point for the degenerate elliptic variational problem $I(\mu) = \int_{\Omega} F(x, u, Xu)\, dx$ where $X = (X_1, \dots, X_m)$ is a system of real smooth vector fields which satisfy the Hörmander's condition. The assumption imposed on $F(x, u, \xi)$ are similar to those for the elliptic case.

1. Introduction

In this paper, we study the existence and the regularity for the minimum points of the following variational problem :

$$I(\mu) = \int_{\Omega} F(x, u, Xu)\, dx, \tag{1.1}$$

where Ω is an open set in $\mathbb{R}^n$, $n \geq 2$, and $X = (X_1, \dots, X_m)$ is a system of real smooth vector fiels in M, which is a bounded domain of $\mathbb{R}^n$ such that $\Omega \subset\subset M$. We assume that $F(x, u, \xi)$ is convex in ξ and that X satisfy the Hörmander's condition in M, i.e.

(H) $\begin{cases} \{X_j\} \textit{ together with their commutators} \\ \textit{up to a certain fixed length } r \textit{ span the} \\ \textit{tangent space at each point of } M. \end{cases}$

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In this case, the Euler's equation of (1.1)

$$\sum_{j=1}^{m} X_j^* F_{\xi_j}(x,u,Xu) + F_u(x,u,Xu) = 0 \tag{1.2}$$

is degenerately elliptic. We assume also, for $j = 1, \dots, m$,

$$\mathrm{Mes}\{x \in \Omega \mid X_j(x) = 0\} = 0.$$

For linear problems of this kind, there is a lot of work after the first appearing of L. HÖRMANDER's (see [1, 2, 4, 5, 7, 8, 9]). In particular, we note that the Hörmander's condition permit us to define a metric $\rho(x,y)$ associated with X in M. Using the geometry of this metric, we can think the Hörmander operator

$$H = \sum_{j=1}^{m} X_j^2 + c(x)$$

as the Laplace operators. Then we can study the existence of weak stationary points of (1.1) by the direct method just as we do for the elliptic problem, and discuss the C^μ regularity of weak solution of (1.2) by Moser's process just as we do for the linear degenerate elliptic problems.

Our result is an extention of those for the elliptic variationnal problem to a certain class of highly degenerate problems. We will consider the C^∞ regularity problems in another paper.

2. Function space $\mathbf{M^{k,p}(\Omega)}$

In order to study the weak solution, we introduce a function space $M^{k,p}(\Omega)$ associated with X, which is analogue to Sobolev's space. For any integer $k \geq 1$, $p \geq 1$ and $\Omega \subset\subset M$, we define

$$M^{k,p}(\Omega) = \Big\{ f \in L^p(\Omega) \mid X^J f \in L^p(\Omega),\ \forall J = (j_1, \dots, j_s),\ |J| \leq k \Big\} \tag{2.1}$$

where $X^J f = X_{j_1} \dots X_{j_s} f$, $|J| = s$ and define the norm in $M^{k,p}(\Omega)$ to be

$$||f||_{M^{k,p}(\Omega)} = \Big(\sum_{|J| \leq k} ||X^J f||^p_{L^p(\Omega)} \Big)^{1/p}. \tag{2.2}$$

We also denote by $M^k(\Omega) = M^{k,2}(\Omega)$. Then we have :

THEOREM 1. — *The function space* $M^{k,p}(\Omega)$ *is a Banach space for* $1 \leq p < +\infty$, *which is reflexive for* $1 < p < +\infty$ *and separable for* $1 \leq p < +\infty$. *Also,* $M^k(\Omega)$ *is a separable Hilbert space.*

Proof. — a) Let $J = (j_1, \ldots, j_s)$, with $1 \leq j_c \leq m$, and denote by X^{J*} the adjoint operator of X^J. Then

$$M^{k,p}(\Omega) = \Big\{ f \in L^p(\Omega) \mid \exists g_J \in L^p(\Omega) \text{ such that } \int_\Omega f \cdot X^{J*}\varphi \, dx = \int_\Omega g_J \, \varphi dx, \ \ \varphi \in C_0^\infty(\Omega), \ \ |J| \leq k \Big\}. \tag{2.3}$$

Suppose $\{u_j\}$ to be a Cauchy sequence of $M^{k,p}(\Omega)$, then $\{X^J u_j\}$, for $|J| \leq k$, are all Cauchy sequence in $L^p(\Omega)$. Hence there exists $u^J \in L^p(\Omega)$ such that $X^J u_j \to u^J$ in $L^p(\Omega)$. On the other hand

$$\int_\Omega u_j X^{J*}\varphi \, dx = \int_\Omega X^J u_j \varphi \, dx, \quad \varphi \in C_0^\infty, \ \ |J| \leq k.$$

Let $j \to \infty$, we have

$$\int_\Omega u^0 X^{J*}\varphi \, dx = \int_\Omega u^J \varphi \, dx, \quad \varphi \in C_0^\infty(\Omega), \ \ |J| \leq k,$$

which proves $u^0 \in M^{k,p}(\Omega)$, $X^J u^0 = u^J$ and $||u_j - u^0||_{M^{k,p}(\Omega)} \to 0$.

b) Setting $E = \prod_{|J| \leq k} L^p(\Omega)$, then E is a reflexive Banach space for $1 < p < +\infty$. Define $T : M^{k,p}(\Omega) \to E$ by $Tu = (X^J u)$, then T is an isometry from $M^{k,p}(\Omega)$ to E. Since $T(M^{k,p}(\Omega))$ is a closed subspace of E and $T(M^{k,p}(\Omega))$ is reflexive, then $M^{k,p}(\Omega)$ is also reflexive. The proof for separability is similar.

We denote by $M_0^{k,p}(\Omega)$ the closure of $C_0^\infty(\Omega)$ in $M^{k,p}(\Omega)$. From the subellipticity of Hörmander's operator H, we have the following lemma :

LEMMA 2. — *Let* Ω *be a bounded subdomain of* M. *Assume that* X *satisfies the Hörmander's condition in* M. *Then, we have the continuous imbedding* $M_0^{k,p}(\Omega) \subset W^{k/r,p}(\Omega)$ *for all* $k \geq 1$, $p \geq 1$ *and there exists* $C = C(p, \Omega, r)$ *such that*

$$||u||_{W^{k/r,p}}(\Omega) \leq C ||u||_{M^{k,p}}(\Omega) \tag{2.4}$$

for all $u \in M_0^{k,p}(\Omega)$. *(Here,* $W^{s,p}(\Omega)$ *is the usual Sobolev's space.)*

For the proof of this LEMMA, see [2, 9]. Using the classical Sobolev inequality in $W^{s,p}(\Omega)$ and imbedding LEMMA above, we obtain the following Sobolev inequality for the function space $M^{k,p}(\Omega)$.

THEOREM 3. — *Assume that Ω is a C^∞ domain. Then, we have continuous imbedding*

$$(2.5)\qquad M_0^{k,p}(\Omega) \subset \begin{cases} L^{np/(n-kp/r)}(\Omega) & \text{for } kp < nr, \\ C^m(\overline{\Omega}) & \text{for } k/r - n/p > m \geq 0. \end{cases}$$

Further, there exists a constant $C = C(n,r,p,k)$ such that for any $u \in M_0^{k,p}(\Omega)$ we have

$$(2.6)\qquad \begin{cases} ||u||_{L^{np/(n-kp/r)}(\Omega)} \leq C||u||_{M^{k,p}(\Omega)} & \text{for } kp < nr, \\ ||u||_{C^m(\overline{\Omega})} \leq C|\Omega|^{k/n-r/p}||u||_{M^{k,p}(\Omega)} & \text{for } k/r - n/p > m \geq 0. \end{cases}$$

By a contradiction argument based on the compactness result of the usual Sobolev's space, we obtain an interpolation inequality for the space $M^{k,p}(\Omega)$.

