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CONTINUOUS DERIVATIONS OF VALUED FIELDS

BY

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SUMMARY. — Let $L \supset K$ be complete ultrametric fields. Every K-linear continuous derivation of L into a Banach space over L factors through the universal K-linear continuous derivation $d : L \to \Omega^b_{L/K}$. The structure of $(d, \Omega^b_{L/K})$ is described in detail for discrete valued fields L/K. For dense valuations there are results on : extensions of continuous derivations; topologically transcendental and algebraic elements; topological p-bases. As application one finds a result on tensor products of Banach algebras over K.

Introduction

The goal of this work is to describe continuous derivations of a valued field L into a Banach space over L. As there are no continuous derivations (other than 0) for archimedean valued fields, we restrict our attention to non-archimedean valued fields.

For complete valued fields $L \supset K$, one constructs in section 2 an universal continuous K-linear derivation $d^b_{L/K} : L \to \Omega^b_{L/K}$. It is shown in (2.5) that the pair $(d^b_{L/K}, \Omega^b_{L/K})$ can be obtained from the universal V_K-linear derivation $d_{V_L/V_K} = V_L \to \Omega^1_{V_L/V_K}$, where V_L and V_K are the rings of integers of L resp. K.

An almost complete description of $(d^b_{L/K}, \Omega^b_{L/K})$ for discrete valued fields is given in sections 3, 4, 5. The results are close to computations of R. BERGER and E. KUNZ on the module of differentials of a discrete valuation ring.

The main result of section 6 is the following :

If $L \supset K$ are complete valued fields with residue-characteristic zero then the canonical map $\Omega^b_K \otimes_K L \to \Omega^b_L$ is isometric (6.1).

In an informal way this means that any continuous derivation of K into a suitable Banach space over L can be extended with the same norm. Using (6.1) one obtains in section 6 a fairly good description of $\Omega^b_{L/K}$ for the case of residue characteristic zero. An application of this yields the following result on tensor products of commutative Banach algebras :

Suppose that K has residue-characteristic zero. Let A and B be commutative Banach algebras with a unit over K such that the norms of A and B are powermultiplicative (i. e. $\| f^n \| = \| f \|^n$ for all f and n). Then the norm on $A \hat{\otimes}_K B$ is also power-multiplicative (8.5).

Section 7 deals with fields of caracteristic $p \neq 0$. A fundamental result on extensions of bounded derivations in that case leads to a description of $\Omega^b_{L/K}$.

As a corollary one can show the existence of topological p-bases for field extensions of countable type. Bases of that type are used by R. KIEHL in his proof of the excellence of affinoid algebras over a field L with $[L : L^p] = \infty$.

Finally, an exposition of the main technical tools used in this work is given in section 1.

1. Preliminaries

To facilitate the reading of the next sections, we present here a number of more or less disjoint an technical topics. At the same time some notation is introduced.

(A) *Non-Archimedean Banach spaces.* — Let L be a complete non-Archimedean valued field. (We will abbreviate this sometimes by " L is a field ".) Its ring of integers $\{ x \in L \mid |x| \leq 1 \}$ is denoted by V_L. The *residue field* of V_L (or L) is denoted by the small letter l. The *value group* of L will be written as $|L^*|$. If $|L^*|$ is discrete (e. g. the valuation is discrete) then π or π_L will denote a *uniformizing parameter* (that is, $0 < |\pi| < 1$, and $|L^*| = \{ |\pi|^n \mid n \in \mathbf{Z} \}$). For discrete valued fields $K \subset L$, $e(L/K)$ will denote the ramification index; so $e(L/K) =$ order of $|L^*| / |K^*|$. A *non-Archimedean Banach space* over L is a vector space E over L provided with a norm $\| \quad \| : E \to \{ r \in \mathbf{R} \mid r \geq 0 \}$ having the properties :

(1) $\| x \| = 0$ if and only if $x = 0$;

(2) $\| \lambda x \| = |\lambda| . \| x \|$ $(\lambda \in L)$;

(3) $\| x + y \| \leq \max (\| x \|, \| y \|)$.

Moreover E is supposed to be complete with respect tot the metric derived from the norm. Of the various possibilities of making a *residue space* of E, we choose :

the residue space τE of E equals

$$\{ x \in E \mid \| x \| \leq 1 \} / \{ x \in E \mid \| x \| < 1 \}.$$

This is in fact a vector space over $l = \tau L$. For additive maps (or L-linear maps) $g : E \to F$, E and F Banach spaces over L, such that $\| g(e) \| \leq 1$ for all $e \in E$, $\| e \| \leq 1$, the induced additive (or l-linear) map : $\tau E \to \tau F$ is denoted by $\tau(g)$. Also the residue map $\{ x \in E \mid \| x \| \leq 1 \} \to \tau E$ will be given the name τ. A Banach space E over L is called *discrete* if $\| E \| = | L |$ and the valuation of L is discrete. For any $\alpha \in \mathbf{R}$, $0 < \alpha \leq 1$, and any subset $X = \{ x_i \mid i \in I \}$ of a Banach space E over L we define :

X is *α-orthogonal* if for any convergent expression $x = \sum \lambda_i x_i$, $\lambda_i \in L$, the inequality $\max_i (| \lambda_i | . \| x_i \|) \geq \| x \| \geq \alpha \max_i (| \lambda_i | . \| x_i \|)$ holds.

X is *α-orthonormal* if for any convergent expression $x = \sum \lambda_i x_i$, $\lambda_i \in L$, the inequality $\max_i | \lambda_i | \geq \| x \| \geq \alpha \max_i |\lambda_i|$ holds. Further 1-orthogonal and 1-orthonormal are abbreviated by orthogonal and orthonormal.

X is called an *α-orthogonal base* (resp. *α-orthonormal base* or *α-base*) if X is α-orthogonal (resp. α-orthonormal) and every $x \in E$ can be expressed as a convergent sum $x = \sum \lambda_i x_i$. One easily shows that a subset $X \subset E$ such that $\| x \| = 1$ for all $x \in X$, E discrete Banach space over L, is an orthonormal base of E if and only if the subset $\tau X \subset \tau E$ is a Hamel-base of τE over l. So a discrete Banach space E is completely determined by its residue space τE.

Further, it is known that any Banach space over L of countable type has for any α, $0 < \alpha < 1$, an α-orthogonal base (*see* [7]). For Banach spaces E and F over L, we define $\mathrm{Hom}_L(E, F)$ or $\mathrm{Hom}(E, F)$ to be the vector space of all bounded L-linear maps from E into F. The definition $\| t \| = \sup \{ \| x \|^{-1} \| t(x) \| \mid x \in E, x \neq 0 \}$ makes $\mathrm{Hom}(E, F)$ into a Banach space over L.

A Banach space F is called *spherically complete* if every sequence of spheres $\{ B_n \mid n \geq 1 \}$ in F, with the property $B_n \supset B_{n+1}$ for all n, has a non-empty intersection.

According to a theorem of A. W. Ingleton, we have : " F is spherically complete if and only if for every Banach space E and every subspace E_1 of E, any bounded L-linear map $l_1 : E_1 \to F$ can be extended to a bounded L-linear map $l : E \to F$ such that $\| l \| = \| l_1 \|$ ".

For a field L, spherically complete is equivalent to maximally complete in the sense of Krull and Kaplansky.

We will need explicitely the following results :

Lemma :

(A 1) *Let E be a vector space over L, provided with two equivalent norms $\| \;\|_1$ and $\| \;\|_2$. Suppose that $(E, \| \;\|_i)$ is a discrete Banach space over L $(i = 1, 2)$. Then the l-linear spaces $\tau(E, \| \;\|_1)$ and $\tau(E, \| \;\|_2)$ are isomorphic.*

(A 2) *Let E be a Banach space over L and F a spherically complete Banach space over L. Then also* Hom (E, F) *is a spherically complete Banach space.*

Proof :

(1) If $\dim E < \infty$, then $\dim_L E = \dim_l \tau(E, \|\ \|_1) = \dim_l \tau(E, \|\ \|_2)$, and (1) is trivial. If $\dim E = \infty$, we take subsets X_1 and X_2 of E such that X_1 is an orthonormal base of $(E, \|\ \|_1)$, and X_2 is an orthonormal base of $(E, \|\ \|_2)$. We have to show that $\operatorname{card} X_1 = \operatorname{card} X_2$. Since the norm $\|\ \|_2$ is equivalent to $\|\ \|_1$, we find that X_2 is an α-orthogonal base of $(E, \|\ \|_1)$, for some α, $0 < \alpha \leq 1$.

By expressing the elements of X_1 as convergent sums in X_2 and using that a convergent expression has a countable support, one finds $\operatorname{card}(X_1) \leq \operatorname{card}(X_2)$. In the same way, $\operatorname{card}(X_2) \leq \operatorname{card}(X_1)$.

(2) Put $G = \operatorname{Hom}(E, F)$, and let $B_n = \{ t \in G \mid \| t - t_n \| \leq r_n \}$ be a sequence of spheres in G such that $B_n \supset B_{n+1}$ for all n. Put $F_n = F$ for all n, and let $r : F \to \prod_{n=1}^{\infty} F_n$ denote the map given by $i(f)_n = f$ for all $f \in F$ and $n \geq 1$.

The map

$$F \xrightarrow{i} \prod F_n \xrightarrow{j} \prod F_n / \sum F_n$$

is isometric and since F is spherically complete, it has a left inverse p with $\| p \| = 1$. Now $t_0 = p_0 j_0 \prod t_n : E \to F$ has the property $t_0 \in B_n$ for all $n \geq 1$. [Part (A 2) of the lemma is due to R. ELLIS.]

More information on Banach spaces can be found in [5], [7] and [8].

(B) *Tensor Product of Banach spaces.* — Let E and F be Banach spaces over L. On $E \otimes F$ we introduce the semi-norm $\|\ \|$, given by

$$\| a \| = \inf \{ \max_{1 \leq i \leq s} \| e_i \| . \| f_i \| \mid a = \textstyle\sum_{i=1}^{s} e_i \otimes f_i \}.$$

Put $T = (E \otimes F, \|\ \|)$.

(B 1) LEMMA. — *T has the following universal property : For every Banach space G over L and every bounded bilinear map $t : E \times F \to G$ the corresponding linear map $t' : E \otimes F \to G$ has the property $\| t \| = \| t' \|$.*

Proof. — First, we note that $\| t \|$ is defined to be the supremum of $\{ \| e \|^{-1} \| f \|^{-1} \| t(e, f) \| \mid e \in E, f \in F \}$. Let $a = \sum e_i \otimes f_i \in E \otimes F$. Then

$$\| t'(a) \| = \| \textstyle\sum t(e_i, f_i) \| \leq \max_i \| t(e_i, f_i) \| \leq \| t \| \max_i \| e_i \| . \| f_i \|.$$

Consequently, $\| t'(a) \| \leq \| t \| . \| a \|$, and so $\| t' \| \leq \| t \|$. On the other hand,

$$\| t(e, f) \| = \| t'(e \otimes f) \| \leq \| t' \| . \| e \otimes f \| \leq \| t' \| . \| e \| . \| f \|.$$

So $\| t \| \leq \| t' \|$.

(B 2) LEMMA.

(1) *Take* $\alpha \in \mathbf{R}$, $0 < \alpha \leq 1$. *If* $\{ e_i \mid 1 \leq i \leq s \} \subset E$ *is* α-*orthogonal then for all* $f_1, \ldots, f_s \in F$,

$$\| \Sigma_{i=1}^{s} e_i \otimes f_i \| \geq \alpha \max (\| e_i \| . \| f_i \|).$$

(2) *The semi-norm on* $E \otimes F$ *is a norm and satisfies* $\| e \otimes f \| = \| e \| . \| f \|$.

(3) *For all suspaces* E_1 *of* E *and* F_1 *of* F *is the map*

$$(E_1 \otimes F_1, \| \quad \|) \rightarrow (E \otimes F, \| \quad \|)$$

isometric.

(4) *If every finite dimensional subspace of* E *has an orthogonal base then every* $a \in E \otimes F$ *can be written as* $a = \Sigma e_i \otimes f_i$ *where* $\| a \| = \max (\| e_i \| . \| f_i \|)$.

Proof :

(1) Let G be a spherically complete field containing L (G exists, we will not go into the details of that). Define $t_1 : L e_1 + \ldots + L e_s \rightarrow G$ by $t_1(e_i) = 1$ and $t_1(e_j) = 0$ if $j \neq i$. Define $t_2 : L f_i \rightarrow G$ by $t_2(f_i) = 1$ (we suppose here, as we may, that $f_i \neq 0$). Extend both mappings to the whole of E, resp. F, with values in G and without increasing their norms. Consider $t : E \times F \rightarrow G$, $t(e, f) = t_1(e) \; t_2(f)$ and let $t' : E \otimes F \rightarrow G$ be the corresponding L-linear map. Then $t'(a) = 1$ and

$$\| t' \| = \| t \| = \| t_1 \| . \| t_2 \| \leq \alpha^{-1} \| e_i \|^{-1} \| f_i \|^{-1}.$$

So $\| a \| \geq \alpha \| e_i \| . \| f_i \|$.

Alternative proof (after T. A. SPRINGER). — Let

$$x = \Sigma_{i=1}^{a} e_i \otimes f_i \qquad \text{and} \qquad x = \Sigma_{j=1}^{b} e'_j \otimes f'_j$$

be another representation of x. We have to show

$$\max \| e'_j \| . \| f'_j \| \geq \alpha \max \| e_i \| . \| f_i \|.$$

Take $\beta \in \mathbf{R}$, $0 < \beta < 1$, and let $g_1, \ldots, g_c$ be an β-orthogonal base of the vector space $L f'_1 + \ldots + L f'_b$. (For every β, $0 < \beta < 1$, such a base exists !). Then $f'_j = \Sigma_{k=1}^{c} \lambda_{jk} g_k$ with

$$\| f'_j \| \geq \beta \max_k (| \lambda_{jk} | . \| g_k \|).$$

Further $x = \Sigma_j e'_j \otimes f'_j = \Sigma_k (\Sigma_j \lambda_{jk} e'_j) \otimes g_k$.

Since the $\{g_1, \ldots, g_c\}$ are linearly independent, we have

$$\Sigma_j \lambda_{jk} e'_j = \Sigma_{i=1}^{a} \mu_{ki} e_i$$

and $f_i = \Sigma_k \mu_{ki} g_k$, for some $\mu_{ki} \in L$. Now

$$\max_j \| e'_j \| . \| f'_j \| \geq \beta \max_{j,k} \| e'_j \| . | \lambda_{jk} | . \| g_k \| \geq \beta \max_k \| \Sigma_j \lambda_{jk} e'_j \| . \| g_k \| \\ \geq \alpha\beta \max_{i,j} | \mu_{ki} | . \| e_i \| . \| g_k \| \geq \alpha\beta \max_i \| e_i \| . \| f_i \|.$$

Since $\beta \in \mathbf{R}$, $0 < \beta < 1$, was arbitrary, it follows that

$$\max \| e'_j \| . \| f'_j \| \geq \alpha \max \| e_i \| . \| f_i \|.$$

(2) Take $a \in E \otimes F$, $a \neq 0$. Write $a = \Sigma e_i \otimes f_i$ where $\{e_i, \ldots, e_s\}$ is linearly independent over L. Then, for some α, $0 < \alpha < 1$, $\{e_i, \ldots, e_s\}$ is α-orthogonal. According to (1), $\| x \| \neq 0$. Hence $\| \; \|$ is a norm. The equality $\| e \otimes f \| = \| e \| . \| f \|$ follows directly from (1).

(3) The norm on $E_1 \otimes F_1$ will be denoted by $\| \; \|_1$. Clearly, $\| x \|_1 \geq \| x \|$ for all x in $E_1 \otimes F_1$. On the other hand : for $x \in E_1 \otimes F_1$ and $\alpha \in \mathbf{R}$, $0 < \alpha < 1$, there are $e_1, \ldots, e_s$ in E_1 and $f_1, \ldots, f_s \in F_1$ such that $e_1, \ldots, e_s$ is α-orthogonal and $x = \Sigma e_i \otimes f_i$.

