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A CLASS OF SYMMETRIC SPACES

BY

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RÉSUMÉ. — Le but de ce travail est d'étudier une variété C^∞ donnée d'une structure projective $\mathcal{P}$ et d'un système de symétries qui laissent $\mathcal{P}$ invariante. Avec une hypothèse supplémentaire d'homogénéité projective, l'auteur classifie tous ces espaces et donne une interprétation géométrique de l'inessentialité de la structure projective symétrique en utilisant des techniques de géométrie affine différentielle.

ABSTRACT. — The aim of this work is to study the situation of a C^∞-manifold endowed with a projective structure $\mathcal{P}$ and with a system of symmetries leaving $\mathcal{P}$ invariant. Under the additional hypothesis of projective homogeneity, the author classifies all such spaces and exhibits a geometrical interpretation for inessentiality of the symmetric projective structure using techniques of affine differential geometry.

Introduction

In a previous paper [8] the author has introduced the class of projectively symmetric spaces : let (M, ∇) be a connected C^∞ manifold with a linear torsion free connection ∇ on its tangent bundle; (M, ∇) is said to be *projectively symmetric* if for every point x of M there is an involutorial projective transformation of M fixing x and whose differential at s is $-\operatorname{Id}$. The assignement of the symmetry s_x at each point x of M is assumed to be not even continuous.

In this work the author gives necessary and sufficient conditions for a projectively symmetric and projectively homogeneous space to be inessential (*i.e.* projectively equivalent to an affine symmetric space, *see* paragraph 1). For complete Riemannian manifolds (M, g) of dimension n ($n \geq 3$) that are projectively symmetric and projective homogeneous (it is shown with an example that projective homogeneity is not implied), the author proves that such spaces are either inessential or isometric to the sphere $S^n(r)$ of radius r or to the projective space $S^n(r)/\pm\operatorname{Id}$ with some choice of symmetries. Some interesting cases are considered, when (M, g)

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is affinely homogeneous or analytic : under these hypotheses the author proves that (M, g) is either a Riemannian symmetric space (that is all projective symmetries are isometries) or (M, g) is isometric to the sphere $S^n(r)$ of radius r or to the projective space $S^n(r)/\pm \mathrm{Id}$ with some choice of symmetries. Finally the case of a smooth distribution of symmetries is considered and the previous results are rivisited from this point of view, exhibiting furthermore a geometrical interpretation for inessentiality.

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1. Projectively symmetric spaces

Let M be a connected real C^∞ manifold whose tangent bundle TM is endowed with a linear torsion free connection ∇. We recall that a diffeomorphism s of M is said to be a projective transformation if s maps geodesics into geodesics when the parametrization is disregarded; equivalently s is projective if the pull back $s^*\nabla$ of the connection is projectively related to ∇, *i.e.* if there exists a global 1-form π on M such that

$$(1.1) \qquad s^*\nabla_X Y = \nabla_X Y + \pi(X)Y + \pi(Y)X \qquad \forall X, Y \in \mathcal{H}(M)$$

where $\mathcal{H}(M)$ denotes the Lie algebra of vector fields on M. If the form π vanishes identically on M, then s is said to be an affine transformation (see *e.g.* [1]). We remark here that all what follows could be made also in the case of a more general projective structure $\mathcal{P}$ defined on the manifold M, but we prefer to work with a projective equivalence class of globally defined linear connections and we shall denote with $[\nabla]$ the projective structure determined by the connection ∇.

Définition 1.1. — $(M, [\nabla])$ is said to be *projectively symmetric* if for every point x in M there exists a projective transformation s_x with the following properties :

(a) $s_x(x) = x$ and x is an isolated fixed point of s_x ;
(b) s_x is involutorial;
(c) $ds_x|_x = -\mathrm{Id}$.

It is easy to see that conditions (a) and (b) imply (c). Moreover we recall that a projective transformation is determined if we fix its value at a point, its differential and its second jet at this point (see [3]), hence a symmetry at x in M is not uniquely determined in general by the conditions (a), (b) (and (c)).

Example 1. — Here is the simplest example of a projectively symmetric space. We consider S^n, the unit sphere in the euclidean space $\mathbb{R}^{n+1}$, endowed with the standard metric g. Then the group of projective transformations of (S^n, g) is naturally identified with $G = \mathrm{GL}(n+1)/\mathbb{R}^+$ under the action $\mu : S^n \times G \to S^n$ given by

$$\mu([g], x) = \frac{gx}{\|gx\|} \qquad \forall x \in S^n \tag{1.2}$$

where $[g]$ denotes the class of an element g of $\mathrm{GL}(n+1)$ in G and $\|\cdot\|$ denotes the euclidiean norm. It is easy to see that the action of G on S^n is effective and C^∞ (for more details, see [4]). We now fix $q = (1, 0, \ldots, 0)$ in S^n; then every projective symmetry s_q turns out be of the form

$$s_q(x) = \mu([A], x) \qquad \forall x \in S^n \tag{1.3}$$

where

$$A = \begin{pmatrix} 1 & \alpha \\ 0 & -\mathrm{Id} \end{pmatrix} \tag{1.4}$$

with ${}^t\alpha \in \mathbb{R}^n$; moreover s_q is an affine transformation if and only if $\alpha = 0$. This simple example shows that no unique choice of s_q is possible.

