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ON ISOTYPE SUBGROUPS OF ABELIAN GROUPS ;

BY

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In his book *Abelian groups*, L. FUCHS asks the following question. Let G be a p-group and H be a subgroup without elements of infinite height. Under what conditions can H be embedded in a pure subgroup of the same power and again without elements of infinite height? (*See* [2], p. 96.) This question has been answered by Charles [1] and IRWIN [3]. Irwin's solution was effected by showing that any subgroup maximal with respect to disjointness from the subgroup of elements of infinite height is pure. For p-groups, the subgroups of element of infinite height is $p^{\omega} G$. Now for any Abelian group G, any prime p, and any ordinal α, one may define $p^{\alpha} G$, and this suggests the following problem. Is any subgroup of G maximal with respect to disjointness from $p^{\alpha} G$ pure in G? Or, more generally, does any such subgroup H of G have the property that $H \cap p^{\beta} G = p^{\beta} H$ for all ordinals β? That is to say, is H p-isotype in G? We will show that indeed any such H is p-isotype, and we will give a partial solution to the problem of determining whether any two such H's are isomorphic. The foregoing considerations will lead to the solution of a more general version of the above mentioned problem of L. FUCHS.

All groups considered in this paper will be Abelian.

DEFINITION 1. — Let G be a group and p be a prime. Define $p^0 G = G$. If $p^{\beta} G$ is defined for all ordinals $\beta < \alpha$, then define $p^{\alpha} G = \bigcap_{\beta < \alpha} p^{\beta} G$ when α is a limit ordinal. If $\alpha = \delta + 1$ for some ordinal δ, let $p^{\alpha} G = p(p^{\delta} G)$.

Thus we have defined $p^{\alpha} G$ for all ordinals α, and clearly the $p^{\alpha} G$'s form a chain of fully invariant subgroups of G.

Definition 2. — Let p be a prime and $g \in G$. The *p-height* $H_p(g)$ of g is the ordinal α such that $g \in p^\alpha G$ and $g \notin p^{\alpha+1} G$. If no such ordinal α exists, then $H_p(g) = \infty$, where the symbol ∞ is considered larger than any ordinal. Let α be an ordinal or ∞. Then a subgroup H of G is p^α*-pure* in G if and only if $H \cap p^\beta G = p^\beta H$ for all ordinals $\beta \leq \alpha$; H is α*-pure* in G if and only if H is p^α-pure in G for all primes p. A subgroup H is *p-isotype* in G if and only if H is p^∞-pure in G. The subgroup H is *isotype* in G if and only if H is p-isotype in G for all primes p.

It follows easily from the définitions that the properties of being isotype, α-pure, or p^α-pure are transitive. Moreover, the union of an ascending chain of subgroups with one of these properties is a subgroup with that property.

It is easy to see that there are groups in which not every pure subgroup is isotype. In fact, there exist reduced p-groups G such that $|p^\beta G| = \aleph_0$ and $|\beta| \geq 2^{\aleph_0}$. (*See* [2], p. 131, Theorem 38.2 for the existence of such a G.) Embed $p^\beta G$ in a pure subgroup K of G with $|K| = \aleph_0$. Clearly K is not isotype since $p^\beta K = 0$ and $K \cap p^\beta G = p^\beta G \neq 0$.

We now state and prove a few facts which will be useful in what follows, and which illustrate the relation between the above definitions and the ordinary notions of purity and height.

Lemma 1. — For a positive integer n, let $n = \prod_{i=1}^{r} p_i^{s_i}$ be its prime decomposition. Then for any group G, $nG = \bigcap_{i=1}^{r} p_i^{s_i} G$.

