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SCHWARTZ'S THEOREM ON MEAN PERIODIC VECTOR-VALUED FUNCTIONS

BY

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RÉSUMÉ. — Nous exposons une preuve plus simple du théorème de SCHWARTZ sur les fonctions continues à valeurs dans $\mathbb{C}^N$.

ABSTRACT. — A simpler proof to SCHWARTZ'S theorem for $\mathbb{C}^N$-valued continuous functions is provided.

1. Introduction and preliminaries

The theorem of L. SCHWARTZ on mean periodic functions of one variable states that every closed translation-invariant subspace of the space of continuous complex functions on $\mathbb{R}$ is spanned by the polynomial-exponential functions it contains [4]. In [2, VII], J.-J. KELLEHER and B.-A. TAYLOR provide a characterization of all closed submodes of $\mathbb{C}^N$-valued entire functions of exponential type which have polynomial growth on $\mathbb{R}$. By duality, their result generalizes Schwartz's Theorem to $\mathbb{C}^N$-valued continuous functions.

Our goal is to provide a simple and a direct proof to this result.

$C(\mathbb{R}, \mathbb{C}^N)$ denotes the space of continuous $\mathbb{C}^N$-valued functions on $\mathbb{R}$, with the topology of uniform convergence on compact sets. By a vector-valued polynomial exponential in $C(\mathbb{R}, \mathbb{C}^N)$, we mean a function of the form $e^{\lambda x}p(x)$, $x \in \mathbb{R}$, where $\lambda \in \mathbb{C}$ and p is a polynomial in $C(\mathbb{R}, \mathbb{C}^N)$.

THEOREM. — *Every translation-invariant closed subspace of $C(\mathbb{R}, \mathbb{C}^N)$ is spanned by the vector-valued polynomial-exponential functions it contains.*

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For the theory of mean-periodic complex functions, we refer the reader to [4], [1], [3]. We need the following notations and results.

Let $M_0(\mathbb{R})$ denote the space of complex Radon measures on $\mathbb{R}$ having compact support. For $\mu \in M_0(\mathbb{R})$, the Laplace transform $\hat{\mu}$ of μ is the entire function defined by $\hat{\mu}(z) = \int e^{-zx} d\mu(x)$, $z \in \mathbb{C}$.

We remind that $f \in C(\mathbb{R})$ is mean periodic if $\mu * f = 0$ for some $\mu \in M_0(\mathbb{R})$, $\mu \neq 0$. For $f \in C(\mathbb{R})$, f^- is the function defined by $f^-(x) = f(x)$ if $x \leq 0$ and $f^-(x) = 0$ if $x > 0$. If f is mean-periodic, $\mu \in M_0(\mathbb{R})$, $\mu \neq 0$ and $\mu * f = 0$, then the function $\mu * f^-$ has compact support and the meromorphic function

$$F = \big(\mu * f^-\big)\hat{}/\hat{\mu},$$

which does not depend on the choice of μ, is defined to be the Laplace transform of f ([3]).

The heart of our proof is the fact that F is entire only if $f = 0$ (see [3, Theorem X]).

The dual of $C(\mathbb{R}, \mathbb{C}^N)$ is the space $M_0(\mathbb{R}, \mathbb{C}^N)$ of $\mathbb{C}^N$-valued Radon measures on $\mathbb{R}$ having compact supports. One notices that $M_0(\mathbb{R})$ is an integral domain under the convolution product and $M_0(\mathbb{R}, \mathbb{C}^N)$ is a module over $M_0(\mathbb{R})$ with the coordinatewise convolution. We denote the duality by

$$\langle \mu, f \rangle = \sum_{j=1}^{N} (\mu_j * f_j)(0)$$

for $\mu = (\mu_j) \in M_0(\mathbb{R}, \mathbb{C}^N)$ and $f = (f_j) \in C(\mathbb{R}, \mathbb{C}^N)$. If f is a vector-valued polynomial-exponential with

$$f_j(x) = \sum_{\ell=0}^{m} \alpha_j^{(\ell)} x^\ell e^{\lambda x} \qquad (1 \leq j \leq N),$$

we have

$$\langle \mu, f \rangle = \sum_{j=1}^{N} \sum_{\ell=0}^{m} \alpha_j^{(\ell)} \hat{\mu}_j^{(\ell)}(\lambda).$$