Similar considerations hold also for the case of the real projective space $\mathbb{RP}^n$.

We now remark that every projective map carries geodesics, but does not preserve in general the affine parameter on the geodesics. However a projective transformation preserves the class of projective parameters (see [1]); nevertheless a projective involution with the properties (a), (b) (and (c)) does not carry necessarily a projective parameter p into $-p$, as it happens for the affine parameter in the classical theory of symmetric spaces. If we look at the previous example, we find that all the geodesics emanating from the point q and with projective parameter t are given by

$$\Gamma(t) = \big((1 + \|\xi\|^2 t^2)^{-1/2},\ t(1 + \|\xi\|^2 t^2)^{-1/2} \xi^i\big) \qquad t \in \mathbb{R},\ i = 1, \ldots, n$$

where $\xi \in \mathbb{R}^n \cong T_q S^n$. If we choose $\xi = (1, 0, \ldots, 0)$ and A is as in (1.2) with ${}^t\alpha = \xi$, then

$$\mu([A], \Gamma(t)) = \Gamma\big(-t(1 + \|\xi\|^2 t)^{-1}\big) \qquad \forall t > -\|\xi\|^{-2}.$$

This simple example shows how different the situation is from the affine case.

Remark 1. — It is clear how to construct examples of projectively but not affinely symmetric spaces; let M be a manifold with a linear torsionfree connection ∇ on its tangent bundle and suppose that (M, ∇) is affinely symmetric; fix a point q in M and denote by s_q the affine symmetry at q. Then if we choose a global 1-form π that is not s_q-invariant, it is enough to define a linear connection ∇^* via the formula

$$\nabla^*_X Y = \nabla_X Y + \pi(X)Y + \pi(Y)X \qquad \forall X, Y \in \mathcal{H}(M)$$

to obtain that M is projectively but not affinely symmetric with respect to the connection ∇^*. A general question is to find conditions under which, given a projectively symmetric space (M, ∇), there exists a projectively related connection ∇^* such that (M, ∇^*) is affinely symmetric; we shall call such spaces inessential projectively symmetric spaces (and essential otherwise). We now show that there is a choice of projective symmetries on the sphere S^n with respect to which S^n is essential : we denote with e the point ${}^t(1, 0, \dots, 0) \in \mathbb{R}^{n+1}$ and choose as symmetry s the transformation induced by an element A of $\mathrm{GL}(n+1)$ as in (1.4) with a fixed $\alpha \in \mathbb{R}^n$, while we put as symmetry σ at the point $-e$ the transformation induced by an element B of $\mathrm{GL}(n+1)$ as in (1.4) with $\alpha' \neq \alpha$. The other symmetries are allowed to be chosen arbitrarily. We claim that S^n with this choice of symmetries is essential : indeed if it were inessential, then we would have that

$$s \circ \sigma = s_{s(-e)} \circ s = \sigma \circ s$$

and this is not the case because $\alpha' \neq \alpha$. Deeply related to this is the question whether projectively symmetric spaces are necessarily projectively homogeneous, since the classical techniques used in the theory of symmetric spaces fail in this case. So we are going to show that there exist Riemannian spaces that are projectively symmetric but not projectively homogeneous.

Example 2. — Example of a Riemannian manifold that is projectively symmetric but not projectively homogeneous.

We consider the real projective space $\mathbb{RP}^n$ ($n \geq 3$) and two distinct point p and q: we claim that the manifold $M = \mathbb{RP}^n \backslash \{p, q\}$ endowed with the restriction of the standard metric of $\mathbb{RP}^n$ is projectively symmetric but not projectively homogeneous. Indeed it is clear that a projective automorphism of M is the restriction to M of a projective transformation of $\mathbb{RP}^n$; so there is no projective transformation carrying a point $x \in M$ belonging to the line ℓ through p and q to a point y not belonging to ℓ. We have now to show that M is projectively symmetric. We fix a point $x \in M$ and consider the canonical projection $\pi : \mathbb{R}^{n+1} \backslash \{0\} \to \mathbb{RP}^n$; pick now x^*, p^*, q^* points of $\mathbb{R}^{n+1}\backslash\{0\}$ that correspond to x, p, q respectively under the map π. We now distinguish two cases :

a) $x \notin \ell$: we put $e = {}^t(1,0,\ldots,0) \in \mathbb{R}^{n+1}\backslash\{0\}$, we may assume that p^* and q^* do not belong to the subspace spanned by e. Pick now any $g \in \mathrm{GL}(n+1)$ with $g(e) = x^*$. We claim that we can construct $s \in \mathrm{GL}(n+1)$ with the following properties :

1) $s^2 = \mathrm{Id}$

2) The map $s^\# : \mathbb{RP}^n \to \mathbb{RP}^n$ induced by s is a symmetry at x and $s^\#(p) = p, s^\#(q) = q$. We note that if we can find $A \in \mathrm{GL}(n+1)$ such that

