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BEHAVIOR OF DIFFUSION SEMI-GROUPS AT INFINITY

BY

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SUMMARY. — Let X be a diffusion process on a differentiable manifold M, with transition semi-group (P_t) and potential kernel $G = \int_0^\infty P_t \, dt$. We ask wether $P_t \, I_K(x)$ and $GI_K(x)$ tend to zero as x tends to infinity in M (K is compact in M). General sufficient conditions yield a positive answer and a speed of convergence when M has negative curvature and X is associated with the Laplace-Beltrami operator; in this case, we study the explosion time of X. When M is a Lie group and X is left invariant, we obtain the speed of convergence to zero for $P_t f(x)$ as x tends to infinity.

0. Introduction

Let M be a differentiable manifold, and D a second order (strictly) elliptic differential operator on M that $D\,1 \leqq 0$, and having Holder continuous coefficients in any local map. The existence and uniqueness of a continuous " diffusion process " X on M patching together all the local diffusion processes defined by D in small open sets with smooth boundaries is well known.

We begin our work by a characterization of the transition semi-group (P_t) of X as the *minimal* sub-Markov semi-group verifying

$$\left(\frac{\partial}{\partial t} - D\right) P_t f = 0,$$
$$\lim_{t \to 0} P_t f(x) = f(x)$$

for all smooth functions f with compact support. In general, P_t transforms bounded Borel functions into continuous functions, but does not leave $C_0(M)$ (continuous functions on M null at ∞) invariant. This is the case if, and only if, (P_t) is the unique strongly continuous contraction semi-group in $C_0(M)$ whose infinitesimal generator coincides with D on smooth functions with compact support.

We characterize this situation by the fact that for $t > 0$,

$$\lim_{x \to \infty} Q_x(T_K < t) = 0,$$

where Q_x is the law of X starting at x, T_K the entrance time of X in a compact set K. Another equivalent property is that, for $\lambda > 0$ the minimal positive solution of $(D - \lambda) u = 0$ on $M - K$, $u \equiv 1$ on ∂K, must tend to zero at infinity.

In dimension 1, this criterion yields explicit necessary and sufficient conditions. In dimension $n > 1$, we show how the maximum principle, and the consideration of minimal positive solutions permit a comparison with the 1-dimensional case, and furnish sufficient criteria for (P_t) to leave or not to leave $C_0(M)$ invariant, and give similar results concerning the behavior at infinity of the potential kernel of (P_t) (renewal theorem). We complete, by minor modifications (dealing with the coefficient $D\,1$, neglected in previous articles on this question) the Khas'minskii criteria for explosion of X and for recurrence.

As a first consequence, we show that, by a proper change of time (" slowing down at infinity ") of the form

$$t = \int_0^{\tau(t)} \frac{1}{f(X_u)} du,$$

it is possible to transform X into another diffusion, whose transition semi-group tends to zero at infinity, and with infinite lifetime.

We then consider the case of simply connected analytic manifolds of negative curvature (not necessarily constant). We show that the minimal semi-group associated with the Laplace-Beltrami operator on M tends to zero at infinity; in dimension $n \geqq 3$, we prove that the corresponding process X (" Brownian motion ") is transient, and that its potential kernel tends to zero at infinity faster than r^{2-n} (where r is the Riemann distance).

Calling $k(t)$ and $K(t)$ the smallest and largest sectional curvatures (in absolute value) at distance t of a fixed origin 0 in M, we show that if $(1/t) \int_0^t K(u)\, du$ is bounded, explosion is impossible for X, while if $k(t) \geqq (\text{Cte})\, t^{2+\varepsilon}$ for a fixed $\varepsilon > 0$ and t large, the probability of explosion is strictly positive.

Finally, if M is an homogeneous space of a Lie group, and D is left invariant, we prove by a probabilistic method that if f has compact support, $P_t f(x)$ is dominated by

$$\exp\left[(\text{Cte})(1+t)\right] \exp\left[-(\text{Cte})\frac{d^2(0, x)}{t}\right] \quad \text{for} \quad d(0, x) \geqq (\text{Cte})(1+t),$$

where all the constants are positive.

The terminology relative to Markov processes is not redefined here; we refer the reader to [6] (Ch. 0 and ch. 1). We point out that

0.1. Définition. — We call a sub-Markov semi-group on M a family of linear positive operators $(P_t)_{t>0}$ of $B(M)$ (the space of bounded Borel functions on M) into itself such that

$$P_t P_s = P_{t+s}; \qquad P_t 1 \leqq 1,$$
$$P_t f(x) \text{ is measurable in } t \quad \text{for } x \in M,\ f \in B(M).$$

1. Existence and uniqueness of diffusion processes

1.1. Let M be a connected differentiable manifold of class 3. Call $C^2(M)$ [resp. $C(M)$; $B(M)$; $C_0(M)$] the spaces of functions on M which are of class 2 (resp. continuous; bounded Borel measurable; continuous and tending to zero at infinity); the subscript c in $C_c^2(M)$, $C_c(M)$, $B_c(M)$ indicates that we consider only functions with compact support.

Let D be a second order elliptic differential operator on M. If $z = (z_1, \ldots, z_n)$ are local coordinates in a neighbourhood V of $x \in M$, we may write in V:

$$D = \sum_{i,j} a_{ij}(z)\frac{\partial^2}{\partial z_i \partial z_j} + \sum_i b_i(z)\frac{\partial}{\partial z_i} - c(z),$$

where the matrix $(a_{ij}(z))$ is symmetric *positive definite*. Throughout the paper, we assume that D has Holder continuous coefficients, i.e. that for each system of local coordinates, $a_{ij}(z)$, $b_i(z)$, $c(z)$ verify a (local) Holder continuity condition; moreover, we assume $c(z) \geqq 0$, so that D verifies the positive maximum principle (*cf.* [4], section I.1).

1.2. Definition. — Dynkin ([9], vol. 1, section 5.18) calls *diffusion on M with differential generator D* any continuous Markov process X on M

such that the characteristic operator of X be defined on $C^2(M)$, and coincide with D on that set. Clearly, this property holds if and only if for some open cover (U_i) of M, the process induced by X on each U_i is a diffusion on U_i with differential generator D.

1.3. DEFINITION. — We say that a sub-Markovian semi-group (P_t) of linear operators on $B(M)$ (*see* section 0) is a C_0-diffusion semi-group with differential generator D, if (P_t) restricted to $C_0(M)$ is a strongly continuous semi-group of operators from $C_0(M)$ *into itself*, and if the strong infinitesimal generator of (P_t) is defined on $C_c^2(M)$ and coincides with D on $C_c^2(M)$.

The maximum principle shows that there is at most one such semi-group for a given D ([11], ch. 2, th. 1 and th. 7). Globally, C_0-diffusion semi-groups need not exist (*cf.* section 4.5), but they always exist locally : if U is an open, relatively compact subset of M, with smooth boundary (*see* def. 1.4 below), then $\overline{U}$ is a manifold with boundary, and as shown in [4], there is an unique C_0-diffusion semi-group on U with differential generator D.

We explicit the notion of smoothness just used :

1.4. DEFINITION. — An open set U in M is said to have *smooth boundary* ∂U if, for any $x \in \partial U$, there is a neighbourhood V of x and a local map h, defined on V, such that $h(U \cap V)$ be the intersection of $h(V)$ with an open half-space in $\mathbf{R}^n$.

As is easily deduced from [19], lemma 4, there is an increasing sequence (U_n) of open, relatively compact subsets of M with smooth boundary such that $M = \bigcup U_n$.

We shall take a limit of the corresponding sequence of C_0-diffusion semi-groups to associate to D a canonical semi-group on M. To characterize it, we introduce the

1.5. DEFINITION. — A sub-Markovian semi-group (P_t) on M is called *minimal* in a set E of semi-groups if $(P_t) \in E$ and if, for any $(Q_t) \in E$ and $f \geqq 0$ in $B(M)$,

$$P_t f \leqq Q_t f \quad \text{for all } t > 0.$$

There is obviously at most one minimal element in E.

1.6. THEOREM. — *Let D be a second order elliptic differential operator on M with Holder coefficients. Call problem* (i), $i = 1, 2, 3$, *the problem* *of finding a sub-Markovian semi-group* (P_t) *on* M *verifying condition* (i), *where* (i) *is any one of the following properties.*

$$(1)\quad \begin{cases} \dfrac{\partial}{\partial t}(P_t f) = DP_t f \\ \lim_{t\to 0} P_t f(x) = f(x) \end{cases} \quad \textit{for all } t>0,\ f\in C_c^2(M),\ x\in M,$$

$$(2)\quad \begin{cases} \dfrac{\partial}{\partial t}(P_t f) = P_t D f \\ \lim_{t\to 0} P_t f(x) = f(x) \end{cases} \quad \textit{for all } t>0,\ f\in C_c^2(M),\ x\in M,$$

$$(3)\quad P_t f(x) - f(x) = \int_0^t P_u D f(x)\,du, \quad \textit{for all } t>0,\ f\in C_c^2(M),\ x\in M.$$

There is a common solution (P_t) *of problems* (1), (2) *and* (3), *which, for each* $i = 1, 2, 3$, *is the unique minimal solution of problem* (i).

Moreover, (P_t) *is the transition semi-group of the unique (up to equivalence) continuous Markov process* X *on* M *whose transition semi-group solves problem* (3) (*the so-called " martingale problem "*, cf. [22], [3]).

The process X *is also the unique continuous strong Markov process which induces on each open relatively compact set with smooth boundary a process whose transition semi-group is a* C_0*-diffusion with differential generator* D. *Finally,* X *is also a diffusion in the sense of definition* 1.2.

The proof is sketched after a few remarks.

1.7. Bibliographical remarks. — The case of R^n and open subsets of R^n, under additional global assumptions on D, is studied thoroughly in Dynkin [9]; the case of compact manifolds with boundary is treated in Bony-Courrège-Priouret [4]. Problem (3) is studied extensively by Stroock-Varadhan [22] in R^n and considered by Molchanov [19] and Azencott [3] on a general manifold, for much less smooth coefficients.

For C^∞-coefficients, Mc Kean [16] (section 4.3) introduces the idea of minimal fundamental solution and gives a result quite similar to theorem 1.6. The question of uniqueness is avoided in [19]; it is solved by adding a compactness hypothesis in [4] or uniform bounds on the coefficients of D ([9], [22]).

We recall a well-known lemma.

1.8. LEMMA. — *Let D be as in the theorem* 1.6. *Let f_n be an increasing, locally bounded, sequence of functions in $C^2(M)$ [resp. $f_n(t, x)$ of class* 1 *in t,* 2 *in $x \in M$] such that $Df_n = g$, (resp. $[(\partial/\partial t)-D]f_n = 0$), where g is Holder continuous. Then the limiting function $f = \lim_{n\to\infty} f_n$ is of class* 2 *(resp.* 2 *in $x \in M$,* 1 *in t), and verifies $Df = g$ (resp. $[(\partial/\partial t)-D]f = 0$).*

Proof. — The lemma is an easy consequence of Schauder's interior estimates ([11], ch. 3, th. 5, and problem 6, p. 89 for the parabolic case, ch. 3, section 8 for the elliptic case).

Q. E. D.

Proof of theorem 1.6. — We have shown in [3], that there is an unique continuous Markov process X on M whose transition semi-group (P_t) solves problem (3), and that moreover, (P_t) is the unique minimal solution of problem (3). Since X is continuous, $P_t Df$ is continuous in t for $f \in C_c^2(M)$ and hence (P_t) is also a solution of problem (2).

Let U be open, relatively compact in M, and let U have smooth boundary. The induced process X_U is (by [3], th. 4.1, th. 9.1, prop. 10.2) the unique continuous Markov process on U whose transition semi-group (P_t^U) solves problem (3).

But, as seen above, there is a C_0-diffusion semi-group (Q_t) on U with differential generator D. In particular, (Q_t) must solve problem (1) on U since its strong infinitesimal generator extends $(D, C_c^2(U))$. *A fortiori*, (Q_t) solves problem (3) on U. Consider a standard Markov process Y on U with transition semi-group Q_t [6]. Since (Q_t) maps $C_0(U)$ into itself, almost all trajectories of Y remain in a compact set during finite time intervals contained in $[0, \zeta[$, where ζ is the lifetime of Y ([6], proof of prop. 9.4). On the other hand, by [3] (prop. 5.1) and [8] (section 6.14, th. 6.6), since (Q_t) solves problem (3), the trajectories of (Q_t) are continuous until the exit-time from any compact. Hence, Y is a continuous Markov process (i.e. continuous until its time of death). Since there is only one such process with transition semi-group solving problem (3), we must have $Q_t = P_t^U$.

Let (U_n) be an increasing sequence of smooth open sets exhausting M, as in definition 1.4. Calling T_n the exit time from U_n for X, and letting $P_t^n = P_t^{U_n}$, we have with standard notations, for $f \geqq 0$ and $f \in B(M)$,

$$P_t f(x) = E_x[f(X_t)] = \lim_{n\to\infty} \uparrow E_x[f(X_t) I_{t<T_n}] = \lim_{n\to\infty} \uparrow P_t^n f(x).$$

The equality $Q_t = P_t^U$ obtained above implies that if $f \in C_c^2(M)$ and if support (f) is included in U_n, then

$$\left(\frac{\partial}{\partial t} - D\right) P_t^n f = 0 \quad \text{on } U_n.$$

Applying now lemma 1.8, we conclude that $P_t f \in C^2(M)$ and that $[(\partial/\partial t) - D]\, P_t f = 0$ on M. Taking account of the continuity of X, we see that (P_t) solves problem (1).

Thus (P_t) is a common solution to problems (1), (2) and (3). Since (P_t) is minimal for problem (3), and since any solution of problem (2) is also a solution of problem (3) (take account of def. 0.1) (P_t) is minimal for problem (2).