LEMMA 4. — *Assume that Ω is a C^∞ subdomain of M and u an element of $M^{k,p}(\Omega)$. Then, for any $\varepsilon > 0$ and $0 < |J| < k$, we have*

$$||X^J u||_{L^p(\Omega)} \leq \varepsilon||u||_{M^{k,p}(\Omega)} + C||u||_{L^p(\Omega)}$$

where $C = C(k, \Omega, \varepsilon)$.

We define now a metric $\rho(x,y)$ associated with X in M as in [7, 9], and take

$$B_R(x) = \{y \in \Omega \mid \rho(x,y) < R\}$$

for $R > 0$ small enough. Then, in the function space $M^{k,p}(\Omega)$, we have also the following Poincaré inequality.

LEMMA 5

(1) *For any $x^0 \in \Omega$, there exists $R_0 > 0$ such that for all $0 < R \leq R_0$, if $\varphi \in M_0^{1,p}(B_R(x^0))$, then*

$$(2.8)\qquad ||\varphi||_{L^p(B_R(x^0))} \leq CR||X\varphi||_{L^p(B_R(x^0))}$$

where C is of independant on φ and R.

(2) *If, in the system of vector field $X = (X_1, \ldots, X_m)$ there exists at last one vector field which can be globally straightened in Ω, then we have*

$$(2.9)\qquad ||\varphi||_{L^p(\Omega)} \leq C \operatorname{diam}\Omega||X\varphi||_{L^p(\Omega)}$$

for all $\varphi \in M_0^{1,p}(\Omega)$, where C is of independant on φ and Ω.

(3) *There exists a constant C and a radius $R_0 > 0$ such that for any $x^0 \in \Omega$ and any R, with $0 < R \leq R_0$, for which $B_{2R}(x^0) \subset \Omega$, we have*

$$\int_{B_R} \left|u(x) - \overline{u}_R\right|^p dx \leq CR^p \int_{B_R} \sum_{j=1}^{m} |X_j u(x)|^p dx \tag{2.10}$$

for all $u \in M^{1,p}(B_R)$, where $\overline{u} = |B_R|^{-1} \int_{B_R} u(y)\, dy$ and $|B_R|$ denote the volume of $B_R(x^0)$.

The proof of (1) is classical and we can find the proof of (3) in [8].

Now, for $u \in M_0^1(\Omega)$ and $k \geq 0$, let

$$A_k = \{x \in \Omega \mid u(x) > k\} \quad \text{and} \quad A'_k = \{x \in \Omega \mid u(x) = k\}.$$

It is readily seen that these sets are mesurable, and

$$A_k = \bigcup_{\varepsilon > 0} A_{k+\varepsilon}, \quad A_k \cup A'_k = \bigcap_{\varepsilon>0} A_{k-\varepsilon},$$
$$\operatorname{Mes}(A_k \setminus A_{k+\varepsilon}) \longrightarrow 0, \quad \operatorname{Mes}(A_{k-\varepsilon} \setminus (A_k \cup A'_k) \longrightarrow 0$$

as $\varepsilon \to 0$. For the functions $u^{(k)}(x) = \max\{u(x) - k, 0\}$, we have

LEMMA 6. — *Let $u \in M_0^1(\Omega)$ and $k \geq 0$. Then $u^{(k)} \in M_0^1(\Omega)$ and for $j = 1, \dots, m$, we have*

$$X_j u^{(k)} = \begin{cases} X_j u(x) & \text{for almost all } x \in A_k, \\ 0 & \text{at other places.} \end{cases} \tag{2.11}$$

Proof. — From the definition, for $u \in M_0^1(\Omega)$, there exists a sequence $\{u_p\} \subset C_0^\infty(\Omega)$ such that $\lim_{p\to\infty} ||u_p - u||_{M^1(\Omega)} = 0$. Thus $u_p \to u$ and $X_j u_p \to X_j u$ in $L^2(\Omega)$ for $j = 1, \dots, m$. We have immediately $u_p^{(k)} \to u^{(k)}$ in $L^2(\Omega)$. Setting $A_k^p = \{x \in \Omega \mid u_p(x) > k\}$, we have, for $p \to \infty$,

$$\operatorname{Mes}\big(A_k \setminus (A_k^p \cap A_k)\big) \longrightarrow 0,$$
$$\operatorname{Mes}\big(A_k^p \setminus (A_k^p \cap (A_k \cup A'_k))\big) \longrightarrow 0.$$

In fact, we know that $\operatorname{Mes}\Omega^{p,\varepsilon} = \operatorname{Mes}\{x \in \Omega\,;\ |u_p(x) - u(x)| \geq \varepsilon\} \to 0$ for all $\varepsilon > 0$ and $p \to \infty$, and $A_{k+\varepsilon} \cap (\Omega \cap \Omega^{p,\varepsilon}) \subset A_k^p$ for $\varepsilon > 0$.

$$A_k \cap \big(A_{k+\varepsilon} \cap (\Omega \setminus \Omega^{p,\varepsilon})\big) = A_{k+\varepsilon} \cap \big(\Omega \setminus \Omega^{p,\varepsilon}\big) \subset A_k^p \cap A_k,$$
$$A \setminus (A_k^p \cap A_k) = (A_k \setminus A_{k+\varepsilon}) \cup (A_{k+\varepsilon} \cap \Omega^{p,\varepsilon}) \cup \big(A_{k+\varepsilon} \cap (\Omega \setminus \Omega^{p,\varepsilon})\big) \setminus (A_k^p \cap A_k).$$

Now; for $\delta > 0$, take $\varepsilon = \varepsilon(\delta)$ such that

$$\operatorname{Mes}(A_k \setminus A_{k+\varepsilon}) \leq \tfrac{1}{2}\delta$$

and $p(\delta)$ such that for $p \geq p(\delta)$, $\operatorname{Mes} \Omega^{p,\varepsilon(\delta)} \leq \frac{1}{2}\delta$, implying that

$$\operatorname{Mes}(A_k \setminus (A_k^p \cap A_k)) \leq \delta.$$

On the other hand, $A_k^p \cap (\Omega \setminus \Omega^{p,\varepsilon}) \subset A_{k-\varepsilon}$, hence is also contained in $A_k^p \cap A_{k-\varepsilon}$. Now

$$\begin{aligned} A_k^p \setminus \big(A_k^p \cap (A_k \cup A_k')\big) = \big(A_k^p \cap (\Omega \setminus \Omega^{p,\varepsilon}) \cup (A_k^p \cap \Omega^{p,\varepsilon}) \setminus (A_k^p \cap A_{k-\varepsilon}) \\ \cup \big(A_k^p \cap (A_{k-\varepsilon} \setminus (A_k \cup A_k'))\big). \end{aligned}$$

For $\delta > 0$, take $\varepsilon = \varepsilon(\delta) > 0$ such that

$$\operatorname{Mes}\big(A_{k-\varepsilon} \setminus (A_k \cup A_k')\big) \leq \tfrac{1}{2}\delta$$

and $p(\delta)$ such that for $p \geq p(\delta)$, $\operatorname{Mes} \Omega^{p,\varepsilon} \leq \frac{1}{2}\delta$. Thus

$$\operatorname{Mes}\big(A_k^p \setminus (A_k^p \cap (A_k \cup A_k'))\big) \leq \delta.$$

Now, since $u_p \in C^\infty(\Omega)$, A_k^p is an open subset of Ω and, for $j = 1, \ldots, m$

$$X_j u_p^{(k)} = \begin{cases} X_j u_p & \text{in } A_k^p, \\ 0 & \text{in } \Omega \setminus \overline{A}_k^p. \end{cases}$$

In fact, denote by $E_j = \{x \in \Omega \mid X_j(x) = 0\}$. Then $\operatorname{Mes} E_j = 0$, and X_j is a nondegenerate vector field on $\Omega \setminus E_j$. Then we can obtain, as in the classical case, $X_j u_p^{(k)} = 0$ for almost $x \in A_k^{p'} \setminus E_j$.

