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RAÚL URES

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## ABUNDANCE OF HYPERBOLICITY IN THE $C^1$ TOPOLOGY

BY RAÚL URES <sup>(1)</sup>

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ABSTRACT. – In this paper we show that the Newhouse's thickness of basic sets as an obstruction to the density of Axiom A diffeomorphisms in the  $C^k$  topology for  $k \geq 2$  is not applicable in the  $C^1$  topology. We prove that the local thickness of saddle-type hyperbolic basic sets vanishes  $C^1$  generically and that, when we consider families with first homoclinic bifurcations, the  $\Omega$ -stability is a "prevalent" phenomena after the unfolding of the tangency.

RÉSUMÉ. – Dans cet article on montre que l'épaisseur de Newhouse des ensembles basiques telle qu'une obstruction pour la densité des difféomorphismes Axiome A dans la topologie  $C^k$ ,  $k \geq 2$ , n'est pas applicable dans la topologie  $C^1$ . On montre que l'épaisseur locale des ensembles basiques hyperboliques de type selle s'annule génériquement dans la topologie  $C^1$  et que, si l'on considère des familles exhibant une première bifurcation homoclinique, l' $\Omega$ -stabilité est un phénomène « prévalent » après le développement de la tangence.

### Introduction

A problem of fundamental importance in the theory of dynamical systems is to determine a dense set of diffeomorphisms whose dynamical behaviour exhibit a fair degree of robustness under perturbations. With this aim and having also as a motivation the construction of structurally stable diffeomorphisms ( $f$  is  $C^r$  structurally stable if it belongs to a  $C^r$  open set such that any diffeomorphism in it is conjugate to  $f$  by a homeomorphism) Smale introduced, in the sixties, the notion of hyperbolic (Axiom A) diffeomorphisms. In fact, deciding whether hyperbolic diffeomorphisms are dense or not in the space of all diffeomorphisms, together with the analog questions for the classes of structurally stable and of  $\Omega$ -stable diffeomorphisms (as structural stable diffeomorphisms but restricting the conjugation to the nonwandering set), became a central problem in the subsequent development of the theory.

It is now well known that if the dimension of the manifold is greater than two then the answer is negative for all classes of diffeomorphisms; *see*, for instance, [AS], [M1], [S]. For diffeomorphisms of surfaces Williams [W] showed that structurally stable diffeomorphisms are not  $C^r$  dense for  $r \geq 1$ . Then, in the early seventies, Newhouse proved that if

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we consider the set of all diffeomorphisms with the  $C^r$  topology,  $r \geq 2$ , hyperbolic diffeomorphisms and  $\Omega$ -stable diffeomorphisms are also not dense. For the  $C^1$  topology, the density or not of  $\Omega$ -stable or hyperbolic diffeomorphisms remains an open question.

In this work we do not give a conclusive answer to this question but we show that Newhouse's obstruction to the density of  $\Omega$ -stability is not valid in the  $C^1$  topology.

The solution of this problem would give a complete panorama of the dynamics of  $C^1$  diffeomorphisms on surfaces. Recall that in the  $C^1$  topology the stability is well characterized by [M2] and [P].

The main tools in Newhouse's proof of the nondensity of Axiom A diffeomorphisms on surfaces are the notions of thickness (*see* the definition in Section 1), a kind of fractional dimension for Cantor sets, and that of homoclinic tangency. He showed that if a diffeomorphism  $\varphi$  exhibits a homoclinic tangency associated to a thick hyperbolic set  $\Lambda$  (thick means  $\tau^s(\Lambda)\tau^u(\Lambda) > 1$  *see* Section 1) then there exists a  $C^2$  open set of diffeomorphisms, nearby the first one, for which a dense set of elements have homoclinic tangencies (persistence of homoclinic tangencies). Even more interesting, such tangencies involving thick hyperbolic sets occur whenever a homoclinic tangency is unfolded (*see* [N3], and also [PT1] for a new and clearer proof).

To obtain these results, a fundamental property of the thickness is its continuity in the  $C^2$  topology. Our first theorem says that continuity breaks down in the  $C^1$  topology and, moreover, the thickness vanishes for a residual set of  $C^1$  diffeomorphisms.

Let  $\tau_{p(\varphi)}^*(\Lambda(\varphi))$ ,  $\star = s, u$ , be the local thickness of  $\Lambda$  at  $p$  (*see* the definition in Section 1).

**THEOREM A.** –  $C^1$  generically  $\tau_{p(\varphi)}^*(\Lambda(\varphi)) = 0$ ,  $\star = s, u$ .

In other words, if  $\mathcal{U}$  is the open set where there exists an “analytic” continuation of  $\Lambda$ , then the stable and unstable thickness are null for the elements of a residual set in  $\mathcal{U}$ . However, we recall that the Hausdorff dimension of hyperbolic basic sets varies continuously in the  $C^1$  topology (*see* [MM] and [PV]).

