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LOCALLY ANALYTIC VECTORS OF UNITARY PRINCIPAL SERIES OF $\mathrm{GL}_2(\mathbb{Q}_p)$

BY RUOCHUAN LIU, BINGYONG XIE AND YUANCAO ZHANG

ABSTRACT. – The p-adic local Langlands correspondence for $\mathrm{GL}_2(\mathbb{Q}_p)$ attaches to any 2-dimensional irreducible p-adic representation V of $G_{\mathbb{Q}_p}$ an admissible unitary representation $\Pi(V)$ of $\mathrm{GL}_2(\mathbb{Q}_p)$. The unitary principal series of $\mathrm{GL}_2(\mathbb{Q}_p)$ are those $\Pi(V)$ corresponding to trianguline representations. In this article, for $p > 2$, using the machinery of Colmez, we determine the space of locally analytic vectors $\Pi(V)_{\mathrm{an}}$ for all non-exceptional unitary principal series $\Pi(V)$ of $\mathrm{GL}_2(\mathbb{Q}_p)$ by proving a conjecture of Emerton.

RÉSUMÉ. – La correspondance de Langlands locale p-adique pour $\mathrm{GL}_2(\mathbb{Q}_p)$ associe à toute représentation irréductible p-adique V de dimension 2 de $G_{\mathbb{Q}_p}$ une représentation admissible unitaire $\Pi(V)$ de $\mathrm{GL}_2(\mathbb{Q}_p)$. Les séries principales unitaires de $\mathrm{GL}_2(\mathbb{Q}_p)$ sont les $\Pi(V)$ correspondant aux représentations triangulines. Dans le présent article, en utilisant la machinerie de Colmez, on détermine l'espace des vecteurs localement analytiques $\Pi(V)_{\mathrm{an}}$ pour toute série principale unitaire non-exceptionnelle $\Pi(V)$ de $\mathrm{GL}_2(\mathbb{Q}_p)$, et on démontre ainsi une conjecture d'Emerton.

1. Introduction

Let F be a finite extension of $\mathbb{Q}_p$. The aim of the p-adic local Langlands programme initiated by Breuil is to look for a "natural" correspondence between certain n-dimensional p-adic representations of $\mathrm{Gal}(\overline{\mathbb{Q}}_p/F)$ and certain Banach space representations of $\mathrm{GL}_n(F)$. Thanks to much recent work, especially that of Colmez and Paškūnas, we have gained a fairly clear picture in the case $F = \mathbb{Q}_p$ and $n = 2$ which is the so-called p-adic local Langlands correspondence for $\mathrm{GL}_2(\mathbb{Q}_p)$ establishing a functorial bijection between 2-dimensional irreducible p-adic representations of $\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$ and non-ordinary irreducible admissible unitary representations of $\mathrm{GL}_2(\mathbb{Q}_p)$.

Although the present version of p-adic local Langlands correspondence is formulated at the level of Banach space representations, it is very useful, as in Breuil's initial work [4], to have information of the subspace of locally analytic vectors. Fix a finite extension L of $\mathbb{Q}_p$ as the coefficient field, and we denote by $\Pi(V)$ the corresponding unitary representation of $\mathrm{GL}_2(\mathbb{Q}_p)$ for any 2-dimensional irreducible L-linear representation V of $\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$.

The *unitary principal series of* $\mathrm{GL}_2(\mathbb{Q}_p)$, which are the simplest ones among these $\Pi(V)$, are those corresponding to trianguline representations. In [13], Emerton made a conjectural description of the subspace of locally analytic vectors $\Pi(V)_{\mathrm{an}}$ for all unitary principal series $\Pi(V)$. We recall his conjecture below.

Let $\mathscr{S}_{\mathrm{irr}}$ be the parameterizing space of 2-dimensional irreducible trianguline representations of $\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$ introduced by Colmez in [7]. For any $s \in \mathscr{S}_{\mathrm{irr}}$, let $V(s)$ be the corresponding trianguline representation. We may write $s = (\delta_1, \delta_2, \mathscr{L})$ so that the étale (φ,Γ)-module $\mathrm{D}_{\mathrm{rig}}(V(s))$ is isomorphic to the extension of $\mathscr{R}(\delta_2)$ by $\mathscr{R}(\delta_1)$ defined by $\mathscr{L}$. For any such s, if $\delta_1\delta_2^{-1} = x^k|x|$ for some $k \in \mathbb{Z}_+$, then we set $\Sigma(s)$ to be the locally analytic $\mathrm{GL}_2(\mathbb{Q}_p)$-representation $\Sigma(k+1,\mathscr{L}) \otimes ((\delta_2|x|^{\frac{2-k}{2}}) \circ \det)$ where $\{\Sigma(k+1,\mathscr{L})\}$ is the family of locally analytic $\mathrm{GL}_2(\mathbb{Q}_p)$-representations introduced by Breuil in [4]. Otherwise, we define $\Sigma(s)$ to be the locally analytic principal series $\left(\mathrm{Ind}_{\mathrm{B}(\mathbb{Q}_p)}^{\mathrm{GL}_2(\mathbb{Q}_p)} \delta_2 \otimes \delta_1(x|x|)^{-1}\right)_{\mathrm{an}}$. The conjecture of Emerton is:

Conjecture 1.1 ([13, Conjecture 6.7.3, 6.7.7]). – *For any $s \in \mathscr{S}_{\mathrm{irr}}$, $\Pi(V(s))_{\mathrm{an}}$ sits in an exact sequence*

$$0 \longrightarrow \Sigma(s) \longrightarrow \Pi(V(s))_{\mathrm{an}} \longrightarrow \left(\mathrm{Ind}_{\mathrm{B}(\mathbb{Q}_p)}^{\mathrm{GL}_2(\mathbb{Q}_p)} \delta_1 \otimes \delta_2(x|x|)^{-1}\right)_{\mathrm{an}} \longrightarrow 0. \tag{1.1}$$

In the special cases when $V(s)$ are twists of crystabelian representations and *non-exceptional*, there is a more precise conjectural description of $\Pi(V(s))_{\mathrm{an}}$ due to Breuil. In [16], the first author proved Breuil's conjecture. The main result of this paper is:

Theorem 1.2 (Theorem 6.13). – *For $p > 2$, Conjecture 1.1 is true if $V(s)$ is non-exceptional.*

In fact, one can easily deduce Breuil's conjecture from Emerton's conjecture. Thus for $p > 2$, the above theorem covers the aforementioned result of the first author.

We now explain the strategy of the proof of Theorem 1.2. The whole proof builds on Colmez's machinery of p-adic local Langlands correspondence for $\mathrm{GL}_2(\mathbb{Q}_p)$ developed in [12]. The key ingredient is Colmez's identification of the locally analytic vectors:

$$(\Pi(\check{V})_{\mathrm{an}})^* = D_{\mathrm{rig}}^{\natural} \boxtimes \mathrm{P}^1 \tag{1.2}$$

where $D = \mathrm{D}(V)$ is Fontaine's étale (φ,Γ)-module associated to V. To apply this formula, for any continuous characters $\delta, \eta : \mathbb{Q}_p^\times \to L^\times$, we construct the objects $\mathscr{R}(\eta) \boxtimes_\delta \mathrm{P}^1$ and $\mathscr{R}^+(\eta) \boxtimes_\delta \mathrm{P}^1$ which are equipped with continuous $\mathrm{GL}_2(\mathbb{Q}_p)$-actions, and the latter is topologically isomorphic to $(\left(\mathrm{Ind}_{\mathrm{B}(\mathbb{Q}_p)}^{\mathrm{GL}_2(\mathbb{Q}_p)} \delta^{-1}\eta \otimes \eta^{-1}\right)_{\mathrm{an}})^*$. In doing so, we are led to modify some of Colmez's constructions to twists of étale (φ,Γ)-modules and rank 1 (φ,Γ)-modules. On the other hand, Colmez [9] (for $p > 2$ and $s \in \mathscr{S}_*^{\mathrm{ng}} \coprod \mathscr{S}_*^{\mathrm{st}}$; this is the only place where we need $p > 2$) and Berger-Breuil [3] (for $s \in \mathscr{S}_{\mathrm{irr}}$ non-exceptional) establish an explicit isomorphism $\mathscr{A}_s : \Pi(V(s)) \cong \Pi(s)$ (for s exceptional, Paškūnas proves $\Pi(V(s)) \cong \Pi(s)$ by an indirect method [17]) where $\Pi(s)$ is the unitary principal series associated to $V(s)$. We deduce from the explicit description of $\mathscr{A}_s$ plus a duality argument the following exact sequence

$$0 \longrightarrow \mathscr{R}(\delta_1) \boxtimes \mathrm{P}^1 \longrightarrow \mathrm{D}_{\mathrm{rig}}(V(s)) \boxtimes \mathrm{P}^1 \longrightarrow \mathscr{R}(\delta_2) \boxtimes \mathrm{P}^1 \longrightarrow 0. \tag{1.3}$$

Then the natural inclusion $\left(\mathrm{Ind}_{\mathrm{B}(\mathbb{Q}_p)}^{\mathrm{GL}_2(\mathbb{Q}_p)}\delta_2\otimes\delta_1(x|x|)^{-1}\right)_{\mathrm{an}}\hookrightarrow\Pi(s)_{\mathrm{an}}$ induces the following commutative diagram

$$\begin{tikzcd}
(\Pi(s)_{\mathrm{an}})^* \arrow[r] \arrow[d] & D^{\natural}_{\mathrm{rig}}(\check{s})\boxtimes\mathrm{P}^1 \arrow[d] \\
\left(\left(\mathrm{Ind}_{\mathrm{B}(\mathbb{Q}_p)}^{\mathrm{GL}_2(\mathbb{Q}_p)}\delta_2\otimes\delta_1(x|x|)^{-1}\right)_{\mathrm{an}}\right)^* \arrow[r] & \mathscr{R}(\check{\delta}_1)\boxtimes\mathrm{P}^1
\end{tikzcd} \tag{1.4}$$

where $\check{s}=(\check{\delta}_2,\check{\delta}_1,\mathscr{L})$ corresponds to the Tate dual of $V(s)$. Using (1.4) together with the fact that $D^{\natural}_{\mathrm{rig}}(s)\boxtimes\mathrm{P}^1$ and $D^{\natural}_{\mathrm{rig}}(\check{s})\boxtimes\mathrm{P}^1$ are orthogonal complements of each other under the paring

$$\{\cdot,\cdot\}_{\mathrm{P}^1}:\mathrm{D}_{\mathrm{rig}}(V(s))\boxtimes\mathrm{P}^1\times\mathrm{D}_{\mathrm{rig}}(V(\check{s}))\boxtimes\mathrm{P}^1\to L,$$

and that (1.3) is dual to

$$0\longrightarrow\mathscr{R}(\check{\delta}_2)\boxtimes\mathrm{P}^1\longrightarrow\mathrm{D}_{\mathrm{rig}}(V(\check{s}))\boxtimes\mathrm{P}^1\longrightarrow\mathscr{R}(\check{\delta}_1)\boxtimes\mathrm{P}^1\longrightarrow 0, \tag{1.5}$$

we deduce that if $\delta_1\delta_2^{-1}\neq x^k|x|$ for any $k\in\mathbb{Z}_+$, then $D^{\natural}_{\mathrm{rig}}(\check{s})$ sits in the exact sequence

$$0\longrightarrow\mathscr{R}^+(\check{\delta}_2)\boxtimes\mathrm{P}^1\longrightarrow D^{\natural}_{\mathrm{rig}}(\check{s})\boxtimes\mathrm{P}^1\longrightarrow\mathscr{R}^+(\check{\delta}_1)\boxtimes\mathrm{P}^1\longrightarrow 0. \tag{1.6}$$

We therefore conclude (1.1) by taking the dual of (1.6). If $\delta_1\delta_2^{-1}=x^k|x|$ for some $k\in\mathbb{Z}_+$, we have that the image of $D^{\natural}_{\mathrm{rig}}(\check{s})$ in $\mathscr{R}(\check{\delta}_1)\boxtimes\mathrm{P}^1$ is a closed subspace of $\mathscr{R}^+(\check{\delta}_1)\boxtimes\mathrm{P}^1$ of codimension k and $\mathscr{R}(\check{\delta}_2)\boxtimes\mathrm{P}^1\cap D^{\natural}_{\mathrm{rig}}(\check{s})\boxtimes\mathrm{P}^1$ contains $\mathscr{R}^+(\check{\delta}_2)\boxtimes\mathrm{P}^1$ as a closed subspace of codimension k. We then conclude (1.1) using Schneider and Teitelbaum's results on the Jordan-Hölder series of locally analytic principal series of $\mathrm{GL}_2(\mathbb{Q}_p)$.

The organization of the paper is as follows. In §2, we recall some necessary background of the theory of (φ,Γ)-modules. In §3, we recall some of Colmez's constructions of the p-adic local Langlands correspondence for $\mathrm{GL}_2(\mathbb{Q}_p)$ especially his identification of the locally analytic vectors, and we make the aforementioned modification. We review the isomorphism $\mathscr{A}_s:\Pi(V(s))\cong\Pi(s)$ in §4. In §5.1, we recall Schneider and Teitelbaum's results on Jordan-Hölder series of the locally analytic principal series of $\mathrm{GL}_2(\mathbb{Q}_p)$. We prove that $\mathscr{R}^+(\eta)\boxtimes_\delta\mathrm{P}^1$ is isomorphic to $(\left(\mathrm{Ind}_{\mathrm{B}(\mathbb{Q}_p)}^{\mathrm{GL}_2(\mathbb{Q}_p)}\delta^{-1}\eta\otimes\eta^{-1}\right)_{\mathrm{an}})^*$ in §5.2. Section 6 is devoted to the proof of Theorem 1.2. We first recall the definition of $\Sigma(s)$ and restate Emerton's conjecture in §6.1. Then we prove the exact sequence (1.3) in §6.2. We finish the proof of Theorem 1.2 in §6.3.

After the work presented in this paper was finished, we learned from Colmez that he had a proof of Conjecture 1.1 for all p and all trianguline representations of $G_{\mathbb{Q}_p}$. The strategy of his proof is different from ours. He constructs the map $\Pi(s)_{\mathrm{an}}\to\left(\mathrm{Ind}_{\mathrm{B}(\mathbb{Q}_p)}^{\mathrm{GL}_2(\mathbb{Q}_p)}\delta_1\otimes\delta_2(x|x|)^{-1}\right)_{\mathrm{an}}$ directly by computing the *module de Jacquet dual* of $\Pi(s)_{\mathrm{an}}$. We refer the reader to his paper [10] for more details.

Notation and conventions

Let p be a prime number, and let v_p denote the p-adic valuation on $\overline{\mathbb{Q}}_p$, normalized by $v_p(p)=1$; the corresponding norm is denoted by $|\cdot|$. Let $G_{\mathbb{Q}_p}$ denote $\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$ for simplicity. Let $\chi:G_{\mathbb{Q}_p}\to\mathbb{Z}_p^\times$ be the p-adic cyclotomic character. The kernel of χ is $H=\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p(\mu_{p^\infty}))$, and let $\Gamma=\mathrm{Gal}(\mathbb{Q}_p(\mu_{p^\infty})/\mathbb{Q}_p)$. For any $a\in\mathbb{Z}_p^\times$, let σ_a be the unique element in Γ such that $\chi(\sigma_a) = a$. For any positive integer h, let $\Gamma_h = \chi^{-1}(1+p^h\mathbb{Z}_p)$. If we regard χ as a character of $\mathbb{Q}_p^\times$ via the local Artin map, then it is equal to $\epsilon(x) = x|x|$. Throughout this paper, we fix a finite extension L of $\mathbb{Q}_p$. We denote by $\mathscr{O}_L$ the ring of integers of L, and by k_L the residue field. Let $\widehat{\mathscr{T}}(L)$ be the set of all continuous characters $\delta : G_{\mathbb{Q}_p} \to L^\times$. For any $\delta \in \widehat{\mathscr{T}}(L)$, the Hodge-Tate weight $w(\delta)$ of δ is defined by $w(\delta) = \frac{\log \delta(u)}{\log u}$ where u is any element of $\mathbb{Q}_p^\times \setminus \mu_{p^\infty}$. For any L-linear representation V of $G_{\mathbb{Q}_p}$, we denote by $\check{V}$ the Tate dual $V^*(\epsilon)$ of V. For any $\mathscr{L} \in L$, let $\log_{\mathscr{L}} : \mathbb{Q}_p^\times \to L$ be the homomorphism defined by $\log_{\mathscr{L}}(p)$=1 and $\log_{\mathscr{L}}(x) = -\sum_{n=1}^{+\infty} \frac{(1-x)^n}{n}$ when $|x-1| < 1$. We put $\log_\infty = v_p$. Hence $\log_{\mathscr{L}}$ is defined for all $\mathscr{L} \in \mathrm{P}^1(L)$.

Let B be the subgroup of upper triangular matrices of GL_2, let $\mathrm{P} = \left[\begin{smallmatrix} * & * \\ 0 & 1 \end{smallmatrix}\right]$ be the mirabolic subgroup of GL_2, let T be the subgroup of diagonal matrices of GL_2, and let Z be the center of GL_2. Put $w = \left[\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right]$. Let $\mathrm{Rep}_{\mathrm{tors}}\mathrm{GL}_2(\mathbb{Q}_p)$ be the category of smooth $\mathscr{O}_L[\mathrm{GL}_2(\mathbb{Q}_p)]$-modules which are of finite lengths and admit central characters. Let $\mathrm{Rep}_{\mathscr{O}_L}\mathrm{GL}_2(\mathbb{Q}_p)$ be the category of $\mathscr{O}_L[\mathrm{GL}_2(\mathbb{Q}_p)]$-modules Π which are separated and complete for the p-adic topology, p-torsion free, and $\Pi/p^n\Pi \in \mathrm{Rep}_{\mathrm{tors}}\mathrm{GL}_2(\mathbb{Q}_p)$ for any $n \in \mathbb{N}$.