Hence, $\{X_j u_p^{(k)}\}_{p=1,\ldots,\infty}$ is a bounded sequence in $L^2(\Omega)$. Then there exists $u_{0,j}^{(k)} \in L^2(\Omega)$ such that a subsequence of $\{X_j u_p^{(k)}\}$ converges weakly to $u_{0,j}^{(k)}$ in $L^2(\Omega)$, i.e. for all $\varphi \in C_0^\infty(\Omega)$, we have

$$\lim_{p\to\infty} \langle X_j u_p^{(k)}, \varphi \rangle = \langle u_{0,j}^{(k)}, \varphi \rangle.$$

On the other hand

$$\begin{aligned} \lim_{p\to\infty} \langle X_j u_p^{(k)}, \varphi \rangle &= \lim_{p\to\infty} \langle u_p^{(k)}, X_j^* \varphi \rangle \\ &= \langle u^{(k)}, X_j^* \varphi \rangle \\ &= \langle X_j u^{(k)}, \varphi \rangle. \end{aligned}$$

Thus $X_j u^{(k)} = u_{0,j}^{(k)} \in L^2(\Omega)$ for $j = 1, \dots, m$, which proves $u^{(k)} \in M_0^1(\Omega)$ because $\operatorname{Supp} u_p^{(k)}$ is compact in Ω and $X_j u_p^{(k)}$ converges to $X_j u^{(k)}$ for almost $x \in \Omega$, hence

$$\lim_{p \to \infty} \operatorname{Mes}\{x \in \Omega\,;\ \big|X_j u_p^{(k)} - X_j u^{(k)}\big| \geq \varepsilon\} = 0$$

for all $\varepsilon > 0$ and $j = 1, \dots, m$. Denote

$$E_p^{(k)} = \{X \in A_k\,;\ |X_j u_p^{(k)} - X_j u| \geq \varepsilon\}.$$

Then we have :

$$E_p^k = \big(E_p^k \cap (A_k^p \cap A_k)\big) \cup \big(E_p^k \cap \big(A^k \setminus (A_k^p \cap A_k)\big)\big)$$

where, for $p \to \infty$,

$$\begin{aligned} \operatorname{Mes}\big(E_p^k \cap (A_k^p \cap A_k)\big) = \operatorname{Mes}\{x \in A_k^p \cap A_k\,;\ |X_j u_p - X_j u| \geq \varepsilon\} \longrightarrow 0, \\ \operatorname{Mes}\big(E_p^k \cap \big(A_k \setminus (A_k^p \cap A_k)\big)\big) \leq \operatorname{Mes}\big(A_k \setminus (A_k^p \cap A_k)\big) \longrightarrow 0. \end{aligned}$$

That implies $X_j u^{(k)}(x) = X_j u(x)$ for almost $x \in A_k$ and $j = 1, \dots, m$. On the other hand, we have

$$\Omega \setminus A_k = \big(\Omega \setminus (A_k^p \cup A_k)\big) \cup \big(A_k^p \setminus (A_k^p \cap (A_k \cup A_k'))\big) \cup (A_k \cup A_k')$$

where, for $p \to \infty$,

$$\begin{aligned} \operatorname{Mes}\{x \in \Omega \setminus (A_k^p \cup A_k)\,;\ |X_j u_p^{(k)}| \geq \varepsilon\} \leq \operatorname{Mes} \partial A_k^p = 0, \\ \operatorname{Mes}\{A_k^p \setminus \big(A_k^p \cap (A_k \cup A_k')\big)\} \longrightarrow 0. \end{aligned}$$

For the term $A_k^p \cap A_k'$, if $\operatorname{Mes} A_k' \neq 0$, denote by $\widetilde{\Omega} = \{x \in \Omega \mid X_j(x) \neq 0\}$. Then $\operatorname{Mes} \widetilde{\Omega} = \operatorname{Mes} \Omega$, and $\widetilde{\Omega}$ is an open subset of Ω by using Hörmander's condition. Hence, just as in the classical case, there exists $\widetilde{A}_k'$ such that $\operatorname{Mes} A_k' = \operatorname{Mes} \widetilde{A}_k$ and $X_j u(x) = 0$ for $x \in \widetilde{A}_k$ and $j = 1, \dots, m$. Then

$$\begin{aligned} &\operatorname{Mes}\{x \in A_k^p \cap A_k'\,;\ \big|X_j u_p^{(k)}(x)\big| \geq \varepsilon\} \\ &\quad = \operatorname{Mes}\{x \in A_k^p \cap \widetilde{A}_k'\,;\ \big|X_j u_p(x) - X_j u(x)\big| \geq \varepsilon\} \\ &\quad \leq \operatorname{Mes}\{x \in \Omega\,;\ \big|X_j u_p(x) - X_j u(x)\big| \geq \varepsilon\} \longrightarrow 0, \end{aligned}$$

for $p \to \infty$, which proves $\lim_{p \to \infty} X_j u_p^{(k)}(x) = 0 = X_j u^{(k)}(x)$ for almost $x \in \Omega \setminus A_k$.

3. Existence of minimizing points

Assume that Ω is a C^∞ bounded subdomain of M. We now consider the problem of minimizing the functionnal

$$I(u) = \int_\Omega F(x, u, Xu)\, dx \tag{3.1}$$

in the function space $M_0^{1,p}(\Omega)$.

For the existence problem, we assume that the function $F(x, u, \xi)$: $\overline{\Omega} \times \mathbb{R} \times \mathbb{R}^m \to \mathbb{R}$ satisfies the following conditions :

(1) $F(x, u, \xi) \geq \lambda |\xi|^p$ for $p > 1$ and $\lambda > 0$;

(2) $F(x, u, \xi)$, $F_u(x, u, \xi)$ and $F_{\xi_j}(x, u, \xi)$ are, for $j = 1, \dots, m$, continuous functions in $\overline{\Omega} \times \mathbb{R} \times \mathbb{R}^m$;

(3) $F(x, u, \xi)$ is convex in ξ for all (x, u).

We will prove the following existence theorem :

THEOREM 7. — *Let X satisfy the condition* (H) *and the assumption* (2) *of* LEMMA 5. *Assume that F satisfies the conditions* (1), (2), (3) *and that there exists a function $\varphi \in M_0^{1,p}(\Omega)$ such that $I(\varphi) < +\infty$. Then the functionnal $I(u)$ attains a minimum in $M_0^{1,p}(\Omega)$.*

Proof. — Assume that $\{u_k\}$ is a minimizing sequence in $M_0^{1,p}(\Omega)$, that is $u_k \in M_0^{1,p}(\Omega)$ and $I(u_k) = d_k \to d = \inf_{v \in M_0^{1,p}(\Omega)} I(v)$. From condition (1), we get

$$\int_\Omega |Xu_k|^p dx \leq I(u_k) \leq \text{constant independant of } k. \tag{3.2}$$

On the other hand, from the point (2) of LEMMA 5, we obtain :

$$\int_\Omega |u_k|^p dx \leq c_1 \int_\Omega |Xu_k|^p dx. \tag{3.3}$$

Hence $||u_k||_{M_0^{1,p}(\Omega)} \leq$ const, independant of k. Now, $M^{1,p}(\Omega)$ is a reflexive Banach space for $p > 1$. Passing to a subsequence when necessary, we know that $\{u_k\}$ converges almost everywhere in Ω, strongly in $L^p(\Omega)$ and weakly in $M^{1,p}(\Omega)$ to a function $u_0 \in M_0^{1,p}(\Omega)$. We have to prove that $I(u_0) = d$. From Egorov's theorem, for any $\varepsilon > 0$, there exists a subdomain $\Omega_\varepsilon \subset \Omega$ such that $\mathrm{Mes}(\Omega \setminus \Omega_\varepsilon) < \varepsilon$ and $\{u_k\}$ converges uniformly to u_0 in Ω_ε. For $N > 0$, we define $\Omega_N = \{x \in \Omega\, ;\ |u_0| + |Xu_0| < N\}$. Since $u_0 \in M_0^{1,p}(\Omega)$, we have $\mathrm{Mes}(\Omega \setminus \Omega_{\varepsilon,N}) \to 0$ when $N \to \infty$. Putting

