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A CONVEXITY THEOREM FOR POISSON ACTIONS OF COMPACT LIE GROUPS

BY HERMANN FLASCHKA AND TUDOR RATIU

ABSTRACT. – When a Poisson Lie group acts on a symplectic manifold, there is a momentum mapping to the dual group. We prove that the image of this momentum mapping meets the positive Weyl chamber in the exponential of a convex set. The result adapts the well-known theorem of Guillemin, Sternberg, and Kirwan to a "multiplicative" (rather than "additive") setting.

1. Introduction

Let (X, ω) be a compact symplectic manifold, and let K be a connected compact Lie group that acts on X and preserves the symplectic form ω. Assume that for every ξ in the Lie algebra $\mathfrak{k}$ of K, the vector field $\xi_X(x) = d/dt|_{t=0} \exp(t\xi) \cdot x$ comes from a Hamiltonian function, say H_ξ. Then the map $\Phi : X \to \mathfrak{k}^*$ (to the dual $\mathfrak{k}^*$ of $\mathfrak{k}$) defined by $H_\xi(x) = \langle \xi, \Phi(x) \rangle$ is called *momentum mapping* for the action of K. If the momentum mapping is equivariant relative to the given action of K on X and the coadjoint action of K on $\mathfrak{k}^*$, the action is called *Hamiltonian*. A fundamental theorem of Guillemin-Sternberg [GS1,GS2] and Kirwan [Ki] asserts that if T is a maximal torus of K and $\mathfrak{t}^*_+$ is a positive Weyl chamber, then $\Phi(X) \cap \mathfrak{t}^*_+$ is a convex polytope.

We will prove a "multiplicative" version of this result in the setting of a Poisson action of a connected compact *semisimple* group K, equipped with a Poisson Lie structure, on a compact connected symplectic manifold X. There is again a momentum mapping which we call J, this time from X to the *dual group* K^{d} of K. There is always a diffeomorphism between K^{d} and the (-1)-eigenspace of the Cartan involution, which we call $\mathfrak{p}$ (it is the set of Hermitean matrices when $K = SU(n)$). The image under the momentum map to $\mathfrak{p}$ intersects the positive Weyl chamber $\mathfrak{a}_+$ (diagonal matrices with non-increasing entries) in a convex polytope.

We conjectured this theorem after reading about convexity properties of singular values of matrices in [MO], and will begin by describing two interesting (and nontrivial) examples in this area.

EXAMPLE 1.1. – Estimating the eigenvalues of the sum of two Hermitean matrices is a basic problem in perturbation theory, and has been studied extensively (*see* [Ka] and [MO] for long lists of references). Let $\mathcal{O}_\lambda$ and $\mathcal{O}_\mu$ denote the sets of $n \times n$ trace-free Hermitean matrices with eigenvalues $\lambda_1 \geq \ldots \geq \lambda_n$, respectively $\mu_1 \geq \ldots \geq \mu_n$. If $x \in \mathcal{O}_\lambda$ and $y \in \mathcal{O}_\mu$, what can one say about the eigenvalues of $x + y$?

This is a question about momentum mappings. The set $X = \mathcal{O}_\lambda \times \mathcal{O}_\mu$ is a connected compact symplectic manifold, and the compact group $K = SU(n)$ acts on it by conjugation. The equivariant momentum mapping is simply $\Phi(x, y) = x + y$. If $z = x + y$ has eigenvalues $\nu_1 \geq \ldots \geq \nu_n$, then by conjugation we can find $x' \in \mathcal{O}_\lambda$, $y' \in \mathcal{O}_\mu$ such that $x' + y' = \operatorname{diag}(\nu_1, \ldots, \nu_n)$. Such a diagonal matrix lies in the positive Weyl chamber $\mathfrak{t}_+$ of the Cartan subalgebra $\mathfrak{t}$ of the Lie algebra $su(n)$, and our perturbation problem can now be phrased as follows: describe the intersection $\Phi(X) \cap \mathfrak{t}_+$. The GSK theorem asserts that it is a convex polytope (as usual, one identifies $\mathfrak{t}$ with its dual). ■

EXAMPLE 1.2. – Linear algebra is also interested in the multiplicative version of Example 1.1: the description of the *singular values* of the product of two matrices. Recall that the singular values of an $n \times n$ matrix g are the eigenvalues of $\sqrt{g^* g}$. Suppose now that g_1 and g_2 have singular values $\lambda_j > 0$ and $\mu_j > 0$, with the same ordering convention as above. What can be said about the singular values ν_j of the product $g_1 g_2$? This problem is also a question about momentum mappings. Let us first translate it into group theory.

The group $G = SL(n, \mathbb{C})$, considered as *real* Lie group $G^{\mathbb{R}}$, has the Iwasawa decomposition (in linear algebra: *Gram-Schmidt orthogonalization*) $G^{\mathbb{R}} = KAN$; here $K = SU(n)$, the compact real form of G, A = the subgroup of positive diagonal matrices, and N= the group of upper triangular matrices with ones on the diagonal. Set $B = AN$. The singular values of $g = kb \in G$ $(k \in K, b \in B)$ are determined by the upper triangular factor b, since $g^* g = b^* k^* k b = b^* b$. Moreover, when $g_1, g_2 \in G$, we can factor g_1 as $k_1 b_1$ and g_2^{-1} as $k_2^{-1} b_2^{-1}$, so that $g_1 g_2 = k_1 b_1 b_2 k_2$, and this has the same singular values as $b_1 b_2$. Describing the possible singular values of $g_1 g_2$, given the singular values of g_1 and g_2, is therefore equivalent to describing the possible singular values of $b_1 b_2$, given the singular values of b_1 and b_2. This problem admits a further reduction. Let Σ_λ denote the set of elements of B with singular values $\lambda_1 \geq \ldots \geq \lambda_n > 0$.

LEMMA 1.3. – *Let $b_1 \in \Sigma_\lambda$ and $b_2 \in \Sigma_\mu$. If $b_1 b_2$ has singular values $\nu_1 \geq \ldots \geq \nu_n > 0$, then there exist $b_3 \in \Sigma_\lambda$ and $b_4 \in \Sigma_\mu$ with $b_3 b_4 = \nu = \operatorname{diag}(\nu_1, \ldots, \nu_n)$.*

Proof. – If $b_1 b_2$ has singular values ν_j, then one can find $k_1, k_2 \in K$ such that $k_1 b_1 b_2 k_2 = \nu$. Write $b_2 k_2 = k_4 b_4$ and $b_1 k_4 = k_3 b_3$. Clearly, $b_3 \in \Sigma_\lambda$ and $b_4 \in \Sigma_\mu$, and we have $k_1 k_3 b_3 b_4 = \nu$. Because $\nu \in A$, the uniqueness of the KAN decomposition shows that $k_1 k_3$ is the identity and $b_3 b_4 = \nu$. ■

So, finally, if we set $X = \Sigma_\lambda \times \Sigma_\mu$, define $J : X \to B$ by $J(b_1, b_2) = b_1 b_2$, and let A_+ denote the set of elements of A with ordered entries, our perturbation problem becomes: describe $J(X) \cap A_+$.

It will be seen in Section 2 that X is a symplectic manifold (coming from a Poisson Lie structure on B), that K acts on it in a Poisson fashion (the action is the map $(k_2^{-1}, (b_1, b_2)) \mapsto (b_3, b_4)$ in the proof of Lemma 1.3), and that J is a momentum mapping for this action. Our theorem, as already advertised, states that $J(X) \cap A_+$ is (the exponential of) a convex polytope. ■

We now state a precise result, deferring all technical background to the next section. The Poisson Lie structures on a semisimple compact group K have been classified in [LS] (see Section 2E). Two Poisson Lie structures are basic: (a) the zero Poisson tensor, which just leads to Hamiltonian actions of K on X, and (b) the Lu-Weinstein [LuWe] Poisson tensor, which we had in mind in the singular value example above. All other Poisson Lie structures are, for our purposes, minor perturbations of these two. We will state our Poisson-Lie version of the GSK convexity theorem for the Lu-Weinstein case below; the adaptation to the general case is given in Section 5. At this stage, we only want to advertise the one observation that is perhaps the main point of our paper: the convexity theorem follows from absolutely "minimal" properties of the momentum mapping–preservation of the symplectic form by the non-abelian group plays no role at all (only the torus action counts). Theorem 4.39 will isolate the essential features.

THEOREM. – *Let K be the compact real form of a connected complex semisimple Lie group G, let $G^{\mathbb{R}}$ be the real Lie group underlying G, and let $G^{\mathbb{R}} = KAN$ be its Iwasawa decomposition. Equip K and $B = AN$ with the Lu-Weinstein Poisson Lie structures (B is the Poisson Lie group dual to K). Let X be a connected compact symplectic manifold, and suppose that K acts on it in a Poisson fashion with equivariant momentum mapping (in the sense of Lu) $J : X \to B$. Let $\mathfrak{a}_+$ be a positive Weyl chamber in the Lie algebra $\mathfrak{a}$ of A (which can be identified with $\mathfrak{t}^*$ via the imaginary part of the Killing form), and set $A_+ = \exp \mathfrak{a}_+$. Then $\log(J(X) \cap A_+)$ is a convex polytope.*

2. Background

In this section, we collect without proof information about Poisson Lie groups, their Poisson actions, and momentum mappings. Proofs of all assertions can be found in [LuWe], [LuRa], [Lu1], [Lu2]. For the Poisson structure on $B = AN$, consult [LuRa], for dressing transformations see [LuWe], and for the momentum mapping see [Lu2] or [Lu1]. To keep the exposition as concrete as possible, we first describe everything for the Lu-Weinstein Poisson tensor on K, and summarize the general case at the end of the section. Readers familiar with Poisson Lie groups should just skim this section to see our notations and conventions, and refer back to specific results as needed. The material in §2E is apparently not spelled out in print, and might be of some interest even to experts.

A. POISSON LIE GROUPS. – Let (M, π) be a Poisson manifold; recall that the bivector field π defines the Poisson bracket $\{\varphi, \psi\} = \pi(d\pi, d\psi)$, $\varphi, \psi \in C^\infty(M)$, satisfying the Jacobi identity. We denote by $\pi^\#$ the bundle map

$$\pi^\# : T^*M \to TM, \quad (\alpha, \pi^\# \beta) = \pi(\alpha, \beta).$$

DEFINITION 2.1. – *A Poisson manifold (H, π_H) is said to be a* Poisson Lie group *if H is a Lie group and the multiplication $(h_1, h_2) \mapsto h_1 h_2$ is a Poisson map from $H \times H$, equipped with the product Poisson structure, to H.*

The Poisson tensor π_H of a Poisson Lie group H must vanish at the identity element e. One can then define the *intrinsic derivative* $\epsilon : \mathfrak{h} \to \mathfrak{h} \wedge \mathfrak{h}$ by $\epsilon(\xi) = (\mathcal{L}_V \pi)(e)$, where V is any vector field with $V_e = \xi$. The dual map $\epsilon^* : \mathfrak{h}^* \wedge \mathfrak{h}^* \to \mathfrak{h}^*$ turns out to satisfy the Jacobi identity, and so makes the dual $\mathfrak{h}^*$ of the Lie algebra $\mathfrak{h}$ into a Lie algebra as well.

The corresponding connected and simply connected Lie group $H^{\rm d}$ is called the *dual group of* H. It has a unique Poisson structure $\pi_{H^{\rm d}}$ making it into a Poisson Lie group such that the intrinsic derivative, at the identity, of $\pi_{H^{\rm d}}$, is the Lie bracket on $\mathfrak{h}$. We note that ϵ is in fact a cocycle, and that it determines uniquely the Poisson Lie structures on H and $H^{\rm d}$ [LuWe]; it will not be necessary to understand the details of this correspondence.