Proof. — Let $T = \bigcap p_i^{s_i} G$. Clearly $nG \subseteq T$. Now let $g \in T$. For $n_i = n/p_i^{s_i}$, there exist integers a_i with $\sum a_i n_i = 1$. But $g \in T$ yields $g = p_i^{s_i} g_i$, $i = 1, \ldots, r$. Hence

$$g = \sum a_i n_i g = \sum a_i n_i p_i^{s_i} g_i = \sum a_i n g_i = n \sum a_i g_i \in nG.$$

Hence $nG = T$, and the proof is complete.

Corollary 1. — *A subgroup H of a group G is pure in G if and only if H is ω-pure.*

Proof. — Suppose H is pure in G. In particular, $H \cap p^m G = p^m H$ for each prime p and non-negative integer m. Now

$$H \cap p^\omega G = H \cap \left(\bigcap_{k<\omega} p^k G \right) = \bigcap_{k<\omega} (H \cap p^k G) = \bigcap_{k<\omega} p^k H = p^\omega H.$$

Hence H is ω-pure. Next suppose H is ω-pure, and n is a positive integer. Then

$$H \cap nG = H \cap \left(\left(\prod p_i^{s_i}\right) G\right) = H \cap \left(\bigcap p_i^{s_i} G\right)$$
$$= \bigcap (H \cap p_i^{s_i} G) = \bigcap p_i^{s_i} H = nH$$

by Lemma 1.

The following definition is standard.

DEFINITION 3. — The subgroup $G^1 = \bigcap_{n<\omega} nG$ is the *subgroup of elements of infinite height* in G.

We are now in a position to prove the following useful

COROLLARY 2. — *Let P be the set of all primes. Then* $G^1 = \bigcap_p p^\omega G$.

PROOF. — Set $T = \bigcap_p p^\omega G$. Then from $p^\omega G = \bigcap_n p^n G$ for each $p \in P$, it follows that $p^\omega G \supset \bigcap_n nG$ for each $p \in P$, and hence $T \supset G^1$. Now for each n we have $nG = \bigcap p_i^{s_i} G \supset T$. Hence $G^1 \supset T$, whence $G^1 = T$.

This corollary shows that the subgroup G^1 of elements of infinite height in G is the set of elements of infinite p-height for each prime p. The following theorem and corollary are generalizations of Kaplansky's Lemma 7 ([5], p. 20)

THEOREM 1. — *Let H be a subgroup of a p-group G, and let α be a limit ordinal or ∞. Then H is p^α-pure in G if and only if whenever $\beta < \alpha$, $h \in H[p]$, and the p-height in G of h is $\geq \beta$, then the p-height in H of h is $\geq \beta$.*

PROOF. — If H is p^α-pure, then clearly the elements in $H[p]$ have the desired property. To prove the converse, it must be established that $H \cap p^\delta G = p^\delta H$ for all $\delta \leq \alpha$. Obviously $H \cap p^\delta G \supset p^\delta H$. Let $P(n)$ be the statement : For $\beta < \alpha$, the elements in H of exponent $\leq n$ have p-height $\geq \beta$ in H if they have p-height $\geq \beta$ in G. We will prove $P(n)$ is true for all n by induction and consequently have that $H \cap p^\delta G \subseteq p^\delta H$ for all $\delta < \alpha$. Now $P(1)$ is true by hypothesis. Assume $P(n)$ holds, and let $h \in H$ with $o(h) = p^{n+1}$, and suppose the p-height of h is $\geq \beta$ in G. Then ph has exponent n and p-height $\geq \beta + 1$ in G. Since $\beta + 1 < \alpha$, our induction hypothesis yields $ph = ph_\beta$ with $h_\beta \in p^\beta H$. Hence $(h - h_\beta) \in H[p]$, has p-height $\geq \beta$ in G, and so p-height $\geq \beta$ in H. Therefore $H \cap p^\delta G \subseteq p^\delta H$ for all $\delta < \alpha$ and since α is a limit ordinal, this holds for all $\delta \leq \alpha$. Thus H is p^α-pure in G.