Hence (1) yields $\| x \| \geq \alpha \max (\| e_i \| . \| f_i \|) \geq \alpha \| x \|_1$. Since $\alpha \in \mathbf{R}$, $0 < \alpha < 1$, was arbitrary, we may conclude $\| x \|_1 \leq \| x \|$.

(4) Take $x \in E \otimes F$. Then $x = \Sigma_{i=1}^{s} e_i \otimes f_i$. Choose an orthogonal base $\{e'_i\}$ of $L e_1 + \ldots + L e_s$. Then x can also be expressed as $\Sigma e'_i \otimes f'_i$ (some $f'_i \in F$). According to (1), we have

$$\| x \| = \max \| e'_i \| . \| f'_i \|.$$

DEFINITION. — The completion of $E \otimes F$ with respect to the norm on the tensorproduction is denoted by $E \hat{\otimes} F$.

(B 3) PROPOSITION (L. GRUSON [4]). — *$\hat{\otimes} F$ is an exact functor for every Banach space F.*

Proof. — Let

$$0 \to E_1 \xrightarrow{\alpha} E_2 \xrightarrow{\beta} E_3 \to 0$$

be an exact sequence of Banach spaces (i. e. the sequence is exact as a sequence of vector spaces over L and α and β are bounded L-linear maps). We have to show that the derived sequence

$$0 \to E_1 \hat{\otimes} F \xrightarrow{\alpha'} E_2 \hat{\otimes} F \xrightarrow{\beta'} E_3 \hat{\otimes} F \to 0$$

is exact. The most difficult part, " α' injective ", follows directly from (B 2) part (3). The rest is left to the reader.

(C) *The structure theorem for complete local rings.* — In this subsection, we gather the components of Cohen's theory on complete local rings which are of particular interest for us.

A complete local ring is a ring R which has precisely one maximal ideal M, and satisfies $\bigcap_{n=1}^{\infty} M^n = 0$, and R is complete with respect to the uniform structure induced by M. We remark that R need not be Noetherean and that M need not be finitely generated. The residue field of R will be written as $K = R/M$. Let $\pi : R \rightarrow K$ be the canonical map. A subfield L of R is called a coefficient field if $\pi(L) = K$. For a coefficient field L is the map $\pi \mid L : L \rightarrow K$ bijective. Suppose that char $K = p \neq 0$. A subring V of R is called a coefficient ring if : (1) V is a discrete complete valuation ring and its maximal ideal is generated by p; (2) $\pi(V) = K$.

(C 1) THEOREM (I. S. COHEN) :

(1) *If* char $K = 0$, *then every maximal subfield of* R *is a coefficient field.*

(2) *If* char $R = p \neq 0$, *then* R *contains a coefficient field.*

(3) *If* char $K = p \neq 0$ *and* $p^n R \neq 0$ *for all* n, *then* R *contains a coefficient ring.*

(4) *If* R *is a complete discrete valuation ring and* char $R =$ char K, *then* R *is isomorphic to* $K[[\pi]]$ *where* $\pi \in R$ *is a uniformizing parameter of* R.

For the proof of this theorem, we refer to [9] volume 2, or " NAGATA, *Local rings* " or [3] EGA IV (première partie), chap. 0, § 19.

(D) *Extensions of fields.* — First of all, we will tacitely use the following criterion for separability ([9], part 1, Th. 42) : " $l \supset k$ is separable if and only if every derivation of k is extendable to l ".

Let us suppose that $l \supset k$ are fields of characteristic $p \neq 0$. By k^+ we mean $k(l^p)$. A *p-independent* (or *p-free*) set $\{ a_i \mid i \in I \} \subset l$ of l/k is a set satisfying : " $a_i \notin k^+(a_j \mid j \in I, j \neq i)$ for all $i \in I$ ". This condition is equivalent with : " the monomials $\{ a_{i_1}^{\alpha_1} \ldots a_{i_n}^{\alpha_n} \mid n \in \mathbf{N};\ i_1, \ldots, i_n$ different elements of I; $0 \leq \alpha_i < p \}$ are linearly independent over k^+ ".

The set $\{ a_i \mid i \in I \}$ is called a p-base of l/k, if this set is p-free in l/k and if moreover $l = k^+(a_i \mid i \in I)$. Every maximal p-independent set in l/k (existence guaranteed by the lemma of Zorn) is a p-base of l/k.

For rings $A \subset B \subset C$, we let $(d_{C/B}, \Omega_{C/B})$ denote the universal module of differentials, and $\gamma_{C/B/A} = \ker(\Omega_{B/A} \otimes C \rightarrow \Omega_{C/A})$. In all cases, where derivations are involved, we keep the notations of [3] (EGA IV, chap. 0, § 20).

Further $\{ a_i \mid i \in I \}$ is a p-base in l/k if and only if $\{ d_{l/k}(a_i) \mid i \in I \}$ is a base of the l-vector space $\Omega_{l/k}$. For the case that k equals the prime field of l one obtains the usual definition and results. (*See* [9], part 1.)

DEFINITION. — Let M be a vector space over l. A symmetric k-bilinear 2-cocycle of l into M is a k-bilinear symmetric map $h : l \times l \to M$ satisfying :

$$h(ab, c) + ch(a, b) = h(a, bc) + ah(b, c) \qquad \text{(for all } a, b, c \in l).$$

This 2-cocycle is called trivial if there exists a k-linear map $h_0 : l \to M$ such that $h(a, b) = h_0(ab) - ah_0(b) - bh_0(a)$ for all $a, b \in l$.

(D 1) PROPOSITION. — *If l/k is separable, then every symmetric k-bilinear 2-cocycle of l is trivial.*

Proof. — Consider $R = l \oplus M$ provided with the ring-structure given by the formulas

$$(a, m) + (a', m') = (a + a', m + m')$$

and

$$(a, m)(a', m') = (aa', am' + a'm + h(a, a')).$$

R is a commutative ring. Its only maximal ideal is M and $M^2 = 0$. Let $\pi : R \to l$ denote the map of R onto its residue field. R is in a trivial way a complete local ring, and we can apply Cohen's structure theorem :

There exist a ring homomorphism $\varphi : l \to R$ such that $\pi \circ \varphi = \text{id}$. The map $D : k \to M$ given by $D(a) = \varphi(a) - (a, 0)$ is a derivation of k into M. This derivation can be extended to $D' : l \to M$. Consider $\varphi' = \varphi - D' : l \to R$. This is again a ring homomorphism satisfying $\pi \circ \varphi' = \text{id}$.

Write φ' in coordinates, $\varphi'(a) = (a, h_0(a))$ (with $a \in l$). Then $h_0 : l \to M$ is a k-linear map. The equation

$$\varphi'(ab) = \varphi'(a)\varphi'(b)$$

written down in coordinates gives

$$h(a, b) = h_0(ab) - ah_0(b) - bh_0(a).$$

2. General theory of continuous derivations

Let L be a complete valued field, and M a Banach space over L. We are interested in continuous derivations $D : L \to M$. If $L = \mathbf{R}$ or $\mathbf{C}$, the only continuous derivation is the zero-derivation. Excluding this case, we assume in the sequel that the valuation of L is non-Archimedean. (The trivial valuation is allowed.)

(2.1) LEMMA. — *Let $D : L \to M$ be a derivation and assume that the valuation of L is non-trivial. The following properties of D are equivalent :*

(1) *D is continuous;*

(2) $\{ \| D(x) \| \mid x \in L, \ | x | \leq 1 \}$ *is bounded;*

(3) $\{ \| D(x) \| / | x | \mid x \in L, \ x \neq 0 \}$ *is bounded.*

Proof. — *See* [1] (3.1.1). We remark that (2) and (3) are equivalent even if the valuation of L is trivial.

DEFINITIONS. — To include the case of a trivial valued field in our theory, we will consider, instead of continuous derivations, bounded derivation in the sense of part (2) and (3) of (2.1). Let $L \supset K$ denote valued fields and let M be a Banach space over L. Then $\mathrm{Derb}_K(L, M)$ denotes the L-linear space of all bounded K-linear derivations of L into M. This vector space is made into a Banach space over L by the norm

$$\| D \| = \sup \{ \| D(x) \| / | x | \mid x \in L, x \neq 0 \}.$$

Further provide $L \otimes_K L$ with the tensorproduct-norm, and let I denote the kernel of the map $\rho : L \otimes_K L \to L$ given by $\rho(\sum a_i \otimes b_i) = \sum a_i b_i$. Let I^2 denote the closure of the ideal in $L \otimes_K L$ generated by $\{ xy \mid x, y \in I \}$, and let $\Omega^b_{L/K}$ denote the completion of the normed space I/I^2. Finally, $d = d^b_{L/K} : L \to \Omega^b_{L/K}$ is the bounded derivation given by

$$d(a) = (a \otimes 1 - 1 \otimes a) + I^2 \in \Omega^b_{L/K}.$$

(2.2) THEOREM. — *The pair $(d^b_{L/K}, \Omega^b_{L/K})$ represents the functor "$M \to \mathrm{Derb}_K(L, M)$" of the category of Banach spaces over L into itself. Moreover the isomorphism $\mathrm{Hom}_L(\Omega^b_{L/K}, M) \to \mathrm{Derb}_K(L, M)$ is isometric.*

Proof. — The derivation $d : L \to \Omega^b_{L/K}$ induces a canonical map α : $\mathrm{Hom}_L(\Omega^b_{L/K}, M) \to \mathrm{Derb}_K(L, M)$ given by $\alpha(l) = l \circ d$. Since I/I^2 is generated by $d(L)$ and is dense in $\Omega^b_{L/K}$, we have that α is injective. For $a \in L$, we find

$$\| d(a) \| \leq \| a \otimes 1 - 1 \otimes a \| \leq | a |.$$

Hence $\| d \| \leq 1$ and $\| \alpha \| \leq 1$. We will construct a map β : $\mathrm{Derb}_K(L, M) \to \mathrm{Hom}_L(\Omega^b_{L/K}, M)$ satisfying $\| \beta \| \leq 1$ and $\alpha \circ \beta = \mathrm{id}$. This implies that α is bijective and isometric.

Construction of β. — Given $D \in \mathrm{Derb}_K(L, M)$, we define $h : L \otimes_K L \to \mathrm{M}$ by $h(\sum a_i \otimes b_i) = \sum a_i D(b_i)$. According to the definition of the norm on $L \otimes_K L$, $\| h \| \leq \| D \|$. The kernel of h is closed and contains

$\{ xy \mid x, y \in I \}$. So h induces an L-linear map $h' : I/I^2 \to M$ with $\| h' \| = \| h \|$. Define $\beta(D)$ to be the unique continuous extension of h' to a map $\Omega^b_{L/K} \to M$. Obviously, $\alpha \circ \beta = \mathrm{id}$ and $\| \beta \| \leqq 1$.

Remarks.

(1) If the valuation of L is trivial, then $\Omega^b_{L/K}$ equals $\Omega_{L/K}$ as a vector space over L, and its norm is trivial.

(2) It is obvious from the construction of $\Omega^b_{L/K}$, that $\Omega^b_{L/K}$ is the completion of $\Omega_{L/K}$ with respect to some semi-norm p. This semi-norm can be described as follows :

Let $D : L \to M$ be a bounded K-derivation and let $l : \Omega_{L/K} \to M$ be the corresponding linear map. The formula $p_D(x) = \| D \|^{-1} \| l(x) \|$ defines a semi-norm on $\Omega_{L/K}$. Now $p(x)$ equals $\sup \{ p_D(x) \mid D$ bounded K-derivation of $L \}$.

This result is not very useful however since one cannot calculate the semi-norm p. A better approach is to connect $\Omega^b_{L/K}$ with Ω_{V_L/V_K}, the universal module of derivations of V_L over V_K. (V_L and V_K are the integers of L and K.) This line of attack is followed in (2.3) up to (2.6).

DEFINITIONS. — Suppose that the valuation of L is non-trivial. Let $V = V_L$ denote the valuation ring of L, and let M be a V-module. The vector space $M \otimes_V L$ has a natural semi-norm induced by the absolute convex subset

$$M' = \{ m \otimes v \mid M \in m, v \in V \}$$

of $M \otimes_V L$. This semi-norm p is given by

$$p(x) = \inf \left\{ | v | \;\middle|\; v \in L, v \neq 0, v^{-1} x \in M' \right\}.$$

Let $F(M)$ denote the (separated) completion of $M \otimes_V L$ with respect to p. Then F is a covariant functor of the category of V-modules into itself.

(2.3) LEMMA. — $F(M) = \varprojlim \{ M \otimes_V (L/\alpha V) \mid \alpha \in V, \alpha \neq 0 \}$.

Proof. — As is well known (or easily checked), the completion of $M \otimes_V L$, with respect to p, equals $\varprojlim M \otimes_V L/O$, where O runs through the set of all convex neighbourhoods of $O \in M \otimes_V L$. A base for those neighbourhoods is

$$O_\alpha = \{ m \otimes \alpha \mid m \in M \}, \qquad \alpha \in V, \quad \alpha \neq 0.$$

Now $M \otimes_V L/O_\alpha$ is isomorphic to $M \otimes_V (L/\alpha V)$ and the lemma follows.

(2.4) PROPOSITION. — *The functor F has the following properties :*

(1) *Let* M *be a V-module and* H *a Banach space over* L. *Then every bounded V-linear map* $l : M \to H$ *factors uniquely through* $F(M)$, *in diagram*

$$\begin{array}{ccc} M & \xrightarrow{l} & H \\ \downarrow & \nearrow_{l'} & \\ F(M) & & \end{array}$$

Moreover $\| l' \| = \sup \| l(M) \| < \infty$.

(2) *If* $a : M_1 \to M_2$ *is a surjective map of V-modules, then* $F(a) : F(M_1) \to F(M_2)$ *is also surjective.*

Proof :

(1) The map $l : M \to H$ extends to $l_1 : M \otimes_V L \to H$, and since $l(M)$ is bounded, l_1 is continuous with respect to p. By continuity l_1 extends uniquely to a continuous $l' : F(M) \to H$. Hence l factors through $F(M)$. According to a result of J. Van Tiel ([8], (2.9), part 1° and 2°) :

(A) If the valuation of L is dense, then

$$\{ x \in M \otimes L \mid p(x) < 1 \} \subset M' \subset \{ x \in M \otimes L \mid p(x) \leqq 1 \}.$$

(B) If the valuation of L is discrete, then

$$M' = \{ x \in M \otimes L \mid p(x) \leqq 1 \}.$$

The set $M \otimes L$ is dense in $F(M)$, and consequently $\| l' \| = \| l_1 \|$.

Case (A). — Let $x \in M \otimes L$, and take $\lambda \in L$ such that $p(x) < | \lambda |$. Then

$$\| l_1(\lambda^{-1} x) \| < \| l_1(x) \| / p(x) = \| l_1(\lambda^{-1} x) \| . | \lambda | / p(x).$$

Because

$$\inf \{ | \lambda | / p(x) \mid \lambda \in L, | \lambda | > p(x) \} = 1,$$

we can conclude that $\| l_1 \| = \sup \| l_1(M') \| = \sup \| l(M) \|$.

Case (B). — The definition of p implies that

$$p(M \otimes L) = | L | = \{ | \pi_L |^n \mid n \in \mathbf{Z} \} \cup \{ 0 \}.$$

Take $x \in M \otimes L$, then $x = \pi_L^n y$, where $p(y) = 1$. Consequently,

$$\| l_1(x) \| / p(x) = \| l_1(y) \| / p(y)$$

and

$$\| l_1 \| = \sup \left\{ \| l_1(y) \| \;\middle|\; y \in M \otimes L, | p(y) | = 1 \right\} = \sup \| l(M) \|.$$

(2). Let M be a V-module and $M^t = \{ m \in M \mid \alpha m = 0$ for some $\alpha \in V, \alpha \neq 0 \}$ its torsion submodule. We show first that the map

$\rho : M \to M/M^t$ induces a bijective map $F(\rho) : F(M) \to F(M/M^t)$.
For every $\alpha \in V$, $\alpha \neq 0$, the sequence

$$0 = M^t \otimes L/\alpha V \to M \otimes L/\alpha V \overset{\rho_\alpha}{\to} M/M^t \otimes L/\alpha V \to 0$$

is exact. So ρ_α is bijective and also $F(\rho) = \varprojlim \rho_\alpha$ is bijective.