B. POISSON LIE STRUCTURES ON K AND B. – We now turn to the pair $H, H^{\rm d}$ of most interest in the sequel. Let G denote a connected complex semisimple Lie group and let K be a compact real form of G; K is therefore connected ([He, p. 256]). If G is thought of as a real Lie group $G^{\mathbb{R}}$, it has the Iwasawa decomposition $G^{\mathbb{R}} = KAN$; we set $B = AN$ (see Example 1.2). The Lu-Weinstein Poisson Lie structure makes K into a Poisson Lie group, and B into its dual group $K^{\rm d}$. Let $\mathfrak{g}^{\mathbb{R}} = \mathfrak{k} + \mathfrak{a} + \mathfrak{n}$ be the Lie algebra decomposition corresponding to $G^{\mathbb{R}} = KAN$, and let T be the connected maximal torus of K with Lie algebra $\mathfrak{t} = i\mathfrak{a}$. Finally, we denote the symmetric space B^*B by P, and we write $\mathfrak{p} = \log P$; the "symmetrization" map $b \mapsto b^*b$ is called Sym, and the map $b \mapsto \log(b^*b)$ is called sym.

NOTATION 2.2. – The symbols R and L denote right and left translation in G. The projections from G to its factors K and B are denoted by ρ_K, ρ_B, and the corresponding Lie algebra projections from $\mathfrak{g} = \mathfrak{k} + \mathfrak{b}$ to $\mathfrak{k}$ and $\mathfrak{b}$ by $\rho_{\mathfrak{k}}, \rho_{\mathfrak{b}}$. Ad and ad denote, respectively, the adjoint actions of G on $\mathfrak{g}$ and of $\mathfrak{g}$ on itself. ■

Let κ denote the Killing form on $\mathfrak{g}$. Its imaginary part $\operatorname{Im}\kappa$ is a nondegenerate invariant symmetric bilinear form on $\mathfrak{g}^{\mathbb{R}}$. One has $\operatorname{Im}\kappa(\mathfrak{k}, \mathfrak{k}) = \operatorname{Im}\kappa(\mathfrak{b}, \mathfrak{b}) = 0$, so that $\mathfrak{k}$ and $\mathfrak{b}$ are dual to each other with respect to $\operatorname{Im}\kappa$. We shall sometimes identify $\mathfrak{k}$ with $\mathfrak{b}^* = T_e^*B$, and we denote this pairing between $\mathfrak{b}$ and $\mathfrak{k} \cong \mathfrak{b}^*$ by $\langle \cdot, \cdot \rangle$. At other times, it will be convenient to identify $\mathfrak{k}^*$ with $\mathfrak{p}$, again via the pairing afforded by $\operatorname{Im}\kappa$.

The Poisson structures on K and B are conveniently described, for our purpose, by defining them at the identity and extending by right translation.

PROPOSITION 2.3. – *The bivector fields π_K and π_B given by*

$$\pi_K(k)(T_k^*R_{k^{-1}}Y_1, T_k^*R_{k^{-1}}Y_2) = -\langle \rho_{\mathfrak{k}}(\operatorname{Ad}_{k^{-1}} Y_1), \rho_{\mathfrak{b}}(\operatorname{Ad}_{k^{-1}} Y_2)\rangle \tag{2.4}$$

for $Y_1, Y_2 \in \mathfrak{b} \cong \mathfrak{k}^*$, and

$$\pi_B(b)(T_b^*R_{b^{-1}}\xi_1, T_b^*R_{b^{-1}}\xi_2) = \langle \rho_{\mathfrak{k}}(\operatorname{Ad}_{b^{-1}} \xi_1), \rho_{\mathfrak{b}}(\operatorname{Ad}_{b^{-1}} \xi_2)\rangle \tag{2.5}$$

for $\xi_1, \xi_2 \in \mathfrak{k} \cong \mathfrak{b}^*$, make K and B into dual Poisson Lie groups.

COROLLARY 2.6. – *The Poisson tensor π_K vanishes on T.*

C. DRESSING ACTIONS. – A left (resp. right) action Ξ of the Poisson Lie group (H, π_H) on a Poisson manifold (X, π_X) is said to be a *Poisson action* if $\Xi : H \times X \to X$ (resp. $\Xi : X \times H \to X$) is a Poisson map. There are always two infinitesimal Poisson actions of H on its dual group $H^{\rm d}$:

$$\begin{aligned} \xi \in (\mathfrak{h}^*)^* &\mapsto \quad \pi_{H^{\rm d}}(\cdot, \xi^\ell) \\ \xi \in (\mathfrak{h}^*)^* &\mapsto -\pi_{H^{\rm d}}(\cdot, \xi^r), \end{aligned}$$

where ξ^ℓ and ξ^r are the left and right invariant one-forms on H^{d} whose values at the identity are ξ. In the Lu-Weinstein setting, the corresponding global Lie group actions are easy to describe.

PROPOSITION 2.7. – *The K-actions on B defined by*

$$\mathrm{D}^\ell(k)(b) = \rho_B(bk^{-1}) \tag{2.8}$$

respectively

$$\mathrm{D}^r(k)(b) = \rho_B(b^{-1}k)^{-1} \tag{2.9}$$

are Poisson actions, called the left *and* right dressing actions. *The orbits of these actions are the symplectic leaves in the Poisson manifold (B, π_B). The infinitesimal actions are given by*

$$\mathrm{d}^\ell(\xi)(b) = -T_e R_b \rho_{\mathfrak{b}}(\mathrm{Ad}_b\,\xi), \quad \mathrm{d}^r(\xi)(b) = T_e L_b \rho_{\mathfrak{b}}(\mathrm{Ad}_{b^{-1}}\,\xi). \tag{2.10}$$

EXAMPLE 2.11. – Example 1.2 fits into this scheme. The iso-singular-value sets $\Sigma_\lambda, \Sigma_\mu$ are just two orbits of the dressing action of K on B: for example, Σ_λ may be identified with the set of all $b^*b \in P$ with fixed spectrum, and this is an orbit of K on P—a dressing orbit, as explained in [LuRa]. Hence $X = \Sigma_\lambda \times \Sigma_\mu \subset B \times B$ is a symplectic leaf in $B \times B$.

It should be noted, however, that the Poisson action of K on X is not the diagonal one. Of course, $(k, (b_1, b_2)) \mapsto (\mathrm{D}^\ell(k)(b_1), \mathrm{D}^\ell(k)(b_2))$ is an action of K on X, but it is not Poisson. Some additional "twisting" is needed, and it is remarkable that the correct action arises completely naturally in the linear algebra problem of Example 1.2. Define (S stands for "singular value")

$$\begin{aligned} S(k)(b_1, b_2) &= (\rho_B(b_1\rho_K(b_2k^{-1})), \rho_B(b_2k^{-1})) \\ &= (\mathrm{D}^\ell(\rho_K(b_2k^{-1})^{-1})(b_1), \mathrm{D}^\ell(k)(b_2)). \end{aligned} \tag{2.12}$$

A short calculation (along the lines of Lemma 1.3) shows that this is indeed a (left) action of K on X, and one can check that it is Poisson. It then follows from (2.10) that the infinitesimal action s is given by

$$s(\xi)(b_1, b_2) = (\mathrm{d}^\ell(\rho_{\mathfrak{k}}(\mathrm{Ad}_{b_2}\,\xi))(b_1), \mathrm{d}^\ell(\xi)(b_2)). \tag{2.13}$$

These actions on a product space are perhaps best understood from the point of view of momentum mappings; we return to this example in the next subsection. ■

D. MOMENTUM MAPPINGS. – The notion of momentum mapping was generalized from the symplectic case to the Poisson case by Lu [Lu1, Lu2]. We recall the definition and some important properties in the Lu-Weinstein setting.

DEFINITION 2.14. – *Let the Poisson Lie group K act on a Poisson manifold (X, π_X) in a Poisson fashion, and for $\xi \in \mathfrak{k}$, let ξ_X be the induced vector field $d/dt|_{t=0}\exp(t\xi)\cdot x$. Via the identification $\mathfrak{k} \cong \mathfrak{b}^* = T_e^*B$, every $\xi \in \mathfrak{k}$ defines a left-invariant one-form ξ^ℓ on B. We say that $J : X \to B$ is a* momentum mapping *for the action of K if*

$$\xi_X = \pi_X^\#(J^*(\xi^\ell)) = \pi_X(\cdot, J^*(\xi^\ell)). \tag{2.15}$$

PROPOSITION 2.16. – *J is a Poisson map from (X, π_X) to (B, π_B) if, and only if, it is equivariant:*

$$J(k \cdot x) = \mathrm{D}^{\ell}(k)(J(x)). \tag{2.17}$$

Conversely, if the Poisson Lie group K acts on the Poisson manifold (X, π_X) and if there is a Poisson mapping $J : X \to B$ satisfying (2.15), then the action of K is Poisson (and J is its momentum mapping).

EXAMPLE 2.18. – The action S in Example 2.11 has a momentum mapping, namely $J(b_1, b_2) = b_1 b_2$ (this will follow from the next lemma). Because J is just the multiplication in the Poisson Lie group B, it is a Poisson map by definition. Hence, it should be equivariant: that is precisely what makes Lemma 1.3 work. ∎

Example 2.18 is a special case of a general fact that completes the Poisson-geometric setting for Example 1.2.

LEMMA 2.19. – *Let (P_1, π_1) and (P_2, π_2) be two Poisson manifolds on which K acts, and suppose that there are equivariant momentum mappings $J_i : P_i \to B$, $i = 1, 2$. For $\xi \in \mathfrak{k}$, let ξ_{P_i} be the infinitesimal action on P_i. Then the action of K on the product manifold $(P, \pi) = (P_1 \times P_2, \pi_1 \times \pi_2)$ defined by*

$$k : (p_1, p_2) \mapsto \left(\rho_K(J_2(p_2)k^{-1}J_2(p_2)^{-1})^{-1} \cdot p_1, k \cdot p_2\right)$$

(or, infinitesimally, for $\xi \in \mathfrak{k}$,

$$\xi_P(p_1, p_2) = ([\rho_{\mathfrak{k}}(\mathrm{Ad}_{b_2} \xi)]_{P_1}(p_1), \xi_{P_2}(p_2)), \quad b_2 = J_2(p_2)),$$

is Poisson and has momentum map $J(p_1, p_2) = J_1(p_1) J_2(p_2)$.
The proof of the lemma is a lengthy but straightforward verification.

E. THE GENERAL POISSON LIE STRUCTURE ON A COMPACT SEMISIMPLE GROUP. – So far, we have seen two Poisson Lie structures on a compact semisimple Lie group K: the trivial one ($\pi_K = 0$) and the Lu-Weinstein structure. All others have been classified in [LS], and for our purposes, they are slight perturbations of these two basic ones. We now summarize the general setting, most of which will only be needed in Section 5.

As is explained after Definition 2.1, the Poisson tensor on K is determined by a cocycle $\epsilon : \mathfrak{k} \to \mathfrak{k} \wedge \mathfrak{k}$. Let ϵ_0 be the cocycle defining the Lu-Weinstein Poisson tensor used so far. We consistently identify $\mathfrak{k}^*$ with $\mathfrak{p}$ via the pairing $\mathrm{Im}\, \kappa$ (see Section 2B). Let $\mathfrak{a}^\perp \cap \mathfrak{p}$ be the orthogonal complement, with respect to the Killing form κ (*not* $\mathrm{Im}\, \kappa$), of $\mathfrak{a}$ in $\mathfrak{p}$.