1′) $A(e) = e$, $A^2 = \mathrm{Id}$ and $dA^\#|_{\pi(e)} = -\mathrm{Id}$

2′) $A(g^{-1}(p^*)) = -g^{-1}(p^*)$, $A(g^{-1}(q^*)) = -g^{-1}(q^*)$

then $s = g \circ A \circ g^{-1}$ will work. We can choose A as in (1.4) with some ${}^t\alpha \in \mathbb{R}^n$ so that 1′) is satisfied; if we write $g^{-1}(p^*) = u = (u_1, u')$, $g^{-1}(q^*) = v = (v_1, v')$ for some $u', v' \in \mathbb{R}^n$, we note that condition 2′) is equivalent to the system

$$\alpha \cdot u' = -2u_1$$
$$\alpha \cdot v' = -2v_1$$

that admits at least one solution α iff rank $(u,v) = \mathrm{rank}(u',v')$; but $\mathrm{rank}(u,v) = 2$ and $\mathrm{rank}(u',v') = 2$ because the plane spanned by the vectors $(0,u')$ and $(0,v')$ does not contain the vector e by hypothesis and we are done.

b) $x \in \ell$: with the same notations as above, we claim that we can construct $A \in \mathrm{GL}(n+1)$ such that

1″) $A(e) = e$, $A^2 = \mathrm{Id}$ and $dA^\#|_{\pi(e)} = -\mathrm{Id}$

2″) $A(u) \in \langle v\rangle$, where $\langle v\rangle$ denotes the subspace spanned by the vector v. If this choice is possible, then $s = g \circ A \circ g^{-1}$ will work (since $s^\#(p) = q$ and $s^\#(q) = p$ automatically because $s^2 = \mathrm{Id}$). We choose A as in (1.4) for some $\alpha \in \mathbb{R}^n$ so that 1″) is satisfied. Since $x \in \ell$, we can find $\lambda \in \mathbb{R}\backslash\{0\}$ with $u' \neq 0$ and $v' \neq 0$) and condition 2″) is equivalent to the equation

$$u_1 + \alpha \cdot u' = -\frac{1}{\lambda} v_1$$

that can be solved for α since $u' \neq 0$ and we are done. □

We note that the Riemannian space that we have just exhibited is not complete and we don't know of any projectively symmetric complete Riemannian space that is not projectively homogeneous.

Question. — Is any complete simply connected Riemannian manifold, that is projectively symmetric, necessarily projectively homogeneous ?

We now make an overview of the results that have been established by the author in his previous work [8] :

THEOREM 1. — *Let (M, ∇) be a projectively symmetric space which is p-hyperbolic (∇ is supposed to be complete). Then there exists a Riemannian metric g on M such that (M, g) becomes a Riemannian symmetric space with the following properties :*

a) *(M, g) is an Einstein space with negative curvature;*

b) *The Levi Civita connection of g is projectively related to ∇.*

THEOREM 2. — *Let (M, g) be a complete Riemannian manifold which is*

a) *locally symmetric;*

b) *projectively homogeneous;*

c) *properly projectively symmetric (i.e. there exists at least a symmetry that is a projective but not affine transformation), then (M, g) is projectively equivalent to the standard sphere S^n ($n = \dim M \geq 3$) or to the real projective space $\mathbb{RP}^n$.*

Remark. — In the proof of THEOREM 2 we show something more, that is that (M, g) is isometric to the sphere $S^n(r)$ of radius r in $\mathbb{R}^{n+1}$ or to the projective space $S^n(r)/\pm \mathrm{Id}$.

We have already observed that the choice of the projective symmetry is in general not unique. We now want to establish the following

PROPOSITION 1.1. — *Let $(M, [\nabla])$ be a projectively symmetric manifold of dimension $n > 2$; if there exist two different projective symmetries at a point q of M, then the projective curvature tensor W vanishes at q.*

Proof. — Let σ_1 and σ_2 be two different projective symmetries at a point q of M. Since σ_1 is involutorial we can find a projectively related affine connection ∇^* that is σ_1-invariant. As a consequence we have that $\nabla^* W = 0$ at the point q. We now denote by Φ the 1-form on M that corresponds to the projective automorphism $\sigma_2 = s$ of (M, ∇^*). Then if X, Y, Z, U are vector fields on M, we have

$$\begin{aligned}
&(\nabla^*_{sU} W)(sX, sY)sZ \\
&\quad = s\big(\nabla^*_U [W(X,Y)Z] + \Phi(U)W(X,Y)Z + \Phi\big(W(X,Y)Z\big)U\big) \\
&\qquad - s\big(W(\nabla^*_U X, Y)Z\big) - \Phi(U)s\big(W(X,Y)Z\big) - \Phi(X)s\big(W(U,Y)Z\big) \\
&\qquad - s\big(W(X, \nabla^*_U Y)Z\big) - \Phi(U)s\big(W(X,Y)Z\big) - \Phi(Y)s\big(W(X,U)Z\big) \\
&\qquad - s\big(W(X,Y)\nabla^*_U Z\big) - \Phi(U)s\big(W(X,Y)Z\big) - \Phi(Z)s\big(W(X,Y)U\big)
\end{aligned}$$