COROLLARY 3. — *Let H be a subgroup of a p-group G. Then H is isotype in G if and only if the elements in $H[p]$ have the same p-height in H as in G.*

PROOF. — Since G is a p-group, we have $qH = H$ for all $q \neq p$, and hence H is q-isotype for all $q \neq p$. To get H p-isotype, let α be ∞ in Theorem 1.

We proceed now to our main results and begin with the following definition :

DEFINITION 4. — Let K and L be subgroups of G. Then H is L-high in K if and only if H is a subgroup of K maximal with respect to the property that $H \cap L = 0$. A *high subgroup H* of G is a subgroup maximal with respect to the property $H \cap G^1 = 0$. (*See* [3].)

The principal result of this paper is the following theorem :

THEOREM 2. — *Let G be a group, let p be a prime, let α be an ordinal, let K be a subgroup of $p^\alpha G$, and let H be K-high in G. Then H is $p^{\alpha+1}$-pure in G, and $p^\beta H$ is K-high in $p^\beta G$ for all ordinals $\beta \leqq \alpha$.*

PROOF. — To show that H is $p^{\alpha+1}$-pure in G we need to establish that $H \cap p^\beta G = p^\beta H$ for all $\beta \leqq \alpha + 1$. We induct on β, and if $\beta = 0$, the equality is trivial. Now suppose $0 < \beta \leqq \alpha + 1$, and suppose the equality holds for all ordinals less than β. If β is a limit ordinal, then

$$H \cap p^\beta G = H \cap \left(\bigcap_{\delta<\beta} p^\delta G \right) = \bigcap_{\delta<\beta} (H \cap p^\delta G) = \bigcap_{\delta<\beta} p^\delta H = p^\beta H.$$

Next suppose β is not a limit ordinal. Then there is an ordinal δ such that $\beta = \delta + 1$. Then

$$p^\beta H \subseteq H \cap p^\beta G = H \cap p(p^\delta G).$$

Let $h = pg_\delta$ with $h \in H$ and $g_\delta \in p^\delta G$. If $g_\delta \in H$, then

$$g_\delta \in H \cap p^\delta G = p^\delta H,$$

and

$$h = pg_\delta \in p(p^\delta H) = p^\beta H.$$

So suppose $g_\delta \notin H$. Since H is K-high in G and $K \subseteq p^\alpha G$, we have

$$h_1 + ng_\delta = k \neq 0,$$

where $h_1 \in H$, $k \in K$, and n an integer. Clearly $(n, p) = 1$, and $k \in p^\alpha G$. Since $\delta \leqq \alpha$, we have $h_1 \in p^\delta G$. The induction hypothesis yields $h_1 \in p^\delta H$. Now

$$ph_1 + npg_\delta = ph_1 + nh = pk = 0.$$

Therefore

$$nh = -ph_1 \in p(p^\delta H) = p^\beta H.$$

Also $ph \in p^{\beta}H$ since $h \in p^{\beta}G \subseteq p^{\delta}G$, consequently $h \in p^{\delta}H$. There exist integers a and b such that $an + bp = 1$. Thus

$$anh + bph = h \in p^{\beta}H.$$

Hence $H \cap p^{\beta}G = p^{\beta}H$ and H is $p^{\alpha+1}$-pure in G as stated.

It remains to show that $p^{\beta}H$ is K-high in $p^{\beta}G$ for $\beta \leqq \alpha$. Suppose this is not the case. Then there exists $g_{\beta} \in p^{\beta}G$, $g_{\beta} \notin p^{\beta}H$ such that the subgroup generated by $p^{\beta}H$ and g_{β} is disjoint from K. If $g_{\beta} \in H$, then since H is $p^{\alpha+1}$-pure in G and $\beta \leqq \alpha$, $g_{\beta} \in p^{\beta}H$ contrary to the choice of g_{β}. Hence $g_{\beta} \notin H$. Since H is K-high in G, we have $h + ng_{\beta} = k \neq 0$, where $h \in H$ and $k \in K \subseteq p^{\alpha}G$. From $\beta \leqq \alpha$ we have that $h \in p^{\beta}G$, and hence $h \in p^{\beta}H$ by $p^{\alpha+1}$-purity of H in G. But this together with the equation $h + ng_{\beta} = k \neq 0$ contradicts the fact that the subgroup generated by $p^{\beta}H$ and g_{β} is disjoint from K. This concludes the proof.