For any subset A of $C(\mathbb{R}, \mathbb{C}^N)$ let

$$A^\perp = \big\{\mu \in M_0(\mathbb{R}, \mathbb{C}^N)\, ;\ \langle \mu, f \rangle = 0 \quad \text{for all f} \in \text{A}\big\}.$$

If V is a translation-invariant closed subspace of $C(\mathbb{R}, \mathbb{C}^N)$, $\mathrm{Sp}(V)$ denotes the set of all vector-valued polynomial-exponentials that belong to V.

By duality, V is spanned by $\mathrm{Sp}(V)$ if and only if $\mathrm{Sp}(V)^\perp \subset V^\perp$. Since V is translation-invariant, $V^\perp$ is a submodule of $M_0(\mathbb{R}, \mathbb{C}^N)$ and $\mu = (\mu_j) \in V^\perp$ if and only if

$$\sum_{j=1}^{N} \mu_j * f_j = 0 \quad \text{for all } f = (f_j) \in V.$$

2. Main result

In this section, V denotes a given translation-invariant closed subspace of $C(\mathbb{R}, \mathbb{C}^N)$. We have to prove $\langle \mu, f \rangle = 0$ for any $\mu \in \mathrm{Sp}(V)^\perp$ and $f \in V$. We need some more notation and three lemmas.

Let $0 \le r \le N$ be the *rank* of $V^\perp$ as a module over $M_0(\mathbb{R})$. That means r is the greatest integer for which there exists a system $(\sigma_\ell)_{1 \le \ell \le r}$ where $\sigma_\ell = (\sigma_{\ell,j})_{1 \le j \le N} \in V^\perp$ for $1 \le \ell \le r$ and with a non-zero determinant of order r. We shall suppose given such a system with, say,

$$\rho = \det\bigl(\sigma_{\ell,j}\ ;\ 1 \le \ell,\, j \le r\bigr) \ne 0.$$

One notices that $\hat\rho$ is the non identically zero entire function given by

$$\hat\rho(\lambda) = \det\bigl(\hat\rho_{\ell,j}(\lambda)\ ;\ 1 \le \ell,\, j \le r\bigr), \qquad \lambda \in \mathbb{C}.$$

If $r = 0$, *i.e.* $V^\perp = \{0\}$, we take for ρ the Dirac measure at 0 and $\hat\rho(\lambda) = 1$, $\lambda \in \mathbb{C}$.

For $\mu = (\mu_j) \in M_0(\mathbb{R}, \mathbb{C}^N)$ let

$$\Delta_j(\mu) = \det \begin{vmatrix} \mu_1 & \cdots & \mu_r & \mu_j \\ \sigma_{1,1} & \cdots & \sigma_{1,r} & \sigma_{1,j} \\ \vdots & \ddots & \vdots & \vdots \\ \sigma_{r,1} & \cdots & \sigma_{r,r} & \sigma_{r,j} \end{vmatrix} \qquad (\text{for } 1 \le j \le N)$$

and

$$\tau_\ell(\mu) = \det \begin{vmatrix} \sigma_{1,1} & \cdots & \sigma_{1,r} \\ \vdots & \ddots & \vdots \\ \sigma_{\ell-1,1} & \cdots & \sigma_{\ell-1,r} \\ \mu_1 & \cdots & \mu_r \\ \sigma_{\ell+1,1} & \cdots & \sigma_{\ell+1,r} \\ \vdots & \ddots & \vdots \\ \sigma_{r,1} & \cdots & \sigma_{r,r} \end{vmatrix} \qquad (\text{for } 1 \le \ell \le r).$$

From the definition of r, for any $\mu \in V^{\perp}$

$$\Delta_j(\mu) = 0 \qquad (\text{for } 1 \leq j \leq N). \tag{1}$$

By expanding the $\Delta_j(\mu)$ along the last column, (1) is equivalent to

$$\rho * \mu_j = \sum_{\ell=1}^{r} \tau_\ell(\mu) * \sigma_{\ell,j} \qquad (\text{for } 1 \leq j \leq N). \tag{2}$$

