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David MARÍN & Jean-François MATTEI

Monodromy and topological classification of germs of holomorphic foliations

MONODROMY AND TOPOLOGICAL CLASSIFICATION OF GERMS OF HOLOMORPHIC FOLIATIONS

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ABSTRACT. – We give a complete topological classification of germs of holomorphic foliations in the plane under rather generic conditions. The key point is the introduction of a new topological invariant called monodromy representation. This monodromy contains all the relevant dynamical information, in particular the projective holonomy representations whose topological invariance was conjectured in the eighties by Cerveau and Sad and is proved here under mild hypotheses.

RÉSUMÉ. – Nous donnons une classification topologique complète des germes de feuilletages holomorphes dans le plan, sous des conditions de type plutôt générique. Le point-clé est l'introduction d'un nouvel invariant topologique appelé représentation de monodromie. Cette monodromie contient toutes les informations dynamiques pertinentes, en particulier les représentations d'holonomie projective dont l'invariance topologique a été conjecturée dans les années quatre-vingt par Cerveau et Sad et est prouvée ici sous des hypothèses faibles.

1. Introduction

The objective of this paper is to provide a complete topological classification of germs of singular non-dicritical holomorphic foliations $\mathscr{F}$ at $(0,0) \in \mathbb{C}^2$ under very generic conditions. To do this we introduce a new topological invariant which is a representation of the fundamental group of the complement of the separatrix curve into a suitable automorphism group. We shall call this representation the *monodromy of the foliation germ*.

In fact, the motivation of this work was the following conjecture of D. Cerveau and P. Sad in 1986, cf. [3, page 246]. Consider two germs of foliations defined by germs of differential holomorphic 1-forms ω and ω' at $(0,0) \in \mathbb{C}^2$.

CONJECTURE 1.1 (Cerveau-Sad). – *If ω and ω' are topologically conjugate and if ω is a generalized curve, then their respective projective holonomy representations are conjugate.*

It was given in two forms, each of them with natural generic hypothesis concerning the germ of the foliation $\mathcal{F}$ along the *exceptional divisor* $\mathcal{E}_{\mathcal{F}} := E_{\mathcal{F}}^{-1}(0)$ of the reduction $E_{\mathcal{F}} : \mathcal{B}_{\mathcal{F}} \to \mathbb{C}^2$ of the singularity of $\mathcal{F}$, cf. [18, 12]. The weak form (named Conjecture A) assumes that the separatrix curve is the union of smooth and transverse branches. In particular, $E_{\mathcal{F}}$ corresponds to a single blow-up. The strong form (named Conjecture B) only asks that the reduced foliation $\underline{\mathcal{F}} = E_{\mathcal{F}}^*(\mathcal{F})$ on $\mathcal{B}_{\mathcal{F}}$ does not have any saddle-node singularity.

Conjecture A was established by one of us in [8]. We give here an affirmative answer to Conjecture B. More precisely, Theorem I below gives a list of topological invariants containing the projective holonomy representations. In turn Theorem II gives a complete topological classification.

It is worthwhile to stress here that through the whole paper all the topological conjugations between foliations that we consider are supposed to preserve the orientations of the ambient space and also the leaves orientations.

As in the situation considered by D. Cerveau and P. Sad, we restrict our attention to a reasonable class of foliations that are going to be called Generic General Type. Let $\mathcal{F}$ be a non-dicritical foliation, i.e., having a finite number n of irreducible analytic germ curves $S_1, \ldots, S_n$ invariant by $\mathcal{F}$, which are called *separatrices*. Here it is worth to recall the celebrated Separatrix Theorem of [2] asserting that $n > 0$. In the sequel we will call $S_{\mathcal{F}} := \bigcup_{i=1}^n S_i$ the *separatrix curve* of $\mathcal{F}$. Following [9] we say that the foliation $\mathcal{F}$ is of *General Type* if all the singularities of $\underline{\mathcal{F}}$ which are not linearizable are resonant, more precisely:

(GT) *for each singularity of $\underline{\mathcal{F}}$ there are local holomorphic coordinates (u, v) such that $\underline{\mathcal{F}}$ is locally defined by a holomorphic 1-form of one of the two following types:*
  (i) *$\lambda_1 u dv + \lambda_2 v du$, with $\lambda_1 \lambda_2 \neq 0$ and $\lambda_1/\lambda_2 \notin \mathbb{Q}_{<0}$ (linearizable singularity),*
  (ii) *$(\lambda_1 u + \cdots)dv + (\lambda_2 v + \cdots)du$, with $\lambda_1, \lambda_2 \in \mathbb{N}^*$ (resonant saddle).*

To introduce the additional genericity condition (G) we recall first that a singularity of $\underline{\mathcal{F}}$ is of *nodal type* if it can be locally written as

$$(\lambda_1 u + \cdots)dv + (\lambda_2 v + \cdots)du,$$

with $\lambda_1 \lambda_2 \neq 0$ and $\lambda_1/\lambda_2 \in \mathbb{R}_{<0} \setminus \mathbb{Q}_{<0}$. Such singularities are always linearizable and consequently the only local analytical invariant of a node is its Camacho-Sad index $-\lambda_1/\lambda_2$. The topological specificity of a nodal singularity s is the existence, in any small neighborhood of s, of a saturated closed set whose complement is an open disconnected neighborhood of the two punctured local separatrices of the node. We call *nodal separator* such a saturated closed set. We denote by $\mathrm{Node}(\underline{\mathcal{F}})$ the set of nodal singularities of $\underline{\mathcal{F}}$. With this notation the genericity condition can be stated as follows:

(G) *The closure of each connected component of $\mathcal{E}_{\mathcal{F}} \setminus (\mathrm{Node}(\underline{\mathcal{F}}) \cap \mathrm{Sing}(\mathcal{E}_{\mathcal{F}}))$ contains an irreducible component of the exceptional divisor $\mathcal{E}_{\mathcal{F}}$ having a non solvable holonomy group.*

Notice that when $\mathrm{Node}(\underline{\mathcal{F}}) = \varnothing$, the genericity condition (G) only asks for a single irreducible component of $\mathcal{E}_{\mathcal{F}}$ having a non solvable holonomy group. In the space of coefficients of the germ of holomorphic 1-form defining the foliation this condition is generic in the sense of the Krull topology, cf. [6].

A foliation satisfying Conditions (G) and (GT) above will be called *Generic General Type*. For such a foliation $\mathcal{F}$, if $\mathrm{Node}(\mathcal{F}) = \varnothing$ Theorem I below provides a list of topological invariants. In the case that $\mathrm{Node}(\mathcal{F}) \neq \varnothing$ we must restrict the class of topological conjugations in order to keep their invariance. In fact, the first version [11] of this work dealt only with Generic General Type foliations $\mathcal{F}$ satisfying the additional requirement $\mathrm{Node}(\mathcal{F}) = \varnothing$. Here this hypothesis is eliminated by modifying slightly the statements and the proofs given in [11]. In practice, this is done by adding a prefix "$\mathcal{N}$-" to some notions whose new meaning is made precise when they appear for the first time. We recommend the reader to ignore all the prefix "$\mathcal{N}$-" in a first reading.

Definitions for the nodal case

– A *nodal separatrix of $\mathcal{F}$* is a separatrix whose strict transform by the reduction map $E_{\mathcal{F}}$ meets the exceptional divisor at a nodal singular point of $\underline{\mathcal{F}}$.
– An *$\mathcal{N}$-separator of $\underline{\mathcal{F}}$* is the union of a system of nodal separators, one for each point in $\mathrm{Node}(\underline{\mathcal{F}}) \cap \mathrm{Sing}(\mathcal{E}_{\mathcal{F}})$ jointly with some tubular neighborhoods of the strict transforms of the nodal separatrices of $\mathcal{F}$. An *$\mathcal{N}$-separator of $\mathcal{F}$* is the image by $E_{\mathcal{F}}$ of an $\mathcal{N}$-separator of $\underline{\mathcal{F}}$. If $\mathrm{Node}(\underline{\mathcal{F}}) = \varnothing$, an $\mathcal{N}$-separator is the empty set.
– An *$\mathcal{N}$-topological conjugation between two foliation germs $\mathcal{F}$ and $\mathcal{F}'$* is a germ of homeomorphism h preserving the orientation of the ambient space as well as the orientation of the leaves, which is a topological conjugation between $\mathcal{F}$ and $\mathcal{F}'$, such that for each nodal separatrix S_j of $\mathcal{F}$, $h(S_j)$ is a nodal separatrix of $\mathcal{F}'$ and the Camacho-Sad indices of $\underline{\mathcal{F}}$ and $\underline{\mathcal{F}}'$ along the strict transforms of S_j and $h(S_j)$ coincide.
– An *$\mathcal{N}$-transversely holomorphic conjugation* between $\mathcal{F}$ and $\mathcal{F}'$ (resp. $\underline{\mathcal{F}}$ and $\underline{\mathcal{F}}'$) is an $\mathcal{N}$-topologically conjugation between these foliations, which is transversely holomorphic on the complementary of some $\mathcal{N}$-separator of $\mathcal{F}$ (resp. $\underline{\mathcal{F}}$).

Clearly the notions of $\mathcal{N}$-topological conjugation and $\mathcal{N}$-transversely holomorphic conjugation coincide with the usual notions of topological conjugation and transversely holomorphic conjugation, when $\mathrm{Node}(\underline{\mathcal{F}}) = \varnothing$. In Section 7.2, cf. Remark 7.2.1, we shall prove that:

– *Any topological conjugation which is transversely holomorphic in a neighborhood of each nodal separatrix minus the origin is an $\mathcal{N}$-topological conjugation.*

In particular, any transversely holomorphic conjugation is an $\mathcal{N}$-topological conjugation[1]. In order to assure the transverse holomorphy of a conjugation we shall use a generalized form of the following theorem of J. Rebelo [16]:

THEOREM 1.2 (Transverse Rigidity Theorem). – *Every topological conjugation between two germs of non-dicritical holomorphic foliations satisfying the genericity condition* (G) *and having singularities, after reduction, of type $(\lambda_1 u + \cdots)dv + (\lambda_2 v + \cdots)du$ with $\lambda_1 \lambda_2 \neq 0$, $\lambda_1/\lambda_2 \notin \mathbb{R}_{<0}$, is transversely conformal.*

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[1] Added in proof: R. Rosas has shown in a recent preprint [17] that every topological conjugation is an $\mathcal{N}$-topological conjugation.

In fact the proof provided in [16] shows that if we allow nodal singularities then each connected component of $\mathcal{E}_{\mathcal{F}} \setminus (\mathrm{Node}(\mathcal{F}) \cap \mathrm{Sing}(\mathcal{E}_{\mathcal{F}}))$ possesses an open neighborhood W such that the restriction of the topological conjugation to $E_{\mathcal{F}}(W) \setminus \{0\}$ is transversely conformal. The extended version of the Transverse Rigidity Theorem asserts:

(TRT)   *For any orientation preserving topological conjugation Φ between two germs of non-dicritical generalized curves satisfying condition (G) we have that Φ is an $\mathcal{N}$-topological conjugation if and only if Φ is an $\mathcal{N}$-transversely holomorphic conjugation.*

THEOREM 1.3 (Theorem I). – *For every non-dicritical Generic General Type foliation $\mathcal{F}$, the analytic type of the projective holonomy representation of each irreducible component of the exceptional divisor $\mathcal{E}_{\mathcal{F}}$ is a topological invariant when $\mathrm{Node}(\mathcal{F}) = \varnothing$. More generally, the semilocal data $\mathcal{SL}(\mathcal{F})$ constituted by*

  – *the topological type of the embedding of the total separatrix curve $S_{\mathcal{F}}$ of $\mathcal{F}$ into $(\mathbb{C}^2, 0)$,*
  – *the collection of local analytic types $[\underline{\mathcal{F}}_s]^{\mathrm{hol}}$ of the reduced foliation $\underline{\mathcal{F}}$ at each singular point $s \in \mathrm{Sing}(\underline{\mathcal{F}})$, codifying in particular the Camacho-Sad index $\mathrm{CS}(\underline{\mathcal{F}}, D, s)$ of $\underline{\mathcal{F}}$ at every singular point s along each irreducible component D of $\mathcal{E}_{\mathcal{F}}$ containing s,*
  – *the analytic type of the holonomy representation $\mathcal{H}_{\underline{\mathcal{F}}, D}$ of each irreducible component D of $\mathcal{E}_{\mathcal{F}}$,*

is an $\mathcal{N}$-topological invariant [2] *of the germ of $\mathcal{F}$ at $0 \in \mathbb{C}^2$.*

Notice that the Camacho-Sad index $\mathrm{CS}(\underline{\mathcal{F}}, D, s)$ determines the analytic type of $\underline{\mathcal{F}}$ at s when s is not a resonant singularity after the assumption (GT). On the other hand, the genericity condition (G) is strictly necessary in Theorem I. Indeed, inside the family of homeomorphisms $\Psi(x, y) = (x|x|^a, y|y|^b)$ there is a topological conjugation between any pair of linear hyperbolic singularities having different Camacho-Sad indices.

Theorem I asserts that $\mathcal{SL}(\mathcal{F})$ is a topological invariant for the class of Generic General Type foliations with $\mathrm{Node}(\mathcal{F}) = \varnothing$. In fact, the equality $\mathcal{SL}(\mathcal{F}) = \mathcal{SL}(\mathcal{F}')$ needs to be specified because the index sets of the families can be different for $\mathcal{F}$ and $\mathcal{F}'$. In order to do this, we recall that a topological conjugation between $\mathcal{F}$ and $\mathcal{F}'$ as above transforms $S_{\mathcal{F}}$ into $S_{\mathcal{F}'}$ and induces a unique homeomorphism

$$\Psi^{\sharp} : \mathcal{E}_{\mathcal{F}} \to \mathcal{E}_{\mathcal{F}'}, \quad \Psi^{\sharp}(\mathrm{Sing}(\underline{\mathcal{F}})) = \mathrm{Sing}(\underline{\mathcal{F}}')$$

between the exceptional divisors up to isotopy. This is a consequence of the following result proved in a previous work [10].

THEOREM 1.4 (Marking Theorem). – *Let S and S' be two germs of analytic curves at the origin in $\mathbb{C}^2$ and let $h : (\mathbb{C}^2, 0) \overset{\sim}{\longrightarrow} (\mathbb{C}^2, 0)$ be a germ of homeomorphism such that $h(S) = S'$. If E_S and $E_{S'}$ denote the minimal reduction of singularities of S and S', then there is a germ of homeomorphism $h_1 : (\mathbb{C}^2, 0) \overset{\sim}{\longrightarrow} (\mathbb{C}^2, 0)$ such that:*

  (i)   *$h_1(S) = S'$ and the restrictions of h and h_1 to the complements of S and S' are homotopic,*

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[2] Added in proof: By applying the previously cited preprint of R. Rosas [17], Theorem I can be rephrased as "$\mathcal{SL}(\mathcal{F})$ is a topological invariant".

(ii) $E_{S'}^{-1} \circ h_1 \circ E_S$ *extends to a homeomorphism from a neighborhood of* $\mathcal{D}_S := E_S^{-1}(S)$ *onto a neighborhood of* $E_{S'}^{-1}(S')$, *being holomorphic over an open neighborhood of* $\mathrm{Sing}(\mathcal{D}_S)$ *and compatible with the Hopf fibrations outside another open neighborhood of* $\mathrm{Sing}(\mathcal{D}_S)$.

The topological invariance of $\mathcal{SL}(\mathcal{F})$ *induced by the conjugation* Ψ *between* $\mathcal{F}$ *and* $\mathcal{F}'$ *means that for each irreducible component* D *of* $\mathcal{E}_{\mathcal{F}}$ *and each singular point* $s \in \mathrm{Sing}(\mathcal{F})$, *the following conditions hold:*

a) $\mathrm{CS}(\mathcal{F}', \Psi^{\sharp}(D), \Psi^{\sharp}(s)) = \mathrm{CS}(\mathcal{F}, D, s))$ and $[\underline{\mathcal{F}}'_{\Psi^{\sharp}(s)}]^{\mathrm{hol}} = [\underline{\mathcal{F}}_s]^{\mathrm{hol}}$,

b) there exists a germ of biholomorphism ψ between two germs (Δ, m) and (Δ', m') of transverse analytic curves to $\mathcal{F}$ and $\mathcal{F}'$ through points $m \in D \backslash \mathrm{Sing}(\underline{\mathcal{F}})$ and $m' := \Psi^{\sharp}(m)$ respectively, such that the following diagram is commutative:

$$
\begin{array}{ccc}
\pi_1(D \setminus \mathrm{Sing}(\underline{\mathcal{F}}),\, m) & \xrightarrow{\;\mathscr{H}_D^{\mathcal{F}}\;} & \mathrm{Diff}(\Delta, m) \\
\Big\downarrow{\scriptstyle \Psi_*^{\sharp}} & \circlearrowleft & \Big\downarrow{\scriptstyle \psi_*} \\
\pi_1(D' \setminus \mathrm{Sing}(\underline{\mathcal{F}'}),\, m') & \xrightarrow{\;\mathscr{H}_{D'}^{\mathcal{F}'}\;} & \mathrm{Diff}(\Delta', m')
\end{array}
$$

(1)

with $\psi_*(\varphi) := \psi \circ \varphi \circ \psi^{-1}$ and $\Psi_*^{\sharp}(\dot{\gamma}) := \Psi^{\sharp} \circ \dot{\gamma}$.

Notice that $\mathcal{SL}(\mathcal{F})$ is a "semi-local" invariant in the sense that it only contains information along the irreducible components of $\mathcal{E}_{\mathcal{F}}$, but it does not provide any information about the combinatorial gluing of these data. Thus, it cannot be reasonably a complete invariant of $\mathcal{F}$. To remedy for this situation, the idea is to consider the separatrix curve $S_{\mathcal{F}}$ as the "organization center" of the topology of $\mathcal{F}$ as was conjectured by René Thom in the seventies. The incompressibility of the leaves inside the complement of $S_{\mathcal{F}}$ proved in [9] plays a major role here and it indicates that the fundamental group of the complement of $S_{\mathcal{F}}$ "controls" the topology of the leaves of $\mathcal{F}$. It also suggests the possibility to replace the usual notion of "holonomy" by that of "monodromy". The "holonomy" consists of the pseudo-group of local automorphisms of the ambient space coming from the ambiguity of the (multivalued) first integrals of $\mathcal{F}$. The "monodromy" reports the automorphisms of the set of (multivalued) first integrals coming from the ambiguity of the ambient space. We precise this notion in a general setting:

DEFINITION 1.5. – *Let* $\mathcal{G}$ *be a differentiable foliation on a manifold* M *and consider the universal covering* $q : \widetilde{M} \to M$ *of* M. *We denote by* $\widetilde{\mathcal{G}}$ *the lift of* $\mathcal{G}$ *in* $\widetilde{M}$ *and by* $\widetilde{M}/\widetilde{\mathcal{G}}$ *the space of leaves of* $\widetilde{\mathcal{G}}$. *Then the* monodromy *of* $\mathcal{G}$ *is the morphism*

$$
\mathfrak{M}_M^{\mathcal{G}} : \mathrm{Aut}(\widetilde{M}, q) \to \mathrm{Aut}(\widetilde{M}/\widetilde{\mathcal{G}})\,,
$$

sending an element φ *of the group of deck transformations of* q *to the automorphism of* $\widetilde{M}/\widetilde{\mathcal{G}}$ *obtained by factorizing* φ, *i.e.,* $\mathfrak{M}_M^{\mathcal{G}}(\varphi) \circ \tau = \tau \circ \varphi$, *where* $\tau : \widetilde{M} \to \widetilde{M}/\widetilde{\mathcal{G}}$ *denotes the natural quotient map.*

For instance, if $(M, \mathcal{G})$ is a foliated bundle over B with simply connected fibre F then $\widetilde{M} = \widetilde{B} \times F$, $\widetilde{M}/\widetilde{\mathcal{G}} = F$ and the monodromy $\mathfrak{M}_M^{\mathcal{G}}$ can be identified with the global holonomy representation $\pi_1(M) \to \operatorname{Aut}(F)$.

In [9] we have constructed a fundamental system $(U_\alpha)_\alpha$ of neighborhoods of $S_{\mathcal{G}}$ in a Milnor ball such that the space of leaves of the universal covering $\widetilde{U}_\alpha^* \to U_\alpha^*$ of $U_\alpha^* := U_\alpha \setminus S_{\mathcal{G}}$ is a (in general non Hausdorff) holomorphic manifold. Thus, the monodromy of the global foliation $\mathcal{G}_{|U_\alpha^*}$ is a representation of $\pi_1(U_\alpha^*)$ to the group of holomorphic automorphisms of the leaf space $\widetilde{Q}_{U_\alpha}^{\mathcal{G}}$ of $\widetilde{U}_\alpha^*$. However, this monodromy representation depends on the choice of the open set where is defined the chosen representative of the germ $\mathcal{G}$. This notion of monodromy admits a reformulation in term of germs by considering the *category of pro-objects*, cf. Section 3.1. This allows us to introduce the notion of *monodromy of a germ of foliation* in Definition 3.4.1. Roughly speaking, the image of a deck transformation g of the universal covering $\widetilde{U}_\alpha^* \to U_\alpha^*$ by the monodromy representation of $\mathcal{G}$ is given by taking the germ over all the open sets U_α of the mappping from $\widetilde{Q}_{U_\alpha}^{\mathcal{G}}$ to itself defined by $\widetilde{L}_\alpha \mapsto g(\widetilde{L}_\alpha)$. The interest of this notion lies in the fact that it takes into account simultaneously the "transverse structure" of the foliation and the topology of the complement of their separatrix curves.

We must also consider the quite technical but highly relevant notions of *geometric conjugation of monodromies* (Definition 3.4.3) *preserving the Camacho-Sad indices* (Definition 3.5.5) and *realizable over transversals* (Definition 3.6.1), which allows us to compare in a precise way the monodromies of two germs of foliations. We refer the reader to Section 3 to have precise definitions of these technical notions. Using them, the statement of the main result of this paper is the following:

THEOREM 1.6 (Theorem II). – *If $\mathcal{G}$ and $\mathcal{G}'$ are Generic General Type foliations, then the following properties are equivalent:*

1. *there exists an $\mathcal{N}$-topological conjugation between the germs $\mathcal{G}$ and $\mathcal{G}'$,*
2. *there exists an $\mathcal{N}$-transversely holomorphic conjugation between the foliations $\underline{\mathcal{G}}$ and $\underline{\mathcal{G}}'$, which is defined on open neighborhoods of the exceptional divisors,*
3. *there exists a geometric $\mathcal{N}$-conjugation of the monodromies of the germs $\mathcal{G}$ and $\mathcal{G}'$, preserving the Camacho-Sad indices, which is realizable over $\mathcal{N}$-collections of transversals Σ and Σ' of $\mathcal{G}$ and $\mathcal{G}'$.*

If $\operatorname{Node}(\underline{\mathcal{G}}) = \varnothing$ then *an $\mathcal{N}$-collection of transversals of $\mathcal{G}$* consists in a single germ of regular holomorphic curve transverse to any separatrix of $\mathcal{G}$, at a regular point. When $\operatorname{Node}(\underline{\mathcal{G}}) \neq \varnothing$ we must precise the location of the connected components of an $\mathcal{N}$-collection of transversals of $\mathcal{G}$ by using the following theorem of [15, Corollary 4.1] generalizing the main result of [2]:

THEOREM 1.7 (Strong Camacho-Sad Separatrix Theorem). – *Each connected component of $\mathcal{E}_{\mathcal{G}} \setminus (\operatorname{Node}(\underline{\mathcal{G}}) \cap \operatorname{Sing}(\mathcal{E}_{\mathcal{G}}))$ contains a singular point of $\underline{\mathcal{G}}$ lying on the strict transform of a separatrix of $\mathcal{G}$ whose Camacho-Sad index has positive real part.*

In particular, for each connected component C of $\mathcal{E}_{\mathscr{F}} \setminus (\text{Node}(\underline{\mathscr{F}}) \cap \text{Sing}(\mathcal{E}_{\mathscr{F}}))$ exists a (not necessarily unique) non-nodal separatrix of $\mathscr{F}$ whose strict transform meets C. We define an $\mathcal{N}$-*collection of transversals of* $\mathscr{F}$ as a collection $\Sigma = \{(\Sigma_1, p_1), \ldots, (\Sigma_m, p_m)\}$ where each (Σ_i, p_i) is the germ at $p_i \in S_{\mathscr{F}} \setminus \{0\}$ of a regular curve Σ_i transverse to $\mathscr{F}$ and the whole collection fulfills the following property:

(σ) *for each connected component C of $\mathcal{E}_{\mathscr{F}} \setminus (\text{Node}(\underline{\mathscr{F}}) \cap \text{Sing}(\mathcal{E}_{\mathscr{F}}))$ there exists a germ (Σ_i, p_i) of Σ with p_i belonging to a non-nodal separatrix of $\mathscr{F}$ whose strict transform meets C.*

In fact, Theorems I and II are easy consequences of Theorems 5.0.1 and 5.0.2 that are going to be proved in Sections 7 and 8. They are more general than Theorems I and II because the genericity hypothesis (G) is replaced by weaker but more technical conditions. Notice also that Corollary 5.0.4 is of interest even in the case $\mathscr{F} = \mathscr{F}'$. If, in addition, we assume that $\text{Node}(\underline{\mathscr{F}}) = \varnothing$ then it implies the following result concerning the automorphism group $\text{Aut}_0(\mathscr{F})$ of orientations preserving homeomorphisms germs conjugating $\mathscr{F}$ to itself.