Now let (R_t) be a solution of problem (1), and let $f \geqq 0$ be in $C_c^2(M)$. If P_t^n is defined as above, with n large enough, the function $h = R_t f - P_t^n f$ verifies

$$\left(\frac{\partial}{\partial t} - D\right) h = 0, \quad \text{on } U_n,$$
$$\lim_{t \to 0} h = f, \quad \text{on } U_n,$$
$$\lim_{x \to a} h \geqq 0, \quad a \in \partial U_n$$

and hence, by the maximum principle ([11], ch. 2, th. 1), $h \geqq 0$ on U_n. As n tends to infinity, we get $R_t f \geqq P_t f$, and (P_t) is minimal for problem (1).

Finally the fact that X is a diffusion in the sense of definition 1.2 need only be checked locally. By a local diffeomorphism, one reduces the situation to the case where M is an open ball of $\mathbf{R}^n$, and X the trace of a " canonical diffusion process " (*cf.* [9], vol. 1, section 5.26) in $\mathbf{R}^n$, which achieves the proof.

Q. E. D.

1.9. Definition and remarks. — We call the semi-group (P_t) and the process X obtained in theorem 1.6 the *minimal semi-group* and the *minimal process* associated to D (on the state space M). We note the fact (*cf.* [3], th. 9.1 and cor. 10.3) that if the lifetime of X is almost surely infinite, then the minimal semi-group is in fact the unique solution of problem (i) above, for each $i = 1, 2, 3$.

Since a continuous strong Markov process X on M is uniquely determined by the collection of induced processes X_{U_i}, where (U_i) is an open cover on M (*cf.* [7], th. 2.4.2), the second characterization of X in theorem 1.6 shows that :

1.10. COROLLARY. — *If X is the minimal process associated to D on M, and if U is any arbitrary open subset of M (not necessarily smooth), the process induced by X on U is the minimal process associated to D on U.*

We recall that a sub-Markovian semi-group (P_t) on M has the *strong Feller property* if each P_t maps $B(M)$ into $C(M)$. The strong Feller property of the semi-groups considered here is proved in MOLCHANOV [19], but we obtain a stronger result :

1.11. PROPOSITION. — *The minimal semi-group (P_t) associated to D has the strong Feller property; in fact, if $f \in B(M)$, $P_t f(x)$ is of class 1 in t, 2 in x, and verifies $[(\partial/\partial t)-D]\,P_t f(x) = 0$, $t > 0$, $x \in M$.*

Proof. — The property described above certainly holds if $f \in C_c^2(M)$. But if $f \in C_c(M)$ there is an *increasing* sequence f_n such that $f_n \in C_c^2(M)$ and $f = \lim_{n\to\infty} f_n$ (when $M = R^p$, approximate $[f-(1/2^n)]$ within $1/2^{n+1}$ by $g_n \in C^2$ and set $f_n = hg_{2n}$, where $0 \leqq h \leqq 1$, $hf = f$, and $h \in C_c^2$). Applying lemma 1.8, we get $[(\partial/\partial t)-D]\,P_t f = 0$. A similar argument gives the result if f is the indicator function of an open set, or if $(-f)$ is the indicator function of a closed set. The result holds then for all simple functions and again by monotone limits for all $f \in B(M)$.

Q. E. D.

1.12. PROPOSITION. — *Let D be as in theorem 1.6. A sub-Markovian semi-group (P_t) on M is the C_0-diffusion semi-group with differential generator D if, and only if, (P_t) is the minimal semi-group associated with D and leaves $C_0(M)$ invariant.*

Proof. — The fact that a C_0-diffusion semi-group with differential generator D is necessarily minimal was proved in the course of the proof of theorem 1.6 (for U instead of M, but the argument is the same). Conversely, if the minimal semi-group (P_t) associated with D leaves $C_0(M)$ invariant, (P_t) is weakly continuous (since the minimal process has continuous trajectories) and hence strongly continuous on $C_0(M)$. Equation (3), in theorem 1.6, shows readily that if $f \in C_c^2(M)$, f belongs to the domain of the weak infinitesimal generator A' of (P_t) and that

$A'f = Df$. But A' is also the strong infinitesimal generator of (P_t) ([9], vol. 1, lemma 2.11), and hence (P_t) is a C_0-diffusion semi-group with differential generator D.

Q. E. D.

1.13. REMARK. — We note that the minimal semi-group (P_t) has a density $p(t, x, y)$ with respect to a natural measure on M (locally equivalent to Lebesgue measure) which as a function of (t, x) verifies $[(\partial/\partial t) - D]\, p(t, x, y) = 0$ for each y (*see* [2] for a sketch of the proof). As shown in [2], (compare also Mc KEAN [16], section 4.3 for C^∞-coefficients), $p(t, x, y)$ is the minimal fundamental solution of $[(\partial/\partial t) - D]$. The main tools used to prove these results are the maximum principle (prop. 1.11), and as a starting point ([3], prop. 6.1 and prop. 9.2).

1.14. LEFT-LIMITS AT EXPLOSION TIME. — Let X be the minimal process and ζ its *lifetime* ($\zeta = \inf\{t;\, X_t = \delta\}$, where δ is a " cemetery " point at infinity adjoined to M). The left-limit of X_t as $t \to \zeta - 0$ exists almost surely (in $M \cup \delta$), on the set $\zeta < +\infty$ after [3] (prop. 5.5 and lemma 8.3). We also note that when $c(x) = -D\,1$ *is identically* 0, then we have $\lim_{t \to \zeta - 0} X_t = \infty$ almost surely, on the set $\zeta < +\infty$. Indeed in that case, let U be an open relatively compact set in M, with smooth boundary, and let T be the entrance time in U^c for X. If Q_x is the law of X starting at x, the function $g(x) = Q_x(T < +\infty)$ is clearly harmonic in DYNKIN's sense ([9], vol. 2, section 12.18) on the set U. Applying locally [9] (vol. 2, th. 13.9), we conclude that $D\,g = 0$ on U. On the other hand, every point of ∂U is regular for U^c ([9], vol. 2, th. 13.8) so that $\lim_{x \to a} g(x) = 1$ for $a \in \partial U$ (*cf.* [9], vol. 2, proof of th. 13.1). Since $D\,1 \equiv 0$, we must have $g(x) \equiv 1$ on U. Apply this result to an increasing sequence U_n of such open sets exhausting M, to see that we have $\zeta = \lim_{n \to \infty} \uparrow T_n$, almost surely ($T_n$ is the entrance time in U_n). In particular, almost surely

$$X_{\zeta - 0} = \lim_{t \nearrow \zeta - 0} X_t = \lim_{n \to \infty} X_{T_n} = \infty$$

if we make sure for instance that U_n contains a ball of radius n, where the distance used is finite on compact sets.

This result (similar to Mc KEAN, [16], section 4.3, which treats the case where D has C^∞-coefficients) justifies the alternate name of *explosion time* for ζ [at least when $c(x) = -D\,1(x) = 0$]. More is said about explosion time in lemma 4.5, proposition 4.6, proposition 5.4, lemma 6.3, remark 6.4.

2. Potential kernel

We need two auxiliary results.

2.1 LEMMA. — *Let* (P_t) *be the minimal semi-group associated to* D. *Then for any* $f \geqq 0$ *in* $C(M)$, $t \geqq 0$, *the function* $P_t f$ *is strictly positive, unless* $f \equiv 0$.

Proof. – Consequence of the maximum principle ([11], ch. 2, th. 1) since by proposition 1.11, we have $[(\partial/\partial t)-D]\, P_t f = 0$.

Q. E. D.

Let X be the minimal process associated to D on M; let U be an open, relatively compact subset of M, with smooth boundary. Let T be the first exit time from U for X. Let Q_x be the law of X, starting at x. For $x \in U$ and $t > 0$, we have $Q_x(T < \infty) > Q_x(X_t \in U^c)$ and hence, by lemma 2.1, $Q_x(T < \infty) > 0$.

Consider U_1 open, relatively compact, with smooth boundary, containing U. Thanks to the preceding remark we may apply successively [9], vol. 2, theorem 13.15, and [9], vol. 1, theorem 5.9, to the induced process X_{U_1} and U. We then obtain the following extension of [9], vol. 2, theorem 13.16 :

2.2. If h is a continuous function on ∂U and if g is a Holder continuous function on U, then the function

$$f(x) = E_x\left[\int_0^T g(X_t)\, dt\right] + E_x[h(X_T)], \qquad x \in U,$$

is in $C^2(U)$ and is the unique solution of

$$Df = -g \quad \text{in } U,$$
$$\lim_{x \to a} f(x) = h(a), \qquad a \in \partial U.$$

Define now the *potential kernel* G of D by

$$Gf(x) = \int_0^\infty P_t f(x)\, dt, \qquad \text{for } f \geqq 0 \text{ and } f \in B(M),$$

where (P_t) is the minimal semi-group associated to D.

2.3. PROPOSITION. — *Let the differential operator D be as in theorem* 1.6. *If $f \geqq 0$ is Holder continuous, then either $Gf(x)$ is infinite for all $x \in M$, or $Gf \in C^2(M)$ and verifies*

$$D(Gf) = -f \quad \text{on } M.$$

In this last case, if $h \geqq 0$ is in $C^2(M)$ and if $Dh \leqq -f$ then $Gf \leqq h$.

Proof. — Let U_n be an increasing sequence of open sets exhausting M (as in def. 1.4). Let X and (P_t) be the minimal process and semi-group associated to D on M. Call T_n the first exit time from U_n for X. Then, with standard notations,

$$P_t f(x) = \lim_{n\to\infty} \uparrow E_x[f(X_t) I_{t<T_n}] \qquad \text{for} \quad f \geqq 0$$

and hence

$$Gf = \lim_{n\to\infty} \uparrow h_n \qquad \text{where} \quad h_n = \int_0^{T_n} f(X_t)\,dt.$$

Since f is Holder continuous, the result 2.2 implies $Dh_n = -f$ in U_n. If for some point x_0, $Gf(x_0)$ is finite, then for $t > 0$, we clearly have

$$P_t\,Gf(x_0) \leqq Gf(x_0) < \infty$$

and hence, since (*cf.* remark 1.13) the measure $P_t(x_0, dy)$ is absolutely continuous with respect to a measure μ locally equivalent to Lebesgue measure, we get $Gf(x) < \infty$ for μ-almost all x, and consequently $Gf(x) < \infty$ for $x \in E$ where E is dense in M. For y fixed and arbitrary in M, the precise Harnack inequality established by SERRIN ([21], th. 4) shows the existence of a neighbourhood U of y such that if $x \in U$ and if the sequence $h_n(x)$ is bounded, then the sequence h_n is uniformly bounded on U. Since $E \cap U$ is not empty, we conclude that the sequence h_n is locally bounded. Then lemma 1.8 implies that $Gf \in C^2(M)$ and $D(Gf) = -f$.

Now if $h \in C^2(M)$ verifies $Dh \leqq -f$ and $h \geqq 0$, the maximum principle implies $h \geqq h_n$ on U_n and hence $h \geqq \lim_{n\to\infty} h_n = Gf$.

Since due to lemma 2.1 any two points in M " communicate " (in the sense of [1], p. 196, p. 199) the following result could be deduced from [1], but we prove it directly.

2.4. COROLLARY. — (AZEMA-DUFLO-REVUZ [1]). — *Let D be as in the theorem* 1.6, *and let G be the potential kernel of D. Either GI_U is identi-*

cally infinite for each open set U in M, or GI_K is everywhere finite for each compact set K in M.

Proof. — If $GI_U(x) < \infty$ for some open set U and some $x \in M$, there is a function $f \geqq 0$, Holder continuous and bounded such that $f \leqq I_U$ and hence by proposition 2.3, Gf is everywhere finite. For $t > 0$ the function $P_t f$ is continuous and strictly positive (prop. 1.11 and lemma 2.1), and we have $GP_t f \leqq Gf$. For each K compact, I_K is dominated by a multiple of $P_t f$, and hence GI_K is everywhere finite.

Q. E. D.

Since P_t has the strong Feller property, and since any two states communicate in the sense of [1], the results of [1] imply that X is *transient* if, and only if, G is finite on compact sets, *recurrent* if, and only if, G is infinite on open sets. Here transience means that for each $x \in M$ the trajectories of X tend Q_x-a.s. to infinity as $t \to +\infty$, and recurrence means that for each $x \in M$ the trajectories of X return (Q_x-a.s.) infinitely often to any given open set in M.

2.5. COROLLARY. — *For $\lambda > 0$, let $R_\lambda = \int_0^\infty e^{-\lambda t} P_t \, dt$ be the resolvent kernel of D, where P_t is the minimal semi-group associated to D. If f is bounded positive and Holder continuous, then $R_\lambda f$ is in $C^2(M)$ and is the minimal positive solution of $[D-\lambda] R_\lambda f = -f$.*

Proof. — Note that $e^{-\lambda t} P_t$ is the minimal semi-group associated to $[D-\lambda]$ (*cf.* [3], th. 8.1 and th. 9.1) and apply proposition 2.3 to $[D-\lambda]$.

Q. E. D.

3. Diffusion semi-groups tending to zero at infinity

Let (P_t) be the minimal semi-group associated to the differential operator D. The simplest case in which (P_t) may be explicitly considered as a continuous contraction semi-group of operators on a Banach space, and where its infinitesimal operator is explicitly described is the case where (P_t) is a C_0-diffusion semi-group, i.e. by proposition 1.12 when (P_t) leaves $C_0(M)$ invariant. Historically, this case was emphasized by DYNKIN, who introduced ad hoc global hypotheses (when $M = \mathbf{R}^n$) on the coefficients of D, inherited from the early theory of parabolic equations ([11], p. 3, and [9], section 5.25) to realize this situation. We give various forms of this property.