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2. Preliminaries on (φ, Γ)-modules

2.1. Dictionary of p-adic functional analysis

Let $\mathscr{O}_{\mathscr{E}}$ be the ring of Laurent series $f = \sum_{i\in\mathbb{Z}} a_i T^i$, where $a_i \in \mathscr{O}_L$, such that $v_p(a_i) \to \infty$ as $i \to -\infty$. Let $\mathscr{E} = \mathscr{O}_{\mathscr{E}}[1/p]$ be the fraction field of $\mathscr{O}_{\mathscr{E}}$.

For any $r \in \mathbb{R}_+ \cup \{+\infty\}$, let $\mathscr{E}^{]0,r]}$ be the ring of Laurent series $f = \sum_{i\in\mathbb{Z}} a_i T^i$, with $a_i \in L$, such that f is convergent on the annulus $0 < v_p(T) \leq r$. For any $0 < s \leq r \leq +\infty$, we define the valuation $v^{\{s\}}$ on $\mathscr{E}^{]0,r]}$ by

$$v^{\{s\}}(f) = \inf_{i\in\mathbb{Z}} \{v_p(a_i) + is\} \text{ if } s \neq \infty;\ v^{\{\infty\}}(f) = v_p(f(0)).$$

We provide $\mathscr{E}^{]0,r]}$ with the Fréchet topology defined by the family of valuations $\{v^{\{s\}} | 0 < s \leq r\}$; then $\mathscr{E}^{]0,r]}$ is complete. We equip the *Robba ring* $\mathscr{R} = \bigcup_{r>0} \mathscr{E}^{]0,r]}$

with the inductive limit topology. We denote $\mathscr{E}^{]0,+\infty]}$, the ring of analytic functions on the open unit disk, by $\mathscr{R}^+$.

Let $\mathscr{E}^\dagger$ be the subring of overconvergent elements of $\mathscr{E}$, i.e. $\mathscr{E}^\dagger$ is the set of $f \in \mathscr{E}$ such that $f(T)$ is convergent over some annulus $0 < v_p(T) \le r$. Let $\mathscr{E}^{(0,r]} = \mathscr{E}^\dagger \cap \mathscr{E}^{]0,r]}$. We equip $\mathscr{E}^\dagger = \bigcup_{r>0} \mathscr{E}^{(0,r]}$ with the inductive limit topology. We denote $\mathscr{E}^{(0,\infty]} = \mathscr{O}_L[[T]][1/p]$ by $\mathscr{E}^+$, and let $\mathscr{O}_{\mathscr{E}^+} = \mathscr{O}_L[[T]]$.

Let R denote any of $\mathscr{O}_{\mathscr{E}^+}, \mathscr{E}^+, \mathscr{O}_{\mathscr{E}}, \mathscr{E}, \mathscr{E}^\dagger, \mathscr{R}^+$ and $\mathscr{R}$. We equip the ring R with commuting continuous actions of φ and Γ defined by

$$\varphi(f(T)) = f((1+T)^p - 1), \qquad \gamma(f(T)) = f((1+T)^{\chi(\gamma)} - 1), \quad \gamma \in \Gamma. \tag{2.1}$$

If we view R as a $\varphi(R)$-module, then R is freely generated by $\{(1+T)^i | i = 0, \ldots, p-1\}$. Thus for any $y \in R$, we may write $y = \sum_{i=0}^{p-1} (1+T)^i \varphi(y_i)$ for some uniquely determined $y_0, \ldots, y_{p-1} \in R$, and we define the operator $\psi : R \to R$ by setting $\psi(y) = y_0$. It follows directly from the definition that ψ is a left inverse to φ, and that ψ commutes with the Γ-action. For any $f = \sum_{i\in\mathbb{Z}} a_i T^i \in \mathscr{R}$, we define the residue of the 1-form $\omega = f \cdot dT$ as $\operatorname{res}(\omega) = a_{-1}$; and for any $f \in \mathscr{R}$, we define $\operatorname{res}_0(f) = \operatorname{res}(f \frac{dT}{1+T})$.

We denote by $\mathscr{C}^0(\mathbb{Z}_p, L)$ the space of continuous functions on $\mathbb{Z}_p$ with values in L; this is an L-Banach space equipped with the supremum norm. Let $\mathrm{LA}(\mathbb{Z}_p, L)$ denote the space of locally analytic functions on $\mathbb{Z}_p$ with values in L. The classical results of Mahler and Amice assert that the set of functions $\{\binom{x}{n}\}_{n\in\mathbb{N}}$ constitutes an orthogonal basis of $\mathscr{C}^0(\mathbb{Z}_p, L)$, and that for $f = \sum_{n\in\mathbb{N}} a_n(f)\binom{x}{n} \in \mathscr{C}^0(\mathbb{Z}_p, L)$, $f \in \mathrm{LA}(\mathbb{Z}_p, L)$ if and only if there exists some $r > 0$ such that $v_p(a_n(f)) - rn \to +\infty$ as $n \to +\infty$.

For any $u \ge 0$, we denote by $\mathscr{C}^u(\mathbb{Z}_p, L)$ the space of all $\mathscr{C}^u$-functions on $\mathbb{Z}_p$; this is an L-Banach space (see [8] for more details). We have $\mathrm{LA}(\mathbb{Z}_p, L) \subset \mathscr{C}^u(\mathbb{Z}_p, L) \subseteq \mathscr{C}^0(\mathbb{Z}_p, L)$, and that $\mathrm{LA}(\mathbb{Z}_p, L)$ is dense in $\mathscr{C}^u(\mathbb{Z}_p, L)$ for any $u \ge 0$.

We denote by $\mathscr{D}(\mathbb{Z}_p, L)$, $\mathscr{D}_u(\mathbb{Z}_p, L)$ the topological dual of $\mathrm{LA}(\mathbb{Z}_p, L)$, $\mathscr{C}^u(\mathbb{Z}_p, L)$ respectively. Note that the natural map $\mathscr{D}_u(\mathbb{Z}_p, L) \to \mathscr{D}(\mathbb{Z}_p, L)$ is injective since $\mathrm{LA}(\mathbb{Z}_p, L)$ is dense in $\mathscr{C}^u(\mathbb{Z}_p, L)$. The elements of $\mathscr{D}(\mathbb{Z}_p, L)$ are called *distributions on* $\mathbb{Z}_p$. A distribution μ is called of *order* u if $\mu \in \mathscr{D}_u(\mathbb{Z}_p, L)$. We define the actions of φ, ψ and Γ on $\mathscr{D}(\mathbb{Z}_p, L)$ by the formulas

$$\int_{\mathbb{Z}_p} f\varphi(\mu) = \int_{\mathbb{Z}_p} f(px)\mu, \quad \int_{\mathbb{Z}_p} f\psi(\mu) = \int_{p\mathbb{Z}_p} f(p^{-1}x)\mu, \quad \int_{\mathbb{Z}_p} f\sigma_a(\mu) = \int_{\mathbb{Z}_p} f(ax)\mu$$

for any $f \in \mathrm{LA}(\mathbb{Z}_p, L), \mu \in \mathscr{D}(\mathbb{Z}_p, L)$ and $a \in \mathbb{Z}_p^\times$.

The *Amice transformation* $\mathscr{A}$ on $\mathscr{D}(\mathbb{Z}_p, L)$ is defined by

$$\mathscr{A} : \mathscr{D}(\mathbb{Z}_p, L) \to L[[T]], \quad \mathscr{A}(\mu) = \int_{\mathbb{Z}_p} (1+T)^x \mu(x) = \sum_{n=0}^{+\infty} T^n \int_{\mathbb{Z}_p} \binom{x}{n} \mu.$$

It is an immediate consequence of the results of Mahler and Amice that the Amice transformation $\mu \mapsto \mathscr{A}(\mu)$ induces topological isomorphisms from $\mathscr{D}_0(\mathbb{Z}_p, L)$ and $\mathscr{D}(\mathbb{Z}_p, L)$ to $\mathscr{E}^+$ and $\mathscr{R}^+$ respectively which are compatible with the actions of φ, ψ and Γ.

We denote by $\mathrm{LA}_c(\mathbb{Q}_p, L)$ the space of compactly supported L-valued locally analytic functions on $\mathbb{Q}_p$, and denote by $\mathscr{D}(\mathbb{Q}_p, L)$ the topological dual of $\mathrm{LA}_c(\mathbb{Q}_p, L)$. The elements of $\mathscr{D}(\mathbb{Q}_p, L)$ are called *distributions on* $\mathbb{Q}_p$. For any $\mu \in \mathscr{D}(\mathbb{Q}_p, L)$, let $\mu^{(n)}$ be the distribution on $\mathbb{Z}_p$ defined by

$$\int_{\mathbb{Z}_p} f\mu^{(n)} = \int_{p^{-n}\mathbb{Z}_p} f(p^n x)\mu$$

for any $f \in \mathrm{LA}(\mathbb{Z}_p, L)$. It follows that $\psi(\mu^{(n+1)}) = \mu^{(n)}$, and that any sequence of distributions $(\mu^{(n)})_{n\in\mathbb{N}}$ on $\mathbb{Z}_p$ so that $\psi(\mu^{(n+1)}) = \mu^{(n)}$ uniquely determines a distribution μ on $\mathbb{Q}_p$. The *Amice transformation* $\mathscr{A}(\mu)$ for $\mu \in \mathscr{D}(\mathbb{Q}_p, L)$ is then defined to be the sequence $(\mathscr{A}(\mu^{(n)}))_{n\in\mathbb{N}}$.

A distribution μ on $\mathbb{Q}_p$ is said to be *of order* u if all $\mu^{(n)}$ are of order u. The distribution μ is said to be *globally of order* u, if there is a constant $C_u(\mu)$ such that $v_{\mathscr{D}_u}(\mu^{(n)}) \geq nu + C_u(\mu)$ for all $n \in \mathbb{N}$. Let $\mathscr{D}_u(\mathbb{Q}_p, L)$ denote the space of distributions on $\mathbb{Q}_p$ globally of order u.

2.2. The category of (φ, Γ)-modules

Keep notations as in §2.1. We define a (φ, Γ)-*module* over R to be a finite free R-module D equipped with commuting continuous semilinear φ, Γ-actions. When $R = \mathscr{O}_{\mathscr{E}}$, the (φ, Γ)-module D is called *étale* if $\varphi(D)$ generates D as an $\mathscr{O}_{\mathscr{E}}$-module. When $R = \mathscr{E}$, the (φ, Γ)-module D is called *étale* if it arises by base change from an étale (φ, Γ)-module over $\mathscr{O}_{\mathscr{E}}$. A (φ, Γ)-module D over $\mathscr{E}^\dagger$ is called *étale* if $D \otimes_{\mathscr{E}^\dagger} \mathscr{E}$ is étale as an (φ, Γ)-module over $\mathscr{E}$. A (φ, Γ)-module D over $\mathscr{R}$ is called *étale* if the underlying φ-module is *pure of slope* 0 in the sense of Kedlaya's slope theory [15].

Example 2.1. – For any $\delta \in \widehat{\mathscr{T}}(L)$, we define $R(\delta)$ to be the rank 1 (φ, Γ)-module over R which has an R-base e satisfying

$$\varphi(e) = \delta(p)e, \qquad \sigma_a(e) = \delta(a)e, \quad a \in \mathbb{Z}_p^\times. \tag{2.2}$$

Such an element e, which is unique up to a nonzero scalar (this is because $R^{\varphi=1,\Gamma=1} = L$ or $\mathscr{O}_L$), is called a *standard basis* of $R(\delta)$.

Let V be a d-dimensional L-linear representation of $G_{\mathbb{Q}_p}$, and let T be a $G_{\mathbb{Q}_p}$-invariant $\mathscr{O}_L$-lattice of V. Let $\widehat{\mathscr{E}^{\mathrm{ur}}}$ be the p-adic completion of the maximal unramified extension of $\mathscr{E}$, and let $\widehat{\mathscr{O}_{\mathscr{E}}^{\mathrm{ur}}}$ be the ring of integers of $\widehat{\mathscr{E}^{\mathrm{ur}}}$. The φ, Γ-actions on $\mathscr{E}$ naturally extend to continuous actions, which we again denote by φ, Γ respectively, on $\widehat{\mathscr{E}^{\mathrm{ur}}}$. We define

$$\mathrm{D}(T) = (T \otimes_{\mathscr{O}_{\mathscr{E}}} \widehat{\mathscr{O}_{\mathscr{E}}^{\mathrm{ur}}})^H \qquad (\text{resp. } \mathrm{D}(V) = (V \otimes_{\mathscr{E}} \widehat{\mathscr{E}^{\mathrm{ur}}})^H),$$

which is a (φ, Γ)-module over $\mathscr{O}_{\mathscr{E}}$ (resp. $\mathscr{E}$). We define $\mathrm{D}^\dagger(V)$ to be the maximal finite dimensional φ, Γ-stable $\mathscr{E}^\dagger$-subspace of $\mathrm{D}(V)$, and we define $\mathrm{D}_{\mathrm{rig}}(V) = \mathrm{D}^\dagger(V) \otimes_{\mathscr{E}^\dagger} \mathscr{R}$; then $\mathrm{D}^\dagger(V)$ and $\mathrm{D}_{\mathrm{rig}}(V)$ are (φ, Γ)-modules over $\mathscr{E}^\dagger$ and $\mathscr{R}$ respectively.

Theorem 2.2 (Fontaine [14], Cherbonnier-Colmez [6], Berger [1], [2])

$\mathrm{D}(T)$ (*resp.* $\mathrm{D}(V)$, $\mathrm{D}^\dagger(V)$, $\mathrm{D}_{\mathrm{rig}}(V)$) *is an étale* (φ, Γ)-*module of rank d. Furthermore, the functor* $T \mapsto \mathrm{D}(T)$ (*resp.* $V \mapsto \mathrm{D}(V)$, $V \mapsto \mathrm{D}^\dagger(V)$, $V \mapsto \mathrm{D}_{\mathrm{rig}}(V)$) *is a rank preserving equivalence of categories from the category of* $\mathscr{O}_L$-*linear* (*resp. L-linear*) $G_{\mathbb{Q}_p}$-*representations to the category of étale* (φ, Γ)-*modules over* $\mathscr{O}_{\mathscr{E}}$ (*resp.* $\mathscr{E}$, $\mathscr{E}^\dagger$, $\mathscr{R}$).

Let D be a (φ,Γ)-module over R. If D is isomorphic to its φ-pullback, then for any $y \in D$, we may write $y = \sum_{i=0}^{p-1}(1+T)^i\varphi(y_i)$ for some uniquely determined $y_i \in D$. We define $\psi : D \to D$ by setting $\psi(y) = y_0$. It follows that ψ commutes with Γ and satisfies

$$\psi(a\varphi(x)) = \psi(a)x, \quad \psi(\varphi(a)x) = a\psi(x)$$

for any $a \in R$, $x \in D$. In particular, ψ is a left inverse to φ. Set $\mathrm{Res}_{p\mathbb{Z}_p}(y) = \varphi\psi(y)$, $\mathrm{Res}_{\mathbb{Z}_p^\times}(y) = (1-\varphi\psi)(y)$, and denote $\mathrm{Res}_{p\mathbb{Z}_p}(D), \mathrm{Res}_{\mathbb{Z}_p^\times}(D)$ by $D\boxtimes p\mathbb{Z}_p, D\boxtimes\mathbb{Z}_p^\times$ respectively.

For an étale (φ,Γ)-module D over $\mathscr{O}_{\mathscr{E}}$, A *treillis* of D is a compact $\mathscr{O}_{\mathscr{E}^+}$-submodule N which $\mathscr{O}_{\mathscr{E}}$-linearly generates D. Colmez [11] proves that the set of ψ-stable treillis admits a unique minimal element $D^\natural$,and that ψ is surjective on $D^\natural$. It follows from the uniqueness that $D^\natural$ is stable under the Γ-action. In the simplest case when $D = \mathscr{O}_{\mathscr{E}}$, we have $D^\natural = \mathscr{O}_{\mathscr{E}^+}$. For an étale (φ,Γ)-module D over $\mathscr{E}$, if D is the base change of an étale (φ,Γ)-module D_0 over $\mathscr{O}_{\mathscr{E}}$, we define $D^\natural = D_0^\natural[1/p]$ which is independent of the choice of D_0.