THEOREM 2.20. – *[LS] Up to Poisson isomorphism, the Poisson Lie structures on a simple compact Lie group K are given by*

$$\epsilon = a\epsilon_0 + u, \quad a \in \mathbb{R}, \ u \in \bigwedge^2 \mathfrak{t}.$$

(Here we think of u as a constant map, sending $\mathfrak{k}$ to an element of $\mathfrak{k} \wedge \mathfrak{k}$ by extending it to be zero on $\mathfrak{a}^\perp \cap \mathfrak{p}$).

Remark 2.21. – When K is a product of simple factors, one would have different a_i and u_i for each simple component. We ignore this trivial modification for simplicity of notation. ■

Remark 2.22. – When K acts symplectically, there is no preferred maximal torus. Nontrivial Poisson Lie tensors, however, cannot even be defined until a maximal torus and a positive Weyl chamber (=positive roots of $\mathfrak{g}$) are chosen. A nonzero u in Theorem 2.20 evidently requires a choice of maximal torus. A nonzero a does also: Proposition 2.3 involves a distinguished KAN factorization. ■

To be able to compute with these Poisson Lie structures, [LS] use *Manin triples* (*see*, for example, [LuWe]). This is a very general notion, and in complete generality, there would be provisos of various sorts. We state the definitions and results only for the case we need: when one of the subalgebras is a simple compact Lie algebra $\mathfrak{k}$ (according to Remark 2.21, this really covers the case of semisimple compact $\mathfrak{k}$ as well).

DEFINITION-THEOREM 2.23. – *Let $\mathfrak{k}$ be a simple compact Lie algebra with connected Lie group K. A* Manin triple *consists of a real Lie algebra $\mathfrak{g}$, two subalgebras $\mathfrak{k}$ and $\mathfrak{k}^{\mathrm{d}}$ of $\mathfrak{g}$, and a symmetric* ad*-invariant pairing $\langle\cdot,\cdot\rangle$ on $\mathfrak{g}$ such that (a) $\mathfrak{g}\cong\mathfrak{k}\oplus\mathfrak{k}^{\mathrm{d}}$ as vector spaces; (b) $\langle\mathfrak{k},\mathfrak{k}\rangle=\langle\mathfrak{k}^{\mathrm{d}},\mathfrak{k}^{\mathrm{d}}\rangle=\{0\}$. To a Manin triple, there corresponds a real Lie group G, Lie subgroups K and K^{d} such that $G=KK^{\mathrm{d}}$ as groups and $G\cong K\times K^{\mathrm{d}}$ as manifolds, and Poisson tensors π_K and $\pi_{K^{\mathrm{d}}}$ turning K and K^{d} into dual Poisson Lie groups. These Poisson tensors are given by analogs of equations (2.4) and (2.5):*

$$\pi_K(k)(T_k^*R_{k^{-1}}Y_1,T_k^*R_{k^{-1}}Y_2)=-\langle\rho_{\mathfrak{k}}(\mathrm{Ad}_{k^{-1}}Y_1),\rho_{\mathfrak{k}^{\mathrm{d}}}(\mathrm{Ad}_{k^{-1}}Y_2)\rangle \tag{2.24}$$

for $Y_1,Y_2\in\mathfrak{k}^{\mathrm{d}}\cong\mathfrak{k}^*$, and

$$\pi_{K^{\mathrm{d}}}(b)(T_b^*R_{b^{-1}}\xi_1,T_b^*R_{b^{-1}}\xi_2)=\langle\rho_{\mathfrak{k}}(\mathrm{Ad}_{b^{-1}}\xi_1),\rho_{\mathfrak{k}^{\mathrm{d}}}(\mathrm{Ad}_{b^{-1}}\xi_2)\rangle \tag{2.25}$$

for $\xi_1,\xi_2\in\mathfrak{k}\cong\mathfrak{k}^{\mathrm{d}*}$. In (2.24) and (2.25), $\rho_{\mathfrak{k}}$ and $\rho_{\mathfrak{k}^{\mathrm{d}}}$ denote the projections governed by the decomposition $\mathfrak{g}\cong\mathfrak{k}\oplus\mathfrak{k}^{\mathrm{d}}$. Furthermore, the left dressing action of K on K^{d} (analog of (2.8)) is given by

$$\mathrm{D}^{\ell}(k)(b)=\rho_{K^{\mathrm{d}}}(bk^{-1}),\quad k\in K, b\in K^{\mathrm{d}}, \tag{2.26}$$

where $\rho_{K^{\mathrm{d}}}$ is the K^{d} factor in the decomposition $G=KK^{\mathrm{d}}$. The momentum mapping $J:X\to K^{\mathrm{d}}$ for a Poisson action of K (equipped with any one of the Poisson tensors (2.23)) on a Poisson manifold X is again characterized by the property (2.15).

Remark. – In the examples of interest in this paper, the special feature is that the factorization $G=KK^{\mathrm{d}}$ will always be global. ■

For later use, we now record the Manin triples and corresponding Lie groups for the cocycles listed in Theorem 2.20. This again comes directly from [LS].

When $\mathfrak{k}^*$ is identified with $\mathfrak{p}$ via $\mathrm{Im}\,\kappa$, the coadjoint action of K on $\mathfrak{k}^*$, $(k,\alpha)\in K\times\mathfrak{k}^*\mapsto\mathrm{Ad}^*_{k^{-1}}\alpha\in\mathfrak{k}^*$ and its linearization, $(\xi,\alpha)\in\mathfrak{k}\times\mathfrak{k}^*\mapsto-\mathrm{ad}^*_{\xi}(\alpha)\in\mathfrak{k}^*$, the coadjoint action of $\mathfrak{k}$ on $\mathfrak{k}^*$, become the usual adjoint actions of K and $\mathfrak{k}$ on $\mathfrak{p}$ respectively. We will also need the semidirect product $K\ltimes\mathfrak{p}$, in which the group multiplication is given by

$$(k_1,U_1)(k_2,U_2)=(k_1k_2,U_1+\mathrm{Ad}_{k_1}(U_2)). \tag{2.27}$$

Its Lie algebra is the semidirect product $\mathfrak{k} \ltimes \mathfrak{p}$, with bracket operation

$$[(\xi_1, U_1), (\xi_2, U_2)] = ([\xi_1, \xi_2], [\xi_1, U_2] - [\xi_2, U_1]). \tag{2.28}$$

Finally, for $u \in \bigwedge^2 \mathfrak{t}$, that is, $u : \mathfrak{a} \times \mathfrak{a} \to \mathbb{R}$ is a bilinear skew-symmetric map, we denote by $\hat{u} : \mathfrak{a} \to \mathfrak{t}$ the map induced by u: $\operatorname{Im} \kappa(\hat{u}(\alpha), \beta) = u(\alpha, \beta)$, for all $\alpha, \beta \in \mathfrak{a}$.

The Poisson Lie structures in Theorem 2.20, parametrized by $a \in \mathbb{R}$ and $u \in \bigwedge^2 \mathfrak{t}$, are all distinct (no Poisson isomorphism takes one to another). For the convexity theorem, however, all that matters is whether a is zero or nonzero, and whether u is zero or nonzero. Thus, there are just four cases of interest:

(2.29 *a*) $a = 0$, $u = 0$. Here, K acts symplectically on X.
(2.29 *b*) $a = 1$, $u = 0$. This is the Lu-Weinstein structure discussed at length above.
(2.29 *c*) $a = 1, u \neq 0$. This is a (minor, for us) perturbation of (2.29 *b*).
(2.29 *d*) $a = 0, u \neq 0$. This is a (minor, for us) perturbation of (2.29 *a*).

First we consider the case $a = 1$, i.e. (2.29 *b*, *c*). Set $\mathfrak{h}^u = \{\hat{u}(\alpha) + \alpha \mid \alpha \in \mathfrak{a}\}$ and set $\mathfrak{k}^{\mathrm{d}} = \mathfrak{h}^u + \mathfrak{n}$, and note that $(\mathfrak{g}^{\mathbb{R}}, \mathfrak{k}, \mathfrak{k}^{\mathrm{d}})$ is a Manin triple relative to the pairing $\operatorname{Im} \kappa$. If H^u is the Lie group corresponding to $\mathfrak{h}^u$, then $K^{\mathrm{d}} = H^u N$. The Poisson tensors on K and K^{d} are defined by (2.24) and (2.25). When $u = 0$, we recover the Lu-Weinstein Poisson structures (2.4), (2.5); when $u \neq 0$, we obtain a "twisting" of that structure. The dressing action of K on K^{d} is defined in (2.26). Note that we have an obvious factorization $G^{\mathbb{R}} = KK^{\mathrm{d}}$, namely,

$$g = k \exp(\alpha) n = (k \exp(-\hat{u}(\alpha)))(\exp(\hat{u}(\alpha)) \exp(\alpha))\, n \in KH^u N. \tag{2.30}$$

The diffeomorphism $hn \mapsto \log((hn)^*(hn)) \in \mathfrak{p}$, for $hn \in H^u N = K^{\mathrm{d}}$, will again be denoted by sym (see §2.B); exactly as in [LuRa], it takes the dressing action of K on K^{d} to conjugation by K on $\mathfrak{p}$.

We now introduce the first of two definitions that replace the momentum mapping $J : X \to K^{\mathrm{d}}$ by an equivalent map $j : X \to \mathfrak{p} \cong \mathfrak{k}^*$, which is the more familiar setting for momentum mappings and convexity theorems.

DEFINITION 2.31. – *Let K be equipped with one of the Poisson structures (2.29 b, c), and suppose it has a Poisson action on a symplectic manifold X, with equivariant momentum mapping $J : X \to K^{\mathrm{d}}$. Write $j = \operatorname{sym} \circ J : X \to \mathfrak{p}$. This map will be K-equivariant (relative to the adjoint action of K on $\mathfrak{p}$).*

Second, we consider the case $a = 0$. Recall that $\mathfrak{a}^\perp \cap \mathfrak{p}$ is the Killing form orthogonal of $\mathfrak{a}$ in $\mathfrak{p}$. Define $\mathfrak{h}^u$ as in the preceding paragraph let and $\mathfrak{k}^{\mathrm{d}} = \mathfrak{h}^u \oplus (\mathfrak{a}^\perp \cap \mathfrak{p})$ considered as a subalgebra of the semidirect product $\mathfrak{k} \ltimes \mathfrak{p}$. Then one sees that $(\mathfrak{k} \ltimes \mathfrak{p}, \mathfrak{k}, \mathfrak{k}^{\mathrm{d}})$ is a Manin triple; the pairing on the underlying vector space $\mathfrak{g}^{\mathbb{R}} = \mathfrak{k} \oplus \mathfrak{p}$ is still given by $\operatorname{Im} \kappa$. The dual group K^{d} is a connected and simply connected subgroup of $K \ltimes \mathfrak{p}$; it may be thought of as the set of pairs

$$K^{\mathrm{d}} = \{(\exp \hat{u}(\alpha), \alpha + v) \mid \alpha \in \mathfrak{a}, v \in \mathfrak{a}^\perp \cap \mathfrak{p}\}. \tag{2.32}$$

When $u = 0$, the Poisson-Lie structure on K is identically zero, the dual group is $\mathfrak{k}^*$ viewed as an abelian group under addition, and its Poisson structure is defined by the usual

Lie-Poisson bracket. The dressing action is the usual coadjoint action [LuRa]. A Poisson action of K on a symplectic manifold X is then Hamiltonian (so this special case will ultimately yield the GSK convexity theorem for Hamiltonian actions of K).