$\Omega_{\varepsilon,N} = \Omega_\varepsilon \cap \Omega_N \subset \Omega$, we have $\mathrm{Mes}(\Omega \setminus \Omega_{\varepsilon,N}) \to 0$ when $\varepsilon \to 0$ and $N \to \infty$. Using the condition (3), we find

$$\int_{\Omega_{\varepsilon,N}} \big(F(x,u_k X u_k) - F(x,u_0,Xu_0)\big)\,dx \geq \int_{\Omega_{\varepsilon,N}} \sum_{j=1}^{m} F_{\xi_j}(x,u_k,Xu_0)\cdot X_j(u_k-u_0)\,dx + \int_{\Omega_{\varepsilon,N}} \big(F(x,u_k,Xu_0) - F(x,u_0,Xu_0)\big)\,dx.$$

Let $k \to \infty$. Since $F_{\xi_j}(x,u_k,Xu_0)$ is bounded and converges uniformly in $\Omega_{\varepsilon,N}$ and $X_j(u_k-u_0) \to 0$ weakly in $L^p(\Omega)$, we deduce

$$\int_{\Omega_{\varepsilon,N}} \sum_{j=1}^{m} F_{\xi_j}(x,u_k,Xu_0)X_j(u_k-u_0)\,dx \longrightarrow 0 \ \text{ as } \ k \to \infty.$$

On the other hand, $\{u_k\}$ converges uniformly to u_0 in Ω_ε,

$$\int_{\Omega_{\varepsilon,N}} \big(F(x,u_k,Xu_0) - F(x,u_0,Xu_0)\big)\,dx \longrightarrow 0 \ \text{ as } \ k \to \infty.$$

Therefore, we have obtained, for any $\varepsilon > 0$ and N :

$$\lim_{k\to\infty} \int_{\Omega_{\varepsilon,N}} F(x,u_k,Xu_k)\,dx \geq \int_{\Omega_{\varepsilon,N}} F(x,u_0,Xu_0)\,dx.$$

That means $d \geq \int_{\Omega_{\varepsilon,N}} F(x,u_0,Xu_0)\,dx$. Now, let $\varepsilon \to 0$ and $N \to \infty$: we have proved the theorem.

REMARK 8. — For the non-homogeneous Dirichlet problem, we have to study the trace of $M^{1,p}(\Omega)$ functions. If $\partial\Omega$ is C^∞ and non-characteristic for X, we know from [2] that for $u \in M^{1,p}(\Omega)$, the function $u|_{\partial\Omega}$ is measurable in $\partial\Omega$. In this case, we can consider in a similar way the minimizing problem of $I(u)$ in $\mathcal{M} = \{v \in M_0^{1,p}(\Omega) \mid v - \varphi \in M^{1,p}(\Omega)\}$ for some function $\varphi \in M^{1,p}(\Omega)$ which take a prescribed value on the boundary $\partial\Omega$.

4. Estimation of Ess sup|u| of weak solutions

In the preceeding section, we have obtained a weak solution in $M^{1,p}$ for the variational problem $I(u)$. We now study the regularity of this solution. For simplifying the notations, we suppose that $p = 2$ and consider non-homogeneous problems. We have :

THEOREM 9. — *Let X satisfy the condition*(H) *and the assumption* (2) *of* LEMMA 5. *Assume that $I(v)$ attains a minimum in $\mathcal{M} = \{v \in M^1(\Omega) \mid v - \varphi \in M_0^1(\Omega)\}$, at $u \in M^1(\Omega)$ and* $\operatorname{Ess\,sup}_{\partial\Omega} |u| \leq M_0$. *Let the function $F(x, u, \xi)$ satisfy, for $|u| \geq M_0$, the following conditions :*

$$F(x,u,\xi) \geq \lambda|\xi|^2 - \mu|u|^\alpha - |u|^2\varphi(x), \tag{4.1}$$

$$F(x,u,0) \geq \mu|u|^\alpha + |u|^2\varphi(x), \tag{4.1}$$

where $\lambda > 0$, $\mu \geq 0$, $\alpha \in]2, \overline{2}[$, $\varphi \in L^q(\Omega)$, $q > \frac{1}{2}nr$ and $\overline{2} = 2nr/(nr-2)$. Then we have

$$\operatorname*{Ess\,sup}_{\Omega} |u(x)| < C \tag{4.3}$$

where C depends on n, r, λ, μ, α, M_0, $||\varphi||_{L^2}$, $||Xu||_{L^2}$ and $\operatorname{Mes}\Omega$.

Proof. — We take $A_k = \{x \in \Omega \mid u(x) > k\}$, and prove the majoration $\operatorname{Ess\,sup}_\Omega u(x) \leq C$. (The proof of $\operatorname{Ess\,sup}_\Omega(-u(x)) \leq C$ is similar, using in this case the set $\widehat{A}'_k = \{x \in \Omega \mid -u(x) > k\}$.) Setting

$$u^k(x) = \begin{cases} u(x) & \text{if } x \in \Omega \setminus A_k, \\ k & \text{if } x \in A_k, \end{cases}$$

then, for $k \geq M_0$ and from the LEMMA 6, we get $u^k = u - u^{(k)} \in \mathcal{M}$, and $I(u) = d = \inf I(v)$:

$$\begin{aligned} \int_\Omega F(x,u,Xu)\,dx &\leq \int_\Omega F(x,u^k,Xu^k)\,dx \\ &= \int_{A_k} F(x,k,0)\,dx + \int_{\Omega\setminus A_k} F(x,u,Xu)\,dx. \end{aligned}$$

Hence

$$\int_{A_k} F(x,u,Xu)\,dx \leq \int_{A_k} F(x,k,0)\,dx.$$

It follows from the conditions (4.1) and (4.2) that

$$\lambda\int_{A_k} |Xu|^2 dx - \int_{A_k} \big(\mu|u|^\alpha + |u|^2\varphi(x)\big)\,dx \leq \int_{A_k} \big(\mu k^\alpha + k^2\varphi(x)\big)\,dx.$$

Taking $k \geq 1$, we have

$$\int_{A_k} |Xu|^2 dx \leq \frac{2}{\lambda} \int_{A_k} \big(\mu |u|^\alpha + |u|^2 \varphi(x)\big)\, dx.$$

From the conditions of the THEOREM, we also have $q > \frac{1}{2}nr$, $\overline{2}/(\alpha-2) > \frac{1}{2}nr$ and $||u||_{L^{\overline{2}}}^{\alpha-2} = ||u^{\alpha-2}||_{L^{\overline{2}/(\alpha-2)}}$, hence

$$\begin{aligned}\int_{A_k} |u|^\alpha dx &\leq ||u^{\alpha-2}||_{L^{\overline{2}/(\alpha-2)}(A_k)} ||u^2||_{L^{\overline{2}/(\overline{2}+2-\alpha)}(A_k)} \\ &\leq 2||u||^{\alpha-2}_{L^{\overline{2}}(A_k)} \big(||u-k||^2_{L^{\ell_1}(A_k)} + k^2 \operatorname{Mes}^{2/\ell_1} A_k\big)\end{aligned}$$

$$\begin{aligned}\int_{A_k} |u(x)|^2 \varphi(x)\, dx &\leq ||\varphi||_{L^q(A_k)} ||u^2||_{L^{q'}(A_k)} \\ &\leq 2||\varphi||_{L^q(A_k)} \big(||u-k||^2_{L^{\ell_2}(A_k)} + k^2 \operatorname{Mes}^{2/\ell_2} A_k\big),\end{aligned}$$

where $\ell_1 = 2 \times \overline{2}/(\overline{2}+2-\alpha)$, $\ell_2 = 2q/(q-1)$, $\ell_1, \ell_2 \in]2, \overline{2}[$. Therefore, we have obtained, for all $k \geq \max\{M_0, 1\}$, that

$$(4.5) \qquad \int_{A_k} |Xu|^2 dx \leq \sum_{i=1,2} C_i \big(||u-k||^2_{L^{\ell_i}(A_k)} + k^2 (\operatorname{Mes} A_k)^{2/\ell_i}\big),$$

where $C_1 = 4\mu/\lambda ||u||^{\alpha-2}_{L^{\overline{2}}(\Omega)}$, $C_2 = 4\lambda^{-1}||\varphi||$ and $2/\ell_i = 1 - 2/(nr) + \varepsilon_i$ with $\varepsilon_1 = 2/(nr) - (\alpha-2)/\overline{2} > 0$ and $\varepsilon_2 = 2/(nr) - 1/q > 0$.