If we compute this at the point q we obtain that

$$\Phi\big(W(X,Y)Z\big)U = 2\Phi(U)W(X,Y)Z + \Phi(X)W(U,Y)Z + \Phi(Y)W(X,U)Z + \Phi(Z)W(X,Y)U \tag{1.5}$$

where all the operators are evaluated at q. If we now fix X, Y, Z tangent vectors at q and take the trace of (1.5) with respect to U, we obtain that

$$n\Phi\big(W(X,Y)Z\big) = 2\Phi\big(W(X,Y)Z\big)$$

because trace $\{V \to W(V,X)Y\} = 0$ and $W(X,Y)Z + W(Z,X)Y + W(Y,Z)X = 0$ (see *e.g.* [1]); so since $n > 2$ we have $\Phi(W(X,Y)Z) = 0$ and by (1.5)

$$2\Phi(U)W(X,Y)Z + \Phi(X)W(U,Y)Z + \Phi(Y)W(X,U)Z + \Phi(Z)W(X,Y)U = 0 \tag{1.6}$$

for every $X, Y, Z, U \in TM_q$. Since σ_1 and σ_2 are different, the 1-form Φ can not vanish identically on TM_q, so the subspace A defined as $A = \{X \in TM_q \mid \Phi(X) = 0\}$ has dimension $n-1$. Let us pick $V \in TM_q$ with $\Phi(V) = 1$. We now devide the proof into steps :

a) if $X, Y \in A$ and $U = V$ then by (1.6) we have that

$$2W(X,Y)Z + \Phi(Z)W(X,Y)V = 0$$

so if $Z \in A$ then $W(X,Y)Z = 0$ and if $Z = V$, then $W(X,Y) = 0$; since TM_q is direct sum of A and the subspace spanned by V, we have that $W(X,Y) = 0$ for every $X, Y \in A$.

b) If $X \in A$, $Y = V = U$, then by (1.6) we have that

$$3W(X,V)Z + \Phi(Z)W(X,V)V = 0$$

so, as above, if $Z \in A$ or if $Z = V$, we have that $W(X,V) = 0$. By a) and b) we obtain that the projective curvature tensor W vanishes at q. □

We now consider a connected real C^∞ manifold (M,∇) with a linear torsion free connection ∇ on its tangent bundle; we suppose that (M,∇) is projectively symmetric and projectively homogeneous and distinguish two cases :

a) there exists a point q at which two different projective symmetries can be defined. Then, by the last PROPOSITION, the projective Weyl tensor of (M,∇) vanishes at q, hence on M thanks to projective homogeneity.

b) At every point x of M the projective symmetry is uniquely determined. Then if $f \in P(M,\nabla)_q$ (= the isotropy subgroup of $P(M,\nabla)$ at q) we have

$$f \circ s_q \circ f^{-1} = s_q. \tag{1.7}$$

Let us denote with Φ_h the 1-form corresponding to an element h of $P(M,\nabla)$; if f and g are projective automorphisms an easy computation shows that

$$\Phi_{fg} = \Phi_g + g^*\Phi_f \tag{1.8}$$

hence by (1.7) and (1.8)

$$\Phi_\sigma = \Phi_{f^{-1}} + {f^{-1}}^*\Phi_\sigma + {f^{-1}}^*\sigma^*\Phi_f$$

where we have put $\sigma = s_q$; so if X is any vector field on M, using (1.8) we obtain that

$$\begin{aligned}\Phi_\sigma(X) &= \Phi_{f^{-1}}(X) + \Phi_\sigma(f^{-1}X) + \Phi_f(\sigma f^{-1}X)\\ &= -\Phi_f(f^{-1}X) + \Phi_\sigma(f^{-1}X) + \Phi_f(\sigma f^{-1}X).\end{aligned}$$

If we now put $X = fY$ where Y is a vector field on M, we have that

$$\Phi_\sigma(fY) = -\Phi_f(Y) + \Phi_\sigma(Y) + \Phi_f(\sigma Y)$$

and if we evaluate the last formula at the point q, we obtain

$$\Phi_f(Y) = \frac{1}{2}\big[\Phi_\sigma(Y) - \Phi_\sigma(fY)\big]. \tag{1.9}$$

So by (1.9) we have that the isotropy representation of $P(M,\nabla)_q$ in $\mathrm{GL}(n)$ given by the differential at q is faithful.

Moreover if p and q are points of M and $z = s_p(q)$, then

$$s_p \circ s_q = s_z \circ s_p \tag{1.10}$$

due to the fact that the symmetries are univoquely determined; by (1.10) and projective homogeneity, we deduce that the map

$$\begin{aligned}S : M \times M &\longrightarrow M\\ (x,y) &\longmapsto s_x(y)\end{aligned}$$

is differentiable. We now put $G = P(M,\nabla)^0$ and $H = G_q$ the isotropy subgroup of G at q, so that we can write $M = G/H$. The involution σ induces an involutorial automorphism of G, als denoted by σ, with the property that if $G_\sigma = \{g \in G \mid \sigma(g) = g\}$, then $G_\sigma \supset H \supset G_\sigma^0$; so (G, H, σ) is a symmetric space and we can find a linear connection ∇^* on TM that is invariant under the action of G and under all the symmetries (see *e.g.* [3]). If ∇ is the Levi-Civita connection of (M, g), we can ask when ∇ is projectively equivalent to ∇^* : if this were the case we could find a 1-form π with

$$\nabla^*_X Y = \nabla_X Y + \pi(X)Y + \pi(Y)X \qquad \forall X, Y \in \mathcal{H}(M).$$