3.1. PROPOSITION. — *Let D be a differential operator on M, as in theorem 1. Let (P_t) and X be the minimal semi-group and processes associated to D. Let Q_x be the law of X starting at x, on the adequate space of trajectories, and let E_x be the corresponding expectation operator. For each compact subset K of M let T_K be the entrance time of X in K. The following properties are equivalent :*

(i) *(P_t) is a C_0-diffusion semi-group;*

(ii) *for each $t > 0$,*

$$\lim_{x\to\infty} P_t f(x) = 0, \qquad \textit{for all } f \in B_c(M);$$

(iii) *for each $\lambda > 0$,*

$$\lim_{x\to\infty} \int_0^\infty e^{-\lambda t} P_t f(x)\, dt = 0, \quad \textit{for all } f \in B_c(M);$$

(iv) *for each $t > 0$,*

$$\lim_{x\to\infty} Q_x(T_K < t) = 0, \quad \textit{for all } K \textit{ compact in } M;$$

(v) *for each $\lambda > 0$,*

$$\lim_{x\to\infty} E_x(e^{-\lambda T_K}) = 0, \quad \textit{for all } K \textit{ compact in } M.$$

Proof. — The equivalence of (i) and (ii) is a direct consequence of proposition 1.11 and 1.12. We now prove (v) $\Rightarrow$ (iv) $\Rightarrow$ (ii) $\Rightarrow$ (iii) $\Rightarrow$ (v).

(v) $\Rightarrow$ (iv) is a consequence of the inequality

$$E_x(e^{-\lambda T_K}) \geqq E_x(e^{-\lambda T_K} I_{T_K<t}) \geqq e^{-\lambda t}\, Q_x(T_K < t).$$

For $f \in E_c(M)$, let K be the support of f; then $|f(X_t)| \leqq \|f\|\, I_{T_K \leqq t}$ for x not in K, and hence

$$P_t f(x) \leqq \|f\|\, Q_x(T_K \leqq t)$$

which proves (iv) $\Rightarrow$ (ii).

The dominated convergence theorem shows that (ii) $\Rightarrow$ (iii). To prove (iii) $\Rightarrow$ (v), we fix $\lambda > 0$ and note (cor. 2.5) that if $f \in C_c^2(M)$, the function $g = \int_0^\infty e^{-\lambda t} p_t f\, dt$ verifies $(D-\lambda)\, g = -f$. Let K be a compact containing the support of f. Using the strong Markov property of X, one sees readily that the function $h(x) = E_x(e^{-\lambda T_K})$ is harmonic in $M - K$

(in the sense of [9], vol. 2, section 12.18) with respect to the process associated to the semi-group $e^{-\lambda t} P_t$. Applying *locally* [9] (vol. 2, th. 13.9), we see that h is in $C^2(M-K)$ and verifies $(D-\lambda)h = 0$ in $M-K$.

Let the open sets U_n and the exit times T_n be as in proposition 2.3. We have $h = \lim_{n\to\infty} \uparrow h_n$, where

$$h_n(x) = E_x(e^{-\lambda T_K} I_{T_K < T_n}).$$

As above one can prove that $(D-\lambda)h_n = 0$ in $U_n - K$. We now study the limit values of h_n on ∂U_n and ∂K. For n large enough, we have $U_n \supset K$ and for each $u > 0$:

$$\begin{aligned} 0 \leqq h_n(x) \leqq Q_x(T_K < T_n) &= Q_x(T_K < T_n < u) + Q_x(T_K < T_n, u < T_n) \\ &\leqq Q_x(T_K < u) + Q_x(T_n \geqq u). \end{aligned}$$

When x remains in a compact set disjoint from K, we have

$$\lim_{u\to 0} Q_x(T_K < u) = 0$$

uniformly in x (*cf.* [3], cor. 5.3). On the other hand, for *fixed* $u > 0$ and $a \in \partial U_n$, $\lim_{x\to a} Q_x(T_n \geqq u) = 0$ since U_n is smooth ([9], vol. 2, proof of th. 13.1, th. 13.8). These two facts and the last inequality show that $\lim_{x\to a} h_n(x) = 0$ for $a \in \partial U_n$. On the other hand, $\overline{\lim}_{x\to b} h_n(x) \leqq 1$ for $b \in \partial K$. Assume now $f \geqq 0$. Since $R_\lambda f$ is continuous and strictly positive (by lemma 2.1) there is a $c > 0$ such that $R_\lambda f \geqq c$ on ∂K; we have

$$\overline{\lim}_{x\to b} h_n(x) \leqq \frac{1}{c} R_\lambda f(b) \qquad \text{for} \quad b \in \partial(U_n - K),$$

and hence, by the maximum principle $h_n \leqq (1/c) R_\lambda f$ on $U_n - K$. As $n \to \infty$, we get $h \leqq (1/c) R_\lambda f$ on $M-K$; the fact that (iii) $\Rightarrow$ (v) is easy to deduce from this inequality.

Q. E. D.

3.2. Remark. — The arguments just used show also that if $K = \overline{V}$ where V is open with smooth boundary, then $h(x) = E_x(e^{-\lambda T_K})$ is the minimal positive solution of the system

$$\begin{cases} (D-\lambda)h = 0 & \text{in } M-K, \\ \lim_{x\to a} h(x) = 1, & a \in \partial K. \end{cases}$$

3.3. REMARK. — Properties (iv) and (v) above show that the situation of proposition 3.1 is realized if the process X needs (with large probability) a large amount of time T_K to come back to K from a faraway starting position x. In particular, the amplitude of the oscillations of X or its drift " backwards " should not be too large when the starting point x tends to infinity. This suggests that (P_t) will be a (C_0)-diffusion if the " coefficients " of D are " bounded " or tend to zero at infinity. This heuristic consideration is completely supported by the criteria of proposition 4.3 and proposition 5.2.

Also this makes it likely that, by slowing down the process when the trajectories approach infinity (change of time), we will transform (P_t) into a C_0-diffusion. This guess is correct as shown in proposition 6.2 below.

We have just seen that (P_t) " tends to zero at infinity " if, and only if, the resolvent kernels R_λ have the same property. This says nothing about the behavior of the potential kernel G at ∞, tied up with the renewal theorem, which we now consider.

3.4. PROPOSITION. — *Let D be a differential operator (as in th. 1.6) on M, G the potential kernel of D, X the minimal process associated to D, Q_x its law starting at x, and T_K its entrance time in a set K. The two following properties are equivalent:*

(i) *G is finite on compact sets (transient case) and* $\lim_{x\to\infty} Gf(x) = 0$ *for all $f \in B_c(M)$;*

(ii) *for each compact K in M,* $\lim_{x\to\infty} Q_x(T_K < \infty) = 0$.

Moreover for each K compact with non-empty interior, and each f non-zero in $C_c(M)$, the ratio $Q_x(T_K < \infty)/Gf(x)$ is then bounded away from 0 *and $\pm\infty$ as x varies in M.*

Proof. — It is easily checked that $g(x) = Q_x(T_K < \infty)$ is harmonic (in DYNKIN's sense) for the process X, in the set $M - K$. Applying locally [9] (vol. 2, th. 13.9), we conclude that $g \in C^2(M - K)$ and verifies $Dg = 0$ in $M - K$. If the potential G is finite on compact sets, we take $f \geqq 0$ in $C_c^2(M)$ and K compact with non-empty interior and smooth boundary. Let the open set U_n and the exit times T_n be as in proposition 2.3. As in proposition 2.3, we have $Gf = \lim_{n\to\infty} \uparrow h_n$, where $Dh_n = -f$ on U_n, and $\lim_{x\to a} h_n(x) = 0$ for $a \in \partial U_n$. As in the proof of proposition 3.1, we have $g = \lim_{n\to\infty} \uparrow g_n$ where $Dg_n = 0$ on

$U_n - K$, and $\lim_{x \to a} g_n(x) = 0$ for $a \in \partial U_n$ [take

$$g_n(x) = Q_x(T_K < \infty,\ T_K < T_n)].$$

If $f \not\equiv 0$ and K has non-empty interior, lemma 2.1 implies that g and Gf are strictly positive. The maximum principle applied to $U_n - K'$, where K' is compact and contains K and the support of f, shows then the existence of strictly positive numbers c, c' such that

$$h_n \leqq cg,\ g_n \leqq c' h \quad \text{on } U_n - K' \quad \text{for all } n.$$

Letting n tend to infinity, we see that g/h is bounded away from 0 and $+\infty$ on $M - K'$. Obvious arguments conclude the proof; one needs nevertheless remark that if we are in the recurrent case, i.e. if G is infinite on open sets, then $Q_x(T_K < \infty)$ is identically equal to 1 (*cf.* [1]).

Q. E. D.

3.5. Remark. — In view of proposition 3.1 (v), when the preceding situation is realized, then (P_t) must be a C_0-diffusion semi-group. Note also that as in remark 3.2, $Q_x(T_K < \infty)$ is the minimal positive solution of $Du = 0$ on $M - K$, $u = 1$ on ∂K, when K is the closure of a smooth open set.

4. The one dimensional case

The more general sufficient criteria, given in section 5, will be obtained by comparison with the one-dimensional case which we now consider.

4.1. Let D be the operator $a(x)(d^2/dx^2) + b(x)(d/dx) - c(x)$ defined on $)s_1, s_2($ where $-\infty \leqq s_1 < s_2 \leqq +\infty$, and where $a(x)$, $b(x)$, $c(x)$ are Holder continuous (locally) with $a > 0$, $c \geqq 0$. Fix s_0 such that $s_1 < s_0 < s_2$. The well-known transformation ([9], vol. 2, section 17.2):

$$z = h(x) = \int_{s_0}^{x} H(x)\,dx \qquad \text{where} \quad H(x) = \exp\left[-\int_0^x \frac{b(t)}{a(t)}\,dt\right]$$

is a diffeomorphism of $)s_1, s_2($ onto $)r_1, r_2($ where

$$r_i = \int_{s_0}^{s_i} H(x)\,dx, \qquad i = 1, 2,$$

and transforms D into the operator

$$A = \alpha(z)\frac{d^2}{dz^2} - \gamma(z),$$

where

$$\alpha \circ h(x) = a(x) H^2(x),$$
$$\gamma \circ h(x) = c(x).$$

Presumably, the one-dimensional criterion which we are looking for could be deduced from FELLER [10], or DYNKIN [9] (vol. 2, ch. 16, 17), or MEYERS-SERRIN [18], or MANDL [(17] (ch. II). But either these works consider only the case $c(x) = 0$, or, as FELLER, or MANDL [17], (ch. II, probem 6), cannot keep track explicitly of that coefficient in the final results. We present a self-contained proof, built to deal with the coefficient $c(x)$.

4.2. LEMMA. — *Fix $\lambda \geqq 0$ and r_0 such that $r_1 < r_0 < r_2$. There is a positive continuous function $u \not\equiv 0$ on $[r_0, r_2[$ such that*

(1) $$\begin{cases} (A-\lambda)u = 0 \quad on\ ]r_0, r_2[, \\ \lim_{z \to r_2} u(z) = 0, \end{cases}$$

if, and only if,

– *either : r_2 is finite;*

– *or : $r_2 = +\infty$ and*

$$\int_{r_0}^{+\infty} dz \int_z^{+\infty} \frac{\lambda + \gamma(t)}{\alpha(t)} dt = +\infty.$$

Proof. – As shown in proposition 3.1 and proposition 3.4 there is a minimal positive solution v of the system

(2) $$\begin{cases} (A-\lambda)v = 0 \quad \text{on }]r_0, r_2[, \\ \lim_{z \to r_0} v(z) = 1, \end{cases}$$

and $v = \lim_{n \to \infty} \uparrow v_n$, where v_n verifies

$$\begin{cases} (A-\lambda)v_n = 0 \quad \text{on }]r_0, r_n[, \\ v_n(r_0) = 1, \\ v_n(r_n) = 0, \end{cases}$$

and where $r_2 = \lim_{n \to \infty} \uparrow r_n$.

Assume first $r_2 < +\infty$; the linear function $f(z) = (r_2 - z)/(r_2 - r_0)$ verifies $(A-\lambda)f \leqq 0$ on $]r_0, r_2[$. The maximum principle shows that $v_n \leqq f$ on $]r_0, r_n[$, and hence

(3) $$v(z) \leqq \frac{r_2 - z}{r_2 - r_0}, \qquad r_0 < z < r_2.$$

In particular, $\lim_{z \to r_2} v(z) = 0$.

Assume now $r_2 = +\infty$. By the maximum principle, v is non-increasing; since $v'' = [(\lambda+\gamma)/\alpha]\, v \geqq 0$, we see that v' is increasing. It isthen easy to see that $\lim_{z \to \infty} v'(z) = 0$.

Then (2) implies by integration

(4) $$-v'(z) = \int_z^{+\infty} \frac{\lambda + \gamma(t)}{\alpha(t)} v(t)\, dt \leqq v(z) \int_z^{+\infty} \frac{\lambda + \gamma(t)}{\alpha(t)} dt$$

and integrating again

$$-\log v(z) \leqq \int_{r_0}^{z} dx \int_x^{+\infty} \frac{\lambda + \gamma(t)}{\alpha(t)} dt.$$

In particular, if $\lim_{z \to +\infty} v(z) = 0$, we get

$$\int_{r_0}^{+\infty} dz \int_z^{+\infty} \frac{\lambda + \gamma(t)}{(t)} dt = +\infty.$$

If $\lim_{z \to +\infty} v(z) = l > 0$, (4) implies

$$-v'(z) \geqq l \int_z^{+\infty} \frac{\lambda + \gamma(t)}{\alpha(t)} dt$$

and by integration from r_0 to $+\infty$:

(5) $$\left[1 + \int_{r_0}^{+\infty} dz \int_z^{+\infty} \frac{\lambda + \gamma(t)}{\alpha(t)} dt\right] l \leqq 1.$$

In particular the integral is finite.

Q. E. D.

As mentioned above, when $c(x) \equiv 0$, the following criteria can probably be extracted from the classical works : [10], [9], [18], [17].