Hence, THEOREM 9 can be proved with the following lemma :

LEMMA 10. — *Let $u \in M^{1,p}(\Omega)$, with $1 < p \leq nr$, and $\operatorname{Ess\,sup}_{\partial\Omega} u(x) \leq k_0 < +\infty$. Assume that for any $k \geq k_0$, we have*

$$(4.6) \qquad \begin{aligned}\int_{A_k} |Xu|^p dx \leq \gamma \sum_{j=1}^{N_1} ||u-k||^p_{L^{\ell_j}(A_k)} \\ + \gamma \sum_{j=1}^{N_2} k^{\alpha_j} (\operatorname{Mes} A_k)^{1-p/(nr)+\xi_j},\end{aligned}$$

where $\ell_j < \overline{p}$, $\varepsilon_j > 0$ and $p \leq \alpha_j < \varepsilon_j \overline{p} + p$. Then, we have

$$(4.7) \qquad \operatorname*{Ess\,sup}_{\Omega} u \leq C\big(n, r, p, k, \gamma, \ell_j, \alpha_j, \varepsilon_j, ||u||_{L^{\overline{p}}}\big).$$

Proof. — First, it follows from the Hölder inequality

$$||u-k||_{L^{\ell_j}(A_k)} \leq (\operatorname{Mes} A_k)^{1/\ell_j - 1/\overline{p}} ||u-k||_{L^{\overline{p}}(A_k)}.$$

Using point (2) of LEMMA 5 and, as in the proof of LEMMA 6, $u^{(k)} \in M_0^{1,p}(\Omega)$, we have

$$||u-k||_{L^{\ell_j}(A_k)} \leq C_0(\mathrm{Mes}\, A_k)^{1/\ell_j - 1/\overline{p}}||Xu||_{L^{\overline{p}}(A_k)}.$$

On the other hand,

$$k(\mathrm{Mes}\, A_k)^{1/\overline{p}} \leq L \equiv ||u||_{L^{\overline{p}}(A_k)}.$$

Thus, for

$$k \geq k' = \max\Big\{k_0, L(2C_0^p N_1\gamma)^{\ell_j/(p(\overline{p}-\ell_j))}\Big\}, \tag{4.8}$$

we have

$$\begin{aligned}\gamma\sum_{j=1}^{N_1}||u-k||^p_{L^{\ell_j}(A_k)} &\leq \gamma\sum_{j=1}^{N_1} C_0(\mathrm{Mes}\, A_k)^{1/\ell_j-1/\overline{p}}||Xu||^p_{L^p(A_k)}\\ &\leq ||Xu||^p_{L^p(A_k)}\gamma\sum_{j=1}^{N_1} C_0^p(LK^{-1})^{\overline{p}(\overline{p}-\ell_j)/(\ell_j\overline{p})p}\\ &\leq \frac{1}{2}||Xu||^p_{L^p(A_k)}.\end{aligned}$$

Taking $\delta = \min\{\varepsilon_j - (\alpha_j - p)/\overline{p} > 0$, we have obtained, for $k \geq k'$, that

$$\int_{A_k}|Xu|^p dx \leq \gamma_1 k^p(\mathrm{Mes}\, A_k)^{1-p/(nr)+\delta}, \tag{4.9}$$

where $\gamma_1 = 2\gamma\sum_{j=1}^{N_2} L^{\alpha_j - p}$. Now, for $u \in M_0^{1,p}(\Omega)$, we use the Hölder inequality and the LEMMA 5 to obtain

$$\begin{aligned}\int_{A_k}(u-k)\,dx &\leq ||u-k||_{L^{\overline{p}}(A_k)}(\mathrm{Mes}\, A_k)^{1-1/\overline{p}}\\ &\leq C_1\Big(\int_{A_k}|Xu|^p dx)^{1/p}(\mathrm{Mes}\, A_k)^{1-1/p+1/(nr)}\\ &\leq C_1\gamma_1^{1/p}k(\mathrm{Mes}\, A_k)^{1/p+\delta/p-1/(nr)+1-1/p+1/(nr)}.\end{aligned}$$

Hence, we have proved, for $k \geq k'$,

$$\int_{A_k}(u-k)\,dx \leq C_2 k(\mathrm{Mes}\, A_k)^{1+\delta/p} \tag{4.10}$$

Therefore, the integrable function u satisfies the conditions of lemma 5.1 of chapter 2 of [13]. We have proved this LEMMA and hence THEOREM 9.

For studying the regularity of weak solution of variational problem $I(u)$, we give some conditions on F such that the weak solution of variational problem is also the weak solution of its Euler's equations. Suppose $F(x,u,\zeta)$ satisfies

$$\begin{cases} \big|F(x,u,\zeta)\big| \le \mu\big(|\zeta|^2 + |u|^{\overline{2}} + \psi_0(x)\big), \\ \big|F_u(x,u,\zeta)\big| \le \mu\big(|\zeta|^{2/\overline{2}'} + |u|^{\overline{2}-1} + \psi_1(x)\big), \\ \big|F_{\zeta_j}(x,u,\zeta)\big| \le \mu\big(|\zeta| + |u|^{\widehat{2}/2} + \psi_2(x)\big), \end{cases} \tag{4.11}$$

where $\psi_0 \in L^1(\Omega)$, $\psi_1 \in L^{\overline{2}}(\Omega)$, $\psi_2 \in L^2(\Omega)$, with $\overline{2}' = \overline{2}/(\overline{2}-1)$ and $\widehat{2} < \overline{2} = 2nr/(nr-2)$.

Now, let $\eta \in M_0^1(\Omega)$ and $I(u) = d = \inf I(v)$. Then, for all $t \in \mathbb{R}$, we have

$$\varphi(t) = I(u + t\eta) \ge I(u).$$

Hence, we have formally

$$\frac{d\varphi(t)}{dt} = \int_\Omega \Big(\sum_{j=1}^m F_{\xi_j}(x, u + t\eta, Xu + tX\eta)X_j\eta + F_u(x, u+t\eta, Xu + tX\eta)\eta\Big)\,dx,$$

and from (4.11)

$$\begin{aligned} |F_{\xi_j}X_j\eta| &\le \frac{1}{2}|F_{\xi_j}|^2 + \frac{1}{2}|X_j\eta|^2 \\ &\le C\big(|\psi_2|^2 + |u|^{\overline{2}} + |\eta|^{\overline{2}} + |Xu|^2 + |X\eta|^2\big), \\ |F_u\eta| &\le \frac{1}{\overline{2}'}|F_u|^{\overline{2}'} + \frac{1}{\overline{p}}|\eta|^{\overline{2}} \\ &\le C\big(|\psi_1|^{\overline{2}} + |u|^{\overline{2}'} + |\eta|^{\overline{2}} + |Xu|^2 + |X\eta|^2\big). \end{aligned}$$