By insisting that σ is a projective automorphism for (M, ∇) and leaves the connection ∇^* invariant, we find that

$$\Phi_\sigma(X) + \pi(\sigma X) = \pi(X) \qquad \forall X \in \mathcal{H}(M)$$

hence at q $\pi|_q(X) = \frac{1}{2}\Phi_\sigma|_q(X)$ $\forall X \in \mathcal{H}(M)$. So let us define a 1-form, still denoted by π, through the following formula

$$\pi|_X(X) = \frac{1}{2}\Phi_{s_x}|_x(x) \tag{1.11}$$

for $x \in M$ and X vector field on M. We note that π is a C^∞ 1-form because the map S is differentiable. We define ∇' a linear torsionfree connection projectively related to ∇ through the 1-form π and prove the following

PROPOSITION 1.2. — *The connection ∇' is invariant under all the symmetries of M.*

Proof. — Let s be any symmetry, say at a point q of M; the condition that ∇' is invariant under s is equivalent to

$$\pi(X) - \pi(sX) = \Phi_s(X) \qquad \forall X \in \mathcal{H}(M). \tag{1.12}$$

We verify (1.12) at a point p of M : if we call $s_p = \sigma$, we have to prove that, by (1.11),

$$\frac{1}{2}\Phi_\sigma|_p(X) - \pi|_{s(p)}(sX) = \Phi_s|_p(X) \qquad \forall X \in TM_p \tag{1.13}$$

so if we call $z = s(p)$ and $s' = s_z$ we have that (1.13) is

$$\frac{1}{2}\Phi_\sigma|_p(X) - \frac{1}{2}\Phi'_s|_z(sX) = \Phi_s|_p(X) \qquad \forall X \in TM_p. \tag{1.14}$$

But we have that $s \circ \sigma = s_q \circ s_p = s_z \circ s$ by (1.10), hence if we apply formula (1.6)

$$\Phi_\sigma(Y) + \Phi_s(\sigma Y) = \Phi_s(Y) + \Phi_{s'}(sY) \qquad \forall Y \in \mathcal{H}(M) \tag{1.15}$$

and if evaluate (1.15) at p we have that $\forall Y \in TM_p$

$$2\Phi_s|_p(Y) = \Phi|_p(Y) - \Phi_{s'}|_Z(sY) \tag{1.16}$$

and we are done. $\square$

We now prove that $\nabla' = \nabla^*$; indeed the difference $\nabla' - \nabla^*$ is a tensor field of type $(1,2)$ on M that is invariant under all symmetries, hence vanishes identically on M. So we have the following

THEOREM 1.1. — *Let (M, ∇) be a projectively symmetric and projectively homogeneous manifold of dimension $n \geq 3$. Then the following three conditions are equivalent*

1) *(M, ∇) is projectively equivalent to an affine symmetric space (M, ∇^*) with the same symmetries and such that $P^0(M, \nabla) = A^0(M, \nabla^*)$ (= the group of affine transformations of (M, ∇^*)).*

2) *The linear isotropy representation $\rho : P(M, \nabla)_q \to \mathrm{GL}(n, \mathbb{R})$ is faithful for every $q \in M$.*

3) *If f and g are projective symmetries at q ($q \in M$), then $f = g$.*

If one of these conditions is not fulfilled, then (M, ∇) is projectively flat. Moreover (M, ∇) is inessential if and only if the following condition is fulfilled

$$\forall p, q \in M \qquad s_p \circ s_q = s_z \circ s_p \qquad (\text{where } z = s_p(q)). \tag{1.17}$$

If this last condition does not hold, then (M, ∇) is projectively flat.

Proof. — 1) $\Rightarrow$ 3) and 2) $\Rightarrow$ 3) are trivial. Let us see that 3) $\Rightarrow$ 2) : if s is the symmetry at q, then for every $f \in P(M, \nabla)_q$, we have that $f \circ s \circ f^{-1} = s$ and our claim follows from formula (1.9). The implication 3) $\Rightarrow$ 1) follows from the arguments stated above. If one of these conditions is not fulfilled, then 3) does not hold and PROPOSITION 1.1 applies.

If (M, ∇) is inessential, it is clear that (1.17) holds. If (1.17) holds, then the map

$$S : M \times M \longrightarrow M$$
$$(x, y) \longmapsto s_X(y)$$

is differentiable (M is projective homogeneous) and we can construct the 1-form π given by (1.11) and the connection ∇' projectively related to

∇ through π. Then PROPOSITION 1.2 applies and we are done. We note that if condition (1.17) does not hold, then for some p and q points of M $s_p \circ s_q$ and s_z (where $z = s_p(q)$) are two different symmetries at z and PROPOSITION 1.1 applies. □

The following proposition, that was kindly suggested to me by Professor K. NOMIZU, gives a sufficient condition for inessentiality.

PROPOSITION 1.2. — *Let $(M, [\nabla])$ be a projectively symmetrix space and suppose that there is a volume form ω that is invariant by any symmetry up to a scalar factor, then $(M, [\nabla])$ is inessential.*

Proof. — We recall that there is one and only one torsion free linear connection ∇^* projectively related to ∇ and such that $\nabla^*\omega = 0$ (see [1]). If s is any summetry, then by assumption ω is $s^*\nabla^*$-parallel and so by uniqueness $s^*\nabla^* = \nabla^*$ and we are done. □

Remark. — The Riemannian symmetric space S^n with the standard metric satisfies condition (1.17) but not condition 1).

