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## RIEMANN-ROCH FOR GENERAL ALGEBRAIC VARIETIES

BY

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**ABSTRACT.** — Grothendieck proved the Riemann-Roch theorem for a morphism between smooth projective varieties, and Baum-Fulton-MacPherson extended it to proper morphisms between quasi-projective, possibly singular, varieties. The Riemann-Roch theorem presented here is valid in the category of proper morphisms between arbitrary (possibly singular, not quasi-projective) algebraic schemes. The construction of the Riemann-Roch transformation uses Chow's lemma and a little higher  $K$ -theory; it is computed, for varieties imbeddable in smooth varieties, using a relative version of Riemann-Roch for local complete intersection morphisms.

**RÉSUMÉ.** — Grothendieck a démontré le théorème de Riemann-Roch pour un morphisme entre variétés projectives et lisses, et Baum-Fulton-MacPherson l'ont étendu aux morphismes propres entre variétés quasi projectives, éventuellement singulières. Le théorème de Riemann-Roch présenté ici vaut dans la catégorie des morphismes propres entre variétés algébriques arbitraires (qu'on ne suppose ni lisses, ni quasi projectives). Dans la construction de la transformation de Riemann-Roch, on utilise le lemme de Chow et un peu de  $K$ -théorie algébrique; on explicite cette transformation pour les variétés plongeables dans les variétés lisses, à l'aide d'une version relative de Riemann-Roch pour les morphismes localement intersection complète.

### 0. Introduction

In this note we show how to free the Riemann-Roch theorems of [4], [1], and [2] from all projective assumptions, thereby completing a program begun in [5] and [9].

The proof, which is remarkably simple, uses ideas most of which have been available for several years. Not surprisingly, the Riemann-Roch theorem for general proper morphisms is deduced from the projective case *via* Chow's

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lemma; however to do this we must introduce the notion of an "envelope" of a variety and use a little higher  $K$ -theory (as far as  $K_1$ ). It is important to point out that even to extend the Grothendieck-Riemann-Roch theorem of [4] to arbitrary proper morphisms between smooth varieties over a field of characteristic  $p > 0$ , we need to use, in the absence of resolution of singularities, the singular Riemann-Roch theorem of [1] together with a simple extension of the Verdier-Riemann-Roch theorem of [13].

Fix an arbitrary ground field  $K$ . All schemes  $X$  will be algebraic  $K$ -schemes (i. e.,  $X$  is of finite type and separated over  $\text{Spec}(K)$ ). Denote by  $K_0 X$  (resp.  $K^0 X$ ) the Grothendieck group of coherent algebraic sheaves (resp. algebraic vector bundles) on  $X$ . Let  $A_* X_0$  be the group of algebraic cycles modulo rational equivalence on  $X$ , with rational coefficients, and let  $A^* X_0$  denote the corresponding cohomology ring, as constructed in [7], paragraph 9.

**RIEMANN-ROCH THEOREM.** — *For every algebraic scheme  $X$  there is a homomorphism*

$$\tau_X : K_0 X \rightarrow A_* X_0.$$

*These homomorphisms satisfy the following properties:*

(1) (Covariance) *If  $f : X \rightarrow Y$  is proper, and  $\alpha \in K_0 X$ , then:*

$$f_* \tau_X(\alpha) = \tau_Y f_*(\alpha).$$

(2) (Module) *If  $\alpha \in K_0 X$ ,  $\beta \in K^0 X$ , then:*

$$\tau_X(\beta \otimes \alpha) = \text{ch}(\beta) \cap \tau_X(\alpha).$$

(3) (Formula) *If  $i : X \rightarrow M$  is a closed imbedding in a scheme  $M$  which is smooth over  $\text{Spec}(K)$ ,  $\mathcal{F}$  a coherent sheaf on  $X$ ,  $E$  a resolution of  $\mathcal{F}$  by a bounded complex of locally free sheaves on  $M$ , then:*

$$\tau_X(\mathcal{F}) = \text{td}(i^* T_M) \cap \text{ch}_X^M(E).$$

*Here  $T_M$  is the tangent bundle of  $M$ ,  $\text{td}$  is the Todd class, and  $\text{ch}_X^M(E) \in A_* X_0$  is the localized Chern character of  $E$ . ([1], § II. 1).*

(4) (Local complete intersections) *Let  $f : X \rightarrow Y$  be a l.c.i. morphism. Assume that there are closed imbeddings  $X \subset M$ ,  $Y \subset P$ , with  $M$*

and  $P$  smooth. Then for all  $\alpha \in K_0 Y$ :

$$\tau_X f^*(\alpha) = \text{td}(T_f) \cdot f^* \tau_Y(\alpha).$$

Here  $T_f$  is the virtual tangent bundle to  $f$ .