Since $u - \varphi$, $\eta \in M_0^1(\Omega) \subset L^{\overline{2}}(\Omega)$, the integrand is finite. Hence the derivative $d\varphi(t)/dt$ exists and is continuous in t. On the other hand, $\varphi(t)$ takes its minimum on $t = 0$, then

$$\begin{aligned} \frac{d\varphi(t)}{dt}\Big|_{t=0} &= \delta I(u,\eta) \\ &= \int_\Omega \Big(\sum_{j=1}^m F_{\xi_j}(x,u,Xu)X_j\eta + F_u(X,u,Xu)\eta\Big)\,dx = 0. \end{aligned}$$

We have thus proved that u is a weak solution of the following Euler's equation

$$(4.13) \qquad \sum_{j=1}^{m} X_j^* F_{\xi_j}(X, u, Xu) + F_u(X, u, Xu) = 0.$$

5. Local properties of weak solutions

In order to study the regularity of weak solutions for the Euler's equation (4.13), consider the following general quasilinear equation

$$Qu \equiv \sum_{j=1}^{m} X_j^* A_j(x, u, Xu) + B(x, u, Xu) = 0$$

in $\Omega \subset\subset M$. Suppose that Ω is bouded and connected, and that Ω can be covered by balls defined by the metric $\rho(x, y)$, i.e. $\Omega = \bigcup B_R(x)$. We also assume that the function $A_j(x, u, \xi)$, for $j = 1, \dots, m$, and $B(x, u, \xi)$ are of class $C^\infty(\overline{\Omega} \times \mathbb{R} \times \mathbb{R}^m)$, and satisfy the following structure conditions. For all $(x, u, \xi) \in \Omega \times \mathbb{R} \times \mathbb{R}^m$:

$$(5.1) \qquad \begin{cases} \sum_{j=1}^m A_j(x, u, \xi)\xi_j \geq |\xi|^2 - g(x)^2, \\ |A_j(x, u, \xi)| \leq \Lambda(|\xi| + g(x)), \\ |B(x, u, \xi)| \leq \Lambda(|\xi|^2 + f(x)), \end{cases}$$

where f, $g \in C^0(\overline{\Omega})$ and ≥ 0, Λ is a constant.

We shall prove the following local estimate :

THEOREM 11. — *Let the operator Q satisfy the structure conditions (5.1). Let $u \in M^1(\Omega)$ satisfy $Qu \geq 0$ in B_R for some $R > 0$ and $u \geq 0$ in B_R. For some $q \in]nr, n(r+1)[$, we set*

$$(5.2) \qquad K = ||f + g^2||_{L^{\tilde{q}}} + ||g||_{L^{q'}}, \quad F_0 = KR^{1-nr/q},$$

with $\tilde{q} = nq/(2n - q + nr) > 1$ and $q' = nq/(nr + n - q) > 1$. Then, for all $p > 0$ and $\frac{1}{2} \leq \theta < 1$, we have

$$(5.3) \qquad \sup_{B_{\theta R}} \tilde{u} \leq C\big((1-\theta)R\big)^{-\alpha/p} |B_R|^{-1/p} ||\tilde{u}||_{L^p(B_R)},$$

where $\tilde{u} = u + F_0$, $B_R = B(x_0, R) \subset \Omega$, $C = C(n, r, \Lambda, p, q, ||u||_L^\infty)$ and $\alpha = \alpha(n, r) > 0$.

Proof. — We first assume that $p \geq 2$ and choose the test function $\varphi = \zeta^2 \tilde{u}^{2p-1} e^{\Lambda u} \in M_0^1(B_R)$, where $\zeta \in C_0^\infty(B_R)$. Using the structure inequality (5.1), we have

$$\begin{aligned}\int_{B_R} \sum_{j=1}^m A_j(x,u,Xu) X_j\big(\zeta^2 \tilde{u}^{2p-1} e^{\Lambda u}\big)\,dx &\\ \leq -\int_{B_R} B(x,u,Xu)\zeta^2\tilde{u}^{2p-1}e^{\Lambda u}dx &\\ \leq \Lambda \int_{B_R} \big(|X\tilde{u}|^2 + f\big)\zeta^2 \tilde{u}^{2p-1} e^{\Lambda u} dx &\end{aligned}$$

and hence from (5.1), we have

$$\begin{aligned}(2p-1)\int_{B_R} &\zeta^2\tilde{u}^{2p-2}e^{\Lambda u}|X\tilde{u}|^2 dx\\ &\leq (2p-1)\int_{B_R}\zeta^2\tilde{u}^{2p-2}e^{\Lambda u}g^2dx + \Lambda\int_{B_R}(g^2+f)\zeta^2\tilde{u}^{2p-2}e^{\Lambda u}dx\\ &\qquad + 2\Lambda\int_{B_R}\big(|X\tilde{u}|+g\big)\zeta^2|X\zeta|\tilde{u}^{2p-1}e^{\Lambda u}dx.\end{aligned}$$

Since $\tilde{u} \geq F_0$ and $||u||_{L^\infty} < +\infty$, we have

$$\begin{aligned}(2p-1)\int_{B_R} &\zeta^2\tilde{u}^{2p-2}e^{\Lambda u}|X\tilde{u}|^2dx\\ &\leq (2p-1)C_1\int_{B_R}\zeta^2\tilde{u}^{2p-2}|g|^2dx + \Lambda C_1\int_{B_R}\zeta^2\tilde{u}^{2p-1}\big(|g|^2+f\big)dx\\ &\qquad + (2p-1)\varepsilon\int_{B_R}\zeta^2\tilde{u}^{2p-2}|X\tilde{u}|^2dx\\ &\qquad\quad + \frac{C_1C_\varepsilon}{2p-1}\int_{B_R}|X\zeta|^2\tilde{u}^{2p}dx + 2\Lambda\int_{B_R} g\zeta|X\zeta|\tilde{u}^{2p-1}dx,\end{aligned}$$

where $C_1 = \max_\Omega e^{2\Lambda u}$. Denote $\nu = \tilde{u}^p$ and let $\varepsilon = \frac{1}{2}$. We have

$$\frac{1}{p}\int_{B_R}\zeta^2|X\nu|^2dx \leq Cp\int_{B_R}h(X)\zeta^2\nu^2dx + \frac{C}{p}\int_{B_R}|X\zeta|^2\nu^2dx,$$

where $h(x) = (g^2+f)/F_0 + g^2/F_0^2$. The choice of $\tilde{q}$ and q' implies $||h||_{L^{q/2}} \leq C(n)$. Using the Hölder inequality, we have

$$\int_{B_R} h\zeta^2\nu^2dx \leq ||h||_{L^{q/2}}||\zeta^2\nu^2||_{L^{q/(q-2)}},$$

and the interpolation inequality for L^p norms

$$||u||_{L^q} \leq \varepsilon ||u||_{L^\ell} + \varepsilon^{-\mu} ||u||_L,$$

where $p \leq q \leq \ell$, $\mu = (1/p - 1/q)/(1/q - 1/\ell)$. We have, with C dependant on n, q, r and $|\Omega|$,

$$\begin{aligned} ||\zeta^2\nu^2||_{L^{q/(q-2)}} &= ||\zeta\nu||^2_{L^{2q/(q-2)}} \\ &\leq \varepsilon ||\zeta\nu||^2_{L^2} + \varepsilon^{-nr/(q-nr)} ||\zeta\nu||^2_{L^2}. \end{aligned}$$

Hence

$$\int_{B_R} h\zeta^2\nu^2 dx \leq C\varepsilon ||X(\zeta\nu)||^2_{L^2} + C\varepsilon^{-nr/(q-nr)} ||\zeta\nu||^2_{L^2}.$$

Let $\varepsilon = 1/(2C)p^{-2}$. Then

$$\int_{B_R} \left|X(\zeta\nu)\right|^2 dx \leq C\big(p^{2nr/(q-nr)+2} + ||X\zeta||^2_{L^\infty}\big) \int_{B_R} \nu^2 dx.$$

Using LEMMA 5, we obtain

$$\Big(\int_{B_R} |\zeta\nu|^{\bar{2}} dx\Big)^{2/\bar{2}} \leq C\big(p^{2nr/(q-nr)+2} + ||X\zeta||^2_{L^\infty}\big) \int_{B_R} \nu^2 dx.$$