LEMMA 1. — *Let $\lambda \in \mathbb{C}$ such that $\hat{\rho}(\lambda) \neq 0$. For $\alpha = (\alpha_j) \in \mathbb{C}^N$, the vector-exponential $e^{\lambda x} \cdot \alpha$ belongs to V if and only if*

$$\sum_{j=1}^{N} \alpha_j \hat{\sigma}_{\ell,j}(\lambda) = 0 \qquad 1 \leq \ell \leq r. \tag{3}$$

Proof. — Let $\alpha \in \mathbb{C}^N$. We have $e^{\lambda x} \cdot \alpha \in V$ if and only if, for every $\mu = (\mu_j) \in V^{\perp}$,

$$\langle \mu, e^{\lambda x} \cdot \alpha \rangle = \sum_{j=1}^{N} \alpha_j \hat{\mu}_j(\lambda) = 0. \tag{4}$$

This proves the "only if" part. Conversly, since $\hat{\rho}(\lambda) \neq 0$, (2) implies that for any $\mu \in V^{\perp}$ the equation in (4) is a linear combination of the equations (3).

LEMMA 2. — *Let $\mu \in M_0(\mathbb{R}, \mathbb{C}^N)$. If $\langle \mu, e^{\lambda x} \cdot \alpha \rangle = 0$ for all $\lambda \in \mathbb{C}$ such $\hat{\rho}(\lambda) \neq 0$ and $\alpha \in \mathbb{C}^N$ such that $e^{\lambda x} \cdot \alpha \in V$, then $\Delta_j(\mu) = 0$ for $1 \leq j \leq N$.*

Proof. — Let $\lambda \in \mathbb{C}$ with $\hat{\rho}(\lambda) \neq 0$. If μ satisfies the hypothesis, the solutions of (3) are solutions of (4), which implies that the determinants $\Delta_j(\mu)\hat{}(\lambda)$ for $1 \leq j \leq N$ are equal to zero. Then, since $\hat{\rho}$ and the $\Delta_j(\mu)\hat{}$ are entire functions and $\hat{\rho} \neq 0$, the $\Delta_j(\mu)\hat{}$ are identically zero. Hence, $\Delta_j(\mu) = 0$ for $1 \leq j \leq N$.

Remark. — LEMMA 2 shows that any $\mu \in \operatorname{Sp}(V)^{\perp}$ satisfies (1) and (2). If $r = 0$, $\Delta_j(\mu) = \mu_j$ for $1 \leq j \leq N$; hence $\operatorname{Sp}(V)^{\perp} = \{0\}$ if $V^{\perp} = \{0\}$.

LEMMA 3. — *Let $\lambda \in \mathbb{C}$, $m \geq 0$ and $\mu \in \operatorname{Sp}(V)^{\perp}$. There exists $\nu \in V^{\perp}$ such that*

$$\hat{\nu}_j^{(\ell)}(\lambda) = \hat{\mu}_j^{(\ell)}(\lambda) \qquad (\text{for } 1 \leq j \leq N,\ 0 \leq \ell < m).$$

Proof. — Suppose the element $(\hat{\mu}_j^{(\ell)}(\lambda))_{1<j<N,\,0<\ell-m}$ of $\mathbb{C}^{Nm}$ does not belong to the subspace

$$M(\lambda, m) = \big\{(\hat{\nu}_j^{(\ell)}(\lambda))_{1\le j\le N,\,0\le\ell<m}\ ;\ \nu \in V^{\perp}\big\}.$$

Then there exists $(\alpha_j^{(\ell)})_{1\le j\le N,\,0\le\ell<m}$ such that

$$\sum_{j=1}^{N}\sum_{\ell=0}^{m-1} \alpha_j^{(\ell)}\hat{\nu}_j^{(\ell)}(\lambda) = 0 \qquad \text{for } \nu \in V^{\perp}$$

and

$$\sum_{j=1}^{N}\sum_{\ell=0}^{m-1} \alpha_j^{(\ell)}\hat{\mu}_j^{(\ell)}(\lambda) \neq 0.$$

Then if

$$f_j(x) = \sum_{\ell=0}^{m-1} \alpha_j^{(\ell)} x^{\ell} \qquad (\text{for } 1 \le j \le N),$$

the polynomial-exponential $f = (f_j)_{1\le j\le N}$ satisfies

$$\langle \nu, f\rangle = 0 \qquad (\text{for } \nu \in V^{\perp}),$$

therefore $f \in \operatorname{Sp}(V)$, and

$$\langle \mu, f\rangle \neq 0,$$

and we have a contradiction, since $\mu \in \operatorname{Sp}(V)^{\perp}$.