As an easy consequence of Theorem 2 we obtain a generalization of Irwin's result mentioned above.

Corollary 4. — *Let K be any subgroup of G^1 and H be K-high in G. Then H is $(\omega+1)$-pure (and hence pure) in G. In particular, if H is high in G, then H is pure in G.*

Proof.— Since $K \subseteq p^{\omega}G$ for each prime p, H is $p^{\omega+1}$-pure for each p. Hence H is $(\omega+1)$-pure.

Another result along these lines is

Corollary. 5. — *Let H be $p^{\alpha}G$-high in G. Then H is p-isotype in G, and $p^{\beta}H$ is $p^{\alpha}G$-high in $p^{\beta}G$ for all β.*

Proof. — Since H is $p^{\alpha}G$-high in G, then $H \cap p^{\beta}G = p^{\beta}H = 0$ for all $\beta \geqq \alpha$, and Theorem 2 yields H is p-isotype. For ordinals $\beta \geqq \alpha$, the only $p^{\alpha}G$-high subgroup in $p^{\beta}G$ is 0 and $p^{\beta}H = 0$ for such β. By Theorem 2, $p^{\beta}H$ is $p^{\alpha}G$-high in p^{β}G for all β.

Lemma 3. — For any group G and any ordinals α and β, $p^{\alpha}(p^{\beta}G) = p^{\beta+\alpha}G$.

Proof. — Induct on α. The assertion is true for $\alpha = 0$. Now assume $\alpha > 0$ and that the assertion is true for all ordinals $\delta < \alpha$. Suppose α is a limit ordinal. Then

$$\begin{aligned} p^{\alpha}(p^{\beta}G) &= \bigcap_{\delta<\alpha} p^{\delta}(p^{\beta}G) \\ &= \bigcap_{\delta<\alpha}(p^{\beta+\delta}G) = \bigcap_{\beta \leqq \lambda < \beta+\alpha}(p^{\lambda}G) = \bigcap_{\lambda<\beta+\alpha}(p^{\lambda}G) = p^{\beta+\alpha}G \end{aligned}$$

since $\beta+\alpha$ is a limit ordinal. Suppose $\alpha=\delta+1$. Then

$$p^{\alpha}(p^{\beta}G)=p(p^{\delta}(p^{\beta}G))=p(p^{\beta+\delta}G)=p^{(\beta+\delta)+1}G=p^{\beta+(\delta+1)}G=p^{\beta+\alpha}G.$$

As a simple application of Lemma 3 we have

COROLLARY 6. — *Let H be $p^{\alpha}G$-high in G. Then $p^{\beta}H$ is p-isotype in $p^{\beta}G$ for all β.*

PROOF. — By Corollary 5, $p^{\beta}H$ is $p^{\alpha}G$-high in $p^{\beta}G$ for all β. If $\alpha\leq\beta$, then $p^{\beta}H=0$ and hence is isotype. If $\beta<\alpha$, then $\alpha=\beta+\delta$ for some δ. By Lemma 3 we have that $p^{\beta}H$ is $p^{\alpha}G=p^{\beta+\delta}G=p^{\delta}(p^{\beta}G)$-high in $p^{\beta}G$, and Corollary 5 completes the proof.