LEMMA 2.33. – (a) $K \ltimes \mathfrak{p} = KK^{\mathrm{d}}$, that is, there is a unique factorization of $(k, U) \in K \ltimes \mathfrak{p}$ given by

$$(k, U) = (k_1, 0) \cdot (\exp \hat{u}(\beta), \beta + w), \tag{2.34}$$

where

$$\begin{aligned} \beta &= \rho_0(\mathrm{Ad}_{k^{-1}} U) \in \mathfrak{a}, \\ k_1 &= k \exp(-\hat{u}(\beta)) \in K, \\ w &= \mathrm{Ad}_{k_1^{-1}} U - \beta \in \mathfrak{a}^{\perp} \cap \mathfrak{p} \end{aligned} \tag{2.35}$$

and $\rho_0 : \mathfrak{p} \to \mathfrak{a}$ is the projection according to the direct sum decomposition $\mathfrak{p} = \mathfrak{a} \oplus (\mathfrak{a}^{\perp} \cap \mathfrak{p})$. In addition, this algebraic factorization defines a diffeomorphism of $K \ltimes \mathfrak{p}$ with the product manifold $K \times K^{\mathrm{d}}$. (b) The dressing action $\mathrm{D}^{\ell} : K \times K^{\mathrm{d}} \to K^{\mathrm{d}}$ is given by

$$\mathrm{D}^{\ell}(k)(\exp \hat{u}(\alpha), \alpha + v) = (\exp \hat{u}(\beta), \mathrm{Ad}_{k_1^{-1}}(\alpha + v)), \tag{2.36}$$

where

$$\beta = \rho_0(\mathrm{Ad}_{k \exp(\hat{u}(-\alpha))}(\alpha + v)) \in \mathfrak{a}, \quad \text{and} \quad k_1 = \exp(\hat{u}(\alpha)) k^{-1} \exp(-\hat{u}(\beta)) \in K. \tag{2.37}$$

Proof. – For formula (2.36) to make sense it is necessary that $\rho_0(\mathrm{Ad}_{k_1^{-1}}(\alpha + v)) = \beta$. The proof of this identity and of (a) is a straightforward computation; the key identity on $\mathfrak{p}$ needed is: $\rho_0 \circ \mathrm{Ad}_t = \rho_0$ for all $t \in T$ (on $\mathfrak{a}$, both sides of the equality are the identity map, and on $\mathfrak{a}^{\perp} \cap \mathfrak{p}$ both sides are zero since $\mathrm{Ad}_t \, \mathfrak{a}^{\perp} \subset \mathfrak{a}^{\perp}$). Part (b) follows from (a) and formula (2.26) for the dressing action. ∎

The analog of Definition 2.31, giving the "modified" momentum mapping $j : X \to \mathfrak{p} \cong \mathfrak{k}^*$, is as follows.

DEFINITION 2.38. – *Let K be equipped with one of the Poisson structures (2.29a,d), and suppose it has a Poisson action on a symplectic manifold X, with equivariant momentum mapping $J : X \to K^{\mathrm{d}}$. Define*

$$\psi : K^{\mathrm{d}} \to \mathfrak{p}, \quad \psi : (\exp(\hat{u}(\alpha)), \alpha + v) \mapsto \mathrm{Ad}_{\exp(\hat{u}(-\alpha))}(\alpha + v),$$

and set $j = \psi \circ J : X \to \mathfrak{p}$. This map will be K-equivariant (relative to the adjoint action of K on $\mathfrak{p}$).

LEMMA 2.39. – *(a) The map ψ is a diffeomorphism. (b) With $\mathrm{D}^{\ell}(k)$ given by (2.36), one has*

$$(\psi \circ \mathrm{D}^{\ell}(k))(\exp(\hat{u}(\alpha)), \alpha + v) = \mathrm{Ad}_{k \exp(\hat{u}(-\alpha))}(\alpha + v). \tag{2.40}$$

(c) The map ψ in Definition 2.38 takes an orbit of K on K^{d} to an adjoint orbit in $\mathfrak{p}$.

Proof. – (*a*) is evident. (*b*) The formula for $\psi \circ \mathrm{D}^{\ell}(k)$ follows from (2.36) and (2.37). (*c*) Consider a dressing orbit (2.36) through an element with $v = 0$. From (2.40), we have

$$\mathrm{Ad}_{k \exp(\hat{u}(-\alpha))}(\alpha) = \mathrm{Ad}_k \, \mathrm{Ad}_{\exp(-\hat{u}(\alpha))}(\alpha) = \mathrm{Ad}_k(\alpha).$$

The image of the dressing orbit under ψ is therefore a conjugation orbit. Because ψ is a diffeomorphism, we get all conjugation orbits in this fashion. ∎

3. Momentum map for the torus action

When a compact group K acts *symplectically* on (X, ω), the properties of the momentum mapping $\Phi : X \to \mathfrak{k}^*$ are strongly influenced by the momentum mapping for the action restricted to a maximal torus. This is also true for Poisson actions of K. Throughout this section, we consider K to be equipped with the Lu-Weinstein structure (for the general case, see §5). According to Corollary 2.6, the Poisson structure on $T = \exp(i\mathfrak{a}) \subset K$ is trivial. It follows that T preserves ω, and this torus action must have a momentum mapping $\phi : X \to \mathfrak{t}^*$ in the usual symplectic sense. We now compute this torus momentum map ϕ.

Let $H : B \to A$ denote the Iwasawa projection, $H : b = an \mapsto a$, and let $\log : A \to \mathfrak{a}$ be the globally defined logarithm that maps A to its Lie algebra.

PROPOSITION 3.1. – *Suppose that $\Xi : K \times X \to X$ is a Poisson action of K on the symplectic manifold (X, ω) admitting an equivariant momentum mapping $J : X \to B$. Then its restriction $\Xi_T : T \times X \to X$ preserves ω, and has a T-equivariant momentum mapping $\phi : X \to \mathfrak{t}^*$ given, after identification of $\mathfrak{t}^*$ with $\mathfrak{a}$ via $\operatorname{Im} \kappa$, by*

$$\phi = \log \circ H \circ J.$$

COROLLARY 3.2. – *Define j as in 2.31. Then on the set $j^{-1}(\mathfrak{a})$, one has $j = 2\phi$.*

Proof of 3.1. – For $\xi \in \mathfrak{t}$, let ξ_X be the induced vector field on X. On the one hand, according to the definition of the momentum mapping J for the K-action, $\xi_X = \pi^\#(J^*(\xi^\ell))$. On the other hand, according to the definition of the momentum mapping for a symplectic torus action, ξ_X is Hamiltonian, with Hamiltonian $\langle \xi, \phi(\cdot) \rangle$. Hence, we must show that

$$d\langle \xi, \phi \rangle = J^*(\xi^\ell). \tag{3.3}$$

For every $v_x \in T_x X$, we have

$$d\langle \xi, \phi(x) \rangle (v_x) = \langle \xi, T_x\phi(v_x) \rangle = \langle T_x^*\phi(\xi), v_x \rangle.$$

Therefore, to prove (3.3), we should show that

$$(T_x^*\phi)(\xi) = T_x^* J(\xi^\ell(J(x))) \tag{3.4}$$

(on the left side of (3.4), the argument $\xi \in \mathfrak{t}$ is a one-form on the linear space $\mathfrak{a} \cong \mathfrak{t}^*$). Since

$$(T_x^*\phi)(\xi) = (T_x^* J \circ T_{J(x)}^* H \circ T_{H \circ J(x)}^* \log)(\xi),$$

we should verify that

$$(T_{J(x)}^* H \circ T_{H \circ J(x)}^* \log)(\xi) \tag{3.5}$$

is exactly the left-invariant one-form ξ^ℓ on B at the point $J(x)$.

This is now a lengthy but straightforward calculation, as is the verification that ϕ is T-equivariant. We omit the details, but we do prove an analogous result in Proposition 5.6, for the less familiar Poisson-Lie structures on K. ■

EXAMPLE 3.6. – Proposition 3.1 is a generalization of Theorem 4.13 in [LuRa]. If K acts on a dressing orbit $\Sigma_\lambda \subset B$ with momentum mapping $J(b) = b$, then the symplectic torus action has momentum mapping $\phi(b) = \log \circ H(b)$. Application of the Atiyah-Guillemin-Sternberg convexity theorem for symplectic torus actions immediately gives Kostant's nonlinear convexity theorem mentioned in the introduction (for the case of complex flag manifolds, at least). ■

4. The convexity theorem: Lu-Weinstein case

A. DERIVATIVE OF THE MOMENTUM MAPPING. – Our first goal will be the Poisson Lie version of the following standard fact ([AMM, theorem 2], [GS1, Lemma 2.1]):

LEMMA 4.1. – *Let $G \times X \to X$ be a Hamiltonian action of the Lie group G on the symplectic manifold (X, ω) with equivariant momentum map $\Phi : X \to \mathfrak{g}^*$. Let $\mathfrak{g}_x = \{\xi \in \mathfrak{g} \mid \xi_X(x) = 0\}$, where ξ_X is the infinitesimal generator of the action defined by $\xi \in \mathfrak{g}$. Then the range of $T_x\Phi$ is the annihilator of $\mathfrak{g}_x$ in $\mathfrak{g}^*$.*

LEMMA 4.2. – *For $x \in X$ let $\mathfrak{k}_x$ be the Lie algebra of the stabilizer group of x; via the identification $\mathfrak{k} \cong (\mathfrak{k}^{\mathrm{d}})^* = T_e^* K^{\mathrm{d}}$ we think of $\mathfrak{k}_x$ as subspace of $T_e^* K^{\mathrm{d}}$. Then the image of $T_x J : T_x X \to T_{J(x)} K^{\mathrm{d}}$ is the annihilator of*

$$(T^*_{J(x)} L_{J(x)^{-1}})(\mathfrak{k}_x) \subset T^*_{J(x)} K^{\mathrm{d}}.$$

Proof. – Let $\xi \in \mathfrak{k}$. Then

$$\langle df, \xi_X \rangle (x) = 0 \tag{4.3}$$

for every $f \in C^\infty(X)$ if, and only if, $\xi \in \mathfrak{k}_x$. By (2.15), relation (4.3) becomes

$$0 = \langle df, \pi^{\#}(J^*(\xi^\ell))\rangle(x) = -\langle \pi^{\#}(df), J^*(\xi^\ell)\rangle(x).$$

Because X is symplectic, the vector fields $\pi^{\#}(df)(x)$ span $T_x X$. Hence for every $v_x \in T_x X$,

$$0 = \langle v_x, T_x^* J(\xi^\ell(J(x)) \rangle = \langle T_x J(v_x), \xi^\ell(J(x))\rangle.$$

The conclusion now follows, because

$$\xi^\ell(J(x)) = T_e^* L_{J(x)^{-1}}(\xi)$$

runs over $T^*_{J(x)} L_{J(x)^{-1}}(\mathfrak{k}_x)$ as ξ runs over $\mathfrak{k}_x$. ■

If E is a vector space and V is a subspace, we shall denote in what follows by V^{ann} its annihilator in the dual space E^*. With this notation, Lemma 4.2 can be simply stated as:

$$T_x J(T_x X) = \left[(T^*_{J(x)} L_{J(x)^{-1}})(\mathfrak{k}_x)\right]^{\mathrm{ann}}. \tag{4.4}$$

What we need for the proof of the convexity theorem, however, is the analog of Lemma 4.1 not for J, but for the map $j = \mathrm{sym} \circ J : X \to \mathfrak{p}$ (*see* Definition 2.31). The calculations that follow are really the key step in the proof of our convexity theorem; once we know their outcome, we can (after changing some language) quote freely from the literature.

For $U \in \mathfrak{p}$, introduce the series (*see* [He])

$$\mathbf{S}(U) := \frac{\sinh \operatorname{ad} U}{\operatorname{ad} U}$$

in $(\operatorname{ad} U)^2$ whose constant term is the identity. This linear map leaves $\mathfrak{p}$ and $\mathfrak{k}$ invariant and is an isomorphism on each of these two spaces. The inverse of $\mathbf{S}$ on $\mathfrak{p}$ or $\mathfrak{k}$ will be denoted by $\mathbf{S}^{-1}$.

