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QUASI-PROJECTIVE ABELIAN GROUPS

BY

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A module M over a ring R is called *quasi-projective* (*see* [3]) if for every submodule N of M and for every R-homomorphism $\varphi : M \to M/N$ there is an R-endomorphism ψ of M making the diagram

$$\begin{array}{ccc} & & M \\ & \overset{\psi}{\swarrow} & \downarrow{\scriptstyle \varphi} \\ M & \xrightarrow{\eta} & M/N \end{array}$$

commute where η denotes the natural map. The aim of this note is to describe explicitly the quasi-projective abelian groups.

It is relatively easy to list the quasi-injective abelian groups, since they are exactly the fully invariant subgroups of injective, i. e. divisible groups, and hence either divisible or torsion groups each p-component of which is the direct sum of isomorphic cyclic or quasicyclic groups $Z(p^n)$ $(n \leqq \infty)$. JANS and WU [3] described the finitely generated quasi-projective abelian groups; the general case seems to be unsettled so far. We shall show that the expected structure theorem holds : an abelian group is quasi-projective exactly if it is either free or a torsion group each p-component of which is the direct sum of isomorphic cyclic groups of orders p^n for some n (which may depend on p).

We shall need a couple of lemmas which we formulate for arbitrary unital R-modules M.

LEMMA 1. — *Every direct summand of a quasi-projective module is quasi-projective.*

LEMMA 2. — *If M is quasi-projective and N is a fully invariant submodule of M, then M/N is likewise quasi-projective.*

For these two lemmas, we refer to JANS and WU [3].

LEMMA 3. — *If M_i $(i \in I)$ are quasi-projective R-modules such that, for every submodule N of the direct sum $M = \bigoplus M_i$, $N_i = \bigoplus (N \cap M_i)$ holds, then M is again quasi-projective.*

Hypothesis implies that every quotient module M/N of M is of the form $\bigoplus (M_i/N_i)$ with $N_i \subseteq M_i$. Every homomorphism $M_i \rightarrow M_j/N_j$ with $i \neq j$ must be trivial, because otherwise there exist submodules N_i' and N_j' such that $M_i/N_i' \simeq N_j'/N_j$ are non-zero modules, and so there is a subdirect sum of M_i and N_j' which is not their direct sum. Thus every $\varphi : \bigoplus M_i \rightarrow \bigoplus (M_i/N_i)$ acts coordinate-wise whence the quasi-projectivity of M is obvious.

LEMMA 4. — *If N is a submodule of a quasi-projective module M such that M/N is isomorphic to a direct summand of M, then N itself is a summand of M.*

Let A be a summand of M with $\pi : M \rightarrow A$, $\rho : A \rightarrow M$ as projection and injection maps, and let $\alpha : A \rightarrow M/N$ be an isomorphism. For the natural map $\eta : M \rightarrow M/N$, there exists a $\psi : M \rightarrow M$ rendering

$$\begin{array}{ccc} M & \xrightarrow{\pi} & A \\ {\scriptstyle\psi}\downarrow & & \downarrow{\scriptstyle\alpha} \\ M & \xrightarrow{\eta} & M/N \end{array}$$

commutative, i. e. $\eta\psi = \alpha\pi$. Define $M/N \rightarrow M$ as $\psi\rho\alpha^{-1}$; then $\eta\psi\rho\alpha^{-1} = \alpha\pi\rho\alpha^{-1}$ is the identity map of M/N. Hence the sequence $0 \rightarrow N \rightarrow M \xrightarrow{\eta} M/N \rightarrow 0$ splits.