We now want to study with particular care the case when (M, ∇) is a complete Riemannian manifold (M, g) and ∇ is the Levi Civita connection. We denote the 1-form relating ∇ and ∇^* in THEOREM 1.1(1) by π and we call it the fundamental 1-form; we recall that π was defined as

$$\pi|_x(X) = \frac{1}{2}\Phi_{s_x}|_x(x) \qquad x \in M,\ X \in TM_x$$

In view of THEOREM 2 we can restate THEOREM 1.1 as follows :

THEOREM 1.2. — *Let (M, g) be a complete Riemannian manifold that is projectively symmetric and projectively homogeneous. Then one of the following statements is true :*

a) *(M, g) is isometric either to $S^n(r)$ for some r with some choice of the symmetries on $S^n(r)$ or to the real projective space $S^n(r)/\pm\mathrm{Id}$ with some choice of the symmetries.*

b) *(M, g) is projectively equivalent to an affine symmetric space (M, ∇^*) with the same symmetries and such that $P^0(M, g) = A^0(M, \nabla^*)$ (= the group of affine transformations of (M, ∇^*)).*

Moreover if b) holds and if the fundamental 1-form π is closed, then either $\nabla^* = \nabla$ or (M, g) is isometric either to $S^n(r)$ for some r or to $S^n(r)/\pm\mathrm{Id}$ with a C^∞ distribution of symmetries.

Proof. — Indeed THEOREM 1.1 applies and we distinguish two cases :

α) if condition 3) does not hold, then (M, g) is projectively flat, hence of constant curvature and so locally symmetric. Moreover by hypothesis at one point q there exist two different projective symmetries, hence one of them, say s, is not an affine transformation. Then (M,g) with the same symmetries at every point different from q and s at q satisfies the conditions of THEOREM 2 and we obtain case a).

β) if condition 3) is fulfilled then we have b). To prove the last assertion, we recall that if Ric and Ric* the Ricci tensors of ∇ and ∇^* respectively, then (see *e.g.* [1])

$$\mathrm{Ric}^*(X,Y) = Ric(X,Y) + \Pi(Y,X) - n\Pi(X,Y) \tag{1.18}$$

for all $X, Y \in \mathcal{H}(M)$, where

$$\Pi(X,Y) = (\nabla_X \pi)(Y) - \pi(X)\pi(Y)$$

so that Ric* is symmetric if and only if π is closed, *i.e.* $d\pi = 0$. By a Theorem of SINJUKOV [9] if a Riemannian space is in projective correspondence with a symmetric space whose Ricci tensor is symmetric, the correspondence is affine unless both spaces are projectively flat; so either π vanishes identically (and so $\nabla^* = \nabla$) or $\pi \neq 0$ and (M,g) is of constant curvature; by the definition of π, (M,g) becomes a properly projectively symmetric space and THEOREM 2 applies. The distribution of symmetries turns out to be automatically C^∞. □

COROLLARY 1.1. — *Let (M,g) be a complete Riemannian manifold of dimension $n \geq 3$, that is projectively symmetric and projectively homogeneous; then every geodesic of (M,g) is, up to parametrisation, an integral curve of a projective Killing vector. Moreover If the isotropy subgroup $P^0(M,g)_q$ at some point q of M is compact, then (M,g) is a Riemannian symmetric space.*

Proof. — The first assertion is clear; for the second one THEOREM 1.2 applies : case a) can not occur since $P^0(S^n)_q$ is not compact; hence we can look at case b) and since $A^0(M,\nabla^*)_q = P^0(M,g)_q$ is compact, the space (M,∇^*) is a Riemannian manifold (M,h) with ∇^* as Levi Civita connection; so the fundamental form is closed and by the previous theorem $\nabla = \nabla^*$, that is our conclusion. □

2. Classification of complete Riemannian manifolds that are projectively symmetric and affinely homogeneous

Let (M,g) be a complete Riemannian manifold supposed to be projectively symmetric and affinely homogeneous. We can apply THEOREM 1.2 and consider only case b) for the moment. By hypothesis $A^0(M,g)$ (= the identity component of the group of affine transformations of (M,g)) acts transitively on M and is a Lie subgroup of $A^0(M, \nabla^*)$, with the same notations of THEOREM 1.1. Now if $j \in A^0(M, g)$ and π is the 1-form relating ∇^* and ∇, then π is j-invariant : indeed if $X \in \mathcal{H}(M)$

$$j\nabla^*_X X = \nabla^*_{jX} jX = \nabla_{jX} jX + 2\pi(jX)jX = j\nabla_X X + 2\pi(jX)jX$$

hence $$j\nabla_X X + 2\pi(X)jX = j\nabla_X X + 2\pi(jX)jX$$

so $$\pi(jX) = \pi(X) \qquad \forall X \in \mathcal{H}(M).$$

We now claim that $\nabla^*\pi = 0$: indeed since π is $A^0(M, g)$-invariant and $A^0(M, g)$ is a subgroup of $A^0(M, \nabla^*)$, it is enough to prove that for some $q \in M$ $\nabla^*_{X}\pi\big|_q = 0$ for all $X \in TM_q$; let us choose $X \in TM_q$: since $A^0(M, g)$ acts transitively on M we can find $Y \in \mathfrak{a}(M, g)$ (the Lie algebra of complete vector fields generating affine transformations of (M, g)) such that $Y_q = X$. Our claim will be proved as soon as we recall that the integral curve $\exp tY$ through q of Y is a geodesic for ∇^* and that the parallel displacement along $\exp sY$ for $0 \le s \le t$ coincides with the differential of $\exp tY$ at q (see [3]). So by $\nabla^*\pi = 0$ we obtain that