LEMMA 4.5. – *If $b \in B$, $\beta \in \mathfrak{b}$, and $p = b^*b \in P$, the derivatives of* Sym *and* sym *are given by*

$$T_b \operatorname{Sym}(T_e L_b(\beta)) = T_e R_p(\beta^*) + T_e L_p(\beta) \in T_p P, \tag{4.6}$$

$$T_b \operatorname{sym}(T_e L_b(\beta)) = \mathbf{S}^{-1}(\log p)\big(\frac{1}{2}(\beta + \beta^* + \operatorname{Ad}_p \beta + \operatorname{Ad}_{p^{-1}} \beta^*)\big) \in T_{\log p}\mathfrak{p}. \tag{4.7}$$

Proof. – For $\beta_b \in T_b B$, we have $T_b \operatorname{Sym}(\beta_b) = T_{b^*} R_b(\beta_b^*) + T_b L_{b^*}(\beta_b)$. Take β_b to be of the form $T_e L_b(\beta)$ for $\beta \in \mathfrak{b}$, and remember that $p = b^*b$. Then the previous formula becomes (4.6).

To prove (4.7), we begin by recalling the formula for the derivative of the exponential map $\exp : \mathfrak{p} \to P$ [He]. If $U \in \mathfrak{p}$ and $V \in T_U\mathfrak{p} \cong \mathfrak{p}$, then

$$\begin{aligned} T_U \exp(V) &= T_{\exp U}\Pi \circ T_e L_{\exp U}\left(\frac{\sinh \operatorname{ad} U}{\operatorname{ad} U}(V)\right) \\ &= T_{\exp U}\Pi \circ T_e L_{\exp U} \circ \mathbf{S}(U)(V). \end{aligned} \tag{4.8}$$

In (4.8), $\Pi : G \to G/K \cong P$ is the natural projection $g \mapsto gg^* \mapsto p_1$, with $p_1^2 = gg^*$; when restricted to P, $\Pi|P$ is a diffeomorphism.

To compute the derivative of the diffeomorphism $\Pi|P$, set $W = \mathbf{S}(U)(V) \in \mathfrak{p}$ in (4.8) and consider the curve $p\exp(tW) \in P$, which is mapped by Π as follows:

$$p\exp(tW) \mapsto p(\exp(2tW))p = p_1(t)^2 \mapsto p_1(t). \tag{4.9}$$

Differentiation of (4.9) gives

$$T_e L_p(W) \mapsto 2T_p R_p \circ T_e L_p(W) = T_p L_p(v_p) + T_p R_p(v_p), \quad v_p = \dot{p}_1(0) \in T_p P. \tag{4.10}$$

Applying the inverses of $T_p R_p$ and $T_e L_p$ to (4.10), we get

$$2W = T_p L_{p^{-1}}(v_p) + T_p R_{p^{-1}}(v_p) \in \mathfrak{p} \tag{4.11}$$

and therefore the derivative of $\Pi|P$ is the following isomorphism of $T_p P$:

$$2T_p\Pi(T_e L_p W) = T_p\Pi\big(T_e L_p(T_p L_{p^{-1}}(v_p) + T_p R_{p^{-1}}(v_p))\big) = 2v_p. \tag{4.12}$$

By the inverse function theorem, for $v_p \in T_p P$, we have

$$T_p \log(v_p) = \big(T_{\log p} \exp\big)^{-1}(v_p)$$

and thus (4.8) and (4.12) give

$$T_p \log(v_p) = (T_{\log p} \exp)^{-1} \circ T_p\Pi \circ T_e L_p(W) = \mathbf{S}^{-1}(\log p)(W), \tag{4.13}$$

with W given by (4.11).

We are now in the position to compute the derivative of $\mathrm{sym} = \log \circ \mathrm{Sym}$ by using (4.6), (4.13), and (4.11). We have for $\beta \in \mathfrak{b}$ and $p = b^* b \in \mathfrak{p}$:

$$(4.14) \qquad \begin{aligned} T_b \,\mathrm{sym}(T_e L_b(\beta)) &= T_p \log \circ T_b \,\mathrm{Sym}(T_e L_b(\beta)) \\ &= T_p \log \big(T_e R_p(\beta^*) + T_e L_p(\beta)\big) = \mathbf{S}^{-1}(\log p)(W'), \end{aligned}$$

where

$$(4.15) \qquad \begin{aligned} 2W' &= T_p L_{p^{-1}}\big(T_e R_p(\beta^*) + T_e L_p(\beta)\big) + T_p R_{p^{-1}}\big(T_e R_p(\beta^*) + T_e L_p(\beta)\big) \\ &= \beta + \beta^* + \mathrm{Ad}_p \beta + \mathrm{Ad}_{p^{-1}} \beta^* \in \mathfrak{p}. \end{aligned}$$

Formulae (4.14) and (4.15) give (4.7). ■

LEMMA 4.16. – *The image of $T_x j : T_x X \to \mathfrak{p} \cong \mathfrak{k}^*$ is the annihilator of $\mathfrak{k}_x$, or*

$$\big[T_x j(T_x X)\big]^{\mathrm{ann}} = \mathfrak{k}_x.$$

Proof. – Let $A : E \to F^*$ be an isomorphism of vector spaces and $V \subset E^*$ a subspace. Then the image of $V^{\mathrm{ann}} \subset E$ under A equals

$$(4.17) \qquad A(V^{\mathrm{ann}}) = \big(A^{-1*}(V)\big)^{\mathrm{ann}} \subset F^*.$$

Since $j = \mathrm{sym} \circ J$, (4.4) and (4.17) imply:

$$(4.18) \qquad \big[T_x j(T_x X)\big]^{\mathrm{ann}} = [T_e^*(\mathrm{sym} \circ L_{J(x)})]^{-1}(\mathfrak{k}_x).$$

Since $j : X \to \mathfrak{p}$ is equivariant, we have $\mathfrak{k}_x \subset \mathfrak{k}_{j(x)}$ (here $\mathfrak{k}_{j(x)}$ is the stabilizer algebra of $j(x) = \log J(x)^* J(x)$ for the adjoint action on $\mathfrak{p}$). We will prove below that for every $b \in B$, with $p = b^* b$,

$$(4.19) \qquad T_e^*(\mathrm{sym} \circ L_b)|\mathfrak{k}_{\log p} = 2 \times \mathfrak{k}_{\log p}.$$

Then (4.18) becomes

$$\big[T_x j(T_x X)\big]^{\mathrm{ann}} = \frac{1}{2}\mathfrak{k}_x = \mathfrak{k}_x$$

and the lemma will follow.

We now prove (4.19). Several preparatory remarks are in order. First, since $\mathfrak{k}_{\log p} = \{\xi \in \mathfrak{k} \,|\, [\xi, \log p] = 0\}$, we conclude that for any $\xi \in \mathfrak{k}_{\log p}$ and any positive integer n we have $(\mathrm{ad}\, \log p)^n \xi = 0$. Therefore,

$$(4.20) \qquad \xi = \mathrm{e}^{\mathrm{ad}\, \log p} \xi = \mathrm{Ad}_{\exp\, \log p}\, \xi = \mathrm{Ad}_p\, \xi$$

for any $\xi \in \mathfrak{k}_{\log p}$. Second, since $\mathbf{S}(\log p)$ is a power series in $(\mathrm{ad}\, \log p)^2$ with constant term the identity, it stabilizes any $\xi \in \mathfrak{k}_{\log p}$, that is

$$(4.21) \qquad \mathbf{S}(\log p)|\mathfrak{k}_{\log p} = \text{identity on } \mathfrak{k}_{\log p}.$$

Third, it also follows that

$$\mathbf{S}(\log p) \text{ is } \kappa\text{-symmetric.} \tag{4.22}$$

Fourth, since $\xi \in \mathfrak{k}_{\log p} \subset \mathfrak{k} = i\mathfrak{p}$, we have $\kappa(\xi, \beta^*) = -\overline{\kappa(\xi, \beta)}$ for any $\beta \in \mathfrak{b}$ (see [He]), and in particular,

$$\operatorname{Im} \kappa(\xi, \beta^*) = \operatorname{Im} \kappa(\xi, \beta). \tag{4.23}$$

Since $T_e(\operatorname{sym} \circ L_b) : \mathfrak{b} \to \mathfrak{p}$, it follows that $T_e^*(\operatorname{sym} \circ L_b) : \mathfrak{k} \cong \mathfrak{p}^* \to \mathfrak{k} \cong \mathfrak{b}^*$, where the identifications with the duals for both $\mathfrak{b}$ and $\mathfrak{p}$ are done by $\operatorname{Im} \kappa$. Thus, for arbitrary $\xi \in \mathfrak{k}_{\log p}$ and $\beta \in \mathfrak{b}$, we get

$$\begin{aligned}
&2\operatorname{Im} \kappa(T_e^*(\operatorname{sym} \circ L_b)(\xi), \beta) \\
&\quad = 2\operatorname{Im} \kappa(\xi, T_e(\operatorname{sym} \circ L_b)(\beta)) \\
&\quad = \operatorname{Im} \kappa\big(\xi, \mathbf{S}^{-1}(\log p)(\beta + \beta^* + \operatorname{Ad}_p \beta + \operatorname{Ad}_{p^{-1}} \beta^*)\big) && \text{(by (4.4))} \\
&\quad = \operatorname{Im} \kappa\big(\mathbf{S}^{-1}(\log p)(\xi), \beta + \beta^* + \operatorname{Ad}_p \beta + \operatorname{Ad}_{p^{-1}} \beta^*\big) && \text{(by (4.22))} \\
&\quad = \operatorname{Im} \kappa\big(\xi, \beta + \beta^* + \operatorname{Ad}_p \beta + \operatorname{Ad}_{p^{-1}} \beta^*\big) && \text{(by (4.21))} \\
&\quad = \operatorname{Im} \kappa(\xi, \beta + \beta^*) + \operatorname{Im} \kappa(\operatorname{Ad}_{p^{-1}} \xi, \beta) + \operatorname{Im} \kappa(\operatorname{Ad}_p \xi, \beta^*) \\
&\quad = 2\operatorname{Im} \kappa(\xi, \beta + \beta^*) && \text{(by (4.20))} \\
&\quad = 2\operatorname{Im} \kappa(\xi, \beta) - 2\operatorname{Im} \overline{\kappa(\xi, \beta)} && \text{(by (4.23))} \\
&\quad = 4\operatorname{Im} \kappa(\xi, \beta),
\end{aligned}$$

which proves (4.19). ■

B. A SYMPLECTIC CROSS-SECTION. – We now prove the existence of a symplectic cross section for the momentum map $j : X \to \mathfrak{p}$. By equivariance of j, the image $j(X) \subset \mathfrak{p}$ consists of a union of coadjoint orbits, each of which intersects $\mathfrak{a}_+$ in a unique point. The cross-section, called Y below, will be the inverse image of a "generic" subset of $j(X) \cap \mathfrak{a}_+$. The argument follows, in various respects, [GS1], [CDM], and [HNP]. We include certain details, because we shall later need to isolate the important steps in our Theorem 4.39.