(5) (Cartesian products) For all  $\alpha \in K_0 X$ ,  $\beta \in K_0 Y$ :

$$\tau_{X \times Y}(\alpha \times \beta) = \tau_X(\alpha) \times \tau_Y(\beta).$$

(6) (Top term) If  $X$  is an  $n$ -dimensional variety (reduced and irreducible), then:

$$\tau_X(\mathcal{O}_X) = [X] + \text{terms of dimension} < n.$$

In addition, these homomorphisms  $\tau_X$  are uniquely determined by properties (1), (4) (for open imbeddings of quasi-projective varieties), and (6) (for  $X = \mathbb{P}^n$ ).

Granting this theorem, define the Todd class  $\text{Td}(X) \in A_* X$ , for any algebraic scheme  $X$  by setting:

$$\text{Td}(X) = \tau_X(\mathcal{O}_X),$$

$\mathcal{O}_X$  the structure sheaf of  $X$ .

COROLLARY 1. — (i) If  $X$  is a smooth scheme (or a l. c. i. scheme which is imbeddable in a smooth scheme) then:

$$\text{Td}(X) = \text{td}(T_X) \cap [X].$$

Here  $T_X$  is the tangent bundle (or virtual tangent bundle) of  $X$ .

(ii) Let  $f: X \rightarrow Y$  be a proper morphism, and let  $\beta \in K^0 X$ . Assume there is an element  $f_*(\beta)$  in  $K^0 Y$  such that:

$$f_*(\beta \otimes \mathcal{O}_X) = f_*(\beta) \otimes \mathcal{O}_Y,$$

in  $K_0 Y$ . Then:

$$f_*(\text{ch}(\beta) \cap \text{Td}(X)) = \text{ch}(f_* \beta) \cap \text{Td}(Y).$$

Note that there is a canonical such element  $f_*(\beta)$  whenever  $f$  is a l. c. i. morphism, or, more generally, a perfect morphism (cf. [3]).

COROLLARY 2 (Grothendieck-Riemann-Roch). — Let  $f: X \rightarrow Y$  be a proper morphism of smooth schemes,  $\beta \in K^0 X$ . Then:

$$f_*(\text{ch}(\beta) \cdot \text{td}(T_X)) = \text{ch}(f_* \beta) \cdot \text{td}(T_Y).$$

The same formula holds for  $X, Y$  l. c. i. schemes which can be imbedded in smooth schemes.

**COROLLARY 3 (Hirzebruch-Riemann-Roch).** — Assume  $X$  is proper over  $\text{Spec}(K)$ , and  $E$  is a vector bundle on  $X$ . Then:

$$\chi(X, E) = \int_X \text{ch}(E) \cap \text{Td}(X).$$

In particular  $\chi(X, \mathcal{O}_X) = \int_X \text{Td}(X)$ . From (5) of the theorem, one has also:

$$\text{Td}(X \times Y) = \text{Td}(X) \times \text{Td}(Y).$$

**COROLLARY 4.** — For all  $X$ ,  $\tau_X$  induces an isomorphism:

$$K_0 X \otimes \mathbb{Q} \rightarrow A_* X_{\mathbb{Q}}.$$

Corollary 1 (i) follows from (3) in the non-singular case, and (4) (for  $f: X \rightarrow Y$ ,  $Y$  smooth,  $\alpha = \mathcal{O}_Y$ ) if  $X$  is a l. c. i.; (ii) then follows from (1) and (2). Corollary 2 is a special case of Corollary 1. Corollary 3 follows from (1) (for  $X \rightarrow \text{Spec}(K)$ ) and (2). Corollary 4 follows from (1) and (6), as in [1], § III (cf. Step 7 below).

For quasi-projective schemes,  $\tau$  was constructed in [1]. The proof that  $\tau$  satisfies (1)-(6), in the category of quasi-projective schemes, was given in [1] and [13]. In paragraph 1 we use Chow's lemma and an exact sequence involving a (first) higher  $K$ -group, to extend  $\tau$  to all algebraic schemes. Paragraph 1 also contains the definition and a discussion of the properties of an envelope. In paragraph 2 we show that for schemes which are imbeddable in smooth schemes, the formula (3) is independent of the imbedding; this fact is deduced from a relative version of (4), for which VERDIÉ's proof in [13] suffices. The proof is then easily completed (§ 3).