Now we may suppose that M_2 has no torsion. This implies (*see* Bourbaki, *Algèbre commutative*) that M_2 is a flat V-module. The exact sequence $0 \to K \to M_1 \to M_2 \to 0$ gives for every $\alpha \in V$, $\alpha \neq 0$, an exact sequence

$$0 \to K \otimes L/\alpha V \to M_1 \otimes L/\alpha V \to M_2 \otimes L/\alpha V \to 0.$$

For any $\beta \in V$, $0 < |\beta| \leq |\alpha|$ and any V-module M, the map $M \otimes L/\beta V \to M \otimes L/\alpha V$ is surjective. Hence the condition of Mittag-Leffler is satisfied, and we may conclude with [3] (EGA III, chap. 0, (13.2.2)) that

$$0 \to F(K) \to F(M_1) \to F(M_2) \to 0$$

is exact.

Remark. — The functor F is neither left- nor right-exact.

(2.5) Theorem. — *Let $M \supset L \supset K$ denote complete valued fields.*

(1) *There is a canonical isomorphism $\alpha : F(\Omega_{V_L/V_K}) \to \Omega^b_{L/K}$. The map α is in general not isometric but satisfies $\rho \| x \| \leq \| \alpha(x) \| \leq \| x \|$ where $\rho = \sup \{ |\lambda| \mid \lambda \in L, |\lambda| < 1 \}$.*

(2) *If the valuation of L is dense, then any $x \in \Omega^b_{L/K}$ with $\| x \| < 1$ can be represented as a convergent sum*

$$x = \sum \lambda_i \, dx_i; \qquad \lambda_i, x_i \in L; \qquad |\lambda_i| < 1, \qquad |x_i| < 1$$

and $\lim \lambda_i = 0$

(3) *If the valuation of L is discrete, then $\| \Omega^b_{L/K} \| = | L |$ and any $x \in \Omega^b_{L/K}$ with $\| x \| \leq 1$ can be represented as a convergent sum*

$$x = \lambda \pi_L^{-1} \, d\pi_L + \sum \lambda_i \, da_i$$

where λ, λ_i, $a_i \in V_L$ and $\lim \lambda_i = 0$.

(4) *The sequence*

$$\Omega^b_{L/K} \hat{\otimes} M \overset{\alpha}{\to} \Omega^b_{M/K} \overset{\beta}{\to} \Omega^b_{M/L}$$

has the following properties :

(*a*) *β is surjective and induces the norm on $\Omega^b_{M/L}$;*

(*b*) *$\beta \circ \alpha = 0$ and* $\ker \beta$ *is the closure of* $\operatorname{im} \alpha$;

(*c*) $\| \alpha \| \leq 1$ *and* $\| \beta \| \leq 1$.

Proof :

(1) The derivation $d = d^b_{L/K}$ restricted to V_L can uniquely be written in the form $d = l \circ d_{V_L|V_K}$ where $l : \Omega_{V_L|V_K} \to \Omega^b_{L/K}$ is a V_L-linear map. Since $\| d \| \leqq 1$, we find sup $\| l(\Omega_{V_L|V_K}) \| \leqq 1$. Using (2.4), we find an L-linear map $\alpha : F(\Omega_{V_L|V_K}) \to \Omega^b_{L/K}$ with norm $\leqq 1$. On the other hand, the map $V_L \to \Omega_{V_L|V_K} \to F(\Omega_{V_L|V_K})$ induces a K-derivation : $L \to F(\Omega_{V_L|V_K})$ with norm $\leqq \rho^{-1}$. The induced L-linear map from $\Omega^b_{L/K}$ into $F(\Omega_{V_L|V_K})$ is the inverse of α and has also norm $\leqq \rho^{-1}$.

(2) $F(M) = \Omega^b_{L/K}$ where $M = \Omega_{V_L|V_K}$ and

$$\{ x \in M \otimes L \mid p(x) < 1 \} \subset M' \subset \{ x \in M \otimes L \mid p(x) \leqq 1 \}.$$

Let $x \in F(M)$, $\| x \| < \varepsilon < 1$. It suffices to show that there exists a finite sum $y = \sum \lambda_i \, dx_i$ such that $| \lambda_i | < \varepsilon$, $| x_i | < 1$, $\| x - y \| < \varepsilon^2$. Indeed, by induction one forms :

$$x_1 = x - y_0; \qquad x_2 = x_1 - y_1; \qquad \ldots; \qquad x_{n+1} = x_n - y_n; \qquad \ldots$$

such that

$$\| x_n \| < \varepsilon^n, \qquad y_n = \sum \lambda_{i,n} \, dx_{i,n} \quad \text{(finite sum)},$$
$$| \lambda_{i,n} | < \varepsilon_n, \qquad | x_{i,n} | < 1.$$

It follows that $x = \sum_n \sum_i \lambda_{i,n} \, dx_{i,n}$, and this is the required expression.

Since $F(M)$ is the completion of $M \otimes L$, there exists $y \in M \otimes L$ with $\| x - y \| < \varepsilon^2$. Hence $\| x \| = \| y \| < \varepsilon$. Take $a, b \in L$, $0 < | a | < \varepsilon$, $0 < | b | < 1$, such that $\tilde{y} = a^{-1} b^{-1} y$ has norm < 1. Now $\tilde{y} \in M'$, and we can write $\tilde{y} = \sum \lambda_i \, dx_i$ with λ_i, $x_i \in V_L$. This implies the required expression for $y = \sum a \lambda_i (d(bx_i) - x_i \, d(b))$.

(3) As in the proof of (2), it is sufficient to show that any $x \in \Omega^b_{L/K}$ with $\| x \| \leqq 1$ can be approximated by a finite sum of the required type. Going back to the definition of $\Omega^b_{L/K}$, this means that we may suppose $x \in I/I^2$. Take $y \in I$ with $\| y \| \leqq 1$ and $y + I^2 = x$. The element y can be written as $\sum a_i \otimes b_i$, with max $\| a_i \| . \| b_i \| = \| y \| \leqq 1$. (*See* section 1.) Further $\sum a_i \, b_i = 0$, and we can write

$$y = \sum (a_i \otimes 1 - 1 \otimes a_i) \, b_i.$$

It follows that $x = \sum b_i \, da_i$ with $| a_i \, b_i | \leqq 1$ for all i. Substituting $a_i = \pi_L^{n(i)} a'_i$ and $b_i = \pi_L^{m(i)} b'_i$ with $| a'_i | = | b'_i | = 1$, and collecting terms one obtains

$$x = \sum b'_i \, da'_i + (\sum b_i \, a_i) \, \pi_L^{-1} \, d\pi_L.$$

This is the expression we are looking for.

(4) The surjective map $\Omega_{V_M|V_K} \to \Omega_{V_M|V_L}$ yields the surjectivity of β after applying (2.5) part (1) and (2.4) part (2). Let H be any Banach space over M then

$$0 \to \operatorname{Hom}(\Omega^b_{M/L}, H) \overset{\beta^*}{\to} \operatorname{Hom}(\Omega^b_{M,K}, H) \overset{\alpha^*}{\to} \operatorname{Hom}(\Omega^b_{L,K} \hat{\otimes}_L M, H)$$

is an exact sequence since

$$\operatorname{Hom}(\Omega^b_{M/L}, H) = \operatorname{Derb}_L(M, H); \qquad \operatorname{Hom}(\Omega^b_{M,K}, H) = \operatorname{Derb}_K(M, H)$$

and

$$\operatorname{Hom}\left(\Omega^b_{L/K} \hat{\otimes}_L M, H\right) = \operatorname{Derb}_K(L, H).$$

It follows from this, by substituting various spaces H, that $\beta \circ \alpha = 0$ and ker β equals the closure of im α. The remaining parts of (4) follow easily from (2) and (3).

(2.6) COROLLARY. — *If $L \supset K$ are discrete valued fields such that $\Omega_{l/k}$ is finite dimensional, then :*

(a) $\Omega_{V_L|V_K} = M \oplus T \oplus D$, where M is a free, finitely generated module; T is a finitely generated torsion module and D is a divisible module.

(b) $\dim_L \Omega^b_{L/K} = \operatorname{rank} M$.

Proof :

(*a*) For this, we refer to [2].

(*b*) $\Omega^b_{L/K} \cong F\left(\Omega_{V_L|V_K}\right) = F(M)$ since $F(T) = F(D) = 0$. Clearly rank $M = \dim_L F(M)$.

Remark. — This proposition gives the link between calculations of rank M, done by R. BERGER-E. KUNZ in [2] (especially, Satz 8,12) and the calculations done in paragraphs 3, 4, 5. The reader can verify that the results on rank M are special cases of results in paragraphs 3, 4, 5.

Comparing of $\Omega^b_{L/K}$ with $\Omega_{l/k}$ and definition of $\varphi_{L/K}$ and $\varepsilon_{L/K}$. — The universal bounded derivation $d = d^b_{L/K} : L \to \Omega^b_{L/K}$ has norm ≤ 1. Hence

$$d(V_L) \subseteq \left\{ x \in \Omega^b_{L/K} \;\middle|\; \|x\| \leq 1 \right\}$$

and

$$d\left(\left\{ x \in V_L \;\middle|\; |x| < 1 \right\}\right) \subseteq \left\{ x \in \Omega^b_{L/K} \;\middle|\; \|x\| < 1 \right\}.$$

So d induces a k-linear derivation $\tau(d) : l \to \tau\, \Omega^b_{L/K}$, which factors uniquely through the universal derivation $d_{l/k} : l \to \Omega_{l/k}$. In diagram

$$
\begin{array}{ccccc}
V_L & \xrightarrow{d} & \{ x \in \Omega^b_{L/K} \mid \| x \| \leqq 1 \} & \longrightarrow & \tau\, \Omega^b_{L/K} \\
\Big\downarrow \tau & & \nearrow \tau(d) & & \Big\uparrow \varphi_{L/K} \\
l & & \xrightarrow{\quad d_{l/k} \quad} & & \Omega_{l/k}
\end{array}
$$

The map $\varphi_{L/K}$ can be described as follows :

$$\varphi_{L/k} (d_{l/K} (\tau\, A)) = \tau\, (dA) \quad \text{for all } A \in L, \qquad | A | \leqq 1.$$

Of course $\varphi_{L/K}$ depends " functorially " on L/K which means that for L'/K', which $L \subset L'$, $K \subset K'$ the following diagram is commutative :

$$
\begin{array}{ccccc}
\tau\, (\Omega^b_{L/K}) \otimes_l l' & \hookrightarrow & \tau\, (\Omega^b_{L/K} \hat{\otimes}_L L') & \longrightarrow & \tau\, (\Omega^b_{L'/K'}) \\
\Big\uparrow \varphi_{L/K} \oplus 1 & & & & \Big\uparrow \varphi_{L'/K'} \\
\Omega_{l/k} \otimes_l l' & & \longrightarrow & & \Omega_{l'/k'}
\end{array}
$$

The cokernel of $\varphi_{L/K}$ is denoted by $\varepsilon_{L/K}$.

(2.7) LEMMA. — *Consider for fields $L \supset K$ the following statements :*

(1) $\alpha : \Omega^b_K \hat{\otimes} L \to \Omega^b_L$ *is isometric.*

(2) *For every spherically complete Banach space B over L and every bounded derivation $D : K \to B$ there exists an extension $D^* : L \to B$ with $\| D \| = \| D^* \|$.*

(3) *Every derivation $D : K \to K$ with norm 1 can be extended to a derivation $D^* : L \to L$ with norm 1.*

(4) $\tau\, (\alpha) : \tau\, (\Omega^b_K \hat{\otimes} L) \to \tau\, (\Omega^b_L)$ *is injective.*

Then :

(*a*) (1) $\Leftrightarrow$ (2) $\Rightarrow$ (4).

(*b*) *If the valuation of L is discrete, then all four statements are equivalent, and moreover*

$$\tau\, (\Omega^b_K) \otimes_k l \xrightarrow{\sim} \tau\, (\Omega^b_K \hat{\otimes}_K L).$$

Proof :

(*a*) (1) $\Rightarrow$ (4) is trivial, and (1) $\Leftrightarrow$ (2) follows from the Hahn-Banach property of spherically complete Banach spaces : Every bounded L-linear map $l : E \to B$, where E is a subspace of the Banach space F can be extended $l^* : F \to B$ with $\| l \| = \| l^* \|$.

(*b*) Discrete Banach spaces are spherically complete and $\Omega_K^b \hat{\otimes} L$, Ω_K^b are discrete. From this, (*b*) follows easily.

(2.8) PROPOSITION. — *Let fields $L \supset K$ be given :*

(1) *The subspace of $\Omega_{L/K}^b$ generated by $d(L)$ is dense.*

(2) *If there exists a countable (resp. finite) set $A \subset L$ such that $K(A)$ is dense in L, then $\Omega_{L/K}^b$ is a Banach space of countable type (resp. of finite dimension).*

(3) *If $L \supset K$ is finite algebraic, then $\Omega_{L/K} = \Omega_{L/K}^b$.*

(4) *Let $\Lambda \subset K$ be a complete subfield, and suppose that L is finite algebraic over K, then*

$$\gamma_{L/K/\Lambda} \to \Omega_{K/\Lambda}^b \hat{\otimes} L \xrightarrow{\alpha} \Omega_{L/\Lambda}^b \xrightarrow{\beta} \Omega_{L/K}^b \to 0$$

is exact.

(5) *If L/K is finite algebraic, then*

$$0 \to \gamma_{L/K} \to \Omega_K^b \hat{\otimes} L \to \Omega_L^b \to \Omega_{L/K}^b \to 0$$

is exact.

Proof. — (1) and (2) are quite obvious. Further (3) follows from the fact that every L-linear map $l : E \to F$, with E finite dimensional and F an arbitrary Banach space is continuous.

(4) There exists a constant $C > 0$ such that for any bounded derivation $D : K \to B$ (B some Banach space over L) which extends to a derivation : $L \to B$, there exists an extension $D^* : L \to B$ with $\| D^* \| \leq C \| D \|$. This follows from the fact that any linear base of L over K is γ-orthonormal for some γ, $0 < \gamma < 1$. It follows that $\operatorname{im} \alpha = \ker \beta$, and that the image of

$$\gamma_{L/K/\Lambda} = \ker (\Omega_{K/\Lambda} \otimes L \to \Omega_{L/\Lambda})$$

under the canonical map $\Omega_{K/\Lambda} \otimes L \to \Omega_{K/\Lambda}^b \hat{\otimes} L$ is the kernel of α. Hence the sequence in (4) is exact.

(5) Let in (4) Λ denote the completion of the prime field of K. Then we have an exact sequence

$$\gamma_{L/K} \to \Omega_K^b \hat{\otimes} L \to \Omega_L^b \to \Omega_{L/K}^b \to 0.$$

So we have only to prove that $\gamma_{L/K} \to \Omega_K^b \otimes L$ is injective. This we do by induction on $[L : K]$. To start induction, we have to consider the case $L = K(x)$; $x^p = a \in K \setminus K^p$, where char $K = p \neq 0$. Since $\gamma_{L/K} \subset \Omega_K \otimes L$ is 1-dimensional and generated by $d_K(a)$, all we have to show is that $d_K^b(a) \neq 0$.

This is true, and proved in paragraph 7, (7.2). We remark that in the proof of (7.2) no use is made of (2.8). So we may use (7.2).

