TOSHIKO KOYAMA

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ON QUASI-CLOSED GROUPS AND TORSION COMPLETE GROUPS

BY

TOSHIKO KOYAMA.

1. Introduction.

Every group, in this paper, is an abelian p-group. We will observe some properties of abelian p-groups using topological methods. Notation and terminology follow FUCHS [1] except that we use the word " torsion complete " instead of " closed " following [4]. Let G be a p-group. Then we can introduce the p-adic topology in G. If G has no elements of infinite height, this topology is a metric topology.

Let G be a p-group without elements of infinite height. If every bounded Cauchy sequence of G has a limit in G with respect to the p-adic topology, G is called torsion complete. A torsion complete group G has following properties.

(I) Let $B = C_1 \oplus C_2$ be a basic subgroup of G. Then $G = C_1^- \oplus C_2^-$, where C_1^- and C_2^- are closures of C_1 and C_2 in G.

(II) Let H be a pure subgroup of G. Then H^- is a direct summand of G.

(III) Let H be a pure subgroup of G. Then H^- is again pure.

(III)′ Let H be a pure subgroup of G. $\left(\frac{G}{H}\right)^1$ is divisible.

(IV) (Strong Purification Property). For a given subgroup P of $G[p]$ and for a given pure subgroup H of G such that $H[p] \subset P$, there exists a pure subgroup K containing H such that $K[p] = P$.

After considering these properties a natural question arises : Is the reduced p-group which satisfies (I) or (II) necessarily torsion complete ? We will give an affirmative answer to this question. This gives rise to a nice characterization of torsion complete groups.

P. HILL and C. MEGIBBEN [2] called the reduced p-group which satisfies (III) quasi-closed group. They have showed an example which is quasi-closed but not torsion complete in [2]. They have also proved in [2] that a quasi-closed group which is not torsion complete is essentially indecomposable. We will show that properties (III), (III)′ and (IV) are equivalent. That is, a reduced p-group is quasi-closed if and only if G satisfies " Strong Purification Property ". Since unbounded direct sum of cyclic groups is neither essentially indecomposable, nor torsion complete, it is not quasi-closed. We will construct a pure subgroup H in unbounded direct sum of cyclic groups such that H^- is not pure.

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2. Topological Preliminaries.

Let G be a p-group and x be an element of G. $h(x)$ denotes the height of x. Set $d(x, y) = p^{-h(x-y)}$ for $x, y \in G$. d defines a pseudo-metric in G. Since d is invariant, G is a topological group with this pseudo-metric. This topology is called p-adic topology. If we assume the condition $G^1 = 0$, then d defines a metric in G. Let H be a subset of G, then we write H^- for the closure of H in G with respect to the p-adic topology of G.

LEMMA 1. — *$\mathcal{B} = \{p^n G, n = 0, 1, 2, \ldots\}$ is a local base at 0 for the p-adic topology of G. Hence $\{0\}^- = G^1$, the p-adic topology in a bounded group is discrete and the p-adic topology in a divisible group is trivial.*

LEMMA 2. — *If H is a pure subgroup of G, then the p-adic topology of H coincides with the relative topology, since $p^n H = H \cap p^n G$. Hence we need not distinguish the relative topology and the p-adic topology in H whenever H is pure in G.*

LEMMA 3. — *Let $G = \sum_{i=1}^{m} G_i$. Then the p-adic topology of G is the product of p-adic topologies in G_i's. Hence a direct summand of G is closed in G whenever $G^1 = 0$.*

LEMMA 4. — *A direct sum of a finite number of torsion complete groups is torsion complete and a direct summand of a torsion complete group is torsion complete.*

LEMMA 5. — *Let G be a torsion complete group and let $B = C_1 \oplus C_2$ be a basic subgroup of G. Then $G = C_1^- \oplus C_2^-$. Hence, if H is a pure subgroup of G, H^- is a direct summand of G.*

LEMMA 6. — $G[p^n] = \{x \in G : p^n x = 0\}$ $(n = 1, 2, 3, \ldots)$ *is closed in G with respect to any compatible Hausdorff topology in G.*

Proof. — $f(x) = p^n x$ is continuous in any topological group, $\{0\}$ is closed in any Hausdorff topological group and $G[p^n]$ is the inverse image of o by $f(x)$. Hence $G[p^n]$ is closed.