4.3. PROPOSITION. — *Let D be the differential operator*

$$a(x) \frac{d^2}{du^2} + b(u) \frac{d}{du} - c(u)$$

defined on $]s_1, s_2[$ *satisfying condition* 4.1. *Fix* s_0 *in* $]s_1, s_2[$; *define*

$$H(x) = \exp\left[-\int_{s_0}^{x} \frac{b(t)}{a(t)}\,dt\right].$$

Let (P_t) *be the minimal semi-group on* $]s_1, s_2[$ *associated to D. Then*

(1) (P_t) *is a* C_0*-diffusion semi-group if, and only if, for each* $i = 1, 2$, *one of the two following conditions holds*

– *either* $\int_{s_0}^{s_i} H(x)\,dx$ *is finite;*

– *or*

$$\int_{s_0}^{s_i} H(x)\,dx \quad \text{and} \quad \int_{s_0}^{s_i} dx\,H(x)\int_{x}^{s_i} \frac{1+c(t)}{a(t)}\,\frac{1}{H(t)}\,dt$$

are infinite.

(2) *The potential kernel* $\int_0^\infty P_t\,dt$ *is finite on compact sets* (*transient case*) *and tends to zero at infinity if, and only if, for each* $i = 1, 2$, *one of the two following conditions holds*

– *either* $\int_{s_0}^{s_i} H(x)\,dx$ *is finite;*

– *or*

$$\int_{s_0}^{s_i} H(x)\,dx \quad \text{and} \quad \int_{s_0}^{s_i} H(x)\int_{x}^{s_i} \frac{c(t)}{a(t)}\,\frac{1}{H(t)}\,dt$$

are infinite.

(3) *The potentiel kernel is infinite on open sets* (*recurrent case*) *if, and only if,* $c(x) \equiv 0$ *and for each* $i = 1, 2$ *the integral* $\int_{s_0}^{s_i} H(x)\,dx$ *is infinite.*

Proof. – To prove (1), apply the change of variable 4.1, and then lemma 4.2 (with $\lambda > 0$), proposition 3.1 and remark 3.2.

To prove (2), after the same change of variable, apply lemma 4.2 (with $\lambda = 0$) and proposition 3.4.

To prove (3), note that the minimal process associated to A on $]r_1, r_2[$ is recurrent if, and only if, $Q_x(T < \infty) \equiv 1$, where T is the first passage through some $r_0 \in]r_1, r_2[$. The minimal positive solution of $A v = 0$

on $]r_1, r_0[\cup]r_0, r_2[$ with $v(r_0) = 1$ is then $v \equiv 1$. In view of lemma 4.2 (with $\lambda = 0$) for each i, r_i is infinite and

$$K_i = \int_{r_0}^{r_i} dz \int_{z}^{r_i} \frac{\gamma(t)}{\alpha(t)} dt$$

is finite. Inequality (5) shows that $l = \lim_{z \to +\infty} v(z)$ is equal to 1 if, and only if, $K_2 = 0$, which forces $\gamma \equiv 0$ on $[r_0, r_2[$. Similarly one gets $\gamma \equiv 0$ on $]r_1, r_0]$. Conversely, in this situation, v is the minimal positive solution of $v'' = 0$ on $[-\infty, r_0[\cup [r_0, +\infty[$ with $v(r_0) = 1$, which is clearly $v \equiv 1$.

4.4. REMARK. — When $c(x) \equiv 0$, a comparison with Feller's criteria for the classification of boundaries (*cf.* [10], ch. II, section 1) shows that (1) is realized when neither s_1 nor s_2 are " entrance boundaries ".

For the sake of completeness, we recall the criteria obtained by KHAS'MINSKII [15] for the explosion of the diffusion. A little extra work is necessary to deal with the coefficient $c(x)$ (assumed to be zero in [15]).

4.5. LEMMA. — *Let D be an elliptic differential operator on a manifold M, with Holder continuous coefficients. Let X be the minimal process associated to D on M, let Q_x be the law of X starting at x, and let ζ be the lifetime (or explosion time) of X. If the term $c(x) = -D\,1(x)$ of order* 0 *in D is not identically zero, then $Q_x(\zeta = +\infty)$ is strictly less than* 1 *for all $x \in M$.*

Proof. — Consider the elliptic operator $L = D + c$ which has no term of order 0, and let Y be the minimal process associated to L. Let F_x (resp. E_x) be the expectation operator associated to Y (resp. X). According to [3] (section 8), we may write

$$Q_x(\zeta > t) = P_t 1 = F_x\left(\exp\left[-\int_0^t c(Y_s)\, ds\right]\right).$$

Now, if $c(x) \not\equiv 0$, we have $c(x) \geqq a > 0$ for x in some open set U. For $x \in U$, the continuity of the trajectories of Y implies $\int_0^t c(Y_s)\, ds > 0$, F_x-almost surely and hence $Q_x(\zeta > t) < 1$. *A fortiori*, we have

$$Q_x(\zeta = +\infty) < 1 \quad \text{for} \quad x \in U. \tag{6}$$

The function $Q_x(\zeta = +\infty)$ is clearly harmonic (DYNKIN's sense) on M and hence it is either constant or does not reach its maximum. Thus (6) implies that $Q_x(\zeta = +\infty)$ is strictly less than 1 for all $x \in M$.

Q. E. D.

4.6. PROPOSITION (after KHAS'MINSKII [15], FELLER [10]). – *Let D be he differential operator of proposition 4.3 on $]s_1, s_2[$; let X be the minimal rocess associated to D. The lifetime of X is almost surely infinite if, and only if, (notation of prop. 4.3) $c(x) \equiv 0$ and, for each $i = 1, 2$:*

$$\int_{s_0}^{s_i} dx\, H(x) \int_{s_0}^{x} \frac{1}{a(t)\, H(t)}\, dt$$

is infinite.

5. The general case

We describe a general method similar to the construction of barriers at ∞ (*c f.* [18]), which can be used to generate sufficent criteria for dimension larger than 1.

5.1. Let D be an elliptic differential operator (as in section 1.1) on a non-compact manifold M. Let K be a compact set in M, and let h be a strictly positive function of class 3 on $M-K$, such that

$$h(M-K) \supset]1, +\infty[\text{ and } \lim_{x \to +\infty} h(x) = +\infty.$$

We consider functions u on M of the form $u = U \circ h$, where U is in $C^2(]0, +\infty[)$. We have

$$D u(x) = A(x)\, U'' \circ h(x) + B(x)\, U' \circ h(x) - c(x)\, U \circ h(x),$$

where the coefficients $A(x)$, $B(x)$, $c(x)$ are Holder continuous, and *depend only* on h and D. They are readily computed using local coordinates, and it is easily seen that $A(x) > 0$.

We now introduce six functions on $]1, +\infty[$:

$$A^+(r) = \sup_{h(x)=r} A(x), \qquad M^+(r) = \sup_{h(x)=r} \frac{B(x)}{A(x)},$$

$$C^+(r) = \sup_{h(x)=r} \frac{c(x)}{A(x)}.$$

A^-, M^-, C^- are defined similarly, replacing supremum by infimum. We assume that these functions are Holder continuous [if this were not the case we would replace A^+, M^+, C^+ (resp. A^-, M^-, C^-) by larger (resp. smaller) Holder continuous functions]. We note that in the following result, part (4) is a result of KHAS'MINSKII [15].

5.2. PROPOSITION. — *Let D and h be as above* (*section* 5.1). *Let* (P_t) *be the minimal semi-group associated with D. Define the functions* $A^{\pm}$, $M^{\pm}$, $C^{\pm}$ *as above and let*

$$H^+(r) = \exp\left[-\int_1^r M^+(t)\,dt\right], \qquad H^-(r) = \exp\left[-\int_1^r M^-(t)\,dt\right].$$

Then :

(1) *if* $\int_1^{+\infty} H^-(r)\,dr < +\infty$,

— or *if both*

$$\int_1^{+\infty} H^-(r)\,dr \quad \text{and} \quad \int_1^{+\infty} dr\,H^-(r)\int_r^{+\infty}\left[\frac{1}{A^+(t)}+C^-(t)\right]\frac{1}{H^-(t)}\,dt$$

are infinite, (P_t) *is a* C_0*-diffusion semi-group;*

(2) *if*

$$\int_1^{+\infty} H^+(r)\,dr = +\infty$$

and

$$\int_1^{+\infty} dr\,H^+(r)\int_r^{+\infty}\left[\frac{1}{A^-(t)}+C^+(t)\right]\frac{1}{H^+(t)}\,dt < +\infty,$$

then (P_t) *is not a* C_0*-diffusion semi-group;*

(3) *if* $\int_1^{+\infty} H^-(r)\,dr < +\infty$:

— or *if*

$$\int_1^{+\infty} H^-(r)\,dr = +\infty$$

and

$$\int_1^{+\infty} dr\,H^-(r)\int_r^{+\infty} C^-(t)\frac{1}{H^-(t)}\,dt = +\infty,$$

the potential kernel $\int_0^\infty P_t\,dt$ *of* D *is finite on compact sets* (*transient case*) *and tends to zero at infinity;*

(4) *if* $c \equiv 0$ *and* $\int_1^\infty H^+(r)\,dr = +\infty$, *the potential kernel is infinite on open sets* (*recurrent case*).

Proof. — Define two differential operators on $]1, +\infty[$ by

$$D_2 U = U'' + M^- U' - \left(C^- + \frac{\lambda}{A^+}\right) U,$$

$$D_1 U = U'' + M^+ U' - \left(C^+ + \frac{\lambda}{A^-}\right) U,$$

where $\lambda \geqq 0$. If $U' \leqq 0$, we clearly have, for $x \in M$:

$$A(x)\, D_1 [U \circ h](x) \leqq (D-\lambda)[U \circ h](x) \leqq A(x)\, D_2 [U \circ h](x).$$

Let U_i be the minimal positive solution of $D_i U_i = 0$ on $]1, +\infty[$, such that $U_i(1) = 1$. Letting u_i be $U_i \circ h$, we get (since $U' \leqq 0$) :

$$(D-\lambda) u_2 \leqq 0 \qquad \text{and} \qquad (D-\lambda) u_1 \geqq 0 \quad \text{on }]1, +\infty[.$$

By the maximum principle, the minimal positive solution u of $(D-\lambda)\, u = 0$ such that $u = 1$ on ∂K verifies then

$$(\text{Cte})\, u_1 \leqq u \leqq (\text{Cte})\, u_2. \tag{5}$$

From then on one needs only to apply the results of section 4 as well as proposition 3.1 and proposition 3.4.

Q. E. D.

For convenience, we reformulate in the present context the results of [15] on the lifetime of the minimal process.

5.3. PROPOSITION (after KHAS'MINSKII [15], MC KEAN [16], BONAMI, KAROUI, ROYNETTE and REINHARDT [5]). — *Same notations and hypotheses as in proposition* 5.2. *Let* X *be the minimal process associated to* D *on* M, *let* ζ *be its lifetime* (*or explosion time*), *and let* Q_x *be the law of* X *starting at* x. *Call* $c(x) = -D\,1$ *the term of order* 0 *in* D. *Then*

(1) *if*

$$c \equiv 0 \qquad \text{and} \qquad \int_1^{+\infty} dr\, H^+(r) \int_1^r \frac{dt}{A^+(t)\, H^+(t)} = +\infty,$$

we have $Q_x(\zeta = +\infty) = 1$ *for all* $x \in M$,

(2) *if*

$$c \not\equiv 0 \qquad \text{or if} \qquad \int_1^{+\infty} dr H^-(r) \int_1^r \frac{dt}{A^-(t) H^-(t)} < +\infty,$$

we have $Q_x(\zeta = +\infty) < 1$ *for all* $x \in M$.

Proof. — Use lemma 4.5 to reduce the situation to the case $c \equiv 0$. The key probabilistic remarks are : if there is on $M-K$ a positive unbounded function u such that $(D-1)u \leqq 0$, with $u = 1$ on ∂K, then $Q_x(\zeta = \infty) \equiv 1$; if there is on $M-K$ a positive bounded function u such that $(D-1)u \geqq 0$ and $\sup_{\partial K}(u) < \lim_{x \to \infty} u(x)$, then $Q_x(\zeta = +\infty) < 1$ for all $x \in M$. For a proof, *see* [16], and [5], p. 58.

In the one dimensional case, such functions are increasing, by the maximum principle. Now if $U' \geqq 0$ on $]1, +\infty[$, we have

$$A(x) L_1[U \circ h](x) \leqq (D-1)[U \circ h](x) \leqq A(x) L_2[U \circ h](x),$$

where

$$L_1 U = U'' + M^- U' - \frac{1}{A^-} U,$$

$$L_2 U = U'' + M^+ U' - \frac{1}{A^+} U.$$

One applies these one-dimensional criterions, as in [16], and [5], p. 60-63, to $A^- L_1$ and $A^+ L_2$ to conclude the proof.

Q. E. D.

6. Application : Change of time

6.1. Let D be an elliptic differential operator on the manifold M, as in section 1.1. Let $f(x)$ be a strictly positive Holder continuous function on M. Let X be the minimal process associated to D. Consider the new time clock $\tau(t)$ defined by

$$t = \int_0^{\tau(t)} \frac{1}{f(X_u)} du.$$

It is easily shown, starting from [9] (vol. 1, th. 10.12), that the process Y, defined by $Y_t = X_{\tau(t)}$, is the minimal process associated to the operator $f.D$.

If we take f small enough at infinity, this operation has smoothing properties as could be expected (*cf.* remark 3.3); indeed we have the following proposition.