DEFINITION 4.24. – *An element $U \in \mathfrak{p}$ is called regular if its stabilizer (under the adjoint action) is a maximal torus of K (not necessarily the distinguished torus T). We say that $b \in B$ is regular if $\operatorname{sym}(b) \in \mathfrak{p}$ is regular. Let $\mathfrak{p}_{\text{reg}}$, respectively B_{reg}, be the sets of regular elements in $\mathfrak{p}$ and B. Let $m = \max\{\dim(K \cdot x) \mid x \in X\}$ and $X_{\text{reg}} = \{x \in X \mid \dim(K \cdot x) = m\}$, that is, X_{reg} consists of all principal and exceptional orbits. Define $X_{\max} = \{x \in X \mid \dim(K \cdot j(x)) \text{is maximal in } j(X)\}$. Let $\mathfrak{a}_+$ be the positive Weyl chamber in $\mathfrak{a} \cong \mathfrak{t}^*$ corresponding to the Borel group B, let $\mathfrak{a}_0$ be the interior of $\mathfrak{a}_+$, and set $A_+ = \exp \mathfrak{a}_+$ and $A_0 = \exp \mathfrak{a}_0$. Let $W^1, W^2, \ldots$ be all the closed walls of varying dimensions of the positive Weyl chamber $\mathfrak{a}_+$; there are only finitely many such walls. If P^i is the subspace of $\mathfrak{a}$ spanned by W^i, then W^i is closed in P^i and W_0^i denotes the interior of W^i in P^i. We will also use the notation $W^0 = \mathfrak{a}_+$ and $W_0^0 = \mathfrak{a}_0$; note that $P^0 = \mathfrak{a}$.*

The following is well known (*see* [Br] or [J]):

LEMMA 4.25. – *Both $X_{\max} \cap X_{\text{reg}}$ and $X_{\max}$ are K-invariant, open, dense, and connected in X.*

Consider the stratification of $\mathfrak{p}$ into submanifolds S consisting of K-orbits in $\mathfrak{p}$ of the same dimension. Connectedness of these strata, the fact that $\mathfrak{a}_+$ is a fundamental domain of this action, and lemma 4.25 immediately implies:

LEMMA 4.26. – *If $j(X) \cap \mathfrak{a}_0 = \emptyset$, then there is a unique closed wall W^i of $\mathfrak{a}_+$ such that $j(X) \cap \mathfrak{a}_+ \subset W^i$ and $j(X) \cap W_0^i \neq \emptyset$. Moreover, $K \cdot j^{-1}(W_0^i) = X_{\max}$.*

Let us now consider $j(X) \cap \mathfrak{a}_+$. There are two possibilities: either $j(X) \cap \mathfrak{a}_0 \neq \emptyset$ or $j(X) \cap \mathfrak{a}_0 = \emptyset$. In the first case, define $Y = J^{-1}(A_0) = j^{-1}(\mathfrak{a}_0)$ and note that $X_{\max} = X_{\text{reg}}$ and that $K \cdot j^{-1}(\mathfrak{a}_0) = X_{\text{reg}}$. In the second case, define $Y = j^{-1}(W_0^i)$, where W^i is given by lemma 4.26, $i \geq 1$. The purpose of the next proposition is to show as in [CDM] that Y is a symplectic cross section of the action.

PROPOSITION 4.27. – *Define $Y = j^{-1}(W_0^i) \subset X_{\max}$ where W^i is as given in Lemma 4.26. If $i = 0$, then $Y \subset X_{\text{reg}}$. Let S^i be the unique stratum (of the coadjoint action of K on $\mathfrak{p}$) containing W_0^i. We have:*

(*a*) *Y is T-invariant and connected.*

(*b*) *Every K-orbit in $X_{\max}$ intersects Y.*

(*c*) *Y is a submanifold of X.*

(*d*) *For $y \in Y$, $\ker T_y j \subset T_y Y$.*

(*e*) *Let $y \in Y$. Then Y and the orbit $K \cdot y$ are transversal at y.*

(*f*) *Y is a symplectic submanifold of X.*

(*g*) *The T-action on Y is Hamiltonian and its momentum map is*

$$(j|Y)/2 = \log \circ H \circ J : Y \to \mathfrak{a}$$

(notation as in Proposition 3.1).

Proof. – (*a*) Y is T-invariant thanks to equivariance of j, and connectedness of Y is an easy verification.

(*b*) Every coadjoint K-orbit in $\mathfrak{p}$ of maximal dimension in $j(X)$ (if $i = 0$ the set of all such elements is $\mathfrak{p}_{\text{reg}}$) intersects W_0^i, and by equivariance of j, every K-orbit in $X_{\max}$ intersects Y.

(*c*) If $\alpha \in W_0^i$, the intersection of $K \cdot \alpha$ (the coadjoint K-orbit through α) with W_0^i is transversal relative to the stratum S^i containing W_0^i and consists of the single point α. More precisely, if P^i is the subspace spanned by W^i in $\mathfrak{a}$, then $K \cdot \alpha \cap W_0^i = \{\alpha\}$ and $T_\alpha(K \cdot \alpha) + P^i = T_\alpha S^i$. A dimension count and $\operatorname{Im} \kappa(\mathfrak{k}_\alpha, T_\alpha(K \cdot \alpha)) = 0$ show that $\mathfrak{k}_\alpha^{\text{ann}} = T_\alpha(K \cdot \alpha)$. Equivariance of $j : X \to \mathfrak{p}$ implies that $\mathfrak{k}_x \subset \mathfrak{k}_{j(x)}$. Thus, when $j(x) = \alpha \in W_0^i$, Lemma 4.16 implies

$$T_\alpha S^i = T_\alpha(K \cdot \alpha) + P^i = \mathfrak{k}_\alpha^{\text{ann}} + P^i \subset \mathfrak{k}_x^{\text{ann}} + P^i = T_x j(T_x X) + P^i.$$

Because $j(X_{\max}) \subset W_0^i \subset S^i$, and $X_{\max}$ is open in X (Lemma 4.25), we conclude that $T_x j(T_x X) \subset T_\alpha S^i$. Since $P^i \subset T_\alpha S^i$ by construction, we have equality above which means that $j : X_{\max} \to S^i$ is transversal to W_0^i at x. Therefore, $Y = j^{-1}(W_0^i)$ is a submanifold of X.

(*d*) In (c), we showed that $j : X_{\max} \to S^i$ is transversal to W_0^i. By the transversality theorem, the tangent space to Y at y equals $(T_y j)^{-1}(T_{j(y)} W_0^i) = (T_y j)^{-1}(P^i)$. Because $0 \in P^i$, we have $\ker T_y j \subset T_y Y$.

NOTATION. – Suppose K acts on a manifold M (which for us may be X, Y, or $\mathfrak{p}$). For $m \in M$ and $\mathfrak{h}$ a subalgebra of $\mathfrak{k}$, let $\widetilde{\mathfrak{h}}(m)$ denote the subspace of $T_m M$ generated by the vectors $\xi_M(m) = d/dt|_{t=0} \exp \xi t \cdot m$ for $\xi \in \mathfrak{h}$.

(*e*) From the proofs of (c) and (d) we know that $K \cdot \alpha \cap W_0^i = \{\alpha\}$, that $T_\alpha(K \cdot \alpha) + P^i = \widetilde{\mathfrak{k}}(\alpha) + P^i = T_\alpha S^i$, and that $T_Y Y = (T_y j)^{-1}(P^i)$. If $y \in Y$, equivariance of j implies $T_y j(\widetilde{\mathfrak{k}}(y)) = \widetilde{\mathfrak{k}}(j(y))$ and hence $(T_y j)^{-1}(\widetilde{\mathfrak{k}}(j(y))) = \widetilde{\mathfrak{k}}(y) + \ker T_y j = T_y(K \cdot y) + \ker T_y j$. Since $j(X_{\max}) \subset S^i$ and $X_{\max}$ is open in X we get for $y \in Y$

$$\begin{aligned} T_y X = T_y X_{\max} &= (T_y j)^{-1}(T_{j(y)} S^i) = (T_y j)^{-1}(\widetilde{\mathfrak{k}}(j(y)) + (T_y j)^{-1}(P^i) \\ &= \widetilde{\mathfrak{k}}(y) + \ker T_y j + T_y Y = T_y(K \cdot y) + T_y Y, \end{aligned}$$

since by (d), $\ker T_y j \subset T_y Y$. This proves that $K \cdot y$ and Y are transversal.

(*f*) Let superscript ω denote the orthogonal complement with respect to the symplectic form ω. The goal is to show that for every $y \in Y$,

$$T_y Y \cap (T_y Y)^\omega = \{0\}. \tag{4.28}$$

The proof requires several subsidiary lemmas. The first one is a general fact about momentum mappings.

LEMMA 4.29. – *(a) Equip K with any of the Poisson Lie structures in Theorem 2.20. The kernel of $T_x J : T_x X \to T_{J(x)} K^{\mathrm{d}}$ is $\widetilde{\mathfrak{k}}(x)^\omega$. (b) Let j be as in Definition 2.31 or 2.38. The kernel of $T_x j : T_x X \to T_{j(x)} \mathfrak{p} = \mathfrak{p}$ is also $\widetilde{\mathfrak{k}}(x)^\omega$.*

Proof. – (*a*) By (2.15), for $v_x \in T_x X$ and $\xi \in \mathfrak{k}$,

$$\omega(x)(\xi_X(x), v_x) = \langle J^*(\xi^\ell)(x), v_x \rangle = \langle \xi^\ell(J(x)), T_x J(v_x) \rangle.$$

If $v_x \in \ker T_x J$ then the right side vanishes, and so $v_x \in \widetilde{\mathfrak{k}}(x)^\omega$. If, conversely, $v_x \in \widetilde{\mathfrak{k}}(x)^\omega$, then the left side vanishes for every ξ, and $T_x J(v_x) = 0$. (b) $T_x j$ has the same kernel as $T_x J$ since $j = \mathrm{sym} \circ J$ or $j = \psi \circ J$, and both $\mathrm{sym}, \psi : K^{\mathrm{d}} \to \mathfrak{p}$ are diffeomorphisms. ■

LEMMA 4.30. – *Let $x \in X$, and let $\mathfrak{h}$ be the stabilizer algebra of $j(x)$ under the adjoint action of $\mathfrak{k}$ on $\mathfrak{p}$. Then*

$$\ker(T_x j) \cap \widetilde{\mathfrak{k}}(x) = \widetilde{\mathfrak{h}}(x).$$

Proof. – Write $\eta = j(x)$. Because j is equivariant, we have $j(k \cdot x) = \mathrm{Ad}_k \eta$. Thus if $\xi \in \mathfrak{k}$, it follows that $T_x j(\xi_X(x)) = [\xi, \eta]$. Hence $\xi_X(x)$ is in $\ker(T_x j)$ if, and only if, ξ commutes with η, which means precisely that $\xi \in \mathfrak{h}$. ■

LEMMA 4.31. – *Let $y \in Y$ and let $\mathfrak{h}$ be the stabilizer algebra of $j(y)$ under the adjoint action of $\mathfrak{k}$ on $\mathfrak{p}$. Then*

$$T_y Y \cap \widetilde{\mathfrak{k}}(y) = \widetilde{\mathfrak{h}}(y).$$

If $i = 0$, the right hand side of this equality is $\widetilde{\mathfrak{t}}(y)$.