Making certain provisions about G, we are able to say when $p^{\alpha}G$-high subgroups are q-isotype for any prime q. In this connection we have

THEOREM 3. — *Let H be $p^{\alpha}G$-high in G, and suppose $p^{\alpha}G$ has no elements of order q, where q is a prime Then H is q-isotype in G.*

PROOF. — If $q=p$, the assertion follows from Corollary 5. Now assume $q\neq p$. We show that $H\cap q^{\beta}G=q^{\beta}H$ for all ordinals β. For this purpose it suffices to verify that $H\cap q^{\beta}G\subseteq q^{\beta}H$. For $\beta=0$ this is trivial. Let $\beta>0$, and suppose the inequality holds for all ordinals $\delta<\beta$. If β is a limit ordinal, then

$$H\cap q^{\beta}G=H\cap\left(\bigcap_{\delta<\beta}(q^{\delta}G)\right)=\bigcap_{\delta<\beta}(H\cap q^{\delta}G)=\bigcap_{\delta<\beta}(q^{\delta}H)=q^{\beta}H.$$

Next suppose $\beta=\delta+1$. Let $h\in H\cap q^{\beta}G=H\cap q(q^{\delta}G)$. Then $h=qg_{\delta}$, where $g_{\delta}\in q^{\delta}G$. By the induction hypothesis, if $g_{\delta}\in H$, then $g_{\delta}\in q^{\delta}H$ and $h=qg_{\delta}\in q(q^{\delta}H)=q^{\beta}H$. Now assume $g_{\delta}\notin H$. Then since H is $p^{\alpha}G$-high in G, we have $h_1+ng_{\delta}=g_{\alpha}\neq 0$, where $h_1\in H$, $g_{\alpha}\in p^{\alpha}G$, and n is an integer. Thus $qh_1+nqg_{\delta}=qh_1+nh=qg_{\alpha}\in H$. Therefore $qg_{\alpha}=0$, and since $p^{\alpha}G$ has no elements of order q, $g_{\alpha}=0$. This contradiction establishes the theorem.

The following two corollaries follow immediately from Theorem 3.

COROLARY 7. — *Let H be $p^{\alpha}G$-high in G, and suppose $p^{\alpha}G$ is torsion-fre Then H is isotype in G, and in particular H is pure in G.*

COROLLARY 8. — *Let H be $p^{\alpha}G$-high in G, and suppose $p^{\alpha}G$ is a p-group. Then H is isotype in G. In particular, H is pure in G.*

If G is a p-group, then the subgroup G^1 of elements of infinite height in G is $p^{\omega}G$. Thus Corollary 8 implies that a high subgroup H of a p-group is isotype, and consequently pure. The answer to Fuchs' question is readily obtained from the purity of H. (*See* [3].) However, we proceed now to derive more general results.

THEOREM 4. — *Let A be a subgroup of G, and let S be a non-void set of primes. For each $p \in S$, let α_p be an ordinal. Suppose that for each $a \in A$, $a \neq 0$, there exists $p \in S$ such that $H_p(a) < \alpha_p$. Then A is contained in a subgroup H of G such that H is p^{α_p+1}-pure in G for each $p \in S$, and for each $h \in H$, $h \neq 0$, there exists $p \in S$ such that $H_p(h) < \alpha_p$.*

PROOF. — Since $A \cap \left(\bigcap_{p \in S} p^{\alpha_p} G \right) = 0$, A is contained in a $\bigcap_{p \in S} p^{\alpha_p}$ G-high subgroup H of G. Now the proof follows immediatley from Theorem 2.

The following result generalizes a theorem of Erdélyi ([2], p. 81).

COROLLARY 9. — *Let H be a subgroup of G, let p be a prime, and let α be an ordinal. Suppose that for each nonzero $h \in H_p$, $H_p(h) < \alpha$. Then H is contained in a p-isotype subgroup A of G such that for each nonzero $a \in A$, $H_p(a) < \alpha$.*

PROOF. — This proof is analogous to the proof of Theorem 4, using Corollary 5.