Now the induction on $[L : K] < \infty$ goes as follows :

Let $K \subsetneqq L_1 \subsetneqq L$, then we have a commutative diagram

$$\begin{array}{lccccccccccc}
 & & & & & & 0 & & 0 & & & \\
 & & & & & & \downarrow & & \downarrow & & & \\
 & & & & & & \gamma_{L/L_1} & \longrightarrow & \gamma_{L/L_1/K} & & & \\
 & & & & & & \downarrow & & \downarrow & & & \\
(1) & 0 & \longrightarrow & \gamma_{L_1/K} \hat{\otimes} L & \longrightarrow & \Omega_K^b \otimes L & \xrightarrow{g} & \Omega_{L_1}^b \otimes L & \longrightarrow & \Omega_{L_1/K} \otimes L & \longrightarrow & 0 \\
 & & & \downarrow & & \downarrow \scriptstyle{\mathrm{id}} & & \downarrow & & \downarrow & & \\
(2) & 0 & \longrightarrow & \gamma_{L/K} & \xrightarrow{f} & \Omega_K^b \otimes L & \longrightarrow & \Omega_L^b & \longrightarrow & \Omega_{L/K} & \longrightarrow & 0 \\
 & & & & & & & \downarrow & & \downarrow & & \\
 & & & & & & & \Omega_{L/L_1} & \xrightarrow{\mathrm{id}} & \Omega_{L/L_1} & & \\
 & & & & & & & \downarrow & & \downarrow & & \\
 & & & & & & & 0 & & 0 & & \\
 & & & & & & & (3) & & (4) & &
\end{array}$$

Now (1) and (3) are exact by induction; (4) is exact and according to [3] [EGA IV, Chap. 0, (21.6.1)],

$$0 \to \gamma_{L_1/K} \otimes L \to \gamma_{L/K} \xrightarrow{d} \gamma_{L/L_1} \to \gamma_{L/L_1/K} \to 0$$

is exact where d is some natural map.

One verifies easily that $\operatorname{im}(g \circ f) \subset \gamma_{L/L_1}$ and that $d = g \circ f$.

This implies that f is injective, since $f(x) = 0$, $x \in \gamma_{L/K}$ implies $d(x) = 0$ and $x \in \gamma_{L_1/K} \otimes L$. Further $\gamma_{L_1/K} \otimes L \to \Omega_K^b \otimes L$ is injective.

(2.9) Lemma. — *Let $L \supset K$ denote complete valued fields and let B be a spherically complete Banach space over L. Suppose that the bounded derivation $D : K \to B$ can be extended to a bounded derivation $D' : L \to B$. Then there exists an extension $D^* : L \to B$ of D with minimal norm.*

Proof. — Consider the exact sequence

$$0 \to \mathrm{Derb}_K(L, B) \xrightarrow{f} \mathrm{Derb}(L, B) \xrightarrow{g} \mathrm{Der}(K, B).$$

Clearly f is isometric, and by section 1 (A 2),

$$\mathrm{Derb}_K(L, B) = \mathrm{Hom}_L(\Omega^b_{L/K}, B)$$

is spherically complete. So f has a left-inverse f_1 of norm ≤ 1. Now $D^* = D' - f \circ f(D')$ is an extension of D with minimal norm.

3. Discrete valued fields (Tame case)

For given discrete valued fields $L \supset K$ the Banach space $\Omega^b_{L/K}$ is discrete and hence determined by its residue space $\tau\,\Omega^b_{L/K}$. So almost all in formation on $\Omega^b_{L/K}$ can be derived from ker $\varphi_{L/K}$ and coker $\varphi_{L/K}$, where $\varphi_{L/K} : \Omega_{l/k} \to \tau\,\Omega^b_{L/K}$ denotes the canonical map introduced in paragraph 2. From (2.5) part (3), it follows that $\varepsilon_{L/K} = \mathrm{coker}\,\varphi_{L/K}$ is generated by the image of $\pi_L^{-1}\,d\pi_L$ in $\tau\,\Omega^b_{L/K}$. So $\dim_l \varepsilon_{L/K}$ can only be 0 or 1.

Besides ker $\varphi_{L/K}$ we are interested in extensions of bounded derivations which means that we want information on the map $\alpha : \Omega^b_K \hat{\otimes} L \to \Omega^b_L$. In this section we deal with the case L/K is tamely ramified [i. e. l/k is separable and $e(L/K)$ is not divisible by the characteristic of k].

(3.1) THEOREM. — *Suppose that $L \supset K$ are discrete valued fields and that $L \supset K$ is tame. Then*

(1) ker $\varphi_{L/K} = 0$ *and* $\varepsilon_{L/K} = 0$; *in other words* $\varphi_{L/K} : \Omega_{l/k} \to \tau(\Omega^b_{L/K})$ *is bijective.*

(2) $0 \to \Omega^b_K \hat{\otimes} L \xrightarrow{\alpha} \Omega^b_L \to \Omega^b_{L/K} \to 0$ *is exact and* α *is isometric.*

Proof :

(1) $\varphi_{L/K}$ injective is equivalent to :

$$\mathrm{Hom}_l(\tau\,\Omega^b_{L/K}, l) \to \mathrm{Hom}(\Omega_{l/k}, l)$$

is surjective; since

$$\mathrm{Hom}_l(\tau\,\Omega^b_{L/K}, l) = \tau\,\mathrm{Hom}_L(\Omega^b_{L/K}, L) = \tau\,\mathrm{Derb}_K(L, L)$$

and

$$\mathrm{Hom}_l(\Omega_{l/k}, l) = \mathrm{Der}_K(l, l),$$

this means :

($\bigstar$) *For every k-derivation $D : l - l$ there exists a V_K-derivation D^e :* $V_L \to V_L$ (of norm ≤ 1) *such that the diagram*

$$\begin{array}{ccc} V_L & \xrightarrow{D^e} & V_L \\ \downarrow{\scriptstyle\tau} & & \downarrow{\scriptstyle\tau} \\ l & \xrightarrow{D} & l \end{array}$$

is commutative.

We split the proof of ($\bigstar$) in two parts.

(1 A) *Reduction to the unramified case* [i. e. l/k separable and $e(L/K)=1$]. — Let $e = e(L/K)$, then $\pi_L^e = a\,\pi_K$ for some $a \in L$, $|a| = 1$. It follows that $e\,\pi_L^{-1}\, d\pi_L = a^{-1}\, da + \pi_K^{-1}\, d\pi_K$. Since $d\pi_K = 0$ and $|e| = 1$, we have $\varepsilon_{L/K} = 0$. So it suffices to show that $\varphi_{L/K}$ is injective.

The reduction of the polynomial $X^e - a \in L[X]$ to $X^e - \tau\, a \in l[X]$ is separable. By Hensel's lemma, the reduction of an irreducible factor $q(X)$ of $X^e - a$ remains irreducible and separable in $l[X]$. Let b, an element in the algebraic closure of L, be a root of $q(X)$. Put $L_1 = L(b)$ and $K_1(b^{-1}\pi_L)$. Then L_1/K_1 is unramified,

$$k_1 = k, \qquad \Omega_{l/k} \otimes l_1 \to \Omega_{l_1/k}$$

is bijective, and $\Omega^b_{L_1/K} = \Omega^b_{L_1/K_1}$ since K_1/K is separable algebraic.

In the following commutative diagram,

$$\begin{array}{ccc} \Omega_{l/k} \otimes l_1 & \xrightarrow{\varphi_{L/K}\otimes 1} & \tau\left(\Omega^b_{L/K} \hat{\otimes} L_1\right) = \tau\left(\Omega^b_{L/K}\right) \otimes_l l_1 \\ \downarrow & & \downarrow \\ \Omega_{l_1/k_1} & \xrightarrow{L_1/K_1} & \tau\left(\Omega^b_{L_1/K_1}\right) = \tau\left(\Omega^b_{L_1/K}\right) \end{array}$$

φ_{L_1/K_1} is injective if the unramified case of ($\bigstar$) is supposed to be true. It follows that $\varphi_{L/K}$ is injective.

(1 B) *The unramified case.* — Let a k-derivation $D : l \to l$ be given. By induction, we will define V_K-derivations $D_n : V_L \to V_L/\pi^n V_L$, $\pi = \pi_K = \pi_L$, such that D_1 equals $V_L \xrightarrow{\tau} l \xrightarrow{D} l = V_L/\pi\, V_L$ and such that all diagrams

$$\begin{array}{ccc} V_L & \xrightarrow{D_{n+1}} & V_L/\pi^{n+1}\, V_L \\ & \searrow^{D_n} & \downarrow \\ & & V_L/\pi^n\, V_L \end{array}$$

are commutative.

If that is done, then $D^e = \varprojlim D_n : V_L \to \varprojlim V_L/\pi^n\, V_L = V_L$ is a V_K-derivation of norm $\leqq 1$. And D^e is clearly a lifting of D in the sense of ($\bigstar$).

Construction of D_{n+1}. — Let $\{ a_i \mid i \in I \}$ be an orthonormal base of L over K. Then every element of V_L can uniquely be expressed as a convergent sum $\sum \lambda_i\, a_i$ with all $\lambda_i \in V_K$. Further we may assume that for some $i_0 \in I$, $a_{i_0} = 1$. Let D_n be given; we define a V_K-linear $f : V_L \to V_L/\pi^{n+1}\, V_L$ by $f(\sum \lambda_i\, a_i) = \sum \lambda_i\, b_i$, where the b_i are chosen such that $b_i \equiv D_n(a_i) \bmod \pi^n$.

The diagram with f on the place of D_{n+1} is commutative but f need not be a derivation. The V_K-bilinear 2-cocyle

$$h: \quad V_L \times V_L \to \pi^n \, V_L/\pi^{n+1} \, V_L,$$

given by $h(a, b) = f(ab) - af(b) - bf(a)$ is zero on the subset $(\pi \, V_L \times V_L) \cup (V_L \times \pi \, V_L)$ and induces a k-bilinear 2-cocycle h' : $l \times l \to \pi^n \, V_L/\pi^{n+1} \, V_L$. This 2-cocycle is trivial (section 1) since l/k is separable. So for some V_K-linear map

$$g: \quad V_L \xrightarrow{\tau} l \to \pi^n \, V_L/\pi^{n+1} \, V_L,$$

we have $h(a, b) = g(ab) - ag(b) - bg(a)$.

Now $D_{n+1} = f - g$ has the required properties.

(2) According to (2.5) and (2.6) the statement is equivalent to : Every derivation $D : K \to K$ of norm 1 extends to a derivation D^* : $L \to L$ with norm 1.

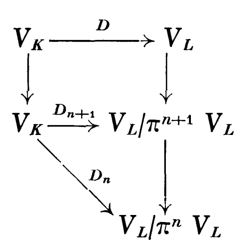
As in (1 A), we make first a reduction to the unramified case.

With the notations of (1 A), $\{1, (b^{-1} \pi_L), \ldots, (b^{-1} \pi_L)^{e-1}\}$ is an orthogonal base of K_1 over K. The unique extension $D_1 : K_1 \to K_1$ of D satisfies

$$D_1 \left(\textstyle\sum_{i=0}^{e-1} \lambda_i \, (b^{-1} \pi_L)^i\right) = \textstyle\sum_{i=0}^{e-1} \left(i \, \lambda_i \, e^{-1} \, \pi_K^{-1} \, D(\pi_K) + D(\lambda_i)\right) (b^{-1} \pi_L)^i.$$

Using the orthogonality of the base, one finds $\| D_1 \| = 1$. Assuming the unramified case, D_1 extends to $D_2 : L_1 \to L_1$, $\| D_2 \| = 1$. Let P : $L_1 \to L$ be an L-linear projection onto L with norm 1, then $D^* = P \circ D_2 \mid L$ has norm 1 and extends D.

(2 A) *The unramified case.* — Again this is done by truncating V_L. By induction we will construct derivations $D_n : V_L \to V_L/\pi^n \, V_L$ $\pi = \pi_L = \pi_K$, such that all diagrams



are commutative.

If we succed it will follow that $D^* = \varprojlim D_n$ is a derivation extending D and $\| D^* \| = 1$.

Construction of D_1. — The derivation $D : V_K \to V_K \subset V_L$, having norm 1, induces a derivation $D' : k \to l$. Since l/k is separable, D' extends to $D'' : l \to l$. Now put

$$D_1 = V_L \overset{\tau}{\to} l \overset{D''}{\to} l = V_L/\pi^{n+1} V_L.$$

Construction of D_{n+1}. — Using the terminology of (1 B), we define $f : V_L \to V_L/\pi^{n+1} V_L$ by the formula

$$f(\Sigma \lambda_i a_i) = \Sigma \lambda_i b_i + \Sigma D(\lambda_i)(a_i \bmod \pi^{n+1}),$$

where the $b_i \in V_L/\pi^{n+1} V_L$ are chosen such that

$$b_i \equiv D_n(a_i) \bmod \pi^n \qquad \text{and} \qquad b_{i_0} = 0.$$

Clearly f on the place of D_{n+1} in the diagram above makes the diagram commutative. From f, we derive the V_K-bilinear 2-cocycle

$$h : \quad V_L \times V_L \to \pi^n V_L/\pi^{n+1} V_L, \; h(a, h) = f(ab) - af(b) - bf(a).$$

This h induces a k-bilinear 2-cocycle : $l \times l \to \pi^n V_L/\pi^{n+1} V_L$ which is trivial since l/k is separable. And we conclude the existence of a V_K-linear

$$g : \quad V_L \to l \to \pi^n V_L/\pi^{n+1} V_L$$

satisfying

$$h(a, b) = g(ab) - ag(b) - bg(a).$$

Now $D_{n+1} = f - g$ has the required properties. *Hence the theorem is proved.*

(2 B) *Alternative proofs.* — If char K = char k, then using Cohen's structure theorem there are coefficient fields k and l of V_K, resp. V_L such that $k \subset l$. So $V_K = k[[x]] \subset l[[y]] = V_L$ and $k \subset l$,

$$x = ay^e + \Sigma_{n>e} a_n y^n$$

where $e = e(L/K)$ and $a \in l$, $a \neq 0$. Moreover e is not divisible by the characteristic of k, and we can change y such that $x = ay^e$. For this explicit situation, it is easy to deduce (1) and (2) of theorem (3.1).

(2 C) The statement of (2 A) can also be proved with the help of [3] (EGA IV, Chap. 0). Using (19.7.1) one finds that V_L is formally smooth over V_K with respect to the π-adic topologies. Now (20.7.2) implies that $\delta : \Omega_{V_K} \otimes V_L \to \Omega_{V_L}$ has formally a left inverse. That implies, in particular, that the restriction map

$$\mathrm{Der}(V_L, V_L) \to \mathrm{Der}(V_K, V_L)$$

is surjective.

Since $e(L/K) = 1$, this means that every derivation $D : K \to L$ with norm ≤ 1 can be extended to $D^* : L \to L$ with norm ≤ 1.

(2 D) The similarity of the proofs (1 B) and (2 A) leads to the conjecture that (1) implies (2) (or conversely). In general, this seems not to be the case. In some examples however (2) follows directly from (1). Example : $\mathbf{Q}_p \subset K \subset L$, $e(L/\mathbf{Q}_p) = 1$ and l/k is separable. We have the commutative diagram :

$$\begin{array}{ccc} \tau(\Omega_K^b) \otimes_k l = \tau(\Omega_K^b \hat{\otimes} L) & \xrightarrow{\tau(\alpha)} & \tau\,\Omega_L^b \\ \uparrow{\scriptstyle \varphi_K \otimes 1} & & \uparrow{\scriptstyle \varphi_L} \\ \Omega_k \otimes l & \xrightarrow{\quad\delta\quad} & \Omega_l \end{array}$$

Since every bounded derivation on $\mathbf{Q}_p$ is zero, $\varphi_K = \varphi_{K/\mathbf{Q}_p}$ and $\varphi_L = \varphi_{L/\mathbf{Q}_p}$. By part (1) of (3.1), φ_K and φ_L are bijective; δ is injective because l/k is separable. Hence $\tau(\alpha)$ is injective, and consequently α is isometric.