Proof. – Let $\xi \in \mathfrak{k}$. If $\xi_X(y) \in T_yY$, then $T_yj(\xi_X(y)) \in P^i$ by the proof of Proposition 4.27(c). On the other hand, as in Lemma 4.30, $T_yj(\xi_X(y)) = [\xi, j(y)]$, which by definition is tangent to the adjoint K-orbit through $\alpha = j(y) \in W_0^i$. This orbit intersects W_0^i transversally at the single point α relative to the stratum S^i. Thus, $T_yj(\xi_X(y))$ can belong to P^i only if it is zero, and hence $\xi_X(y) \in \ker T_yj$. By Lemma 4.30, ξ stabilizes α and we proved $T_yY \cap \widetilde{\mathfrak{k}}(y) \subset \widetilde{\mathfrak{h}}(y)$. The opposite inclusion is obvious: by Lemma 4.30 and Proposition 4.27(d), $\widetilde{\mathfrak{h}}(y) = \ker(T_yj) \cap \widetilde{\mathfrak{k}}(y) \subset T_yY \cap \widetilde{\mathfrak{k}}(y)$. ∎

We can now prove part (f) of Proposition 4.27. By Lemma 4.29 and the inclusion $\ker T_yj \subset T_yY$ in Proposition 4.27(d) we get

$$(T_yY)^\omega \subset (\ker(T_yJ))^\omega = \widetilde{\mathfrak{k}}(y).$$

Together with Lemma 4.31, this shows that

$$T_yY \cap (T_yY)^\omega \subset T_yY \cap \widetilde{\mathfrak{k}}(y) = \widetilde{\mathfrak{h}}(y), \tag{4.32}$$

where $\mathfrak{h}$ is the adjoint stabilizer subalgebra of $\mathfrak{k}$ on $\mathfrak{p}$. Therefore,

$$\begin{aligned} \widetilde{\mathfrak{h}}(y) &= \ker(T_yj) \cap \widetilde{\mathfrak{k}}(y) \quad \text{(by Lemma 4.30)} \\ &= (\widetilde{\mathfrak{k}}(y))^\omega \cap \widetilde{\mathfrak{k}}(y) \quad \text{(by Lemma 4.29)} \\ &= (\widetilde{\mathfrak{k}}(y))^\omega \cap (\ker(T_yj))^\omega. \end{aligned} \tag{4.33}$$

Since the left hand side of (4.32) lies in $(T_yY)^\omega \subset (\ker(T_yj))^\omega$, (4.33) implies

$$T_yY \cap (T_yY)^\omega \subset (\widetilde{\mathfrak{k}}(y))^\omega \cap (T_yY)^\omega = (\widetilde{\mathfrak{k}}(y) + T_yY)^\omega = (T_yX)^\omega = \{0\},$$

by the transversality in Proposition 4.27(e).

(*g*) Proposition 3.1 shows that the symplectic manifold X is a Hamiltonian T-space with equivariant momentum map $\phi = \log \circ H \circ J : X \to \mathfrak{a}$. The rest is Corollary 3.2. ∎

Proceeding as in [GS1], one now proves the following:

PROPOSITION 4.34. – *The closure of $j(Y)$ in $\mathfrak{a}_+$ is $j(X) \cap \mathfrak{a}_+$.*

C. THE CONVEXITY PROOF. – The first ingredient is a local convexity property of momentum mappings for torus actions ([GS1, Theorem 4.8], [CDM, II.1.2]), which we state in the form given in [HNP, Theorem 2.3].

THEOREM 4.35. – *Let M be a (not necessarily compact) symplectic manifold, and let a torus T act symplectically with momentum mapping $\phi : M \to \mathfrak{t}^*$. Then*

for every $m \in M$ there exists an arbitrarily small open neighborhood U of m and a convex polyhedral cone $C_m \subset \mathfrak{a}$ with vertex $\phi(m)$, such that:

(*a*) $\phi(U)$ is an open neighborhood of $\phi(m)$ in C_m;

(*b*) $\phi : U \to C_m$ is an open map;

(*c*) $\phi^{-1}(\phi(u))$ is connected for all $u \in U$.

DEFINITION 4.36. – *[HNP] Let Z be a connected Hausdorff space, and let $\phi : Z \to \mathfrak{a}$ be a map satisfying the conclusions of the preceding theorem. The correspondence $z \in Z \mapsto C_z$ is said to determine* local convexity data.

The next result converts local convexity to global convexity. It is convenient that the space Z need not be a manifold.

THEOREM 4.37. – *[CDM], [HNP, Theorem 3.10] Let ϕ be as described in Definition 4.36, and suppose it is a proper map. Then $\phi(Z)$ is a closed locally polyhedral subset of $\mathfrak{a}$, all fibers $\phi^{-1}(\alpha)$ are connected, and $\phi : Z \to \phi(Z)$ is an open mapping.*

We can now complete the proof of the convexity theorem as stated in the introduction. Theorem 4.35 is applied with $M = Y$ and the momentum mapping for the torus action, ϕ, equal to $j/2$ by Proposition 4.27(g). This gives local convexity data in the sense of Definition 4.36, but unfortunately Theorem 4.37 may not apply directly because $j|Y$ need not be a proper map [HNP]. A more circuitous argument is needed. Following [CK] and [HNP], consider the wall $W^i \subset \mathfrak{a}_+$ that contains the image $j(Y)$ (see Proposition 4.27; W^i may be all of $\mathfrak{a}_+$). For all small $\delta > 0$, let W^i_δ be the set of points in W^i at distance $\geq \delta$ from the boundary of W^i; W^i_δ is a closed convex polyhedral cone. Let $Z_\delta = j^{-1}(W^i_\delta)$. This is a compact Hausdorff space, and according to [HNP, Prop. 6.11], the restriction $j|Z_\delta$ determines local convexity data. Now Theorem 4.37 may be applied, and we see that $j(Y) \cap W^i_\delta$ is a convex, locally polyhedral set. Thus $j(Y)$ looks like a polytope away from the boundary of W^i. Letting $\delta \to 0$, we see ([CK] and [HNP]) that $\overline{j(Y)} \cap \mathfrak{a}_+$ is the union of these nested convex polyhedral sets and is therefore convex. It does not seem automatic that it is a convex polytope, however, since faces may accumulate at the boundary of W^i. To exclude this possibility, we follow [CK] and apply a lemma from [GS1]. We explicitly use compactness of X in the proof, but properness of j should in principle give the same result.

PROPOSITION 4.38. – *The number of faces of $\overline{j(Y)}$ is finite.*

Proof. – By Proposition 4.27(g), the compact symplectic manifold X is a Hamiltonian T-space with equivariant momentum map ϕ, which has the property that $\phi|Y = (j|Y)/2$; Y is a symplectic submanifold of X and a Hamiltonian T-space as well. If T' is a closed subgroup of T, then $X_{T'} = \{x \in X \,|\, T' \text{ is the stabilizer group of } x \text{ in } K\}$ is a symplectic submanifold of X ([GS1], Lemma 3.6). It is also T-invariant: if $y \in X_{T'}$ and $t \in T$, then $T_{t\cdot y} = tT_y t^{-1} = tT't^{-1} = T'$ since T is abelian. Since X is compact, $X_{T'}$ has finite connectivity (*see* [Br] or [J]). For X compact, there are only finitely many orbit types (*see* [Br] or [J]), that is, only finitely many closed subgroups $T_1, T_2, ..., T_M$ of T are stabilizer groups of points in X. The manifolds

$$X_i = \{x \in X \mid \text{ the stabilizer of } x \text{ in } T \text{ is } T_i\}, \quad i = 1, 2, ..., M$$

are symplectic and have finite connectivity. Allowing repetitions amongst the T_i, we may assume that all X_i are connected. Thus the symplectic manifold X is a disjoint union of connected symplectic T-invariant submanifolds $X_i, i = 1, 2, ..., M$ and each one of them is a Hamiltonian T-space.

If $\mathfrak{t}_i$ denotes the Lie algebra of T_i, Lemma 4.1 implies that $T_x\phi(T_x X_i) = \mathfrak{t}_i^{\mathrm{ann}}$ for all $x \in X_i$. Now we need the following remark: if V is a topological vector space, $W \subset V$ a closed subspace, and $t \in [0,1] \mapsto v(t) \in V$ is a differentiable path, then $c'(t) \in W$ if, and only if, $c(t) \in c(0) + W$ for all $t \in [0,1]$. Fix $x_0 \in X_i$ and let $x_1 \in X_i$ be arbitrary. Since X_i is connected, there is a smooth path $c(t) \in X_i$ with $c(0) = x_0$ and $c(1) = x_1$. Then $(\phi \circ c)(t)$ is a smooth path in $\phi(X_i) \subset \mathfrak{t}^* \cong \mathfrak{a}$ connecting $\alpha_i = \phi(x_0)$ to $\phi(x_1)$ and whose tangent vector is always in $\mathfrak{t}_i^{\mathrm{ann}}$. The prior remark guarantees then that $\phi(x_1) \in \alpha_i + \mathfrak{t}_i^{\mathrm{ann}}$, that is, $\phi(X_i) \subset \alpha_i + \mathfrak{t}_i^{\mathrm{ann}}$. Therefore $\phi|X_i : X_i \to \alpha_i + \mathfrak{t}_i^{\mathrm{ann}}$ is a submersion and hence an open map. Thus there is an open set $U_i \subset \mathfrak{a}$ such that X_i is mapped submersively by ϕ onto $U_i \cap (\alpha_i + \mathfrak{t}_i^{\mathrm{ann}})$ for each $i = 1, 2, \ldots, M$.

We want to draw a similar conclusion with X replaced by Y. This is not immediately possible since Y is not compact. To get around the problem, we define

$$Y_i = X_i \cap Y = \{y \in Y \mid \text{ the stabilizer of } y \text{ in } T \text{ is } T_i\}$$

and conclude as above that the collection of the sets Y_i forms a partition of Y by symplectic submanifolds. Y_i is T-invariant since both X_i and Y are T-invariant. In addition, Y_i is a Hamiltonian T-space with equivariant momentum mapping $\phi|Y_i$.

It follows from Lemma 4.1 that $T_y\phi(T_yY_i) = \mathfrak{t}_i^{\mathrm{ann}}$. However, as we already saw above, $T_y\phi$ maps T_yX_i onto the same space surjectively. Therefore there are open sets $V_i \subset \mathfrak{a}$ such that each Y_i is mapped submersively by ϕ onto $V_i \cap (\alpha_i + \mathfrak{t}_i^{\mathrm{ann}})$. Thus, $\overline{j(Y)}$ is a finite union of closures of open subsets of affine spaces, and being convex, it is a convex polytope. ■

D. A GENERAL CONVEXITY THEOREM. – The convexity proof at the end of the last subsection depends on a few properties of the map $j : X \to \mathfrak{p}$. Some of them have so far only been established for the Lu-Weinstein stucture on K. We now abstract them in a general theorem; Section 5 will verify the hypotheses of this theorem for all Poisson Lie structures on K.

THEOREM 4.39. – *Let a compact connected Lie group K act on a compact connected symplectic manifold X, and let a maximal torus T act in a Hamiltonian fashion with equivariant momentum mapping ϕ. Suppose there exists a map $j : X \to \mathfrak{p}$ satisfying:*

(*a*) *j is equivariant (with respect to the adjoint action of K on $\mathfrak{p}$);*

(*b*) *for every $x \in X$, $T_xj(T_xX) = \mathfrak{k}_x^{\mathrm{ann}}$;*

(c) *for every $x \in X$, the kernel of T_xj is $\widetilde{\mathfrak{k}}(x)^{\omega}$;*

(*d*) *the restriction of j to $j^{-1}(\mathfrak{a}_+)$ is proportional to ϕ.*

Then $j(X) \cap \mathfrak{a}_+$ is a convex polytope.

Remark 4.40. – Theorem 4.37 implies that the fibers of $j : X \to \mathfrak{p}$ are connected. ■

Proof. – Equivariance of j is important throughout. Once the analog of Proposition 4.27 is proved, the convexity proof follows from the argument in Subsection 4B; it requires only that the torus action be Hamiltonian, and that $j|Y$ have the local convexity property of Theorem 4.35. This is assured by assumption (*d*).

For the construction of the cross-section Y, we need Lemmas 4.25 and 4.26. The crucial ingredient there was constancy of rank T_xj on $j^{-1}(W_0^i)$; above, this followed from Lemma 4.16, which is now assured by assumption (*b*). Assumption (*b*) is used once more, in Prop. 4.27 (*c*), the transversality proof that shows Y to be a submanifold.