6.2. PROPOSITION. — *If X is the minimal process associated to D (elliptic differential operator on the manifold M), it is possible to transform X, by a change of time, into the minimal process Y associated to $f.D$ (f as above) in such a way that the transition semi-group (Q_t) of Y be a C_0-diffusion semi-group. If moreover $c(x) = -D\,1(x) \equiv 0$, it is possible to grant simultaneously that the lifetime of Y be almost surely infinite.*

Proof. — Take $h(x)$ as in section 5.1 and f as a function of $h(x)$. Use the notations of section 5.1. Note that the function $H^{\pm}$ are the same for D and $f.D$ while on the other hand $A^{\pm}$ is multiplied by f, if f is a function of $h(x)$, as one passes from to D to $f D$. This result is then an easy corollary of proposition 5.2 and 5.3.

Q. E. D.

We point out that conversely it is not always possible to destroy the property of being a C_0-diffusion semi-group by " speeding up " the process at infinity. Indeed if we start from a transient minimal process X for which the potential kernel tends to zero at infinity, this property is invariant by the changes of time considered above (this is an easy consequence of prop. 3.4 and remark 3.5), and implies, by proposition 3.1, that the corresponding semi-group is a diffusion semi-group.

Similarly, it is not always possible to force an explosion of the process by speeding it up at infinity. Indeed if X is recurrent, this property is invariant by change of time and,

6.3. LEMMA. — *If the minimal process X is recurrent, then*

$$c(x) = -D\,1(x)$$

is identically zero, and the lifetime ζ of X is almost surely infinite.

Proof. — We first note that the function $g(x) = Q_x(\zeta = +\infty)$ is harmonic in DYNKIN's sense on the whole of M. Indeed, let U be an open relatively compact subset of M, and let T be the entrance time in U^c; since the process is recurrent and any two states " communicate " T is finite Q_x−a.s. for any x (*see* [1]), and we have, with standard notations $\zeta = T + \zeta \circ \theta_T$ so that

$$Q_x(g(X_T)) = Q_x(E_x[I_{\zeta \circ \theta_T = +\infty} \mid F_T]) = Q_x(\zeta \circ \theta_T = +\infty)$$
$$= Q_x(\zeta - T = +\infty) = Q_x(\zeta = +\infty) = g(x).$$

The same argument proves that $h(x) = Q_x(\zeta < +\infty)$ is harmonic, and hence the function $1 = h(x) + g(x)$ is harmonic. Applying locally [9] (vol. 2, th. 13.9), we get $D\,1 \equiv 0$, that is $c(x) \equiv 0$.

But in that case, we know (*cf.* section 1.14) that, Q_x−a.s. the relation $\zeta < +\infty$ implies that X_t tends to $+\infty$ as t increases to $\zeta - 0$. This contradicts the recurrence of the paths of X unless $Q_x(\zeta < +\infty) \equiv 0$.

Q. E. D.

6.4. REMARK. — In general, that is, when $c(x) \not\equiv 0$, it can still be proved that Q_x-a,s, the relation $\zeta = +\infty$ implies $T < \infty$, and then, that $g(x) = Q_x(\zeta = +\infty)$ is a solution of $Dg \equiv 0$.

But, contrary to an assertion of MC KEAN [16] (section 4.4), the function $h(x) = 1 - g(x) = Q_x(\zeta < +\infty)$ is *not* a solution of $D\,h \equiv 0$ *unless* $c(x) \equiv 0$, an obvious counterexample is the case of $D = (d^2/dx^2) - 1$ on the real line, where $g(x) \equiv 0$ and $h(x) \equiv 1$.

6.5. CONJECTURES. — The arguments just given make quite plausible the following two conjectures :

(*a*) it is always possible to transform X into a process whose transition semi-group is *not* a C_0-diffusion semi-group by " speeding it up at infinity ", unless the potential kernel of X is finite on compact sets and tends to zero at infinity;

(*b*) it is always possible to transform X into a process with positive probability of explosion by speeding it up at infinity, unless X is recurrent.

These two conjectures are readily checked in dimension 1, using the criteria of section 4.

6.6. We also note the smoothing effect of killing the process X with high probability at infinity, through the multiplicative functional $\exp\left[-\int_0^t m(X_s)\,ds\right]$ where m is positive, Holder continuous, and large at infinity. This replaces D by $D-m$ (*see* [3], section 7), and as proposition 5.2 implies immediately,

– there are functions m for which the new process X^m has a potential kernel tending to zero at infinity (and hence its transition semi-group is a C_0-diffusion semi-group).

Intuitively, when X^m starts from far away, it has large probability of being killed soon, and hence a small probability of returning to any compact set in a finite time.

7. Application : Manifolds of negative curvature

7.1. In this paragraph M is assumed to be a complete *analytic* simply connected manifold of negative curvature (*cf.* [13]). If in a system of local coordinates $(z_1, \ldots, z_n)$ the quadratic form defining the metric is given by the matrix $[g_{ij}(z)]$, the so-called *Laplace-Beltrami operator* Δ of M is defined by

$$\Delta = \frac{1}{\sqrt{\bar{g}}} \sum_k \frac{\partial}{\partial z_k}\left(\sum_i g^{ik} \sqrt{\bar{g}} \frac{\partial}{\partial z_i}\right), \tag{1}$$

where (g^{ik}) is the inverse matrix of (g_{ij}), and $\bar{g} = \det(g_{ij})$ (*see* [13], p. 387). We assume that the Riemann structure is analytic so that Δ has analytic coefficients.

The minimal process X associated to Δ on M may be called the *Brownian motion on M*, by analogy with the case of $\mathbf{R}^n$.

7.2. Let S be the unit sphere in $\mathbf{R}^n$, and let $R^+ =]0, +\infty[$. If $0 \in M$ the exponential map at 0 :

$$\text{Exp} : \quad R^+ \times S \to M,$$

defined by $(t, X) \to \text{Exp}(tX)$ (*cf.* [13], ch. 1), is an analytic diffeomorphism of $R^+ \times S$ onto M.

For $a \in S \subset \mathbf{R}^n$, let $a = (a_1, \ldots, a_n)$ with $\sum_i a_i^2 = 1$. In polar coordinates $(t, a_1, \ldots, a_n)$ the Riemannian structure is given by ([13], ch. 1, lemma 9.2) :

$$ds^2 = dt^2 + \sum_{i=1}^n (\bar{\omega}^i)^2,$$

where the $\bar{\omega}^i$ are one-forms in $da_1, \ldots, da_n$ not containing dt and verify the structural equations ([13], p. 71) :

$$\frac{\partial^2 \bar{\omega}^i}{\partial t^2} = \sum_{j,k,l} R^i_{ljk}\, a_l\, a_j\, \bar{\omega}^k \tag{2}$$

with the initial values

$$\bar{\omega}^i = 0 \qquad \text{for} \quad t = 0, \quad (a_1, \ldots, a_n) \in S,$$

$$\frac{\partial \bar{\omega}^i}{\partial t} = da_i \qquad \text{for} \quad t = 0, \quad (a_1, \ldots, a_n) \in S,$$

and where R^i_{ljk} represents the curvature tensor in a proper basis of the tangent space ([13], p. 70). This basis is obtained from a fixed one in $T_0(M)$ by parallel translation along the geodesics starting at 0.

7.3. PROPOSITION. — *If M is a complete analytic simply connected Riemannian manifold of negative curvature (not necessarily constant), of dimension $n \geqq 2$, the transition semi-group of the Brownian motion X on M is a C_0-diffusion semi-group. Moreover, if $n \geqq 3$, X is transient, and its potential kernel G tends to zero at infinity : indeed, for each fixed $f \in B_c(M)$, then $|Gf(x)| \leqq$ (Cte) $r(x)^{2-n}$ where $r(x)$ is the distance between x and a fixed point $0 \in M$.*

7.4. REMARK. — The transcience of X was proved by other methods by PRAT [20], but only for manifolds where the (negative) curvature is bounded in absolute value. It is proved in general (under another name!) in [23].

Proof of proposition 7.3 (notations of section 7.2). — Let $a \in S$, and let $T_a(S)$ be the tangent space to S at a. For fixed $t \in R^+$, the ds^2 given above defines a quadratic form $Q_{t,a}$ on $T_a(S)$, and we have (by definition of the symmetric product of two 1-forms, *cf.* [13], p. 49) :

$$Q_{t,a}(v) = \textstyle\sum_1^n \langle \bar{\omega}_i \otimes \bar{\omega}_{i'}\, v \otimes v \rangle \qquad \text{for} \quad v \in T_a(S),$$

$$Q_{t,a}(v) = \textstyle\sum_1^n \langle \bar{\omega}_i, v \rangle^2. \tag{3}$$

Let U be a fixed small open set in S. Let $A_1, A_2, \ldots, A_{n-1}$ be $n-1$ smooth fixed vector fields on U such that for each a in U, $A_1(a), \ldots, A_{n-1}(a)$ are a basis of $T_a(S)$. If $v \in T_a(S)$, we associate to v its matrix V of coordinates in this basis of $T_a(S)$. We note $\|v\| = (V^* V)^{1/2}$, and we call $B_{t,a}$ the matrix of $Q_{t,a}$, defined by $Q_{t,a}(v) = V^* B_{t,a} V$. For " a " *fixed*, $B_{t,a}$ is an analytic function of t, and hence according to the technical lemma 7.5 below, there is a countable subset E of R^+ (eventually empty and which may depend on a) and $n-1$ analytic vector functions $v_i(t)$, $i = 1, \ldots, (n-1)$, from $R^+ - E$ to $T_a(S)$ such that for each $t \in R^+ - E$, the $v_i(t)$, $i = 1, \ldots, (n-1)$, form a basis of $T_a(S)$ made up of eigenvectors of $B_{t,a}$. The corresponding eigenvalues $\lambda_i(t)$ are of course analytic on $R^+ - E$, and we may assume $\| v_i(t) \| \equiv 1$.

We then have, for " a " fixed in U :

$$\begin{aligned}\frac{d\lambda_i}{dt}(t) &= \frac{d}{dt}(V_i^*(t) B_{t,a} V_i(t)) \\ &= 2\frac{dV_i^*}{dt}(t) B_{t,a} V_i(t) + V_i^*(t)\frac{\partial B_{t,a}}{\partial t} V_i(t).\end{aligned}$$

But since $V_i^*(t) V_i(t) = 1$, we have

$$\frac{dV_i^*}{dt}(t) B_{t,a} V_i(t) = \lambda_i(t)\frac{dV_i^*}{dt}(t)\, V_i(t) = 0$$

and finally

$$\frac{d\lambda_i}{dt}(t) = V_i^*(t)\frac{\partial B_{t,a}}{\partial t} V_i(t), \qquad t \in R^+ - E.$$

In particular, if we denote by $f(a, t, v)$ the function

(4) $$f(a, t, v) = [Q_{t,a}(v)]^{1/2} = [V^* B_{t,a} V]^{1/2}, \qquad (a, t, v) \in S \times R^+ \times T_a(S),$$

we get

(5) $$\frac{d\sqrt{\lambda_i(t)}}{dt} = \frac{\partial f}{\partial t}(a, t, v_i(t)) \qquad \text{for} \quad t \in R^+ - E \quad (\text{and } a \text{ fixed}).$$

But the curvature being negative, the function $f(a, t, v)$ verifies for a and v fixed ($v \neq 0$) (use (3) and [13], p. 72-73) :

$$f(a, 0, v) = 0, \qquad \frac{\partial f}{\partial t}(a, 0, v) > 0, \qquad \frac{\partial^2 f}{\partial t^2}(a, t, v) \geqq 0.$$

Hence we must have

$$f(a, t, v) - t\frac{\partial f}{\partial t}(a, t, v) \leqq 0 \qquad \text{for} \quad t \in R^+$$

and consequently

$$\frac{\partial f}{\partial t}(a, t, v)\frac{1}{f(a, t, v)} \geqq \frac{1}{t} \qquad \text{for} \quad t \in R^+$$

which due to (5) implies

$$\frac{1}{\sqrt{\lambda_i(t)}}\, \frac{d\sqrt{\lambda_i(t)}}{dt} \geqq \frac{1}{t} \qquad \text{for} \quad t \in R^+ - E \quad (\text{and } a \text{ fixed}).$$

Since $\det B_{t,a} = \lambda_1(t)\ldots\lambda_{n-1}(t)$, we obtain

$$(6) \qquad \frac{1}{\sqrt{\det B_{t,a}}}\, \frac{d}{dt}\sqrt{\det B_{t,a}} \geqq \frac{n-1}{t} \qquad \text{for} \quad t \in R^+ - E \quad (\text{and } a \text{ fixed}).$$

But both sides are continuous on R^+, and hence

$$(7) \qquad (\det B_{t,a})^{-1/2}\frac{\partial}{\partial t}[(\det B_{t,a})^{1/2}] \geqq \frac{n-1}{t} \qquad \text{for} \quad t \in R^+,\ a \in U.$$

Now, if $(t, a) \in R^+ \times U$, then the tangent space to $R^+ \times S$ at (t, a) is $R \times T_a(S)$, and in the natural basis of $R \times T_a(S)$ obtained by completing the basis $A_1(a), \ldots, A_{n-1}(a)$ the matrix (g_{ij}) of the Riemannian metric becomes (notation of section 7.1) :

$$(g_{ij}) = \begin{bmatrix} 1 & 0 \\ 0 & B_{t,a} \end{bmatrix}, \qquad (g^{ij}) = \begin{bmatrix} 1 & 0 \\ 0 & B_{t,a}^{-1} \end{bmatrix}.$$

If $z_1, \ldots, z_{n-1}$ are local coordinates in U such that

$$A_i(a)\, h = \frac{\partial h}{\partial z_i}(a) \qquad \text{for} \quad i = 1, \ldots, (n-1), \quad a \in U, \quad h \in C^2(U),$$

the Laplace-Beltrami operator Δ is given by (1). We have, if F is a smooth function on $R^+ \times U$:

$$\Delta F(t, a) = \frac{1}{\sqrt{g}}\, \frac{\partial}{\partial t}\left(\sqrt{g}\frac{\partial F}{\partial t}\right) + \frac{1}{\sqrt{g}}\sum_{k=1}^{n-1} \frac{\partial}{\partial z_k} \sum_{i=1}^{n-1} \sqrt{g}\, g^{ik}\frac{\partial F}{\partial z_i}$$

In particular, if F depends on t only, we get

$$(8) \quad \Delta F(t, a) = \frac{\partial^2 F}{\partial t^2}(t) + \frac{1}{\sqrt{g(t, a)}}\, \frac{\partial\sqrt{g}}{\partial t}(t, a)\frac{\partial F}{\partial t}(t), \qquad t \in R^+, \quad a \in U.$$

But clearly $\bar{g} = \det(g_{ij}) = \det(B_{t,a})$, and in view of (7), we see that for functions F of t only

$$\Delta F(t, a) = \frac{\partial^2 F}{\partial t^2}(t) + b(t, a)\frac{\partial F}{\partial t}(t), \qquad (t, a) \in R^+ \times U,$$

where $b(t, a) \geqq (n-1)/t$ for $(t, a) \in R^+ \times U$.