When  $K = \mathbb{C}$ , the same construction determines, for all complex algebraic schemes  $X$ , a homomorphism:

$$\alpha_X : K_0 X \rightarrow K_0^{\text{top}}(X),$$

where  $K_0^{\text{top}}(X)$  is the homology topological  $K$ -theory of  $X$ , satisfying analogues of (1)-(6). This generalizes [3] and [7] PRR precisely as the above Riemann-Roch theorem generalizes [1] and [13]. In particular, every

complex algebraic variety has a canonical orientation  $\{X\}$  in  $K_0^{\text{top}}(X)$ , namely  $\{X\} = \alpha_X(\mathcal{O}_X)$ . This is compatible with the previous theorem: there is a commutative diagram:

$$\begin{array}{ccc} K_0 X & \xrightarrow{\alpha_X} & K_0^{\text{top}}(X) \\ \tau_X \downarrow & & \downarrow \text{ch}_* \\ A_* X_0 & \longrightarrow & H_*(X; \mathbb{Q}) \end{array}$$

where  $\text{ch}_*$  is the homology Chern character, and the lower horizontal map is the cycle map.

In § 5 of the second author's paper [9] it was implicitly assumed that if  $X$  is a singular scheme quasi-projective over two different base schemes  $S$  and  $T$  then the two, *a priori* different, Riemann-Roch transformations  $\tau^S, \tau^T : K_0 \rightarrow A_0^*$  on the categories  $\mathcal{C}_S, \mathcal{C}_T$  of schemes quasi-projective over  $S$  and  $T$  respectively, coincide on  $X$ . The equality  $\tau_X^S = \tau_X^T$  follows from Proposition 2 of the present paper.

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1. The construction of  $\tau$ .

PROPOSITION 1. — Consider a fibre square:

$$\begin{array}{ccc} Y' & \xrightarrow{j} & X' \\ q \downarrow & & \downarrow p \\ Y & \xrightarrow{i} & X \end{array}$$

with  $i$  a closed imbedding and  $p$  projective. Assume that  $p$  maps  $X' - Y'$  isomorphically onto  $X - Y$ . Then the sequence:

$$K_0 Y' \xrightarrow{a} K_0 Y \oplus K_0 X' \xrightarrow{b} K_0 X \rightarrow 0$$

is exact, where  $a(\alpha) = (q_* \alpha, -j_* \alpha)$ , and  $b(\alpha, \beta) = i_* \alpha + p_* \beta$ .

Proof. — If  $f : Z \rightarrow W$  is any proper morphism, let  $\mathbf{F}(Z, f)$  be the full subcategory of the exact category  $\mathbf{M}(Z)$  of coherent sheaves on  $Z$  consisting

of those  $\mathcal{F}$  for which  $R^i f_* \mathcal{F} = 0$  for  $i > 0$ . Writing  $U' = X' - Y'$ ,  $U = X - Y$ ,  $u : U' \rightarrow X'$  and  $v : U \rightarrow X$  for the natural inclusions, and  $r : U' \rightarrow U$  for the natural isomorphism, we have a commutative diagram of exact functors:

$$\begin{array}{ccccc}
 F(Y', q) & \xrightarrow{j_*} & F(X', p) & \xrightarrow{u^*} & M(U') \\
 \downarrow q_* & & \downarrow p_* & & \downarrow r_* \\
 M(Y) & \xrightarrow{i_*} & M(X) & \xrightarrow{v^*} & M(U).
 \end{array}$$

By QUILLEN [12], paragraph 7, 2.7 we know that if  $p$  is projective then  $K_i(F(X', p)) \simeq K_i(X') (= K_i(M(X'))$  by definition) and  $K_i(F(Y', q)) \simeq K_i(Y')$ . Hence by [12], paragraph 7, Proposition 3.2, applying  $BQ$  to this diagram gives a map of fibration sequences and hence a map of long exact sequences:

$$\begin{array}{ccccccc}
 \rightarrow & K_1 U' & \longrightarrow & K_0 Y' & \xrightarrow{j_*} & K_0 X' & \longrightarrow & K_0 U' & \longrightarrow & 0 \\
 & \downarrow \simeq r_* & & \downarrow q_* & & \downarrow p_* & & \downarrow \simeq r_* & & \\
 \rightarrow & K_1 U & \longrightarrow & K_0 Y & \xrightarrow{i_*} & K_0 X & \longrightarrow & K_0 U & \longrightarrow & 0
 \end{array}$$

The proposition then follows by a simple diagram chase.