COROLLARY 10. — *Let G be a p-group, and let A be a subgroup of G such that A has no nonzero elements of infinite height. Then A is contained in an isotype subgroup H of G such that H has no nonzero elements of infinite height.*

PROOF. — The proof is similar to the proof of Corollary 9, using Corollary 8.

COROLLARY 11. — *Let A be a subgroup of G with no elements of infinite height; i. e., $A \cap G^1 = 0$. Then A is contained in a pure subgroup K of G such that K has no elements of infinite height and such that $|K| \leq \aleph_0 |A|$.*

PROOF. — The subgroup A is contained in a high subgronp H of G, and H is pure in G by Corollary 4. Now A can be embedded in a pure subgroup K of H such that $|K| \leq \aleph_0 |A|$. (*See* [2], p. 78.) Clearly K has no elements of infinite height and is pure in G.

We will now discuss the question of how isomorphic the $p^\alpha G$-high subgroups are. In particular we will show that if G is a countable p-group, then any two $p^\alpha G$-high subgroups of G are isomorphic. When any two such subgroups of an arbitrary group G are isomorphic is not known. However, we will state and prove an interesting theorem concerning the relationship of the Ulm invariants of these subgroups to those of G when G is a p-group.

LEMMA 4. — Let L be a subgroup of a group G with H and K both L-high subgroups of G. Then

$$((H \oplus L)/L)[p] = ((K \oplus L)/L)[p]$$

for each prime p.

Proof. — For $h \in H$ we have that $o(h+L) = p$ if and only if $o(h) = p$. If $h \in (H \cap K)[p]$, then $h+L$ is in $((K \oplus L)/L)[p]$. Suppose $h \in H[p] \setminus K \cap H$. Then there exists $k \in K$, $x \in L$ with $h - k = x$, whence $o(k) = p$. Thus

$$h + L = k + L \in ((K \oplus L)/L)[p];$$

and since h was arbitrary, we have by symmetry that

$$((H \oplus \mathrm{L})/L([p] = ((K \oplus L)/L)[p]$$

as stated.

Lemma 5. — Let H and K be $p^{\beta} G$-high in a reduced p-group G. Then $|H| = |K|$.

Proof. — If $p^{\beta} G = 0$, $H = K$, When β is finite, then $H \cong K$. (*See* [2], p. 99 and 104). When β is infinite and $p^{\beta} G \neq 0$, embed G in a divisible hull E of G. (A divisible hull of G is a minimal divisible group containing G.) Then $r(H) = r(E/D) = r(K)$, where D is a divisible hull of $p^{\beta} G$ in E. That $|H| = |K|$ follows now from easy set theoretic considerations.

Lemma 6. — Let H be $p^{\beta} G$-high in G. Then for each ordinal α we have

$$(p^{\alpha} H \oplus p^{\beta} G)/p^{\beta} G = p^{\alpha}((H \oplus p^{\beta} G)/p^{\beta} G).$$

Proof. — If $\alpha \geq \beta$, then both sides are zero. We prove the assertion for $\alpha < \beta$ by induction on α. So assume the equation holds for all ordinals $\delta < \alpha$. (If $\alpha = 0$, then the equality is trivial.) If $\alpha = \delta + 1$, then

$$\begin{aligned}(p^{\alpha} H \oplus p^{\beta} G)/p^{\beta} G &= (p(p^{\delta} H) \oplus p^{\beta} G)/p^{\beta} G \\ &= p((p^{\delta} H \oplus p^{\beta} G)/p^{\beta} G) \\ &= p(p^{\delta}((H \oplus p^{\beta} G)/p^{\beta} G)) = p^{\alpha}((H \oplus p^{\beta} G)/p^{\beta} G).\end{aligned}$$

Now assume α is a limit ordinal. Set

$$L = \left(\left(\bigcap_{\delta < \alpha} p^{\delta} H\right) \oplus p^{\beta} G\right)/p^{\beta} G \quad \text{and} \quad R = \bigcap_{\delta < \alpha} p^{\delta}((H \oplus p^{\beta} G/p^{\beta} G).$$