COROLLARY 1.8. – *If $\mathscr{F}$ is a Generic General Type foliation with $\text{Node}(\underline{\mathscr{F}}) = \varnothing$, then for each $h \in \text{Aut}_0(\mathscr{F})$ there exists a homeomorphism $h_1 \in \text{Aut}_0(\mathscr{F})$ satisfying for $S = S' = S_{\mathscr{F}}$ the properties (i) and (ii) of Marking Theorem.*

2. Preliminary notions

Through all the paper we will use the following notations:

$$\mathbb{B}_r = \{(x,y) \in \mathbb{C}^2, \; |x|^2 + |y|^2 \leq r\}, \quad \mathbb{D}_r = \{z \in \mathbb{C}, \; |z| \leq r\},$$

and if $B \subset A$, $B' \subset A'$, $f : (A, B) \to (A', B')$ will denote the germ of a map f_1 defined on a neighborhood of B in A into A', such that $f_1(B) \subset B'$.

In this section $S \subset \mathbb{C}^2$ denotes a holomorphic curve with an isolated singularity at the origin $0 = (0,0)$ of $\mathbb{C}^2$ and $\mathbb{B} := \mathbb{B}_{r_0}$ is a closed *Milnor ball* for S, i.e., each sphere $\partial \mathbb{B}_r$, $0 < r \leq r_0$, is transverse to S, cf. [13]. We denote by $E_S : \mathscr{B}_S \to \mathbb{B}$ the *minimal desingularization map* of S such that the *total divisor* $\mathscr{D}_S := E_S^{-1}(S)$ has normal crossings. We denote by $\mathcal{E}_S := E_S^{-1}(0)$ the *exceptional divisor* and by $\mathscr{S} := \overline{\mathscr{D}_S \setminus \mathcal{E}_S}$ the *strict transform* of S. We will also use the following conventions along all the paper: for $A \subset \mathbb{B}$ and for $\mathscr{A} \subset \mathscr{B}_S$ we put

(2) $$A^* := A \setminus S \quad \text{and} \quad \mathscr{A}^* := \mathscr{A} \setminus \mathscr{D}_S.$$

2.1. Incompressibility of the leaves

Let $\mathscr{F}$ be a singular non-dicritical holomorphic foliation defined in a neighborhood of $\mathbb{B}$, having 0 as the unique singularity and S as the *separatrix curve in* $\mathbb{B}$, i.e., S is invariant by $\mathscr{F}$ and every analytic invariant curve passing through 0 is contained in S. Denote by $E_{\mathscr{F}} : \mathscr{B}_{\mathscr{F}} \to \mathbb{B}$ the minimal reduction of singularities map of $\mathscr{F}$ and by $\underline{\mathscr{F}} = E_{\mathscr{F}}^* \mathscr{F}$ the reduced foliation over $\mathscr{B}_{\mathscr{F}}$. The hypothesis (GT) on $\mathscr{F}$ implies that $\underline{\mathscr{F}}$ does not have any saddle-node singularity, so that $\mathscr{F}$ is a *generalized curve* and consequently $\mathscr{B}_{\mathscr{F}} = \mathscr{B}_S$ and $E_S = E_{\mathscr{F}}$, cf. [1].

Now we fix an $\mathscr{N}$-collection of holomorphic transversal curves $\Sigma \subset \mathbb{B}$ (i.e., satisfying Condition (σ) given in the introduction). The saturation of a small transversal to one of the separatrices of a non-nodal (GT)-singularity s jointly with the two local separatrices of s, is a neighborhood of s. Consequently the following property holds:

a) *there exists an $\mathscr{N}$-separator N of $\mathscr{F}$, such that for any open neighborhood W of S in $\mathbb{B}$, the closure of*

$$\operatorname{Sat}_{\underline{\mathscr{F}}}(E_S^{-1}(\Sigma), E_S^{-1}(W) \setminus N)$$

is a neighborhood of $\mathscr{D}_S \setminus \operatorname{Node}(\underline{\mathscr{F}})$, in $E_S^{-1}(W \setminus N)$,

where $\operatorname{Sat}_{\underline{\mathscr{F}}}(A, B)$ denotes the union of all the leaves of $\underline{\mathscr{F}}_{|B}$ meeting A and it is called *the saturation of A in B by $\underline{\mathscr{F}}$.*

We say that an open neighborhood U of S in $\mathbb{B}$ is $(\mathscr{F}, \Sigma)$-*admissible*, if for each leaf L of the regular foliation $\mathscr{F}_{|U^*}$, the following properties hold:

b) *L is incompressible in U^*, i.e., the inclusion $L \subset U^*$ induces a monomorphism $\pi_1(L, p) \hookrightarrow \pi_1(U^*, p)$, $p \in L$; in addition the map $\pi_1(U^*, p) \to \pi_1(\mathbb{B}^*, p)$ induced by the inclusion $U^* \subset \mathbb{B}^*$ is an isomorphism;*

c) *every path in L whose ends are in Σ^* and which is homotopic in U^* to a path contained in Σ^*, is a null homotopic loop in L.*

Notice that an admissible open set $U \subset \mathbb{B}$ is not necessarily saturated in $\mathbb{B}$. We denote by $\mathfrak{U}_{\mathscr{F}, \Sigma}$ the collection of connected open neighborhoods of S which are $(\mathscr{F}, \Sigma)$-admissible. Property c) above will play a key role in this work. In fact, it is equivalent to the *foliated 1-connexity* of $\Sigma \cap U^*$ in U^*. This notion introduced in [9] plays a major role in the proof of the main result of [9, Theorem 6.1.1] which can be stated as follows:

THEOREM 2.1.1. – *If $\mathscr{F}$ is a foliation of General Type, then $\mathfrak{U}_{\mathscr{F}, \Sigma}$ is a fundamental system of neighborhoods of S in the closed Milnor ball $\mathbb{B}$.*

2.2. Leaf spaces

We fix once for all a universal covering $q : \widetilde{\mathbb{B}}^* \to \mathbb{B}^*$ of $\mathbb{B}^*$ and for every subset $A \subset \mathbb{B}$ we will denote

$$(3) \qquad \widetilde{A}^* := q^{-1}(A^*) \quad \text{and} \quad q_A := q_{|\widetilde{A}^*} : \widetilde{A}^* \longrightarrow A^*.$$

If $U \in \mathfrak{U}_{\mathscr{F}, \Sigma}$ then q_U is a universal covering of U^*. The group $\Gamma := \operatorname{Aut}_q(\widetilde{\mathbb{B}}^*)$ of deck transformations of the covering q can be identified with the group Γ_U of deck transformations of the covering q_U by the restriction map $g \mapsto g_{|\widetilde{U}^*}$. Hence, we can also identify Γ with

$$(4) \qquad \Gamma_\infty := \varprojlim_{U \in \mathfrak{U}_{\mathscr{F}, \Sigma}} \Gamma_U .$$

On $\widetilde{\mathbb{B}}^*$ we consider the regular foliation $\widetilde{\mathscr{F}}$, pull-back of $\mathscr{F}$ by q. For $U \in \mathfrak{U}_{\mathscr{F}, \Sigma}$, we denote by $\widetilde{\mathscr{F}}_U$ its restriction to $\widetilde{U}^*$ and for an arbitrary subset W of U, we denote by

$$(5) \qquad \widetilde{Q}_W^{\mathscr{F}} := \left(\widetilde{W}^* \Big/ \widetilde{\mathscr{F}}_W \right) , \quad \varrho_W : \widetilde{W}^* \to \widetilde{Q}_W^{\mathscr{F}} \subset \widetilde{Q}_U^{\mathscr{F}} ,$$

the leaf space of the restriction $\widetilde{\mathscr{F}}_W$ of $\widetilde{\mathscr{F}}_U$ to $\widetilde{W}^*$, endowed with the quotient topology and the quotient map ϱ_W. It turns out that if W is saturated in U then the natural map $\widetilde{Q}_W^{\mathscr{F}} \hookrightarrow \widetilde{Q}_U^{\mathscr{F}}$

is a *topological embedding*, i.e., a homeomorphism onto its image, and we may consider $\widetilde{Q}_W^{\mathscr{F}}$ as a subset of $\widetilde{Q}_U^{\mathscr{F}}$.

Properties *a)*, *b)* and *c)* satisfied by the open sets of $\mathfrak{U}_{\mathscr{F},\Sigma}$ can be understood as geometric properties of the foliation $\widetilde{\mathscr{F}}_U$ with respect to the transverse section $\widetilde{\Sigma}_U^* := \widetilde{\Sigma}^* \cap \widetilde{U}^*$, cf. [9, §6.2]:

- *every leaf of $\widetilde{\mathscr{F}}_U$ is simply connected;*
- *the intersection of every leaf of $\widetilde{\mathscr{F}}_U$ with each connected component of $\widetilde{\Sigma}_U^*$ is either empty or consists in a single point;*
- *the restriction of ϱ_U to each connected component of $\widetilde{\Sigma}_U^*$, is a topological embedding.*

Let N be an $\mathscr{N}$-separator of $\mathscr{F}$ satisfying Property *a)* of the previous section. The inverse maps of the ϱ_U's restrictions to the connected components of $\widetilde{\Sigma}_U^*$ form a holomorphic atlas on $\widetilde{Q}_{U \setminus N}^{\mathscr{F}}$, defining a structure of one dimensional complex manifold (non necessarily Hausdorff) over it. It is easy to check that this structure extends to a unique structure of complex manifold over $\widetilde{Q}_U^{\mathscr{F}}$, such that:

- *for every holomorphic map $g : \mathbb{D}_1 \to \widetilde{U}^*$, the composition*

$$\varrho_U \circ g : \mathbb{D}_1 \to \widetilde{Q}_U^{\mathscr{F}}$$

is also holomorphic.

It is clear that each element g of Γ_U preserves the foliation $\widetilde{\mathscr{F}}_U$ and factorizes by an element $g_U^\flat$ of the analytic automorphism group $\mathrm{Aut}_{\mathrm{An}}(\widetilde{Q}_U^{\mathscr{F}})$. In [9] we have defined the *monodromy* of $\mathscr{F}_U$ as the morphism

$$(6) \qquad \mathfrak{M}_U^{\mathscr{F}} : \Gamma_U \longrightarrow \mathrm{Aut}_{\mathrm{An}}(\widetilde{Q}_U^{\mathscr{F}}), \qquad g \mapsto g_U^\flat.$$

This representation of Γ_U is clearly an analytic invariant of the foliation $\mathscr{F}_U$ by biholomorphisms preserving the open set U. In order to obtain an analytic invariant of the germ of $\mathscr{F}$ at 0, or along S, we need to "germify" this notion. This will be done in the following section.

3. Monodromy of a germ of foliation

3.1. Germification

The set $\mathfrak{U}_{\mathscr{F},\Sigma}$ is cofiltered by the partial order

$$U \preceq V :\Longleftrightarrow U \supset V.$$

The maps

$$(7) \qquad \rho_{UV} : \widetilde{Q}_V^{\mathscr{F}} \longrightarrow \widetilde{Q}_U^{\mathscr{F}}, \quad V \subset U, \quad U, V \in \mathscr{U}_{\mathscr{F},\Sigma},$$

sending each leaf L of $\widetilde{\mathscr{F}}_V$ into the unique leaf of $\widetilde{\mathscr{F}}_U$ containing L, are open and holomorphic. They form a projective or inverse system of complex manifolds

$$\widetilde{Q}_\infty^{\mathscr{F}} := \left(\left(\widetilde{Q}_U^{\mathscr{F}} \right)_{U \in \mathfrak{U}_{\mathscr{F},\Sigma}}, \; (\rho_{UV})_{U,V \in \mathfrak{U}_{\mathscr{F},\Sigma}, \, U \preceq V} \right)$$

called the *leaf pro-space of* $\widetilde{\mathcal{G}}$. It is an object in the category $\underleftarrow{\mathrm{An}}$ of pro-objects associated to the category An of (non necessarily Hausdorff) complex manifolds and holomorphic maps. We recall that the objets of $\underleftarrow{\mathrm{An}}$ are the projective families of complex manifolds; on the other hand, if $\mathfrak{A}$ and $\mathfrak{B}$ are cofiltered sets and

$$M = \big((M_\alpha)_{\alpha \in \mathfrak{A}}, \, (\zeta_{\alpha\alpha'})_{\alpha \geq \alpha'}\big) \quad \text{and} \quad M' = \big((M'_\beta)_{\beta \in \mathfrak{B}}, \, (\zeta'_{\beta\beta'})_{\beta \geq \beta'}\big)$$

are two objects of $\underleftarrow{\mathrm{An}}$ then the set of $\underleftarrow{\mathrm{An}}$-*morphisms* of M into M' is by definition

(8) $$\mathrm{Hom}_{\underleftarrow{\mathrm{An}}}(M, M') := \varprojlim_{\beta \in \mathfrak{B}} \varinjlim_{\alpha \in \mathfrak{A}} \, \mathcal{O}(M_\alpha, M'_\beta),$$

where $\mathcal{O}(M_\alpha, M'_\beta)$ denotes the set of holomorphic maps of M_α into M'_β. For further details, see [4].

3.2. Pro-germs at infinity

Let M be a complex submanifold of $\widetilde{\mathbb{B}}^*$ such that $\overline{q(M)} \cap S \neq \varnothing$. The projective system

$$(M, \infty) := \Big((M \cap \widetilde{U}^*)_{U \in \mathfrak{U}_{\mathcal{G}, \Sigma}}, \, (\iota_{UV})_{U, V \in \mathfrak{U}_{\mathcal{G}, \Sigma}, V \subset U}\Big) \,,$$

formed by the inclusion maps ι_{UV} of $M \cap \widetilde{V}^*$ into $M \cap \widetilde{U}^*$ is a pro-object in $\underleftarrow{\mathrm{An}}$. Let T be a complex manifold. Every element g of the set

$$\mathcal{O}((M, \infty), T) := \varinjlim_{U \in \mathfrak{U}_{\mathcal{G}, \Sigma}} \mathcal{O}(M \cap \widetilde{U}^*, T)$$

will be called *germ at infinity* of M into T and denoted by $g : (M, \infty) \to T$. Identifying T to the constant projective system, $\mathcal{O}((M, \infty), T)$ can be naturally identified with $\mathrm{Hom}_{\underleftarrow{\mathrm{An}}}((M, \infty), T)$.

REMARK 3.2.1. – Endowing M with the induced topology of $\widetilde{\mathbb{B}}^*$, the pre-sheaf $W \mapsto \mathcal{O}((W, \infty), T)$ is not a sheaf and two different germs at infinity $f, g \in \mathcal{O}((M, \infty), T)$ can coincide as elements of $\mathcal{O}((M \cap V_j, \infty)$, over the intersection of M with each open set of a covering $(V_j)_{j \in J}$. In particular, if M has infinitely many connected components M^α, each of them satisfying $\overline{q(M^\alpha)} \cap S \neq \varnothing$, then the restriction map $\mathcal{O}(M, \infty) \to \prod_\alpha \mathcal{O}(M^\alpha, \infty)$ is never surjective.

Assume now that T is contained in $\widetilde{\mathbb{B}}^*$. Each element f of the set

$$\mathrm{Hom}_{\underleftarrow{\mathrm{An}}}((M, \infty), (T, \infty))$$

is called *pro-germ at infinity* of M into T and it shall be denoted by $f : (M, \infty) \to (T, \infty)$. Thus, f is a family of germs at infinity

$$f = (f_V)_{V \in \mathfrak{U}_{\mathcal{G}, \Sigma}} \in \prod_{V \in \mathfrak{U}_{\mathcal{G}, \Sigma}} \mathcal{O}((M, \infty), T \cap \widetilde{V}^*) \,,$$

such that $\varsigma_{VW} \circ f_W = f_V$, $W \subset V$, where ς_{WV} denotes the inclusion map of $T \cap \widetilde{W}^*$ into $T \cap \widetilde{V}^*$. The same notions in the category Top of topological spaces and continuous maps define the set of continuous pro-germs at infinity

$$\mathrm{Hom}_{\underleftarrow{\mathrm{Top}}}((M, \infty), (T, \infty)) \,.$$

Notice that the group of pro-germs at infinity of deck transformations of the covering can be canonically identified with the group Γ_∞ defined by (4):

$$\Gamma_\infty \simeq \left\{ \varphi \in \mathrm{Aut}_{\underleftarrow{\mathrm{An}}}(\widetilde{\mathbb{B}}^*, \infty) \mid q_\infty \circ \varphi = q_\infty \right\},$$

where $q_\infty : (\widetilde{\mathbb{B}}^*, \infty) \to \mathbb{B}$ denotes the germ at infinity of the covering map q.

3.3. Canonical pro-germs

For every $U \in \mathfrak{U}_{\mathscr{G},\Sigma}$ we denote by

$$\tau_{M,U} : (M, \infty) \to \widetilde{Q}_U^{\mathscr{G}}$$

the germ at infinity of the quotient map $\widetilde{U} \to \widetilde{Q}_U^{\mathscr{G}}$ restricted to $M \cap \widetilde{U}^*$. The element

$$\tau_M := (\tau_{M,U})_{U \in \mathfrak{U}_{\mathscr{G},\Sigma}} \in \mathrm{Hom}_{\underleftarrow{\mathrm{An}}}((M, \infty), \widetilde{Q}_\infty^{\mathscr{G}}) \subset \prod_{U \in \mathfrak{U}_{\mathscr{G},\Sigma}} \mathcal{O}((M, \infty), \widetilde{Q}_U^{\mathscr{G}})$$

will be called the *canonical pro-morphism* associated to M. Next proposition follows easily from the geometric properties of the foliation $\widetilde{\mathscr{G}}_U$ relatively to $\widetilde{\Sigma}_U^*$ stated in Section 2.1.

PROPOSITION 3.3.1. – *If M is a connected component of $\widetilde{\Sigma}^*$ then τ_M is a monomorphism in the category* $\underleftarrow{\mathrm{An}}$.

3.4. Monodromy of a germ

Let g be an element of Γ_∞ and consider $U, V \in \mathcal{U}_{\mathscr{G},\Sigma}$ with $V \subset U$. With the notations (6) and (7) we have the following commutation relations:

$$g_U^\flat \circ \rho_{UV} = \rho_{UV} \circ g_V^\flat \in \mathcal{O}(\widetilde{Q}_V^{\mathscr{G}}, \widetilde{Q}_U^{\mathscr{G}}).$$

Hence, by denoting $\mathcal{O}(\widetilde{Q}_\infty^{\mathscr{G}}, \widetilde{Q}_U^{\mathscr{G}}) := \varinjlim_{V \in \mathfrak{U}_{\mathscr{G},\Sigma}} \mathcal{O}(\widetilde{Q}_V^{\mathscr{G}}, \widetilde{Q}_U^{\mathscr{G}})$, the elements

$$g_{U\infty}^\flat := \varinjlim_V (g_U^\flat \circ \rho_{UV}) \in \mathcal{O}(\widetilde{Q}_\infty^{\mathscr{G}}, \widetilde{Q}_U^{\mathscr{G}})$$

form a projective family. The $\underleftarrow{\mathrm{An}}$-endomorphism

$$g^\flat := (g_{U\infty}^\flat)_{U \in \mathfrak{U}_{\mathscr{G},\Sigma}} \in \mathrm{End}_{\underleftarrow{\mathrm{An}}}(\widetilde{Q}_\infty^{\mathscr{G}}) \subset \prod_{U \in \mathfrak{U}_{\mathscr{G},\Sigma}} \mathcal{O}(\widetilde{Q}_\infty^{\mathscr{G}}, \widetilde{Q}_U^{\mathscr{G}}),$$

is invertible and its inverse is $(g^{-1})^\flat$. More generally, we have the following covariance relations:

$$(g \circ h)^\flat = g^\flat \circ h^\flat, \quad g, h \in \Gamma_\infty.$$

DEFINITION 3.4.1. – *The morphism of groups*

$$\mathfrak{M}_S^{\mathscr{G}} : \Gamma_\infty \longrightarrow \mathrm{Aut}_{\underleftarrow{\mathrm{An}}}(\widetilde{Q}_\infty^{\mathscr{G}}), \qquad g \mapsto \mathfrak{M}_S^{\mathscr{G}}(g) := g^\flat$$

is called the monodromy of the germ of $\mathscr{G}$ along S.

We now fix once for all a second curve $S' \subset \mathbb{C}^2$ with an isolated singularity at the origin as well as a closed Milnor ball $\mathbb{B}'$ for S'. We denote by $E_{S'} : \mathcal{B}'_{S'} \to \mathbb{B}'$, $\mathcal{D}_{S'}$, $\mathcal{E}_{S'}$, $\mathcal{S}'$, the minimal desingularization map, the total transform, the exceptional divisor and the strict transform of S' respectively. In order to avoid any ambiguity with the notation (2), we set

$$(9) \qquad A^\star := A \setminus S', \quad \mathcal{A}^\star := \mathcal{A} \setminus \mathcal{D}_{S'}, \quad \text{for} \quad A \subset \mathbb{B}' \quad \text{and} \quad \mathcal{A} \subset \mathcal{B}'_{S'}.$$

We also fix a singular non-dicritical holomorphic foliation $\mathcal{F}'$ of General Type defined in a neighborhood of $\mathbb{B}'$ having S' as total separatrix curve. Let $\Sigma' \subset \mathbb{B}'$ be an $\mathcal{N}$-collection of transversals of $\mathcal{F}'$, we denote by $\mathfrak{U}_{\mathcal{F}', \Sigma'}$ the set of open neighborhoods of S' in $\mathbb{B}'$ which are $(\mathcal{F}', \Sigma')$-admissible and we fix a universal covering $q' : \widetilde{\mathbb{B}'^\star} \to \mathbb{B}'^\star$. For $A \subset \mathbb{B}'$, we put

$$(10) \qquad \widetilde{A}^\star := q'^{-1}(A^\star), \quad \text{and} \quad q'_A := q'_{|\widetilde{A}^\star} : \widetilde{A}^\star \to A^\star.$$

As in (4), we identify the projective limit of the groups Γ'_U of deck transformations of the covering $q'_{U'}$ with the group of pro-automorphisms at infinity of $\mathbb{B}'$ preserving the germ at infinity q'_∞ of q':

$$\Gamma'_\infty := \varprojlim_{U \in \mathfrak{U}_{\mathcal{F}', \Sigma'}} \Gamma'_U \simeq \left\{ \varphi \in \mathrm{Aut}_{\underline{\mathrm{An}}}(\widetilde{\mathbb{B}'^\star}, \infty) \mid q'_\infty \circ \varphi = q'_\infty \right\}.$$

When $\mathrm{Node}(\underline{\mathcal{F}}) \neq \varnothing$, we also need to consider a mixed class of pro-germs:

DEFINITION 3.4.2. – *An $\mathcal{N}$-analytic pro-germ is an element of*

$$\mathrm{Hom}_{\mathcal{N}-\underline{\mathrm{An}}}(\widetilde{Q}^{\mathcal{F}}_\infty, \widetilde{Q}^{\mathcal{F}'}_\infty) \subset \mathrm{Hom}_{\underline{\mathrm{Top}}}(\widetilde{Q}^{\mathcal{F}}_\infty, \widetilde{Q}^{\mathcal{F}'}_\infty)$$

consisting in a collection $f = (f_{U'})'_U$ of germs which can be represented by continuous maps $\underline{f}_{U'} : \widetilde{Q}^{\mathcal{F}}_U \to \widetilde{Q}^{\mathcal{F}'}_{U'}$, $U \in \mathfrak{U}_{\mathcal{F}, \Sigma}$, $U' \in \mathfrak{U}_{\mathcal{F}', \Sigma'}$, for which there are $\mathcal{N}$-separators N and N' of $\mathcal{F}$ and $\mathcal{F}'$, such that $\underline{f}_{U'}(\widetilde{Q}^{\mathcal{F}}_{U \cap N}) \subset \widetilde{Q}^{\mathcal{F}}_{U' \cap N'}$, and $\underline{f}_{U'}$ is holomorphic in $\widetilde{Q}^{\mathcal{F}}_{U \setminus N}$, cf. (5).