We define the φ,Γ-actions on the rank 1 R-module $R\frac{dT}{1+T}$ formally by

$$\varphi(x\frac{dT}{1+T}) = \varphi(x)\frac{dT}{1+T}, \qquad \gamma(x\frac{dT}{1+T}) = \chi(\gamma)\gamma(x)\frac{dT}{1+T},\ x\in R.$$

Then the rank 1 (φ,Γ)-module $R\frac{dT}{1+T}$ is isomorphic to $R(\epsilon)$. For any étale (φ,Γ)-module D over R, the étale (φ,Γ)-module $\check{D} = \mathrm{Hom}_R(D, R\frac{dT}{1+T})$ is called the *Tate dual* of D. We define the pairing $\{\cdot,\cdot\} : \check{D}\times D \to L$ by setting $\{x,y\} = \mathrm{res}_0(\sigma_{-1}(x)(y))$. It follows that $\{\cdot,\cdot\}$ is perfect and satisfies

$$\{x,\varphi(y)\} = \{\psi(x),y\}. \tag{2.3}$$

3. p-adic local Langlands correspondence for $\mathrm{GL}_2(\mathbb{Q}_p)$

3.1. Operator w_δ

For an étale (φ,Γ)-module D over $\mathscr{O}_{\mathscr{E}}$ or $\mathscr{E}$ and a continuous character $\delta : \mathbb{Q}_p^\times \to \mathscr{O}_L^\times$, Colmez constructs the involution $w_\delta : D\boxtimes\mathbb{Z}_p^\times \to D\boxtimes\mathbb{Z}_p^\times$ defined by

$$w_\delta(z) = \lim_{n\to+\infty}\sum_{i\in\mathbb{Z}_p^\times \bmod p^n}\delta(i^{-1})(1+T)^i\sigma_{-i^2}\cdot\varphi^n\psi^n((1+T)^{-i^{-1}}z). \tag{3.1}$$

Note that the right hand side of (3.1) only involves $\delta|_{\mathbb{Z}_p^\times}$. Since $\delta(\mathbb{Z}_p^\times) \subseteq \mathscr{O}_L^\times$ for any $\delta \in \widehat{\mathscr{T}}(L)$, (3.1) is still convergent for any D which is a twist of an étale (φ,Γ) one and $\delta \in \widehat{\mathscr{T}}(L)$. From now on, we suppose D is a twist of an étale (φ,Γ)-module over $\mathscr{O}_{\mathscr{E}}$ or $\mathscr{E}$ and $\delta \in \widehat{\mathscr{T}}(L)$, and define $w_\delta : D\boxtimes\mathbb{Z}_p^\times \to D\boxtimes\mathbb{Z}_p^\times$ by (3.1). Let $D^\dagger, D_{\mathrm{rig}}$ denote the (φ,Γ)-modules over $\mathscr{E}^\dagger,\mathscr{R}$ corresponding to D.

Example 3.1. – It follows from (3.1) that $\mathscr{O}_{\mathscr{E}^+}\boxtimes\mathbb{Z}_p^\times \subset \mathscr{O}_{\mathscr{E}}\boxtimes\mathbb{Z}_p^\times$ is stable under w_δ. By [11, V], one can describe the w_δ-action on $(\mathscr{O}_{\mathscr{E}^+})^{\psi=0}$ more explicitly. Namely, for any $f\in\mathscr{C}^0(\mathbb{Z}_p, L)$ and $z\in\mathscr{O}_{\mathscr{E}}^+\boxtimes\mathbb{Z}_p^\times$,

$$\int_{\mathbb{Z}_p^\times} f(x)\mathscr{A}^{-1}(w_\delta(z)) = \int_{\mathbb{Z}_p^\times}\delta(x)f(1/x)\mathscr{A}^{-1}(z). \tag{3.2}$$

For any abelian profinite group C, we denote by Λ_C the complete group algebra

$$\mathcal{O}_L[[C]] = \varprojlim \mathcal{O}_L[C/C']$$

where C' goes through all open subgroups of C. If C is pro-p cyclic, and if c is a topological generator of C, then Λ_C is canonically isomorphic to the ring consisting of $g(c-1)$ for all $g(T) \in \mathcal{O}_L[[T]]$, and we further define $R(C)$ to be the ring consisting of $g(c-1)$ for all $g(T) \in R$ for any R of $\mathscr{E}^+, \mathscr{E}^\dagger, \mathcal{O}_{\mathscr{E}}^\dagger, \mathscr{E}, \mathcal{O}_{\mathscr{E}}, \mathscr{R}^+$ and $\mathscr{R}$; this is independent of the choice of c. Now we choose $d \geq 1$ such that Γ_d is pro-p cyclic (in fact, we can choose $d = 1$ if p is odd, and $d = 2$ if $p = 2$), then we define $R(\Gamma) = \Lambda_\Gamma \otimes_{\Lambda_{\Gamma_d}} R(\Gamma_d)$ which is independent of the choice of d. The topological rings $\mathcal{O}_{\mathscr{E}}(\Gamma)$ or $\mathscr{E}(\Gamma)$, $\mathscr{E}^\dagger(\Gamma)$ and $\mathscr{R}(\Gamma)$ naturally act on $D \boxtimes \mathbb{Z}_p^\times$, $D^\dagger \boxtimes \mathbb{Z}_p^\times$ and $D_{\mathrm{rig}} \boxtimes \mathbb{Z}_p^\times$ respectively. The following Lemma follows from the proof of [12, Lemme V.2.2].

Lemma 3.2. – *For any $\gamma \in \Gamma$ and $z \in D \boxtimes \mathbb{Z}_p^\times$, $w_\delta(\gamma(z)) = \delta(\chi(\gamma))\gamma^{-1}(w_\delta(z))$.*

Let $\iota_\delta : R(\Gamma) \to R(\Gamma)$ denote the involution defined by $\gamma \to \delta(\chi(\gamma))\gamma^{-1}$. It is an immediate consequence of Lemma 3.2 that the action of w_δ on $D \boxtimes \mathbb{Z}_p^\times$ is $\mathcal{O}_{\mathscr{E}}(\Gamma)$-semilinear with respect to ι_δ, i.e.

$$w_\delta(\lambda(z)) = \iota_\delta(\lambda)(w_\delta(z)), \quad \lambda \in \mathcal{O}_{\mathscr{E}}(\Gamma),\ z \in D \boxtimes \mathbb{Z}_p^\times. \tag{3.3}$$

Let $\eta \in \widehat{\mathscr{T}}(L)$.

Proposition 3.3. – *$\mathscr{E}^+(\eta) \boxtimes \mathbb{Z}_p^\times$ is a free $\mathscr{E}^+(\Gamma)$-module of rank 1. Furthermore, we have*

$$\mathscr{R}^+(\eta) \boxtimes \mathbb{Z}_p^\times = \mathscr{R}^+(\Gamma) \otimes_{\mathscr{E}^+(\Gamma)} \mathscr{E}^+(\eta) \boxtimes \mathbb{Z}_p^\times, \quad \mathscr{E}^\dagger(\eta) \boxtimes \mathbb{Z}_p^\times = \mathscr{E}^\dagger(\Gamma) \otimes_{\mathscr{E}^+(\Gamma)} \mathscr{E}^+(\eta) \boxtimes \mathbb{Z}_p^\times.$$

As a consequence, $\mathscr{E}^\dagger(\eta) \boxtimes \mathbb{Z}_p^\times$ is stable under w_δ.

Proof. – It suffices to treat the case $\eta = 1$. By [12, Lemme V.1.16], $\mathscr{E}^\dagger \boxtimes \mathbb{Z}_p^\times$ is a free $\mathscr{E}^\dagger(\Gamma)$-module of rank 1 generated by $1+T$. By [18, B.2.8], $\mathscr{R}^+ \boxtimes \mathbb{Z}_p^\times$ is a free $\mathscr{R}^+(\Gamma)$-module also generated by $1 + T$. We thus deduce that $\mathscr{E}^+ \boxtimes \mathbb{Z}_p^\times = (\mathscr{E}^\dagger)^{\psi=0} \cap (\mathscr{R}^+)^{\psi=0}$ is a free $\mathscr{E}^+(\Gamma)$-module generated by $1 + T$. The last assertion follows from (3.3) and the fact that $\mathscr{E}^+ \boxtimes \mathbb{Z}_p^\times$ is stable under w_δ. □

3.2. Construction of the correspondence

We define

$$D \boxtimes_\delta \mathrm{P}^1 = \left\{ z = (z_1, z_2) \in D \times D \mid \mathrm{Res}_{\mathbb{Z}_p^\times}(z_2) = w_\delta(\mathrm{Res}_{\mathbb{Z}_p^\times}(z_1)) \right\},$$

and we equip $D \boxtimes_\delta \mathrm{P}^1$ with the subspace topology of $D \times D$.

Proposition 3.4. – *There exists a unique continuous action of $\mathrm{GL}_2(\mathbb{Q}_p)$ on $D \boxtimes_\delta \mathrm{P}^1$ satisfying the following conditions:*

(i) *$w(z_1, z_2) = (z_2, z_1)$;*
(ii) *if $a \in \mathbb{Q}_p^\times$, then $\left[\begin{smallmatrix} a & 0 \\ 0 & a \end{smallmatrix}\right](z_1, z_2) = (\delta(a)z_1, \delta(a)z_2)$;*
(iii) *if $a \in \mathbb{Z}_p^\times$, then $\left[\begin{smallmatrix} a & 0 \\ 0 & 1 \end{smallmatrix}\right](z_1, z_2) = (\sigma_a(z_1), \delta(a)\sigma_{a^{-1}}(z_2))$;*
(iv) *if $z = (z_1, z_2)$ and if $z' = \left[\begin{smallmatrix} p & 0 \\ 0 & 1 \end{smallmatrix}\right](z)$, then $\mathrm{Res}_{p\mathbb{Z}_p} z' = \varphi(z_1)$ and $\mathrm{Res}_{\mathbb{Z}_p}(wz') = \delta(p)\psi(z_2)$;*

(v) *for* $b \in p\mathbb{Z}_p$, *if* $z = (z_1, z_2)$ *and if* $z' = \left[\begin{smallmatrix} 1 & b \\ 0 & 1 \end{smallmatrix}\right] z$, *then* $\mathrm{Res}_{\mathbb{Z}_p} z' = \left[\begin{smallmatrix} 1 & b \\ 0 & 1 \end{smallmatrix}\right] z_1$ *and* $\mathrm{Res}_{p\mathbb{Z}_p}(wz') = u_b(\mathrm{Res}_{p\mathbb{Z}_p}(z_2))$, *where*

$$u_b = \delta(1+b) \left[\begin{smallmatrix} 1 & -1 \\ 0 & 1 \end{smallmatrix}\right] \circ w_\delta \circ \left[\begin{smallmatrix} (1+b)^{-2} & b(1+b)^{-1} \\ 0 & 1 \end{smallmatrix}\right] \circ w_\delta \circ \left[\begin{smallmatrix} 1 & 1/(1+b) \\ 0 & 1 \end{smallmatrix}\right].$$

Proof. – It is easy to see that the matrices given in the proposition generate $\mathrm{GL}_2(\mathbb{Q}_p)$. This implies the uniqueness. The existence follows from [12, Proposition II.1.8]. (Its proof applies to our more general situation.) □

We extend $\{\cdot,\cdot\} : \check{D} \times D \to L$ to a pairing $\{\cdot,\cdot\}_{\mathrm{P}^1} : (\check{D} \boxtimes_{\delta^{-1}} \mathrm{P}^1) \times (D \boxtimes_\delta \mathrm{P}^1) \to L$ by setting

$$\{(z_1, z_2), (z_1', z_2')\}_{\mathrm{P}^1} = \{z_1, z_1'\} + \{\mathrm{Res}_{p\mathbb{Z}_p}(z_2), \mathrm{Res}_{p\mathbb{Z}_p}(z_2')\}. \tag{3.4}$$

Proposition 3.5. – *The pairing* $\{\cdot,\cdot\}_{\mathrm{P}^1} : (\check{D} \boxtimes_{\delta^{-1}} \mathrm{P}^1) \times (D \boxtimes_\delta \mathrm{P}^1) \to L$ *is perfect and* $\mathrm{GL}_2(\mathbb{Q}_p)$*-equivariant.*

Proof. – This follows immediately from [12, Théorème II.3.1]. □

Now Let D be of rank two. Then $\det D$ is of the form $\mathcal{O}_{\mathscr{E}}(\delta_D')$ or $\mathscr{E}(\delta_D')$ for some $\delta_D' \in \widehat{\mathscr{T}}(L)$. Let $\delta_D = \epsilon^{-1}\delta_D'$, and we denote $w_{\delta_D}, D \boxtimes_{\delta_D} \mathrm{P}^1$ by $w_D, D \boxtimes \mathrm{P}^1$ for simplicity. For $z = (z_1, z_2) \in D \boxtimes \mathrm{P}^1$, set $\mathrm{Res}_{\mathbb{Z}_p}(z_1, z_2) = z_1$. We then define

$$D^\natural \boxtimes \mathrm{P}^1 = \{z \in D \boxtimes \mathrm{P}^1 \mid \mathrm{Res}_{\mathbb{Z}_p}(\left[\begin{smallmatrix} p^n & 0 \\ 0 & 1 \end{smallmatrix}\right] z) \in D^\natural, \forall n \in \mathbb{N}\}.$$

Theorem 3.6 ([12, Théorème II.3.1]). – *Let D be an irreducible rank 2 étale (φ, Γ)-module over $\mathcal{O}_{\mathscr{E}}$. Then the following hold:*

(i) *The submodule $D^\natural \boxtimes \mathrm{P}^1$ of $D \boxtimes \mathrm{P}^1$ is stable under the action of* $\mathrm{GL}_2(\mathbb{Q}_p)$.
(ii) *The quotient* $\mathrm{GL}_2(\mathbb{Q}_p)$*-representation* $\Pi(D) = D \boxtimes \mathrm{P}^1/D^\natural \boxtimes \mathrm{P}^1$ *is an object of* $\mathrm{Rep}_{\mathcal{O}_L}\mathrm{GL}_2(\mathbb{Q}_p)$*, and has central character δ_D. The continuous* $\mathrm{GL}_2(\mathbb{Q}_p)$*-representation $D^\natural \boxtimes \mathrm{P}^1$ is naturally isomorphic to $\Pi(D)^* \otimes \delta_D$. Hence we have the following exact sequence*

$$0 \longrightarrow \Pi(D)^* \otimes \delta_D \longrightarrow D \boxtimes \mathrm{P}^1 \longrightarrow \Pi(D) \longrightarrow 0.$$

Note that $\check{D} \cong D(\delta_D^{-1})$ because D is of rank two. Hence $\Pi(D)^* \otimes \delta_D$ is isomorphic to

$$(\Pi(\check{D}) \otimes \delta_D)^* \otimes \delta_D = \Pi(\check{D})^*.$$

Corollary 3.7. – *If D is an irreducible étale (φ, Γ)-module of rank 2 over $\mathscr{E}$, then $D^\natural \boxtimes \mathrm{P}^1$ is stable under the* $\mathrm{GL}_2(\mathbb{Q}_p)$*-action and the quotient representation $\Pi(D) = D \boxtimes \mathrm{P}^1/D^\natural \boxtimes \mathrm{P}^1$ is an admissible unitary representation of* $\mathrm{GL}_2(\mathbb{Q}_p)$*. Moreover, $D^\natural \boxtimes \mathrm{P}^1$ is naturally isomorphic to the contragredient representation $\Pi(\check{D})^*$.*

3.3. Locally analytic vectors

Now suppose that D is a twist of an irreducible rank 2 étale (φ,Γ)-module over $\mathscr{E}$. The following proposition follows immediately from ([12, Lemme V.2.4]).

Proposition 3.8. – *$D^\dagger \boxtimes \mathbb{Z}_p^\times$ is stable under the action of w_D.*

By [12, Théorème V.1.12(iii)], we have $D_{\mathrm{rig}} \boxtimes \mathbb{Z}_p^\times = \mathscr{R}(\Gamma) \otimes_{\mathscr{E}^\dagger(\Gamma)} (D^\dagger \boxtimes \mathbb{Z}_p^\times)$. Then for $\Delta = \mathscr{R}(\eta), * = \delta$, or $\Delta = D_{\mathrm{rig}}, * = \delta_D$, we extend the w_*-action to $\Delta \boxtimes \mathbb{Z}_p^\times$ by setting

$$w_*(\lambda \otimes z) = \iota_*(\lambda) \otimes w_*(z).$$

For $\Delta = \mathscr{E}^\dagger(\eta), \mathscr{R}^+(\eta), \mathscr{R}(\eta), * = \delta$, or $\Delta = D^\dagger, D_{\mathrm{rig}}, * = \delta_D$, we set

$$\Delta \boxtimes_* \mathrm{P}^1 = \{(z_1, z_2) \in \Delta \times \Delta, \mathrm{Res}_{\mathbb{Z}_p^\times}(z_2) = w_*(\mathrm{Res}_{\mathbb{Z}_p^\times}(z_1))\},$$

and we equip $\Delta \boxtimes_* \mathrm{P}^1$ with the subspace topology of $\Delta \times \Delta$. Henceforth we denote $D^\dagger \boxtimes_{\delta_D} \mathrm{P}^1$, $D_{\mathrm{rig}} \boxtimes_{\delta_D} \mathrm{P}^1$ by $D^\dagger \boxtimes \mathrm{P}^1$, $D_{\mathrm{rig}} \boxtimes \mathrm{P}^1$ for simplicity.

By Proposition 3.4, it is clear that both $\mathscr{E}^\dagger(\eta) \boxtimes_\delta \mathrm{P}^1$ and $D^\dagger \boxtimes \mathrm{P}^1$ are stable under $\mathrm{GL}_2(\mathbb{Q}_p)$. Since $D^\dagger$ is dense in D_{rig} for any (φ,Γ)-module D over $\mathscr{E}$, we extend the $\mathrm{GL}_2(\mathbb{Q}_p)$-actions on $\mathscr{E}^\dagger(\eta) \boxtimes_\delta \mathrm{P}^1$ and $D^\dagger \boxtimes \mathrm{P}^1$ to continuous $\mathrm{GL}_2(\mathbb{Q}_p)$-actions on $\mathscr{R}(\eta) \boxtimes_\delta \mathrm{P}^1$ and $D_{\mathrm{rig}} \boxtimes \mathrm{P}^1$ which satisfy the formulas listed in Proposition 3.4 by continuity. This yields the following proposition.