(3.2) Corollary. — *Let L be a discrete valued field, and K a subfield on which the valuation is trivial. Suppose that l/k is separable. Then :*

(1) $\ker \varphi_{L/K} = 0$ *and* $\dim \varepsilon_{L/K} = 1$;

(2) $\alpha : \Omega_K^b \hat{\otimes} L \to \Omega_L^b$ *is isometric.*

Proof. — This follows easily from (3.1) if one replaces $K = k$ by $K_1 = k((\pi_L))$.

(3.3) Corollary. — *Let L be a discrete valued field :*

(1) *If* char L = char l, *then* $\tau\,\Omega_L^b \simeq \Omega_l \oplus \varepsilon_L$ *and* $\dim \varepsilon_L = 1$.

(2) *If* char $L = 0 \neq p =$ char l, *then* $\Omega_L^b = \Omega_{L/\mathbf{Q}_p}^b$. *Moreover* $\dim\ker \varphi_{L/\mathbf{Q}_p} = \dim \varepsilon_{L/\mathbf{Q}_p}$. *If $e(L/\mathbf{Q}_p)$ is not divisible by p, then* $\varepsilon_{L/\mathbf{Q}_p} = 0$.

Proof :

(1) Follows at once from (3.2) by taking K = the prime field of L.

(2) The first part follows from : every bounded derivation on $\mathbf{Q}_p$ is zero. If $e(L/\mathbf{Q}_p) = 1$, then second part follows from (3.1).

Using Cohen's structure theorem, there exists a complete subfield K of L such that

$$e(L/K) = e(L/\mathbf{Q}_p) = e, \qquad e(K/\mathbf{Q}_p) = 1, \qquad l = k.$$

Hence $[L : K] = e$. The map $\alpha : \Omega_K^b \hat{\otimes} L \to \Omega_L^b$ has norm ≤ 1 and is bijective since L is a finite separable extension of K. But α need not be isometric. In the commutative diagram :

$$\begin{array}{ccc} \tau(\Omega_K^b \hat{\otimes} L) & \xrightarrow{\tau(\alpha)} & \tau\,\Omega_L^b \\ \uparrow{\scriptstyle \varphi_K} & & \uparrow{\scriptstyle \varphi_L} \\ \Omega_k & = & \Omega_l \end{array}$$

φ_K is bijective by (3.1). Hence

$$\ker \varphi_L \cong \ker \tau(\alpha) \qquad \text{and} \qquad \operatorname{coker} \varphi_L \cong \operatorname{coker} \tau(\alpha).$$

Under the assumption that Ω_l is finite dimensional we get at once $\dim \ker \tau(\alpha) = \dim \operatorname{coker} \tau(\alpha)$.

The general case reduces easily to the finite dimensional case by means of the following trick :

Let $\pi = \pi_L$ satisfy the equation $\pi^e = \sum_{i=0}^{e-1} a_i \pi^i$ over K. Consider the set $\{ K_j \}$ of all closed subfields of K satisfying $\{ a_0, \ldots, a_{e-1} \} \subset K_j$, and $\Omega^b_{K_j}$ is finite dimensional. Put $L_j = K_j(\pi)$. Then

$$K = \bigcup K_j; \qquad L = \bigcup L_j; \qquad \tau\,\Omega^b_K = \varinjlim \tau\,\Omega^b_{K_j};$$

$$\tau\,\Omega^b_L = \varinjlim \tau\,\Omega^b_{L_j}; \qquad \tau(\alpha) = \varinjlim \tau(\alpha_j)$$

where α_j is the canonical map $\Omega^b_{K_j} \otimes L_j \to \Omega^b_{L_j}$. Since for every j,

$$\dim \ker \tau(\alpha_j) = \dim \operatorname{coker} \tau(\alpha_j) \qquad (= 0 \text{ or } 1),$$

the same holds for $\tau(\alpha)$.

Example. — If $L \supset \mathbf{Q}_p$ is a discrete valued field with $e(L/\mathbf{Q}_p) = e$ divisible by p, then both cases $\dim \varepsilon_{L/\mathbf{Q}_p} = 0$ and 1 actually occur. Take $K = \overline{\mathbf{Q}_p(t)}$, where $t \in K$ has absolute value 1 and its residue $\tau t \in k$ is transcendental over $\mathbf{F}_p$. Put $L = K(\pi)$, where $\pi^p = pa$.

(*a*) $a = t$. Clearly $l = k = \mathbf{F}_p(t)$ and $\Omega_l = l\,dt$. Since $p\,\pi^{p-1}\,d\pi = p\,dt$, we have $\| dt \| < 1$, and $\varphi_L : \Omega_l \to \tau\,\Omega^b_L$ is the zero-map. So $\varepsilon_{K/\mathbf{Q}_p}$ has dimension 1.

(*b*) $a = 1$. Now $\pi^p = p$ implies $d\pi = 0$. Consequently $\varepsilon_{L/\ \ p} = 0$.

Remark. — Theorem (3.1) leaves us two cases of non-tame extension $L \supset K$ of discrete fields for study :

Section 4 : Discrete fields of characteristic p. (So char $K = p$.)

Section 5 : Discrete fields of mixed characteristic. (So char $K = 0 \neq p =$ char K.)

4. Discrete valued fields of characteristic p

In this section, $L \supset K$ are discrete valued fields of characteristic $p \neq 0$. The field $K^+ = \overline{K(L^p)}$ is the smallest complete field containing both K and L^p. Since any bounded K-linear derivation of L is zero on K^+, we have $\Omega^b_{L/K} = \Omega^b_{L/K^+}$. The residue field of K^+ is denoted by k^0. Clearly, $k^0 \supset k^+ = k(l^p)$. As we shall see, in general, $k^0 \neq k^+$. Further, $e(L/K^+)$ can only be p or 1.

The best approximation of theorem (3.1) seems to be :

(4.1) THEOREM. — *With the notations above :*

(1) $\varphi_{L/K^+} : \Omega_{l/k^0} \to \tau\, \Omega^b_{L/K}$ *is injective and*

$$\ker \varphi_{L/K} = \ker (\Omega_{l/k^+} \to \Omega_{l/k^0}).$$

Further $\varepsilon_{L/K} = \varepsilon_{L/K^+}$ *has dimension* 0 *or* 1 *according to* $e(L/K^+) = 1$ *or* p.

(2) $\alpha : \Omega^b_{K^+/L^p} \hat{\otimes} L \to \Omega^b_L$ *is isometric.*

Proof :

(1) Let $\{ a_i \mid i \in I \}$ be a p-base of l/k^0, and choose a set of representatives $\{ A_i \mid i \in I \} \subset L$. Put $e = e(L/K^+)$ and $\pi = \pi_L$. Then the set

$$B = \{ \pi^\alpha A_{i_1}^{\alpha_1} \ldots A_{i_n}^{\alpha_n} \mid n \in \mathbf{N};\ i_1, \ldots, i_n \text{ different elements in } I;\ 0 \leq \alpha < e;\ 0 \leq \alpha_i < p \}$$

is an orthogonal base of L over K^+.

We show first that φ_{L/K^+} is injective. Arguing as in (3.1), we have to show that a k^0-derivation $D : l \to l$ lifts to a K^+-derivation $D^e : L \to L$ with norm 1.

Define D^e by

$$D^e (\textstyle\sum_{b \in B} \lambda_b\, b) = \sum \lambda_b\, D^e(b),$$

for any convergent sum $\sum \lambda_b\, b$ with coefficients in K^+. Further for

$$b = \pi^\alpha A_{i_1}^{\alpha_1} \ldots A_{i_n}^{\alpha_n} \in B,$$

we define

$$D^e(b) = b \textstyle\sum_{j=1}^n A_{i_j}^{-1} \alpha_{i_j} c_i,$$

where $c_i \in L$ are chosen such that $\tau\, c_i = D(a_i)$. It is almost immediate that D^e has the required properties. So we have shown that φ_{L/K^+} is injective. It follows at once that $\ker \varphi_{L/K} = \ker (\Omega_{l/k^+} \to \Omega_{l/k^0})$. Further, if $e = p$ then the K^+-linear bounded derivation $D^* : L \to L$, which is given for $b = \pi^\alpha A_{i_1}^{\alpha_1} \ldots A_{i_n}^{\alpha_n} \in B$ by $D^*(b) = \alpha\, b$, has clearly norm 1 and $\tau(D^*) = 0$. So $\varepsilon_{L/K^+} \neq 0$.

(2) Let the L^p-derivation $D : K^+ \to K^+$ be given, $\| D \| = 1$. It suffices to show that D extends to a derivation $D^* : L \to L$ with the same norm. With the notations above, define D^* by

$$D^* (\textstyle\sum_{b \in B} \lambda_b\, b) = \sum D(\lambda_b)\, b.$$

One easily checks that D^* is a derivation extending D. The orthogonality of the base B implies $\| D^* \| = 1$.

(4.2) COROLLARY. — *If* $L \supset K$ *is tame then* $k^0 = k^+$.

Proof. — This follows from (4.1), (3.1) and the observation that $\Omega_{l/k} = \Omega_{l/k^+} \to \Omega_{l/k^0}$ is injective if, and only if, $k^0 = k^+$. Also a *direct proof* of (4.2) is of interest :

Using (2 B) from the proof of (3.1), we have

$$V_K = k[[x]] \subset l[[y]] = V_L$$

with $k \subset l$ and $x = ay^e$, $a \in l$, $a \neq 0$, $(e, p) = 1$. For some $n, m \in \mathbf{Z}$, we have $ne + mp = 1$, hence $z = x^n y^{pm} = a^n y \in V_{K^+}$. Since V_{K^+} contains also k^+, we have $V_{K^+} \supseteq k^+[[z]]$. Further

$$y^p = a^{-np} z^p \in k^+[[z]]; \qquad x = ay^e = a^{mp} z^e \in k^+[[z]].$$

Hence $k^+[[z]] \supseteq k[[x]]$ and $k^+[[z]] \supseteq l^p[[y^p]]$. Consequently $k^+[[z]] = V_{K^+}$ and the residue field of K^+ is k^+.

Remark. — Also for non-tame extensions $L \supset K$ it is possible that $k^0 = k^+$ (or equivalently ker $\varphi_{L/K} = 0$). More specific :

(4.3) COROLLARY. — *Given fields* $k \subset l$ *of characteristics* $p \neq 0$, *and a positive integer* e *such that :*

(1) $\Omega_{l/k} \neq 0$.

(2) *Either* l/k *is inseparable or* e *is divisible by* p.

Then :

(*a*) *There are discrete valued fields* $K \subset L$ *of characteristic* p, *with residue fields* $k \subset l$ *and* $e = e(L/K)$ *such that* ker $\varphi_{L/K} = 0$.

(*b*) *There are discrete valued fields* $K \subset L$ *of characteristic* p, *with residue fields* $k \subset l$ *and* $e = e(L/K)$ *such that* ker $\varphi_{L/K} \neq 0$.

Proof :

(1) Suppose $p \mid e$. Let $V_K = k[[x]] \subset V_L = l[[y]]$ such that $k \subset l$ and $x = ay^e$.

Case (*a*). — Put $a = 1$. Clearly, $V_{K^+} = k^+[[y^p]]$, and so $k^0 = k^+$.

Case (*b*). — Take $a \in l \setminus k^+$; $k^+ \neq l$ since $\Omega_{l/k} \neq 0$. Then $a = xy^{-e}$ belongs to K^+. Hence $a \in k^0 \setminus k^+$.

(2) Suppose that p does not divide e, and that l/k is inseparable. Again $V_K = k[[x]]$ and $V_L = l[[y]]$. The embedding $\psi : k[[x]] \to l[[y]]$ is given in *case* (*a*) by :

$$\psi(x) = y^e; \qquad \psi \text{ maps } k \text{ into } l.$$

Clearly $V_{K^+} = k^+[[y]]$ and $k^0 = k^+$.

Case (*b*). — This is somewhat tricky. Since l/k is inseparable, there are elements $x_1, \ldots, x_n \in k$ which are p-free over k^p but p-dependent in l/l^p. Without loss of generality, we may assume $x_n \in l^p [x_1, \ldots, x_{n-1}]$. There exists a ring homomorphism $\psi : k \to l[[y]]$ such that

$$\psi(a) \equiv a \bmod (y) \quad \text{for all } a \in k$$

and

$$\psi(x_i) \equiv x_i + \mu_i y \bmod (y^2) \qquad \text{where} \quad \mu_1 = \ldots = \mu_{n-1} = 0$$

and

$$\mu_n \in l \setminus k^+.$$

Define further $\psi : k[[x]] \to l[[y]]$ *by* $\psi(x) = y^e$.

Now

$$x_n = \sum \lambda_\alpha^p x_{x_1}^{\alpha_1} \ldots x_{n-1}^{\alpha_{n-1}}; \qquad \lambda_\alpha \in l; \qquad 0 \leq \alpha_i < p$$

and

$$z = \psi(x_n) - \sum \lambda_\alpha^p \psi(x_1^{\alpha_1} \ldots x_{n-1}^{\alpha_{n-1}})$$

belongs to V_{K^+}. Further $z \equiv \mu_n y \bmod (y^2)$. Since also $y \in V_{K^+}$, because $V_{K^+} \supseteq l^p [[y^p, y^e]]$, we have $\mu_n \in k^0$. The choice of μ_n implies $k^0 \neq k^+$.

Remark. — The moral of (4.3) is that ker $\varphi_{L/K}$ and $\varepsilon_{L/K}$ depend not only on $k \subset l$ and $e(L/K)$ but also on the embedding ψ of K into L. An *open problem* is to obtain from an embedding ψ suitable (linear) data which determine ker $\varphi_{L/K}$ and $\varepsilon_{L/K}$. We conclude this section by showing a converse of (3.1) part (2) for fields of characteristic p.

(4.4) Proposition. — *Let $L \supset K$ be discrete valued fields (of characteristic p) such that $\alpha : \Omega_K^b \hat{\otimes} L \to \Omega_L^b$ is isometric. Then $L \supset K$ is tamely ramified.*

Proof. — We prove that l/k is separable by showing that every derivation $D : k \to k$ extends to a derivation $D^* : l \to l$. From (3.2), it follows that D lifts to a derivation $D_1 : K \to K$ with $\| D_1 \| = 1$. It is given (namely α isometric) that D_1 extends to a derivation $D_2 : L \to L$ with $\| D_2 \| = 1$. Then $D^* = \tau(D_2)$ does the job.

To show that $e = e(L/K)$ is not divisible by p is more complicated. Since dim $\varepsilon_K = 1$ by (3.2), there exists a k-linear map $m : \tau \Omega_K^b \to k$ with kernel im φ_K. The map m lifts to a K-linear map $M : \Omega_K^b \to K$ with norm 1. The derivation $D_1 = M \circ d_K^b : K \to K$ extends to a derivation $D_2 : L \to L$ with norm 1. The reduction of D_2, $\tau(D_2) : l \to l$ is obviously a k-derivation and lifts to a K-derivation $D_3 : L \to L$ with norm ≤ 1. So $D_4 = D_2 - D_3$ is an extension of D_1, $\| D_4 \| = 1$ and this time the reduction $\tau(D_4)$ is zero.

Applying D_4 to the equation $\pi_L^e = a\,\pi_K$, $a \in L$, $|\,a\,| = 1$, one finds

$$e\,\pi_L^{-1} D_4(\pi_L) = a^{-1} D_4(a) + \pi_K^{-1} D_4(\pi_K).$$

Now $\|\,a^{-1} D_4(a)\,\| < 1$ and $\|\,\pi_L^{-1} D_4(\pi_K)\,\| = 1$. So e cannot be divisible by p.

5. Discrete fields of mixed characteristic

In the most general case of inseparable residue field extension $k \subset l$, we have no method to compute $\Omega^b_{L/K}$. For reasonable extension $l \supset k$ however, the situation is as follows :

(5.1) THEOREM. — *Given discrete valued fields* $\mathbf{Q}_p \subset K \subset L$ *such that there exists a field* l_0 *with :*

(1) $k \subset l_0 \subset l$;

(2) l_0/k *is separable*;

(3) l/l_0 *is finite algebraic.*

Then *:*

(a) $0 \to \Omega^b_K \hat{\otimes} L \overset{\alpha}{\to} \Omega^b_L \to \Omega^b_{L/K} \to 0$ *is exact;*

(b) $\gamma_{l/k}$ *has finite dimension and* $\gamma_{l/k} \oplus \tau\,\Omega^b_{L/K} \cong \Omega_{l/k}$.