LEMMA 7. — *Let H be a subgroup of G. Then* $\left(\frac{G}{H}\right)^1 = \frac{H^-}{H}$.

Proof. — Let φ be a canonical homomorphism : $G \to G/H$. $h(\varphi(x)) = \infty$ if and only if $(x + p^n G) \cap H \neq \Phi$ for all n. That is, $x \in H^-$.

LEMMA 8. — *A subgroup H is dense in G if and only if G/H is divisible.*

LEMMA 9. — *Let H be a pure subgroup of G. Then H^- is pure if and only if* $\left(\frac{G}{H}\right)^1$ *is divisible* (*i. e. reduced part of* $\frac{G}{H}$ *has no elements of infinite height*).

LEMMA 10. — *Let G be a p-group without elements of infinite height and let H be a pure subgroup of G. Then*

(1) $(H[p])^- = H^-[p]$;

(2) $H[p] = H^-[p]$ *if and only if* $H = H^-$, *i. e.* $H[p]$ *is closed if and only if H is closed.*

(3) $H^-[p] = G[p]$ *if and only if* $H^- = G$, *i. e.* $H[p]$ *is dense in* $G[p]$ *if and only if H is dense in G.*

Proof.

1. $(H[p])^- \subset (G[p])^- \cap H^- = G[p] \cap H^- = H^-[p]$, by Lemma 6.

Suppose $x \in H^-[p]$. $H \cap (x + p^n G) \neq \Phi$ for all n and $px = 0$. That is, there exist $h_n \in H$ and $g_n \in G$ such that

$$h_n = x + p^n g_n \qquad \text{and} \qquad ph_n = p^{n+1} g_n.$$

Since H is pure, there exists $h'_n \in H$ such that $ph_n = p^{n+1} h'_n$.

$$h_n - p^n h'_n = x + p^n(g_n - h'_n), \qquad \text{where} \quad h_n - p^n h'_n \in H[p].$$

That is, $H[p] \cap (x + p^n G) \neq \Phi$ for all n. Hence $x \in (H[p])^-$.

2. H is pure in H^-, since H is pure in G. By Lemma 12, KAPLANSKY [5],

$$H[p] = H^-[p] \quad \text{implies} \quad H = H^-.$$

3. Suppose $H^-[p] = G[p]$. It suffices to show that $pg \in H^-$ implies $g \in H^-$.

If $pg \in H^-$, then $(pg + p^n G) \cap H \neq \Phi$ for all n. Write

$$h_n = pg + p^n g_n, \qquad \text{where} \quad h_n \in H, \quad g_n \in G.$$

Since H is pure, there exists $h'_n \in H$ such that

$$h_n = ph'_n, \quad g + p^{n-1} g_n - h'_n \in G[p].$$

Since

$$G[p] = H^-[p], \qquad g + p^{n-1} g_n \in H^-,$$

i. e. $(g + p^{n-1} G) \cap H^- \neq \Phi$. Therefore $g \in H^-$.

3. **Main Results.**

The following is a characterization of quasi-closed groups.

THEOREM 1. — *Let G be a reduced p-group. Following three conditions are equivalent :*

(III) *Let H be a pure subgroup of G, then H^- is again pure;*

(III)′ *Let H be a pure subgroup of G, then $\left(\frac{G}{H}\right)^1$ is divisible;*

(IV) *Strong Purification Property.* (*See* Introduction.)

Proof. — (III) $\Leftrightarrow$ (III)′ by Lemma 9.

(III)′ $\Rightarrow$ (IV). Since G is reduced, $G^1 = \{o\}^- = o$ by the condition (III). By Zorn's Lemma there exists a maximal pure subgroup K such that $H \subset K$ and $K[p] \subset P$. Suppose $x \in P$ and $x \notin K[p]$. Let φ be a canonical homomorphism $G \rightarrow G/K$. $\varphi(x) \in \frac{G}{K}[p]$ and $\varphi(x) \neq o$. Suppose $h(\varphi(x)) = \infty$. Since $\left(\frac{G}{K}\right)^1$ is divisible, there exists K' containing K such that

$$\frac{K'}{K} \cong Z(p^\infty) \qquad \text{and} \qquad \frac{K'}{K}[p] = \langle \varphi(x) \rangle.$$

K' is pure by Lemma 2, KAPLANSKY [5].