Proposition 3.9. – *There exists a unique continuous action of $\mathrm{GL}_2(\mathbb{Q}_p)$ on $\Delta \boxtimes_* \mathrm{P}^1$ satisfying the formulas listed in Proposition 3.4.*

For $\Delta = \mathscr{R}(\eta), * = \delta$, or $\Delta = D_{\mathrm{rig}}, * = \delta_D$, we set the pairing

$$\{\cdot,\cdot\}_{\mathrm{P}^1} : \check{\Delta} \boxtimes_{*^{-1}} \mathrm{P}^1 \times \Delta \boxtimes_* \mathrm{P}^1 \to L$$

by formula (3.4).

Proposition 3.10. – *The pairing $\{\cdot,\cdot\}_{\mathrm{P}^1} : \check{\Delta} \boxtimes_{*^{-1}} \mathrm{P}^1 \times \Delta \boxtimes_* \mathrm{P}^1 \to L$ is perfect and $\mathrm{GL}_2(\mathbb{Q}_p)$-equivariant.*

Proof. – The restriction of $\{\cdot,\cdot\}_{\mathrm{P}^1}$ on $\mathscr{E}^\dagger(\check{\eta}) \boxtimes_{\delta^{-1}} \mathrm{P}^1 \times \mathscr{E}^\dagger(\eta) \boxtimes_\delta \mathrm{P}^1$ or $\check{D}^\dagger \boxtimes \mathrm{P}^1 \times D^\dagger \boxtimes \mathrm{P}^1$ is $\mathrm{GL}_2(\mathbb{Q}_p)$-equivariant by Proposition 3.5. Hence $\{\cdot,\cdot\}_{\mathrm{P}^1}$ itself is $\mathrm{GL}_2(\mathbb{Q}_p)$-equivariant by the density of $\mathscr{E}^\dagger(\eta) \boxtimes_\delta \mathrm{P}^1$ or $D^\dagger \boxtimes \mathrm{P}^1$. The perfectness of $\{\cdot,\cdot\}_{\mathrm{P}^1}$ follows from the perfectness of $\{\cdot,\cdot\}$ on $\check{\Delta} \times \Delta$ and $\check{\Delta} \boxtimes p\mathbb{Z}_p \times \Delta \boxtimes p\mathbb{Z}_p$. □

If D is étale, it follows from [11, Corollaire II.7.2] that $D^\natural \subset D^\dagger$; hence we may view $D^\natural \boxtimes \mathrm{P}^1$ as a submodule of $D_{\mathrm{rig}} \boxtimes \mathrm{P}^1$. Colmez shows that the inclusion $\Pi(\check{D})^* = D^\natural \boxtimes \mathrm{P}^1 \subset D_{\mathrm{rig}} \boxtimes \mathrm{P}^1$ extends naturally to a $\mathrm{GL}_2(\mathbb{Q}_p)$-equivariant embedding $(\Pi(\check{D})_{\mathrm{an}})^* \hookrightarrow D_{\mathrm{rig}} \boxtimes \mathrm{P}^1$, and he further shows that the image of this embedding, which is denoted by $D^\natural_{\mathrm{rig}} \boxtimes \mathrm{P}^1$, is the orthogonal complement of $\check{D}^\natural \boxtimes \mathrm{P}^1$ under the pairing $\{\cdot,\cdot\}_{\mathrm{P}^1} : \check{D}_{\mathrm{rig}} \boxtimes \mathrm{P}^1 \times D_{\mathrm{rig}} \boxtimes \mathrm{P}^1 \to L$ ([12, Remarque V.2.21(ii)]). The following proposition is the key ingredient for our determination of locally analytic vectors of unitary principal series.

Proposition 3.11 ([12, Remarque V.2.21(i)]). – *$D^\natural_{\mathrm{rig}} \boxtimes \mathrm{P}^1$ and $\check{D}^\natural_{\mathrm{rig}} \boxtimes \mathrm{P}^1$ are orthogonal complements of each other under the pairing $\{\cdot,\cdot\}_{\mathrm{P}^1} : \check{D}_{\mathrm{rig}} \boxtimes \mathrm{P}^1 \times D_{\mathrm{rig}} \boxtimes \mathrm{P}^1 \to L$.*

4. Unitary principal series and 2-dimensional trianguline representations

4.1. 2-dimensional irreducible trianguline representations

A (φ, Γ)-module over $\mathscr{R}$ is called *triangulable* if it is a successive extensions of rank 1 (φ, Γ)-modules over $\mathscr{R}$; an L-linear representation V of $G_{\mathbb{Q}_p}$ is called *trianguline* if $\mathrm{D}_{\mathrm{rig}}(V)$ is triangulable.

Proposition 4.1 ([7, Proposition 3.1]). – *If D is a rank 1 (φ, Γ)-module over $\mathscr{R}$, then there exists a unique $\delta \in \widehat{\mathscr{T}}(L)$ such that D is isomorphic to $\mathscr{R}(\delta)$.*

It follows that if V is a 2-dimensional irreducible trianguline representation, then $\mathrm{D}_{\mathrm{rig}}(V)$ sits in a short exact sequence

$$0 \longrightarrow \mathscr{R}(\delta_1) \longrightarrow \mathrm{D}_{\mathrm{rig}}(V) \longrightarrow \mathscr{R}(\delta_2) \longrightarrow 0 \tag{4.1}$$

for some $\delta_1, \delta_2 \in \widehat{\mathscr{T}}(L)$. Furthermore, we have that (4.1) is non-split by Kedlaya's slope theory. Therefore V is uniquely determined by the triple (δ_1, δ_2, c) where

$$c \in \mathrm{Proj}(\mathrm{Ext}^1(\mathscr{R}(\delta_2), \mathscr{R}(\delta_1))) = \mathrm{Proj}(H^1(\mathscr{R}(\delta_1\delta_2^{-1})))$$

is the element representing the extension (4.1). Let

$$\mathscr{S} = \{(\delta_1, \delta_2, c) | \delta_i \in \widehat{\mathscr{T}}(L), c \in \mathrm{Proj}(H^1(\mathscr{R}(\delta_1\delta_2^{-1})))\}$$

be the set of all such triples; then each element of $\mathscr{S}$ naturally defines a rank 2 triangulable (φ, Γ)-module: the non-split extension of $\mathscr{R}(\delta_2)$ by $\mathscr{R}(\delta_1)$ defined by c. In the rest of this section we assume $p > 2$. The following calculation is due to Colmez.

Proposition 4.2 ([7, Théorème 2.9]). –
(i) *If $\delta = x^{-k}$ or $x^{k+1}|x|$ for $k \in \mathbb{N}$, then* $\dim_L H^1(\mathscr{R}(\delta)) = 2$.
(ii) *Otherwise,* $\dim_L(\mathscr{R}(\delta)) = 1$.

Colmez further specifies an explicit basis of $H^1(\mathscr{R}(\delta))$ in case (i), and identifies $\mathrm{Proj}(H^1(\mathscr{R}(\delta)))$ with $\mathrm{P}^1(L)$ via this basis. By this identification, we may write any $s \in \mathscr{S}$ as $s = (\delta_1, \delta_2, \mathscr{L})$ where $\mathscr{L} \in \mathrm{P}^1(L)$ if $\delta_1\delta_2^{-1} = x^{-k}$ or $x^{k+1}|x|$ for some $k \in \mathbb{N}$, or $\mathscr{L} = \infty$ otherwise. Let $D(s)$ denote the rank 2 triangulable (φ, Γ)-module defined by s. We set $\check{s} \in \mathscr{S}$ to be the triple $(\check{\delta_1}, \check{\delta_2}, \mathscr{L}) = (\epsilon\delta_2^{-1}, \epsilon\delta_1^{-1}, \mathscr{L})$. Then $\check{D}(s)$ is isomorphic to $D(\check{s})$ under Colmez's identification.

Let $\mathscr{S}_*$ be the set of all $s = (\delta_1, \delta_2, \mathscr{L}) \in \mathscr{S}$ such that

$$v_p(\delta_1(p)) + v_p(\delta_2(p)) = 0, \quad v_p(\delta_1(p)) > 0.$$

For $s \in \mathscr{S}_*$, set $u(s) = v_p(\delta_1(p)) = -v_p(\delta_2(p)), w(s) = w(\delta_1) - w(\delta_2), \delta_s = \delta_1\delta_2^{-1}(x|x|)^{-1}$, and we define

$$\begin{aligned}
\mathscr{S}_*^{\mathrm{ng}} &= \{s \in \mathscr{S}_* \mid w(s) \text{ is not a positive integer}\},\\
\mathscr{S}_*^{\mathrm{cris}} &= \{s \in \mathscr{S}_* \mid w(s) \text{ is a positive integer}, u(s) < w(s), \mathscr{L} = \infty\},\\
\mathscr{S}_*^{\mathrm{st}} &= \{s \in \mathscr{S}_* \mid w(s) \text{ is a positive integer}, u(s) < w(s), \mathscr{L} \neq \infty\},
\end{aligned}$$

and $\mathscr{S}_{\mathrm{irr}} = \mathscr{S}_*^{\mathrm{ng}} \coprod \mathscr{S}_*^{\mathrm{cris}} \coprod \mathscr{S}_*^{\mathrm{st}}$. Note that if $s \in \mathscr{S}_*^{\mathrm{st}}$, we must have $\delta_s = x^{k-1}$ for some $k \in \mathbb{Z}_+$.

Proposition 4.3 ([7, Proposition 3.5, 3.7]). – *If $s \in \mathscr{S}_{\mathrm{irr}}$, then $D(s)$ is étale, and the corresponding L-linear representation, which we denote by $V(s)$, is irreducible. Conversely, every 2-dimensional irreducible trianguline representation is isomorphic to $V(s)$ for some $s \in \mathscr{S}_{\mathrm{irr}}$. Moreover, for $s = (\delta_1, \delta_2, \mathscr{L}), s' = (\delta_1', \delta_2', \mathscr{L}') \in \mathscr{S}_{\mathrm{irr}}$, if $\delta_1 = \delta_1'$, then $V(s) \cong V(s')$ if and only if $\delta_2 = \delta_2'$ and $\mathscr{L} = \mathscr{L}'$; if $\delta_1 \neq \delta_1'$, then $V(s) \cong V(s')$ if and only if $s, s' \in \mathscr{S}_*^{\mathrm{cris}}$ and $\delta_1' = x^{w(s)}\delta_2,\ \delta_2' = x^{-w(s)}\delta_1$.*

We call $s \in \mathscr{S}_{\mathrm{irr}}$ *exceptional* if $s \in \mathscr{S}_*^{\mathrm{cris}}$ and $V(s)$ is not Frobenius semi-simple; this is equivalent to $\delta_s = x^{k-1}|x|^{-1}$ for some $k \in \mathbb{Z}_+$.

4.2. Unitary principal series

Throughout this subsection, let $s = (\delta_1, \delta_2, \mathscr{L}) \in \mathscr{S}_{\mathrm{irr}}$. Let $\mathscr{C}^u(\mathrm{P}^1(\delta))$ be the L-vector space of $\mathscr{C}^u$ functions $f : \mathbb{Q}_p \to L$ such that $\delta(x)f(1/x)|_{\mathbb{Q}_p - \{0\}}$ extends to a $\mathscr{C}^u$-function on $\mathbb{Q}_p$. In other words, $\mathscr{C}^{u(s)}(\mathrm{P}^1(\delta_s))$ is the L-vector space of functions $f : \mathbb{Q}_p \to L$ satisfying:

- $f|_{\mathbb{Z}_p}$ is of class $\mathscr{C}^{u(s)}$.
- $\delta_s(z)f(1/z)|_{\mathbb{Z}_p - \{0\}}$ extends to a $\mathscr{C}^{u(s)}$-function on $\mathbb{Z}_p$.

We thus have an isomorphism

$$\mathscr{C}^{u(s)}(\mathrm{P}^1(\delta_s)) \simeq \mathscr{C}^{u(s)}(\mathbb{Z}_p, L) \oplus \mathscr{C}^{u(s)}(\mathbb{Z}_p, L), \quad f \mapsto (f_1, f_2)$$

where $f_1(z) = f(pz)$ and f_2 is the extension of $\delta_s(z)f(1/z)$. By this isomorphism, we may equip $\mathscr{C}^{u(s)}(\mathrm{P}^1(\delta_s))$ with a Banach space structure by defining

$$||f|| = \max\Big(||f_1||_{\mathscr{C}^{u(s)}},\ ||f_2||_{\mathscr{C}^{u(s)}} \Big).$$

We define a $\mathrm{GL}_2(\mathbb{Q}_p)$-representation $B(s)$ on $\mathscr{C}^{u(s)}(\mathrm{P}^1(\delta_s))$ by

$$\left(\left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right] \star_s f \right)(x) = \delta_2(ad - bc)\delta_s(-cx + a) f\left(\frac{dx - b}{-cx + a} \right);$$

then $B(s)$ is a Banach space representation. We define a subspace $M(s)$ of $B(s)$ as below:

- If $\delta_s \neq x^k$ for any $k \in \mathbb{N}$, we define $M(s)$ to be the subspace generated by $\{x^i | 0 \leq i < u(s)\}$ and $\{(x-a)^{-i}\delta_s(x-a) | a \in \mathbb{Q}_p, 0 \leq i < u(s)\}$.
- If $\delta_s = x^{k-1}$ for some $k \in \mathbb{Z}_+$, let $M(s)'$ be the space of functions of the form

$$f = \sum_{u \in U} \lambda_u (x - a_u)^{j_u} \log_{\mathscr{L}}(x - a_u)$$

  where U is a finite set, j_u are integers between $[\frac{k+1}{2}]$ and k, $\lambda_u \in L$ and $a_u \in \mathbb{Q}_p$ such that $\deg(\sum_{u \in U} \lambda_u (x - a_u)^{j_u}) < u(s)$. By [5, Lemme 3.3.2], $M(s)'$ is a subspace of $B(s)$. We define $M(s)$ to be the subspace generated by $M(s)'$ and x^i for $0 \leq i \leq k-1$.

An easy computation shows that $M(s)$ is stable under $\mathrm{GL}_2(\mathbb{Q}_p)$ in both cases. We set $\Pi(s) = B(s)/\widehat{M}(s)$ where $\widehat{M}(s)$ is the closure of $M(s)$ in $B(s)$.

Let $D^\natural(s)$ denote $(\mathrm{D}(V(s)))^\natural$ for simplicity. We fix a standard basis e_2 of $\mathscr{R}(\delta_2)$. For any $z \in D^\natural(s) \boxtimes \mathrm{P}^1$, suppose that the image of $\mathrm{Res}_{\mathbb{Z}_p}(\left[\begin{smallmatrix} p^n & 0 \\ 0 & 1 \end{smallmatrix}\right] z)$ in $\mathscr{R}(\delta_2)$ is $z_2^{(n)}e_2$. The following theorem follows from [12, Théorème IV.4.12].

THEOREM 4.4. – *For $s \in \mathscr{S}_{\mathrm{irr}}$ non-exceptional and $z \in D^{\natural}(s) \boxtimes \mathrm{P}^1$, there exists $\mu_z \in \mathscr{D}_{u(s)}(\mathbb{Q}_p)$ such that*

$$\mathscr{A}^{(n)}(\mu_z) = (\delta_2(p))^{-n} z_2^{(n)}.$$

Furthermore, the map $z \mapsto \mu_z$ is a $\mathrm{GL}_2(\mathbb{Q}_p)$-equivariant topological isomorphism from $D^{\natural}(s) \boxtimes \mathrm{P}^1$ to $\Pi(\check{s})^*$.

We denote the converse of this isomorphism by $\mathscr{A}_s$.

5. Locally analytic principal series and rank 1 (φ, Γ)-modules

5.1. Locally analytic principal series

For any $\delta \in \widehat{\mathscr{T}}(L)$, we denote by $\mathrm{LA}(\mathrm{P}^1(\delta))$ the L-vector space of locally analytic functions $f : \mathbb{Q}_p \to L$ such that $\delta(x) f(1/x)|_{\mathbb{Q}_p - \{0\}}$ extends to a locally analytic function on $\mathbb{Q}_p$. As in the case of $\mathscr{C}^u(\mathrm{P}^1(\delta))$, for any $f \in \mathrm{LA}(\mathrm{P}^1(\delta))$, if we set $f_1(pz) = f|_{p\mathbb{Z}_p}$ and f_2 to be the extension of $\delta_s(x) f(1/x)|_{\mathbb{Z}_p - \{0\}}$, then the map $f \mapsto f_1 \oplus f_2$ induces an isomorphism $\mathrm{LA}(\mathrm{P}^1(\delta)) \cong \mathrm{LA}(\mathbb{Z}_p, L) \oplus \mathrm{LA}(\mathbb{Z}_p, L)$. We then equip $\mathrm{LA}(\mathrm{P}^1(\delta))$ with the topology induced from $\mathrm{LA}(\mathbb{Z}_p, L) \oplus \mathrm{LA}(\mathbb{Z}_p, L)$.