To solve the problem of the density of Axiom A diffeomorphisms it seems to be important to know what happens after the unfolding of homoclinic tangencies. This fits one of Palis' well known conjectures (*see* [PT1]), that diffeomorphisms displaying homoclinic tangencies are dense in the complement of the  $C^k$  hyperbolic ones, for any  $k \geq 1$ . The result of Araújo and Mañé [AM], showing that  $C^2$  diffeomorphisms either have finitely many hyperbolic attractors that attract almost every point or they are  $C^1$  approximated by one with a homoclinic tangency, encourages this point of view.

In the unfolding of homoclinic tangencies of  $C^r$  one-parameter families ( $r \geq 2$ ) there are different possibilities according to the fractional dimension (Hausdorff dimension and thickness) of the hyperbolic set involved in the creation of the tangency:

1. if the Hausdorff dimension of the hyperbolic set is less than one, hyperbolicity corresponds to a set of Lebesgue density one at this parameter value, *see* [PT2],
2. if the Hausdorff dimension of the hyperbolic set is bigger than one, the set of parameter values corresponding to hyperbolic diffeomorphisms is not of density one at the initial bifurcation value. Indeed there are “plenty” of parameter values corresponding to diffeomorphisms exhibiting homoclinic tangencies, *see* [PY],

3. if the hyperbolic set is thick, *i.e.* the product of its stable and unstable thickness is bigger than one, the bifurcating parameter value is in the boundary of an interval of persistence of homoclinic tangencies, *see* [N1-2].

Our second theorem shows that after the unfolding of first homoclinic tangencies (*see* Section 1 for the definition) we will always find hyperbolic diffeomorphisms  $C^1$  near the bifurcating one. Even more, if we consider  $C^1$  one-parameter families that unfold a first homoclinic tangency, generically, the situation is quite similar to the first case above.

Let us state the theorem more precisely. Let  $b$  be the first bifurcating parameter value and

$$H_\delta(\{\varphi_\mu\}) = \{b < \mu < b + \delta; \varphi_\mu \text{ is hyperbolic and has the nocycle property}\}.$$

THEOREM B. – *For  $C^1$  generic families with a first homoclinic bifurcation*

$$\limsup_{\delta \searrow 0} \frac{m(H_\delta(\{\varphi_\mu\}))}{\delta} = 1.$$

Thanks to its continuous variation there exist families with hyperbolic sets having large Hausdorff dimension (bigger than one) that verify Theorem B (compare with [PY]).

Our results raise some questions beyond the one concerning Lebesgue density of hyperbolic or  $\Omega$ -stable diffeomorphisms.

Question 1: Let  $b$  be the first bifurcating parameter value of a one-parameter family. Are the families with topological density of hyperbolic diffeomorphisms in an interval  $[b, b + \delta]$   $C^1$  generic? It would even be interesting to give examples of this kind of families.

Question 2: The same question, replacing hyperbolic diffeomorphisms by diffeomorphisms that stably have only finitely many sinks (*i.e.* are the parameter intervals such that any diffeomorphism corresponding to a parameter value in them has only finitely many sinks dense in  $[b, b + \delta]$ ?).

Question 3: Is it possible to give examples of diffeomorphisms with infinitely many sinks which are  $C^1$  approximated by hyperbolic diffeomorphisms?

The contents of the next sections of this paper are as follows. In Section 1 we give the definitions of thickness and families with a first homoclinic bifurcation. The proof of Theorem A is in Section 2 and the proof of Theorem B is in Section 3. It involves the technique of the proof of Theorem A and arguments of [PT2].

### 1. Preliminaries

DEFINITION 1.1. – *Let  $K \subset \mathbb{R}$  be a Cantor set. A gap is a connected component of  $\mathbb{R} \setminus K$ . Let  $U$  be any bounded gap and  $k$  be a boundary point of  $U$ , so  $k \in K$ . Let  $C$  be the bridge of  $K$  at  $k$ , *i.e.* the maximal interval in  $\mathbb{R}$  such that*

- $k$  is a boundary point of  $C$ ;
- $C$  contains no point of a gap  $U'$  whose length  $l(U')$  is at least the length of  $U$ .

Let  $\tau(K, k) = \frac{l(C)}{l(U)}$ . Then the thickness of  $K$ , denoted by  $\tau(K)$ , is the infimum of these  $\tau(K, k)$  over all boundary points  $k$  of bounded gaps.

Let  $k \in K$ ; the local thickness of  $K$  at  $k$ , denoted by  $\tau_k(K)$ , is the supremum of  $\tau(K \cap [k - \varepsilon, k + \varepsilon])$  over all  $\varepsilon > 0$ . This is not the usual definition of local thickness (see [PT1]) but, as we are mostly interested in Theorem D, we use it to make our exposition clearer.

DEFINITION 1.2. – Let  $\varphi$  be a two dimensional diffeomorphism,  $\Lambda$  be a basic set of saddle type for  $\varphi$  and  $p \in \Lambda$  be a fixed (periodic) point. We define the stable thickness of  $(\Lambda, p)$  as  $\tau_p^s(\Lambda) = \tau(W_{\text{loc}}^u(p) \cap \Lambda)$  and analogously for the unstable thickness. Observe that this definition is invariant by differentiable changes of coordinates.