Remarks.

(1) The map α need not be isometric; the statement in (a) can be translated as follows : « there exists a constant $C > 0$ such that every bounded derivation $D : K \to K$ has an extension $D^e : L \to L$ with $\|\,D^e\,\| \leq C\,\|\,D\,\|$ ».

(2) The isomorphism mentioned in (b) is not canonical. The statement merely says that the cardinal of a Hamel-base of $\gamma_{l/k} \oplus \tau\,\Omega^b_{L/K}$ is equal to the cardinal of a Hamel-base of $\Omega_{l/k}$.

(5.2) COROLLARY. — *Let* $\mathbf{Q}_p \subset K \subset L$ *be discrete valued fields such that* l *is finitely generated over* k. *Then* $\dim\,\tau\,\Omega^b_{L/K} = [l : k]_{\text{tr}}$. ($[\ ]_{\text{tr}}$ *means transcendence degree.*)

Proof. — Of course l_0 with the properties of (5.1) exists in this case. Further by [3] [EGA IV, Chap. 0, (21.7.1)], we have

$$\dim \Omega_{l/k} - \dim \gamma_{l/k} = [l : k]_{\text{tr}}.$$

Proof of (5.1) *:*

(1) First we show : *There exists a complete field* L_0, $K \subset L_0 \subset L$ *such that* $e\,(L_0/K) = 1$, and L_0 *has residue field* l_0.

Let $L_0 \supset K$ be a field extension such that $e(L_0/K) = 1$ and L_0 has residue field l_0. We have to show that there exists a K-linear embedding of L_0 into L, or what is equivalent a V_K-linear embedding of V_{L_0} into V_L. This is done by constructing for each $n \geqq 1$ a V_K-linear ring homomorphism $\varphi_n : V_{L_0} \to V_L/\pi_L^n V_L$ such that all triangles

$$\begin{array}{ccc} V_{L_0} & \xrightarrow{\varphi_n} & V_L/\pi_L^n V_L \\ & {\scriptstyle \varphi_{n+1}} \searrow & \uparrow \\ & & V_L/\pi_L^{n+1} V_L \end{array}$$

are commutative, and $\varphi_1 = V_{L_0} \xrightarrow{\tau} l_0 \hookrightarrow l = V_L/\pi_L V_L$. Then $\varphi = \lim \varphi_n$ is a V_K-linear embedding of V_{L_0} into V_L.

So all we have to do is to give *a construction* of φ_{n+1} : Let $\{ a_i \mid i \in I \}$ be an orthonormal base of L_0 over K. Define $f : V_{L_0} \to V_L/\pi_L^{n+1} V_L$ by the formula $f(\sum \lambda_i a_i) = \sum \lambda_i b_i$ $(\lambda_i \in V_K)$ where the b_i are chosen such that $b_i \equiv \varphi_n(a_i) \bmod (\pi_L^n)$.

Of course f need not be a ring homomorphism, but the 2-cocycle $(a, b) \mapsto f(ab) - f(a) f(b)$ of $V_{L_0} \times V_{L_0} \to \pi_L^n V_L/\pi_L^{n+1} V_L$ is trivial since l_0/k is separable. Hence there exists a V_K-linear map

$$g : \quad V_{L_0} \xrightarrow{\tau} l_0 \to l \cong \pi_L^n V_L/\pi_L^{n+1} V_L$$

with $g(ab) - ag(b) - bg(a) = f(ab) - f(a) f(b)$ for all $a, b \in V_{L_0}$. Then $\varphi_{n+1} = f - g$ has the required properties.

(2) *Proof of* (*a*). — By (3.1) and (2.8), we have a commutative diagram, with exact top-row :

$$\begin{array}{ccccccccc} 0 & \to & \Omega_K^b \hat{\otimes} L & \to & \Omega_{L_0}^b \hat{\otimes} L & \to & \Omega_{L_0/K}^b \hat{\otimes} L & \to & 0 \\ & & \downarrow {\scriptstyle \mathrm{id}} & & \downarrow \wr & & \downarrow \wr & & \\ 0 & \to & \Omega_K^b \hat{\otimes} L & \to & \Omega_L^b & \to & \Omega_{L/K}^b & \to & 0 \end{array}$$

Hence the bottom-row is also exact.

(3) *Proof of* (*b*). — Using section 1, (A 1), (3.1) and $\Omega_{L_0/K}^b \hat{\otimes} L \xrightarrow{\sim} \Omega_{L/K}^b$, we see that $\tau\, \Omega_{L/K}^b \cong \Omega_{l_0/k} \otimes l$. So we have only to show that $(\Omega_{l_0/k} \otimes l) \oplus \gamma_{l/k} \cong \Omega_{l/k}$.

This follows from the following exact sequences :

(A) $$0 \to \gamma_{l/l_0/k} \to \Omega_{l_0/k} \otimes l \to \Omega_{l/k} \to \Omega_{l/l_0} \to 0,$$

(B) $$\Omega_{l/l_0} \cong \gamma_{l/l_0} \text{ since } l/l_0 \text{ is finite algebraic,}$$

(C) $$0 \to \gamma_{l/k} \to \gamma_{l/l_0} \to \gamma_{l/l_0/k} \to 0$$

in the following way :

$$\gamma_{l/k} \oplus (\Omega_{l_0/k} \otimes l) \oplus \Omega_{l/l_0} \cong \gamma_{l/l_0/k} \oplus \gamma_{l/k} \oplus \Omega_{l/k} \quad \text{Use (A).}$$
$$\text{By (C),} \quad \gamma_{l/l_0/k} \oplus \gamma_{l/k} \cong \gamma_{l/l_0} \cong \Omega_{l/l_0} \quad \text{by (B).}$$

So

$$\gamma_{l/k} \oplus (\Omega_{l_0/k} \otimes l) \oplus \Omega_{l/l_0} \cong \Omega_{l/k} \oplus \Omega_{l/l_0}.$$

Dividing by Ω_{l/l_0} one obtains the formula

$$\gamma_{l/k} \oplus (\Omega_{l_0/k} \otimes l) \cong \Omega_{l/k}.$$

Examples :

(1) Let $K \supset \mathbf{Q}_p$ be a complete field such that $e(K/\mathbf{Q}_p) = 1$ and k has a countable p-base $\{ a_n \mid n \geq 1 \}$ over k^p. Take a set of representatives $\{ A_n \mid n \geq 1 \} \subset V_K$ and let B_n denote an element in the algebraic closure of K satisfying $B_n^{p^n} = A_n$.

The complete field

$$L = \overline{\bigcup_{n=1}^{\infty} K(B_1, B_2, \ldots, B_n)}$$

has the properties $e(L/K) = 1$ and $l = k(a_1^{1/p}, a_2^{1/p^2}, \ldots)$. Certainly the condition of (5.1) is not satisfied for $L \supset K$.

According to (3.3), $\{ dA_n \mid n \geq 1 \}$ is an orthonormal base of Ω_K^b, and $\{ dB_n \mid n \geq 1 \}$ is an orthonormal base of Ω_L^b. The map α : $\Omega_K^b \hat{\otimes} L \to \Omega_L^b$ maps $dA_n \otimes 1$ onto $p^n B_n^{p^n - 1} dB_n$. It follows that $\ker \alpha = 0$ and that

$$\operatorname{im} \alpha = \{ \Sigma \lambda_n dB_n \in \Omega_L^b \mid \lambda_n \in L, \lim \lambda_n p^{-n} = 0 \}$$
$$\neq \Omega_L^b = \ker (\Omega_L^b \to \Omega_{L/K}^b = 0).$$

So in this case, $\operatorname{im} \alpha$ is not closed and the sequence

$$\Omega_K^b \hat{\otimes} L \xrightarrow{\alpha} \Omega_L^b \xrightarrow{\beta} \Omega_{L/K}^b \to 0$$

is not exact. It shows moreover that the functor F, introduced in section 1, is neither left- nor right-exact.

(2) Let $K \supset \mathbf{Q}_p$ and $L \supset \mathbf{Q}_p$ be complete discrete valued fields such that $1 = e(L/\mathbf{Q}_p) = e(K/\mathbf{Q}_p)$; $k = \mathbf{F}_p(x)$ and $l = k_a(y)$ where k_a denotes the algebraic closure of k. Also in this case, l/k does not satisfy the condition of (5.1). Moreover, given a $\mathbf{Q}_p$-linear embedding $\varepsilon : K \to L$, one can form $\dim \Omega_{L/K}^b$ and $\ker \varphi_{L/K}$. They depend essentially on the embedding ε.

Proof. — It follows from (3.3) that $\dim \Omega_L^b = \dim \Omega_l = 1$. Let D : $L \to L$ be a derivation with norm 1. The set

$$L_0 = \{ a \in L \mid D(a) = 0 \}$$

is a complete subfield of L. One easily sees that the residue field of L_0 equals k_a. Take representatives $X \in K$ of $x \in k$ and $X_0 \in L_0$ of $x \in k_a$, $Y \in L$ of $y \in l$. The embedding ε depends only on $\varepsilon(X)$, and all possible values of $\varepsilon(X)$ are $X_0 + pu$, where $u \in V_L$.

Case (*a*). — Take $u = 0$; $\varepsilon(X) = X_0$. Then D is a K-derivation;

$$\dim \Omega^b_{L/K} = 1 \quad \text{and} \quad \varphi_{L/K} \text{ is bijective.}$$

Case (*b*). — Take $u = Y$; $\varepsilon(X) = X_0 + p\,Y$. Then D is not a K-derivation;

$$\Omega^b_{L/K} = 0 \quad \text{and} \quad \varphi_{L/K} = 0.$$

6. Fields of residue-characteristic zero

In this section, we deal with fields L such that char $l = 0$. The main result on extensions of bounded derivations is :

(6.1) THEOREM. — *Suppose that* $K \subset L$ *are complete valued fields and that* char $l = 0$. *Then*

$$0 \to \Omega^b_K \hat{\otimes} L \overset{\alpha}{\to} \Omega^b_L \to \Omega^b_{L/K} \to 0$$

is exact and α *is isometric.*

Proof. — According to (2.7), we have to show that any bounded derivation $D : K \to B$ where B is a spherically complete Banach space over L, can be extended to a derivation $D^* : L \to B$ with $\| D \| = \| D^* \|$.

Without loss of generality, we may suppose that L is algebraically closed. Using Zorn's lemma, there exists a maximal extension $D_1 : L_1 \to B$, $K \subset L_1 \subset L$ with $\| D_1 \| = 1$. Of course the field L_1 is also complete. In several steps we will show that $L_1 = L$.

(*a*) *The residu field of* L_1 *is algebraically closed.* — Suppose that this is not the case. Applying Hensel's lemma, we find $\mathfrak{I} \in L$ satisfying a polynomial

$$P(X) = X^n + a_{n-1} X^{n-1} + \ldots + a_0,$$

all $| a_i | \leqq 1$ which is irreducible and such that is residue in $l_1 [X]$ is irreducible. This implies $[L_1 (\mathfrak{I}) : L_1] = [l_1 (\tau\, \mathfrak{I}) : l_1]$ and $1, \mathfrak{I}, \ldots, \mathfrak{I}^{n-1}$ is an orthonormal base of $L_1(\mathfrak{I})$ over L_1. Let $D_2 : L_1(\mathfrak{I}) \to B$ be the unique extension of D_1. Then

$$0 = D_2(P(\mathfrak{I})) = P'(\mathfrak{I})\, D_2(\mathfrak{I}) + (\mathfrak{I}^{n-1} D_1(a_{n-1}) + \ldots + D_1(a_0)).$$

Since $| P'(\mathfrak{I}) | = 1$ $(\overline{P} \in l_1 [X]$ is irreducible and separable$)$, we have $\| D_2(\mathfrak{I}) \| \leqq | \mathfrak{I} | . \| D_1 \|$. Also

$$\| D_2(\mathfrak{I}^i) \| = \| i\, \mathfrak{I}^{i-1} D_2(\mathfrak{I}) \| \leqq | \mathfrak{I}^i | . \| D_1 \|.$$

Now for an arbitrary element $x \in L_1(\mathfrak{I})$, $x = \sum_{i=0}^{n-1} a_i \mathfrak{I}^i$, we find

$$\| D_2(x) \| = \| \Sigma\, a_i D_2(\mathfrak{I}^i) + \Sigma\, \mathfrak{I}^i D_1(a_i) \| \leqq \| D_1 \| \max | a_i \mathfrak{I}^i | = \| D_1 \| . | x |.$$

Hence $\| D_2 \| = \| D_1 \|$. Contradiction.

(*b*) *The value group* $| L_1^* |$ *is divisible.* — Suppose the contrary. Then one can find an extension $L_2 = L_1(\sqrt[n]{\pi})$ of L_1 such that $| L_2^* | / | L_1^* |$ has order n and $\pi \in L_1$. Put $\mathfrak{I} = \sqrt[n]{\pi}$. As before $1, \mathfrak{I}, \ldots, \mathfrak{I}^{n-1}$ is an orthogonal base of L_2 over L_1 and it suffices to show that the unique extension $D_2 : L_2 \rightarrow B$ has the property $\| D_2(\mathfrak{I}) \| \leqq \| D_1 \| . | \mathfrak{I} |$. But this is obvious since

$$\mathfrak{I}^{-1} D_2(\mathfrak{I}) = n^{-1} \pi^{-1} D_1(\pi).$$

Contradiction.

(*c*) L_1 *is algebraically closed.* — Suppose the contrary, and let $L_2 \supsetneqq L_1$ be a finite extension. Then $t = [L_2 : L_1]^{-1} \operatorname{Tr}_{L_2/L_1} : L_2 \rightarrow L_1$ is an L_1-linear projection of L_2 onto L_1 with norm 1. Let $x \neq 0$, $x \in \ker t$. There is an element $x_0 \in L_1$ with $| x - x_0 | < | x_0 |$ because $l_2 = l_1$ and $| L_1^* | = | L_2^* |$. Now

$$| x_0 | = | t(x_0) | = | t(x_0 - x) | \leqq | x - x_0 | < | x_0 |$$

is a contradiction.

(*d*) *We show finally that* $L_1 = L$. — Suppose ortherwise. Take $\mathfrak{I} \in L \setminus L_1$ and $\alpha \in \mathbf{R}$, $0 < \alpha < 1$. Put $L_2 = L_1(\mathfrak{I})$ and $\mathfrak{I}' = \mathfrak{I} - a'$, $a' \in L_1$ chosen such that $| \mathfrak{I}' | \leqq \alpha \inf \{ | \mathfrak{I}' - b | \mid b \in L_1 \}$.

Define $D_2 : L_2 \rightarrow B$ by $D_2 | L_1 = D_1$ and $D_2(\mathfrak{I}') = 0$. Since L_1 is algebraically closed any element of L_2 has the form

$$a_0 (\mathfrak{I}' - a_1)^{n_1} \ldots (\mathfrak{I}' - a_m)^{n_m} \qquad \text{with} \quad n_i \in \mathbf{Z}.$$

It follows that $\| D_2 \| = \sup \{ | \mathfrak{I}' - a |^{-1} \| D_2(\mathfrak{I}' - a) \| \mid a \in L_1 \}$. The latter expression is less or equal to

$$\| D_1 \| . | \mathfrak{I}' | \left(\inf \{ | \mathfrak{I}' - a | \mid a \in L_1 \}\right)^{-1} \leqq \alpha^{-1} \| D_1 \|.$$

So $\| D_2 \| \leqq \alpha^{-1} \| D_1 \|$.

Using lemma (2.9) and the fact that α, $0 < \alpha < 1$, was arbitrary, one concludes that D_1 is extendable to L_2 with the same norm. This contradicts the maximality of L_1.