Now, take $R_k = R(\theta + (1-\theta)/(2k))$ for $k = 0, 1, 2, \ldots$ with $\theta \in]0, 1[$. Using the geometry of the metric ρ (see [11]), we can choose a cut-off function $\zeta_k \in C_0^\infty(B_{R_k})$ such that $0 \leq \zeta_k \leq 1$ and $\zeta_k(x) = 1$ in $B_{R_{k+1}}$ and satisfying $|X\zeta_k| \leq 2^k C/((1-\theta)R)$. Thus

$$\begin{aligned} &\Big(\int_{B_{R_{k+1}}} \tilde{u}^{2npr/(nr-2)} dx\Big)^{(nr-2)/(nr)} \\ &\qquad\qquad \leq C\Big(p^{2nr/(q-nr)+2} + \frac{4k}{(1-\theta)^2R^2}\Big) \int_{B_{R_k}} u^{2p} dx. \end{aligned}$$

Take $p_k = 2p(nr/(nr-2))^k$, with $k \geq p$, and replace p by $\frac{1}{2}p_k$. Then

$$\begin{aligned} &||\tilde{u}||_{L^{p_{k+1}}(B_{R_{k+1}})} \\ &\quad \leq C\Big[\Big(p\big(nr/(nr-2)\big)^k\Big)^{2nr/(q-nr)+2} \\ &\qquad\qquad + \frac{4k}{(1-\theta)^2R^2}\Big]^{1/p_k} ||\tilde{u}||_{L^{p_k}}(B_{R_k}) \\ &\quad \leq Ca^{k/p_k}\big((1-\theta)R\big)^{-2/p_k} ||\tilde{u}||_{L^{p_k}(B_{R_k})}, \end{aligned}$$

where $C = C(n,r,q,p,||u||_{L^\infty})$ and $a = 2\big(nr/(nr-2)\big)^{2nr/(q-nr)+2}+8$. Hence

$$||\tilde{u}||_{L^{p_{k+1}}(B_{R_{k+1}})} \le Ca^{\sum_{j=0}^{\infty} j/p_j}\big((1-\theta)R\big)^{-\sum_{j=0}^{\infty} 2/p_j}||\tilde{u}||_{L^{2p}(B_R)},$$

where

$$\sum_{j=0}^{\infty}\frac{j}{p_j} = C(n,p,r), \quad \sum_{j=0}^{\infty}\frac{2}{p_j} = \frac{1}{p}\sum_{j=0}^{\infty}\Big(1-\frac{2}{nr}\Big)^j = \frac{\alpha}{p},$$

and $\alpha = \alpha(n,r) > 0$. Since $|B_R| < 1$, (for $R > 0$ small enough), we have

$$\begin{aligned}|B_{R_{k+1}}|^{-1/p_{k+1}}|B_R|^{1/(2p)} &\le |B_{R_{k+1}}|^{-1/p_{k+1}}|B_R|^{1/p_{k+1}}\\ &= \Big(\frac{|B_R|}{|B_{R_{k+1}}|}\Big)^{1/p_{k+1}} = C(p,r)^{1/p_{k+1}} \le 2\end{aligned}$$

when k is large enough. Hence

$$\begin{aligned}&|B_{R_{k+1}}|^{-1/p_{k+1}}||\tilde{u}||_{L^{p_{k+1}}(B_{R_{k+1}})}\\ &\qquad\qquad \le C\big((1-\theta)R\big)^{-\alpha/p}|B_R|^{-1/(2p)}||\tilde{u}||_{L^{2p}(B_R)}.\end{aligned}$$

Let $k \to \infty$. We have proved, for all $p \ge 2$,

$$\sup_{B_{\theta R}}\tilde{u} \le C\big((1-\theta)R\big)^{-\alpha/p}|B_R|^{-1/2p}||\tilde{u}||_{L^{2p}(B_R)}.$$

Further, for $0 < p < 2$, we have

$$\begin{aligned}\sup_{B_{\theta R}}\tilde{u} &\le C\big((1-\theta)R\big)^{-\alpha/2}|B_R|^{-1/4}||\tilde{u}||_{L^4(B_R)}\\ &\le C\big((1-\theta)R\big)^{-\alpha/2}|B_R|^{-1/4}(\sup_{B_R}\tilde{u})^{1-2p/4}\Big(\int_{B_R}\tilde{u}^{2p}dx\Big)^{1/4}\\ &\le \frac{1}{2}\sup_{B_R}\tilde{u} + 2C^{2/p}\big((1-\theta)R\big)^{-\alpha/p}|B_R|^{-1/2p}||\tilde{u}||_{L^{2p}(B_R)}.\end{aligned}$$

Now, fix p and set $\varphi(s) = \sup_{B_R}\tilde{u}$. For $\frac{1}{2} \le s < t < 1$, using the results of [7], $|B_{tR}| \ge |B_{1/2}R| \ge C|B_R|$, and by monotonicity, $||\tilde{u}||_{L^{2p}(B_{tR})} \le ||\tilde{u}||_{L^{2p}(B_R)}$. We have

$$\varphi(s) \le \frac{1}{2}\varphi(t) + CA(t-s)^{-2/p},$$

where $A = R^{-\alpha/p}|B_R|^{-1/(2p)}||\tilde{u}||_{L^{2p}(B_R)}$. Repeating this inequality yields, for any sequence $\theta \le p_0 < p_1 < \cdots < p_k < 1$, the inequality

$$\varphi(p_0) \le \frac{1}{2^k}\varphi(p_k) + CA\sum_{j=0}^{k}\frac{1}{2^j}(p_{j+1}-p_j)^{-2/p}.$$

By monotonicity, we have $\varphi(p_k) \le \varphi(1) \le ||\tilde{u}||_{L^\infty} < +\infty$. Let $k \to \infty$. We can prove

$$\varphi(\theta) \le \varphi(p_0) \le CA\sum_{j=0}^{\infty}\frac{1}{2^j}(p_{j+1}-p_j)^{-2/p}.$$

If we choose $p_0 = \theta$, $p_{j+1} = p_j + (1-\tau)\tau^i(1-\theta)$, $j = 0,1,2,\ldots$, with $1 > \tau > (\frac{1}{2})^{p/2}$, then the right hand side converges, which proves that

$$\varphi(\theta) = \sup_{B_{\theta R}} \tilde{u} \le C\big((1-\theta)R\big)^{-\alpha/p}|B_R|^{-1/(2p)}||\tilde{u}||_{L^{2p}(B_R)}.$$

THEOREM 11 is thus proved.

6. Harnack inequality and Hölder continuity

We first introduce two lemmas.

LEMMA 12. — *Under the assumptions of* THEOREM 11, *assume that* $u \in M^1(\Omega)$ *satisfies* $Qu = 0$ *in* Ω *and* $u \ge 0$ *in* B_R. *Then, for all* $0 < \theta < 1$ *and* $s > 0$, *we have*

$$(6.1)\qquad \begin{cases} \big|\{x \in B_{\theta R}\,;\ \log\tilde{u} > s + \beta_0\}\big| \le Cs^{-1}|B_{\theta R}|, \\ \big|\{x \in B_{\theta R}\,;\ \log\tilde{u} < -s + \beta_0\}\big| \le Cs^{-1}|B_{\theta R}|, \end{cases}$$

where $\beta_\theta = |B_{\theta R}|^{-1}\int_{B_{\theta R}}\log\tilde{u}\,dx$, *and* $B_{2R} \subset \Omega$.