LEMMA 5. — *Let N be a submodule of the quasi-projective module M such that there exists an epimorphism $\varepsilon : N \rightarrow M$. Then M is isomorphic to a direct summand of N.*

Write $K = \operatorname{Ker} \varepsilon$. Let $\bar{\varepsilon} : N/K \rightarrow M$ be the isomorphism induced by ε, α the injection $M \rightarrow M/K$ with $\alpha\bar{\varepsilon}$ the identity on N/K, and $\eta : M \rightarrow M/K$ the natural map. By quasi-projectivity, some $\psi : M \rightarrow M$ satisfies $\eta\psi = \alpha$ where $\psi(M) \subseteq \eta^{-1}(N/K) = N$. For $\psi\bar{\varepsilon} : N/K \rightarrow N$, $\eta\psi\bar{\varepsilon} = \alpha\bar{\varepsilon}$ acts identically on N/K, therefore $0 \rightarrow K \rightarrow N \xrightarrow{\eta} N/K \rightarrow 0$ is splitting.

Notice that lemma 5 can also be derived from a result of DE ROBERT [2]; it follows that $\operatorname{Hom}_R(M, N) \rightarrow \operatorname{Hom}_R(M, M)$ is epic whenever M is quasi-projective and $N \subseteq M$, and it suffices to look at a preimage of 1_M to obtain lemma 5.

By $E(M)$ we denote the ring of all R-endomorphisms of M.

LEMMA 6. — *If N is a submodule in a quasi-projective module M, then the cardinality of $E(M/N)$ does not exceed that of $E(M)$.*

To every $\alpha \in E(M/N)$ there exists a $\psi_\alpha \in E(M)$ such that $\eta\psi_\alpha = \alpha\eta$ where again $\eta : M \to M/N$ is the natural map. If $\alpha, \beta \in E(M/N)$ are distinct, then $\alpha\eta \neq \beta\eta$ (since η is epic), and hence $\psi_\alpha \neq \psi_\beta$ in $E(M)$.

We are now ready to prove our result (for the needed facts on abelian groups we refer to [1]) :

THEOREM. — *An abelian group A is quasi-projective if, and only if, it is :*

1° *free, or*

2° *a torsion group such that every p-component A_p is a direct sum of cyclic groups of the same order p^n.*

Free groups F are quasi-projective, so by lemma 2, the groups F/p^nF are likewise quasi-projective. By lemma 3, a direct sum of groups F/p^nF with different primes p is quasi-projective. Since F/p^nF is a direct sum of cyclic groups of order p^n, the sufficiency is evident.

Conversely, assume A is quasi-projective. If A is torsion, then by lemma 1, every A_p is quasi-projective. If A_p is not reduced, then it contains a summand of type $Z(p^\infty)$. By lemmas 1 and 4, every proper subgroup of $Z(p^\infty)$ must be a summand of $Z(p^\infty)$ which is absurd, thus A_p is reduced. It cannot have a summand of the form $Z(p^n) \oplus Z(p^m)$ with $n < m$, because this cannot be quasi-projective in view of the existence of an epimorphism $Z(p^m) \to Z(p^n)$ whose kernel is not a summand. Therefore, the basic subgroups B_p of A_p are direct sums of cyclic groups of the same orders p^n, and so $A_p = B_p$ (namely, B_p is now a summand of A_p, and A_p is reduced).

If A is torsion-free, then we distinguish two cases according as A has finite or infinite rank. If A is of finite rank r, then let F be a free subgroup of rank r in A. Now $E(A)$ is countable, hence $E(A/F)$ is at most countable (lemma 6). Since A/F is torsion, this can happen only if A/F is finite in which case A too is free. If A is of infinite rank, then let F be a free subgroup of A of the same rank as A. The existence of an epimorphism $F \to A$ and lemma 5 lead us to conclude that A is isomorphic to a summand of F and hence A is free.

Finally, we show that A can not be mixed. If T is the torsion part of A, then A/T is quasi-projective by lemma 2, and hence free by what has been proved, i. e. $A = T \oplus F$ with quasi-projective T and free F. If neither $T = 0$ nor $F = 0$, then there exist a cyclic direct summand $Z(p^n)$ of T and an epimorphism $\varepsilon : F \to Z(p^n)$ whose kernel is not a summand of F, in contradiction to lemma 4. This completes the proof.

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