$$(\nabla_X \pi)(Y) = 2\pi(X)\pi(Y). \tag{2.1}$$

From (3.1) and the completeness of (M, g) we obtain that π vanishes identically on M; indeed if $\gamma : \mathbb{R} \to M$ is any geodesic on M with affine parameter $t \in \mathbb{R}$ for ∇, the function

$$\Psi(t) = \pi(\gamma'(t)) \qquad \forall t \in \mathbb{R}$$

satisfies the following differential equation

$$\Psi'(t) = 2[\Psi(t)]^2 \qquad \forall t \in \mathbb{R}$$

that does not admit any global solution other than the trivial one $\Psi(t) = 0$ $\forall t \in \mathbb{R}$. Since π vanishes along any geodesic, π vanishes identically on M. We have proved the following

THEOREM 2.1. — *Let (M, g) be a complete Riemannian manifold of dimension $n \ge 3$, which is projectively symmetric and affinely homogeneous, then either*

a) *(M, g) is isometric to $S^n(r)$ or to $S^n(r)/\pm\mathrm{Id}$ with some choice of symmetries, or*

b) *(M, g) is a Riemannian symmetric space.*

COROLLARY 2.1. — *Let (M, g) be a complete analytic Riemannian manifold of dimension $n \ge 3$, which is projectively symmetric and projectively homogeneous, then either*

a) (M,g) *is isometric to* $S^n(r)$ *or to* $S^n(r)/\pm\mathrm{Id}$ *with some choice of symmetries, or*

b) (M,g) *is a Riemannian symmetric space.*

Proof. — The proof follows from THEOREM 2.1 and from a result of SOLODOVNIKOV [10], stating that a complete analytic Riemannian manifold every projective Killing vector field is affine unless (M,g) has constant sectional curvature. □

3. Some final remarks

LEDGER and OBATA in [5] have studied the case of a differentiable distribution of affine symmetries in an affinely connected manifold. We now want to consider the analogue situation in the projective case.

Let us start with the following

Definition 3.1. — (M,∇) is said to be *smoothly projectively symmetric* (say s.p.s.) if there exists a differentiable map $S : M \to P(M,\nabla)$ such that for every x in M $S(x)$ is a projective symmetry at x (in this definition $P(M,\nabla)$ is considered as a Lie group with the compact open topology; for this fact we refer to [3]. Following LEDGER and OBATA [5] and KOWALSKI [4], we have the following

PROPOSITION 3.1. — *If* (M,∇) *is* s.p.s. *then it is projectively homogeneous.*

Proof. — We fix any x_0 M and consider the C^∞ map $f : M \to M$ given by $f(x) = s_x(x_0)$; since $s_x(x) = x$ for every x in M, an easy computation shows that $df(x_0) = -2\,\mathrm{Id}$, so that f is locally invertible around the point x_0. If now K denotes the closure in $P(M,\nabla)$ of group generated by all the symmetries, then the previous argument shows that the orbit of K through the point x_0 is open and this implies that K acts transitively on M. □

In the study of s.p.s. spaces the fundamental form π plays an important role (see (1.11) for the definition).

Remark.

For the sphere S^n with the standard metric there is a bijection between $\Omega^1(S^n)$ and the smoothly projectively symmetric structures.

It will be very useful to consider the affine torsionfree connection ∇^* projectively related to ∇ through the one form π, in order to find necessary and sufficient conditions for (M,∇) to be inessential. We now recall the following fact that follows immediately from THEOREM 1.1.

If there is a point q in M at which the Weyl curvature tensor does not vanish, then every smoothly projectively symmetric structure on M

is inessential and the discussion is reduced to the affine case. So we may restrict ourselves to the projectively flat case.

PROPOSITION 3.2. — *Let (M, ∇) be projectively flat and s.p.s. Then (M, ∇) is locally inessential (i.e. there is a projectively related connection for which all the symmetries are locally affine transformations) if and only if $\nabla^* \operatorname{Ric}^* = 0$, where Ric^* is the Ricci tensor of ∇^*.*

Proof. — If (M, ∇) is locally inessential, then the projectively related connection that is locally invariant under all the symmetries is necessarily ∇^*, so our conclusion follows. On the other hand let us suppose that Ric^* is ∇^* parallel; since (M, ∇^*) is projectively flat, we have that the curvature tensor of ∇^* is parallel too, hence at each point p we have a local affine symmetry σ_p for ∇^*; but the projective transformation s_p of (M, ∇^*) has its corresponding 1-form vanishing at p by construction and so σ_p and s_p coincide (where σ_p is defined) because their 2-jets are equal at p and we are done. $\square$

Remark. — The condition $\nabla^* \operatorname{Ric}^* = 0$ can be written down in terms of $\nabla \operatorname{Ric}$ and π only of course, but this expression is quite complicated and we prefer the first one.