The usual notion of conjugation of group representations induces the notion of *conjugation* (resp. *$\mathcal{N}$-conjugation*) between the monodromies $\mathfrak{M}^{\mathcal{F}}_S$ and $\mathfrak{M}^{\mathcal{F}'}_{S'}$ as a pair $(\mathfrak{g}, h)$ where $\mathfrak{g} : \Gamma_\infty \xrightarrow{\sim} \Gamma'_\infty$ is an isomorphism of groups and $h \in \mathrm{Isom}_{\underline{\mathrm{An}}}(\widetilde{Q}^{\mathcal{F}}_\infty, \widetilde{Q}^{\mathcal{F}'}_\infty)$ (resp. $h \in \mathrm{Isom}_{\mathcal{N}-\underline{\mathrm{An}}}(\widetilde{Q}^{\mathcal{F}}_\infty, \widetilde{Q}^{\mathcal{F}'}_\infty)$) satisfying the commutation relation $h_* \circ \mathfrak{M}^{\mathcal{F}}_S = \mathfrak{M}^{\mathcal{F}'}_{S'} \circ \mathfrak{g}$ where

$$(11) \qquad h_* : \mathrm{Aut}_{\underline{\mathrm{An}}}(\widetilde{Q}^{\mathcal{F}}_\infty) \xrightarrow{\sim} \mathrm{Aut}_{\underline{\mathrm{An}}}(\widetilde{Q}^{\mathcal{F}'}_\infty)$$
$$\varphi \mapsto h \circ \varphi \circ h^{-1}.$$

This notion of conjugation is algebraic and does not take into account the many topological information contained in the monodromy morphisms. Then we introduce the following more specific notion:

DEFINITION 3.4.3. – *A geometric conjugation (resp. geometric $\mathcal{N}$-conjugation) between the monodromies $\mathfrak{M}^{\mathcal{F}}_S$ and $\mathfrak{M}^{\mathcal{F}'}_{S'}$, is a conjugation (resp. $\mathcal{N}$-conjugation) $(\mathfrak{g}, h)$ such that there are a homeomorphism germ g not necessarily foliated from $(\mathbb{B}, S)$ into $(\mathbb{B}', S')$ preserving the orientations of $\mathbb{B}, \mathbb{B}'$ and S, S' and a pro-germ at infinity $\widetilde{g}$ from $(\widetilde{\mathbb{B}}^*, \infty)$ into $(\widetilde{\mathbb{B}'^*}, \infty)$ lifting g, i.e., $q'_\infty \circ \widetilde{g} = g \circ q_\infty$, such that $\mathfrak{g}$ equals the conjugation morphism defined by $\widetilde{g}$:*

$$(12) \qquad \mathfrak{g} = \widetilde{g}_* : \Gamma_\infty \to \Gamma'_\infty, \quad \varphi \mapsto \widetilde{g} \circ \varphi \circ \widetilde{g}^{-1}.$$

We then have the following commutative diagram:

$$
\begin{array}{ccc}
\Gamma_\infty & \xrightarrow{\mathfrak{M}_S^{\mathscr{G}}} & \mathrm{Aut}_{\underleftarrow{\mathrm{An}}}(\widetilde{\underleftarrow{Q}}_\infty^{\mathscr{G}}) \ , \\
{\scriptstyle g=\widetilde{g}_*}\downarrow & \circlearrowleft & \downarrow{\scriptstyle h_*} \\
\Gamma_\infty' & \xrightarrow{\mathfrak{M}_{S'}^{\mathscr{G}'}} & \mathrm{Aut}_{\underleftarrow{\mathrm{An}}}(\widetilde{\underleftarrow{Q}}_\infty^{\mathscr{G}'})
\end{array}
$$

and we will say that the triple $(g, \widetilde{g}, h)$ *represents geometrically* the ($\mathscr{N}$-)conjugation $(\mathfrak{g}, h)$.

REMARK 3.4.4. – Let $\Theta_t : U \xrightarrow{\sim} \Theta_t(U), U \subset \mathbb{B}$, be a *S-isotopy*, i.e., a continuous family of homeomorphisms depending on a parameter $t \in [0,1]$ such that for each $t \in [0,1]$, $\Theta_t(U)$ is an open neighborhood of S in $\mathbb{B}$, $\Theta_t(S) = S$, $\Theta_t(p) = p$ for all $p \in U \cap \partial\mathbb{B}$ and $\Theta_0 = \mathrm{id}_U$. We denote by $\widetilde{\Theta}_t$ the open embedding of $\widetilde{U}^*$ into $\widetilde{\mathbb{B}}^*$ lifting Θ_t, such that $\widetilde{\Theta}_t(\widetilde{p}) = \widetilde{p}$ if $q(\widetilde{p}) \in \partial\mathbb{B}$. It depends continuously on t, and $\widetilde{\Theta}_0 = id_{\widetilde{U}^*}$. Since the induced maps $(\widetilde{g} \circ \widetilde{\Theta}_t)_* : \Gamma_\infty \to \Gamma_\infty'$ depend continuously on t, they are constant and consequently if $(g, \widetilde{g}, h)$ is a geometric representation of an ($\mathscr{N}$-)conjugation $(\mathfrak{g}, h)$, then $(g \circ \Theta_1, \widetilde{g} \circ \widetilde{\Theta}_1, h)$ is also a geometric representation of $(\mathfrak{g}, h)$. In the same way, if $\Theta_t' : U' \xrightarrow{\sim} \Theta_t'(U'), U' \in \mathbb{B}'$, $t \in [0,1]$, is a S'-isotopy, then $(\Theta_1' \circ g, \widetilde{\Theta}_1' \circ \widetilde{g}, h)$ is also a geometric representation of $(\mathfrak{g}, h)$.

3.5. Marking a germ of curve

In [10] we have introduced the notion of marking of a germ of curve $(S', 0)$ by another germ of curve $(S, 0)$ as a fundamental equivalence class of germs of homeomorphisms of $(\mathbb{C}^2, 0)$ conjugating $(S, 0)$ to $(S', 0)$. From Proposition 2.8 of [10] it follows that two homeomorphic germs ϕ_0 and ϕ_1 are fundamentally equivalent if and only if one of the following equivalent properties is satisfied:

1. *there exist $\varepsilon > 0$ and a homotopy $\Phi \in \mathscr{C}^0(\mathbb{B}_\varepsilon^* \times [0,1], \mathbb{B}'^\star)$ such that $\Phi(\cdot, 0)$ and $\Phi(\cdot, 1)$ are representatives of the restrictions to the complement of S of the germs ϕ_0 and ϕ_1 respectively;*
2. *there exist representatives $\underline{\phi}_0, \underline{\phi}_1$ of the germs ϕ_0 and ϕ_1 on a small ball $\mathbb{B}_\varepsilon$ such that for all $p \in \mathbb{B}_\varepsilon^*$ there is a path α contained in $\mathbb{B}'^\star$ with endpoints $\underline{\phi}_0(p)$ and $\underline{\phi}_1(p)$ such that the morphism $\alpha_* : \dot{\gamma} \mapsto \dot{\alpha}^{-1} \vee \dot{\gamma} \vee \dot{\alpha}$ makes commutative the following diagram:*

$$
\begin{array}{ccc}
\pi_1(\mathbb{B}_\varepsilon^*, p) & \xrightarrow{\underline{\phi}_{0*}} & \pi_1(\mathbb{B}'^\star, \underline{\phi}_0(p)) \\
{\scriptstyle \underline{\phi}_{1*}}\searrow & & \swarrow{\scriptstyle \alpha_*} \\
& \pi_1(\mathbb{B}'^\star, \underline{\phi}_1(p)). &
\end{array}
$$

REMARK 3.5.1. – The morphism $\mathfrak{g}$ of a geometric ($\mathscr{N}$-)conjugation $(\mathfrak{g}, h)$ between the monodromies $\mathfrak{M}_S^{\mathscr{G}}$ and $\mathfrak{M}_{S'}^{\mathscr{G}'}$ determines a marking of $(S', 0)$ by $(S, 0)$, that we will denote again by $\mathfrak{g}$.

In [10] we have seen that every marking can be represented by a homeomorphism having good regularity properties. In order to precise these properties we need to consider some auxiliary geometric data. For each irreducible components D of $\mathscr{D}_S$ and D' of $\mathscr{D}_{S'}$ we fix germs of submersions

$$\pi_D : (\mathscr{B}_S, D) \to D \quad \text{and} \quad \pi_{D'} : (\mathscr{B}_{S'}, D) \to D'$$

whose respective restrictions to D and D' are the identity. They will be called the *Hopf fibrations of D and D'*. For each singular point $s \in \operatorname{Sing}(\mathscr{D}_S)$ and $s' \in \operatorname{Sing}(\mathscr{D}_{S'})$, we fix holomorphic charts $(x_s, y_s) : \mathscr{W}_s \xrightarrow{\sim} \mathbb{D}_1^2$, $(x_{s'}, y_{s'}) : \mathscr{W}_{s'} \xrightarrow{\sim} \mathbb{D}_1^2$, with disjoint domains and such that the local equations of $\mathscr{D}_S$ and $\mathscr{D}_{S'}$ are monomial in these coordinates. The collections

(13) $$\mathscr{L} := ((\pi_D)_D, (x_s, y_s)_s), \quad \mathscr{L}' := ((\pi_{D'})_{D'}, (x_{s'}, y_{s'})_{s'})$$

will be called *local data* for S and S'.

DEFINITION 3.5.2. – *A germ of homeomorphism g from $(\mathbb{B}, S)$ into $(\mathbb{B}', S')$ will be called* excellent *(resp. $\mathscr{N}$-excellent) with respect to the local data $\mathscr{L}$ and $\mathscr{L}'$ (resp. and a foliation $\mathscr{F}$ with separatrix set $S_\mathscr{F} = S$), if it admits a homeomorphic lifting G from $(\mathscr{B}_S, \mathscr{D}_S)$ into $(\mathscr{B}'_{S'}, \mathscr{D}_{S'})$ satisfying the following conditions:*

1. $G(\mathscr{D}_S) = \mathscr{D}_{S'}$ *and* $G(\mathscr{D}_S \cap \mathscr{W}_s) = \mathscr{D}_{S'} \cap \mathscr{W}'_{G(s)}$, $s \in \operatorname{Sing}(\mathscr{D}_S)$,
2. G *is holomorphic in a neighborhood of* $\operatorname{Sing}(\mathscr{D}_S)$
   (resp. $\operatorname{Sing}(\mathscr{D}_S) \setminus (\operatorname{Node}(\mathscr{F}) \cap \operatorname{Sing}(\mathscr{E}_S)))$,
3. *the restriction of G to a neighborhood of the adherence of*

$$\mathscr{D}_S \setminus \bigcup_{s \in \operatorname{Sing}(\mathscr{D}_S)} \mathscr{W}_s$$

commutes with the Hopf fibrations, i.e., $\pi_{G(D)} \circ G = G \circ \pi_D$,

Once we fix the local data $\mathscr{L}$ and $\mathscr{L}'$ we can precise the Marking Theorem stated in the introduction, in the following way.

THEOREM 3.5.3 ([10]). – *Every marking of S' by S possesses an excellent representative with respect to the local data $\mathscr{L}$ and $\mathscr{L}'$.*

COROLLARY 3.5.4. – *Every geometric ($\mathscr{N}$-)conjugation between the monodromies $\mathfrak{M}_S^{\mathscr{F}}$ and $\mathfrak{M}_{S'}^{\mathscr{F}'}$ can be geometrically represented by a triple $(g, \widetilde{g}, h)$, where g is excellent with respect to the local data $\mathscr{L}$ and $\mathscr{L}'$.*

DEFINITION 3.5.5. – *We say that a geometric ($\mathscr{N}$-)conjugation $(\mathfrak{g}, h)$ between the monodromies of two foliation germs $\mathscr{F}$ and $\mathscr{F}'$ preserves the Camacho-Sad indices if once we represent it geometrically by a triple $(g, \widetilde{g}, h)$ with g an excellent homeomorphism germ, then its lifting $G : (\mathscr{B}_S, \mathscr{D}_S) \to (\mathscr{B}'_{S'}, \mathscr{D}_{S'})$ satisfies $\operatorname{CS}(\underline{\mathscr{F}}, D, s) = \operatorname{CS}(\underline{\mathscr{F}}', G(D), G(s))$ for every irreducible component D of $\mathscr{D}_S$ and every singular point $s \in D$ of $\underline{\mathscr{F}}$.*

We will see in Section 7.3 that an ($\mathcal{N}$-)conjugation $(\mathfrak{g}, h)$ between the monodromies of $\mathcal{F}$ and $\mathcal{F}'$ preserves the Camacho-Sad indices if and only if there is a representative g of the marking determined by $\mathfrak{g}$ (cf. Remark 3.5.1) such that for each irreducible component $\check{S}$ of S we have the equality $\mathrm{CS}(\underline{\mathcal{F}}, \check{\mathcal{S}}, \check{s}) = \mathrm{CS}(\underline{\mathcal{F}}', \check{\mathcal{S}}', \check{s}')$ where $\check{\mathcal{S}}$ and $\check{\mathcal{S}}'$ denote respectively the strict transforms of $\check{S}$ and $\check{S}'$ and $\check{s}$ and $\check{s}'$ are their corresponding attaching points in the exceptional divisors.

3.6. Realizations of conjugations

Consider subsets $V \subset \mathbb{B}$ and $V' \subset \mathbb{B}'$ such that $\overline{V^*} \cap S$ and $\overline{V'^*} \cap S'$ are non-empty. We denote by $\Gamma_{\widetilde{V}^*, \infty}$ (resp. $\Gamma'_{\widetilde{V}'^*, \infty}$), the group of germs at infinity φ of deck transformations of the (possibly non-connected) covering $(\widetilde{V}^*, \infty)$ (resp. $(\widetilde{V}'^*, \infty)$), i.e., satisfying $q_\infty \circ \varphi = q_\infty$ (resp. $q' \circ \varphi = q'_\infty$). Clearly, the restriction maps define group monomorphisms $\iota : \Gamma_\infty \hookrightarrow \Gamma_{\widetilde{V}^*, \infty}$ and $\iota' : \Gamma'_\infty \hookrightarrow \Gamma'_{\widetilde{V}'^*, \infty}$.

DEFINITION 3.6.1. – *A geometric conjugation (resp. geometric $\mathcal{N}$-conjugation) $(\mathfrak{g}, h)$ between the monodromies $\mathfrak{M}_S^{\mathcal{F}}$ and $\mathfrak{M}_{S'}^{\mathcal{F}'}$ is called* realizable over the germs (V, S) and (V', S') *if there exist a germ of homeomorphism ψ from (V, S) into (V', S'), and a continuous pro-germ at infinity*

$$\widetilde{\psi} \in \mathrm{Hom}_{\underleftarrow{\mathrm{Top}}}((\widetilde{V}^*, \infty), (\widetilde{V}'^*, \infty))$$

lifting ψ, i.e., $q'_\infty \circ \widetilde{\psi} = \psi \circ q_\infty$, such that the following diagrams commute:

$$(\star) \quad \begin{array}{ccc} (\widetilde{V}^*, \infty) & \xrightarrow{\widetilde{\psi}} & (\widetilde{V}'^*, \infty) \\ \tau_{\widetilde{V}^*} \downarrow & \circlearrowright & \downarrow \tau_{\widetilde{V}'^*} \\ \widetilde{Q}_\infty^{\mathcal{F}} & \xrightarrow{h} & \widetilde{Q}_\infty^{\mathcal{F}'} \end{array} \qquad (\star\star) \quad \begin{array}{ccc} \Gamma_\infty & \xhookrightarrow{\iota} & \Gamma_{\widetilde{V}^*, \infty} \\ \mathfrak{g} \downarrow & \circlearrowright & \downarrow \widetilde{\psi}_* \\ \Gamma'_\infty & \xhookrightarrow{\iota'} & \Gamma'_{\widetilde{V}'^*, \infty} \end{array}$$

where $\tau_{\widetilde{V}^*}$ and $\tau_{\widetilde{V}'^*}$ are the canonical pro-germs defined in Section 3.3 and $\widetilde{\psi}_*$ is the conjugation morphism $\varphi \mapsto \widetilde{\psi} \circ \varphi \circ \widetilde{\psi}^{-1}$. We will say then that the triple $(\psi, \widetilde{\psi}, h)$ is a realization *of $(\mathfrak{g}, h)$ over the germs of V and V'.*

REMARK 3.6.2. – If $g : (\mathbb{B}, S) \to (\mathbb{B}', S')$ is an ($\mathcal{N}$-)transversely holomorphic conjugation between $\mathcal{F}$ and $\mathcal{F}'$, then every lifting $\widetilde{g} : (\widetilde{\mathbb{B}}^*, \infty) \to (\widetilde{\mathbb{B}}'^*, \infty)$ of g determines a ($\mathcal{N}$-)$\underleftarrow{\mathrm{An}}$-isomorphism h from $\widetilde{Q}_\infty^{\mathcal{F}}$ onto $\widetilde{Q}_\infty^{\mathcal{F}'}$. The pair $(\widetilde{g}_*, h)$ constituted by the conjugation isomorphism (12) is a geometric ($\mathcal{N}$-)conjugation between the monodromies of these germs of foliations and $(g, \widetilde{g}, h)$ is a geometric realization of this conjugation over $(\mathbb{B}, S)$ and $(\mathbb{B}', S')$.

REMARK 3.6.3. – If W is a submanifold of V such that $\overline{W^*} \cap S \neq \varnothing$, then the restriction $(\psi_{|W}, \widetilde{\psi}_{|\widetilde{W}^*}, h)$ of a realization $(\psi, \widetilde{\psi}, h)$ of $(\mathfrak{g}, h)$ over V and V', is a realization of $(\mathfrak{g}, h)$ over W and $\psi(W)$.

An S-isotopy $\Theta_t : U \to \Theta_t(U)$, $U \subset \mathbb{B}$, in the sense of Remark 3.4.4 will be called *$\mathcal{F}$-isotopy*, if for all $p \in U$ the path $[0, 1] \ni t \mapsto \Theta_t(p)$ is contained in a leaf of $\mathcal{F}$. We have the following invariance property:

PROPOSITION 3.6.4. – *Let* $\Theta_t : U \to \Theta_t(U)$ *(resp.* $\Theta_t' : U' \to \Theta_t'(U')$*),* $t \in [0,1]$*, be a* $\mathscr{F}$*-isotopy (resp.* $\mathscr{F}'$*-isotopy), defined in an open neighborhood* $U \supset S$ *of* $\mathbb{B}$ *(resp.* $U' \supset S'$ *of* $\mathbb{B}'$*) and let* $(g, \widetilde{g}, h)$ *be a realization of a geometric (*$\mathscr{N}$*-)conjugation* $(\mathfrak{g}, h)$ *between the monodromies of* $\mathscr{F}$ *and* $\mathscr{F}'$*, over subsets* V *and* V' *of* $\mathbb{B}^*$ *and* $\mathbb{B}'^\star$*. Then* $(\Theta_1' \circ g \circ \Theta_1^{-1}, \widetilde{\Theta}_1' \circ \widetilde{g} \circ \widetilde{\Theta}_1^{-1}, h)$ *is a realization of* $(\mathfrak{g}, h)$ *over* $\Theta_1(V)$ *and* $\Theta_1'^{-1}(V')$*, where the lifting* $\widetilde{\Theta}_t$ *and* $\widetilde{\Theta}_t'$ *are defined as in Remark 3.4.4.*

Proof. – The idea is quite simple: on one hand $\widetilde{\Theta}_t$ preserves the leaves of $\widetilde{\mathscr{F}}$ and consequently the corresponding diagram $(\star)$ is commutative; on the other hand, Remark 3.4.4 implies the equality $(\widetilde{\Theta}_1' \circ \widetilde{g} \circ \widetilde{\Theta}_1^{-1})_* = \widetilde{g}_*$ and therefore the diagram $(\star\star)$ is also commutative. We leave the details of the proof to the reader. $\qquad\square$

4. Monodromy and projective holonomy

4.1. Holonomy representation of a JSJ block

First of all, we recall some classical notions which will be used in the sequel. For a curve $\mathscr{D}$, the *valence* of an irreducible component D of $\mathscr{D}$ is the number $v(D)$ of irreducible components of $\mathscr{D}$ other than D meeting D. We call *dead branch of the exceptional divisor* $\mathscr{E}_S$ *any* maximal connected union of components of $\mathscr{E}_S$ having valence two in $\mathscr{D}_S$, except for one of them which has valence one in $\mathscr{D}_S$. Every dead branch $\mathscr{M}$ has a single *attaching point* belonging to a unique component of $\mathscr{E}_S$ having valence at least three.

Let $F : \mathbb{B} \to \mathbb{C}$ (resp. $F' : \mathbb{B}' \to \mathbb{C}$) be a reduced equation of the separatrix curve S (resp. S'), and consider $\mathscr{L}$, $\mathscr{L}'$ two local data as in (13). For $\varepsilon > 0$ small enough, the composition of these equations by the reduction morphisms E_S and $E_{S'}$ define the real smooth hypersurfaces $\{|F \circ E_S| = \varepsilon\}$ and $\{|F' \circ E_{S'}| = \varepsilon\}$. These hypersurfaces bound the *Milnor tubes* of $\mathscr{D}_S$ and $\mathscr{D}_{S'}$,

$$(14) \qquad \mathscr{T}_\varepsilon := \{|F \circ E_S| \le \varepsilon\} \subset \mathscr{B}_S, \quad \mathscr{T}_\varepsilon' := \{|F' \circ E_{S'}| \le \varepsilon\} \subset \mathscr{B}_{S'},$$

and they are transverse to the hypersurfaces $\{|x_s| = 1\}$, $\{|y_s| = 1\}$ and $\{|x_{s'}| = 1\}$, $\{|y_{s'}| = 1\}$, for all $s \in \mathrm{Sing}(\mathscr{D}_S)$, $s' \in \mathrm{Sing}(\mathscr{D}_{S'})$, as well as to the spheres $E_S^{-1}(\partial\mathbb{B})$ and $E_{S'}^{-1}(\partial\mathbb{B}')$. We extend for subset of these Milnor tubes, the convenient notation introduced in (2).

CONVENTIONS 4.1.1. – For $\mathscr{A} \subset \mathscr{T}_\varepsilon$ and $\mathscr{A}' \subset \mathscr{T}_{\varepsilon'}'$, we set

$$\widetilde{\mathscr{A}}^* := \underline{q}^{-1}(E_S(\mathscr{A})), \qquad \widetilde{\mathscr{A}}'^\star := \underline{q}'^{-1}(E_{S'}(\mathscr{A}')),$$

and we consider the following universal coverings:

$$\underline{q} := E_S^{-1} \circ q_{|\widetilde{\mathscr{T}}_\varepsilon^*} : \widetilde{\mathscr{T}}_\varepsilon^* \longrightarrow \mathscr{T}_\varepsilon^*, \quad \underline{q}' := E_{S'}^{-1} \circ q'_{|\widetilde{\mathscr{T}}_{\varepsilon'}'^\star} : \widetilde{\mathscr{T}}_{\varepsilon'}'^\star \longrightarrow \mathscr{T}_{\varepsilon'}'^\star.$$

Given a germ of homeomorphism $\phi_S : (\mathscr{A}, \mathscr{D}_S) \to (\mathscr{A}', \mathscr{D}_{S'})$ and a pro-germ at infinity $\widetilde{\phi}_\infty : (\widetilde{\mathscr{A}}^*, \infty) \to (\widetilde{\mathscr{A}}'^\star, \infty)$ lifting ϕ_S, we will say that the triple $(\phi_S, \widetilde{\phi}_\infty, h)$ *carries out over* $\mathscr{A}$ *and* $\mathscr{A}'$ *a geometric conjugation (resp. a geometric* $\mathscr{N}$*-conjugation)* $(\mathfrak{g}, h)$, if the triple $(\phi_S^\flat, \widetilde{\phi}_\infty, h)$, with $\phi_S^\flat := E_{S'} \circ \phi_S \circ E_{S|\mathscr{A}^*}^{-1}$, is a realization of the geometric conjugation (resp. geometric $\mathscr{N}$-conjugation) $(\mathfrak{g}, h)$ over $E_S(\mathscr{A})$ and $E_{S'}(\mathscr{A}')$.

Let D be an irreducible component of $\mathcal{D}_S$ having valence $v = v(D) \geq 3$. If D has $v - r$ attaching points belonging to dead branches, we numerate the points $s_1, \ldots, s_v$ of $\mathrm{Sing}(\mathcal{D}_S) \cap D$ in such a way that $\{s_j \mid j > r\}$ is the set of attaching points of the adjacent dead branches to D. We also set

(15) $$D^\sharp := D \setminus \cup_{j=1}^r \{|x_{s_j}| < 1\}, \quad D^\circ := D \setminus \cup_{j=1}^v \{|x_{s_j}| < 1\}.$$

We assume that for $j = 1, \ldots, v$, $y_{s_j} = 0$ is a local reduced equation of D.

DEFINITION 4.1.2. – *We define the* Jaco-Shalen-Johannson block *(JSJ for short)* $B_D(\varepsilon)$ *of* $\mathcal{T}_\varepsilon$ *associated to D as the adherence of the connected component of* $\mathcal{T}_\varepsilon \setminus \cup_{j=1}^r \{|x_{s_j}| = 1\}$ *which contains $D^\sharp$.*

We fix $\varepsilon > 0$ small enough and we denote $B_D(\varepsilon)$ and $\mathcal{T}_\varepsilon$ simply by B_D and $\mathcal{T}$. Then the following properties hold, cf. [9]:

- $\mathcal{T}$ is a deformation retract of $\mathcal{B}_S$, and for $0 < \varepsilon' \leq \varepsilon$, $B_D(\varepsilon')$ and $\mathcal{T}_{\varepsilon'}$ are deformation retracts of B_D and $\mathcal{T}$ respectively;
- B_D^* is incompressible in $\mathcal{T}^*$, hence it is so in $E_S^{-1}(\mathbb{B}^*)$;
- a presentation of the fundamental group of B_D^* by generators and relations can be obtained in the following way. We consider loops $\gamma_1, \ldots, \gamma_v$ in $B_D \cap \pi_D^{-1}(D^\circ)$ having the same origin m and such that the projections $\pi_D \circ \gamma_j$ are the boundaries of closed conformal disks $\mathcal{V}_j \subset D$ satisfying $\overset{\circ}{\mathcal{V}}_j \cap \mathrm{Sing}(\mathcal{D}_S) = \mathcal{V}_j \cap \mathrm{Sing}(\mathcal{D}_S) = \{s_j\}$; then we consider a loop c in the fiber $\Delta := \pi_D^{-1}(m_0)$, $m_0 := \pi_D(m)$, with the same origin m and having rotation index one with respect to the point m_0. Then we have

$$\pi_1(B_D^*, m) = \langle \dot{c}, \dot{\gamma}_1, \ldots \dot{\gamma}_v \mid [\dot{\gamma}_j, \dot{c}] = 1, \dot{\gamma}_k^{p_k} = \dot{c}^{q_k} \rangle_{\substack{j=1,\ldots,v \\ k=r+1,\ldots,v}},$$

where $\gcd(p_k, q_k) = 1$ and $-\frac{q_k}{p_k}$ is the Camacho-Sad index of $\underline{\mathcal{G}}$ along D at the point s_k;
- the germ of $\underline{\mathcal{G}}$ at each point $s = s_{r+1}, \ldots, s_v$ possesses a holomorphic first integral that can be written as $x_s^{p_k} y_s^{q_k} A(x_s, y_s)$, $A(0,0) \neq 0$.