For any pair $(\delta_1, \delta_2) \in \widehat{\mathscr{T}}(L) \times \widehat{\mathscr{T}}(L)$, let $\widetilde{\Sigma}(\delta_1, \delta_2)$ denote the locally analytic parabolic induction

$$\begin{aligned}\left(\mathrm{Ind}_{\mathrm{B}(\mathbb{Q}_p)}^{\mathrm{GL}_2(\mathbb{Q}_p)} \delta_2 \otimes \delta_1 \epsilon^{-1}\right)_{\mathrm{an}} = \{&\text{locally analytic functions } F : \mathrm{GL}_2(\mathbb{Q}_p) \to L \text{ such} \\ &\text{that } F(bg) = (\delta_2 \otimes \delta_1 \epsilon^{-1})(b) F(g) \text{ for all } b \in \mathrm{B}(\mathbb{Q}_p)\},\end{aligned}$$

which is equipped with the left $\mathrm{GL}_2(\mathbb{Q}_p)$-action $(gF)(g') = F(g'g)$ for any $g, g' \in \mathrm{GL}_2(\mathbb{Q}_p)$. Put $\delta = \delta_1 \delta_2^{-1} \epsilon^{-1}$. We may identify the underlying topological space of $\widetilde{\Sigma}(\delta_1, \delta_2)$ with $\mathrm{LA}(\mathrm{P}^1(\delta))$ by the map

$$F \mapsto f(x) := F(\left[\begin{smallmatrix} 0 & 1 \\ -1 & x \end{smallmatrix}\right])$$

for any $F \in \widetilde{\Sigma}(\delta_1, \delta_2)$. In addition, the corresponding $\mathrm{GL}_2(\mathbb{Q}_p)$-action on $\mathrm{LA}(\mathrm{P}^1(\delta))$ is given by the formula

$$\left(\left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right] \cdot f\right)(x) = \delta_2(ad - bc)\delta(-cx + a) f\left(\frac{dx - b}{-cx + a}\right). \tag{5.1}$$

If $k = w(\delta_1 \delta_2^{-1}) = w(\delta) + 1$ is a positive integer, then the k-th differential map

$$I_k : \mathrm{LA}(\mathrm{P}^1(\delta)) \to \mathrm{LA}(\mathrm{P}^1(x^{-2k}\delta)), \quad f(x) \mapsto \left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^k f(x),$$

induces an intertwining between $\widetilde{\Sigma}(\delta_1, \delta_2)$ and $\widetilde{\Sigma}(x^{-k}\delta_1, x^k \delta_2)$. The kernel of I_k, which consists of locally polynomial functions of degree $\leq k - 1$, is isomorphic to

$$(\delta_2 \circ \det) \otimes \mathrm{Sym}^{k-1} L^2 \otimes \mathrm{Ind}_{\mathrm{B}(\mathbb{Q}_p)}^{\mathrm{GL}_2(\mathbb{Q}_p)} (1 \otimes (x^{-k+1}\delta))_{\mathrm{sm}} \tag{5.2}$$

as a locally analytic representation. Moreover, if $\delta = x^{k-1}$, the L-vector subspace generated by $\{x^i | 0 \leq i \leq k-1\}$ is $\mathrm{GL}_2(\mathbb{Q}_p)$-invariant, and is isomorphic to $(\delta_2 \circ \det) \otimes \mathrm{Sym}^{k-1} L^2$ as a $\mathrm{GL}_2(\mathbb{Q}_p)$-representation. The quotient of $\ker I_k$ by this subspace is isomorphic to $(\delta_2 \circ \det) \otimes \mathrm{Sym}^{k-1} L^2 \otimes \mathrm{St}$.

We define

$$(5.3)\quad \Sigma(\delta_1,\delta_2)=\begin{cases}\widetilde{\Sigma}(\delta_1,\delta_2)/(\delta_2\circ\det)\otimes\mathrm{Sym}^{k-1}L^2 & \text{if } \delta=x^{k-1} \text{ for some integer } k\geq 1;\\ \widetilde{\Sigma}(\delta_1,\delta_2) & \text{otherwise.}\end{cases}$$

The following proposition, which follows by the main results of [21], [20], determines the Jordan-Hölder series of $\Sigma(\delta_1,\delta_2)$.

Proposition 5.1. – *With notations as above, the following are true.*

(i) *If* $w(\delta)\notin\mathbb{N}$, *then* $\Sigma(\delta_1,\delta_2)=\widetilde{\Sigma}(\delta_1,\delta_2)$ *is a topological irreducible locally analytic representation of* $\mathrm{GL}_2(\mathbb{Q}_p)$.

(ii) *If* $w(\delta)\in\mathbb{N}$ *and* $\delta\neq x^{k-1}$, *then* I_k *is surjective, and* $\Sigma(\delta_1,\delta_2)=\widetilde{\Sigma}(\delta_1,\delta_2)$ *is a non-split extension of* $\widetilde{\Sigma}(x^{-k}\delta_1,x^k\delta_2)$ *by* $(\delta_2\circ\det)\otimes\mathrm{Sym}^{k-1}L^2\otimes\mathrm{Ind}_{\mathrm{B}(\mathbb{Q}_p)}^{\mathrm{GL}_2(\mathbb{Q}_p)}(1\otimes(x^{-k+1}\delta))_{\mathrm{sm}}$, *and both* $\widetilde{\Sigma}(x^{-k}\delta_1,x^k\delta_2)$ *and* $(\delta_2\circ\det)\otimes\mathrm{Sym}^{k-1}L^2\otimes\mathrm{Ind}_{\mathrm{B}(\mathbb{Q}_p)}^{\mathrm{GL}_2(\mathbb{Q}_p)}(1\otimes(x^{-k+1}\delta))_{\mathrm{sm}}$ *are topological irreducible.*

(iii) *If* $\delta=x^{k-1}$ *for some integer* $k\geq 1$, *then* $\Sigma(\delta_1,\delta_2)$ *is a non-split extension of* $\widetilde{\Sigma}(x^{-k}\delta_1,x^k\delta_2)$ *by* $(\delta_2\circ\det)\otimes\mathrm{Sym}^{k-1}L^2\otimes\mathrm{St}$, *and both* $\widetilde{\Sigma}(x^{-k}\delta_1,x^k\delta_2)$ *and* $(\delta_2\circ\det)\otimes\mathrm{Sym}^{k-1}L^2\otimes\mathrm{St}$ *are topological irreducible.*

5.2. $\tilde{\Sigma}(\eta^{-1}\epsilon,\delta^{-1}\eta)^*\cong\mathscr{R}^+(\eta)\boxtimes_\delta \mathrm{P}^1$.

For any $\delta_1,\delta_2\in\widehat{\mathscr{T}}(L)$, let $\mathrm{GL}_2(\mathbb{Q}_p)$ acts on $\widetilde{\Sigma}(\delta_1,\delta_2)^*$ by the formula $\langle f,g\cdot\mu\rangle=\langle g^{-1}\cdot f,\mu\rangle$ for any $f\in\widetilde{\Sigma}(\delta_1,\delta_2)$, $\mu\in\widetilde{\Sigma}(\delta_1,\delta_2)^*$ and $g\in\mathrm{GL}_2(\mathbb{Q}_p)$. Thus by (5.1), we have

$$(wf)(x)=\eta(-1)\delta\eta^{-2}(x)f(1/x)$$

for any $f\in\widetilde{\Sigma}(\eta^{-1}\epsilon,\delta^{-1}\eta)$. Therefore, by the description of $\mathrm{LA}(\mathrm{P}^1(\delta))$ given in §5.1, we see that the map $\mu\mapsto(\mu|_{\mathbb{Z}_p},w\mu|_{\mathbb{Z}_p})$ is a homeomorphism from $\widetilde{\Sigma}(\eta^{-1}\epsilon,\delta^{-1}\eta)^*$ to

$$(5.4)\qquad \{(\mu_1,\mu_2)\in\mathscr{D}(\mathbb{Z}_p,L)\oplus\mathscr{D}(\mathbb{Z}_p,L)|\int_{\mathbb{Z}_p^\times}f(x)\mu_2=\int_{\mathbb{Z}_p^\times}\eta(-1)(\delta\eta^{-2})(x)f(1/x)\mu_1\},$$

where the latter object is equipped with the subspace topology of $\mathscr{D}(\mathbb{Z}_p,L)\oplus\mathscr{D}(\mathbb{Z}_p,L)$.

We fix a standard basis $e_\eta\in\mathscr{E}^+(\eta)$.

Lemma 5.2. – $\mathscr{A}(w\mu|_{\mathbb{Z}_p^\times})\otimes e_\eta=w_\delta(\mathscr{A}(\mu|_{\mathbb{Z}_p^\times})\otimes e_\eta)$ *for any* $\mu\in\widetilde{\Sigma}(\eta^{-1}\epsilon,\delta^{-1}\eta)^*$.

Proof. – The case $\mathscr{A}(\mu|_{\mathbb{Z}_p^\times})\in\mathscr{E}^+\boxtimes\mathbb{Z}_p^\times$ follows directly from (3.2) and (5.4). Since $\mathscr{E}^+$ is dense in $\mathscr{R}^+$, we deduce the case $\mathscr{A}(\mu|_{\mathbb{Z}_p^\times})\in\mathscr{R}^+\boxtimes\mathbb{Z}_p^\times$ by the continuity of w and $\mathscr{A}$. We then conclude the general case by Proposition 3.3. □

As a consequence, $\mathscr{A}_{\delta,\eta}(\mu)=(\mathscr{A}(\mu|_{\mathbb{Z}_p})\otimes e_\eta,\mathscr{A}(w\mu|_{\mathbb{Z}_p})\otimes e_\eta)$ is an element of $\mathscr{R}^+(\eta)\boxtimes_\delta\mathrm{P}^1$.

Proposition 5.3. – *The map* $\mathscr{A}_{\delta,\eta}:\widetilde{\Sigma}(\eta^{-1}\epsilon,\delta^{-1}\eta)^*\to\mathscr{R}^+(\eta)\boxtimes_\delta\mathrm{P}^1$ *is a* $\mathrm{GL}_2(\mathbb{Q}_p)$*-equivariant topological isomorphism.*

Proof. – By the description of $\widetilde{\Sigma}(\eta^{-1}\epsilon, \delta^{-1}\eta)^*$ given in (5.4), one sees easily that $\mathscr{A}_{\eta,\delta}$ is an embedding. On the other hand, for any $z = (z_1 \otimes e_\eta, z_2 \otimes e_\eta) \in \mathscr{R}^+(\eta) \boxtimes_\delta \mathrm{P}^1$, if we put $\mu = \mathscr{A}^{-1}(z_1) + w\mathscr{A}^{-1}(\mathrm{Res}_{p\mathbb{Z}_p}(\mu_2))$, then $\mathscr{A}_{\eta,\delta}(\mu) = z$. Hence $\mathscr{A}_{\eta,\delta}$ is a topological isomorphism.

To prove that $\mathscr{A}_{\eta,\delta}$ is $\mathrm{GL}_2(\mathbb{Q}_p)$-equivariant, we only need to show

$$\mathscr{A}_{\eta,\delta}(g \cdot \mu) = g \cdot \mathscr{A}_{\eta,\delta}(\mu) \tag{5.5}$$

for (1) $g = \left[\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right]$; (2) $g = \left[\begin{smallmatrix} a & 0 \\ 0 & a \end{smallmatrix}\right], a \in \mathbb{Q}_p^\times$; (3) $g = \left[\begin{smallmatrix} a & 0 \\ 0 & 1 \end{smallmatrix}\right], a \in \mathbb{Z}_p^\times$; (4) $g = \left[\begin{smallmatrix} p & 0 \\ 0 & 1 \end{smallmatrix}\right]$; (5) $g = \left[\begin{smallmatrix} 1 & b \\ 0 & 1 \end{smallmatrix}\right], b \in p\mathbb{Z}_p$.

Case (1) is trivial. Both $\widetilde{\Sigma}(\eta^{-1}\epsilon, \delta^{-1}\eta)^*$ and $\mathscr{R}^+(\eta) \boxtimes_\delta \mathrm{P}^1$ have central characters δ; this proves (2). For any $a \in \mathbb{Z}_p^\times$, we have

$$\int_{\mathbb{Z}_p} f(x)(\left[\begin{smallmatrix} a & 0 \\ 0 & 1 \end{smallmatrix}\right])\mu) = \int_{\mathbb{Z}_p} (\left[\begin{smallmatrix} a^{-1} & 0 \\ 0 & 1 \end{smallmatrix}\right] f(x))\mu = \int_{\mathbb{Z}_p} \eta(a^{-1})f(ax)\mu = \int_{\mathbb{Z}_p} f(x)(\eta(a^{-1})(\sigma_a(\mu)));$$

this yields (3). For case (4), we have

$$\begin{aligned} \int_{p\mathbb{Z}_p} f(x)(\left[\begin{smallmatrix} p & 0 \\ 0 & 1 \end{smallmatrix}\right] \mu) &= \int_{p\mathbb{Z}_p} \left[\begin{smallmatrix} p^{-1} & 0 \\ 0 & 1 \end{smallmatrix}\right] (f(x)1_{\mathbb{Z}_p}(x))\mu \\ &= \int_{\mathbb{Z}_p} \eta(p)f(px)\mu \\ &= \int_{\mathbb{Z}_p} f(x)(\eta(p)\varphi(\mu|_{\mathbb{Z}_p})), \end{aligned}$$

yielding $\left[\begin{smallmatrix} p & 0 \\ 0 & 1 \end{smallmatrix}\right] \mu|_{p\mathbb{Z}_p} = \eta(p)\varphi(\mu|_{\mathbb{Z}_p})$. This implies

$$\mathrm{Res}_{p\mathbb{Z}_p}(\mathscr{A}_{\delta,\eta}(\left[\begin{smallmatrix} p & 0 \\ 0 & 1 \end{smallmatrix}\right] \mu)) = \varphi(\mathrm{Res}_{\mathbb{Z}_p}(\mathscr{A}_{\delta,\eta}(\mu))). \tag{5.6}$$

A similar computation shows that

$$\begin{aligned} \mathrm{Res}_{\mathbb{Z}_p}(w(\mathscr{A}_{\delta,\eta}(\left[\begin{smallmatrix} p & 0 \\ 0 & 1 \end{smallmatrix}\right] \mu))) &= \mathrm{Res}_{\mathbb{Z}_p}(\mathscr{A}_{\delta,\eta}(\left[\begin{smallmatrix} 1 & 0 \\ 0 & p \end{smallmatrix}\right] w\mu)) = \delta(p)\mathrm{Res}_{\mathbb{Z}_p}(\mathscr{A}_{\delta,\eta}(\left[\begin{smallmatrix} p^{-1} & 0 \\ 0 & 1 \end{smallmatrix}\right] w\mu)) \\ &= \delta(p)\psi(\mathrm{Res}_{\mathbb{Z}_p}((\mathscr{A}_{\delta,\eta}(w\mu)))) = \delta(p)\psi(\mathrm{Res}_{\mathbb{Z}_p}(w(\mathscr{A}_{\delta,\eta}(\mu)))). \end{aligned} \tag{5.7}$$

This proves (4).