It can be shown more generally that this proposition is true if  $p$  is proper rather than projective.

Let us define an *envelope* of a scheme  $X$  to be a proper morphism  $p : X' \rightarrow X$  such that for every closed subvariety  $V$  of  $X$  there is a closed subvariety  $V'$  of  $X'$  such that  $p$  maps  $V'$  birationally onto  $V$ . We call  $p$  a *Chow envelope* if, in addition,  $X'$  is quasi-projective over  $\text{Spec}(K)$ .

LEMMA. — (1) If  $p : X' \rightarrow X$  and  $q : X'' \rightarrow X'$  are envelopes, then  $qp : X'' \rightarrow X$  is an envelope.

(2) If  $p : X' \rightarrow X$  is an envelope, and  $f : Y \rightarrow X$  is an arbitrary morphism, then the fibre product  $X' \times_X Y \rightarrow X'$  is an envelope.

(3) For any scheme  $X$  there is a closed subscheme  $Y \subset X$  with  $X - Y$  dense in  $X$ , and a Chow envelope  $p : X' \rightarrow X$  such that  $p$  maps  $X' - p^{-1}(Y)$  isomorphically onto  $X - Y$ .

(4) If  $p_1 : X'_1 \rightarrow X$ ,  $p_2 : X'_2 \rightarrow X$  are envelopes, then there is a Chow envelope  $p : X' \rightarrow X$ , with morphisms  $q_i : X' \rightarrow X'_i$  such that  $p_i q_i = p$  for  $i = 1, 2$ .

(5) For any morphism  $f: Y \rightarrow X$ , and any Chow envelope  $p: X' \rightarrow X$ , there is a Chow envelope  $q: Y' \rightarrow Y$  and a morphism  $f': Y' \rightarrow X'$  such that  $pf' = fq$ . If  $f$  is proper,  $f'$  may also be taken to be proper.

(6) If  $p: X' \rightarrow X$  is an envelope, then the induced morphism  $p_*: K_0 X' \rightarrow K_0 X$  is surjective.

*Proof.* — (1) and (2) are straightforward. For (3), by Chow's lemma [10], paragraph 5.6, there is a  $Y \subset X$  with  $X - Y$  dense in  $X$ , and a proper morphism  $p'_1: X'_1 \rightarrow X$ , with  $X'_1$  quasi-projective, such that  $p'_1$  restricts to an isomorphism over  $Y - X$ . By noetherian induction there is a Chow envelope  $p'_2: X'_2 \rightarrow Y$ . Then the disjoint union of  $X'_1$  and  $X'_2$ , with its canonical map to  $X$ , is a Chow envelope of  $X$ . (4) and (5) follow from (1), (2) and (3). (6) follows from the fact that  $K_0 X$  is generated by classes of structure sheaves of closed subvarieties of  $X$  (cf. [5]).

In [1], within the category of quasi-projective schemes over  $\text{Spec}(K)$ , homomorphisms:

$$\tau_X: K_0 X \rightarrow A_* X_{\mathbb{Q}}$$

were defined, satisfying properties (1)-(3), (5), (6) of the Riemann-Roch Theorem stated in the introduction. Property (4) was proved in [13]. Some additional argument was needed in [1], paragraph II.1.2 to handle the case where  $K$  is not algebraically closed, but subsequent improvements in intersection theory have taken care of this (see [6]).

In the rest of this section we show how to extend the construction of  $\tau_X$  to all algebraic schemes  $X$ , so that properties (1), (2), (5) and (6) of the Riemann-Roch Theorem hold. There are several steps in the argument.

*Step 1.* — Let us say that a homomorphism  $\tau: K_0 X \rightarrow A_* X_{\mathbb{Q}}$  is compatible with a Chow envelope  $p: X' \rightarrow X$  if the diagram:

$$\begin{array}{ccc} K_0 X' & \xrightarrow{\tau_{X'}} & A_* X'_{\mathbb{Q}} \\ p_* \downarrow & & \downarrow p_* \\ K_0 X & \xrightarrow{\tau} & A_* X_{\mathbb{Q}} \end{array}$$

commutes. Here  $\tau_{X'}$  is the homomorphism previously constructed for the quasi-projective scheme  $X'$ . From Lemma (6) it follows that there can be at most one  $\tau$  compatible with  $p$ .