Since U is arbitrary, the preceding result holds for $(t, a) \in R^+ \times S$. This is a well known result (*see* [20] for instance), but the details of the proof will be needed later.

In particular, applying the method of section 5, with a function $h(x)$ defined as the distance from 0 to x on M we may take (notation of proposition 5.2) :

$$M^-(t) = \frac{n-1}{t} \leqq \inf_{a \in S} b(t, a), \qquad t > 1,$$

$$H^-(t) = \exp \int_1^t -M^-(r)\, dr = t^{1-n}, \qquad t > 1.$$

Clearly if $n \geqq 3$, the integral $\int_1^{+\infty} H^-(t)\, dt$ is finite, and by proposition 5.2, the potential kernel of Δ is finite on compact sets and tends to zero at infinity. The other assertions are consequences of this one. The inequalities (5) in the proof of proposition 5.2 prove also that the minimal positive solution of $\Delta u = 0$ in $M - K$ (K compact with smooth boundary) and $u = 1$ on ∂K is bounded by (Cte) $\times v$, where v is the minimal positive solution of $(\partial^2/\partial t^2) + [(n-1)/t\,](\partial/\partial t)$. Since this is the radial component of the usual Laplace operator on $\mathbf{R}^n$, in polar coordinates, we see, taking account of proposition 3.4 and remark 3.5 that, for fixed $f \in B_c(M)$:

$$Gf(x) \leqq (\text{Cte})\, G_1 f_1(h(x)),$$

where G_1 is the potential kernel of the usual Brownian motion on $\mathbf{R}^n$, and f_1 some function in $B_c]1, +\infty[$.

Hence $Gf(x) \leqq (\text{Cte})\, h(x)^{2-n}$, for $x \in M$.

Now i $n = 2$, then $H^-(t) = t^{-1}$, and since $A^+(t) = 1$, we get

$$\int_1^{+\infty} H^-(t)\, dt = +\infty = \int_1^{+\infty} dr\, H^-(r) \int_r^{+\infty} \frac{1}{H^-(t)}\, dt$$

so that the transition semi-group of the Brownian motion is a C_0-diffusion semi-group (prop. 5.2).

Q. E. D.

We now prove the technical lemma used in the preceding proof.

7.5. LEMMA. — *Let $B(t)$, $0 < t < +\infty$, be a symmetric positive matrix fif order p whose coefficients are analytic functions of t. One can then ond a countable subset E of R^+ and p vector-valued analytic functions $v_i(t)$, $1 \leqq i \leqq p$, on $R^+ - E$ such that for each $t \in R^+ - E$ the $v_i(t)$, $i = 1, \ldots, p$, form a basis of $\mathbf{R}^p$ made up of eigenvectors of $B(t)$.*

Proof. — The characteristic polynomial $P(t, \lambda)$ of $B(t)$ has coefficients analytic in t and may be written

$$P(t, \lambda) = (-1)^p \lambda^p + \ldots = \det(B(t) - \lambda I).$$

Call $\lambda_1(t), \ldots, \lambda_{q(t)}(t)$ the *distinct* roots of $P(t, \lambda)$. Obviously they are *exactly* the roots of the polynomial $P(t, \lambda)/Q(t, \lambda) = Z(t, \lambda)$ where $Q(t, \lambda)$ is the largest common divisor of $P(t, \lambda)$ and $(\partial/\partial\lambda) P(t, \lambda)$. Moreover all the roots of $Z(t, \lambda)$ are simple.

To compute $Q(t, \lambda)$, for t fixed, we use the classical algorithm

$$R_0(\lambda) = P(t, \lambda), \qquad R_1(\lambda) = \frac{\partial}{\partial\lambda} P(t, \lambda),$$

$$R_n = R_{n+1} A_n + R_{n+2}, \qquad n = 0, 1, \ldots, \qquad \text{degree } R_{n+2} < \text{degree } R_{n+1},$$

where A_n, R_n are polynomials in λ; if $n(t)$ is the first integer or which $R_{n(t)+2} = 0$, we have $Q(t, \lambda) = R_{n(t)+1}(\lambda)$. An immediate proof by induction shows that there is some countable set $E \subset R^+$ such that, for $t \notin E$, each $R_i(\lambda)$, $i = 1, \ldots, n(t)$, has constant degree and coefficients analytic in t; in particular, for $t \notin E$, $n(t)$ is constant, and $Q(t, \lambda)$ has coefficients analytic in t and fixed degree. The same property is then true for $Z(t, \lambda)$. But the roots of $Z(t, \lambda)$ being distinct and the coefficients being analytic in $t \in R^+ - E$ (with non-zero leading coefficient), we see that $q = q(t)$ is constant on $R^+ - E$ and that the roots $\lambda_1(t), \ldots, \lambda_q(t)$ are analytic in t on $R^+ - \text{E}$. Since $B(t)$ is symmetric the corresponding eigenspaces $E_1(t), \ldots, E_q(t)$ span $\mathbf{R}^p$ and if we define

$$B_i(t) = \prod\nolimits_{1 < j < q,\, j \neq i} (B(t) - \lambda_j(t) I),$$

the range of $B_i(t)$ is $E_i(t)$. Since $B_i(t)$ is analytic on $R^+ - E$, there is a countable set E' containing E, such that for $t \in \mathbf{R}^+ - E'$, and for each $i = 1, \ldots, q$, $E_i(t)$ has constant dimension r_i and a basis formed of columns of $B_i(t)$ with *fixed* indices. The union of all these bases forms an independent family of analytic eigenvectors of $B(t)$ spanning $\mathbf{R}^p$.

Q. E. D.

7.6. REMARK. — In the case $n = 2$, the Brownian motion may well be recurrent if the curvature is small enough at infinity. Indeed (*cf.* [20]) in the case $n = 2$, we may write in polar coordinates $(t, \theta) \in \,]0, \infty[\, \times [0, 2\pi]$:

$$\Delta = \frac{\partial^2}{\partial t^2} + \frac{1}{F^2}\frac{\partial^2}{\partial \theta^2} + \frac{1}{F}\frac{\partial F}{\partial t}\frac{\partial}{\partial t} - \frac{1}{F^3}\frac{\partial F}{\partial \theta}\frac{\partial}{\partial \theta}, \tag{9}$$

where $F = F(t, \theta)$ is strictly positive for $t > 0$ and verifies

$$\begin{cases} k(t, \theta) = -\dfrac{1}{F}\dfrac{\partial^2 F}{\partial t^2} \leqq 0 \qquad (k \text{ is the curvature}), \\ F(0, \theta) = 0, \\ \dfrac{\partial F}{\partial t}(0, \theta) = 1. \end{cases} \tag{10}$$

Conversely if $F(t, \theta)$ is any function verifying (10), the formula $ds^2 = dt^2 + F\, d\theta^2$ defines a Riemann metric of negative curvature on $\mathbf{R}^2$, in particular take

$$F(t, \theta) = f(t)\, g(\theta),$$

with f, g analytic and such that

$$-\frac{f''(t)}{f(t)} = k(t) \leqq 0, \qquad f(0) = 0, \quad f'(0) = 1, \quad g(\theta) > 0.$$

With the notation of proposition 5.2, we have

$$H^+(t) = \exp\left[-(\text{Cte})\int_1^t \frac{f'(u)}{f(u)}\, du\right] = \frac{\text{Cte}}{f(t)}$$

and similarly

$$H^-(t) = \frac{\text{Cte}}{f(t)}.$$

Thus the Brownian motion is recurrent if and only if

$$\int_1^{\infty} \frac{dt}{f(t)} = +\infty.$$

As shown by the maximum principle, the relation $|k(t)| \leqq 1/(t^2 \log t)$ for t large implies $f(t) \leqq [t \log t] \times \text{Cte}$ and hence implies recurrence.

Clearly if $k(t) = \text{Cte} = -k$ with $k > 0$, $f(t) = (1/-k)\,\text{sh}(-kt)$ and the Brownian motion is transient (first proved in PRAT [20]); in fact we can say that its potential kernel tends to zero at infinity, by proposition 5.2.

7.7. In the preceding set up [case $n = 2$, $F(t, \theta) = f(t)\,g(\theta)$], we can also see that if the absolute value of the curvature is large enough at infinity, the probability of explosion is strictly positive. Indeed proposition 5.3 shows that explosion at a finite time is a.s. impossible if, and only if

$$\int_1^{+\infty} \frac{dr}{f(r)} \int_1^r f(t)\,dt = +\infty. \tag{11}$$

This is clearly the case if $k(t) = \text{Cte} = +k$ since then

$$f(t) = \frac{1}{-k}\,\text{sh}(-kt).$$

Now take $-k(t) = 9\,c^2\,t^4 + 6\,ct$ for $t \geqq 1$, $c > 0$ and hence $f(t) = (\text{Cte}) \exp(ct^3)$ for $t \geqq 1$. An easy computation shows that the integral in (11) converges and hence the probability of explosion is strictly positive.

7.8. We now look at the same question in dimension higher than 2. For each point $x \in M$, and each pair X, Y of noncollinear tangent vectors at x, let $k(x, X, Y)$ be the *absolute value* of the sectional curvature of M at x corresponding to the tangent plane X, Y (*c f.* [13], ch. 1, section 12). Define then for $t \geqq 0$

$$\text{(12)} \quad \left\{ \begin{aligned} K(t) &= \sup\{k(x, X, Y);\ d(0, x) = t; \\ &\qquad X, Y \in T_x(M), X, Y \text{ noncollinear}\}, \\ k(t) &= \inf\{k(x, X, Y);\ d(0, x) = t; \\ &\qquad X, Y \in T_x(M), X, Y \text{ noncollinear}\}. \end{aligned} \right.$$

In particular if the sectional curvature is constant, or if the manifold M is homogeneous (i.e. the isometries are transitive) we have $K(t) \leqq$ Cte for all $t > 0$.

7.9. PROPOSITION. — *Let M be a complete analytic simply connected Riemannian manifold of negative curvature, t the distance on M (from a fixed origin 0), $K(t)$ and $k(t)$ the "largest" and "smallest" sectional curvature at distance t (see 7.8 above), X the Brownian motion on M.*

(i) *If* $(1/t)\int_0^t K(u)\,du \leqq$ Cte *for t large, then the lifetime of X is almost surely infinite (this is in particular the case if the sectional curvature is globally bounded — for instance if M is homogeneous).*

(ii) *If for some $\varepsilon > 0$, we have $k(t) \geqq$ (Cte) $t^{2+\varepsilon}$ for all $t \geqq t_0$, the lifetime of X is finite with strictly positive probability, for every starting point $p \in M$.*

NOTE. — We point out that in view of remark 1.9, in case (i) there is an unique Markov semi-group associated to Δ by problems (1), (2) and (3) of theorem 1.6, and hence an unique *positive* fundamental solution for Δ, while in case (ii), an infinity of such semi-groups, and hence an infinity of such fundamental solutions may be constructed, by extending the trajectories of the minimal process after their time of death.

Proof. — Use the notation of the proof of proposition 7.3 and section 7.2. Let $X_0 = (a_1, \ldots, a_n)$ and $Y_0 = (b_1, \ldots, b_n)$ be two fixed distinct *unit* vectors in $T_0(M)$. For $q = (t, a) \in R^+ \times S = M$, let X_q and Y_q be the vectors obtained from X_0 and Y_0 by parallel translation along the geodesic from 0 to q. As shown by [13] (p. 70, p. 72, and th. 12.2, p. 65), the sectional curvature $k(q, X_q, Y_q)$ corresponding to the tangent plane (X_q, Y_q) is

$$(13) \qquad k(q, X_q, Y_q) = \left(-\sum_{i,j,k,l} R^i_{ljk}\, a_l\, a_j\, b_i\, b_k\right) |X_q \wedge Y_q|^{-2},$$

where $|X_q \wedge Y_q|$ is simply the area of the parallelogram spanned by (X_q, Y_q). In particular, since $q = (t, a)$, and since parallel translation preserves the norm we have, using definition

$$(14) \qquad \left|\sum_{i,j,k,l} R^i_{ljk}\, a_l\, a_j\, b_i b_k\right| \leqq K(t)\, |X_q \wedge Y_q|^2 \leqq K(t),$$

where R^i_{ljk} is computed at $(t, a) \in R^+ \times S$.

Let now

$$A_{ik} = \sum_{l,j} R^i_{l,j,k}\, a_l\, a_j. \tag{15}$$

We presently prove that $A_{ik} = A_{ki}$. Indeed, calling g the metric tensor on M, R the curvature tensor, and X, Y, Z, T four vector fields on M, we have ([13], lemma 12.5, p. 69) :

$$g(R(X, Y)Z, T) = g(R(Z, T)X, Y).$$

On the other hand, if (X_i) is the orthonormal moving frame obtained by parallel translation along rays from the origin, we have ([13], p. 44) :

$$R^i_{ljk} = g(R(X_j, X_k)X_l, X_i), \tag{16}$$

and hence $R^i_{ljk} = R^k_{jli}$ for all i, j, k, l. This readily implies that $A_{ik} = A_{ki}$.