For case (5), first note that

$$\int f(x) \left[\begin{smallmatrix} 1 & b' \\ 0 & 1 \end{smallmatrix}\right] \mu = \int \left[\begin{smallmatrix} 1 & -b' \\ 0 & 1 \end{smallmatrix}\right] f(x)\mu = \int f(x+b')\mu.$$

This implies

$$\mathscr{A}(\left[\begin{smallmatrix} 1 & b' \\ 0 & 1 \end{smallmatrix}\right] \mu|_U) = \left[\begin{smallmatrix} 1 & b' \\ 0 & 1 \end{smallmatrix}\right] \mathscr{A}(\mu|_U) \tag{5.8}$$

for any $b' \in U \subseteq \mathbb{Z}_p$. Hence $\mathrm{Res}_{\mathbb{Z}_p}(\mathscr{A}_{\delta,\eta}(\left[\begin{smallmatrix} 1 & b \\ 0 & 1 \end{smallmatrix}\right] \mu)) = \left[\begin{smallmatrix} 1 & b \\ 0 & 1 \end{smallmatrix}\right] \mathrm{Res}_{\mathbb{Z}_p}(\mathscr{A}_{\delta,\eta}(\mu))$. It remains to check that

$$\mathrm{Res}_{p\mathbb{Z}_p}(\mathscr{A}_{\delta,\eta}(w\left[\begin{smallmatrix} 1 & b \\ 0 & 1 \end{smallmatrix}\right] \mu)) = u_b(\mathrm{Res}_{p\mathbb{Z}_p}(\mathscr{A}_{\delta,\eta}(w\mu))), \tag{5.9}$$

where

$$u_b = \delta(1+b) \left[\begin{smallmatrix} 1 & -1 \\ 0 & 1 \end{smallmatrix}\right] \circ w_\delta \circ \left[\begin{smallmatrix} (1+b)^{-2} & b(1+b)^{-1} \\ 0 & 1 \end{smallmatrix}\right] \circ w_\delta \circ \left[\begin{smallmatrix} 1 & 1/(1+b) \\ 0 & 1 \end{smallmatrix}\right].$$

By (5.8), Lemma 5.2 and case (2),

$$\begin{aligned} &u_b(\mathrm{Res}_{p\mathbb{Z}_p}(\mathscr{A}_{\delta,\eta}(w\mu))) \\ &\quad = \mathrm{Res}_{p\mathbb{Z}_p}(\mathscr{A}_{\delta,\eta}(\delta(1+b)\left[\begin{smallmatrix} 1 & -1 \\ 0 & 1 \end{smallmatrix}\right] w \left[\begin{smallmatrix} (1+b)^{-2} & b(1+b)^{-1} \\ 0 & 1 \end{smallmatrix}\right] w \left[\begin{smallmatrix} 1 & 1/(1+b) \\ 0 & 1 \end{smallmatrix}\right] w \left[\begin{smallmatrix} 1/(1+b) & 0 \\ 0 & 1/(1+b) \end{smallmatrix}\right] \mu)) \\ &\quad = \mathrm{Res}_{p\mathbb{Z}_p}(\mathscr{A}_{\delta,\eta}(\left[\begin{smallmatrix} 0 & 1 \\ 1 & b \end{smallmatrix}\right] \mu)) \\ &\quad = \mathrm{Res}_{p\mathbb{Z}_p}(\mathscr{A}_{\delta,\eta}(w \left[\begin{smallmatrix} 1 & b \\ 0 & 1 \end{smallmatrix}\right] \mu)). \end{aligned}$$

This proves (5.12). □

6. Determination of locally analytic vectors

6.1. $\Sigma(s)$ and Emerton's conjecture

We first recall the locally analytic representations $\Sigma(k,\mathscr{L})$ of $\mathrm{GL}_2(\mathbb{Q}_p)$ which are originally constructed by Breuil (in the case $\mathscr{L} \neq \infty$). We refer the reader to [4, 2.1] and [13, 5.1] for more details. Fix an integer $k \geq 2$. Given $\mathscr{L} \in \mathrm{P}^1(L)$, let $\sigma(\mathscr{L})$ denote the representation of $\mathrm{B}(\mathbb{Q}_p)$ on $L^2 = Le_1 \oplus Le_2$ defined by $\left[\begin{smallmatrix} a & b \\ 0 & d \end{smallmatrix}\right] e_1 = e_1$, $\left[\begin{smallmatrix} a & b \\ 0 & d \end{smallmatrix}\right] e_2 = e_1 + (\log_{\mathscr{L}} a - \log_{\mathscr{L}} d)e_2$. One thus has a non-split extension

$$0 \longrightarrow 1 \longrightarrow \sigma(\mathscr{L}) \longrightarrow 1 \longrightarrow 0. \tag{6.1}$$

We put $\sigma(k,\mathscr{L}) = \sigma(\mathscr{L}) \otimes \chi_k$ where $\chi_k : \mathrm{B}(\mathbb{Q}_p) \to L^\times$ is the character $\left[\begin{smallmatrix} a & b \\ 0 & d \end{smallmatrix}\right] \mapsto |ad|^{\frac{k-2}{2}} d^{k-2}$. Twisting (6.1) by χ_k, and then taking locally analytic parabolic induction, one obtains an exact sequence of locally analytic representations

$$0 \longrightarrow (\mathrm{Ind}_{\mathrm{B}(\mathbb{Q}_p)}^{\mathrm{GL}_2(\mathbb{Q}_p)} \chi_k)_{\mathrm{an}} \longrightarrow (\mathrm{Ind}_{\mathrm{B}(\mathbb{Q}_p)}^{\mathrm{GL}_2(\mathbb{Q}_p)} \sigma(k,\mathscr{L}))_{\mathrm{an}} \xrightarrow{s_{\mathscr{L}}} (\mathrm{Ind}_{\mathrm{B}(\mathbb{Q}_p)}^{\mathrm{GL}_2(\mathbb{Q}_p)} \chi_k)_{\mathrm{an}} \longrightarrow 0. \tag{6.2}$$

Note that $\chi_k = |x|^{\frac{k-2}{2}} \otimes x^{k-2}|x|^{\frac{k-2}{2}}$. Thus $(\mathrm{Ind}_{\mathrm{B}(\mathbb{Q}_p)}^{\mathrm{GL}_2(\mathbb{Q}_p)} \chi_k)_{\mathrm{an}} = \widetilde{\Sigma}(x^{k-1}|x|^{\frac{k}{2}}, |x|^{\frac{k-2}{2}})$ which has $(|x|^{\frac{k-2}{2}} \circ \det) \otimes \mathrm{Sym}^{k-2} L^2$ as a subrepresentation following the discussion above. We define

$$\Sigma(k,\mathscr{L}) = s_{\mathscr{L}}^{-1}((|x|^{\frac{k-2}{2}} \circ \det) \otimes \mathrm{Sym}^{k-2} L^2)/(|x|^{\frac{k-2}{2}} \circ \det) \otimes \mathrm{Sym}^{k-2} L^2. \tag{6.3}$$

One thus has an extension of locally analytic representations

$$0 \longrightarrow \Sigma(x^{k-1}|x|^{\frac{k}{2}}, |x|^{\frac{k-2}{2}}) \longrightarrow \Sigma(k,\mathscr{L}) \longrightarrow (|x|^{\frac{k-2}{2}} \circ \det) \otimes \mathrm{Sym}^{k-2} L^2 \longrightarrow 0. \tag{6.4}$$

From now on, let $s = (\delta_1, \delta_2, \mathscr{L}) \in \mathscr{S}_{\mathrm{irr}}$. We define

(6.5)

$$\Sigma(s) = \begin{cases} \Sigma(k+1,\mathscr{L}) \otimes ((\delta_2|x|^{\frac{2-k}{2}}) \circ \det) & \text{if } w(s) = k \text{ is a positive integer and } \delta_s = x^{k-1}; \\ \widetilde{\Sigma}(\delta_1,\delta_2) & \text{otherwise.} \end{cases}$$

It follows that in the first case $\Sigma(s)$ sits in the exact sequence

$$0 \longrightarrow \Sigma(\delta_1,\delta_2) \longrightarrow \Sigma(s) \longrightarrow (\delta_2 \circ \det) \otimes \mathrm{Sym}^{k-1} L^2 \longrightarrow 0. \tag{6.6}$$

Following [4, 2.2], we now give a geometric model of $\Sigma(s)$ in the first case. Let $\mathrm{LA}(\mathrm{P}^1(x^{k-1},\mathscr{L}))$ be the space of locally analytic functions H on $\mathbb{Q}_p$ with values in L such that

$$H(z) = z^{k-1}(\sum_{n=0}^{+\infty} \frac{a_n}{z^n}) + P(z)\log_{\mathscr{L}}(z), \tag{6.7}$$

for $|z| \gg 0$, where $a_n \in L$, $P(z)$ is a polynomial of degree $\leq k-1$ with coefficients in L. Let $\mathrm{GL}_2(\mathbb{Q}_p)$ act on this space by

$$([\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}]\star_s H)(z) = \delta_2(ad-bc)(-cz+a)^{k-1}\left[H\left(\frac{dz-b}{-cz+a}\right) - \frac{1}{2}P\left(\frac{dz-b}{-cz+a}\right)\log_{\mathscr{L}}\left(\frac{ad-bc}{(-cz+a)^2}\right)\right].$$

Note that $\mathrm{LA}(\mathrm{P}^1(x^{k-1}))$ is exactly the subspace consisting of functions H with $P=0$ in the expression (6.7). An easy computation shows that the L-vector subspace generated by x^i, $0 \leq i \leq k-1$, is $\mathrm{GL}_2(\mathbb{Q}_p)$-invariant. We define $C(x^{k-1},\mathscr{L})$ to be the quotient of $\mathrm{LA}(\mathrm{P}^1(x^{k-1},\mathscr{L}))$ by this subspace. It turns out that the resulting representation of $\mathrm{GL}_2(\mathbb{Q}_p)$ on $C(x^{k-1},\mathscr{L})$ is topologically isomorphic to $\Sigma(s)$, and the natural map $\mathrm{LA}(\mathrm{P}^1(x^{k-1})) \to C(x^{k-1},\mathscr{L})$ gives rise to the inclusion $\Sigma(\delta_1,\delta_2) \hookrightarrow \Sigma(s)$. We denote by $C(x^{k-1})$ the image of the map $\mathrm{LA}(\mathrm{P}^1(x^{k-1})) \to C(x^{k-1},\mathscr{L})$. Then the quotient $C(x^{k-1},\mathscr{L})/C(x^{k-1})$ is a k-dimensional L-vector space spanned by $1_{D(\infty,1)}\cdot x^n\log_{\mathscr{L}} x$, $0 \leq n \leq k-1$, which is isomorphic to $(\delta_2\circ\det)\otimes\mathrm{Sym}^{k-1}L^2$. By this geometric model, one can show that (cf. [4, Lemme 2.4.2]) $(\delta_2\circ\det)\otimes\mathrm{Sym}^{k-1}L^2\otimes\mathrm{St}$ (resp. $(\delta_2\circ\det)\otimes\mathrm{Sym}^{k-1}L^2$) is the only topologically irreducible subrepresentation (resp. quotient representation) of $\Sigma(s)$. In particular, the extension (6.6) is non-split.

Although it is known to experts that there is a natural morphism $\Sigma(s) \to \Pi(s)$ which realizes $\Pi(s)$ as the universal completion of $\Sigma(s)$, we cannot find a reference for this result. For our purpose, we rephrase the work of Breuil and Emerton in the following proposition to construct the desired morphism. We first note that the natural inclusion $\mathrm{LA}(\mathrm{P}^1(\delta_s)) \subset \mathscr{C}^{u(s)}(\mathrm{P}^1(\delta_s))$ induces a $\mathrm{GL}_2(\mathbb{Q}_p)$-equivariant continuous map

$$\iota_s : \widetilde{\Sigma}(\delta_1,\delta_2) \to \Pi(s), \quad f \mapsto \bar{f}.$$

Proposition 6.1. – *For $s \in \mathscr{S}_{\mathrm{irr}}$ non-exceptional, the $\mathrm{GL}_2(\mathbb{Q}_p)$-equivariant continuous map $\iota_s : \widetilde{\Sigma}(\delta_1,\delta_2) \to \Pi(s)$ induces an injection $\iota_s : \Sigma(\delta_1,\delta_2) \to \Pi(s)$. Moreover, in the case when $\delta_s = x^{k-1}$ for some positive integer k, the map $\iota_s : \Sigma(\delta_1,\delta_2) \to \Pi(s)$ extends naturally to an injective map $\iota_s : \Sigma(s) \to \Pi(s)$ which is continuous and $\mathrm{GL}_2(\mathbb{Q}_p)$-equivariant.*

Proof. – If $\delta_s = x^{k-1}$, since the subrepresentation $(\delta_2\circ\det)\otimes\mathrm{Sym}^{k-1}L^2$, which consists of polynomials of degree $\leq k-1$, is contained in $M(s)$, ι_s induces a map $\Sigma(\delta_1,\delta_2) \to \Pi(s)$. The injectivity of ι_s on $\Sigma(\delta_1,\delta_2)$ is proved by Emerton in [13, Lemma 6.7.2]. We rewrite his proof in our set up as below for the reader's convenience.

We first have that $\iota_s(\Sigma(\delta_1,\delta_2))$ is dense in $\Pi(s)$ since $\mathrm{LA}(\mathrm{P}^1(\delta_s))$ is dense in $\mathscr{C}^{u(s)}(\mathrm{P}^1(\delta_s))$. Hence ι_s is nonzero because $\Pi(s) \neq 0$ by Theorem 4.4. If $w(\delta_s) \notin \mathbb{N}$, $\Sigma(\delta_1,\delta_2)$ is topologically irreducible by Proposition 5.1. Thus ι_s is either injective or zero. It therefore follows that ι_s must be injective.

In case $w(\delta_s) \in \mathbb{N}$, we put $k = w(s) = w(\delta_s)+1$. We see from Proposition 5.1 that all the proper admissible subrepresentations of $\Sigma(\delta_1,\delta_2)$ are contained in the image of $\ker(I_k)$. Thus it reduces to show that ι_s is injective on the image of $\ker(I_k)$. Note that $k = w(s) > u(s)$. Therefore $\mathrm{LP}^{[0,k-1]}(\mathbb{Z}_p, L)$, the space of locally polynomial functions of degree $\leq k-1$ on $\mathbb{Z}_p$, is dense in $\mathscr{C}^{u(s)}(\mathbb{Z}_p, L)$ by the classical theorem of Amice-Vélu and Vishik. We thus deduce that $\iota_s(\ker(I_k))$ is dense in $\Pi(s)$. It follows that $\iota_s(\ker(I_k))$ is infinite dimensional because $\Pi(s)$ is infinite dimensional. If $\delta_s \neq x^{k-1}$, the only possible nontrivial quotient of $\ker(I_k)$ is the finite dimensional representation $(\delta_2|x|^{-1}\circ\det)\otimes\mathrm{Sym}^{k-1}L^2)$. Hence ι_s must be injective on $\ker(I_k)$ in this case. If $\delta_s = x^{k-1}$, note that the image of $\ker(I_k)$ in $\Sigma(\delta_1,\delta_2)$, which is isomorphic to $(\delta_2\circ\det)\otimes\mathrm{Sym}^{k-1}L^2$, is irreducible. It follows that ι_s is injective on the image of $\ker(I_k)$ as well.

Now suppose $\delta_s = x^{k-1}$. The extension of ι_s to $\Sigma(s)$ is actually due to Breuil who identifies $\Pi(s)$ with the universal unitary completion of $\Sigma(s)$ and shows that the natural map $\Sigma(s)\to\Pi(s)$ is injective as long as $\Pi(s)\neq 0$ ([4, Proposition 4.3.5], [5, Corollaire 3.3.4]). We briefly recall his construction of the natural map $\Sigma(s)\to\Pi(s)$ as below. One easily sees that Breuil's map extends ι_s. For any $0\le i\le k-1$, and

$$l_i(x)=\sum_{u\in U}\lambda_u(x-a_u)^{j_u}\log_{\mathscr{L}}(x-a_u)$$

where U is a finite set, j_u are integers between $[\frac{k+1}{2}]$ and $k-1$, $\lambda_u\in L$, $a_u\in\mathbb{Q}_p$ such that $\deg(\sum_{u\in U}\lambda_u(x-a_u)^{j_u}+x^i) < u(s)$, it follows from [5, Lemme 3.3.2] that $l_i(x)+x^i\log_{\mathscr{L}}(x)1_{D(\infty,n)}\in\mathscr{C}^{u(s)}(\mathrm{P}^1(\delta_s))$ for $n\in\mathbb{Z}$. We thus define

$$\int_{D(\infty,n)}x^i\log_{\mathscr{L}}(x)\mu(x)=\int_{\mathrm{P}^1(\mathbb{Q}_p)}(l_i(x)+x^i\log_{\mathscr{L}}(x)1_{D(\infty,n)})\mu(x)$$

for any $\mu\in\Pi(s)^*$; this is independent of the choice of $l_i(x)$ because the difference of any such two $l_i's$ lies in $M(s)'$ which is killed by μ. By this way, we extend μ to an element of $C(\delta_s,\mathscr{L})^*$. This yields a continuous $\mathrm{GL}_2(\mathbb{Q}_p)$-equivariant morphism $\Pi(s)^*\to\Sigma(s)^*$. Taking dual of this morphism, we get Breuil's map $\Sigma(s)\to\Pi(s)$. $\square$

We are now in the position to reformulate Emerton's conjecture for non-exceptional s. Note that $\iota_s(\Sigma(s))\subset\Pi(s)_{\mathrm{an}}$ since $\Sigma(s)$ is a locally analytic representation.

Conjecture 6.2. – *For $s\in\mathscr{S}_{\mathrm{irr}}$ non-exceptional, the cokernel of the inclusion $\iota_s:\Sigma(s)\to\Pi(s)_{\mathrm{an}}$ is isomorphic to $\widetilde{\Sigma}(\delta_2,\delta_1)$ as locally analytic $\mathrm{GL}_2(\mathbb{Q}_p)$-representations. Thus the space of locally analytic vectors $\Pi(s)_{\mathrm{an}}$ sits in a short exact sequence of locally analytic $\mathrm{GL}_2(\mathbb{Q}_p)$-representations*

$$0\longrightarrow\Sigma(s)\longrightarrow\Pi(s)_{\mathrm{an}}\longrightarrow\widetilde{\Sigma}(\delta_2,\delta_1)\longrightarrow 0. \tag{6.8}$$

Remark 6.3. – Emerton shows that if the above conjecture is true, then the extension (6.8) must be non-split ([13]).