*Step 2.* — Suppose  $\tau : K_0 X \rightarrow A_* X_0$  is a homomorphism compatible with some Chow envelope  $p : X' \rightarrow X$ . Then for any proper morphism  $f : Y \rightarrow X$ , with  $Y$  quasi-projective, the diagram:

$$\begin{array}{ccc} K_0 Y & \xrightarrow{\tau_Y} & A_* Y_0 \\ \downarrow f_* & & \downarrow f_* \\ K_0 X & \xrightarrow{\tau} & A_* X_0 \end{array}$$

commutes. In particular,  $\tau$  is compatible with every Chow envelope of  $X$ . To see this, choose  $q : Y' \rightarrow Y, f' : Y' \rightarrow X'$  as in Lemma (5). Given  $\alpha \in K_0 Y$ , choose  $\alpha' \in K_0 Y'$  with  $\alpha = q_* \alpha'$  (Lemma (6)). Then:

$$f_* \tau_Y(\alpha) = f_* \tau_Y(q_* \alpha') = f_* q_* \tau_{Y'}(\alpha') = p_* f'_* \tau_{Y'}(\alpha') = p_* \tau_{X'}(f'_*(\alpha')),$$

by the covariance property for quasi-projective schemes. By the compatibility of  $\tau$  with  $p$  :

$$p_* \tau_{X'}(f'_*(\alpha')) = \tau p_*(f'_* \alpha') = \tau f_*(\alpha).$$

as required.

*Step 3.* — We construct a homomorphism  $\tau_X : K_0 X \rightarrow A_* X_0$ , compatible with some (and hence any) Chow envelope of  $X$ , by induction on the dimension, the cases of dimensions  $-1$  and  $0$  being trivial. Given  $X$ , choose a Chow envelope  $p : X' \rightarrow X$ , with  $Y \subset X'$  as in Lemma (3). Form the fibre square

$$\begin{array}{ccc} Y' & \xrightarrow{j} & X' \\ q \downarrow & & \downarrow p \\ Y & \xrightarrow{i} & X \end{array}$$

Since  $j$  is a closed imbedding,  $Y'$  is quasi-projective. By induction, there is a homomorphism  $\tau_{Y'} : K_0 Y' \rightarrow A_* Y'_0$  compatible with the Chow envelope  $q$  (cf. Lemma (2)). Consider the diagram:

$$\begin{array}{ccccc} K_0 Y' & \xrightarrow{a} & K_0 Y \oplus K_0 X' & \xrightarrow{b} & K_0 X \rightarrow 0 \\ \downarrow \tau_{Y'} & & \downarrow \tau_Y \oplus \tau_{X'} & & \\ A_* Y'_0 & \xrightarrow{a} & A_* Y_0 \oplus A_* X'_0 & \xrightarrow{b} & A_* X_0 \end{array}$$

where  $a$  and  $b$  are defined as in Proposition 1. The square commutes by the covariance of  $\tau$  for the inclusion  $j$  of quasi-projective schemes, and by the induction hypothesis for  $q$ . By Proposition 1 the top row is exact. The composite of the two homomorphisms in the lower row is clearly zero (in fact, this row is also exact). There is therefore a unique homomorphism  $\tau_X$  from  $K_0 X$  to  $A_* X_0$  making the right square commute. In particular,  $\tau_X$  is compatible with  $p$ .

*Step 4.* — Proof of covariance (1). Given  $f: Y \rightarrow X$  proper, choose Chow envelopes  $p: X' \rightarrow X$ ,  $q: Y' \rightarrow Y$ , and a proper  $f': Y' \rightarrow Y$  as in Lemma (5). The proof that  $f_* \tau_{Y'} = \tau_X f_*$  is exactly the same as in Step 2.

*Step 5.* — Proof of the module property (2). Choose a Chow envelope  $p: X' \rightarrow X$ , and  $\alpha' \in K_0 X'$  with  $p_* \alpha' = \alpha$ . Using the projection formula and the known result on  $X'$ :

$$\begin{aligned} \tau_X(\beta \otimes \alpha) &= \tau_X p_* (p^* \beta \otimes \alpha) = p_* \tau_{X'} (p^* \beta \otimes \alpha') \\ &= p_* (\text{ch}(p^* \beta) \cap \tau_{X'}(\alpha')) = \text{ch}(\beta) \cap p_* \tau_{X'}(\alpha') = \text{ch}(\beta) \cap \tau_X(\alpha). \end{aligned}$$

*Step 6.* — Proof of the Cartesian product property (5). Choose Chow envelopes  $p: X' \rightarrow X$ ,  $q: Y' \rightarrow Y$ ,  $p_*(\alpha') = \alpha$ ,  $q_*(\beta') = \beta$ . As in the previous step, the required equation for  $\alpha \times \beta$  follows from the known result for  $\alpha' \times \beta'$ .