Since $\mathcal{H}_D(\dot{\gamma})$ is the map sending a point p of Δ to the end of the path having origin p and lifting γ^{-1} in the leaf of $\underline{\mathcal{G}}$ passing through p, the kernel of the holonomy representation of $\underline{\mathcal{G}}$ along D°,

(16) $$\mathcal{H}_D : \pi_1(D^\circ, m_0) = \mathbb{Z}\dot{\gamma}_{r+1} * \cdots * \mathbb{Z}\dot{\gamma}_v \longrightarrow \mathrm{Diff}(\Delta, m_0),$$

contains the normal subgroup generated by the elements $\dot{\gamma}_k^{p_k}$, for $k = r+1, \ldots, v$. The morphism $\mathcal{H}_D$ factorizes through a morphism $\mathcal{H}_D^{\mathrm{orb}}$ defined on the quotient group

$$\pi_1^{\mathrm{orb}}(D^\sharp, m_0) := \pi_1(D^\circ, m_0)/ \ll \dot{\gamma}_{r+1}^{p_{r+1}}, \ldots, \dot{\gamma}_v^{p_v} \gg .$$

The morphism π_{D*} from $\pi_1(B_D \cap \pi_D^{-1}(D^\circ), m)$ onto $\pi_1(D^\circ, m_0)$ induced by the fibration π_D, determines a morphism π_{D*}^{orb} which enters in the following exact sequence

(17) $$1 \longrightarrow \pi_1(\Delta^*, m) = \mathbb{Z}\dot{c} \longrightarrow \pi_1(B_D^*, m) \overset{\pi_{D*}^{\mathrm{orb}}}{\longrightarrow} \pi_1^{\mathrm{orb}}(D^\sharp, m_0) \longrightarrow 1 .$$

DEFINITION 4.1.3. – *We call the morphism* $\mathcal{H}_{B_D} := \mathcal{H}_D^{\mathrm{orb}} \circ \pi_{D*}^{\mathrm{orb}}$,

$$\mathcal{H}_{B_D} : \pi_1(B_D^*, m) \longrightarrow \mathrm{Diff}(\Delta, m_0), \quad \dot{\gamma} \mapsto \mathcal{H}_{B_D}(\dot{\gamma}) = \mathcal{H}_D(\pi_D \circ \dot{\gamma}),$$

the holonomy representation *of* $\mathcal{F}$ *along* B_D *realized on the transverse section* Δ.

4.2. Extended holonomy and monodromy

With Conventions 4.1.1 and the precedent notations, we denote by $(\widetilde{\Delta}^{*\alpha})_{\alpha \in \pi_0(\widetilde{\Delta}^*)}$ and $(\widetilde{B}_D^{*\beta})_{\beta \in \pi_0(\widetilde{B}_D^*)}$ the collection of connected components of $\widetilde{\Delta}^*$ (resp. of $\widetilde{B}_D^*$). Thanks to the incompressibility of B_D^* in $E_S^{-1}(\mathbb{B}^*)$, the restriction of $\underline{q}$ to each connected component $\widetilde{B}_D^{*\beta}$ is a universal covering of B_D^*. Thus, once we fix a point $m \in B_D^*$ and $\widetilde{m} \in \underline{q}^{-1}(m) \cap \widetilde{B}_D^{*\beta}$, the group Γ_∞ can be canonically identified with $\pi_1(\mathcal{T}^*, m) \simeq \pi_1(\mathbb{B}^*, m)$ and the subgroup $\Gamma_\infty(\beta)$ consisting of the elements φ in Γ_∞ preserving $B_D^{*\beta}$, can be identified with $\pi_1(B_D^*, m)$. If $\widetilde{m}$ belongs to $\widetilde{\Delta}^{*\alpha} \subset \widetilde{B}_D^{*\beta}$ then we have the following exact sequence of groups:

$$1 \longrightarrow \Gamma_\infty(\beta, \alpha) \longrightarrow \Gamma_\infty(\beta) \xrightarrow{\sigma} \pi_1^{\mathrm{orb}}(D^\sharp, m_0) \longrightarrow 1,$$

with $\Gamma_\infty(\beta, \alpha) \simeq \pi_1(\Delta^*, m)$ denoting the subgroup consisting of those $\varphi \in \Gamma_\infty(\beta)$ which preserve $\widetilde{\Delta}^{*\alpha}$ and σ is the well-defined morphism determined by

$$\sigma(\varphi) := \pi_D^{\mathrm{orb}}(\underline{q} \circ \mu_\varphi), \qquad \varphi \in \Gamma_\infty^\beta,$$

where μ_φ is a path in $\widetilde{B}_D^{*\beta}$ whose endpoints are $\widetilde{m}$ and $\varphi(\widetilde{m})$. Notice that $\Gamma_\infty(\beta, \alpha)$ is a normal subgroup in $\Gamma_\infty(\beta)$, because $\pi_1(\Delta^*, m)$ is the center of $\pi_1(B_D^{*\beta})$; this means that each $\varphi \in \Gamma_\infty(\beta, \alpha)$ preserves every component $\widetilde{\Delta}^{*\alpha'} \subset B_D^{*\beta}$.

PROPOSITION 4.2.1. – *If* $\widetilde{\Delta}^{*\alpha}$ *and* $\widetilde{\Delta}^{*\alpha'}$ *are contained in the same connected component* $\widetilde{B}_D^{*\beta}$*, then there is a unique pro-germ* $h_{\alpha'\alpha} : (\widetilde{\Delta}^{*\alpha}, \infty) \to (\widetilde{\Delta}^{*\alpha'}, \infty)$ *commuting with the canonical pro-germs, i.e.,* $\tau_{\widetilde{\Delta}^{*\alpha'}} \circ h_{\alpha'\alpha} = \tau_{\widetilde{\Delta}^{*\alpha}}$.

Proof. – For all $\breve{U} \in \mathfrak{U}_{\mathcal{F}, \Sigma}$ put $U := E_S^{-1}(\breve{U})$ and denote by $W_U^{\alpha'}$ the $\widetilde{\mathcal{F}}$-saturation of $\widetilde{\Delta}^{*\alpha'} \cap \widetilde{U}^*$ inside $\widetilde{B}_D^{*\beta} \cap \widetilde{U}^*$. The map

$$h_U : \mathfrak{O}_U^{\alpha'\alpha} := W_U^{\alpha'} \cap \widetilde{\Delta}^{*\alpha} \to \widetilde{\Delta}^{*\alpha'}$$

obtained by following the leaves of $\widetilde{\mathcal{F}}_{|W_U^{\alpha'}}$ is defined without ambiguity because every leaf meets each transversal $\widetilde{\Delta}^{*\alpha}$ and $\widetilde{\Delta}^{*\alpha'}$ in at most one point, cf. Section 2.2. We will see that $\mathfrak{O}_U^{\alpha'\alpha}$ always contains a non-empty open set of type $\widetilde{V}^* \cap \widetilde{\Delta}^{*\alpha}$, with $V := E_S^{-1}(\breve{V})$, $\breve{V} \in \mathfrak{U}_{\mathcal{F}, \Sigma}$. In order to conclude, it suffices to put

$$h_{\alpha'\alpha} := (\varinjlim_{\breve{V}} h_{VU})_{\breve{U} \in \mathfrak{U}_{\mathcal{F}, \Sigma}}, \quad \text{with} \quad h_{VU} := h_{U|\widetilde{V}^* \cap \widetilde{\Delta}^{*\alpha}} : \widetilde{V}^* \cap \widetilde{\Delta}^{*\alpha} \to \widetilde{\Delta}^{*\alpha'}.$$

Consider $\varphi \in \Gamma_\infty(\beta, \alpha)$ and choose $\breve{V} \in \mathfrak{U}_{\mathcal{F}, \Sigma}$, $\breve{V} \subset \breve{U}$, small enough so that all $p \in V \cap \Delta^*$ be the origin of a path γ_p ending in Δ^*, which is contained in a leaf of the restriction of $\mathcal{F}$ to $B_D^* \cap U \cap \pi_D^{-1}(D^\circ)$, and such that the homotopy class of $\pi_D \circ \gamma_p$ in $\pi_1^{\mathrm{orb}}(D^\sharp, m_0)$ coincides with $\sigma(\varphi)$. The lift $\mu_{\widetilde{p}}$ of γ_p in $\widetilde{B}_D^{*\beta}$, passing through an arbitrary point $\widetilde{p}$ of $\underline{q}^{-1}(p) \cap \widetilde{\Delta}^{*\alpha}$, is contained in a leaf of $\widetilde{\mathcal{F}}$. We will see that its endpoint, which coincides with $h_U(\widetilde{p})$, always belongs to $\widetilde{\Delta}^{*\alpha'}$; then the inclusion $\mathfrak{O}_U^{\alpha'\alpha} \supset \widetilde{V}^* \cap \widetilde{\Delta}^{*\alpha}$ will follow.

We consider a path ξ in $\widetilde{\Delta}^*$ having origin $\mu_{\widetilde{p}}(1)$ and endpoint in $\underline{q}^{-1}(m)$, as well as a path δ having origin in $\underline{q}^{-1}(m)$ and endpoint $\widetilde{p}$. The homotopy class $\dot{\zeta} \in \pi_1(B_D^*, m)$ of the loop $\zeta := \underline{q} \circ (\delta \vee \mu_{\widetilde{p}} \vee \xi)$ satisfies:

$$\pi_{D*}^{\mathrm{orb}}(\dot{\zeta}) = \overline{\pi_D \circ \gamma_p} = \sigma(\varphi),$$

where $\overline{\pi_D \circ \gamma_p}$ denotes the class of $\pi_D \circ \gamma_p$ in $\pi_1^{\mathrm{orb}}(D^\sharp, m_0)$. Let μ_φ be a path in $\widetilde{B}^{*\beta}$ joining $\widetilde{m}$ to $\varphi(\widetilde{m})$. Thanks to (17), the homotopy class of the loop $\underline{q} \circ \mu_\varphi$ in $\pi_1(B_D^*, m)$ differs from $\dot{\zeta}$ in an element of $\pi_1(\Delta^*, m)$; therefore the paths $\delta \vee \mu_{\widetilde{p}} \vee \xi$ and μ_φ have their endpoints on the same connected component of $\widetilde{\Delta}^*$; the same property holds for $\mu_{\widetilde{p}}$ and μ_φ. $\qquad\square$

For three components $\widetilde{\Delta}^{*\alpha}, \widetilde{\Delta}^{*\alpha'}, \widetilde{\Delta}^{*\alpha''}$ contained in $\widetilde{B}_D^{*\beta}$, we clearly have the relation

$$h_{\alpha''\alpha'} \circ h_{\alpha'\alpha} = h_{\alpha''\alpha}.$$

On the other hand, the above constructions are "compatible" with the action of $\Gamma_\infty(\beta)$, because Γ_∞ preserves $\widetilde{\mathscr{F}}$. More precisely, with the above notations, if $\varphi \in \Gamma_\infty(\beta)$ then the paths $\varphi \circ \mu_{\widetilde{p}}$ and $\mu_{\varphi(\widetilde{p})}$ coincide. Thus, denoting also by $\varphi : \pi_0(\widetilde{\Delta}^*) \xrightarrow{\sim} \pi_0(\widetilde{\Delta}^*)$ the bijection induced by a deck transformation φ, we have that

$$\varphi \circ h_{\alpha'\alpha} \circ \varphi^{-1} = h_{\varphi(\alpha')\varphi(\alpha)}.$$

We easily deduce that the map

$$\widetilde{\mathscr{H}}_D^\alpha : \Gamma_\infty(\beta) \longrightarrow \mathrm{Aut}(\widetilde{\Delta}^{*\alpha}, \infty), \quad \varphi \mapsto h_{\alpha\varphi(\alpha)} \circ \varphi = h_{\varphi(\alpha)\alpha}^{-1} \circ \varphi,$$

is a morphism of groups. In the case that $E_{\mathscr{F}}$ consists in a single blow-up this morphism was considered already in [14] where it was called *extended holonomy*. We adopt their definition in the general context.

DEFINITION 4.2.2. – *We will call* $\widetilde{\mathscr{H}}_D^\alpha$, *the* extended holonomy morphism of D over $\widetilde{\Delta}^{*\alpha}$.

In order to justify this definition we note that if $\dot{\mu} \in \pi_1(B_D^*, m)$ and $\varphi \in \Gamma_\infty(\beta)$ satisfy $\pi_{D*}^{\mathrm{orb}}(\dot{\mu}) = \sigma(\varphi)$, then $\widetilde{\mathscr{H}}_D^\alpha(\varphi)$ is the lifting of the holonomy diffeomorphism $\mathscr{H}_{B_D}(\dot{\mu})$ on the connected component $\widetilde{\Delta}^{*\alpha}$, considered as a universal covering of Δ^*. We finally obtain the following commutative diagram:

$$
\begin{array}{ccccc}
(\Delta, m_0) & \xleftarrow{\ \underline{q}_\infty\ } & (\widetilde{\Delta}^{*\alpha}, \infty) & \xhookrightarrow{\ \tau_{\widetilde{\Delta}^{*\alpha}}\ } & \widetilde{Q}_\infty^{\mathscr{F}} \\[2pt]
{\scriptstyle(\diamond)_\varphi} \quad \mathscr{H}_{B_D}(\dot{\mu})\Big\downarrow & & \widetilde{\mathscr{H}}_D^\alpha(\varphi)\Big\downarrow & & \Big\downarrow\mathfrak{M}_S^{\mathscr{F}}(\varphi) \ . \\[2pt]
(\Delta, m_0) & \xleftarrow{\ \underline{q}_\infty\ } & (\widetilde{\Delta}^{*\alpha}, \infty) & \xhookrightarrow{\ \tau_{\widetilde{\Delta}^{*\alpha}}\ } & \widetilde{Q}_\infty^{\mathscr{F}}
\end{array}
$$

4.3. Relationship between holonomy and monodromy conjugations

We fix a geometric ($\mathcal{N}$-)conjugation $(\mathfrak{g}, h)$ between the monodromies $\mathfrak{M}_S^{\mathcal{F}}$ and $\mathfrak{M}_{S'}^{\mathcal{F}'}$ of the General Type foliations $\mathcal{F}$ and $\mathcal{F}'$. Thanks to Corollary 3.5.4, there exists a geometric representation $(g, \widetilde{g}, h)$ of $(\mathfrak{g}, h)$, cf. Definition 3.4.3, where $g : (\mathbb{B}, S) \to (\mathbb{B}', S')$ is the germ of an excellent homeomorphism and $G : (\mathscr{B}_S, \mathscr{D}_S) \to (\mathscr{B}'_{S'}, \mathscr{D}_{S'})$ is its lifting to the total spaces of their reductions of singularities. Let (Δ, m_0), $m_0 \notin \mathrm{Sing}(\mathscr{D}_S)$, be a fibre of the Hopf fibration of an irreducible component D of $\mathscr{D}_S$, such that $(\Delta', m_0') := (G(\Delta), G(m_0))$ is also a fibre of the Hopf fibration associated to the component $D' := G(D)$ of $\mathscr{D}_{S'}$. Finally, we will identify Δ and Δ' with their images $E_S(\Delta)$ and $E_{S'}(\Delta')$ by the reduction of singularities maps.

THEOREM 4.3.1. – *If there is a realization $(\psi, \widetilde{\psi}, h)$ of $(\mathfrak{g}, h)$ over Δ and Δ', then ψ and the restriction $G_{|D} : D \xrightarrow{\sim} D'$ conjugate the holonomy representations associated to the irreducible components D and D', i.e., the following diagram is commutative:*

$$
\begin{array}{ccc}
\pi_1(D^\circ, m_0) & \xrightarrow{\;\mathscr{H}_D\;} & \mathrm{Diff}(\Delta, m_0) \\[2pt]
G_* \downarrow & \circlearrowleft & \downarrow \psi_* \\[2pt]
\pi_1(D'^\circ, m_0) & \xrightarrow{\;\mathscr{H}_{D'}\;} & \mathrm{Diff}(\Delta', m_0')
\end{array}
$$

where $\psi_*(\varphi) := \psi \circ \varphi \circ \psi^{-1}$ and G_* is induced by the restriction of G to D°.

Proof. – Consider $\dot{\gamma} \in \pi_1(D^\circ, m_0)$, $\dot{\mu} \in \pi_1(\widetilde{B}_D^*, m)$ and $\varphi \in \Gamma_\infty(\beta)$ such that $\pi_{D*}^{\mathrm{orb}}(\dot{\mu}) = \sigma(\varphi) = \dot{\gamma}$ and therefore $\mathscr{H}_{B_D}(\dot{\mu}) = \mathscr{H}_D(\dot{\gamma})$. Consider also the diagram of Figure 1. Both frontal sides (behind and ahead) are constituted by the commutative diagrams $(\diamond)_\varphi$ and $(\diamond)_{\mathfrak{g}(\varphi)}$; both lateral sides are constituted by the commutative diagram $(\star)$ and that one expressing that $\widetilde{\psi}$ lifts ψ; the commutativity of the bottom horizontal diagram, $h \circ \mathfrak{M}_S^{\mathcal{F}}(\varphi) = \mathfrak{M}_{S'}^{\mathcal{F}'}(\mathfrak{g}(\varphi)) \circ h$, follows from the fact that $(\mathfrak{g}, h_*)$ is a conjugation between $\mathfrak{M}_S^{\mathcal{F}}$ and $\mathfrak{M}_{S'}^{\mathcal{F}'}$. Since the canonical pro-germs $\tau_{\widetilde{\Delta}*\alpha}$ and $\tau_{\widetilde{\Delta}'*\alpha'}$ are monomorphisms, cf. Proposition 3.3.1, the median horizontal diagram is also commutative, i.e., $\widetilde{\psi} \circ \widetilde{\mathscr{H}}_D^\alpha(\varphi) = \widetilde{\mathscr{H}}_{D'}^{\alpha'} \circ \widetilde{\psi}$. Finally, the commutativity of the top horizontal diagram,

$$\psi \circ \mathscr{H}_D(\dot{\gamma}) = \mathscr{H}_{D'}(G_*(\dot{\gamma})) \circ \psi \,,$$

follows from the fact that the pro-germs $\underline{q}_\infty$ and $\underline{q}'_\infty$ are epimorphisms. □

5. Statements and proofs of Theorems I and II

We keep the notations introduced in the precedent sections about the germs of foliations $\mathcal{F}$ et $\mathcal{F}'$, in particular (2), (3), (9) and (10). We shall introduce here two statements that imply Theorems I and II in the introduction.
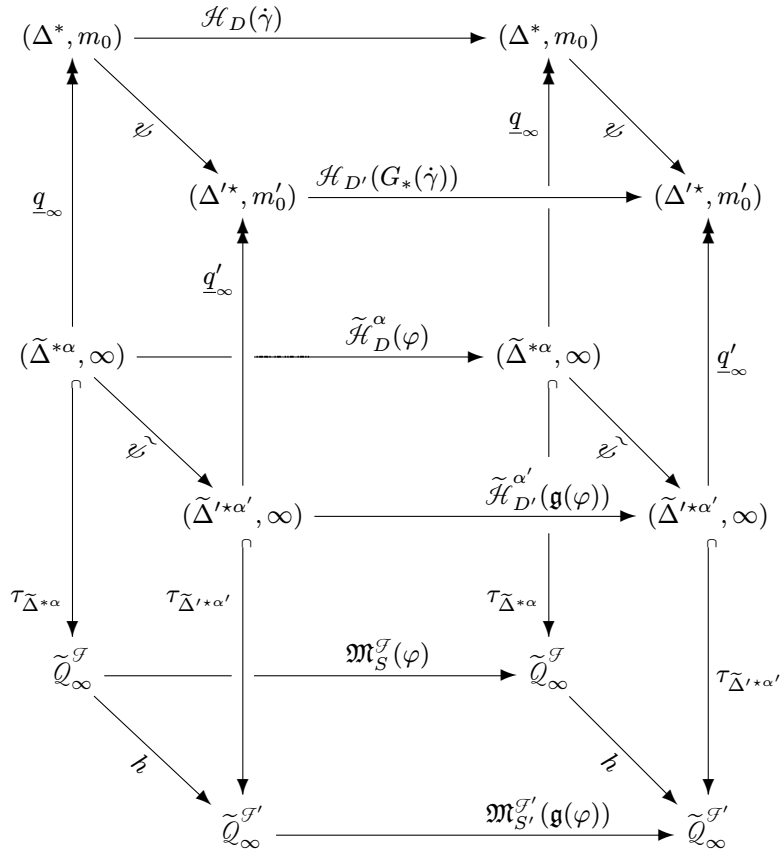
FIGURE 1. Diagram concerning the conjugation between the holonomies and monodromies.

THEOREM 5.0.1 (Theorem of invariance). – *Assume that $\mathcal{F}$ and $\mathcal{F}'$ are of General Type and they are conjugate by a germ of transversely holomorphic (resp. $\mathcal{N}$-transversely holomorphic) homeomorphism $\Psi : (\mathbb{C}^2, 0) \xrightarrow{\sim} (\mathbb{C}^2, 0)$. Consider a germ $g : (\mathbb{C}^2, 0) \xrightarrow{\sim} (\mathbb{C}^2, 0)$ of excellent homeomorphism fundamentally equivalent to Ψ, cf. Section 3.5, and denote by $G : (\mathcal{B}_S, \mathcal{D}_S) \to (\mathcal{B}'_{S'}, \mathcal{D}_{S'})$ its lifting. Then,*

1. *for each irreducible component D of $\mathcal{D}_S$ and for each singular point $s \in \mathrm{Sing}(\underline{\mathcal{F}}) \cap D$, we have equality of Camacho-Sad indices:*

(18) $$\mathrm{CS}(\underline{\mathcal{F}}, D, s) = \mathrm{CS}(\underline{\mathcal{F}'}, G(D), G(s));$$

2. *there is a geometric conjugation (resp. a geometric $\mathcal{N}$-conjugation) $(\mathfrak{g}, h)$ between the monodromies $\mathfrak{M}_S^{\mathcal{F}}$ and $\mathfrak{M}_{S'}^{\mathcal{F}'}$ realizable over some $\mathcal{N}$-collections of transversals (i.e., satisfying Condition (σ) in the introduction) of $\mathcal{F}$ and $\mathcal{F}'$, such that Ψ is a representative of the marking determined by $\mathfrak{g}$.*

The following theorem can be considered as a sort of converse of the precedent result.

THEOREM 5.0.2 (Theorem of classification). – *Let $(\mathfrak{g}, h)$ be a geometric $\mathcal{N}$-conjugation between the monodromies $\mathfrak{M}_S^{\mathcal{F}}$ and $\mathfrak{M}_{S'}^{\mathcal{F}'}$ of two General Type foliations, which possesses a realization $(\psi, \widetilde{\psi}, h)$ over $\mathcal{N}$-collections of transversals Σ and Σ' of $\mathcal{F}$ and $\mathcal{F}'$. Denoting by $(g, \widetilde{g}, h)$ a geometric representation of $(\mathfrak{g}, h)$, assume that the following conditions hold:*

1. *for each connected component $\check{\Sigma}$ of Σ intersecting an irreducible component $\check{S}$ of S we have $\psi(\check{\Sigma}) \cap g(\check{S}) \neq \varnothing$,*

2. *along each irreducible component $\check{S}$ of S, we have coincidence of Camacho-Sad indices: $\mathrm{CS}(\underline{\mathcal{F}}, \breve{\mathcal{S}}, \breve{s}) = \mathrm{CS}(\underline{\mathcal{F}'}, \breve{\mathcal{S}}', \breve{s}')$, where $\breve{\mathcal{S}}$ (resp. $\breve{\mathcal{S}}'$) denotes the strict transform of $\check{S}$ (resp. $g(\check{S})$), and $\breve{s} \in \mathrm{Sing}(\underline{\mathcal{F}})$ (resp. $\breve{s}' \in \mathrm{Sing}(\underline{\mathcal{F}'})$) are their attaching points in the exceptional divisors.*

Then there exists a homeomorphism Ψ defined on an open neighborhood U of S onto an open neighborhood U' of S' and there is a lift $\widetilde{\Psi} : \widetilde{U}^* \overset{\sim}{\longrightarrow} \widetilde{U}'^*$ of Ψ, such that:

(a) *$\Psi(\Sigma \cap U) \subset \Sigma' \cap U'$, the germ of $\Psi_{|\Sigma \cap U}$ at the finite set $\Sigma \cap S$ equals ψ and the germ at infinity of $\widetilde{\Psi}_{|\widetilde{\Sigma} \cap \widetilde{U}}$ coincides with $\widetilde{\psi}$,*

(b) *Ψ is $\mathcal{N}$-excellent, conjugates $\mathcal{F}_{|U}$ to $\mathcal{F}'_{|U'}$, and it is $\mathcal{N}$-transversely holomorphic,*

(c) *denoting by Ψ_S the germ of Ψ along S and by $\widetilde{\Psi}_\infty$ the pro-germ at infinity of $\widetilde{\Psi}$, then $(\Psi_S, \widetilde{\Psi}_\infty, h)$ is a realization of $(\mathfrak{g}, h)$ over the complement of some $\mathcal{N}$-separators of $\mathcal{F}$ and $\mathcal{F}'$.*

In fact we will prove in Section 7.3 that Condition (2) in Classification Theorem 5.0.2 is equivalent to Assertion (1) in Invariance Theorem 5.0.1. The relationship between these two results and Theorems I and II in the introduction come from the following property of transversal rigidity.