Proof of the THEOREM. — Let $\mu = (\mu_j) \in \operatorname{Sp}(V)^{\perp}$, $f = (f_j) \in V$ and

$$g = \sum_{j=1}^{N} \mu_j * f_j.$$

We have to prove that $g = 0$. By LEMMA 2, $\Delta_j(\mu) = 0$ for $1 \le j \le N$ and μ verifies (2); therefore

$$\rho * \sum_{j=1}^{N} \mu_j * f_j = \sum_{\ell=1}^{r}\Big(\tau_\ell(\mu) * \sum_{j=1}^{N} \sigma_{\ell,j} * f_j\Big).$$

For $1 \le \ell \le r$, since $\sigma_\ell \in V^{\perp}$, we have $\sum_{j=1}^{N} \sigma_{\ell,j} * f_j = 0$. So

$$\rho * g = 0.$$

Hence g is mean-periodic and the Laplace transform G of g may be defined by

$$G = \big(\rho * g^{-}\big)\hat{}\,/\hat{\rho}.$$

By ([3, Theorem X]) it is enough to prove that G is entire.

If $[a,b]$ is any interval that contains the supports of the μ_j $(1 \le j \le N)$, $\sum \mu_j * f_j^{-}(x)$ is equal to $g(x)$ for $x < a$ and 0 for $x > b$. Thus the function

$$s = g^{-} - \sum_{j=1}^{N} \mu_j * f_j^{-}$$

has compact support. For $1 \le \ell \le r$, let

$$h_\ell = \sum_{j=1}^{N} \sigma_{\ell,j} * f_j^{-}.$$

By the same argument, the functions h_ℓ have compact supports and, by (2),

$$\rho * \sum_{j=1}^{N} \mu_j * f_j^{-} = \sum_{\ell=1}^{r} \tau_\ell(\mu) * h_\ell.$$

So

$$\rho * g^{-} = \sum_{\ell=1}^{r} \tau_\ell(\mu) * h_\ell + \rho * s\,;$$

$$G = \frac{1}{\hat{\rho}} \sum_{\ell=1}^{r} \tau_\ell(\mu)\hat{}\cdot \hat{h}_\ell + \hat{s}. \tag{5}$$

The functions $\hat{s}$ and $\hat{h}_\ell$ $(1 \le \ell \le r)$ are entire, as Laplace transforms of compactly supported functions.

For any $\nu \in V^{\perp}$, since $\sum \nu_j * f_j = 0$, $\sum \nu_j * f_j^{-}$ has compact support, and it follows by (2) that the function

$$\frac{1}{\hat{\rho}} \sum_{\ell=1}^{r} \tau_\ell(\nu)\hat{}\cdot \hat{h}_\ell \qquad \text{is entire.} \tag{6}$$

Let $\lambda \in \mathbb{C}$ and let m be the order of $\hat{\rho}$ at λ. By LEMMA 3, we can choose $\nu \in V^{\perp}$ so that $\hat{\nu}_j^{(k)}(\lambda) = \hat{\mu}_j^{(k)}(\lambda)$ for $1 \le j \le N$, $0 \le k < m$. Then the functions $(\hat{\nu}_j - \hat{\mu}_j)/\hat{\rho}$ for $1 \le j \le N$ and the functions

$$\frac{1}{\hat{\rho}}\big(\tau_\ell(\nu)\hat{} - \tau_\ell(\mu)\hat{}\big) \qquad (\text{for } 1 \le \ell \le r)$$

are analytic at λ. It follows from (5) and (6) that G is analytic at λ.

Since λ is arbitrary, G is entire. That completes the proof of the Theorem.

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