*Step 7.* — Proof of property (6). Let  $F_k K_0 X$  denote the subgroup of  $K_0 X$  generated by coherent sheaves whose support has dimension at most  $k$ , or, equivalently, by structure sheaves of closed subvarieties of dimension at most  $k$ . From the covariance property it follows that  $\tau_X$  maps  $F_k K_0 X$  into  $\sum_{i=0}^k A_i X_0$ . Choose a quasi-projective variety  $X'$  and a proper birational morphism  $p: X' \rightarrow X$ . The result follows from the known result for  $X'$  and the fact that  $p_* [\mathcal{C}_{X'}] = [\mathcal{C}_X] + \alpha$ ,  $\alpha \in F_{n-1} K_0 X$ .

*Step 8.* — Proof of uniqueness. This was proved in the category of quasi-projective schemes in [1], paragraph III.2. We saw in Step 3 that the extension to general algebraic schemes was uniquely determined by the covariance property (1).

The remaining properties (3) and (4) will be proved in paragraph 3.

## 2. Imbeddings in smooth varieties

For the proposition and corollary to be proved in this section, we explicitly ignore the constructions made for non-projective varieties in the preceding section. Instead we make use of a relative version of the theorems of [1] and [13]. Namely, fix a base scheme  $S$  which is smooth, but not necessarily quasi-projective over  $\text{Spec}(K)$ . Consider the category  $\mathcal{C}_S$  of schemes which are quasi-projective over  $S$ . For  $X$  in  $\mathcal{C}_S$ , choose a closed imbedding  $i : X \rightarrow M$ ,  $M \in \mathcal{C}_S$ ,  $M$  smooth over  $S$ . Define:

$$\tau_X : K_0 X \rightarrow A_* X_{\mathbf{Q}},$$

by setting:

$$\tau_X(\mathcal{F}) = \text{td}(i^* T_M) \cap \text{ch}_X^M(E),$$

where  $E$  is a resolution of the coherent sheaf  $i_*(\mathcal{F})$  by a complex of locally free sheaves,  $T_M$  is the tangent bundle to  $M$ , and  $\text{ch}_X^M(E) \in A_* X_{\mathbf{Q}}$  is the localized Chern character [1] paragraph II. 1.

The proofs of [1] and [13] extend without essential change, to show <sup>(1)</sup> that  $\tau_X$  is independent of the imbedding and the resolution, and that properties (1)-(4) hold with all schemes and morphisms in  $\mathcal{C}_S$ . (One small change is needed, to show that every vector bundle  $E$  on  $X \in \mathcal{C}_S$  is the restriction of a vector bundle on a smooth scheme in  $\mathcal{C}_S$ . Choose an imbedding  $i : X \rightarrow M$ ,  $M$  smooth in  $\mathcal{C}_S$ , and choose a surjection of a vector bundle  $F$  on  $M$  onto the coherent sheaf  $i_* E$ . Let  $\pi : G \rightarrow M$  be the Grassmann bundle of  $e$ -dimensional quotients of  $F$ ,  $e = \text{rank } E$ . Then  $G$  is smooth and in  $\mathcal{C}_S$ , and there is a morphism  $s : X \rightarrow G$  such that  $E$  is the pull-back of the universal quotient bundle.) For details, see [6], paragraph 18.

As in [1], if  $X \subset M$ ,  $\mathcal{F}, E$  are as above we write  $\text{ch}_X^M(\mathcal{F})$  in place of  $\text{ch}_X^M(E)$ ;  $\text{ch}_X^M$  determines a homomorphism from  $K_0 X$  to  $A_* X_{\mathbf{Q}}$ .

**PROPOSITION 2.** — *Let  $i : X \rightarrow M$ ,  $j : X \rightarrow P$  be closed imbeddings of a scheme  $X$  in smooth schemes  $M$  and  $P$ . Let  $\mathcal{F}$  be a coherent sheaf on  $X$ . Then:*

$$\text{td}(i^* T_M) \cap \text{ch}_X^M(\mathcal{F}) = \text{td}(j^* T_P) \cap \text{ch}_X^P(\mathcal{F}).$$

---

<sup>(1)</sup> It will follow from our Riemann-Roch theorem that this  $\tau_X$  agrees with that constructed in paragraph 1, but this fact cannot be used at this point in the argument.