The cross-section Y is symplectic thanks to Lemma 4.29, which is now assured by assumption (*c*). This concludes the proof. ■

5. The general Poisson-Lie structure

We resume the discussion of the general Poisson Lie structure on K as in section 2E. Equivariance of j (which is very important) was already noted in 2.31 and 2.38. To verify the hypotheses of Theorem 4.39, we need to check three items.

$$\text{The } T \text{ action is Hamiltonian} \tag{5.1}$$

$$T_x j(T_x X) = \mathfrak{k}_x^{\mathrm{ann}} \text{ and } \ker T_x j = \tilde{\mathfrak{k}}(x)^\omega, \tag{5.2}$$

$$j | j^{-1}(\mathfrak{a}_+) \text{ is proportional to } \phi. \tag{5.3}$$

PROPOSITION 5.4. – *Let $a = 1$, and equip K with the Poisson Lie structure 2.29 b or 2.29 c. Suppose there is a Poisson action of K on a symplectic manifold X, with equivariant momentum mapping $J : X \to K^{\mathrm{d}} = H^u N$. Write $j = \mathrm{sym} \circ J : X \to \mathfrak{p}$ as in Definition 2.31. Then: (a) the Poisson structure on K vanishes on T, so that T acts symplectically on X; (b) the equivariant momentum mapping for the T-action is $\phi = \log \circ H \circ J : X \to \mathfrak{a}$, where H is the Iwasawa projection,*

$$H : KH^u N \to A, \quad H : k \exp(\hat{u}(\alpha)) \exp(\alpha) n \mapsto \exp(\alpha); \tag{5.5}$$

in particular, if $y \in j^{-1}(\mathfrak{a}_+)$, then $j(y) = 2\phi(y)$; (c) the image of $T_x j$ is $\mathfrak{k}_x^{\mathrm{ann}}$; (d) the kernel of $T_x j$ is $\tilde{\mathfrak{k}}(x)^\omega$.

Proof. – These assertions, for the case $u = 0$, are covered by Corollary 2.7, Propositions 3.1 and 4.27 (g), Lemma 4.16, and Lemma 4.29. The proofs apply also when $u \neq 0$, provided, of course, one replaces $\mathfrak{b}$ by $\mathfrak{k}^{\mathrm{d}}$ and B by K^{d} throughout. ■

Let us now turn to the case $a = 0$. Introduce the notation $A^u = \{(\exp(\hat{u}(\alpha)), \alpha) \mid \alpha \in \mathfrak{a}\}$ for the "diagonal subgroup" of ■.

PROPOSITION 5.6. – *Let $a = 0$, and equip K with the Poisson Lie structure described in 2.29(a) and 2.29(d). Suppose there is a Poisson action of K on a compact connected symplectic manifold X, with equivariant momentum mapping*

$$J : X \to K^{\mathrm{d}} = \{(\exp(\hat{u}(\alpha), \alpha + v) \mid \alpha \in \mathfrak{a}, v \in \mathfrak{a}^\perp \cap \mathfrak{p}\} \subset K \ltimes \mathfrak{p}.$$

Write $j = \psi \circ J : X \to \mathfrak{p}$, with ψ as defined in 2.38. Then: (a) the Poisson structure on K vanishes on T, so that T acts symplectically on X; (b) the equivariant momentum mapping for the T-action is $\phi = \psi \circ F \circ J : X \to \mathfrak{a}$, where F is the "diagonal projection"

$$K : K^{\mathrm{d}} \to A^u, \quad F : (\exp(\hat{u}(\alpha)), \alpha + v) \mapsto (\exp(\hat{u}(\alpha)), \alpha); \tag{5.7}$$

in particular, if $y \in j^{-1}(\mathfrak{a})$ then $j(y) = \phi(y)$; (c) the image of $T_x j$ is $\mathfrak{k}_x^{\mathrm{ann}}$; (d) the kernel of $T_x j$ is $\tilde{\mathfrak{k}}(x)^\omega$.

Proof. – (*a*) According to (2.24), the Poisson tensor on K is defined by

$$\pi_K(k)(T_k^* R_{k^{-1}} Y_1, T_k^* R_{k^{-1}} Y_2) = -\langle \rho_{\mathfrak{k}}(\mathrm{Ad}_{k^{-1}} Y_1), \rho_{\mathfrak{k}^{\mathrm{d}}}(\mathrm{Ad}_{k^{-1}} Y_2)\rangle. \tag{5.8}$$

Here $Y_1 = (\hat{u}(\alpha), \alpha + v), Y_2 = (\hat{u}(\beta), \beta + w) \in \mathfrak{k}^{\mathrm{d}}$, with $\alpha, \beta \in \mathfrak{a}$ and $v, w \in \mathfrak{a}^\perp \cap \mathfrak{p}$. The adjoint action in (5.8) is that of the semidirect product $K \ltimes \mathfrak{p}$ on its Lie algebra $\mathfrak{k} \ltimes \mathfrak{p}$ with k identified with the pair $(k, 0)$. If we temporarily denote the adjoint actions of K on $\mathfrak{k}$ and on $\mathfrak{p} = i\mathfrak{k}$ by $\widetilde{\mathrm{Ad}}$, we find that

$$\mathrm{Ad}_{k^{-1}}(\hat{u}(\alpha), \alpha + v) = (\widetilde{\mathrm{Ad}}_{k^{-1}}\hat{u}(\alpha), \widetilde{\mathrm{Ad}}_{k^{-1}}(\alpha + v)). \tag{5.9}$$

When $k \in T$, the right side of (5.9) becomes

$$(\hat{u}(\alpha), \alpha + \widetilde{\mathrm{Ad}}_{k^{-1}} v) \in \mathfrak{k}^{\mathrm{d}},$$

and therefore the projection $\rho_{\mathfrak{k}}(\mathrm{Ad}_{k^{-1}} Y_1)$ in (5.8) is zero. (b) Let $m = (\exp(\hat{u}(\alpha)), \alpha+v) \in K^{\mathrm{d}}$, and let $\zeta = (\hat{u}(\beta), \beta + w) \in \mathfrak{k}^{\mathrm{d}} = T_e K^{\mathrm{d}}$. A short calculation shows that

$$T_e L_m(\zeta) = (T_e L_{\exp(\hat{u}(\alpha))}\hat{u}(\beta), \mathrm{Ad}_{\exp(\hat{u}(\alpha))}(\beta + w)) \in T_m K^{\mathrm{d}} \subset T_{\exp(\hat{u}(\alpha))} K \times \mathfrak{p}. \tag{5.10}$$

One then finds from Definition 2.38 and (5.10) that

$$T_e(\psi \circ L_m)(\zeta) = -\mathrm{Ad}_{\exp(-\hat{u}(\alpha))}\,\mathrm{ad}_{\hat{u}(\beta)}(\alpha + v)) + \beta + w. \tag{5.11}$$

With these formulae, we check the analog of (3.3): writing $J(x) = m$ and taking $\xi \in \mathfrak{t}$, we want to show that $(T_m^* F \circ T_{\psi(m)}^* \psi)(\xi)$ is the value at m of the left-invariant one-form ξ^ℓ on K^{d}. Applying (5.11) to $m = (\exp(\hat{u}(\alpha)), \alpha) \in A^u$ and $\zeta = (\hat{u}(\beta), \beta) \in T_e A^u$, one finds that

$$\langle T^*_{(\exp(\hat{u}(\alpha)),\alpha)}\psi(\xi), T_e L_{(\exp(\hat{u}(\alpha)),\alpha)}(\beta)\rangle = \langle \xi, \beta\rangle.$$

It follows that $T^*_{(\exp(\hat{u}(\alpha)),\alpha)}\psi(\xi)$ is the value at $(\exp(\hat{u}(\alpha)), \alpha)$ of the left-invariant one-form on A^u whose value at the identity is ξ. Call this one-form $\xi^\ell_{A^u}$.

Now let $m = J(x) = (\exp(\hat{u}(\alpha)), \alpha+v)$. We first check that $T_m^* F(\xi^\ell_{A^u}((\exp(\hat{u}(\alpha)), \alpha)))$ is the value at m of the one-form ξ^ℓ on K^{d}, and to this end, we work out

$$\langle \xi, T_e(L_{\exp(-\hat{u}(\alpha), -\alpha)} \circ F \circ L_m)(\hat{u}(\beta), \beta + w)\rangle. \tag{5.12}$$

It is easily seen that the second entry in (5.12) is simply $(\hat{u}(\beta), \beta)$. Then (5.12) becomes $\langle \xi, (\hat{u}(\beta), \beta)\rangle = \mathrm{Im}\,\kappa(\xi, \hat{u}(\beta) + \beta)$, but because $w \in \mathfrak{a}^\perp \cap \mathfrak{p}$ is orthogonal to $\mathfrak{t}$, this is the same as $\mathrm{Im}\,\kappa(\xi, \hat{u}(\beta) + \beta + w)$. Thus, (5.12) is completely determined by its value at the identity, and the desired conclusion follows.

The identity $j(y) = \phi(y)$ follows once one observes that the diffeomorphsim ψ maps A^u onto $\mathfrak{a}$. Finally, equivariance of ϕ is a direct verification.

(c) This statement is the analog of Lemma 4.16, whose proof depends on two assertions. The first is Lemma 4.2, which is a general fact about momentum mappings for Poisson Lie group actions, and holds in the present case as well. The second crucial claim is (4.19); we show that in our new setting,

$$T_e^*(\psi \circ L_m)|\mathfrak{k}_{\psi(m)} = \text{ identity on } \mathfrak{k}_{\psi(m)}. \tag{5.13}$$

Let $\xi \in \mathfrak{k} \cong T_e^* K^{\mathrm{d}}$, and note from (5.11) that

$$\langle T_e^*(\psi \circ L_m)(\xi), (\hat{u}(\beta), \beta + w)\rangle = -\langle \xi, \mathrm{ad}_{\hat{u}(\beta)}(\psi(m))\rangle + \langle \xi, \beta + w\rangle. \tag{5.14}$$

We will check that the first term on the right side of (5.14) is zero when $\xi \in \mathfrak{k}_{\psi(m)}$, and then (5.13) will follow from (5.14). Indeed,

$$-\langle \xi, \mathrm{ad}_{\hat{u}(\beta)}(\psi(m))\rangle = \mathrm{Im}\,\kappa([\hat{u}(\beta), \xi], \psi(m)) = \mathrm{Im}\,\kappa(\hat{u}(\beta), \mathrm{ad}_\xi(\psi(m))$$

and $\mathrm{ad}_\xi(\psi(m)) = 0$ when $\xi \in \mathfrak{k}_{\psi(m)}$.

The proof of (d) was already given in Lemma 4.29. ∎

We can now deduce the following general result from Theorem 4.39, Section 2E, and the earlier part of the present section.

THEOREM 5.15. – *Let K be a compact connected semisimple Lie group, equipped with one of the Poisson Lie structures listed in Theorem 2.20. Let X be a compact connected symplectic manifold and suppose that there is a Poisson action of K on X with equivariant momentum mapping $J : X \to K^{\mathrm{d}}$. Define $j : X \to \mathfrak{p}$ as in 2.31 or 2.38. Then $j(X) \cap \mathfrak{a}_+$ is a (compact) convex polytope.*

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Remarks added in proof. – Professor G. Heckman pointed out to us that Example 1.1 is contained in Section 7 of this paper "Projections of Orbits and Asymptotic Behavior of Multiplicities for Compact Connected Lie Groups", *Invent. Math.*, Vol. 67, 1982, pp. 333-356. A short proof of our main theorem has been given by A. Yu. Alekseev, "On Poisson actions of compact Lie groups on symplectic manifolds", February 1996, preprint.

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