In the case when $s\in\mathscr{S}_*^{\mathrm{cris}}$, there is a more explicit description of $\Pi(s)_{\mathrm{an}}$ which is due to Breuil. Recall that $D(s)$ is isomorphic to $D(s')$ for $s'=(x^{w(s)}\delta_2,x^{-w(s)}\delta_1,\mathscr{L})$. We thus obtain a morphism $\Sigma((x^{w(s)}\delta_2,x^{-w(s)}\delta_1)\to\Pi(s')_{\mathrm{an}}\cong\Pi(s)_{\mathrm{an}}$. On the other hand, if $\alpha,\beta:\mathbb{Q}_p^\times\to L^\times$ are smooth characters such that $|\alpha(p)|\le|\beta(p)|$, there is an intertwining from $\mathrm{Ind}_{\mathrm{B}(\mathbb{Q}_p)}^{\mathrm{GL}_2(\mathbb{Q}_p)}(\alpha\otimes\beta|x|^{-1})_{\mathrm{sm}}$ to $\mathrm{Ind}_{\mathrm{B}(\mathbb{Q}_p)}^{\mathrm{GL}_2(\mathbb{Q}_p)}(\beta\otimes\alpha|x|^{-1})_{\mathrm{sm}}$, yielding an intertwining from $\mathrm{Sym}^{k-1}L^2\otimes\mathrm{Ind}_{\mathrm{B}(\mathbb{Q}_p)}^{\mathrm{GL}_2(\mathbb{Q}_p)}(\alpha\otimes\beta|x|^{-1})_{\mathrm{sm}}$ to $\mathrm{Sym}^{k-1}L^2\otimes\mathrm{Ind}_{\mathrm{B}(\mathbb{Q}_p)}^{\mathrm{GL}_2(\mathbb{Q}_p)}(\beta\otimes\alpha|x|^{-1})_{\mathrm{sm}}$. It thus follows that if we set $\Sigma(\delta_1,\delta_2)_{\mathrm{lalg}}$ to be the image of $\ker I_k$ in $\Sigma(\delta_1,\delta_2)$, then there exists an intertwining between $\Sigma(\delta_1,\delta_2)_{\mathrm{lalg}}$ and $\Sigma(x^{w(s)}\delta_2,x^{-w(s)}\delta_1)_{\mathrm{lalg}}$ which is always injective (but the direction can be either way). We therefore get a morphism

$$\Sigma(\delta_1,\delta_2)\,\widetilde{\oplus}\,\Sigma(x^{w(s)}\delta_2,x^{-w(s)}\delta_1)\to\Pi(s)_{\mathrm{an}} \tag{6.9}$$

where $\widetilde{\oplus}$ denotes the amalgamated sum of two summands over the intertwining between $\Sigma(\delta_1,\delta_2)_{\mathrm{lalg}}$ and $\Sigma(x^{w(s)}\delta_2,x^{-w(s)}\delta_1)_{\mathrm{lalg}}$.

Conjecture 6.4 ([3, Conjectures 4.4.1, 5.3.7]). – *For $s\in\mathscr{S}_*^{\mathrm{cris}}$ non-exceptional, (6.9) is a topological isomorphism.*

Proposition 6.5. – *For $s\in\mathscr{S}_*^{\mathrm{cris}}$ non-exceptional, Emerton's conjecture is equivalent to Breuil's conjecture.*

Proof. – The generic case that $V(s)$ does not admit an $\mathscr{L}$-invariant (this is equivalent to $\delta_s \neq x^{w(s)-1}, x^{w(s)-1}|x|^{-2}$) is already proved in [13, 6.7.5]. We now prove the remaining cases. The injectivity of (6.9) is already ensured by [3, Corollaires 5.3.6, 5.4.3]. It reduces to show that $\Sigma(\delta_1,\delta_2)\,\widetilde{\oplus}\,\Sigma(x^{w(s)}\delta_2,x^{-w(s)}\delta_1)$ and $\Sigma(s)\oplus\widetilde{\Sigma}(\delta_2,\delta_1)$ have same constitutes. If $\delta_s = x^{w(s)-1}|x|^{-2}$, then $\Sigma(s)=\Sigma(\delta_1,\delta_2)$ and the intertwining is $\Sigma(x^{w(s)}\delta_2,x^{-w(s)}\delta_1)_{\mathrm{lalg}}\to\Sigma(\delta_1,\delta_2)_{\mathrm{lalg}}$. Therefore

$$\left(\Sigma(\delta_1,\delta_2)\,\widetilde{\oplus}\,\Sigma(x^{w(s)}\delta_2,x^{-w(s)}\delta_1)\right)/\Sigma(s)$$
$$=\Sigma(x^{w(s)}\delta_2,x^{-w(s)}\delta_1)/\Sigma(x^{w(s)}\delta_2,x^{-w(s)}\delta_1)_{\mathrm{lalg}}\cong\widetilde{\Sigma}(\delta_2,\delta_1)$$

by Proposition 5.1. If $\delta_s = x^{w(s)-1}$, the intertwining is $\Sigma(\delta_1,\delta_2)_{\mathrm{lalg}}\to\Sigma(x^{w(s)}\delta_2,x^{-w(s)}\delta_1)_{\mathrm{lalg}}$ and the quotient is isomorphic to $(\delta_2\circ\det)\otimes\mathrm{Sym}^{w(s)-1}L^2$. Thus

$$\left(\Sigma(\delta_1,\delta_2)\,\widetilde{\oplus}\,\Sigma(x^{w(s)}\delta_2,x^{-w(s)}\delta_1)\right)/\Sigma(\delta_1,\delta_2)$$

is an extension of $\Sigma(x^{w(s)}\delta_2,x^{-w(s)}\delta_1)/\Sigma(x^{w(s)}\delta_2,x^{-w(s)}\delta_1)_{\mathrm{lalg}}\cong\widetilde{\Sigma}(\delta_2,\delta_1)$ by $(\delta_2\otimes\det)\otimes\mathrm{Sym}^{w(s)-1}L^2$. We thus obtain that $\Sigma(\delta_1,\delta_2)\,\widetilde{\oplus}\,\Sigma(x^{w(s)}\delta_2,x^{-w(s)}\delta_1)$ and $\Sigma(s)\oplus\widetilde{\Sigma}(\delta_2,\delta_1)$ have same constitutes in both cases. $\square$

6.2. An exact sequence

From now on, we suppose $p>2$. Recall that for any $s=(\delta_1,\delta_2,\mathscr{L})\in\mathscr{S}_{\mathrm{irr}}$, there is an exact sequence

$$0\longrightarrow\mathscr{R}(\delta_1)\xrightarrow{\ i\ }D(s)\xrightarrow{\ j\ }\mathscr{R}(\delta_2)\longrightarrow 0.$$

For $i=1,2$, we denote $\mathscr{A}_{\delta_s,\delta_i},\mathscr{R}^+(\delta_i)\boxtimes_{\delta_s}\mathrm{P}^1,\mathscr{R}(\delta_i)\boxtimes_{\delta_s}\mathrm{P}^1$ by $\mathscr{A}_i,\mathscr{R}^+(\delta_i)\boxtimes\mathrm{P}^1,\mathscr{R}(\delta_i)\boxtimes\mathrm{P}^1$ for simplicity. Recall that $\mathscr{A}_s:\Pi(\check{s})^*\to D^\natural(s)\boxtimes\mathrm{P}^1$ is the topological $\mathrm{GL}_2(\mathbb{Q}_p)$-equivariant isomorphism given by Theorem 4.4.

Proposition 6.6. – *If s is non-exceptional, then the $\mathrm{GL}_2(\mathbb{Q}_p)$-equivariant morphism*

$$\mathscr{A}_2\circ\iota_{\check{s}}^*\circ\mathscr{A}_s^{-1}:D^\natural(s)\boxtimes\mathrm{P}^1\to\mathscr{R}^+(\delta_2)\boxtimes\mathrm{P}^1$$

satisfies $\mathscr{A}_2\circ\iota_{\check{s}}^*\circ\mathscr{A}_s^{-1}((z,z'))=(j(z),j(z'))$ for any $(z,z')\in D^\natural(s)\boxtimes\mathrm{P}^1$.

Proof. – Suppose that the images of z,z' in $\mathscr{R}(\delta_2)$ are $z_2e_2,z_2'e_2$ where e_2 is the standard basis of $\mathscr{R}(\delta_2)$ fixed in §4.2. Suppose $\mathscr{A}_s^{-1}((z,z'))=\mu$. Then by the definition of $\mathscr{A}_s$, we see that $\mathscr{A}(\mu|_{\mathbb{Z}_p})=z_2,\mathscr{A}(w\mu|_{\mathbb{Z}_p})=z_2'$. Hence $\mathscr{A}_2\circ\iota_s^*\circ\mathscr{A}_s^{-1}((z,z'))=(z_2\otimes e_2,z_2'\otimes e_2)=(j(z),j(z'))$. $\square$

Corollary 6.7. – *If s is non-exceptional, then $j(w_D(x))=w_{\delta_s}(j(x))$ for any $x\in D(s)\boxtimes\mathbb{Z}_p^\times$.*

Proof. – If $x \in (1-\varphi)D(s)^{\psi=1}$, by the proof of [12, Proposition V.2.1], there exists $z = (z_1, z_2) \in D^{\natural}(s) \boxtimes \mathrm{P}^1$ such that $\mathrm{Res}_{\mathbb{Z}_p^\times} z_1 = x$. Since j commutes with $\mathrm{Res}_{\mathbb{Z}_p^\times}$ and $(j(z_1), j(z_2)) \in \mathscr{R}^+(\delta_2) \boxtimes \mathrm{P}^1$, we get

$$w_{\delta_s}(j(x)) = w_{\delta_s}(j(\mathrm{Res}_{\mathbb{Z}_p^\times}(z_1))) = w_{\delta_s}(\mathrm{Res}_{\mathbb{Z}_p^\times}(j(z_1)))$$
$$= \mathrm{Res}_{\mathbb{Z}_p^\times}(j(z_2)) = j(\mathrm{Res}_{\mathbb{Z}_p^\times}(z_2)) = j(w_D(x)).$$

By [12, Corollaire V.1.13], $D(s) \boxtimes \mathbb{Z}_p^\times$ is generated by $(1-\varphi)D(s)^{\psi=1}$ as an $\mathscr{R}(\Gamma)$-module. We conclude the corollary from the case $x \in (1-\varphi)D(s)^{\psi=1}$ and the fact that both w_D, w_{δ_s} are $\mathscr{R}(\Gamma)$-antilinear. □

Proposition 6.8. – *The maps*

$$i_{\mathrm{P}^1} : \mathscr{R}(\delta_1) \boxtimes \mathrm{P}^1 \to D(s) \boxtimes \mathrm{P}^1, \ \ (z_1, z_2) \mapsto (i(z_1), i(z_2))$$

and

$$j_{\mathrm{P}^1} : D(s) \boxtimes \mathrm{P}^1 \to \mathscr{R}(\delta_2) \boxtimes \mathrm{P}^1, \ \ (z_1, z_2) \mapsto (j(z_1), j(z_2))$$

are well-defined morphisms of continuous $\mathrm{GL}_2(\mathbb{Q}_p)$*-representations. Moreover, we have the short exact sequence*

$$0 \longrightarrow \mathscr{R}(\delta_1) \boxtimes \mathrm{P}^1 \xrightarrow{i_{\mathrm{P}^1}} D(s) \boxtimes \mathrm{P}^1 \xrightarrow{j_{\mathrm{P}^1}} \mathscr{R}(\delta_2) \boxtimes \mathrm{P}^1 \longrightarrow 0. \tag{6.10}$$

Proof. – By Propositions 3.4, 3.9, the $\mathrm{GL}_2(\mathbb{Q}_p)$-actions on $\mathscr{R}(\delta_1) \boxtimes \mathrm{P}^1, \mathscr{R}(\delta_2) \boxtimes \mathrm{P}^1$ and $D(s) \boxtimes \mathrm{P}^1$ satisfy the same set of formulas. It thus follows that $i_{\mathrm{P}^1}, j_{\mathrm{P}^1}$ are $\mathrm{GL}_2(\mathbb{Q}_p)$-equivariant as long as they are well-defined. In general, the formula (3.1) of w_D is not convergent on $D^\dagger \boxtimes \mathbb{Z}_p^\times$. Hence the well-defineness of i_{P^1} and j_{P^1} are not obvious from their definitions.

The well-defineness of j_{P^1} follows from Corollary 6.7. To show that i_{P^1} is well-defined, we use the pairing $\{\cdot,\cdot\}$. First note that $i : \mathscr{R}(\delta_1) \to D(s)$ is dual to $j : D(\check{s}) \to \mathscr{R}(\check{\delta}_1)$ with respect to $\langle\cdot,\cdot\rangle$. It therefore follows that $i : \mathscr{R}(\delta_1) \boxtimes \mathbb{Z}_p^\times \to D(s) \boxtimes \mathbb{Z}_p^\times$ is the dual of $j : D(\check{s}) \boxtimes \mathbb{Z}_p^\times \to \mathscr{R}(\check{\delta}_1) \boxtimes \mathbb{Z}_p^\times$ with respect to $\langle\cdot,\cdot\rangle$. Hence $i : \mathscr{R}(\delta_1) \boxtimes \mathbb{Z}_p^\times \to D(s) \boxtimes \mathbb{Z}_p^\times$ is the dual of $j : D(\check{s}) \boxtimes \mathbb{Z}_p^\times \to \mathscr{R}(\check{\delta}_1) \boxtimes \mathbb{Z}_p^\times$ with respect to $\{\cdot,\cdot\}$. Since $\{\cdot,\cdot\}$ is w-invariant and we have already proved that j commutes with w on $D(\check{s}) \boxtimes \mathbb{Z}_p^\times$, we thus deduce that i commutes with w on $\mathscr{R}(\delta_1) \boxtimes \mathbb{Z}_p^\times$. Hence $i_{\mathrm{P}^1} : \mathscr{R}(\delta_1) \boxtimes \mathrm{P}^1 \to D(s) \boxtimes \mathrm{P}^1$ is well-defined.

For (6.10), the injectivity of i_{P^1} and the exactness at $D(s) \boxtimes \mathrm{P}^1$ are obvious. To show the surjectivity of j_{P^1}, for any $(z, z') \in \mathscr{R}(\delta_2) \boxtimes \mathrm{P}^1$, we pick $y \in D(s)$ and $y' \in D(s) \boxtimes p\mathbb{Z}_p$ which lift z and $\mathrm{Res}_{p\mathbb{Z}_p} z'$ respectively. Then an easy computation shows that

$$j_{\mathrm{P}^1}(y, y' + w_D(\mathrm{Res}_{\mathbb{Z}_p^\times} y)) = (z, z').$$

This proves that j_{P^1} is surjective. □

Henceforth we identify $\mathscr{R}(\delta_1) \boxtimes \mathrm{P}^1$ with a submodule of $D(s) \boxtimes \mathrm{P}^1$ via i_{P^1}.

6.3. Proof of Emerton's conjecture

We prove Conjecture 6.2 for $p > 2$ in this subsection. From now on, let $s \in \mathscr{S}_{\mathrm{irr}}$ be non-exceptional. Since $\mathscr{A}_s$ is a topological $\mathrm{GL}_2(\mathbb{Q}_p)$-equivariant isomorphism between the contragredient representation $\Pi(\check{s})^*$ and $D^\natural(\check{s}) \boxtimes \mathrm{P}^1$, it induces an isomorphism

$$\mathscr{A}_{s,\mathrm{an}} : (\Pi(\check{s})_{\mathrm{an}})^* \to ((D^\natural(s) \boxtimes \mathrm{P}^1)_{\mathrm{an}})^* = D^\natural_{\mathrm{rig}}(s) \boxtimes \mathrm{P}^1$$

between coadmissible $D(\mathrm{GL}_2(\mathbb{Z}_p))$-modules $(\Pi(\check{s})_{\mathrm{an}})^*$ and $D^\natural_{\mathrm{rig}}(s) \boxtimes \mathrm{P}^1$, where $D(\mathrm{GL}_2(\mathbb{Z}_p))$ denotes the algebra of locally analytic distributions on $\mathrm{GL}_2(\mathbb{Z}_p)$.

PROPOSITION 6.9. – *The diagram*

$$\begin{array}{ccc} (\Pi(\check{s})_{\mathrm{an}})^* & \xrightarrow{\mathscr{A}_{s,\mathrm{an}}} & D^\natural_{\mathrm{rig}}(s) \boxtimes \mathrm{P}^1 \\ \downarrow{\scriptstyle \iota_{\check{s}}^*} & & \downarrow{\scriptstyle j_{\mathrm{P}^1}} \\ \widetilde{\Sigma}(\check{\delta}_2, \check{\delta}_1)^* & \xrightarrow{\mathscr{A}_2} & \mathscr{R}(\delta_2) \boxtimes \mathrm{P}^1 \end{array} \tag{6.11}$$

is commutative. As a consequence, we have $j_{\mathrm{P}^1}(D^\natural_{\mathrm{rig}}(s) \boxtimes \mathrm{P}^1) = \mathscr{A}_2(\Sigma(\check{\delta}_2, \check{\delta}_1)^*)$.

Proof. – Recall that for an admissible Banach space representation U of $\mathrm{GL}_2(\mathbb{Q}_p)$, U_{an} is dense in U ([22, Theorem 7.1]); hence U^* is dense in U^*_{an}. The diagram (6.11) commutes on $\Pi(\check{s})^* \subset (\Pi(\check{s})_{\mathrm{an}})^*$ following Proposition 6.6. We thus conclude the commutativity of (6.11) by the density of $\Pi(\check{s})^*$ in $(\Pi(\check{s})_{\mathrm{an}})^*$. It thus follows that $j_{\mathrm{P}^1}(D^\natural_{\mathrm{rig}}(s) \boxtimes \mathrm{P}^1) = \mathscr{A}_2(\iota_{\check{s}}^*((\Pi(\check{s})_{\mathrm{an}})^*)) = \mathscr{A}_2(\Sigma(\check{\delta}_2, \check{\delta}_1)^*)$. □

LEMMA 6.10. – $\mathscr{R}^+(\check{\eta}) \boxtimes_{\delta^{-1}} \mathrm{P}^1$ *and* $\mathscr{R}^+(\eta) \boxtimes_\delta \mathrm{P}^1$ *are orthogonal complements of each other under the pairing* $\mathscr{R}(\check{\eta}) \boxtimes_{\delta^{-1}} \mathrm{P}^1 \times \mathscr{R}(\eta) \boxtimes_\delta \mathrm{P}^1 \to L$.