*Proof.* — Consider the diagram:

$$\begin{array}{ccccc}
 X & \xrightarrow{f} & M \times X & \xrightarrow{k} & M \times P \\
 & & \downarrow q & & \downarrow p \\
 & & X & \xrightarrow{j} & P
 \end{array}$$

where  $f=(i, 1_X)$ ,  $k=1_X \times j$ , and  $p$  and  $q$  are the projections. Since  $q$  is smooth,  $f$  is a regular imbedding, with normal bundle  $i^*(T_M)$ . Let  $E$  be a resolution of  $j_* (\mathcal{F})$  by locally free sheaves on  $P$ . Then  $p^*(E)$  is a resolution of  $q^*(\mathcal{F})$  by locally free sheaves on  $M \times P$ , so by [1], paragraph II.2.6:

(i) 
$$q^* \text{ch}_X^p(\mathcal{F}) = \text{ch}_{M \times P}^{M \times X}(q^* \mathcal{F}).$$

Now let  $S = M \times P$ , and regard  $X$  and  $M \times X$  as quasi-projective schemes over  $S$  by means of the morphisms  $kf$  and  $k$ . We apply the formula of property (4) to the morphism  $f$ , using (3) to express  $\tau_X$  and  $\tau_{M \times X}$  explicitly. This gives the formula:

(ii) 
$$(kf)^*(\text{td}(T_{M \times P})) \cap \text{ch}_X^{M \times P}(f^* \alpha) = \text{td}(i^* T_M)^{-1} \cap f^*(k^*(\text{td}(T_{M \times P})) \cap \text{ch}_{M \times X}^{M \times P}(\alpha)),$$

for any  $\alpha \in K_0(M \times X)$ . Apply this to  $\alpha = q^*[\mathcal{F}] = [q^* \mathcal{F}]$ . Then:

$$f^* \alpha = f^* q^* [\mathcal{F}] = (qf)^* [\mathcal{F}] = [\mathcal{F}]$$

and:

$$\text{td}(i^* T_M)^{-1} \cdot (kf)^* \text{td}(T_{M \times P}) = j^* \text{td}(T_P).$$

Using (i) to rewrite  $\text{ch}_{M \times X}^{M \times P}(\alpha)$ , (ii) becomes:

(iii) 
$$(kf)^* \text{td}(T_{M \times P}) \cap \text{ch}_X^{M \times P}(\mathcal{F}) = j^* \text{td}(T_P) \cap f^*(q^* \text{ch}_X^p(\mathcal{F})) = j^* \text{td}(T_P) \cap \text{ch}_X^p(\mathcal{F}).$$

From (iii) it follows that the diagonal imbedding  $kf$  of  $X$  in  $M \times P$  determines the same class as the imbedding  $j$  of  $X$  in  $P$ . By symmetry, the required equality for  $X \subset M$  and  $X \subset P$  follows.

**COROLLARY.** — *Let  $Y$  be a quasi-projective scheme over  $\text{Spec}(K)$ ,  $X$  a scheme,  $i : X \rightarrow M$  a closed imbedding in a scheme  $M$  which is smooth over*

$\text{Spec}(K)$ . Let  $f: Y \rightarrow X$  be a proper morphism,  $\alpha \in K_0 Y$ . Then:

$$\text{td}(i^* T_M) \cap \text{ch}_X^M(f_* \alpha) = f_* \tau_Y(\alpha).$$

Here  $\tau_Y$  is the Riemann-Roch map constructed for the quasi-projective scheme  $Y$  in [1], paragraph II.

*Proof.* — Choose a closed imbedding  $j: Y \rightarrow U$ , with  $U$  open in some  $\mathbb{P}^n$ . Then  $(f, j)$  is a closed imbedding of  $Y$  in  $M \times U$ . By Proposition 2 :

$$\tau_Y(\alpha) = (f, j)^*(\text{td}(T_{M \times U}) \cap \text{ch}_Y^{M \times U}(\alpha)).$$

Then the required equation amounts to the covariance property for the Riemann-Roch theorem in the category  $\mathcal{C}_M$  of schemes quasi-projective over  $M$ .

### 3. Conclusion of the proof

We now return to the proof of the theorem in progress in paragraph 1. Thus  $\tau_X$  is defined for all schemes, extended by Chow envelopes from the explicitly constructed  $\tau$  for schemes quasi-projective over  $\text{Spec}(K)$ .