DEFINITION 5.0.3. – *We say that the germ of $\mathcal{F}$ at $(0,0)$ is* transversely rigid *(resp. $\mathcal{N}$-transversely rigid), if every germ of homeomorphism preserving the orientations of $(\mathbb{C}^2, 0)$ and those of the leaves, and conjugating $\mathcal{F}$ to a General Type foliation, is necessarily transversely (resp. $\mathcal{N}$-transversely) holomorphic.*

After the extended version (TRT) of the Transverse Rigidity Theorem of J. Rebelo [16] stated in the introduction, the hypothesis (G) on $\mathcal{F}$ implies the $\mathcal{N}$-transverse rigidity of $\mathcal{F}$. As we have already pointed out in the introduction, the genericity of this property in the sense of the Krull topology was proved in [6]. We remark that there are other interesting, although more particular, situations inducing also the transverse rigidity of the foliation, cf. [7].

COROLLARY 5.0.4. – *Let $\mathcal{F}$ and $\mathcal{F}'$ be two germs of General Type $\mathcal{N}$-transversely rigid foliations which are conjugated by a germ of orientations preserving homeomorphism Ψ_0. Then there is a germ of $\mathcal{N}$-excellent homeomorphism Ψ conjugating $\mathcal{F}$ and $\mathcal{F}'$. In particular, denoting by $\Psi^\sharp : \mathcal{D}_S \overset{\sim}{\longrightarrow} \mathcal{D}_{S'}$ the restriction to the total divisors of the lifting of Ψ over the reduction of singularities, the following properties hold:*

(a) *for each $s \in \mathrm{Sing}(\mathcal{F})$, the germs of $\underline{\mathcal{F}}$ at s and that of $\underline{\mathcal{F}'}$ at $\Psi^\sharp(s)$ are holomorphically equivalent;*

(b) *for each irreducible component D of $\mathcal{D}_S$, the holonomy representations $\mathcal{H}_D$ and $\mathcal{H}_{\Psi^\sharp(D)}$ defined in (16) are holomorphically conjugate via $\Psi^\sharp$.*

In addition, if $\mathrm{Node}(\mathcal{F}) = \varnothing$ *then* Ψ *is fundamentally equivalent to* Ψ_0, *cf. Section* 3.5.

Assertion (b) in Corollary 5.0.4 means that if Δ and $\Delta' := \Psi(\Delta)$ are holomorphic curves transverse to D and $D' := \Psi^{\sharp}(D)$ at the points $m \in D \setminus \mathrm{Sing}(\mathcal{F})$ and $m' := \Psi^{\sharp}(m)$, then the diagram (1) in the introduction commutes taking $\psi = \Psi_{|\Delta}$.

Proof. – We can apply first Theorem of Invariance 5.0.1, because Ψ is $\mathcal{N}$-transversely holomorphic; then we apply Classification Theorem 5.0.2. $\qquad\square$

Notice that this corollary is a more precise statement of Theorem I. On the other hand, thanks to the transverse rigidity theorem, Theorem II follows from the invariance and classification theorems below, using the additional generic hypothesis (G) considered in the introduction.

6. Peripheral structure of a germ of curve

Before proving Theorems 5.0.1 and 5.0.2, we shall examine some auxiliary topological notions that we will need in the sequel. In fact, this section deals uniquely with curves and there is no foliation there.

6.1. Peripheral groups

Following the previously introduced notations (2), (9), (14) and Conventions 4.1.1, let $\check{S}$ be an irreducible component of $S \subset \mathbb{B}$. We consider a tubular neighborhood $W_{\check{S}}$ of $\check{S}^{\circ} := \check{S} \setminus \{0\}$ in $\mathbb{B} \setminus \{0\}$. The pair $(W_{\check{S}}, \check{S}^{\circ})$ is homeomorphic to $(\check{S}^{\circ} \times \mathbb{D}_1, \check{S}^{\circ} \times \{0\})$. Let $s \in \mathrm{Sing}(\mathcal{D}_S)$ be the attaching point of the strict transform $\check{\mathcal{S}}$ of $\check{S}$. Up to permutation of the coordinates (x_s, y_s) of the local datum fixed in (13), we assume that $x_s = 0$ is a local reduced equation of $\check{\mathcal{S}}$. We choose $\varepsilon > 0$ small enough so that $\mathcal{W}_{\check{\mathcal{S}}}^* := E_S^{-1}(W_{\check{S}}^*)$ retracts over the 2-torus $\{|x_s| = \varepsilon, |y_s| = 1\}$.

PROPOSITION 6.1.1. – $W_{\check{S}}^*$ *is incompressible in* $\mathbb{B}^*$.

Proof. – It suffices to show the incompressibility of the torus

$$\{|x_s| = \varepsilon, |y_s| = 1\}$$

inside $\mathcal{B}_S^*$. This can be done by using Van Kampen's Theorem, see for instance the construction of an open neighborhood of $\mathcal{D}_S$ by "boundary assembly" made in [9]. $\qquad\square$

Consider the loops m (resp. p) in $\mathcal{W}_s^*$, having the same origin, defined by $(x_s, y_s) \circ m(t) = (\varepsilon e^{2i\pi t}, 1)$ (resp. $(x_s, y_s) \circ p(t) = (\varepsilon, e^{2i\pi t})$). At the point $\check{c} := E_S(m(0))$, the homotopy classes $\mathfrak{m}_{\check{c}}$ (resp. $\mathfrak{p}_{\check{c}} \in \pi_1(W_{\check{S}}, \check{c})$) of the loops $E_S \circ m$ (resp. $E_S \circ p$), allow to decompose $\pi_1(W_{\check{S}}^*, \check{c}) = \mathbb{Z}\mathfrak{m}_{\check{c}} \oplus \mathbb{Z}\mathfrak{p}_{\check{c}}$. The abelianity of this group implies that the isomorphism from $\pi_1(W_{\check{S}}^*, c_1)$ onto $\pi_1(W_{\check{S}}^*, c_2)$, induced by a path joining the points c_1 and c_2 inside $W_{\check{S}}^*$, does not depend on the particular choice of this path. Thus, the direct sum decomposition of $\pi_1(W_{\check{s}}^*, \check{c})$ is canonical, i.e., it can be unambiguously defined for every base point in $W_{\check{S}}^*$:

$$\mathscr{P}_{\check{S}, c} := \pi_1(W_{\check{S}}^*, c) = \mathbb{Z}\mathfrak{m}_c \oplus \mathbb{Z}\mathfrak{p}_c \subset \pi_1(\mathbb{B}^*, c), \quad c \in W_{\check{S}}^*.$$

DEFINITION 6.1.2. – *We will call* $\mathfrak{m}_c$ *the* meridian, $\mathfrak{p}_c$ *the* parallel *and* $\mathscr{P}_{\check{S},\,c}$ *the* peripheral subgroup *associated to the component* $\check{S}$ *at the point c.*

The following geometric property states that this decomposition is "intrinsic", see [5, 10].

PROPOSITION 6.1.3. – *The subgroup* $\mathscr{P}_{\check{S},\,c}$ *coincides with its normalizer inside* $\pi_1(\mathbb{B}^*, c)$, *i.e.,* $\left(\zeta \in \pi_1(\mathbb{B}^*, c) \text{ and } \zeta\,\mathscr{P}_{\check{S},\,c}\,\zeta^{-1} \subset \pi_1(W_{\check{S}}^*, c)\right) \Rightarrow \zeta \in \mathscr{P}_{\check{S},\,c}$.

We immediately deduce:

COROLLARY 6.1.4. – *The direct sum decomposition*

$$P = \mathbb{Z}\mathfrak{m}_P \oplus \mathbb{Z}\mathfrak{p}_P, \qquad \mathfrak{m}_P := \zeta\,\mathfrak{m}_c\,\zeta^{-1}, \quad \mathfrak{p}_P := \zeta\,\mathfrak{p}_c\,\zeta^{-1},$$

of every subgroup $P = \zeta\,\mathscr{P}_{\check{S},c}\,\zeta^{-1}$, $\zeta \in \pi_1(\mathbb{B}^*, c)$ *conjugated to* $\mathscr{P}_{\check{S},c}$ *is intrinsic, i.e., it does not depend on* $\zeta \in \pi_1(\mathbb{B}^*, c)$.

6.2. Conjugation of peripheral structures

We will see that the canonical meridians and parallels introduced in Definition 6.1.2, associated to the irreducible components of S, are topological invariants.

THEOREM 6.2.1. – *Let U be an open neighborhood of S in $\mathbb{B}$ and let Φ be a homeomorphism from U onto a neighborhood U' of 0 in $\mathbb{B}'$, such that $\Phi(S) = S' \cap U'$. Then for each irreducible component $\check{S}$ of S and for all point c in a tubular neighborhood of $\check{S} \setminus \{0\}$ inside $\mathbb{B}$, the isomorphism Φ_* from $\pi_1(\mathbb{B}^*, c)$ onto $\pi_1(\mathbb{B}'^*, c')$ induced by Φ, sends respectively the meridian $\mathfrak{m}_c$ and the parallel $\mathfrak{p}_c$ associated to $\check{S}$ to the meridian $\mathfrak{m}'_{c'}$ and the parallel $\mathfrak{p}'_{c'}$ associated to the component $\Phi(\check{S})$, at the point $c' := \Phi(c)$.*

Proof. – First we note that Φ_* induces an isomorphism from the peripheral group $\mathscr{P}_{\check{S},\,c}$ of $\check{S}$, onto the peripheral group $\mathscr{P}'_{\check{S}',\,c'}$ of $\check{S}' := \Phi(\check{S})$. Indeed, we can consider a tubular neighborhood W of $\check{S}$ and two tubular neighborhoods W' and W'' of $\check{S}' \setminus \{0\}$ in $\mathbb{B}'$, as well as a ball $\mathbb{B}'' \subset \mathbb{B}'$ centered at the origin, such that $W'' \cap \mathbb{B}'' \subset \Phi(W) \subset W'$. These inclusions induce two $\mathbb{Z}$-linear morphisms at the fundamental group level,

$$\mathbb{Z}^2 \simeq \mathscr{P}'_{\check{S}',\,c'} \to \Phi_*(\mathscr{P}_{\check{S},\,c}) \to \mathscr{P}'_{\check{S}',\,c'} \simeq \mathbb{Z}^2,$$

whose composition is an isomorphism. Hence

(19) $$\Phi_*(\mathscr{P}_{\check{S},\,c}) = \mathscr{P}'_{\check{S}',\,c'}.$$

The Marking Theorem 3.5.3 provides the existence of an excellent homeomorphism g fundamentally equivalent to Φ. We can assume that $W'' \cap \mathbb{B}'' \subset g(W) \subset W'$ and g induces an isomorphism g_* from $\mathscr{P}_{\check{S},\,c}$ onto $\mathscr{P}'_{\check{S}',g(c)}$. Clearly $g_*(\mathfrak{m}_c) = \mathfrak{m}'_{g(c)}$ and $g_*(\mathfrak{p}_c) = \mathfrak{p}'_{g(c)}$, because the lifting $E_{S'}'^{-1} \circ g \circ E_S$ extends to the exceptional divisor. The fundamental equivalence between g and Ψ implies the existence of an element ζ of $\pi_1(\mathbb{B}'^*, c')$ such that

(20) $$I_\zeta \circ \Phi_* = \kappa \circ g_* : \pi_1(\mathbb{B}^*, c') \longrightarrow \pi_1(\mathbb{B}^*, c'),$$

where I_ζ denotes the interior automorphism of $\pi_1(\mathbb{B}^*, c')$ determined by ζ and κ is the canonical isomorphism from $\pi_1(\mathbb{B}^*, g(c))$ onto $\pi_1(\mathbb{B}^*, c')$, determined by an arbitrary path in W' joining $g(c)$ to c'. The relations (19) and (20) give the equality $\zeta\,\mathscr{P}'_{\check{S}',\,c'}\,\zeta^{-1} = \mathscr{P}'_{\check{S}',\,c'}$. By

applying Proposition 6.1.3 we obtain that ζ belongs to $\mathscr{P}_{\check{S}', c'}$. The restriction of I_ζ to $\mathscr{P}_{\check{S}', c'}$ is the identity because this group is abelian. By restricting (20) to $\mathscr{P}_{\check{S}, c}$ we obtain the relation $\Phi_* = \kappa \circ g_*$. Hence

$$\Phi_*(\mathfrak{m}_c) = \kappa(g_*(\mathfrak{m}_c)) = \kappa(\mathfrak{m}'_{g(c)}) = \mathfrak{m}'_{c'} \, ;$$

and analogously $\Phi_*(\mathfrak{p}_c) = \mathfrak{p}'_{c'}$. $\qquad \square$

7. Proof of Invariance Theorem 5.0.1

7.1. Proof of Assertion (2)

By Proposition 3.6.4, we can compose Ψ on the left by a germ of homeomophism $\Theta_1 : (\mathbb{B}', S') \xrightarrow{\sim} (\mathbb{B}', S')$ which is $\mathscr{F}'$-isotopic to the identity. Let Σ and Σ' be $\mathscr{N}$-collections of transversals of $\mathscr{F}$ and $\mathscr{F}'$ respectively, such that for each connected component $\check{\Sigma}$ of Σ intersecting an irreducible component $\check{S}$ of S we have that $\Psi(\check{\Sigma})$ and $g(\check{\Sigma})$ meet the same irreducible component $\check{S}'$ of S'. We can construct an $\mathscr{F}'$-isotopy Θ_1 such that $\Theta_1(\Psi(\check{\Sigma}))$ and $\check{\Sigma}'$ define the same germ at $\check{\Sigma}' \cap \check{S}'$. Assertion (2) of Theorem 5.0.1 follows directly from Remarks 3.6.2 and 3.6.3.

7.2. Invariance of the Camacho-Sad indices associated to the separatrix curve

We will prove the equality (18) when s is the attaching point of the strict transform $\check{\mathscr{S}}$ of an irreducible component $\check{S}$ of S. Following the notations of Section 6.1 we denote by $\rho : \mathcal{W}_s \to \check{\mathscr{S}} \cap \mathcal{W}_s$ the disk fibration such that $y_s \circ \rho = y_s$, by γ_n the loop contained in $x_s = 0$, such that $y_s \circ \gamma_n(t) := e^{2i\pi nt}, 0 \leq t \leq 1$, by q the point having coordinates $(0, 1/2)$ and finally by T the transverse section $\rho^{-1}(q)$.

Consider a sequence $(q_n)_{n \in \mathbb{N}}$ of points in T tending to q, such that the loop γ_n lifts, via ρ, to a path Γ_n contained in a leaf of $\underline{\mathscr{F}}$. We can see that such a sequence always exists and that

$$(21) \qquad \mathrm{CS}(\underline{\mathscr{F}}, \check{\mathscr{S}}, s) = \lim_{n \to \infty} \frac{1}{2i\pi n} \int_{\Gamma_n} \frac{dx_s}{x_s} \, .$$

Fix a real number $\theta_n \in]-\pi, \pi]$ different from the arguments of q_n and that of $\Gamma_n(1)$ and choose a path ξ_n in

$$T \cap \{\arg(x_s) \neq \theta_n \, , \, 0 < |x_n| < 1/n\} \, ,$$

having endpoints q_n and $\Gamma_n(1)$. Since the real part of $\frac{1}{2i\pi n} \int_{\xi_n} \frac{dx_s}{x_s}$ is bounded, the real part of the Camacho-Sad index (21) is given by

$$\mathrm{Re}(\mathrm{CS}(\underline{\mathscr{F}}, \check{\mathscr{S}}, s)) = \lim_{n \to \infty} \frac{I_n}{n} \, , \quad \text{where} \quad I_n := \frac{1}{2i\pi} \int_{\Gamma_n \vee \xi_n} \frac{dx_s}{x_s}.$$

Using the peripheral structure of $\mathscr{P}_{\check{S}, c_n}$ given in Corollary 6.1.4, the homotopy class of the loop $\Gamma_n \vee \xi_n$ in $\pi_1(\mathbb{B}^*, c_n)$, $c_n := E_S(q_n)$ can be decomposed as

$$\overline{\Gamma_n \vee \xi_n} = I_n \mathfrak{m}_{c_n} + n \mathfrak{p}_{c_n} \, .$$

If α_n is an arbitrary path in $T \setminus \{q\}$ with endpoints q_0 and q_n then the homotopy class of the loop $\lambda_n := \alpha_n \vee \Gamma_n \vee \xi_n \vee \alpha_n^{-1}$ in $\pi_1(\mathbb{B}^*, c_0)$ is

$$\overline{\lambda_n} = I_n \mathfrak{m}_{c_0} + n \mathfrak{p}_{c_0} \, .$$

Fix now the same data at the attaching point $s' := G(s)$ of the strict transform $\breve{\mathscr{S}}'$ of $\breve{S}' := \Psi(S)$. We denote by $(x_{s'}, y_{s'}) : \mathscr{W}'_{s'} \xrightarrow{\sim} \mathbb{D}^2_1$ the local coordinates at s' determined by the local datum $\mathscr{L}'$, by $\rho' : \mathscr{W}'_{s'} \to \breve{\mathscr{S}}' \cap \mathscr{W}'_{s'}$ the disk fibration defined by $y_{s'} \circ \rho' = y_{s'}$, and by $q' \in \breve{\mathscr{S}}$ the point having coordinates $(0, 1/2)$. We also set $T' := \rho'^{-1}(q')$. It is easy to see that, after composing it by a homeomorphism $\mathscr{F}'$-isotopic to the identity, Ψ satisfies the following properties:

- $\Psi(V) \subset V'$, where V is the image by E_S of a tubular neighborhood of the circle $\{x_s = 0, |y_s| = 1\}$ inside $\{|x_s| \leq 1, |y_s| = 1\}$ and V' is the image by $E_{S'}$ of the torus $\{|x'_{s'}| \leq 1, |y_{s'}| = 1\}$;
- $\Psi_{|V}$ conjugates the fibrations, i.e., $E'_{S'} \circ \rho' \circ E_{S'}^{-1} \circ \Psi_{|V} = \Psi \circ E_S \circ \rho \circ E_S^{-1}$.

As in (21) we have the following equality:

$$CS(\underline{\mathscr{F}}', \breve{\mathscr{S}}', s') = \lim_{n \to \infty} \frac{1}{2i\pi n} \int_{\Psi \circ \Gamma_n} \frac{dx'_{s'}}{x'_{s'}}.$$

The variation of the argument of $x'_{s'} \circ \Psi \circ \xi_n$ is bounded because the restriction of Ψ to T is holomorphic (recall that Ψ is transversely holomorphic on V). Thus,

(22) $$\mathrm{Re}(CS(\underline{\mathscr{F}}', \breve{\mathscr{S}}', s')) = \lim_{n \to \infty} \frac{J_n}{n}, \quad J_n := \frac{1}{2i\pi} \int_{\Psi \circ (\Gamma_n \vee \xi_n)} \frac{dx'_{s'}}{x'_{s'}}.$$

Clearly the homotopy class of $\Psi \circ \lambda_n$ in $\pi_1(\mathbb{B}^*, \Psi(q_0))$ is

$$J_n \mathfrak{m}'_{\Psi(c_0)} + n\mathfrak{p}'_{\Psi(c_0)} = \Psi_*(I_n \mathfrak{m}_{c_0} + n\mathfrak{p}_{c_0}).$$

Theorem 6.2.1 provides the equality $I_n = J_n$. Thanks to (22), $CS(\underline{\mathscr{F}}, \breve{\mathscr{S}}, s)$ and $CS(\underline{\mathscr{F}}', \breve{\mathscr{S}}', s')$ have the same real part. Hence, they coincide because on the other hand, their difference is an integer number. Indeed, the exponential of each of them is the linear part of the holonomies of $\breve{S}$ and $\breve{S}'$, which are analytically conjugated by the biholomorphism $\Psi_{|T}$.

REMARK 7.2.1. – Notice that the above proof only uses the transverse holomorphy of Ψ on a neighborhood V of the strict transform of each punctured separatrix of $\mathscr{F}$. Hence we deduce that any topological conjugation which is transversely holomorphic in a neighborhood of each punctured nodal separatrix is an $\mathscr{N}$-topological conjugation.

7.3. Invariance of all the Camacho-Sad indices

The proof is based in the Camacho-Sad index formula, which claims that the auto-intersection number of an irreducible component D of the exceptional divisor equals the sum of the Camacho-Sad indices along D, at the singular points of the foliation lying on D. We consider filtrations of the exceptional divisors $\mathscr{E}_S := E_S^{-1}(0)$ and $\mathscr{E}_{S'} := E_{S'}^{-1}(0)$,

$$\mathscr{E}_0 := \mathscr{E}_S \supset \mathscr{E}_1 \supset \mathscr{E}_2 \supset \cdots \quad \text{and} \quad \mathscr{E}'_0 := \mathscr{E}'_S \supset \mathscr{E}'_1 \supset \mathscr{E}'_2 \supset \cdots,$$

defined by induction in the following way: $\overline{\mathscr{E}_{j-1} \setminus \mathscr{E}_j}$ is the union of the components D of $\mathscr{E}_{j-1}$ having valence 1 in $\mathscr{E}_{j-1}$. Since the dual graphs of these divisors are trees, we eventually obtain the empty set. Clearly $G(\mathscr{E}_j) = \mathscr{E}'_j$, for all j. In order to obtain the equalities (18) in Theorem 5.0.1 for every singular point $s \in \mathrm{Sing}(\mathscr{F})$ and each irreducible component D of $\mathscr{D}_S$, it suffices to show the following assertion for all $j \geq 1$:

$(\divideontimes)_j$ *the equality* (18) *holds at every point* $s \in (\mathrm{Sing}(\mathcal{D}_S) \setminus \mathrm{Sing}(\mathcal{E}_j))$, *for each irreducible component D containing s.*

We conclude by noting that the index formula provides the implication $(\divideontimes)_j \Rightarrow (\divideontimes)_{j+1}$ and that $(\divideontimes)_0$ expresses the invariance of the Camacho-Sad indices of the separatrix curve, proved in Section 7.2.

8. Proof of Classification Theorem 5.0.2

We keep the notations (2), (9), (14), Conventions 4.1.1 and we assume the hypotheses of Theorem 5.0.2. We shall construct a global $\mathcal{N}$-transversely holomorphic conjugation between $\underline{\mathcal{F}}$ and $\underline{\mathcal{F}}'$ inducing a realization of $(\mathfrak{g}, h)$ outside some $\mathcal{N}$-separators of $\mathcal{F}$ and $\mathcal{F}'$ in $\mathbb{B}$, satisfying Assertions (a), (b) and (c) of Theorem 5.0.2. We proceed by induction, by constructing the desired homeomorphism step by step, over "elementary pieces" of an appropriated decomposition of a neighborhood of the total divisor $\mathcal{D}_S$ in $\mathcal{B}_S$ that we will describe in Section 8.2. These elementary pieces are associated to each singular point $s \in \mathrm{Sing}(\mathcal{D}_S)$ and to each irreducible component D of $\mathcal{D}_S$ and they are denoted by K_s and K_D respectively.

8.1. Description of the induction

The construction of the elementary pieces is done in Section 8.2. Extension Lemma 8.3.2 is the key tool which allows us to make the inductive step and to begin the process. Given a realization $(\phi, \widetilde{\phi}, h)$ of $(\mathfrak{g}, h)$ over a fiber T of the Hopf fibration contained in a boundary component of an elementary piece K, this lemma gives a simple topological condition (23) which allows to extend $(\phi, \widetilde{\phi}, h)$ to a realization defined over the whole piece. In addition we have that

1. the restriction of this extension to every Hopf fiber contained in ∂K also satisfies the condition (23),
2. when T is contained in the intersection of two adjacent elementary pieces, then the realizations given by Lemma 8.3.2 over each of these pieces coincide over their common intersection.

Thus, if $\mathrm{Node}(\mathcal{F}) = \varnothing$ and we have a realization over an elementary piece K^0, we can extend it step by step over a whole neighborhood of $\mathcal{D}_S$ in $\mathcal{B}_S$. In order to achieve the proof in this case, it suffices to be sure that we can apply Lemma 8.3.2 in the context where D is the strict transform of the irreducible component of S meeting Σ, $T = E^{-1}(\Sigma)$ and letting K^0 be the elementary piece associated to D. In Section 8.4 we will prove the existence of a realization $(g_1, \widetilde{g}_1, h)$ of the conjugation $(\mathfrak{g}, h)$ satisfying the condition (23) in Lemma 8.3.2 in this context. This will achieve the proof of the theorem in the case $\mathrm{Node}(\mathcal{F}) = \varnothing$.