Proof. — Use the test function $\varphi = \zeta^2\tilde{u}^{-1}e^{-\Lambda u}$, where $\zeta \in C_0^\infty(B_{\sigma R})$ for some $\sigma > 1$, with $0 \le \zeta \le 1$ and $\zeta(x) = 1$ in B_R and with $|X\zeta| \le C((\sigma-1)R)^{-1}$. Under the structures conditions (5.1), we have as in the proof of THEOREM 11

$$\begin{aligned}
&\int_{B_{\sigma R}}\zeta^2|X\log\tilde{u}|^2dx \\
&\qquad \le C\int_{B_{\sigma R}}\big(h(x) + |X\zeta|^2\big)\,dx \\
&\qquad \le C||h||_{L^{q/2}}|B_{\sigma R}|^{1-2/q} + ||X\zeta||^2_{L^\infty}|B_{\sigma R}| \\
&\qquad \le C|B_R|R^{-2nr/q} + C|B_R|\big((\sigma-1)R\big)^{-2} \\
&\qquad \le CR^{-2}|B_R|,
\end{aligned}$$

where $C = C(n, r, \Lambda, ||h||_{L^{q/2}}, (\sigma - 1)^{-1})$. Hence

$$\int_{B_R} |X \log \tilde{u}|^2 dx \leq CR^{-2}|B_R|.$$

Since $\log \tilde{u} \in M^1(\Omega)$, by LEMMA 5, we have

$$\int_{B_R} |\log \tilde{u} - \beta_1|^2 dx \leq C|B_R|$$

for all $0 < R \leq R_0$ with $B_{2R_0} \subset \Omega$. Denote

$$Q^-(s) = \{x \in B_{\theta R}\,;\ \log \tilde{u} < -s + \beta_\theta\}.$$

Then

$$C|B_{\theta R}| \geq \int_{B_{\theta R}} |\log \tilde{u} - \beta_\theta|\, dx \geq \int_{Q^-(s)} (\beta_\theta - \log \tilde{u}) \geq s\big|Q^-(s)\big|.$$

Another inequality is proved in a similar way. The proof of LEMMA 12 is completed.

LEMMA 13. — *Let $\{Q(t) : t \in [\frac{1}{2}, 1]\}$ be a family of domains Ω, satisfying $Q(t) \subset Q(\tau)$ for $0 < t < \tau$. Let $w > 0$ be a continuous function defined in a neighborhood of $Q(1)$ and such that*

$$\sup_{Q(t)} w^p \leq C_0(\tau - t)^{-a}\big|Q(1)\big|^{-1} \int_{Q(\tau)} w^p dx$$

for all $\frac{1}{2} \leq t < \tau \leq 1$ and $p \in]0, 1[$. Further, assume that

$$\big|\{x \in Q(1)\,;\ \log w > s\}\big| \leq C_0\big|Q(1)\big|s^{-1}$$

for all $s > 0$. Then there exists a constant $C = C(\alpha, r, C_0)$ such that

$$\sup_{Q(1/2)} w \leq C. \tag{6.2}$$

Since LEMMA 13 is just a modification of LEMMA 3 of [12], the proof is omitted. We can now prove the Harnack inequality for the weak solution of equation $Qu = 0$.

THEOREM 14. — *Under the assumptions of* THEOREM 11, *if* $u \in M^1(\Omega)$ *is a weak solution of the equation* $Qu = 0$ *and if* $u \geq 0$ *in* B_R, *then, for all* $0 < \theta < 1$, *we have*

$$\sup_{B_{\theta R}} \leq C\big\{\inf_{B_{\theta R}} u + F_0\big\}, \tag{6.3}$$

where F_0 *is defined by* (5.2) *and* $C = C(n, r, \Lambda, q, ||u||_{L^\infty})$.

Proof. — Without lose of generality, we can assume that $u \geq k > C$ in B_R. By LEMMA 12, if $Q(t) = B_{tR}$, with $\frac{1}{2} \leq t \leq 1$ and $w = e^{-\beta}\tilde{u}$ or $w = e^{\beta}\tilde{u}^{-1}$, then THEOREM 11 and LEMMA 12 give that the function w and the family of domains $Q(t)$ satisfy the conditions of LEMMA 13. Then, (6.2) implies

$$\sup_{B_{R/2}} \tilde{u} \leq C^2 \inf_{B_{R/2}} \tilde{u}.$$

THEOREM 14 is proved.

REMARK 15. — From (6.3), we have also the following inequality

$$\inf_{B_{\theta R}} \tilde{u} \geq c\Big(|B_R|^{-1} \int_{B_R} |\tilde{u}|^2 dx\Big)^{1/2}. \tag{6.4}$$

From the Harnack inequality (6.3), we have now :

THEOREM 16. — *Under the assumptions of* THEOREM 11, *if* u *is an* $M^1(\Omega)$ *solution of the equation* $Qu = 0$ *in* Ω, *then* u *is locally continous in* Ω *and, for any ball* $B_{R_0} \subset \Omega$ *and* $0 < R \leq R_0$, *we have*

$$\operatorname*{osc}_{B_R} u \leq CR^\alpha\big(R_0^{-\alpha} \sup_{B_{R_0}} |u| + K\big), \tag{6.4}$$

where K *is defined by* (5.2), $C = C(n, r, \Lambda, q, R_0)$ *and* $\alpha = \alpha(n, r, q, \Lambda, R_0)$.

Proof. — Set $M(R) = \sup_{B_R} u$, $m(R) = \inf_{B_R} u$ and $\omega(R) = M(R) - m(R)$, $v = u - m(R)$. Then $v \geq 0$ is a weak solution in B_R of the following equation

$$\sum_{j=1}^m X_j^* \overline{A}_j(x, v, xv) + \overline{B}(x, v, xv) = 0,$$

where $\overline{A}_j(x, v, Xv) = A_j(x, u - m(R), Xu)$, $\overline{B} = B(x, u - m(R), Xu)$. Then, it is obvious that $\overline{A}$ and $\overline{B}$ satisfy the structure conditions (5.1). By THEOREM 14, we have, for all $0 < \theta < 1$

$$\sup_{B_{\theta R}} v \leq C\big\{\inf_{B_{\theta R}} v + F_0\big\},$$

that is

$$M(\theta R) - m(R) \leq C\big(m(\theta R) - m(R) + F_0\big).$$

In the same way, we have for $\tilde{v} = M(R) - u$,

$$\sup_{B_{\theta R}} \tilde{v} \leq C\big\{ \inf_{B_{\theta R}} \tilde{v} + F_0 \big\}$$

that is

$$M(R) - m(\theta R) \leq C\big(M(R) - M(\theta R) + F_0\big).$$

Thus we have

$$\omega(\theta R) \leq \frac{C-1}{C+1}\omega(R) + \frac{2CF_0}{C+1}.$$

Since $0 < \theta$, $(C-1)/(C+1) < 1$. Using lemma 8.23 of [6], we have proved (6.4).

From Hörmander condition (H) for (X), for any $\Omega' \subset\subset \Omega$, there exists a constant $C > 0$ sucht that, for any $x^0 \in \Omega'$, any $B_{R_0}(x^0) \subset \Omega$ and $0 < R \leq R_0$, we have

$$B_R(x^0) \supset A(x^0, CR^r) = \big\{ x \in \Omega\,;\ |x - x^0| < CR^r \big\}. \tag{6.5}$$

The proof can be found in [4, 13]. Hence, from (6.4), we have obtained

$$\operatorname*{osc}_{A(x^0, CR^r)} u \leq CR^\alpha \big(R_0^{-\alpha} \sup_{B_{R_0}} |u| + K\big). \tag{6.6}$$

From this inequality, we have proved the following interior Hölder estimates for the weak solutions of quasilinear equation.

THEOREM 17. — *Under the assumptions of* THEOREM 11, *if* $u \in M^1(\Omega)$ *satisfies* $Qu = 0$ *in* Ω, *then, for any* $\Omega' \subset\subset \Omega$, *we have the following estimate*

$$||u||_{C^\delta(\Omega')} \leq C\big(\sup_{\Omega} |u| + K\big), \tag{6.7}$$

where $C = C(n, r, \Lambda, q, d')$, $d' = (\Omega', \partial\Omega)$ *and* $\delta = \delta(n, r, \Lambda, q, d') > 0$.

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