*Step 9.* — Proof of property (3). Given  $i: X \rightarrow M, \mathcal{F}, E$ , as in (3), choose a Chow envelope  $p: Y \rightarrow X$ , and choose  $\alpha \in K_0 Y$  such that  $p_* \alpha = [\mathcal{F}]$ . By the corollary in paragraph 2:

$$\text{td}(i^* T_M) \cap \text{ch}_X^M(\mathcal{F}) = f_* \tau_Y(\alpha).$$

By Step 2,  $f_* \tau_Y(\alpha) = \tau_X(\mathcal{F})$ , which proves (3).

*Note.* — It now follows from property (3) that the homomorphism  $\tau_X$  defined in paragraph 2 for a scheme  $X$  which is quasi-projective over some smooth base  $S$  is the same as the  $\tau_X$  constructed in paragraph 1.

*Step 10.* — Proof of property (4). Given  $f: X \rightarrow Y, X \subset M, Y \subset P$ , form the commutative diagram:

$$\begin{array}{ccccc} X & \xrightarrow{h} & M \times Y & \xrightarrow{k} & M \times P \\ & \searrow f & \downarrow q & & \downarrow p \\ & & Y & \xrightarrow{j} & P \end{array}$$

As usual, the truth of (4) for  $g$  and  $h$  implies (4) for the composite  $f$ . For  $g$ , (4) follows from the commutativity of localized Chern character with flat pull-back, as in the proof of Proposition 2, equation (ii). For  $h$ , (4) follows from the known property (4) in the category of quasi-projective schemes over  $M \times P$ . In both cases the note preceding this step is used to know that the  $\tau$ 's constructed relative to  $M \times P$  or  $P$  agree with the  $\tau$ 's constructed in paragraph 1. This concludes the proof of the theorem.

*Remarks 1.* — In characteristic zero, the GRR formula (Corollary 2) for a proper morphism  $f: X \rightarrow Y$  of smooth schemes can be deduced easily from the case of quasi-projective morphisms of smooth varieties, using Hironaka's resolution of singularities to construct a Chow envelope  $X' \rightarrow X$  with  $X'$  smooth. Without resolution of singularities we need the whole singular Riemann-Roch theorem to prove GRR for non-singular varieties.

2. When  $K = \mathbb{C}$ , the construction of:

$$\alpha_X: K_0 X \rightarrow K_0^{\text{top}} X,$$

proceeds by exactly the same plan, using [2] and [7] PRR in place of [1] and [13]. We leave the details to the reader.

3. For complex *analytic* spaces, if  $K_0 X$  denotes the Grothendieck group of coherent analytic sheaves, there should be homomorphisms:

$$K_0 X \rightarrow K_0^{\text{top}} X$$

and therefore  $K_0 X \rightarrow H_*(X; \mathbb{Q})$  satisfying the corresponding properties. For complex manifolds the best result so far is the theorem of O'BRIAN, TOLEDO, and TONG [11] which gives a version of GRR, but with values in the cohomology  $H^*(X, \Omega_X^*)$ .

4. We still do not know how to extend the ideas of this paper to higher  $K$ -theory, thus completing the work started in [8]. The essential difference between the  $K_0$  and  $K_i (i > 0)$  situations is that instead of the exact sequence of Proposition 1 we have (for  $i > 0$ ):

$$K_i Y' \xrightarrow{a} K_i Y \oplus K_i X' \xrightarrow{b} K_i X \xrightarrow{d} K_{i-1} Y' \rightarrow$$

hence even if  $\tau_Y, \tau_{Y'}, \tau_{X'}$  are all defined, this does not determine  $\tau_X$ .

5. The notion of an envelope (*see* paragraph 1) is distilled from the rather murky concept of a projective decomposition ([9], § 4). One can make the relationship more explicit by showing that envelopes have the

property of “universal homological descent” (this is analogous to the universal cohomological descent of SGA 4 V bis). Projective decompositions are then hypercoverings for the Grothendieck topology constructed using envelopes as covering maps. For example one may prove that if  $\pi : \tilde{X} \rightarrow X$  is an envelope then:

$$K_0(\tilde{X} \times_X \tilde{X}) \xrightarrow{d} K_0(\tilde{X}) \xrightarrow{\pi_*} K_0(X) \rightarrow 0$$

(where  $d = (p_1)_* - (p_2)_*$  with  $p_i : \tilde{X} \times_X \tilde{X} \rightarrow \tilde{X}$  the natural projections) is an exact sequence.

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