If $\mathrm{Node}(\mathcal{F}) \neq \varnothing$, we begin this induction process in each connected component of Σ and we stop it when it would require to make an extension to an elementary piece containing a singular point s belonging to $\mathrm{Node}(\mathcal{F}) \cap \mathrm{Sing}(\mathcal{E}_{\mathcal{F}})$. For a nodal singularity s (resp. s') belonging to the strict transform $\check{\mathscr{J}}$ (resp. $\check{\mathscr{J}}'$) of a nodal separatrix of $\mathcal{F}$ (resp. $\mathcal{F}'$), Extension Lemma 8.3.2 provides a foliated homeomorphism $K_s \rightarrow K_{s'}$ which can be easily extended to the adjacent elementary pieces $K_{\check{\mathscr{J}}} \rightarrow K_{\check{\mathscr{J}}'}$ by using the product structures of $\underline{\mathcal{F}}$ and $\underline{\mathcal{F}}'$

in $K_{\breve{\mathcal{J}}} \cong (K_s \cap K_{\breve{\mathcal{J}}}) \times [0,1]$ and $K_{\breve{\mathcal{J}}'} \cong (K_{s'} \cap K_{\breve{\mathcal{J}}'}) \times [0,1]$. To complete the process in this case, it suffices to glue the realizations obtained in this way, by constructing in Section 8.5 suitable foliated homeomorphisms defined on the elementary pieces associated to the singular points $s \in \mathrm{Node}(\mathcal{F}) \cap \mathrm{Sing}(\mathcal{E}_{\mathcal{F}})$.

8.2. Elementary pieces

Recall that we have fixed an $\mathcal{N}$-conjugation $(\mathfrak{g}, h)$ in Theorem 5.0.2. By Corollary 3.5.4 there exists an excellent homeomorphism germ $(\mathbb{B}, S) \to (\mathbb{B}, S')$ representing the marking determined by $\mathfrak{g}$. Let $G : (\mathcal{B}, S) \to (\mathcal{B}', S')$ be its lifting. We fix two Milnor tubes, cf. (14), $\mathcal{T}_\varepsilon$ for S and $\mathcal{T}'_{\varepsilon'}$ for S', where $\varepsilon, \varepsilon' > 0$ are chosen small enough so that $G(\mathcal{T}_\varepsilon) \subset \mathcal{T}'_{\varepsilon'}$ and each real hypersurface $\{|x_s| = 1\}$ and $\{|y_s| = 1\}$, $s \in \mathrm{Sing}(\mathcal{F})$, as well as $\{|x_{s'}| = 1\}$ and $\{|y_{s'}| = 1\}$, $s' \in \mathrm{Sing}(\mathcal{F}')$, separates the tube in two connected components and intersects transversely the boundary in a 2-torus. We set

$$\mathcal{H} := \bigcup_{s \in \mathrm{Sing}(\mathcal{F})} \{|x_s| = 1\} \cup \{|y_s| = 1\}, \quad \mathcal{H}' := \bigcup_{s' \in \mathrm{Sing}(\mathcal{F}')} \{|x_{s'}| = 1\} \cup \{|y_{s'}| = 1\}.$$

We call *elementary piece* of $\mathcal{T}_\varepsilon$ (resp. $\mathcal{T}'_{\varepsilon'}$), every intersection $K := \mathcal{K} \cap \mathcal{T}_\varepsilon$ (resp. $K' := \mathcal{K}' \cap \mathcal{T}'_{\varepsilon'}$), where $\mathcal{K}$ (resp. $\mathcal{K}'$) is the adherence of a connected component of $\mathcal{T}_\varepsilon \setminus \mathcal{H}$ (resp. $\mathcal{T}'_{\varepsilon'} \setminus \mathcal{H}'$). For each elementary piece K (resp. K'), one and only one of the following assertions holds:

  – K (resp. K') contains a (unique) point s of $\mathrm{Sing}(\mathcal{F})$ (resp. $s' \in \mathrm{Sing}(\mathcal{F}')$), and it is contained in the domain $\mathcal{W}_s$ (resp. $\mathcal{W}_{s'}$) of the coordinate chart (x_s, y_s) (resp. $(x_{s'}, y_{s'})$);
  – K (resp. K') contains a compact set $D^\circ := \overline{D \setminus \bigcup_s \mathcal{W}_s}$, where D denotes an irreducible component of $\mathcal{D}_S$ (resp. $\mathcal{D}_{S'}$), and s ranges the set of singular points of $\mathcal{F}$ (resp. $\mathcal{F}'$); in addition, if $\varepsilon, \varepsilon' > 0$ are small enough then the restriction of the fibration π_D to $K \cap \pi_D^{-1}(\partial D^\circ)$ is still a disk fibration;

In the first case the elementary piece will be denoted by K_s (resp. $K_{s'}$), and in the second case it will be denoted by K_D. The intersection of two different elementary pieces is either empty or a 3-dimensional solid torus.

8.3. Extension of realizations

By Corollary 3.5.4 there is a geometric representation $(g, \tilde{g}, h)$ of the $\mathcal{N}$-conjugation $(\mathfrak{g}, h)$, with g excellent. We assume that $\varepsilon > 0$ is small enough so that $G := E_{S'}^{-1} \circ g \circ E_{S|\mathcal{T}_\varepsilon}$ is defined on $\mathcal{T}_\varepsilon$ into $\mathcal{T}'_{\varepsilon'}$. Thanks to properties of excellent maps stated in Definition 3.5.2, the restriction of G to the total divisor fulfills the following equalities:

$$G(K_\alpha \cap \mathcal{D}_S) = K_{G(\alpha)} \cap \mathcal{D}_{S'}, \quad \alpha \in \mathrm{Comp}(\mathcal{D}_S) \sqcup \mathrm{Sing}(\mathcal{F}),$$

and $G(K_\alpha)$ is a neighborhood of $K_{G(\alpha)} \cap \mathcal{D}_{S'}$ in $K_{G(\alpha)}$.

We consider an irreducible component D of $\mathcal{D}_S$ and a Hopf fiber $T := \pi_D^{-1}(c) \cap \mathcal{T}_\varepsilon$, over a point c in the boundary of D°, cf. (15). The connected component C of ∂D° containing c, is a circle that bounds a disk $\mathcal{W}_s \cap D$, $s \in \mathrm{Sing}(\mathcal{F})$. The point $c' := G(c)$ belongs to the boundary of D'°, $D' := G(D)$. We denote $T' := \pi_{D'}^{-1}(c')$ and we assume that a germ of biholomorphism $\phi_S : (T, c) \to (T', c')$ is given, as well

as a pro-germ at infinity $\widetilde{\phi}_\infty : (\widetilde{T}^*, \infty) \to (\widetilde{T}'^\star, \infty)$ lifting ϕ_S, such that $(\phi_S, \widetilde{\phi}_\infty, h)$ is a realization of $(\mathfrak{g}, h)$ over T and T'.

CONVENTIONS 8.3.1. – For $B \subset \widetilde{\mathbb{B}}^*$ and $B' \subset \widetilde{\mathbb{B}}'^\star$, we denote:

$$\pi_0(B, \infty) := \varprojlim\nolimits_{U \in \mathfrak{U}_{\mathscr{G}, \Sigma}} \pi_0(B \cap \widetilde{U}^*), \quad \pi_0(B', \infty) := \varprojlim\nolimits_{U \in \mathfrak{U}_{\mathscr{G}', \Sigma'}} \pi_0(B' \cap \widetilde{U}^\star).$$

LEMMA 8.3.2 (of extension of realizations). – *Let K (resp. K') be one of the elementary pieces K_D or K_s (resp. $K_{G(D)}$ or $K_{G(s)}$) and let us denote $Z := K \cap \mathscr{D}_S$ and $Z' := K' \cap \mathscr{D}_{S'}$. We assume that $\widetilde{\phi}_\infty$ and the restriction of $\widetilde{g}$ to $\widetilde{T}^*$ induce the same map*

(23) $$\widetilde{\phi}_\infty = \widetilde{g} : \pi_0(\widetilde{T}^*, \infty) \longrightarrow \pi_0(\widetilde{T}'^\star, \infty),$$

and, in the case that $K = K_s$, we also assume the following equality concerning the Camacho-Sad indices:

$$\mathrm{CS}(\underline{\mathscr{G}}, D, s) = \mathrm{CS}(\underline{\mathscr{G}}', D', G(s)).$$

Then there are homeomorphisms $\Phi : V \to V'$ and $\widetilde{\Phi} : \widetilde{V}^* \to \widetilde{V}'^\star$ such that for each component $\check{D}$ of $\mathscr{D}_S$ meeting K, the following properties hold:

(a) *V (resp. V') is an open neighborhood of $\mathscr{D}_S \cap K$ in K (resp. $\mathscr{D}_{S'} \cap K'$ in K'), $\widetilde{\Phi}$ lifts Φ, i.e., $\underline{q}' \circ \widetilde{\Phi} = \Phi \circ \underline{q}_{|\widetilde{V}^*}$ and $\Phi_{|Z \cap \check{D}^\circ} = G_{|Z \cap \check{D}^\circ}$;*

(b) *Φ preserves the Hopf fibres over $K \cap \check{D}^\circ$, more precisely,*

$$\pi_{\check{D}'} \circ \Phi_{|V \cap \pi_{\check{D}}^{-1}(V \cap \check{D}^\circ)} = G \circ \pi_{\check{D}|V \cap \check{D}^\circ},$$

   with $\check{D}' := G(\check{D})$;

(c) *the germ of the restriction of Φ to T coincides with ϕ_S and the pro-germ at infinity of the restriction of $\widetilde{\Phi}$ to $\widetilde{T}^*$ coincides with $\widetilde{\phi}_\infty$;*

moreover, except in case $K = K_s$, $s \in \mathrm{Node}(\mathscr{G})$, we have:

(d) *if we denote by Φ_S the germ of Φ along $K \cap \mathscr{D}_S$ and by $\widetilde{\Phi}_\infty$ the pro-germ at infinity of $\widetilde{\Phi}$, then $(\Phi_S, \widetilde{\Phi}_\infty, h)$ is a realization of the geometric conjugation $(\mathfrak{g}, h)$ over V and V', in the sense given in Conventions 4.1.1; in particular Φ is a transversely holomorphic conjugation between the restricted foliations $\underline{\mathscr{G}}_{|V}$ and $\underline{\mathscr{G}}'_{|V'}$;*

(e) *for each $t \in \check{D}^\circ \cap K$, the restrictions of $\widetilde{g}$ and $\widetilde{\Phi}$ to $T_t := \pi_{\check{D}}^{-1}(t)$ induce the same map*

$$\widetilde{g}_{|\widetilde{T}_t^*} = \widetilde{\Phi}_{|\widetilde{T}_t^\star} : \pi_0(\widetilde{T}_t^*, \infty) \to \pi_0(\widetilde{T}'^\star_{t'}, \infty), \quad T'_{t'} := \pi_{\check{D}'}^{-1}(t'), \; t' := G(t).$$

In addition, the homeomorphisms obtained above by extension along $K = K_D$ and $K = K_s$, coincide over the intersection of their domains of definition.

8.4. Beginning of the induction

We begin with a realization $(\psi, \widetilde{\psi}, h)$ over $\mathscr{N}$-collections of transversals Σ and Σ' of a geometric $\mathscr{N}$-conjugation $(\mathfrak{g}, h)$ between the monodromies of $\mathscr{G}$ and $\mathscr{G}'$ geometrically represented by $(g, \widetilde{g}, h)$. For each connected component $\check{\Sigma}$ of Σ we consider the irreducible component $\check{S}$ of S meeting $\check{\Sigma}$. From Condition (σ) in the introduction and Condition (2) in Theorem 5.0.2, we deduce that $\check{S}$ (resp. $\check{S}' := g(\check{S})$) is a non-nodal separatrix of $\mathscr{G}$ (resp. $\mathscr{G}'$). Denote by $\check{\mathscr{S}}$ and s (resp. $\check{\mathscr{S}}'$ and s') the strict transform of $\check{S}$ (resp. $\check{S}'$) and its attaching point in the exceptional divisor $\mathscr{E}_S$ (resp. $\mathscr{E}_{S'}$). Denote by $\check{\Sigma}' \supset \psi(\check{\Sigma})$ the connected component of Σ' meeting $\check{S}'$. Thanks to the invariance by $\mathscr{G}$ or $\mathscr{G}'$-isotopies stated in Proposition 3.6.4,

we can assume that $\check{\Sigma}$ (resp. $\check{\Sigma}'$) is contained in the image $E_S(T_0)$ (resp. $E_{S'}(T_0')$) of a Hopf fiber T_0 (resp. T_0') in the intersection of the elementary pieces $K_{\check{\mathscr{A}}}$ and K_s (resp. $K_{\check{\mathscr{A}}'}$ and $K_{s'}$). Analogously, up to composing the geometric representation $(g, \widetilde{g}, h)$ of the $\mathscr{N}$-conjugation $(\mathfrak{g}, h)$ with S or S'-isotopic homeomorphisms, cf. Remark 3.4.4, we can assume the equality of germs $(g(\check{\Sigma}), g(\check{\Sigma} \cap \check{S})) = (\check{\Sigma}', \check{\Sigma}' \cap \check{S}')$. Finally, we also assume that $x_s = 0$ (resp. $x_{s'} = 0$) is a reduced local equation of $\check{\mathscr{A}}$ (resp. $\check{\mathscr{A}}'$) and we set:

$$\mathbb{T} := E_S(\{|y_s| = 1, x_s = 0\}) \quad \text{and} \quad \mathbb{T}' := E_{S'}(\{|y_{s'}| = 1, x_{s'} = 0\}).$$

In order to simplify the notations in the rest of this section we will write $\widetilde{\Sigma}^*$ and $\widetilde{\Sigma}'^{\star}$ instead of $\widetilde{\check{\Sigma}^*}$ and $\widetilde{\check{\Sigma}'^{\star}}$.

Now notice that ψ and $\widetilde{\psi}$ are necessarily holomorphic. This follows from the commutativity of the diagram $(\star)$ in Definition 3.6.1, using Proposition 3.3.1. Then, in order to begin with the inductive process described in Section 8.1, we will construct a geometric representation $(g_1, \widetilde{g}_1, h)$ of the $\mathscr{N}$-conjugation $(\mathfrak{g}, h)$ which will satisfy the topological condition (23) required by the Extension Lemma 8.3.2 in the case $D = \check{\mathscr{A}}$, $T = T_0$ and $(\phi_S, \phi_\infty, h) = (\psi, \widetilde{\psi}, h)$. This condition is equivalent to the following equality:

$$(24) \qquad \widetilde{\psi} = \widetilde{g}_1 : \pi_0(\widetilde{\Sigma}^*, \infty) \longrightarrow \pi_0(\widetilde{\Sigma}'^{\star}, \infty).$$

We can see that (24) holds if the equality $\widetilde{\psi}(\sigma_0) = \widetilde{g}_1(\sigma_0)$ is satisfied by a fixed element σ_0 of $\pi_0(\widetilde{\Sigma}^*, \infty)$. Thanks to the Lemma 8.4.1 below, $\widetilde{g}(\sigma_0)$ and $\widetilde{\psi}(\sigma_0)$ belong to the same fiber of the map $\iota : \pi_0(\widetilde{\Sigma}'^{\star}, \infty) \to \pi_0(\widetilde{\mathbb{T}}'^{\star}, \infty)$ induced by the inclusion $\iota : \widetilde{\Sigma}'^{\star} \hookrightarrow \widetilde{\mathbb{T}}'^{\star}$. This property will allow us to construct a homeomorphism $\Theta : \mathbb{B}' \xrightarrow{\sim} \mathbb{B}'$ which is S'-isotopic to the identity and whose lift $\widetilde{\Theta}$ satisfies $\widetilde{\Theta}(\widetilde{g}(\sigma_0)) = \widetilde{\psi}(\sigma_0)$. We achieve the beginning of the inductive process by taking $g_1 = \Theta \circ g$ and $\widetilde{g}_1 := \widetilde{\Theta} \circ \widetilde{g}$. The homeomorphism Θ is a sort of foliated Dehn twist that we will describe before proving Lemma 8.4.1 below, which is needed to construct it.

On the disk $\check{\mathscr{A}}' \cap \mathcal{W}_{s'}'$, we consider the real vector field ϑ whose flow is $(t, y_{s'}) \mapsto e^{2i\pi t} y_{s'}$. We fix a smooth function u with support contained in $\{x_{s'} = 0, \varsigma \le |y_{s'}| \le 1\}$, $0 < \varsigma < 1$ taking the value 1 on the circle $C' := \{x_{s'} = 0, |y_{s'}| = 1\}$ and we denote by Y the vector field on V' tangent to $\underline{\mathscr{G}}'$ and projecting over $u\vartheta$ by $\pi_{\check{\mathscr{A}}'}$. The flow Υ_t of Y is defined on a open neighborhood U_I' of $\check{\mathscr{A}}'$ in $\mathcal{W}_{s'}'$, once we fix an interval $I \subset \mathbb{R}$ where we allow the time t to vary. Hence, we can lift the flow Υ_t to a unique map $\widetilde{\Upsilon}_t : \widetilde{U_I'}^{\star} \to \widetilde{V}'^{\star}$ being the identity on $|y_{s'} \circ \underline{q}'| \le \varsigma$, and defining consequently a germ at infinity $\widetilde{\Upsilon}_{t\infty} : (\widetilde{V}'^{\star}, \infty) \to (\widetilde{V}'^{\star}, \infty)$. Clearly the germ $\widetilde{\Upsilon}_{n\infty}$ fibers over C', i.e., $\pi_{\check{\mathscr{A}}'} \circ \underline{q}' \circ \widetilde{\Upsilon}_{n\infty}|_{\widetilde{\mathbb{T}}'^{\star}} = \pi_{\check{\mathscr{A}}'} \circ \underline{q}'_{\infty}|_{\widetilde{\mathbb{T}}'^{\star}}$, for each $n \in \mathbb{Z}$. It defines a deck transformation $\widetilde{\Upsilon}_n : \Pi' \xrightarrow{\sim} \Pi'$ of the natural covering

$$\rho' : \Pi' \to C', \quad \rho'^{-1}(p) := \pi_0(\widetilde{T}_p'^{\star}, \infty), \quad T_p' := \pi_{\check{\mathscr{A}}'}^{-1}(p).$$

We put $\Theta := \Upsilon_{n_0}$ and $\widetilde{\Theta} := \widetilde{\Upsilon}_{n_0}$, choosing the integer n_0 in the following way. First, we fix a point a in the circle $C := \{x_s = 0, |y_s| = 1\}$ and an element ν of $\pi_0(\widetilde{T}_a^{\star}, \infty)$, $T_a := \pi_{\check{\mathscr{A}}}^{-1}(a)$. We consider the natural covering $\rho : \Pi \to C$ of C with fibers $\rho^{-1}(p) := \pi_0(\widetilde{T}_p^{\star}, \infty)$ and the

following two covering morphisms over $G_{|C}$

$$
\begin{array}{ccc}
\Pi & \xrightarrow{\Lambda} & \Pi' \\
\rho \downarrow & \circlearrowleft & \downarrow \rho' \\
C & \xrightarrow{G_{|C}} & C'
\end{array}
\qquad \Lambda = \widetilde{\psi} \quad \text{and} \quad \Lambda = \widetilde{g},
$$

defined by $\widetilde{\psi}$ and $\widetilde{g}$. Using Lemma 8.4.1, $\sigma' := \widetilde{\psi}(\sigma_0)$ and $\sigma'' := \widetilde{g}(\sigma_0)$ belong to the same fiber of the map $\iota : \pi_0(\widetilde{T}_{a'}'^{,\star}, \infty) \to \pi_0(\widetilde{\mathbb{T}}'^{\star}, \infty)$, induced by the inclusion map $\iota : \widetilde{T}_{a'}'^{,\star} \hookrightarrow \widetilde{\mathbb{T}}'^{\star}$. On the other hand, the action

$$
\mathbb{Z} \times \pi_0(\widetilde{T}_{a'}'^{,\star}, \infty) \longrightarrow \pi_0(\widetilde{T}_{a'}'^{,\star}, \infty) \qquad (n, \sigma) \mapsto \widetilde{\Upsilon}_n(\sigma),
$$

of $\mathbb{Z}$ on the fiber of ρ' at the point $a' := \psi(a) = G(a) \in C'$ coincide with the action of $\pi_1(C', a') \simeq \mathbb{Z}$ induced by the covering ρ' on this fiber. The orbits of that action correspond to the fibers of ι. We choose n_0 to be the unique integer number such that $\widetilde{\Upsilon}_{n_0}(\sigma') = \sigma''$.

LEMMA 8.4.1. – *The following diagram is commutative:*

$$
\begin{array}{ccc}
\pi_0(\widetilde{\Sigma}^*, \infty) & \xrightarrow{\widetilde{\psi}} & \pi_0(\widetilde{\Sigma}'^{\star}, \infty) \\
\imath \downarrow & \circlearrowleft & \downarrow \imath \\
\pi_0(\widetilde{\mathbb{T}}^*, \infty) & \xrightarrow{\widetilde{g}} & \pi_0(\widetilde{\mathbb{T}}'^{\star}, \infty)
\end{array}
$$

Proof. – Let $\widetilde{\Sigma}_0^*$ be a connected component of $\widetilde{\Sigma}^*$ and consider the subgroup $\Gamma_0 \subset \Gamma_\infty$ consisting of those elements $\varphi \in \Gamma_\infty$ such that $\varphi(\widetilde{\Sigma}_0^*) = \widetilde{\Sigma}_0^*$. Let us denote $\widetilde{\Sigma}_0'^{\star} := \widetilde{g}(\widetilde{\Sigma}_0^*)$ and $\widetilde{\Sigma}_1'^{\star} := \widetilde{\psi}(\widetilde{\Sigma}_0^*)$ and for $i = 0, 1$ consider also the subgroups Γ_i' of Γ_∞' consisting of those elements φ such that $\varphi(\widetilde{\Sigma}_i'^{\star}) = \widetilde{\Sigma}_i'^{\star}$. There exists $\gamma \in \Gamma_\infty'$ such that $\widetilde{\Sigma}_1'^{\star} = \gamma(\widetilde{\Sigma}_0'^{\star})$. A straightforward computation shows that $\Gamma_1' = \gamma\Gamma_0'\gamma^{-1}$. On the other hand, if $\varphi \in \Gamma_0$ then $\widetilde{g}_*(\varphi) \in \Gamma_0'$. Using $(\star\star)$ we also deduce that $\widetilde{g}_*(\varphi) \in \Gamma_1'$ for all $\varphi \in \Gamma_0$. Consequently $\Gamma_0' \cap \gamma\Gamma_0'\gamma^{-1} \neq \{1\}$. We fix a point $\widetilde{c} \in \widetilde{\Sigma}_0^{\star}$ and we identify Γ_∞' to $\pi_1(\mathbb{B}'^{\star}, c)$, $c := q'(\widetilde{c})$, by means of the isomorphism $\chi : \varphi \mapsto q' \circ \dot{\gamma}_\varphi$, $\varphi \in \Gamma_\infty'$, where γ_φ denotes a path in $\widetilde{\mathbb{B}}'^{\star}$, having endpoints $\widetilde{c}$ and $\varphi(\widetilde{c})$. Clearly, $\chi(\Gamma_0') = \pi_1(\Sigma_0'^{\star}, c) = \langle \mathfrak{m}_c \rangle \subset \langle \mathfrak{m}_c, \mathfrak{p}_c \rangle = \pi_1(\mathbb{T}'^{\star}, c)$, cf. Definition 6.1.2. Since $H_1(\mathbb{B}'^{\star}, \mathbb{Z})$ is torsion free, if $\mathfrak{m}_c^\alpha = \chi(\gamma)\mathfrak{m}_c^\beta\chi(\gamma)^{-1}$ then $\alpha = \beta$ and consequently $[\mathfrak{m}_c^\alpha, \chi(\gamma)] = 1$. It remains to see that if $\alpha \neq 0$ then $\chi(\gamma)$ belongs to $\pi_1(\mathbb{T}'^{\star}, c)$. Indeed, if this is the case then $\widetilde{\Sigma}_0'^{\star}$ and $\widetilde{\Sigma}_1'^{\star}$ are contained in the same connected component of $\widetilde{\mathbb{T}}'^{\star}$. That $\chi(\gamma) \in \pi_1(\mathbb{T}'^{\star}, c)$ follows easily from the Sub-Lemma 8.4.2 below. Indeed, we apply it first to $A = \pi_1(W, c)$, where $W \subset \mathbb{B}'^{\star}$ is the JSJ block containing $\Sigma'^{\star}$. We deduce that $\chi(\gamma) \in A$ and we pass to the quotient $\bar{A} = A/\langle \mathfrak{p}_c \rangle$ by its center $\langle \mathfrak{p}_c \rangle$. We express $\bar{A} = \mathbb{Z}\overline{\mathfrak{m}_c} * G$ and we apply again Sub-Lemma 8.4.2 in order to have that $\overline{\chi(\gamma)} \in \langle \overline{\mathfrak{m}_c} \rangle$. Hence $\chi(\gamma) \in \langle \mathfrak{m}_c, \mathfrak{p}_c \rangle$. $\qquad\square$

SUB-LEMMA 8.4.2. – *Let $\Gamma = A *_C B$ be an amalgamated product of groups A and B over a common subgroup C. Fix an element $\alpha \in A \setminus \bigcup_{a \in A} aCa^{-1}$. If $\gamma \in \Gamma$ commutes with α then $\gamma \in A$.*

Proof of the sub-lemma. – For each element γ in Γ, there is a unique natural number $n \geq 1$ such that γ can be written in only one of the following ways

(i) $a_1 b_1 a_2 \cdots a_n b_n$,
(ii) $b_1 a_1 b_2 \cdots b_n a_n$,
(iii) $b_1 a_1 b_2 \cdots a_{n-1} b_n$,
(iv) $a_1 b_1 a_2 \cdots b_{n-1} a_n$,

provided that $a_i \in A \setminus C$ and $b_i \in B \setminus C$. If $[\alpha, \gamma] = 1$ then necessarily γ is of type (iv). Since $a_1^{-1} \alpha a_1 b_1 a_2 \cdots b_{n-1} a_n \alpha^{-1} = b_1 a_2 \cdots b_{n-1} a_n$ and $a_1^{-1} \alpha a_1 \notin C$ it follows that $n = 1$. $\qquad\square$

8.5. Gluing at the nodal singularities

For $s \in \mathrm{Node}(\mathcal{G}) \cap \mathrm{Sing}(\mathcal{E}_S)$ both germs of $\underline{\mathcal{G}}$ at s and of $\underline{\mathcal{G}}'$ at $s' := G(s)$ are holomorphicaly conjugated to the germ, at the origin in $\mathbb{C}^2$, of the same linear complex foliation $\mathcal{L}$ having a multivaluated first integral xy^λ, where $\lambda \in \mathbb{R}_{<0} \setminus \mathbb{Q}$ and $1/\lambda$ are the common Camacho-Sad indices at these singularities. This foliation $\mathcal{L}$ has a conical structure over the induced (real) foliation by lines $\mathcal{L}_\mathbb{R}$ on the sphere $\partial \mathbb{P}$ with $\mathbb{P} = \mathbb{D}_1 \times \mathbb{D}_1$. Outside the knots defined by the coordinate axis, the leaves of $\mathcal{L}_\mathbb{R}$ are contained in the tori constituted by the level sets of $|x||y|^\lambda$ in that sphere. Restricted to any torus, $\mathcal{L}_\mathbb{R}$ can be seen as the suspension of the rotation of angle $2\pi\lambda$. The following property of $\mathcal{L}$ allows us to easily build a topological conjugation between $\underline{\mathcal{G}}$ and $\underline{\mathcal{G}}'$, which is defined on neighborhoods of the divisors in the elementary pieces associated to s and s' and which coincides with the conjugations previously obtained on the adjacent elementary pieces.