Definition. — Given complete fields $L \supset K$ with char $l = 0$. An element $x \in L$ is called *almost algebraic* over K if there exists for every $\varepsilon > 0$ a monic polynomial $p \in K[X]$ such that $| p(x) | \leqq \varepsilon | p'(x) |$. L is called *almost algebraic* over K if every element of L is almost algebraic over K.

Remark. — If L is algebraic over K, then L is almost algebraic over K. Consider L_a, the completion of the algebraic closure of L, and let $\tilde{K}$ denote the closure of the algebraic closure of K in L_a. Since L_a is algebraically closed, $\tilde{K}$ is isomorphic to the completion of the algebraic closure of K, which is K_a. For convenience we write $\tilde{K} = K_a \subset L_a$.

(6.2) PROPOSITION. — *Let $L \supset K$ be complete valued fields and suppose* char $l = 0$. *The following properties of an element $x \in L$ are equivalent :*

(1) *x is almost algebraic;*

(2) $d^b_{L/K}(x) = 0$;

(3) $x \in K_a \cap L$.

Proof :

(1) implies (2). Put $d = d^b_{L/K}$, and let $\varepsilon > 0$. There is a polynomial $p \in K[X]$ such that $|p(x)| \leqq \varepsilon |p'(x)|$. Then

$$\varepsilon |p'(x)| \geqq \| d(p(x)) \| = \| p'(x)\, dx \| = |p'(x)| . \| dx \|;$$

hence $dx = 0$;

(2) implies (3). Take $y \in L \setminus (K_a \cap L)$. It suffices to show that $d^b_{L_a/K_a}(y) \neq 0$. Using (6.1), it is enough to prove $d^b_{K_a(y)/K_a}(y) \neq 0$. Let $D : K_a(y) \to K_a(y)$ denote the K_a-derivation given by $D(y) = 1$. Then D is bounded since

$$\begin{aligned} &|D(a_0 (y - a_1)^{n_1} \dots (y - a_m)^{n_m}| \\ &\quad \leqq |a_0 (y - a_1)^{n_1} \dots (y - a_m)^{n_m}| . |\textstyle\sum_i n_i (y - a_i)^{-1} D(y - a_i)| \end{aligned}$$

and

$$|y - a_i|^{-1} |D(y - a_i)| = |y - a_i| \leqq \left(\inf \{ |y - a| \mid a \in K_a \}\right) < \infty.$$

It follows that $d_{K_a(y)/K_a}(y) \neq 0$;

(3) implies (1). Take $x \in L$ and suppose that

$$\sup \{ |p'(x)|^{-1} |p(x)| \mid p \in K[X] \} = C < \infty.$$

The K-derivation $D : K(x) \to K(x)$ given by $D(x) = 1$ is then bounded. Indeed, one calculates easily that $\| D \| = C$. Hence $d^b_{K(x)/K}(x) \neq 0$. Using (4.1), $d^b_{L_a/K}(x) \neq 0$, and certainly $x \notin K_a$.

(6.3) COROLLARY. — *Let $L \supset K$ denote complete valued fields and let* char $l = 0$. *The following properties are equivalent :*

(1) $\Omega^b_{L/K} = 0$;

(2) $K_a = L_a$;

(3) *L is almost algebraic over K.*

Proof. — Follows directly from (6.2).

DEFINITIONS. — Given complete valued fields $L \supset K$, char $l = 0$, and $\alpha \in \mathbf{R}$, $0 < \alpha < 1$.

$A = \{ x_i \mid i \in I \} \subset L$ is called *α-transcendental* over K if, for any polynomial $p \in K[X_1, \ldots, X_n]$ and different elements $i_1, \ldots, i_n \in I$, the following inequality holds :

$$| p(x_{i_1}, \ldots, x_{i_n}) | \geqq \alpha \max_j | x_{i_j} (\partial p/\partial X_j)(x_{i_1}, \ldots, x_{i_n}) |.$$

The set A is called *topological transcendental* if A is β-transcendental over K for some $\beta \in \mathbf{R}$, $0 < \beta < 1$. The set A is called an *α-transcendance base* of L over K if A is α-transcendental and L is almost algebraic over $\overline{K(A)}$. The set A is called a *topological transcendance base* of L over K if it is a β-transcendance base for some $\beta \in \mathbf{R}$, $0 < \beta < 1$.

Remark. — If A is α-transcendental over K then A is certainly algebraically independent over K.

(6.4) PROPOSITION. — *With the above notations :*

(1) *A is α-transcendental if and only if $\{ x_i^{-1} dx_i \mid i \in I \} \subset \Omega^b_{L/K}$ is α-orthonormal,*

(2) *A is an α-transcendance base if and only if $\{ x_i^{-1} dx_i \mid i \in I \} \subset \Omega^b_{L/K}$ is an α-base,*

(3) *L has a topological transcendance base over K if $\Omega^b_{L/K}$ is a Banach space over L of countable type.*

Proof :

(1) Using (6.1), we may suppose that $L = \overline{K(A)}$. Now « only if » : We have to show « $\| \Sigma a_j x_j^{-1} dx_j \| \geqq \alpha | a_i |$ ». Consider the K-derivation $D_i : K(A) \to K(A)$ given by $D_i(x_j) = \delta_{ij} x_i$. Clearly

$$\| D_i \| = \sup \left\{ | p |^{-1} | D_i(p) | \;\middle|\; 0 \neq p \in K[A] \right\}.$$

It follows from the definition of α-transcendental that $\| D_i \| \leqq \alpha^{-1}$. Extending D_i by continuity to a K-derivation of L and using (2.2), we find an L-linear map $t_i : \Omega^b_{L/K} - L$ such that

$$\| t_i \| \leqq \alpha^{-1}, \qquad t_i(dx_j) = \delta_{ij} x_i.$$

Hence

$$\alpha^{-1} \| \Sigma a_j x_j^{-1} dx_j \| \geqq \| t_i \| . \| \Sigma a_j x_j^{-1} dx_j \| \geqq | a_i |$$

and the required inequality is proved. The proof of the « if-part » is analogous.

(2) This follows from (1) and (6.3).

(3) The Banach space $\Omega^b_{L/K}$ is of countable type over L, and we suppose for convenience that $\Omega^b_{L/K}$ is infinite dimensional. Using (2.5) part (3), one can find a sequence of elements $\{ x_n \mid n \in \mathbf{N} \}$ in L such that $V_n = L\, dx_1 + \ldots + L\, dx_n$ has dimension n and $\bigcup_{n=1}^{\infty} V_n$ is dense in $\Omega^b_{L/K}$. Take $\alpha \in \mathbf{R}$, $0 < \alpha < 1$.

Let $t_n : \Omega^b_{L/K} \to L$ be a bounded linear map satisfying $t_n (V_{n-1}) = 0$, $\| t_n \mid V_n \| \geq \sqrt{\alpha}\, \| t_n \|$, and let $D_n : K(x_1, \ldots, x_n) \to L$ be the corresponding $K(x_1, \ldots, x_{n-1})$-derivation. Choose $y_n \in K(x_1, \ldots, x_n)$ with $| D_n (y_n) | \geq \sqrt{\alpha}\, \| D_n \| . | y_n |$. It follows easily that

$$V_n = L\, dy_1 + \ldots + L\, dy_n$$

and that $\{ y_i^{-1}\, dy_i \mid i \in \mathbf{N} \}$ is α-orthonormal. It is also an α-base of $\Omega^b_{L/K}$ because $\bigcup_{n=1}^{\infty} V_n$ is dense in $\Omega^b_{L/K}$.

(6.5) LEMMA and DEFINITION. — *Let $L \supset K$ be valued fields and let* char $k = 0$. *The map*

$$L^* \to \left\{ x \in \Omega^b_{L/K} \;\middle|\; \| x \| \leq 1 \right\} \to \tau\, \Omega^b_{L/K} \to \varepsilon_{L/K};$$

$\varepsilon_{L/K}$ *being* coker $\varphi_{L/K}$; *given by $a \mapsto a^{-1}\, d^b_{L/K}(a)$ is additive and factors uniquely through $| L^* | / | K^* |$. The induced map $| L^* | / | K^* | \otimes_{\mathbf{Z}} l \to \varepsilon_{L/K}$ will be denoted by $\psi_{L/K}$.*

Proof. — We can restrict ourselves to showing that the kernel of $L^* \to \varepsilon_{L/K}$ contains K^* and $\left\{ x \in L^* \;\middle|\; | x | = 1 \right\}$. The first statement is obvious. Further, take $x \in L^*$, $| x | = 1$, and let $\tau x \in l$ denote its residue. Then $\varphi_{L/K} (d (\tau x)) = \tau (d^b_{L/K} (x))$ and $\tau (x^{-1}\, d^b_{L/K} (x)) \in \operatorname{im} \varphi_{L/K}$.

(6.6) PROPOSITION. — *Let $L \supset K$ be complete valued fields and suppose that* char $k = 0$. *Then $\varphi_{L/K}$ and $\psi_{L/K}$ are injective.*

Proof. — Take a transcendance base $\{ \bar{x}_i \mid i \in I \}$ of l/k, and let $\{ x_i \mid i \in I \} \subset L$ be a set of representatives. Take further a linear base $\{ \bar{y}_j \mid j \in J \}$ of $| L^* | / | K^* | \otimes_{\mathbf{Z}} l$ consisting of elements of the type $a \otimes 1$, $a \in | L^* | / | K^* |$. Let $\{ y_j \mid j \in J \} \subset L$ be a set of representatives.

As is easily seen the monomials in $\{ x_i \mid i \in I \} \cup \{ y_j \mid j \in J \}$ are orthogonal over K. By (6.4) part (1), it follows that $\{ dx_i \mid i \in I \} \cup \{ y_j^{-1}\, dy_j \mid j \in J \}$ is an orthonormal subset of $\Omega^b_{L/K}$. Their images in $\tau\, \Omega^b_{L/K}$ are linearly independent, so $\{ \tau\, dx_i \mid i \in I \} \cup \{ \tau (y_j^{-1}\, dy_j) \mid j \in J \}$ is linearly independent.

Clearly $\varphi = \varphi_{L/K}$ is injective since $\varphi (d\bar{x}_i) = \tau\, dx_i$, and $\{ d\bar{x}_i \mid i \in I \}$ is a base of $\Omega_{l/k}$. Further $\operatorname{im} \varphi$ is the subspace of $\tau\, \Omega^b_{L/K}$ generated by $\{ \tau (dx_i) \mid i \in I \}$, and the formula

$$\psi_{L/K} (\bar{y}_j) = \tau (y_j^{-1} \, dy_j) + \text{im} \, \varphi \in \varepsilon_{L/K}$$

shows that also $\psi_{L/K}$ is injective.

Example. — Let l be a field of characteristic zero, G a subgroup of $\mathbf{R}$ and $\xi : G \times G \to l^*$ a symmetric 2-cocycle (where G acts trivially on l^*). The set $L_0 = l \langle G, \xi \rangle$ of all functions $x : G \to l$ such that

$$\text{supp} (x) = \{ g \in G \mid x (g) \neq 0 \}$$

has limit $+\infty$, becomes a complete valued field under addition, multiplication and valuation given by the formulas :

$$(x + y) (g) = x (g) + y (g);$$
$$(xy) (g) = \Sigma_h \, x (h) \, y (g - h) \, \xi (h, g - h)$$

and

$$| x | = \exp (- \min (\text{supp} (x))).$$

The set $L_1 = l \langle\langle G, \xi \rangle\rangle$ of all functions $x : G \to l$ such that supp (x) is a well-ordered subset of G, becomes in the same way a complete valued field. Kaplansky (Maximal fields with valuations, *Duke math. J.*, vol. 9, 1942, p. 303-321) has shown that every complete valued field, of residue-characteristic zero, is isomorphic (as valued field) to a field L', $L_0 \subset L' \subset L_1$, for suitable l, G, ξ. Further the field L_1 is maximally (= spherically) complete.

As an illustration of (6.6), we calculate $\Omega^b_{L_0}$. To do so, we introduce the following notation : π_g is the element of L_0 given by $\pi_g (h) = \delta_{g, h}$ for all $g, h \in G$.

Every element x of L_0 can now be written as a convergent sum

$$x = \Sigma_{g \in G} \, \lambda_g \, \pi_g \qquad \text{with} \quad \lambda_g \in l.$$

Since

$$dx = \Sigma \, \lambda_g \, \pi_g \, (\pi_g^{-1} \, d\pi_g) + \Sigma \, d (\lambda_g) \, \pi_g,$$

it follows that $\Omega^b_{L_0}$ is topologically generated by

$$\{ d (\lambda) \mid \lambda \in l \} \cup \{ \pi_g^{-1} \, d\pi_g \mid g \in G \}.$$

Hence $\Omega^b_{L_0}$ is also topologically generated by

$$\{ d (\lambda_i) \mid i \in I \} \cup \{ \pi_{g_j}^{-1} \, d\pi_{g_j} \mid j \in J \},$$

where $\{ \lambda_i \mid i \in I \}$ is a transcendance base of $l/\mathbf{Q}$ and $\{ g_j \mid j \in J \}$ is a maximal $\mathbf{Z}$-independent subset of G. Proposition (6.6) asserts that this set is orthonormal and hence it must be an orthonormal base of $\Omega^b_{L_0}$.

7. Fields of characteristic $p \neq 0$

In this section, we deal with fields $L \supset K$ of characteristic $p \neq 0$. Let $K^+ = \overline{K(L^p)}$, which is the smallest complete field containing both K and L^p. Clearly $\Omega^b_{L/K}$ equals Ω^b_{L/K^+}. By means of a result on extensions of bounded derivations we will show that $\Omega^b_{L/K} \neq 0$ if $L \neq K^+$.

(7.1) PROPOSITION. — *Suppose $L \supset K \supset L^p$. Then*

$$0 \to \Omega^b_{K/L^p} \hat{\otimes} L \xrightarrow{\alpha} \Omega^b_L \to \Omega^b_{L/K} \to 0$$

is exact and α is an isometry.

Proof. — It suffices to show [according to (2.7)] that any bounded L^p-derivation $D : K \to B$, where B is a spherically complete Banach space over L, is extendable to L with the same norm.

The lemma of Zorn yields the existence of an intermediate field L_1, $K \subset L_1 \subset L$, such that D extends to $D_1 : L_1 \to B$ with $\| D \| = \| D_1 \|$ and such that any extension of D_1 to a bigger field has a norm greater than $\| D \|$.

We will lead the assumption $L_1 \neq L$ to a contradiction. Take $y \in L \setminus L_1$ and $\alpha \in \mathbf{R}$, $0 < \alpha < 1$. Let $L_2 = L_1(y)$ and let $D_0 : L_2 \to L_2$ be a non-zero L_1-derivation. Take $z \in L_2$ such that $| D_0(z) | \geq \alpha \| D_0 \| . | z |$. We may suppose that $D_0(z) = z$. We want to show that $\{ 1, z, \ldots, z^{p-1} \}$ is an α^{p-1}-orthogonal base of L_2 over L_1. Take $t = \sum_{i=0}^{p-1} a_i z^i$, with all $a_i \in L_1$. Then we have

$$D_0(t) = \sum_{i=0}^{p-1} i a_i z^i, \qquad \ldots, \qquad D_0^{p-1}(t) = \sum_{i=0}^{p-1} i^{p-1} a_i z^i.$$

It follows that

$$a_i z^i = \sum_{j=0}^{p-1} \lambda_{ij} D_0^j(t) \qquad \text{where} \quad \lambda_{ij} \in \mathbf{F}_p.$$

Hence

$$| a_i z^i | \leq \max | D_0^j(t) | \leq \alpha^{-p+1} | t |.$$

Consequently $| t | \geq \alpha^{p-1} \max | a_i z^i |$. This means that $\{ 1, z, \ldots, z^{p-1} \}$ is an α^{p-1}-orthogonal base of L_2 over L_1.

Now define $D_2 : L_2 \to B$ such that $D_2 \mid L_1 = D_1$ and $D_2(z) = 0$. Then $\| D_2 \| \leq \alpha^{-p+1} \| D_1 \|$. Applying lemma (2.9), one concludes that D_1 is extendable to L_2 with the same norm. This is a contradiction.