Proof. – It suffices to show that $\mathscr{R}^+$ is the orthogonal complements of itself under the pairing $\{\cdot,\cdot\} : \mathscr{R} \times \mathscr{R} \to L$. It is obvious that $\{\mathscr{R}^+, \mathscr{R}^+\} = 0$. On the other hand, if $f = \sum_{i\in\mathbb{Z}} a_i T^i \in (\mathscr{R}^+)^\perp$, then for any $j \in \mathbb{N}$, $\{\sigma_{-1}(T^j), f\} = a_{-j-1}$ implies $a_{-j-1} = 0$, yielding $f \in \mathscr{R}^+$. □

LEMMA 6.11. – $j_{\mathrm{P}^1}(D^\natural_{\mathrm{rig}}(s) \boxtimes \mathrm{P}^1) \subset \mathscr{R}(\delta_2) \boxtimes \mathrm{P}^1$ *and* $D^\natural_{\mathrm{rig}}(\check{s}) \boxtimes \mathrm{P}^1 \cap \mathscr{R}(\check{\delta}_2) \boxtimes \mathrm{P}^1$ *are orthogonal complements of each other under* $\{\cdot,\cdot\}_{\mathrm{P}^1} : \mathscr{R}(\delta_2) \boxtimes \mathrm{P}^1 \times \mathscr{R}(\check{\delta}_2) \boxtimes \mathrm{P}^1 \to L$.

Proof. – By the constructions of i_{P^1} and j_{P^1}, one easily checks that $i_{\mathrm{P}^1} : \mathscr{R}(\check{\delta}_2) \boxtimes \mathrm{P}^1 \to D(\check{s}) \boxtimes \mathrm{P}^1$ is dual to $j_{\mathrm{P}^1} : D(s) \boxtimes \mathrm{P}^1 \to \mathscr{R}(\delta_2) \boxtimes \mathrm{P}^1$ with respect to $\{\cdot,\cdot\}_{\mathrm{P}^1}$. Thus by Proposition 3.11, we deduce that

$$\{j_{\mathrm{P}^1}(x), y\}_{\mathrm{P}^1} = \{x, y\}_{\mathrm{P}^1} = 0$$

for any $x \in D^\natural_{\mathrm{rig}}(s) \boxtimes \mathrm{P}^1$ and $y \in D^\natural_{\mathrm{rig}}(\check{s}) \boxtimes \mathrm{P}^1 \cap \mathscr{R}(\check{\delta}_2) \boxtimes \mathrm{P}^1$. This proves $j_{\mathrm{P}^1}(D^\natural_{\mathrm{rig}}(s) \boxtimes \mathrm{P}^1) \subseteq (D^\natural_{\mathrm{rig}}(\check{s}) \boxtimes \mathrm{P}^1 \cap \mathscr{R}(\check{\delta}_2) \boxtimes \mathrm{P}^1)^\perp$.

On the other hand, since $\Sigma(\check{\delta}_1, \check{\delta}_2)$ and $\widetilde{\Sigma}(\check{\delta}_1, \check{\delta}_2)$ are admissible locally analytic representations, $\Sigma(\check{\delta}_1, \check{\delta}_2)^*$ and $\widetilde{\Sigma}(\check{\delta}_1, \check{\delta}_2)^*$ are coadmissible $D(\mathrm{GL}_2(\mathbb{Z}_p))$-modules. Therefore $\Sigma(\check{\delta}_1, \check{\delta}_2)^*$ is a closed subspace of $\widetilde{\Sigma}(\check{\delta}_1, \check{\delta}_2)^*$ by [22, Lemma 3.6]. This implies that $j_{\mathrm{P}^1}(D^\natural_{\mathrm{rig}}(\check{s}) \boxtimes \mathrm{P}^1) = \mathscr{A}_2(\Sigma(\check{\delta}_1, \check{\delta}_2)^*)$ is a closed subspace of $\mathscr{R}^+(\check{\delta}_1) \boxtimes \mathrm{P}^1$ by Proposition 6.9; hence $j_{\mathrm{P}^1}(D^\natural_{\mathrm{rig}}(\check{s}) \boxtimes \mathrm{P}^1)$ is Fréchet complete with the subspace topology of $\mathscr{R}(\check{\delta}_1) \boxtimes \mathrm{P}^1$. By the open mapping theorem for Fréchet type spaces ([19, Proposition 8.8]), we deduce that $j_{\mathrm{P}^1} : D^\natural_{\mathrm{rig}}(\check{s}) \boxtimes \mathrm{P}^1 \to j_{\mathrm{P}^1}(D^\natural_{\mathrm{rig}}(\check{s}) \boxtimes \mathrm{P}^1)$ is open. Therefore the quotient topology and the subspace topology on $j_{\mathrm{P}^1}(D^\natural_{\mathrm{rig}}(\check{s}) \boxtimes \mathrm{P}^1)$ coincide.

Now for any $x \in (D^\natural_{\mathrm{rig}}(\check{s}) \boxtimes \mathrm{P}^1 \cap \mathscr{R}(\check{\delta}_2) \boxtimes \mathrm{P}^1)^\perp \subset \mathscr{R}(\delta_2) \boxtimes \mathrm{P}^1$, we pick $\tilde{x} \in D(s) \boxtimes \mathrm{P}^1$ lifting x. The continuous linear functional $f(y) = \{\tilde{x}, y\}_{\mathrm{P}^1}$ on $D^\natural_{\mathrm{rig}}(\check{s}) \boxtimes \mathrm{P}^1$ induces a continuous linear functional $\bar{f}$ on $j_{\mathrm{P}^1}(D^\natural_{\mathrm{rig}}(\check{s}) \boxtimes \mathrm{P}^1)$. Applying Hahn-Banach theorem for Fréchet type spaces ([19, Corollary 9.4]), we extend $\bar{f}$ to a continuous linear functional on $\mathscr{R}(\check{\delta}_1) \boxtimes \mathrm{P}^1$. Since the pairing $\mathscr{R}(\check{\delta}_1) \boxtimes \mathrm{P}^1 \times \mathscr{R}(\delta_1) \boxtimes \mathrm{P}^1 \to L$ is perfect, we may suppose that the extension of $\bar{f}$ is defined by some $x' \in \mathscr{R}(\delta_1) \boxtimes \mathrm{P}^1$. It therefore follows that for any $y \in D^\natural_{\mathrm{rig}}(\check{s}) \boxtimes \mathrm{P}^1$,

$$\{\tilde{x} - x', y\}_{\mathrm{P}^1} = \{\tilde{x}, y\}_{\mathrm{P}^1} - \{x', j_{\mathrm{P}^1}(y)\}_{\mathrm{P}^1} = \bar{f}(j_{\mathrm{P}^1}(y)) - \{x', j_{\mathrm{P}^1}(y)\}_{\mathrm{P}^1} = 0,$$

yielding $\tilde{x} - x' \in D^\natural_{\mathrm{rig}}(s) \boxtimes \mathrm{P}^1$. We thus conclude that $x \in j_{\mathrm{P}^1}(D^\natural_{\mathrm{rig}}(s) \boxtimes \mathrm{P}^1)$ because $j_{\mathrm{P}^1}(\tilde{x} - x') = x$. This proves $(D^\natural_{\mathrm{rig}}(\check{s}) \boxtimes \mathrm{P}^1 \cap \mathscr{R}(\check{\delta}_2) \boxtimes \mathrm{P}^1)^\perp \subseteq j_{\mathrm{P}^1}(D^\natural_{\mathrm{rig}}(s) \boxtimes \mathrm{P}^1)$. □

Proposition 6.12. – *The following are true:*

(i) *if $\delta_s = x^{k-1}$, where k is an integer ≥ 2, then $\mathscr{R}(\delta_1) \boxtimes \mathrm{P}^1 \cap D^\natural_{\mathrm{rig}}(s) \boxtimes \mathrm{P}^1$ contains $\mathscr{R}^+(\delta_1) \boxtimes \mathrm{P}^1$ as a closed subspace of codimension k, and $j_{\mathrm{P}^1}(D^\natural_{\mathrm{rig}}(s) \boxtimes \mathrm{P}^1)$ is a closed subspace of $\mathscr{R}^+(\delta_2) \boxtimes \mathrm{P}^1$ of codimension k;*

(ii) *otherwise, $\mathscr{R}(\delta_1) \boxtimes \mathrm{P}^1 \cap D^\natural_{\mathrm{rig}}(s) \boxtimes \mathrm{P}^1 = \mathscr{R}^+(\delta_1) \boxtimes \mathrm{P}^1$ and $j_{\mathrm{P}^1}(D^\natural_{\mathrm{rig}}(s) \boxtimes \mathrm{P}^1) = \mathscr{R}^+(\delta_2) \boxtimes \mathrm{P}^1$.*

Proof. – We prove (i) only. The proof of (ii) is similar. For (i), it follows from Proposition 6.9 that $j_{\mathrm{P}^1}(D^\natural_{\mathrm{rig}}(s) \boxtimes \mathrm{P}^1) = \mathscr{A}_2(\Sigma(\check{\delta}_2, \check{\delta}_1)^*)$. Recall that $\Sigma(\check{\delta}_2, \check{\delta}_1)$ is a quotient of $\widetilde{\Sigma}(\check{\delta}_2, \check{\delta}_1)$ by a k-dimensional subrepresentation. Hence $\Sigma(\check{\delta}_2, \check{\delta}_1)^*$ is a closed subspace of $\widetilde{\Sigma}(\check{\delta}_2, \check{\delta}_1)^*$ of codimension k, yielding that $j_{\mathrm{P}^1}(D^\natural_{\mathrm{rig}}(s) \boxtimes \mathrm{P}^1)$ is a closed subspace of $\mathscr{R}^+(\delta_2) \boxtimes \mathrm{P}^1$ of codimension k.

On the other hand, as

$$\mathscr{R}(\delta_1) \boxtimes \mathrm{P}^1 \cap D^\natural_{\mathrm{rig}}(s) \boxtimes \mathrm{P}^1 = j_{\mathrm{P}^1}(D^\natural_{\mathrm{rig}}(\check{s}) \boxtimes \mathrm{P}^1)^\perp \quad \text{and} \quad \mathscr{R}^+(\delta_1) \boxtimes \mathrm{P}^1 = (\mathscr{R}^+(\check{\delta}_1) \boxtimes \mathrm{P}^1)^\perp$$

by Lemmas 6.11, 6.10, we deduce that $\mathscr{R}^+(\delta_1) \boxtimes \mathrm{P}^1$ is a codimension k closed subspace of $\mathscr{R}(\delta_1) \boxtimes \mathrm{P}^1 \cap D^\natural_{\mathrm{rig}}(s)$. □

Theorem 6.13. – *Conjecture 6.2 is true for $p > 2$.*

Proof. – By Proposition 6.12, $\mathscr{R}^+(\check{\delta}_2) \boxtimes \mathrm{P}^1$ is contained in $D^\natural_{\mathrm{rig}}(\check{s}) \boxtimes \mathrm{P}^1$. Let Σ be the locally analytic representation such that Σ^* is isomorphic to $D^\natural_{\mathrm{rig}}(\check{s}) \boxtimes \mathrm{P}^1 / \mathscr{R}^+(\check{\delta}_2) \boxtimes_{\check{\delta}} \mathrm{P}^1$. Since $\mathscr{R}^+(\check{\delta}_2) \boxtimes \mathrm{P}^1$ is isomorphic to $\widetilde{\Sigma}(\delta_2, \delta_1)^*$, we thus have an exact sequence of locally analytic L-representations of $\mathrm{GL}_2(\mathbb{Q}_p)$

$$0 \longrightarrow \Sigma \xrightarrow{\iota_1} \Pi(s)_{\mathrm{an}} \xrightarrow{\iota_2} \widetilde{\Sigma}(\delta_2, \delta_1) \longrightarrow 0. \tag{6.12}$$

If δ_s is not of the form x^{k-1} for any $k \in \mathbb{Z}_+$, then $\Sigma^* \cong \mathscr{R}^+(\check{\delta}_1) \boxtimes \mathrm{P}^1$ by Proposition 6.12(ii), which in turn is isomorphic to the dual of $\widetilde{\Sigma}(\delta_1, \delta_2)$. We thus have that Σ is isomorphic to $\widetilde{\Sigma}(\delta_1, \delta_2) = \Sigma(s)$, yielding (6.8) in this case. Now suppose $\delta_s = x^{k-1}$ for some integer $k \geq 1$. Since $\widetilde{\Sigma}(\delta_2, \delta_1)$ is topologically irreducible, and it is not isomorphic to any topological irreducible subquotients of $\Sigma(s)$ by Proposition 5.1, we deduce that $\iota_2(\iota_s(\Sigma(s))) = 0$. Hence $\iota_s(\Sigma(s)) \subseteq \iota_1(\Sigma)$. On the other hand, by Proposition 6.12 (i), we see that Σ^* is an extension of $\Sigma(\delta_1, \delta_2)^*$ by a k-dimensional L-vector space. Hence Σ contains $\Sigma(\delta_1, \delta_2)$ as a subrepresentation of codimension k. Since $\Sigma(\delta_1, \delta_2)$ is a subrepresentation of $\Sigma(s)$ of codimension k as well, we conclude that $\iota_s(\Sigma(s)) = \iota_1(\Sigma)$. $\square$

REMARK 6.14. – As a consequence of Theorem 6.13 and Proposition 6.12, we see that in case $\delta_s = x^{k-1}$ for some $k \in \mathbb{Z}_+$, the dual of the quotient $\Pi(s)_{\mathrm{an}}/\Sigma(\delta_1, \delta_2)$, which is an extension of $\widetilde{\Sigma}(\delta_2, \delta_1)$ by $(\delta_2 \circ \det) \otimes \mathrm{Sym}^{k-1} L^2$, is isomorphic to $\mathscr{R}(\check{\delta}_2) \boxtimes \mathrm{P}^1 \cap D^{\natural}_{\mathrm{rig}}(\check{s}) \boxtimes \mathrm{P}^1$.

REFERENCES

[1] L. BERGER, Représentations p-adiques et équations différentielles, *Invent. Math.* **148** (2002), 219–284.

[2] L. BERGER, Équations différentielles p-adiques et (ϕ, N)-modules filtrés, *Astérisque* **319** (2008), 13–38.

[3] L. BERGER, C. BREUIL, Sur quelques représentations potentiellement cristallines de $\mathrm{GL}_2(\mathbf{Q}_p)$, *Astérisque* **330** (2010), 155–211.

[4] C. BREUIL, Invariant $\mathscr{L}$ et série spéciale p-adique, *Ann. Sci. École Norm. Sup.* **37** (2004), 559–610.

[5] C. BREUIL, Série spéciale p-adique et cohomologie étale complétée, *Astérisque* **331** (2010), 65–115.

[6] F. CHERBONNIER, P. COLMEZ, Représentations p-adiques surconvergentes, *Invent. Math.* **133** (1998), 581–611.

[7] P. COLMEZ, Représentations triangulines de dimension 2, *Astérisque* **319** (2008), 213–258.

[8] P. COLMEZ, Fonctions d'une variable p-adique, *Astérisque* **330** (2010), 13–59.

[9] P. COLMEZ, La série principale unitaire de $\mathrm{GL}_2(\mathbf{Q}_p)$, *Astérisque* **330** (2010), 213–262.

[10] P. COLMEZ, La série principale unitaire de $\mathrm{GL}_2(\mathbf{Q}_p)$: vecteurs localement analytiques, preprint, 2010.

[11] P. COLMEZ, (ϕ, Γ)-modules et représentations du mirabolique de $\mathrm{GL}_2(\mathbf{Q}_p)$, *Astérisque* **330** (2010), 61–153.

[12] P. COLMEZ, Représentations de $\mathrm{GL}_2(\mathbf{Q}_p)$ et (ϕ, Γ)-modules, *Astérisque* **330** (2010), 281–509.

[13] M. EMERTON, A local-global compatibility conjecture in the p-adic Langlands programme for $\mathrm{GL}_{2/\mathbb{Q}}$, *Pure Appl. Math. Q.* **2** (2006), 279–393.

[14] J.-M. FONTAINE, Représentations p-adiques des corps locaux. I, in *The Grothendieck Festschrift, Vol. II*, Progr. Math. **87**, Birkhäuser, 1990, 249–309.

[15] K. S. KEDLAYA, Slope filtrations for relative Frobenius, *Astérisque* **319** (2008), 259–301.

[16] R. LIU, Locally analytic vectors of some crystabelian representations of $\mathrm{GL}_2(\mathbb{Q}_p)$, *Compositio math.* **148** (2012), 28–64.

[17] V. PAŠKŪNAS, On some crystalline representations of $\mathrm{GL}_2(\mathbb{Q}_p)$, *Algebra Number Theory* **3** (2009), 411–421.

[18] B. Perrin-Riou, Théorie d'Iwasawa des représentations p-adiques semi-stables, *Mém. Soc. Math. Fr. (N.S.)* **84** (2001).

[19] P. Schneider, *Nonarchimedean functional analysis*, Springer Monographs in Math., Springer, 2002.

[20] P. Schneider, J. Teitelbaum, $U(\mathfrak{g})$-finite locally analytic representations, *Represent. Theory* **5** (2001), 111–128.

[21] P. Schneider, J. Teitelbaum, Locally analytic distributions and p-adic representation theory, with applications to GL_2, *J. Amer. Math. Soc.* **15** (2002), 443–468.

[22] P. Schneider, J. Teitelbaum, Algebras of p-adic distributions and admissible representations, *Invent. Math.* **153** (2003), 145–196.

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