– *let $\phi(x, y) := (\underline{\phi}(x, y), y)$ be a germ along the circle $\mathcal{C} := \{0\} \times \partial \mathbb{D}_1$ of a biholomorphism defined on a neighborhood of $\mathcal{C}$ in the solid torus $\mathbb{D}_1 \times \partial \mathbb{D}_1$, which preserves the foliation $\mathcal{L}_\mathbb{R}$. Then there exist positive real numbers $\varepsilon_1 < \varepsilon_2 < 1$ and a global homeomorphism $\Phi : \partial \mathbb{P} \overset{\sim}{\longrightarrow} \partial \mathbb{P}$ which coincide with ϕ for $|x||y|^\lambda \leq \varepsilon_1$, and is the identity map for $|x||y|^\lambda \geq \varepsilon_2$.*

The proof is based on the linearity of the maps $\phi(x, \cdot)$ resulting of their commutations with the holonomy map of $\mathcal{L}$ along the circle $\mathcal{C}$ -which is a non periodic rotation. We can construct Φ by a suitable interpolation, along the family of tori $(\mathbb{T}_r := \partial \mathbb{D}_r \times \partial \mathbb{D}_1)_{\varepsilon_1 \leq r \leq \varepsilon_2}$ between the restriction of ϕ to $\mathbb{T}_{\varepsilon_1}$ and the identity map on $\mathbb{T}_{\varepsilon_2}$.

8.6. Proof of Extension Lemma 8.3.2 for $K = K_D$

Thanks to Theorem 4.3.1, ϕ conjugates the holonomy representation of $\underline{\mathcal{G}}$ along D° and that of $\underline{\mathcal{G}}'$ along D'°. Therefore, by using the classical lifting path method we have the following:

(▲) *there are fundamental systems $(W_k)_{k \in \mathbb{N}}$ and $(W'_k)_{k \in \mathbb{N}}$ of neighborhoods of D° and D'° in K and K' and there are homeomorphisms Φ_k from W_k onto W'_k, such that:*
  1) *for all k, l, Φ_k and Φ_l coincide over $W_k \cap W_l$;*
  2) *the intersection of W_k (resp. W'_k) with the fibers of π_D (resp. $\pi_{D'}$) are conformal disks and hence W_k^* and K^* (resp. $W'^\star_k$ and $K^\star$) are homotopic;*
  3) *$\mathrm{Sat}_{\underline{\mathcal{G}}}(T \cap W_k, W_k) = W_k$ and $\mathrm{Sat}_{\underline{\mathcal{G}}'}(T' \cap W'_k, W'_k) = W'_k$;*
  4) *$\pi_{D'} \circ \Phi_k = G \circ \pi_{D|W_k}$;*
  5) *the restrictions of ϕ and Φ_k to $W_k \cap T$ coincide;*
  6) *Φ_k conjugates $\underline{\mathcal{G}}_{|W_k}$ to $\underline{\mathcal{G}}'_{|W'_k}$.*

Set $V := W_0$, $V' := W_0'$, $\Phi := \Phi_0$, $\Delta := V \cap T$ and $\Delta' := V \cap T'$. From Property
(▲) 2) above it follows that the restriction of $\underline{q}$ to each connected component $\widetilde{V}^{*\beta}$ of $\widetilde{V}^*$,
$\beta \in \pi_0(\widetilde{V}^*)$ (resp. $\widetilde{V}'^{*\beta'}$ of $\widetilde{V}'^\star$, $\beta' \in \pi_0(\widetilde{V}'^*)$) is a universal covering of V^* (resp. V'^*); the
same is true for the connected components $\widetilde{\Delta}^{*\alpha}$ of $\widetilde{\Delta}^*$ over Δ^*, $\alpha \in \pi_0(\widetilde{\Delta}^* \cap \widetilde{V}^{*\beta})$ and for
those $\widetilde{\Delta}'^{*\alpha'}$ of $\widetilde{\Delta}'^*$ over Δ'^*, $\alpha' \in \pi_0(\widetilde{\Delta}'^*)$. We fix $\beta \in \pi_0(\widetilde{V}^*)$, $\alpha \in \pi_0(\widetilde{\Delta}^* \cap \widetilde{V}^{*\beta})$ and we
denote by $\widetilde{\Phi}^\beta : \widetilde{V}^{*\beta} \to \widetilde{V}'^{*\beta'}$ the unique homeomorphism that lifts Φ over $\widetilde{V}^{*\beta}$ coinciding
with $\widetilde{\phi}$ on $\widetilde{\Delta}^{*\alpha}$, where β' corresponds to the connected component of $\widetilde{V}'^*$ containing $\widetilde{\phi}(\widetilde{\Delta}^{*\alpha})$.
We will prove the following assertions:

  (i) $\widetilde{\Phi}^\beta$ does not depend on the choice of $\alpha \in \pi_0(\widetilde{\Delta}^* \cap \widetilde{V}^{*\beta})$;
  (ii) the homeomorphism $\widetilde{\Phi} : \widetilde{V}^* \to \widetilde{V}'^\star$ defined by requiring that its restriction to $\widetilde{V}^{*\lambda}$,
      $\lambda \in \pi_0(\widetilde{V}^*)$, coincides with $\widetilde{\Phi}^\lambda$ satisfies Conditions (a)-(e) of Lemma 8.3.2.

8.6.1. *Proof of Assertion* (i). – Using the uniqueness of the lifting of Φ over the covering
$\widetilde{V}^{*\beta}$, it suffices to show that for each $\kappa \in \pi_0(\widetilde{\Delta}^* \cap \widetilde{V}^{*\beta})$ the maps $\widetilde{\Phi}^\beta$ and $\widetilde{\phi}$ coincide at one
particular point of $\widetilde{\Delta}^{*\kappa}$. In fact, we will prove the equality of the germs at infinity of their
restrictions to $\widetilde{\Delta}^{*\kappa}$.

To do that, it suffices to see that $\widetilde{\Phi}^\beta(\widetilde{\Delta}^{*\kappa})$ and $\widetilde{\phi}(\widetilde{\Delta}^{*\kappa})$ are contained in the same connected
component $\widetilde{\Delta}'^{*\kappa'}$ of $\widetilde{\Delta}'^* \cap \widetilde{V}'^{*\beta'}$. Indeed, $\widetilde{\Phi}^\beta$ conjugates $\widetilde{\mathcal{G}}_{|\widetilde{V}^{*\beta}}$ to $\widetilde{\mathcal{G}}'_{|\widetilde{V}'^{\beta'}}$ and it factorizes
through the leaf spaces of these foliations. Hence the following diagram is commutative:

$$
\begin{array}{ccc}
(\widetilde{\Delta}^{*\kappa}, \infty) & \xrightarrow{\ \Upsilon\ } & (\widetilde{\Delta}'^{*\kappa'}, \infty) \\
\tau_{\widetilde{\Delta}^{*\kappa}} \downarrow & \circlearrowleft & \downarrow \tau_{\widetilde{\Delta}'^{*\kappa'}} \\
\widetilde{Q}^{\mathcal{G}}_\infty & \xrightarrow{\ h\ } & \widetilde{Q}^{\mathcal{G}'}_\infty
\end{array}
$$

where Υ denotes the germ at infinity $\widetilde{\Phi}^\beta_{|\widetilde{\Delta}^{*\kappa}\,\infty}$ of the restriction of $\widetilde{\Phi}^\beta$ to $\widetilde{\Delta}^{*\kappa}$, and the vertical
arrows are the canonical pro-germs defined in Section 3.3. On the other hand, this diagram
still commutes when we take as Υ the germ at infinity $\widetilde{\phi}_{|\widetilde{\Delta}^{*\kappa}\,\infty}$ of the restriction of $\widetilde{\phi}$ to $\widetilde{\Delta}^{*\kappa}$,
because $(\phi_S, \widetilde{\phi}_\infty, h)$ is a realization of $(\mathfrak{g}, h)$ over T. The equality $\widetilde{\Phi}^\beta_{|\widetilde{\Delta}^{*\kappa}\,\infty} = \widetilde{\phi}_{|\widetilde{\Delta}^{*\kappa}\,\infty}$ follows
from Proposition 3.3.1 which asserts that $\tau_{\widetilde{\Delta}^{*\kappa}}$ et $\tau_{\widetilde{\Delta}'^{*\kappa'}}$ are $\underleftarrow{\mathrm{An}}$-monomorphisms.

Consider now the natural connected coverings

$$
\chi_\beta : \bigsqcup_{t \in D^\circ} \pi_0\left(\widetilde{\Delta}^*_t \cap \widetilde{V}^{*\beta}, \infty\right) =: \Pi^\beta_0 \longrightarrow D^\circ, \quad \Delta_t := \pi_D^{-1}(t) \cap V,
$$

$$
\text{and} \quad \chi'_{\beta'} : \bigsqcup_{t \in D'^\circ} \pi_0\left(\widetilde{\Delta}'^*_t \cap \widetilde{V}'^{*\beta'}, \infty\right) =: \Pi'^{\beta'}_0 \longrightarrow D'^\circ, \quad \Delta'_t := \pi_{D'}^{-1}(t) \cap V'.
$$

The maps $\widetilde{g}$ and $\widetilde{\Phi}^\beta$ send each connected component of $\widetilde{\Delta}^*_t \cap \widetilde{V}^{*\beta}$ onto a connected com-
ponent of $\widetilde{\Delta}'^*_{G(t)} \cap \widetilde{V}'^{*\beta'}$, defining in this way covering morphisms $\widetilde{g}^\beta$ and $\widetilde{\Phi}^\beta$ over $G_{|D^\circ}$,
i.e.,

$$
\begin{array}{ccc}
\Pi^\beta_0 & \xrightarrow{\ \Lambda\ } & \Pi'^{\beta'}_0 \\
\chi_\beta \downarrow & \circlearrowleft & \downarrow \chi'_{\beta'} \\
D^\circ & \xrightarrow{\ G_{|D^\circ}\ } & D'^\circ
\end{array} \quad \Lambda = \widetilde{g}^\beta \text{ and } \Lambda = \widetilde{\Phi}^\beta,
$$

Notice that over the point c, the actions of $\widetilde{\Phi}^\beta$ and $\widetilde{g}^\beta$ coincide at the previously fixed point $\alpha \in \pi_0(\widetilde{\Delta}^*, \infty) \subset \Pi_0^\beta$. Indeed, we have the equalities:

$$\widetilde{\Phi}^\beta(\alpha) = \widetilde{\phi}(\alpha) = \widetilde{g}^\beta(\alpha),$$

the first one comes from the construction of $\widetilde{\Phi}^\beta$ and the second one from the hypothesis (23). We deduce the identity $\widetilde{\Phi}^\beta = \widetilde{g}^\beta$ on Π_0^β. Using again (23) we obtain that $\widetilde{\Phi}^\beta(\kappa) = \widetilde{g}^\beta(\kappa) = \widetilde{\phi}(\kappa)$, for all $\kappa \in \pi_0(\widetilde{\Delta}^* \cap \widetilde{V}^{*\beta})$, concluding thus the proof of (i).

8.6.2. *Proof of Assertion* (ii). – Here $\check{D} = D$ and $K \cap \check{D}^\circ = D^\circ$. Assertions (a), (b) and (c) of Lemma 8.3.2 are satisfied by construction. The proof of (i) above also shows Assertion (e). It remains to show (d), i.e., the pro-germ at infinity $\widetilde{\Phi}_\infty$ defined by $\widetilde{\Phi}$ makes commutative the diagrams ($\star$) and ($\star\star$) in Definition 3.6.1.

8.6.2.1. *Proof of the commutativity of* ($\star$).– We fix cofinal families $U_n \in \mathfrak{U}_{\mathcal{F}, \Sigma \cup E_S(T)}$ and $U'_n \in \mathfrak{U}_{\mathcal{F}', \Sigma \cup E_{S'}(T')}$, $n \in \mathbb{N}$, and we represent $h \in \mathrm{Hom}_{\underline{\mathrm{An}}}(\widetilde{Q}_\infty^{\mathcal{F}}, \widetilde{Q}_\infty^{\mathcal{F}'})$ by a family of holomorphic maps $h_{n,p}$ from $\widetilde{Q}_{\widetilde{U}_n}^{\mathcal{F}}$ into $\widetilde{Q}_{\widetilde{U}'_p}^{\mathcal{F}'}$, cf. (8). By definition, the commutativity of ($\star$) means the commutativity of all the diagrams

$$
\begin{array}{ccc}
(\widetilde{V}^*, \infty) & \xrightarrow{\widetilde{\Phi}_\infty} & (\widetilde{V}'^\star, \infty) \\
\tau_n \downarrow & & \downarrow \tau'_p \\
\widetilde{Q}_{U_n}^{\mathcal{F}} & \xrightarrow{h_{n,p}} & \widetilde{Q}_{U'_p}^{\mathcal{F}'}
\end{array}
\qquad , \qquad n \geq p,
$$

where $\tau_n := \tau_{\widetilde{V}^*, U_n}$ and $\tau'_p := \tau_{\widetilde{V}'^\star U'_p}$ are the canonical pro-germs defined in Section 3.3. In order to prove this, thanks to Theorem 2.1.1, it suffices to determine open neighborhoods $V_{n,p}$ of D° in K_D, satisfying

$$(25) \qquad V_{n,p} \subset U_n, \quad \Phi(V_{n,p}) \subset U'_p \quad \text{and} \quad \tau'_{n,p} \circ \widetilde{\Phi}_{|\widetilde{V}_{n,p}} = h_{n,p} \circ \tau_{n,p|\widetilde{V}_{n,p}},$$

where

$$\tau_{n,p} : \widetilde{V}^*_{n,p} \longrightarrow \widetilde{Q}_{U_n}^{\mathcal{F}} \quad \text{and} \quad \tau'_{n,p} : \widetilde{V}'^\star_{n,p} \to \widetilde{Q}_{U'_p}^{\mathcal{F}'}$$

denote the quotient maps. Take for $V_{n,p}$ an open set $W_{\kappa(n,p)}$ of the fundamental system given by ($\blacktriangle$), where the index $\kappa(n,p) \in \mathbb{N}$ is chosen big enough so that the above inclusions hold. The equality in (25) is an equality of maps and it can be checked locally using the open cover $(\widetilde{V}_{n,p}^\sigma)_\sigma$ of $\widetilde{V}^*_{n,p}$. Notice that such an argument is not valid if we want to check the equality of pro-germs $\tau'_p \circ \widetilde{\Phi}_\infty = h_{n,p} \circ \tau_n$ without realizing them previously over the open sets $V_{n,p}$, cf. Remark 3.2.1. Put

$$\widetilde{V}_{n,p}^\sigma := \mathrm{Sat}_{\underline{\widetilde{\mathcal{F}}}}(\widetilde{\Delta}_{n,p}^\sigma, \widetilde{V}_{n,p}), \quad \sigma \in \pi_0(\widetilde{\Delta}_{n,p}^*, \infty), \quad \Delta_{n,p} := V_{n,p} \cap T,$$

where $\widetilde{\Delta}_{n,p}^\sigma$ is the connected component of $\widetilde{\Delta}_{n,p}^*$ corresponding to σ. Property 3) in ($\blacktriangle$) implies that the open sets $\widetilde{V}_{n,p}^\sigma$ cover $V_{n,p}$. On the other hand, Property $c)$ in the definition of $(\mathcal{F}, \Sigma \cup E_S(T))$-admissibility of $V_{n,p}$ stated in Section 2.1, implies that each leaf of the restriction of $\underline{\widetilde{\mathcal{F}}}$ to $\widetilde{V}_{n,p}^\sigma$ meets (transversely) $\widetilde{\Delta}_{n,p}^\sigma$ in exactly one point, defining thus a holomorphic submersion-retraction-first integral r^σ from $\widetilde{V}_{n,p}^\sigma$ onto $\widetilde{\Delta}_{n,p}^\sigma$. By using ($\blacktriangle$) 4), it is clear that $\widetilde{\Phi}(\widetilde{\Delta}_{n,p}^\sigma)$ is a connected component of $\widetilde{\Delta}'^\star_{n,p}$, where $\Delta'_{n,p} := V'_{n,p} \cap T' = \Phi(\Delta_{n,p})$. Analogously, we also have a holomorphic submersion-retraction-first integral r'^σ defined

over the saturated set $\widetilde{V}'^\sigma_{n,p}$ of $\widetilde{\Phi}(\widetilde{\Delta}^\sigma_{n,p})$ in $\widetilde{V}'^\star_{n,p}$ onto $\widetilde{\Phi}(\Delta^\sigma_{n,p})$. Clearly $\widetilde{V}'^\sigma_{n,p} = \widetilde{\Phi}(\widetilde{V}'_{n,p})$ and the following diagram commutes

(26)
$$
\begin{array}{ccc}
\widetilde{V}^\sigma_{n,p} & \xrightarrow{\ \widetilde{\Phi}_{|\widetilde{V}^\sigma_{n,p}}\ } & \widetilde{V}'^\sigma_{n,p} \\
r^\sigma \downarrow & \circlearrowleft & \downarrow r'^\sigma \\
\widetilde{\Delta}^\sigma_{n,p} & \xrightarrow{\ \widetilde{\Phi}_{|\widetilde{\Delta}^\sigma_{n,p}}\ } & \widetilde{\Phi}(\widetilde{\Delta}^\sigma_{n,p})
\end{array}
\;,
$$

thanks to ($\blacktriangle$) 6). On the other hand, $\widetilde{\Phi}$ and $\widetilde{\phi}$ coincide on $\widetilde{\Delta}^*$. Therefore, the commutativity of the diagram $(\star)$ for $\psi = \phi$ in Definition 3.6.1 implies the commutativity of the following diagram if the index $\kappa(n,p)$ is chosen big enough:

(27)
$$
\begin{array}{ccc}
\widetilde{\Delta}^\sigma_{n,p} & \xrightarrow{\ \widetilde{\Phi}_{|\widetilde{\Delta}^\sigma_{n,p}}\ } & \widetilde{\Phi}(\widetilde{\Delta}^\sigma_{n,p}) \\
\tau \downarrow & \circlearrowleft & \downarrow \tau' \\
\widetilde{Q}^{\mathcal{I}}_{U_n} & \xrightarrow{\ h_{n,p}\ } & \widetilde{Q}^{\mathcal{I}'}_{U'_p}
\end{array}
\;,
$$

τ and τ' denoting the quotient maps as usual. It is clear that the commutativity of the diagrams (26) and (27) gives us the relation

$$
\tau'_{n,p} \circ \widetilde{\Phi}_{|\widetilde{V}_{n,p}} = h_{n,p} \circ \tau_{n,p|\widetilde{V}_{n,p}}
$$

of (25). This finish the proof of the commutativity of $(\star)$.

8.6.2.2. *Proof of the commutativity of $(\star\star)$ for $\psi = \Phi_\infty$.–* Notice that $\widetilde{\phi}_\infty$ makes the corresponding diagram $(\star\star)$ commutative in Definition 3.6.1:

$$
\begin{array}{ccc}
\Gamma_\infty & \overset{\iota}{\hookrightarrow} & \Gamma_{\widetilde{T}^*,\infty} \\
\mathfrak{g} \downarrow & \circlearrowleft & \downarrow \widetilde{\phi}_{\infty*} \\
\Gamma'_\infty & \overset{\iota'}{\hookrightarrow} & \Gamma_{\widetilde{T}'^\star,\infty}
\end{array}
\;,
$$

where $\iota(\varphi) := \varphi_{|\widetilde{T}^*}$ (resp. $\iota'(\varphi) := \varphi_{|\widetilde{T}'^\star}$). By applying the following sub-lemma with $W_1 := E_S(T)$, $W_2 := E_S(V)$, $W'_1 := E_{S'}(T')$ and $W'_2 := E_{S'}(V')$, we obtain directly the commutativity of $(\star\star)$ in our context.

SUB-LEMMA 8.6.1. – *Let $W_1 \subset W_2$ (resp. $W'_1 \subset W'_2$) be submanifolds of $\mathbb{B}$ (resp. $\mathbb{B}'$) not contained in S (resp. S') such that $W_1 \cap S \neq \varnothing$ (resp. $W'_1 \cap S' \neq \varnothing$). Consider $\widetilde{\Psi}_2 : \widetilde{W}_2 \to \widetilde{W}'_2$ a lift of a homeomorphism $\Psi_2 : W_2 \to W'_2$ such that $\Psi_2(W_1) = W'_1$. We denote $\widetilde{\Psi}_1 := \widetilde{\Psi}_{2|\widetilde{W}^*_1}$ and we keep the notations introduced in Section 3.6. Then the following diagram $(\star\star)$ commutes for $k = 1$ if and only if it commutes for $k = 2$:*

$$
\begin{array}{ccc}
\Gamma_\infty & \overset{\iota_k}{\hookrightarrow} & \Gamma_{\widetilde{W}^*_k,\infty} \\
\mathfrak{g} \downarrow & & \downarrow \widetilde{\Psi}_{k*} \\
\Gamma'_\infty & \overset{\iota'_k}{\hookrightarrow} & \Gamma_{\widetilde{W}'^\star_k,\infty}
\end{array}
\;, \quad k = 1,2,
$$

where $\iota_k(\varphi) := \varphi_{|\widetilde{W}^*_k}$ and $\iota'_k(\varphi) := \varphi_{|\widetilde{W}'^*_k}$.

Proof of the sub-lemma. – It suffices to observe that the horizontal lines of the diagram below are composition of monomorphisms:

$$
\begin{array}{ccccc}
\Gamma_\infty & \overset{\iota_2}{\hookrightarrow} & \Gamma_{\widetilde{W}_2^*,\infty} & \overset{\iota_{12}}{\hookrightarrow} & \Gamma_{\widetilde{W}_1^*,\infty} \\
\mathfrak{g}\downarrow & & \downarrow \widetilde{\Psi}_{2*} \quad \circlearrowleft & & \downarrow \widetilde{\Psi}_{1*} \\
\Gamma'_\infty & \overset{\iota'_2}{\hookrightarrow} & \Gamma'_{\widetilde{W}'_2{}^\star,\infty} & \overset{\iota'_{12}}{\hookrightarrow} & \Gamma'_{\widetilde{W}'_1{}^\star,\infty},
\end{array}
$$

with $\iota_{12}(\varphi) := \varphi_{|\widetilde{W}_1^*}$ and $\iota'_{12}(\varphi) := \varphi_{|\widetilde{W}'_1{}^\star}$. $\qquad\square$

This achieves the proof Assertion (*ii*) and consequently we have proved Lemma 8.3.2 in the case $K = K_D$.

8.7. Proof of Extension Lemma 8.3.2 for $K = K_s$

We assume that $y_s = 0$ (resp. $y_{s'} = 0$) is a reduced local equation of D (resp. D') supporting the transverse fibers $T = \pi_D^{-1}(c)$ and $T' = \pi_{D'}^{-1}(c')$, $c' = G(c)$. We denote by $\check{D}$ (resp. $\check{D}'$) the irreducible component of $\mathcal{D}_S$ (resp. $\mathcal{D}_{S'}$) meeting K (resp. K') whose reduced local equation is $x_s = 0$ (resp. $x_{s'} = 0$). We also adopt the following notations:

$$\mathbb{P}_{\lambda,\mu} := \{|x_s| \le \lambda, |y_s| \le \mu\}, \quad \mathbb{P}'_{\lambda,\mu} := \{|x_{s'}| \le \lambda, |y_{s'}| \le \mu\},$$

$$\mathbb{T}_{\lambda,\mu} := \{|x_s| \le \lambda, |y_s| = \mu\}, \quad \mathbb{T}'_{\lambda,\mu} := \{|x_{s'}| \le \lambda, |y_{s'}| = \mu\},$$

$$\mathfrak{T}_{\lambda,\mu} := \{|x_s| = \lambda, |y_s| \le \mu\} \quad \text{and} \quad \mathfrak{T}'_{\lambda,\mu} := \{|x_{s'}| = \lambda, |y_{s'}| \le \mu\}.$$

Denote by X (resp. X') the real vector field tangent to $\mathcal{G}$ (resp. $\mathcal{G}'$), whose flow writes as $(x_s, y_s, t) \mapsto (F(x_s, y_s, t), e^t y_s)$ (resp. $(x_{s'}, y_{s'}, t) \mapsto (F'(x_{s'}, y_{s'}, t), e^t y_{s'})$).