Since α is limit ordinal it suffices to prove $L = R$. Clearly $L \subseteq R$. Now let $h + p^{\beta} G \in R$. Then there exists $h_{\delta} \in p^{\delta} H$ such that $h + p^{\beta} G = h_{\delta} + p^{\beta} G$ for each $\delta < \alpha$. This means that for each $\delta < \alpha$ we have $h = h_{\delta} + g_{\beta\delta}$ for some $g_{\beta\delta} \in p^{\beta} G$. Thus since $\alpha < \beta$ and H is isotype, we have $h \in p^{\delta} H$ for each $\delta < \alpha$. Hence $h \in \bigcap_{\delta < \alpha} p^{\delta} H$, and $h + p^{\beta} G \in L$. This concludes the proof.

Corollary 12. — *Let H and K be $p^\beta G$-high in G. Then for each ordinal α we have*

$$(p^\alpha((H \oplus p^\beta G)/p^\beta G)[p] = (p^\alpha((K \oplus p^\beta G)/p^\beta G)[p].$$

Proof. — This follows from Lemma 6, the fact that $p^\alpha H$ and $p^\alpha K$ are $p^\beta G$ high in $p^\alpha G$, and Lemma 4.

Theorem 5. — *Let H and K be $p^\beta G$-high in a p-group G. Then H and K have the same Ulm invariants (as defined by* Kaplansky *in [5]). Moreover for all $\alpha < \beta$, the α-th Ulm invariant of H is the same as the α-th Ulm invariant of G.*

Proof. — First observe that $H \cong (H \oplus p^\beta G/p^\beta G) = \tilde{H}$, and similarly $K \cong \tilde{K}$. We will show that $\tilde{H}$ and $\tilde{K}$ have the same Ulm invariants. From Corollary 12 we have for each ordinal α that

$$(p^\alpha((H \oplus p^\beta G)/p^\beta G))[p] = (p^\alpha((K \oplus p^\beta G)/p^\beta G))[p]$$

so that

$$((p^\alpha \tilde{H})[p])/(p^{\alpha+1} \tilde{H})[p] = ((p^\alpha \tilde{K})[p])/(p^{\alpha+1} \tilde{K})[p].$$

This shows that H and K have the same Ulm invariants. To prove the second part of the theorem notice that for $\alpha < \beta$ we have

$$\begin{aligned}((p^\alpha G)[p])/(p^{\alpha+1} G)[p] &= ((p^\alpha H)[p] \oplus (p^\beta G)[p])/((p^{\alpha+1} H)[p] \oplus (p^\beta G))[p] \\ &\cong (p^\alpha H)[p]/(p^{\alpha+1} H)[p].\end{aligned}$$

The equality follows from Corollary 5 and the fact that $\alpha < \beta$. The isomorphism is the natural one.

As an easy application of Theorem 5 we have

Theorem 6. *Let H and K be $p^\beta G$-high in G, and let G be a p-group. If H is countable, then $H \cong K$. Moreover if H and K are both direct sums of countable groups, then $H \cong K$.*

Proof. — Clearly H and K are reduced. For the first part, $|H| = |K| = \aleph_0$ by Lemma 5. Hence by Theorem 5 and Ulm's theorem, $H \cong K$. If H and K are both direct sums of countable groups, we have by a theorem of Kolettis (*see* [6]) that $H \cong K$.

We conclude with a corollary to Theorem 5.

Theorem 7. — *Let G be a group of type β. (G is a p-group.) Then for each ordinal $\alpha \leq \beta$, there exists an isotype subgroup H of G such that the first α Ulm invariants of G coincide with the Ulm invariants of H.*

Proof. — Let H be $p^\alpha G$-high in G and apply Theorem 5

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