Remark. — For the case of a trivial valued field one finds back a result of [3] [EGA IV, première partie, Chap. 0, (21.4.7)] :

$$\gamma_{L/K/L^p} = 0 \qquad \text{if} \quad L \supset K \supset L^p.$$

(7.2) THEOREM. — *Let* $L \supset K$ *be complete valued fields of characteristic* $p \neq 0$. *If* $x \in L \setminus K^+$, *then* $d^b_{L/K}(x) \neq 0$. *In particular,* $\Omega^b_{L/K} = 0$ *implies* $L = K^+$.

Proof. — Form $L_1 = K^+(x)$, and the bounded K^+-derivation $D: L_1 \to L_1$ given by $D(x) = 1$. Hence $d^b_{L/K^+}(x) \neq 0$. According to (7.1) also $d^b_{L/K}(x) \neq 0$.

DEFINITIONS. — A subset $A = \{ x_i \mid i \in I \}$ of L is called an α-p-free ($\alpha \in \mathbf{R}$, $0 < \alpha \leq 1$) set in L/K if the monomials

$$M = \{ x_{i_1}^{\alpha_1} \ldots x_{i_n}^{\alpha_n} \mid n \in \mathbf{N};\ i_1, \ldots, i_n \in I;\ 0 \leq \alpha_i < p \}$$

are α-orthogonal in L viewed as a Banach space over K^+. The set A is called an *α-p-base* of L/K if M is an α-orthogonal base of L over K^+. If A is β-p-free in L/K for some β then A is called *topologically p-free* in L/K, and if A is a β-p-base of L/K for some β then A is called a *topological p-base* of L/K.

(7.3) PROPOSITION. — *With the above notations :*

(1) $A \subset L$ *is* α*-p-free in* L/K *if and only if* $\{ a^{-1} d^b_{L/K}(a) \mid a \in A \}$ *is* α*-orthonormal.*

(2) $A \subset L$ *is an* α*-p-base of* L/K *if and only if* $\{ a^{-1} d^b_{L/K}(a) \mid a \in A \}$ *is an* α*-orthonormal base of* $\Omega^b_{L/K}$.

(3) *If* $\Omega^b_{L/K}$ *is a Banach space of countable type over L, then L has a topological p-base over K.*

Proof :

(1) Let $A = \{ x_i \mid i \in I \}$ be α-p-free. Using (7.1), we may assume that

$$L = \overline{K^+(x_i \mid i \in I)}.$$

Consider the K^+-derivation D_i of L into L given by $D_i(x_j) = \delta_{ij} x_j$. The norm of D_i is $\leq \alpha^{-1}$ because the set of monomials M is supposed to be α-orthogonal.

Let $t_i : \Omega^b_{L/K} \to L$ be the linear map corresponding to D_i. The elements $x_i^{-1} dx_i \in \Omega^b_{L/K}$ have the property $t_j(x_i^{-1} dx_i) = \delta_{ij}$. From this one deduces :

$$\alpha^{-1} \| \Sigma a_i x_i^{-1} dx_i \| \geq \| t_j \| . \| \Sigma a_i x_i^{-1} dx_i \| \geq | a_j |$$

and the required inequality is proved.

The second part of (1) can be proved in the same way.

(2) This follows from (1) and (7.2).

(3) The proof is verbally the same as the proof of (6.4) part (3).

Remarks :

(1) If the field L is spherically complete or if the Banach space $\Omega^b_{L/K}$ is of countable type then $\Omega^b_{L/K} \neq 0$ implies that $\mathrm{Derb}_K(L, L) \neq 0$. However in the case that L is not spherically complete, we can not draw this conclusion. It seems likely that there are complete valued fields $L \supset K$ of characteristic $p \neq 0$ such that $L \neq K^+$ and

$$\mathrm{Derb}_K(L, L) = 0.$$

The author does not know any example of this type.

(2) Topological p-bases form an essential part of Kiehl's proof [6] of the exellence of affinoid algebras and analytic rings over a complete valued field L with $[L : L^p] = \infty$. Proposition (7.3) gives an alternative proof of the existence of p-base in the countable case.

8. Application to tensor products

In this section, we deal with a problem raised in the theory of affinoid algebras : « Let A and B be Banach algebras (commutative and with a unit element) over the field K. Suppose that the norms on A and B are power-multiplicative (which means $\| f^n \| = \| f \|^n$ for all f and all n). Is the norm on $A \otimes_K B$ (or on $A \hat{\otimes}_K B$) also power-multiplicative ? »

First of all we reduce this question to a problem on tensor products of fields.

(8.1) Proposition (T. A. Springer). — *Let A be a Banach algebra over K, which is commutative and has a unit element. The norm on A is power-multiplicative if, and only if A can be embedded in a product of complete fields $L_i \supset K$;* so $A \subset \prod_{i \in I} L_i$.

Proof. — We remark that $\prod_{i \in I} L_i$ is defined to be the set of all elements $(l_i)_{i \in I}$ such that $\sup_i \| l_i \| < \infty$ and the norm

$$\| (l_i)_{i \in I} \| = \sup \| l_i \|$$

makes $\prod_{i \in I} L_i$ into a Banach algebra over K. Its norm is clearly power-multiplicative. The other part of (8.1) will be shown in a number of lemmata.

(8.2) Lemma. — *In the set $\Phi = \Phi(A)$ of all maps $\varphi : A \to \{ r \in \mathbf{R} \mid r \geq 0 \}$ satisfying :*

(1) $\varphi(1) = 1$ *and* $\varphi(a) \leq \| a \|$;

(2) $\varphi(ab) \leq \varphi(a)\, \varphi(b)$;

(3) $\varphi(a + b) \leq \max(\varphi(a), \varphi(b))$;

every element lies above a minimal element (in the natural order of Φ). Every minimal element φ of Φ satisfies $\varphi(ab) = \varphi(a)\varphi(b)$ for all $a, b \in A$.

Proof. — We apply Zorn. Let Φ_0 be a totally ordered subset of φ, then it is easily checked that $\varphi_0(x) = \inf\{\varphi(x) \mid \varphi \in \Phi_0\}$ is also an element of Φ. Further let φ be a minimal element of Φ. Take $a \in A$ with $\varphi(a) \neq 0$. Then φ^* defined by

$$\varphi^*(b) = \inf\{\varphi(a^n b)\varphi(a)^{-n} \mid n \geq 0\}$$

is also an element of Φ and satisfies $\varphi^* \leq \varphi$. Consequently $\varphi^* = \varphi$ and $\varphi(ab) = \varphi(a)\varphi(b)$ for all $a, b \in A$.

(8.3) LEMMA. — *Let $\| \ \|_{sp}$ be the semi-norm on A given by*

$$\| a \|_{sp} = \lim (\| a^n \|)^{1/n}.$$

Then $\| \ \|_{sp} \in \Phi$. Take $a \in A$ with $\| a \|_{sp} = \rho$. Then there exists a complete field extension L of K and a K-algebra homomorphism $f : A \to L$ satisfying $\| f \| = 1$ and $| f(a) | = \rho$.

Proof. — The first statement, $\| \ \|_{sp} \in \Phi$, is classical. Consider the algebra B consisting of all power series $\sum_{i=0}^{\infty} a_i T^i$ with coefficients in A and such that $\lim \| a_i \|_{sp} \rho^{-i} = 0$. With the norm

$$\| \textstyle\sum_{i=0}^{\infty} a_i T^i \| = \max(\| a_i \|_{sp} \rho^{-i})$$

B becomes a normed algebra over K. The element $x = aT - 1 \in B$ has the property $\| xy \| = \| y \|$ for all $y \in B$ and $\| x \| = 1$.

Let $\hat{B}$ denote the completion of B then $\hat{B}x$ is a proper closed ideal in $\hat{B}$. Let C be the Banach algebra $\hat{B}/\hat{B}x$. Take a minimal $\varphi \in \Phi(C)$ and let $\hat{C}_\varphi$ be the completion of C with respect to φ. Since $\varphi(ab) = \varphi(a)\varphi(b)$ for all a and b, $\hat{C}_\varphi$ is in fact a subring of a complete valued field $L \supset K$. The composed map $f : A \to \hat{C}_\varphi \subset L$ has certainly norm 1. Further the map $A \to B$ maps onto an element of norm $\| a \|_{sp} = \rho$, hence $| f(a) | \leq \rho$. The image of T in L is t and satisfies $tf(a) = 1$. Since $| t | \leq \| T \| = \rho^{-1}$, we have $| f(a) | = \rho$.

End of the proof of (8.1). — Suppose that the norm on A is power-multiplicative. Then $\| \ \| = \| \ \|_{sp}$. For each $a \in A$, $a \neq 0$, we can take an extension L_a of K and a K-algebra homomorphism $f_a : A \in L_a$ such that $| f_a(x) | = \| a \|$ and $\| f_a \| = 1$. Then the map

$$f = \prod f_a : A \to \prod L_a$$

is an embedding of A in a product complete field extensions of K.

Remark. — Suppose that the norm on both A and B is power-multiplicative. Then $A \subset \prod L_i$ and $B \subset \prod M_j$. Using (B 2) part (2), one finds that the problem : « Is the norm on $A \otimes_K B$ power-multiplicative ? » reduces to : « Is the norm on $L_i \otimes_K M_j$ for each i and j power-multiplicative ? ». As can be expected, in the case char $k = 0$ the answer is yes *:*

(8.4) THEOREM. — *Let $L \supset K$ and $M \supset K$ be valued fields then the norm on $L \otimes_K M$ is power-multiplicative provided that* char $k = 0$.

Proof. — Using (B 2) part (2), we may suppose that L is algebraically closed and without loss of generality we may, according to (8.1), at any time in this proof replace a power-multiplicative normed algebra by a valued field. Further for any intermediate field E, $K \subset E \subset L$, the normed algebras $L \otimes_K M$ and $L \otimes_E (E \otimes_K M)$ are isometrically isomorphic. The lemma of Zorn yields the existence of a maximal intermediate field E such that $E \otimes_K M$ has a power-multiplicative norm. If $E = L$,we are done. If not we have the following cases :

(1) $\Omega^b_{L/E} \neq 0$, and there exists $x \in L$ with $\Omega^b_{\overline{E(x)}/E} \neq 0$.

(2) $\Omega^b_{L/E} = 0$ and according to (6.2), $L = E_a$.

So it suffices to show the validity of (8.4) in the following two cases :

(1) $L = \overline{K(x)}$, x is topological transcendental and K is algebraically closed.

(2) $L = K_a$ (= the completion of the algebraic closure of K).

Case (1). — Let f, $g \in K[x] \otimes_K M$ have degree $\leq n$, and let a real number δ, $0 < \delta < 1$, be given. Let $D : L \to L$ be the bounded K-derivation satisfying $D(x) = 1$. Since K is algebraically closed there exists a polynomial p of degree 1 in $K[x]$ such that $|D(p)| \geq \delta^{1/2n} \|D\| . |p|$. Then $\{1, p, p^2, \ldots, p^{2n}\}$ is a δ-orthogonal base of the K-vector space of all polynomials in $K[x]$ of degree $\leq 2n$. Let

$$f = \Sigma_{i=0}^{n} p^i \otimes m_i;$$

$$g = \Sigma_{j=0}^{n} p^j \otimes m_j^* \qquad \text{and} \qquad fg = \Sigma_{k=0}^{2n} p^k \otimes \Sigma_{i+j=k} m_i m_j^*.$$

So

$$\begin{aligned} \| fg \| &\geq \delta \max_k | \Sigma_{i+j=k} m_i m_j^* | . | p^k | \\ &= \delta (\max_i | m_i | . | p^i |) . (\max_j | m_j^* | . | p^j |) \geq \delta \| f \| . \| g \|. \end{aligned}$$

Hence the norm on $K[x] \otimes_K M$ is multiplicative and consequently also on $L \otimes_K M$.

Case (2). — First we enlarge M such that $| M | \supset | K_a |$, and we consider special cases of algebraic extensions of K :

(A) If $[L : K] = [l : k] < \infty$ then the norm on $L \otimes_K M$ is power-multiplicative.

(B) If $L = K(\mathfrak{I})$, $\mathfrak{I}^n = a \in K$ and $| \mathfrak{I}^i | \notin | K^* |$ for $i = 1, \ldots, n-1$, then the norm on $L \otimes_K M$ is power-multiplicative.

Proof of (A). —Since for

$$x = a \otimes b \quad \text{and} \quad y \in L \otimes_K M, \quad \| xy \| = \| x \| . \| y \|$$

it is sufficient to show that $\| y \| = 1$ implies that $\| y^n \| = 1$ for all n. Put in a different way, we have to show that the ring

$$R = \{ y \in L \otimes M \mid \| y \| \leq 1 \} / \{ y \in L \otimes M \mid \| y \| < 1 \}$$

has no nilpotent elements. Now L has an orthonormal base over K and it follows from (B 2) that $R = l \otimes_k m$. This ring is known to have no nilpotents.

Proof of (B). — Again it suffices to show that R has no nilpotents. Take an element $b \in M$ such that $\| \mathfrak{I} \otimes b \| = 1$. Then $1 \otimes 1$, $\mathfrak{I} \otimes b, \ldots, \mathfrak{I}^{n-1} \otimes b^{n-1}$ is an orthonormal base of $L \otimes_K M$ over the field M. It follows that $R = m[T]/(T^n - a)$ where $a \in m$ is that residue of the element $\mathfrak{I}^n b^n \in M$. Since $a \neq 0$, again R has no nilpotents.

Proof of the case (2). — Let K_1 be a maximal subfield of L, containing K, such that the norm on $K_1 \otimes_K M$ is power-multiplicative.

(*a*) *The residue field* k_1 *of* K_1 *is algebraically closed.* — Suppose not, then there exists a field K_2 with $K_1 \subsetneq K_2 \subset L$, $[K_2 : K_1] = [k_2 : k_1] < \infty$. Now $K_2 \otimes_K M$ is isomorphic to $K_2 \otimes_{K_1} (K_1 \otimes_K M)$ and using (6.1) and (A), we find the contradiction that the norm on $K_2 \otimes_K M$ is power-multiplicative.

(*b*) *The value-group of* K_1 *is divisible.* — Suppose not, then there is an extension $K_1(\mathfrak{I})$ of K_1 of the type described in (B). In the same way as (*a*), this leads ot a contradiction.

(*c*) $K_1 = L$. If not, then K_1 is not algebraically closed. Let $K_2 \subset L$ be finite extension of K_1. According to (*a*) and (*b*),

$$f(K_2/K_1) = e(K_2/K_1) = 1.$$

And hence for any $x \in K_2$, there exists an element $y \in K_1$ with $| x - y | < | x | = | y |$. On the other hand, let $0 \neq x \in K_2$ be such that $\mathrm{Tr}(x) = 0$. Then, for any $y \in K_1$, we have

$$| x - y | \geq | \mathrm{Tr}(x - y) | = | \mathrm{Tr}(-y) | = | y |.$$

This is a contradiction.

(8.5) COROLLARY. — *Suppose* char $k = 0$, *and let* A *and* B *be Banach algebras (commutative and with a unit element) over* K. *If the norms on* A *and* B *are power-multiplicative, then so is the norm on* $A \hat{\otimes}_K B$.

Remark. — In case char $k = p \neq 0$, the statements corresponding to (8.4) and (8.5) are obviously false. However, one can prove the following : « If $L \supset K$ is a tamely ramified extension of discrete valued fields, then the norm on $L \otimes_K M$ is power-multiplicative for every field $M \supset K$ ».

This leads to the conjecture that the following statements concerning $L \supset K$ are equivalent :

(1) For all $M \supset K$, the norm on $L \otimes_K M$ is power-multiplicative.

(2) For all $M \supset K$, with $[M : K] < \infty$, the norm on $L \otimes_K M$ is power-multiplicative.

(3) The map $\alpha : \Omega_K^b \hat{\otimes} L \to \Omega_L^b$ is isometric.

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