First, we will construct Φ and $\widetilde{\Phi}$ satisfying Assertions (a)-(d) of Lemma 8.3.2. Next we will modify these two homeomorphisms, without affecting Properties (a)-(d), in order to satisfy also Assertion (e).

8.7.1. *First step: construction of Φ.* – Thanks to Theorem 4.3.1, the germ ϕ_S conjugates the holonomies of $\mathcal{G}$ and $\mathcal{G}'$. Therefore it extends to neighborhoods of the punctured disks

$$D_s^\diamond := \{0 < |x_s| \le 1, y_s = 0\}, \quad D'_{s'}{}^\diamond := G(D_s^\diamond) = \{0 < |x_{s'}| \le 1, y_{s'} = 0\},$$

defining a unique germ of homeomorphism $\Phi_{D_s^\diamond}$ from $(\mathcal{W}_s, D_s^\diamond)$ onto $(\mathcal{W}'_{s'}, D'_{s'}{}^\diamond)$ which conjugates $\mathcal{G}$ to $\mathcal{G}'$ and commutes with the Hopf fibrations, i.e., $\pi_{D'} \circ \Phi_{D_s^\diamond} = G \circ \pi_D$. Since the germ ϕ_S is holomorphic and G is excellent, cf. Definition 3.5.2, and consequently holomorphic on a polydisk $\mathbb{P}_{\alpha,\beta}$, we deduce that the germ $\Phi_{D_s^\diamond}$ is holomorphic for $|x_s| \le \alpha$ if $\alpha > 0$ is small enough. In fact, following the construction given in [10] we can assume that G satisfies also the relations

$$x_{s'} \circ G_{|\mathbb{P}_{\alpha,\beta}} = x_{s|\mathbb{P}_{\alpha,\beta}}, \qquad y_{s'} \circ G_{|\mathbb{P}_{\alpha,\beta}} = y_{s|\mathbb{P}_{\alpha,\beta}}.$$

The equality of the Camacho-Sad indices will allow us to extend the germ $\Phi_{D_s^\diamond}$ to the singular point s and this extension can be represented by a homeomorphism Φ_{D_s} defined on a polydisk $\mathbb{P}_{1,\epsilon}$, $0 < \epsilon < \alpha$ into an open set containing another polydisk $\mathbb{P}'_{1,\epsilon'}$. This homeomorphism Φ_{D_s} conjugates $\mathcal{G}$ to $\mathcal{G}'$ and verifies $\pi_{D'} \circ \Phi_{D_s} = G \circ \pi_{D|\mathbb{P}_{1,\epsilon}}$. In addition, Φ_{D_s} is holomorphic on $\mathbb{P}_{\alpha,\epsilon}$.

When s is not a nodal singular point of $\underline{\mathcal{G}}$, we can apply the conjugation Theorem stated in [12] (see also [7, §5.2.1]) to construct Φ_{D_s}. Otherwise, the nodal singularities are linearizable and these linearizations can specifically be done without changing the Hopf fibrations of D and D'. Indeed, there exists a coordinate $\widetilde{y}_s$ (resp. $\widetilde{y}_{s'}$) such that $\underline{\mathcal{G}}$ (resp. $\underline{\mathcal{G}}'$) is given a linear vector field in the coordinates $(x_s, \widetilde{y}_s)$ (resp. $(x_s, \widetilde{y}_{s'})$). On the transversals T and T' the holonomy maps around the singularities s and s' are irrational rotations when expressed in the coordinates $\widetilde{y}_{s|T}$ and $\widetilde{y}_{s'|T'}$ respectively. Thus the conjugation germ Φ_S is linear in these coordinates and any linear extension is a conjugation between $\underline{\mathcal{G}}$ and $\underline{\mathcal{G}}'$. In both cases up to restricting α and $\epsilon > 0$, we can also assume that

- the vector field X is transverse to the torus $\mathbb{T}_{\alpha,\epsilon}$ and X' is transverse to the real analytic hypersurface $H := \Phi_{D_s}(\mathbb{T}_{\alpha,\epsilon})$,
- H does not intersect the torus $\mathbb{T}'_{1,\beta}$,
- for $\alpha' > 0$ small enough, $(\mathbb{P}'_{\alpha',1} \setminus H)$ possesses two connected components and that one not containing s, does contain $\mathbb{T}'_{\alpha',\beta}$; we denote by $\mathbb{P}'^{+}_{\alpha',1}$ its adherence.

We fix such a α' and we "glue" Φ_{D_s} with $\Phi_{\mathcal{C}}$ given by applying the following sub-lemma. We denote by D_1 (resp. D'_1) the irreducible component of $\mathcal{D}_S$ (resp. $\mathcal{D}_{S'}$), containing s (resp. s'), which is different from D (resp. D').

SUB-LEMMA 8.7.1. – *There is a homeomorphism $\Phi_{\mathcal{C}}$ defined on an open neighborhood $V_{\mathcal{C}}$ of the annulus $\mathcal{C} := \{x_s = 0, \epsilon \leq |y_s| \leq 1\}$ in $\{|x_s| \leq \alpha', \epsilon \leq |y_s| \leq 1\}$, into an open neighborhood $V'_{\mathcal{C}'}$ of $\mathcal{C}' := \mathbb{P}'^{+}_{\alpha',1} \cap \{x_{s'} = 0\}$ in $\mathbb{P}'^{+}_{\alpha',1}$, conjugating $\underline{\mathcal{G}}_{|V_{\mathcal{C}}}$ to $\underline{\mathcal{G}}'_{|V'_{\mathcal{C}'}}$, which coincides with Φ_{D_s} when restricted to $V_{\mathcal{C}} \cap \mathbb{T}_{1,\epsilon}$, satisfying the relation $\pi_{D'_1} \circ \Phi_{\mathcal{C}}(p) = G \circ \pi_{D_1}(p)$, for $|y_s(p)| \geq \beta$ and verifying $\mathrm{Sat}_{\underline{\mathcal{G}}}(V_{\mathcal{C}} \cap \mathbb{T}_{1,\epsilon}, V_{\mathcal{C}}) = V_{\mathcal{C}}$.*

The proof can be done using the classical lifting path method and the vector fields X and X'.

Thus, we obtain a homeomorphism $\underline{\Phi}$ defined on $\mathbb{P}_{1,\epsilon} \cup V_{\mathcal{C}}$. We choose $\zeta > 0$ small enough so that the set $V := \{|x_s y_s| \leq \zeta\} \subset \mathcal{W}_s$ is contained in $\mathbb{P}_{1,\epsilon} \cup V_{\mathcal{C}}$ and we put $\Phi := \underline{\Phi}_{|V} : V \to V' := \underline{\Phi}(V)$.

8.7.2. *Second step: construction of $\widetilde{\Phi}$.* – With Conventions 8.3.1, we have that the natural maps

(28) $$\pi_0(\widetilde{\mathfrak{T}}^{*}_{1,1}, \infty) \longrightarrow \pi_0(\widetilde{V}^{*}, \infty), \quad \pi_0(\widetilde{\mathfrak{T}}'^{*}_{1,1}, \infty) \longrightarrow \pi_0(\widetilde{V}'^{*}, \infty),$$

induced by the inclusions $\widetilde{\mathfrak{T}}^{*}_{1,1} \subset \widetilde{V}^{*}$ and $\widetilde{\mathfrak{T}}'^{*}_{1,1} \subset \widetilde{V}'^{*}$, are bijective. Hence, each lift $\widetilde{\Phi}_{\mathfrak{T}_{1,1}} : \widetilde{\mathfrak{T}}^{*}_{\zeta,1} \to \widetilde{V}'^{*}$ of the restriction of Φ to $\mathfrak{T}^{*}_{\zeta,1}$, extends in a unique way to a homeomorphism $\widetilde{\Phi} : \widetilde{V}^{*} \to \widetilde{V}'^{*}$ lifting Φ. In order to choose $\widetilde{\Phi}_{\mathfrak{T}_{\zeta,1}}$, we first apply the Lemma 8.3.2 in the case $K = K_D$, for which it is already proved. We obtain a homeomorphism $\Phi_{D^{\circ}}$ defined on a neighborhood $V_{D^{\circ}}$ of D° and a lift $\widetilde{\Phi}_{D^{\circ}}$ defined on $\widetilde{V}^{*}_{D^{\circ}}$ satisfying the assertions of the lemma. Since Conditions (b) and (c) of Lemma 8.3.2 give the unicity of these homeomorphisms and Φ also satisfies (c), we deduce that its restriction to $\mathfrak{T}_{\zeta,1}$ coincide with $\Phi_{D^{\circ}}$, after taking germs. Therefore we can define $\widetilde{\Phi}_{\mathfrak{T}_{\zeta,1}}$ as the extension of $\widetilde{\Phi}_{D^{\circ}}$ onto $\widetilde{V}^{*}$, up to restricting $\zeta > 0$ so that $\mathfrak{T}_{\zeta,1}$ be contained in $V_{D^{\circ}}$.

8.7.3. *Third step: proof of Assertions* (a)-(d). – Assertions (a)-(c) are trivially satisfied. In order to show (d) we must prove the commutativity of the two diagrams appearing in Definition 3.6.1. The commutativity of $(\star\star)$ follows exactly in the same way that in the case $K = K_D$. It only remains to prove the commutativity of $(\star)$. Using the previous notations we remark that since the triple $(\Phi_{D^\circ}, \widetilde{\Phi}_{D^\circ}, h)$ is a realization of the $\mathcal{N}$-conjugation $(\mathfrak{g}, h)$ over V, its restriction $(\Phi_{|\mathfrak{T}_{\zeta,1}}, \widetilde{\Phi}_{|\widetilde{\mathfrak{T}}^*_{\zeta,1}}, h)$ is a realization of $(\mathfrak{g}, h)$ over $\mathfrak{T}_{\zeta,1}$. Hence, we have the following commutative diagram

$$(29) \qquad \begin{array}{ccc} (\widetilde{\mathfrak{T}}^*_{\zeta,1}, \infty) & \xrightarrow{\widetilde{\Phi}_{|\mathfrak{T}_{\zeta,1}}} & (\widetilde{\mathfrak{T}}'^\star_{\zeta,1}, \infty) \\ \tau \downarrow & \circlearrowleft & \downarrow \tau \\ \widetilde{\mathcal{Q}}^{\mathcal{G}}_\infty & \xrightarrow{h} & \widetilde{\mathcal{Q}}^{\mathcal{G}'}_\infty \end{array} \quad ,$$

where τ and τ' denote the canonical pro-germs. On the other hand, using the techniques appearing in [9, §4.2] in the case of a resonant singularity, and a straightforward computation in the case of a linearizable singularity (excluding the case of a nodal singularity by hypothesis in Lemma 8.3.2), we can prove the following result.

SUB-LEMMA 8.7.2. – *There are open neighborhoods U of $\mathcal{D}_S \cap K$ in K and U' of $\mathcal{D}_{S'} \cap K'$ in K' and there are deformation retractions $R : U^* \to U \cap \mathfrak{T}^*_{1,1}$ and $R' : U'^\star \to U' \cap \mathfrak{T}'^\star_{1,1}$ such that if we set $U_\lambda := (\mathcal{D}_S \cap K) \cup R^{-1}(\mathfrak{T}^*_{1,\lambda}))$ and $U'_\lambda := (\mathcal{D}_{S'} \cap K') \cup R'^{-1}(\mathfrak{T}'^\star_{1,\lambda}))$, then*

1. *the family $(U_\lambda)_{0<\lambda\ll 1}$ (resp. $(U'_\lambda)_{0<\lambda\ll 1}$) is a fundamental system of neighborhoods of $\mathcal{D}_S \cap K$ in K (resp. of $\mathcal{D}_{S'} \cap K'$ in K');*
2. *every point p of U_λ (resp. U'_λ) belongs to the same leaf of $\underline{\mathcal{F}}_{|U^*_\lambda}$ (resp. $\underline{\mathcal{F}}'_{|U'^\star_\lambda}$) than $R(p)$, (resp. $R'(p)$).*

Thanks to Properties (1) and (2) in Sub-Lemma 8.7.2 above, the (unique) lifts of these retractions define germs at infinity R_∞ and R'_∞ which make commutative the following diagrams

$$\begin{array}{ccc} (\widetilde{V}^*, \infty) & \xrightarrow{\quad R_\infty \quad} & (\widetilde{\mathfrak{T}}^*, \infty) \\ & \tau_{\widetilde{V}^*} \searrow \quad \swarrow \tau_{\widetilde{\mathfrak{T}}^*} & \\ & \widetilde{\mathcal{Q}}^{\mathcal{G}}_\infty & \end{array} \qquad \begin{array}{ccc} (\widetilde{V}'^\star, \infty) & \xrightarrow{\quad R'_\infty \quad} & (\widetilde{\mathfrak{T}}'^\star, \infty) \\ & \tau_{\widetilde{V}'^\star} \searrow \quad \swarrow \tau_{\widetilde{\mathfrak{T}}'^\star} & \\ & \widetilde{\mathcal{Q}}^{\mathcal{G}'}_\infty & \end{array}$$

as well as

$$\begin{array}{ccc} (\widetilde{V}^*, \infty) & \xrightarrow{\widetilde{\Phi}_\infty} & (\widetilde{V}'^\star, \infty) \\ R_\infty \downarrow & \circlearrowleft & \downarrow R'_\infty \\ (\widetilde{\mathfrak{T}}^*, \infty) & \xrightarrow{\widetilde{\Phi}_{|\widetilde{\mathfrak{T}}^* \infty}} & (\widetilde{\mathfrak{T}}'^\star, \infty). \end{array}$$

The commutativity of $(\star\star)$ follows directly from the commutativity of these three diagrams and that of (29):

$$
\begin{array}{ccc}
(\widetilde{V}^*,\infty) & \xrightarrow{\ \widetilde{\Phi}_\infty\ } & (\widetilde{V}'^\star,\infty) \\
\end{array}
$$
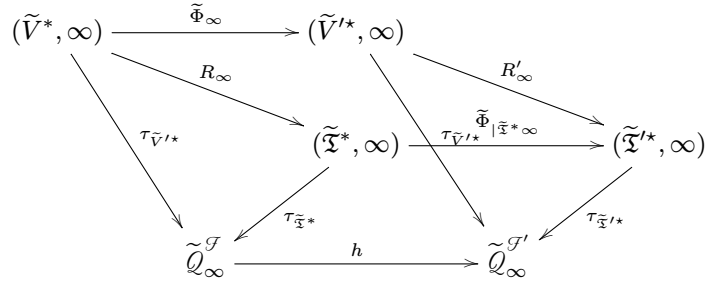
8.7.4. *Fourth step: modification of Φ and $\widetilde{\Phi}$ to satisfy* (e). – A priori the homeomorphism $\widetilde{\Phi}$ that we have constructed does not have to satisfy (e). However, it induces on $\pi_0(\widetilde{\mathbb{T}}^*_{1,1},\infty)$ the same map as $\widetilde{g}$,

$$(30) \qquad \widetilde{\Phi} = \widetilde{g} : \pi_0(\widetilde{\mathbb{T}}^*_{1,1},\infty) \longrightarrow \pi_0(\widetilde{\mathbb{T}}'^\star_{1,1},\infty).$$

Indeed, $\widetilde{\Phi}$ and $\widetilde{g}$ induce the same map from $\pi_0(\widetilde{\mathfrak{T}}^*_{1,1},\infty)$ onto $\pi_0(\widetilde{\mathfrak{T}}'^*_{1,1},\infty)$, because $\widetilde{\Phi}_{D\circ}$ satisfies (e). Then the equality (30) follows from the bijections (28) and

$$(31) \qquad \pi_0(\widetilde{\mathbb{T}}^*_{1,1},\infty)\xrightarrow{\ \sim\ }\pi_0(\widetilde{V}^*,\infty) \quad \text{and} \quad \pi_0(\widetilde{\mathbb{T}}'^\star_{1,1},\infty)\xrightarrow{\ \sim\ }\pi_0(\widetilde{V}^\star,\infty).$$

We will compose $\widetilde{\Phi}$ to the left with the germ at infinity $\widetilde{\Theta}_\infty$ of the lift of a "Dehn twist" Θ along the leaves of $\underline{\mathscr{G}}'$ with support contained in a small neighborhood of $\mathbb{T}'_{1,1}$:

$$
\begin{array}{ccc}
(\widetilde{V}'^\star,\infty) & \xrightarrow{\ \widetilde{\Theta}_\infty\ } & (\widetilde{V}'^\star,\infty) \\
\underline{q}'_\infty \downarrow & \circlearrowleft & \downarrow \underline{q}'_\infty \\
(V',\mathscr{D}_S) & \xrightarrow{\ \Theta_S\ } & (V',\mathscr{D}_S).
\end{array}
$$

We will see then that $\widetilde{\Theta}_\infty \circ \widetilde{\Phi}$ satisfies (e) as well as (a)-(d), already verified by $\widetilde{\Phi}$.

8.7.4.1. *A) Construction of Θ and $\widetilde{\Theta}$.*– Recall that $\check{D}'$ is the irreducible component of $\mathscr{D}_{S'}$ whose local equation is $x_{s'} = 0$. On the disk $\check{D}' \cap \mathscr{W}'_{s'}$, we consider the real vector field ϑ whose flow is $(t,y_{s'}) \mapsto e^{2i\pi t}y_{s'}$. We fix a smooth function u with support contained in $\{x_{s'} = 0, \varsigma \leq |y_{s'}| \leq 1\}$, $0 < \varsigma < 1$ taking the value 1 on $C' := \{x_{s'} = 0, |y_{s'}| = 1\} = K' \cap \check{D}'^\circ$ and we denote by Y the vector field on V' tangent to $\underline{\mathscr{G}}'$ and projecting over $u\vartheta$ by $\pi_{\check{D}'}$. The flow Υ_t of Y is defined on an open neighborhood U_I of $\check{D}$ in $\mathscr{W}_{s'}$, once we fix an interval $I \subset \mathbb{R}$ where we allow the time t to vary. Hence, we can lift the flow Υ_t to a unique map $\widetilde{\Upsilon}_t : \widetilde{U}_I^\star \to \widetilde{V}'^\star$ being the identity on $|y_{s'} \circ \underline{q}'| \leq \varsigma$, and defining consequently a germ at infinity $\widetilde{\Upsilon}_{t\infty} : (\widetilde{V}'^\star,\infty) \to (\widetilde{V}'^\star,\infty)$. Clearly the germ $\widetilde{\Upsilon}_{n\infty}$ fibers over C', i.e., $\pi_{D'} \circ \underline{q}' \circ \widetilde{\Upsilon}_{n\infty|\widetilde{\mathbb{T}}'^\star} = \pi_{D'} \circ \underline{q}'_{\infty|\widetilde{\mathbb{T}}'^\star}$, for each $n \in \mathbb{Z}$. It defines a deck transformation $\widetilde{\Upsilon}_n : \Pi' \xrightarrow{\sim} \Pi'$ of the natural covering

$$\rho' : \Pi' \to C', \quad \rho'^{-1}(p) := \pi_0(\widetilde{T}'^\star_p,\infty), \quad T'_p := \pi_{\check{D}'}^{-1}(p).$$

We put $\Theta := \Upsilon_{n_0}$ and $\widetilde{\Theta} := \widetilde{\Upsilon}_{n_0}$, choosing the integer n_0 in the following way. First, we fix a point a in the circle $C := \{x_s = 0, |y_s| = 1\} = K \cap \check{D}^\circ$ and an element ν

of $\pi_0(\widetilde{T}_a^\star, \infty)$, $T_a := \pi_{\check{D}}^{-1}(a)$. We consider the natural covering $\rho : \Pi \to C$ of C with fibers $\rho^{-1}(p) := \pi_0(\widetilde{T}_p^*, \infty)$ and the following two covering morphisms over $G_{|C}$

$$
(\blacklozenge)_\Lambda \quad
\begin{array}{ccc}
\Pi & \xrightarrow{\Lambda} & \Pi' \\
\rho \downarrow & \circlearrowleft & \downarrow \rho' , \\
C & \xrightarrow{G_{|C}} & C'
\end{array}
\qquad \Lambda = \widetilde{\Phi} \quad \text{and} \quad \Lambda = \widetilde{g},
$$

defined by $\widetilde{\Phi}$ and $\widetilde{g}$. After the equality (30) and the bijections (28) and (31), the images $\sigma' := \widetilde{\Phi}(\nu)$ and $\sigma'' := \widetilde{g}(\nu)$ belong to the same fiber of the map $\iota : \pi_0(\widetilde{T}'_{a'}{}^\star, \infty) \to \pi_0(\widetilde{\mathbb{T}}'^\star_{1,1}, \infty)$, induced by the inclusion map $\iota : \widetilde{T}'_{a'}{}^\star \hookrightarrow \widetilde{\mathbb{T}}'^\star_{1,1}$. On the other hand, the action

$$
\mathbb{Z} \times \pi_0(\widetilde{T}'_{a'}{}^\star, \infty) \longrightarrow \pi_0(\widetilde{T}'_{a'}{}^\star, \infty) \qquad (n, \sigma) \mapsto \widetilde{\Upsilon}_n(\sigma) ,
$$

of $\mathbb{Z}$ on the fiber of ρ' at the point $a' := \Phi(a) = G(a) \in C'$ coincides with the action of $\pi_1(C', a') \simeq \mathbb{Z}$ induced by the covering ρ' on this fiber. The orbits of that action correspond to the fibers of ι. We choose n_0 to be the unique integer number such that $\Upsilon_{n_0}(\sigma') = \sigma''$.

8.7.4.2. *B) Proof of Assertions* (a)-(e) *for* $\widetilde{\Theta} \circ \widetilde{\Phi}_\infty$.– Properties (a)-(c) are trivially satisfied. Property (d) follows from Proposition 3.6.4. It only remains to prove (e), that is, to show the equalities

$$
(\Diamond)_t \quad (\Theta \circ \widetilde{\Phi}_{\widetilde{T}_t^*}) = \widetilde{g}_{\widetilde{T}_t^*} : \pi_0(\widetilde{T}_t^*, \infty) \longrightarrow \pi_0(\widetilde{T}'^\star_{G(t)}, \infty) ,
$$

for all $t \in C$ and for all $t \in D^\circ \cap K$. In the last case, $\widetilde{\Theta}$ is the identity near $\widetilde{T}'^\star_t$ and therefore $\widetilde{\Theta} \circ \widetilde{\Phi}$ coincides with the homeomorphism $\widetilde{\Phi}_{D^\circ}$ given by Lemma 8.3.2 in the context $K = K_D$, which satisfies (e) as we have already seen. In the case $t \in C$, $(\Diamond)_t$ is equivalent to the equality $(\widetilde{\Theta} \circ \widetilde{\Phi}) = \widetilde{g}$ of the covering morphisms $(\blacklozenge)_{(\widetilde{\Theta} \circ \widetilde{\Phi})}$ and $(\blacklozenge)_{\widetilde{g}}$ defined by $\widetilde{\Theta} \circ \widetilde{\Phi}_\infty$ and $\widetilde{g}_\infty$. Thus, it suffices to show the equality on a single fiber, i.e., $(\Diamond)_a$. This equality is satisfied for the element σ of $\pi_0(\widetilde{T}_t^*, \infty)$ that we have previously fixed to define n_0: $(\widetilde{\Theta} \circ \widetilde{\Phi})(\sigma) = \widetilde{g}(\sigma)$. Since Γ_∞ acts transitively on $\pi_0(\widetilde{T}_t^*, \infty)$, we must show the equality $(\widetilde{\Theta} \circ \widetilde{\Phi})(\varphi(\sigma)) = \widetilde{g}(\varphi(\sigma))$, for all $\varphi \in \Gamma_\infty$, which follows directly from the commutativity of $(\star\star)$ for $\psi = \Phi \circ \Phi_{|T_a}$. Indeed,

$$
(32) \quad
\begin{aligned}
(\widetilde{\Theta} \circ \widetilde{\Phi})(\varphi(\sigma)) &= [(\widetilde{\Theta} \circ \widetilde{\Phi})_*(\varphi)]((\widetilde{\Theta} \circ \widetilde{\Phi})(\sigma)) = (\widetilde{g}_*(\varphi))(\widetilde{g}(\sigma)) \\
&= (\widetilde{g}_*(\varphi) \circ \widetilde{g})(\sigma) = (\widetilde{g} \circ \varphi)(\sigma) = \widetilde{g}(\varphi(\sigma)) .
\end{aligned}
$$

This concludes the proof of Extension Lemma 8.3.2 for $K = K_s$.

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David Marín
Departament de Matemàtiques
Universitat Autònoma de Barcelona
E-08193 Bellaterra (Barcelona), Spain
E-mail: davidmp@mat.uab.es

Jean-François Mattei
Institut de Mathématiques de Toulouse
Université Paul Sabatier
118, route de Narbonne
F-31062 Toulouse Cedex 9, France
E-mail: jean-francois.mattei@math.univ-toulouse.fr