$\varphi(K'[p]) = (\varphi K')[p] = \langle \varphi(x) \rangle$ by Lemma 1, KAPLANSKY [5]. Hence

$$K'[p] = \langle x \rangle \oplus K[p] \subset P.$$

This contradicts to the maximality of K. If $h(\varphi(x)) = n < \infty$. Then we can find K' such that $\frac{K'}{K} = \langle \bar{y} \rangle$, where $\varphi(x) = p^n \bar{y}$. Therefore $K[p] = P$.

(IV)$\Rightarrow$(III). Since G satisfies socle purification property, $G^1 = 0$. Let H be a pure subgroup of G. By the strong purification property, there exists a pure subgroup K such that $H \subset K$ and $K[p] = H^-[p]$. By Lemma 10, (1) and (2), K is closed. Hence $H^- \subset K$. Since K is pure, we can apply Lemma 10, (3). Therefore $H^- = K$.

Definition. — Let $B = \sum_{n=1}^{\infty} \langle x_n \rangle$. If $o(x_n) = p^n$, B is called a standard group. If $\{o(x_n)\}$ is a strictly increasing sequence, B is called a substandard group.

Theorem 2. — *Let B be a substandard group. There exists a pure subgroup H of B such that $\left(\frac{B}{H}\right)^1 \simeq C(p)$, i. e. H^- is not pure.*

Remark. — The fact that B is not quasi-closed follows immediately from Theorem 4 of [2], since B can be decomposed into a direct sum of two unbounded components.

Proof. — Let $B = \sum_{i=1}^{\infty} \langle x_i \rangle$, $o(x_i) = p^{n_i}$, $1 \leq n_1 < n_2 < n_3 \ldots$. Set

$$y_i = x_{2i} + p^{n_{2i+1} - n_{2i} + 1} x_{2i+1} - p^{n_{2i+2} - n_{2i}} x_{2i+2}.$$

Then $o(y_i) = p^{n_{2i}}$. Let H be a subgroup of B generated by $\{y_i : i = 1, 2, \ldots\}$. This H is a desired subgroup. We can verify that H is pure and $H^- = \langle p^{n_2 - 1} x_2 \rangle \oplus H$.

Following theorem gives us a characterization of torsion complete groups. There is a direct proof for the corollary to this theorem in [6].

Theorem 3. — *A reduced p-group G is torsion complete if and only if H^- is a direct summand of G whenever H is a pure subgroup of G.*

Proof. — The necessity follows from Lemma 5.

Suppose that H^- is a direct summand of G whenever H is pure. G is quasi-closed. Let $B = \sum_{n=1}^{\infty} B_n$, where $B_n \simeq \sum C(p^n)$ be a basic subgroup of G. We can decompose B into a direct sum of two unbounded components

$$C_1 = \sum_{n \in N_1} B_n \quad \text{and} \quad C_2 = \sum_{n \in N_2} B_n, \quad \text{where} \quad N_1 \cap N_2 = \Phi.$$

Set $G = \bar{C_1} \oplus K$. K must be unbounded since the basic subgroup of K is isomorphic to C_2. G is torsion complete by Theorem 4, Hill and Megibben [2].

COROLLARY. — *A reduced p-group G is torsion complete if and only if G satisfies following condition :*

(I) *If $B = C_1 \oplus C_2$ is any basic subgroup of G and its decomposition, then $G = C_1^- \oplus C_2^-$, where C_1^- and C_2^- are closures of C_1 and C_2 in G.*

Remark. — The exercise 16 in KAPLANSKY [5] shows us how a standard group does not satisfy the property (I) in above corollary.

Let $B = \sum_{i=1}^{\infty} \langle x_i \rangle$, where $o(x_i) = p^i$ and let

$$S_0 = \sum_{i=1}^{\infty} \langle y_{2i-1} \rangle \quad \text{and} \quad S_e = \sum_{i=1}^{\infty} \langle y_{2i} \rangle, \quad \text{where} \quad y_i = x_i - px_{i+1}.$$

Then $S = S_0 \oplus S_e$ is a basic subgroup of B. On the other hand S_0 and S_e are the direct summands of B. Hence $S_0^- \oplus S_e^- = S_0 \oplus S_e \neq B$.

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Mrs. Toshiko KOYAMA,
Department of Mathematics,
Ochanomizu University,
Ootsuka-machi, Bunkyo-ku,
Tokyo (Japon).