Equality (14) becomes

$$\sum_{i,k} A_{ik}\, b_i\, b_k \leqq K(t),$$

and, leaving q fixed while varying X_0, we conclude

$$|A_{ik}| \leqq (\mathrm{Cte})\, K(t) \quad \text{for all } q = (t, a),\ i,\ k, \tag{17}$$

where the constant depends only on the dimension.

Fix a point a on the sphere S and a vector $v \in T_a(S)$. Denoting

$$\alpha_i(t, a, v) = \overline{\omega}^i(v), \qquad \gamma_i = da_i(v),$$

we have, using the notation (15), and the equation (2) :

$$\frac{\partial^2 \alpha_i}{\partial t^2} = \sum_k A_{ik}\,\alpha_k, \quad \alpha_i(0, a, v) = 0, \qquad \frac{\partial \alpha_i}{\partial t}(0, a, v) = \gamma_i. \tag{18}$$

Define as in equation (4) :

$$f(a, t, v) = [\textstyle\sum_i \overline{\omega}^i(v)^2]^{1/2} = (\sum_i \alpha_i^2)^{1/2}.$$

Then

$$f\frac{\partial^2 f}{\partial t^2} + \left(\frac{\partial f}{\partial t}\right)^2 = \sum_i \left[\alpha_i \frac{\partial^2 \alpha_i}{\partial t^2} + \left(\frac{\partial \alpha_i}{\partial t}\right)^2\right] \tag{19}$$

and hence taking account of (18) :

$$f\frac{\partial^2 f}{\partial t^2} \leqq \sum_{i,k} A_{ik}\,\alpha_i\,\alpha_k + \sum_i \left(\frac{\partial \alpha_i}{\partial t}\right)^2. \tag{20}$$

But (18) implies, using inequality (17) :

$$\left|\frac{\partial \alpha_i}{\partial t} - \gamma_i\right| = \left|\int_0^t \sum_{i,k} A_{ik}\,\alpha_k\right| \leqq (\mathrm{Cte})\int_0^t K(u) f(a, u, v)\,du.$$

Thus, since f is an increasing function of t (*cf.* proof of prop. 7.3) we get, provided $K(t)$ is not identically zero :

$$\left|\frac{\partial \alpha_i}{\partial t}\right| \leqq (\mathrm{Cte}) f(a, t, v)\int_0^t K(u)\,du \qquad \text{for} \quad t \geqq t_0. \tag{21}$$

Inequality (21) yields

$$f\frac{\partial f}{\partial t} = \sum \alpha_i \frac{\partial \alpha_i}{\partial t} \leqq (\mathrm{Cte}) f^2 \int_0^t K(u)\,du, \qquad t \geqq t_0,$$

and finally

$$\frac{1}{f}\,\frac{\partial f}{\partial t} \leqq (\mathrm{Cte})\int_0^t K(u)\,du, \qquad t \geqq t_0.$$

As is clear from the argument the constant as well as t_0 depend only on $|v|$ but not on t, or $a \in S$. As in the proof of proposition 7.3 we may then conclude (notation of section 7.1) :

$$\frac{1}{\sqrt{g}}\,\frac{\partial}{\partial t}\sqrt{g} \leqq (\mathrm{Cte})\int_0^t K(u)\,du, \qquad t \geqq t_0, \quad a \in S. \tag{22}$$

The Laplace-Beltrami operator Δ is written as in (8) when acting on functions of t only; with the notations of proposition 5.2, we have

$$M^+(t) = (\mathrm{Cte})\int_0^t K(u)\,du,$$

$$H^+(t) = \exp\left[-\int_1^t M^+(r)\,dr\right].$$

Clearly if $(1/t)\int_0^t K(u)\,du \leqq \mathrm{Cte}$, we have

$$\int_1^{+\infty} dr\, H^+(r)\int_1^r \frac{dt}{H^+(t)} = \int_1^{+\infty} dr \int_1^r dt \exp\left[-\int_t^r M^+(u)\,du\right]$$

$$\geqq \int_1^{+\infty} dr \int_1^r dt \exp\left[-\mathrm{Cte}\,(r^2 - t^2)\right] = +\infty$$

and hence (prop. 5.3), almost surely, explosion is impossible for the corresponding Brownian motion.

Now, (19) implies

$$(23)\qquad f\frac{\partial^2 f}{\partial t^2} = \sum_{i,k} A_{ik}\,\alpha_i\,\alpha_k + \sum_i\left(\frac{\partial \alpha_i}{\partial t}\right)^2 - \left(\frac{\partial f}{\partial t}\right)^2 \geqq \sum_{i,k} A_{ik}\,\alpha_i\,\alpha_k$$

since, by Schwartz' inequality :

$$\left|\frac{\partial f}{\partial t}\right| = \frac{1}{f}\left|\sum \alpha_i\,\frac{\partial \alpha_i}{\partial t}\right| \leqq \left(\sum\left(\frac{\partial \alpha_i}{\partial t}\right)^2\right)^{1/2}.$$

We now note that the vectors $(\alpha_1, \ldots, \alpha_n)$ and $(a_1, \ldots, a_n)$ are orthogonal. Indeed (18) implies

$$\frac{\partial^2}{\partial t^2}(\sum_i a_i\,\alpha_i) = \sum_{i,j,k,l} R^i_{ljk}\,a_l\,a_j\,a_i\,\alpha_k.$$

But the relation (*cf.* [13], lemma 12.5, p. 69) :

$$g(R(X, Y)Z, T) = -g(R(X, Y)\,T, Z)$$

implies, in view of (16) :

$$R^i_{ljk} = -R^l_{ijk} \quad \text{for all } i, j, k, l$$

so that

$$\sum_{i,l,j} R^i_{ljk}\,a_i\,a_j\,a_l = 0 \quad \text{for each } k$$

and hence

$$\frac{\partial^2}{\partial t^2}(\sum a_i\,\alpha_i) \equiv 0.$$

Thus $(\partial/\partial t)(\sum a_i\,\alpha_i)$ is constantly equal to its initial value

$$\sum a_i\frac{\partial \alpha_i}{\partial t}(0, a, v) = \sum a_i\,da_i(v) = 0,$$

(since $\sum_i a^2 \equiv 1$ on S). Finally $\sum a_i\,\alpha_i$ is constant, and initially equal to 0 by (18).

In particular, the area of the parallelogram spanned by $(a_1, \ldots, a_n)$ and $(\alpha_1, \ldots, \alpha_n)$ is simply $(\sum \alpha_i^2)^{1/2} = f$. The evaluation (13) of the sectional curvature, and definition (12) yield then

$$\sum_{i,k} A_{ik}\,\alpha_i\,\alpha_k \geqq k(t)f^2(a, t, v)$$

so that finally, by (23) :

$$(24)\qquad \frac{\partial^2 f}{\partial t^2} \geqq k(t)f.$$

Take $\alpha > 0$; the function $u(t) = \exp(\alpha t^{2+\varepsilon})$ verifies

$$\frac{u''}{u} = \alpha^2(2+\varepsilon)^2 t^{2+2\varepsilon} + \alpha(2+\varepsilon)(1+\varepsilon)t^{\varepsilon},$$

$$\frac{u'}{u} = \alpha(2+\varepsilon)t^{1+\varepsilon}.$$

If $k(t) \geqq \text{Cte}\, t^{2+2\varepsilon}$ for $t \geqq t_0$ (t_0 and ε fixed), we have, for a proper choice of $\alpha > 0$, $(1/f)(\partial^2 f/\partial t^2) \geqq (u''/u)$ for $t \geqq t_0$, and hence by lemma 7.10 below

$$\frac{1}{f}\frac{\partial f}{\partial t} \geqq \alpha(2+\varepsilon)t^{1+\varepsilon} - \left|\frac{1}{f}\frac{\partial f}{\partial t}(t_0, a, v) - \alpha(2+\varepsilon)t_0^{1+\varepsilon}\right|.$$

As is easily shown the supremum of $(1/f)(t_0, a, v)(\partial f/\partial t)(t_0, a, v)$ for $a \in S$, $|v| \leqq 1$ is finite, and we get

$$\frac{1}{f}\frac{\partial f}{\partial t} \geqq \alpha_1 t^{1+\varepsilon} - \alpha_2 \qquad \text{for} \quad t \geqq t_0, \quad a \in S, \quad |v| \leqq 1$$

for some choice of the constants $\alpha_1 > 0$, $\alpha_2 > 0$.

The same computations as in proposition 7.3 yield then

$$\frac{1}{\sqrt{g}}\frac{\partial\sqrt{g}}{\partial t} \geqq (n-1)\alpha_1 t^{1+\varepsilon} - (n-1)\alpha_2 \qquad \text{for} \quad t \geqq t_0, \quad a \in S, \quad |v| \leqq 1.$$

We may now apply the criteria of proposition 5.3 to the Laplace-Beltrami operator in polar coordinates [equation (8)], and we may take (notation of prop. 5.3) for some $\alpha_3 > 0$:

$$M^-(t) = \alpha_3 t^{1+\varepsilon}, \qquad t \geqq t_1,$$

$$H^-(t) = \exp\left[-\int_{t_1}^{t} M^-(u)\,du\right]$$

so that with $\alpha_4 = \alpha_3(1+\varepsilon)^{-1}$:

$$\int_{t_1}^{+\infty} dr \int_{t_1}^{r} \frac{H^-(r)}{H^-(t)}\,dt \leqq \int_{t_1}^{+\infty} dr \int_{t_1}^{r} \exp[-\alpha_4(r^{2+\varepsilon} - t^{2+\varepsilon})] < +\infty.$$

To check that the integral is finite one may set $t = r+u$, note that $(t+u)^{2+\varepsilon} - t^{2+\varepsilon} \geqq u(2+\varepsilon)t^{2+\varepsilon}$, and bound the integral by [denoting $\alpha_5 = \alpha_4(2+\varepsilon)$] :

$$\int_0^1 du \int_{t_1}^{+\infty} dt \exp(-\alpha_5 u t^{1+\varepsilon}) + \int_{t_1}^{+\infty} dt \int_1^{+\infty} du \exp(-\alpha_5 u t^{1+\varepsilon}) < +\infty.$$

Thus the probability of explosion is strictly positive at every point of M, in this case.

Q. E. D.

We now prove the technical lemma just used :

7.10. LEMMA. — *Let v and u be two strictly positive increasing continuous functions on* $[1, +\infty[$. *Assume that*

$$\frac{v''}{v} \geqq \frac{u''}{u} \quad \text{on }]1, +\infty[.$$

Then we have

$$\frac{v'}{v} \geqq \frac{u'}{u} - c \qquad \text{where} \quad c = \left| \frac{v'}{v}(1) - \frac{u'}{u}(1) \right|.$$

Proof. — Let $Z = v'/v$, $z = u'/u$, then

$$Z' + Z^2 = \frac{v''}{v} \geqq \frac{u''}{u} = z' + z^2$$

and hence

$$(Z' - z') \geqq (z - Z)(z + Z).$$

Let $w = z - Z$; we have

$$w' \leqq -w(z+Z). \tag{25}$$

Assume that at some point t_0, $w(t_0) < 0$; let

$$t_1 = \inf\{t > t_0;\ w(t) = 0\}.$$

We get from (25) :

$$\log[-w(t_0)] - \log[-w(t)] \leqq \log \frac{v(t)u(t)}{v(t_0)u(t_0)}, \qquad t_0 \leqq t < t_1$$

so that, letting $t \nearrow t_1$, we must have $t_1 = +\infty$ and hence

$$w(t) < 0 \quad \text{for all } t \geqq t_0.$$

In particular, if $w(1) \leqq 0$, then $w \leqq 0$ on $[1, +\infty[$. If $w(1) > 0$, let t_0 be the first point (possibly $+\infty$) such that $w(t_0) = 0$. By (25), w decreases on $1, t_0$. Clearly, we then have $w \leqq |w(1)|$ in all cases, on $[1, +\infty[$.

Q. E. D.

7.11. REMARK. — Inequalities (22), (24) and the method used above show that when the sectional curvature has a strictly negative upper bound on M, the potential $Gf(x)$ (f with compact support) is bounded by Cte $e^{-ar(x)}$ (where $a > 0$ is constant) as $x \to \infty$.

8. Application : Homogeneous spaces

8.1. Let G be a connected Lie group and $M = G/H$ a homogeneous space of G (where H is a closed subgroup of G). Let D be a second order elliptic differential operator on M, invariant by left translations by elements of G. Assume that on M there is a riemannian structure invariant by left translations, and let d be the corresponding distance on M; if $\mathfrak{H}$ and $\mathfrak{G}$ are the Lie algebras of H and G, there is such a Riemann structure on M if, and only if, the natural representation of Ad (H) into the group of automorphisms of $\mathfrak{G}/\mathfrak{H}$ maps Ad (H) into a relatively compact group. This is in particular the case if Ad (H) is compact (for instance if $H = \{ e \}$).

It is well-known, when H is compact (*cf.* [14]) that there is a C_0-diffusion semi-group (P_t) on M with infinitesimal generator D. We prove that this is still true in the situation considered here, and in fact give an estimate of the speed at which P_t tends to zero at infinity generalizing and refining results due to L. Gårding, when D is self-adjoint.