Under some additional assumption we are now going to give a geometrical interpretation or the previous result.

We suppose that M is simply connected and that the Ricci tensor of ∇ is symmetric : the second assumption is not too special because there is always a projective change of ∇ so that the Ricci tensor becomes symmetric (see [1]).

We recall now the following facts (*see* VEBLEN [11] and NAGANO [6]) : we consider the direct product $M \times \mathbb{R}$ covered by the coordinate system (x^i, x^0) where (x^i) is a coordinate system on M and letus denote with (Γ^i_{jk}) the Christoffel symbolds for ∇ : consider now on $M \times \mathbb{R}$ the connection ∇^0 whose Christoffel symbols are given by (Greek indices run over $0, 1, \ldots, n$)

$$\Gamma^{0\lambda}_{\mu\nu} = \delta^\lambda_i \delta^j_\mu \delta^k_\nu \Gamma^i_{jk} + \delta^\lambda_\mu \delta^0_\nu + \delta^\lambda_\nu \delta^0_\mu - g_{\mu\nu} \delta^0_\lambda$$

where

$$g_{\mu\nu} \xi^\mu \xi^\nu = (\xi^0)^2 + (n-1)^{-1} R_{ij} \xi^i \xi^j \qquad \forall (\xi^\alpha) \in T(M \times \mathbb{R})$$

and (R_{ij}) are the local components of the Ricci tensor of ∇.

We state without proof the following properties of ∇^0 :

a) if ∇ is projectively flat, then ∇^0 is flat;

b) every projective transformation f of (M, ∇) can be lifted to an affine transformation of $M \times \mathbb{R}$ with respect to ∇^0. Indeed if Φ is its corresponding 1-form, then by simply connectedness of M and Ricci symmetry, Φ is the differential of a C^∞ function ρ and it is a lenghty but straighforward calculation to verify that

$$f^0(x, x^0) = \big(f(x), x_0 - \rho(x)\big)$$

works.

So let us return to a s.p.s. space : for every q in M take the C^∞ function ρ_q with $\rho_q(q) = 0$ and $d\rho_q$ equal to the corresponding 1-form of s_q ; then for every $t \in \mathbb{R}$

$$s^0_{(q,t)}(x, x^0) = \big(s_q(x), x_0 - \rho_q(x)\big)$$

is an involutorial affine transformation of $(M \times \mathbb{R}, \nabla^0)$ lifting s_q and fixing (q, t). At (q, t) we have

$$T(M \times R)_{(q,t)} = \langle \partial/\partial x^0 \rangle \bigoplus D_{(q,t)}$$

where $D_{(q,t)} = \{(X, V) \in T(M \times R)_{(q,t)} \mid \pi(X) = V\}$ is the eigenspace of $ds^0_{(q,t)}$ relative to the eigenvector -1, while $\langle \partial/\partial x^0 \rangle$ is the eigenspace relative to the eigenvalue $+1$.

So we have obtained a C^∞ distribution D which is integrable if and only if π is closed, as one can easly check. From now on we will suppose that π is closed in order to get maximal integral submanifolds for the distribution D : we fix a connected maximal integral submanifold M' and take along this the vector field $\xi = \partial/\partial x^0$ as normal vector field; according to K. NOMIZU and U. PINKALL [7] we consider on M' the induced connection ∇' by means of

$$\nabla^0_X Y = \nabla'_X Y + h(X, Y)\xi \qquad \forall X, Y \in \mathcal{H}(M')$$

and if we put $X = X_1 + \pi(X_1)\xi$ and $Y = Y_1 + \pi(Y_1)\xi$ with $X_1, Y_1 \in TM$, then

$$\nabla'_X Y = \nabla^*_{X1} Y_1 + \pi(\nabla^*_{X1} Y_1)\xi$$

and

$$h(X, Y) = -(n-1)^{-1} \operatorname{Ric}^*(X_1, Y_1)$$

moreover $\nabla^0_X \xi = X \ \forall X \in \mathcal{H}(M')$, so the shape operator S is equal to $-\mathrm{Id}$ and the transversal connection form vanishes (following the same notations as in [7]).

Since the ambient space $M \times \mathbb{R}$ is flat, the cubic form C for the affine immersion $M' \to M \times \mathbb{R}$ is given by

$$C(X, Y, Z) = (\nabla'_X h)(Y, Z)$$

and with the same notations as above we find that

$$C(X,Y,Z) = -(n-1)^{-1}(\nabla^*_{X_1} \operatorname{Ric}^*)(Y_1, Z_1).$$

So we may reformulate PROPOSITION 3.2 as follows :

PROPOSITION 3.3. — *Let (M, ∇) be a simply connected, projectively flat and s.p.s. manifold with symmetric Ricci tensor; suppose furthermore that the fundamental form π is closed. Then (M, ∇) is locally inessential if and only if the cubic form of any integral submanifold M' for the distribution D on $M \times \mathbb{R}$ described above vanishes.*

This characterization of local inessentiality of s.p.s. manifolds in terms of the vanishing of the cubic form of an affinely immersed submanifold of $M \times \mathbb{R}$ could be useful to obtain some deeper results, because of the importance of the cubic form in the investigation of the geometry of affine immersions (see [7]); we hope this will be the object of a further paper.

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