8.2. PROPOSITION. — *Let M be a noncompact connected homogeneous space of a Lie group G; assume that there is on M a Riemannian distance d invariant by left translations* (cf. 8.1 *above*). *Fix a point* 0 *in M. Let D be a left invariant second order elliptic differential operator on M. Let (P_t) be the C_0-diffusion semi-group on M with infinitesimal generator D. For each function with compact support $f \in B_c(M)$, for each $\alpha > 0$, there are constants $\beta > 0$, $\gamma > 0$ such that*

$$|P_t f(x)| \leqq \exp[\beta(1+t)] \exp\left[-\gamma \frac{d^2(0, x)}{t}\right] \quad \text{for} \quad d(0, x) \geqq \alpha(1+t).$$

Proof. — It is clearly enough to consider only the case where $c = -D\,1 = 0$. Since the domain of the infinitesimal generator of P_t contains the constants (*cf.* [14], section 6), we have

$$\frac{\partial}{\partial t}[P_t 1] = P_t D 1 = 0$$

so that $P_t 1 \equiv 1$; the lifetime of the minimal process must then be infinite.

Let K be a fixed compact set in M; let Q_x be the law of X starting at $x \in M$, and T_K the entrance time in K for X.

Consider the balls B_n, $n = 1, 2, \ldots$, in M of radii n and center 0 where 0 is a fixed point in M. Call T_n the first entrance time in B_n for X. Assume (which does not restrict the generality) that $K \subset B_1$. For $x \in M - B_n$, we write, when T_K is finite

$$T_K \geqq T_1 \geqq T_1 - T_2 + T_2 - T_3 + \ldots + T_{n-1} - T_n.$$

Let

$$S_i = T_i - T_{i+1}, \quad i = 1, \ldots, n-1, \quad \text{on } T_K < \infty.$$

Let ε_i, $i = 1, \ldots$, be an infinite, decreasing sequence of positive numbers such that $\sum_{i=1}^{\infty} \varepsilon_i = +\infty$, $\lim_{n \to \infty} \varepsilon_n = 0$. Fix $t > 0$. If $T_K < t$, we have $\sum_{i=1}^{n-1} S_i < t$. If moreover exactly k of the events $\{ S_i \geqq \varepsilon_i \}$, $i = 1, \ldots, n-1$, are realized (say those corresponding to $i_1, i_2, \ldots, i_k$), we get

$$\varepsilon \quad + \ldots + \varepsilon_{i_k} < t,$$

and hence since (ε_i) is a decreasing sequence

(1) $$\varepsilon_{n-k} + \varepsilon_{n-k+1} + \ldots + \varepsilon_{n-1} \leqq \varepsilon_{i_1} + \varepsilon_{i_2} + \ldots + \varepsilon_{i_k} < t.$$

Denoting

(2) $$F(n) = \sum_{i=1}^{n} \varepsilon_i$$

we get from (1) :

(3) $$F(n-k) > F(n-1) - t.$$

Defining a function $G(u)$ on $[0, +\infty[$ such that

(4) $$G(u) \leqq \inf\{ n \geqq 1;\ F(n) > u \}.$$

(3) becomes $n-k > G[F(n-1)-t]$. Consequently, if $\{ T_K < t \}$ occurs, then at least $p \geqq G[F(n-1)-t]$ events among the $\{ S_i \geqq \varepsilon_i \}$, $i = 1, 2, \ldots, n-1$, are not realized, that is

(5) $$\{ T_K < t \} \subset \{ S_{j_1} < \varepsilon_{j_1} \} \cap \ldots \cap \{ S_{j_p} < \varepsilon_{j_p} \},$$

where

$$1 \leqq j_1 \leqq \ldots \leqq j_p \leqq n-1 \qquad \text{and} \qquad p \geqq G[F(n-1)-t].$$

An easy application of the strong Markov property shows that for $x \in M - B_n$:

$$Q_x(S_{j_p} < \varepsilon_{j_p}, \ldots, S_{j_1} < \varepsilon_{j_1}) = E_x\{I_{S_{j_d} < \varepsilon_{j_d}} \ldots I_{S_{j_2} < \varepsilon_{j_2}} Q_{X T_{j_2}}(T_{j_1} < \varepsilon_{j_1})\}.$$

Denoting

$$m_i = \sup_{x \in \partial Bi} Q_x(T_i < \varepsilon_i)$$

we get by an easy induction (since $X_{T_i} \in$ à ∂B_i for $T_i < +\infty$) :

$$(6) \qquad Q_x(S_{j_p} < \varepsilon_{j_p}, \ldots, S_{j_1} < \varepsilon_{j_1}) \leqq m_{j_p} \ldots m_{j_1} \qquad \text{for } x \in M - B_n.$$

Let B be a small ball of center 0, f a diffeomorphism of a neighbourhood of $\overline{B}$ onto an open set in $\mathbf{R}^n$. Call $B(x)$ the image of B by a left translation L_x transforming 0 in x, and $f_x = f \circ L_x^{-1}$ the corresponding diffeomorphism of B_x into $\mathbf{R}^n$. Since $f_x(D) = f \circ L_x^{-1}(D) = f(D)$, it is clear from [3] (cor. 5.3), that, if $\sigma_{B(x)}$ is the exit time from $B(x)$ for X, then

$$Q_x(\sigma_{B(x)} < u) \leqq A \exp\left(-\frac{c}{u}\right) \qquad \text{for } 0 < u < \alpha \text{ and all } x \in M,$$

where A, c, α are three positive constants. Then *a fortiori* we get [assuming radius $(B) < 1$] :

$$m_i \leqq \sup_{x \in \partial Bi} Q_x(\sigma_{B(x)} < \varepsilon_i) \leqq A \exp\left(-\frac{c}{\varepsilon_i}\right)$$

for $\varepsilon_i < \alpha$.

Call now r the first subscript such that $\varepsilon_r < \alpha$; then since the sequence ε_i is decreasing, and since for n large enough $p > r$:

$$(7) \qquad m_{j_p} \ldots m_{j_1} \leqq A^p \exp\left(-c \sum_{i=r}^{p} \frac{1}{\varepsilon_i}\right).$$

Finally using (5), (6) and (7), we get

$$Q_x(T_K < t) \leqq \exp\left(-c \sum_{i=k}^{p} \frac{1}{\varepsilon_i} + p \log A\right) \qquad \text{for } x \in M - B_n,$$

where $p \geqq G(F(n-1) - t)$.

In particular, take $\varepsilon_n = t/n$. Then

$$|F(n) - t \log n| \leqq \alpha_1 t \qquad \text{for some } \alpha_1 > 0.$$

Then

$$\inf\{n \geqq 1;\ F(n) > u\} \geqq \inf\{n \geqq 1;\ t \log n + \alpha_1 t > u\},$$

and we may take by (4) :

$$G(u) = \exp\left[\frac{u}{t} - \alpha_1\right],$$

$$G(F(n-1)-t) \geqq G[t \log(n-1) - \alpha_1 t - t],$$

$$G(F(n-1)-t) \geqq (n-1)\exp[-(2\alpha_1+1)].$$

Now

$$\sum_{i=r}^{p} \frac{1}{\varepsilon_i} = \frac{1}{t}\sum_{i=r}^{p} i = \frac{1}{t}\left[\frac{p(p+1)}{2} - \frac{r(r-1)}{2}\right]$$

$$\geqq \frac{1}{t}\,\frac{(n-1)^2}{2}\exp[-4\alpha_1+2)] - \frac{r(r-1)}{2t}.$$

Since $t/\alpha \leqq r \leqq (t/\alpha)+1$, we get for $x \in M - B_n$:

$$Q_x(T_K < t) \leqq \exp[\alpha_2(1+t)]\exp\left[-\frac{(n-1)^2}{t}\alpha_3 + (n-1)\alpha_4\right],$$

where $\alpha_2, \alpha_3, \alpha_4$ are positive constants. Thus if $d(0, x) \geqq 2+\alpha_5 t$, we get

$$Q_x(T_K < t) \leqq \exp[\alpha_6(1+t)]\exp\left[-\alpha_7\frac{d(0, x)^2}{t}\right],$$

where $\alpha_5, \alpha_6, \alpha_7$ are positive constants.

This achieves the proof since for $g \in B_c(M)$, we have, for K containing the support of f (*cf.* proof of prop. 3.1) :

$$|P_t g(x)| \leqq \|g\|\, Q_x(T_K < t).$$

One must also note that a similar result holds if d is replaced by a multiple of itself, so that the condition $d(0, x) \geqq 2+\alpha_5 t$ may be replaced by $d(0, x) \geqq \alpha_8(1+t)$ where α_8 is an arbitrary fixed positive number. Of course α_6 and α_7 must be modified accordingly.

Q. E. D.

9. Application : Case of $\mathbf{R}^n$

9.1. We now take $M = \mathbf{R}^n$ and

$$D = \sum_{i,j} a_{ij}(x)\frac{\partial^2}{\partial x_i \partial x_j} + \sum_i b_i(x)\frac{\partial}{\partial x_i} - c(x),$$

where the coefficients are Holder continuous (locally), (a_{ij}) is positive definite, $c(x) \geqq 0$. We sketch a couple of *rough* results obtained by applying our criteria, to evaluate the method in this case. To apply the results of section 5 take

$$h(x) = |x| = (\sum_i x_i^2)^{1/2}.$$

With the notations of section 5.1, we get

$$A(x) = |x|^{-2} \sum_{i,j} a_{ij} x_i x_j,$$
$$B(x) = |x|^{-1} [\sum_i a_{ii} + \sum_i b_i x_i - A(x)].$$

Call $\lambda(x)$ and $\Lambda(x)$ the smallest and largest eigenvalues of $(a_{ij}(x))$ and let

$$c^-(r) = \inf_{|x|=r} c(x), \qquad c^+(r) = \sup_{|x|=r} c(x)$$
$$b^-(r) = \inf_{|x|=r} \frac{1}{r} \sum b_i(x) x_i, \qquad b^+(r) = \sup_{|x|=r} \frac{1}{r} \sum b_i(x) x_i.$$

Geometrically we may say that when $b^-(r) < 0$, $(-b^-(r))$ is the largest inward drift and when $b^+(r) > 0$, $b^+(r)$ is the largest outward drift, at distance r.

9.2. Assume $b^-(r) \geqq 0$ (no inward drift) and for each $i = 1, \ldots, n$, $a_{ii}(x) \leqq \mathrm{Cte}\,|x|^2$ (for $|x|$ large). The minimal semi-group P_t is then a C_0-diffusion semi-group.

Indeed in this case (notations of section 5.1), we may take $M^-(r) = 0$ since

$$\frac{B(x)}{A(x)} \geqq |x|^{-1}\left(\frac{\sum a_{ii}}{A(x)} - 1\right) \geqq |x|^{-1}\left(\frac{(n-1)\lambda(x) + \Lambda(x)}{\Lambda(x)} - 1\right)$$
$$= |x|^{-1} \frac{\lambda(x)}{\Lambda(x)} (n-1).$$

Then $H^-(r) \equiv 1$ and $A^+(r) = \mathrm{Cte}\, r^2$ since

$$A(x) \leqq \Lambda(x) \leqq \sum_i a_{ii}(x).$$

Apply proposition 5.2 to conclude.

9.3. Assume $n = 2$, $b^+(r) \leqq 0$ (no outward drift), and, for some $\varepsilon > 0$ and for $|x|$ large

$$\frac{\Lambda(x)}{\lambda(x)} \leqq 1+\text{Cte}\,|x|^{-\varepsilon}, \qquad \lambda(x) \geqq \text{Cte}\,|x|^{2+\varepsilon}(1+c^+(\|x\|)).$$

Then the minimal semi-group is *not* a C_0-diffusion semi-group.

Indeed, for any n,

$$\frac{B(x)}{A(x)} \leqq |x|^{-1}\left(\frac{\sum a_{ii}}{A(x)}-1\right) \leqq |x|^{-1}\frac{\lambda(x)+(n-1)\Lambda(x)}{\lambda(x)}$$
$$\leqq |x|^{-1}\frac{\Lambda(x)}{\lambda(x)}(n-1)$$

and one can then apply proposition 5.2 with $M^+ = |x|^{-1}+\text{Cte}\,|x|^{-1-\varepsilon}$ and $\text{Cte}\,r^{-1} \leqq H^+(r) \leqq r^{-1}$.

Examples of such semi-groups for $n \geqq 2$ can also be constructed easily, taking for instance $\sum a_{ii}(x)+\sum b_i(x)\,x_i = 0$, while $\lambda(x)$ is larger than $\text{Cte}\,|x|^{2+\varepsilon}(1+c^+(\|x\|))$ (this corresponds to a very large inward drift at infinity). All these examples correspond of course to recurrent processes since our sufficient criteria for P_t not to be a C_0-diffusion semi-group implies the classical sufficient criterion for recurrence (*cf.* prop. 5.2). Thus in dimension $b \geqq 2$, we have no example of diffusion semi-groups on $\mathbf{R}^n$ which would *not* tend to zero at infinity and which would also be *transient* (such examples are easily obtained in dimension 1 using proposition 4.3). As was pointed out to us by J. TAYLOR, if U is an open set in $\mathbf{R}^n$ with irregular boundary points (DYNKIN's sense) the induced Brownian motion on U is not a C_0-diffusion semi-group; of course such a process is transient if U is relatively compact. By a suitable diffeomorphism, this may well take care of the case of $\mathbf{R}^n$, $n \geqq 2$, but we have not checked it.

9.4. Assume that $b^-(r) \geqq 0$ (no inward drift) and that, for each i

$$a_{ii}(x) \leqq \text{Cte}\,|x|^2\,c^-(|x|) \qquad (\text{for } |x| \text{ large}).$$

Then the potential kernel of P_t is finite and tends to zero at infinity.

The same conclusion holds if we assume instead that $n \geqq 3$, $b^-(r) \geqq 0$ and $\lambda(x)/\Lambda(x) \geqq (1+\varepsilon)/(n-1)$ for $|x|$ large, where $\varepsilon > 0$ is fixed.

Of course, in these two situations, P_t is a C_0-diffusion semi-group. The proof is a direct application of proposition 5.2.

For applications of the recurrence criteria and the explosion criteria to the case of $\mathbf{R}^n$, we refer to [15] and [16].

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