Let us recall the main properties of the thickness.

(1) Gap Lemma, (see [N1-2]). Let  $K_1, K_2 \subset \mathbb{R}$  be Cantor sets with thickness  $\tau_1$  and  $\tau_2$ . If  $\tau_1 \cdot \tau_2 > 1$ , then one of the following three alternatives occurs:  $K_1$  is contained in a gap of  $K_2$ ;  $K_2$  is contained in a gap of  $K_1$ ;  $K_1 \cap K_2 \neq \emptyset$ .

(2)  $\tau_p^{u(s)}(\Lambda(\varphi))$  depend continuously on  $\varphi$  with respect to the  $C^2$  topology. This means that if  $U \subset \text{Diff}^2(M)$  is open and if, for  $\varphi \in U$ ,  $\Lambda(\varphi)$  is a basic set of saddle type with fixed (or periodic) point  $p(\varphi)$ , both depending continuously on  $\varphi \in U$ , then  $\tau_p^{u(s)}(\Lambda(\varphi))$  are continuous functions on  $U$  (see [N1-2] and [PT1]).

(3) If  $\varphi \in \text{Diff}^2(M)$  and  $\Lambda$  is a hyperbolic basic set of saddle type for  $\varphi$  then  $\tau_p^{u(s)}(\Lambda)$  do not depend on the point  $p$  and, moreover,  $\tau_p^{u(s)}(\Lambda) > 0$ .

Properties (1) and (2) are the main ingredients in Newhouse's proof of the non-density of hyperbolic (Axiom A) diffeomorphisms in  $\text{Diff}^2(M^2)$ .

DEFINITION 1.3. – We say that a diffeomorphism  $\varphi$  is hyperbolic if the nonwandering set of  $\varphi$ ,  $\Omega(\varphi)$ , is hyperbolic and the periodic points of  $\varphi$  are dense in it.

Let  $\Omega(\varphi) = \Lambda_1 \cup \dots \cup \Lambda_k$  be the spectral decomposition of a hyperbolic diffeomorphism. A  $j$ -cycle on  $\Omega(\varphi)$  is a string of  $j$  pairs of points  $x_1, y_1 \in \Lambda_{i_1}, \dots, x_j, y_j \in \Lambda_{i_j}$ , with not all  $i_1, \dots, i_j$  equal, such that  $W^u(y_1) \cap W^s(x_2) \neq \emptyset, \dots, W^u(y_j) \cap W^s(x_1) \neq \emptyset$ . We say that  $\varphi$  has the nocycle property if there are no cycles on  $\Omega(\varphi)$ .

When there are no cycles we can construct a filtration for  $\varphi$ : a sequence  $M_0 = \emptyset \subset M_1 \subset \dots \subset M_k = M$  of compact submanifolds with boundary for  $0 < i < k$  such that  $\varphi(M_i) \subset \text{int } M_i$  and in  $M_{i+1} \setminus M_i$  the maximal invariant set is  $\Lambda_i$ .

DEFINITION 1.4. – We say that a  $C^1$  one parameter family  $(\varphi_\mu)_{\mu \in (-\varepsilon, \varepsilon)}$  has a first homoclinic bifurcation at  $b$  if:

(1)  $\varphi_\mu$  is hyperbolic and has the nocycle property for  $\mu < b$  (i.e. it is  $\Omega$ -stable for  $\mu < b$ ).

(2)  $\varphi_b$  is hyperbolic on  $\lim_{\mu \nearrow b} \Omega(\varphi_\mu) = \tilde{\Omega}(\varphi_b)$  ( $\Omega(\varphi)$  means the nonwandering set of  $\varphi$ ).

(3) There exist a fixed (or periodic) point  $p_b$  of  $\tilde{\Omega}(\varphi_b)$  and fundamental domains  $S_b$  of  $W^s(p_b)$  and  $U_b$  of  $W^u(p_b)$  such that

$$S_b \cap W^u(p_b) = U_b \cap W^s(p_b) = \text{int}(S_b) \cap \text{int}(U_b)$$

where  $\text{int}(S_b)$  and  $\text{int}(U_b)$  denote the interior of these sets in the topology induced by  $M$ . Recall that a fundamental domain is a "segment" of the corresponding invariant manifold such that any orbit of this manifold has only one point of intersection with it.

(4) The  $\Omega$ -set of  $\varphi_b$  is the union of  $\tilde{\Omega}(\varphi_b)$  and the orbit of  $S_b \cap U_b$ .



of the hyperbolic set  $\Lambda$  when it contains a point of  $\Lambda$  (see [PT1]). Then, we can define, in the usual way,  $C^1$  linearizing coordinates such that  $Q$  is the square  $[0, 1] \times [0, 1]$ . That is, we can find a  $C^1$  diffeomorphism  $\psi$  from a neighborhood of  $Q$  onto a neighborhood of  $[0, 1] \times [0, 1]$  such that

- $\psi(p) = (0, 0)$ ,
- $\psi^{-1}(\{0\} \times [0, 1] \cup \{1\} \times [0, 1]) \subset W^u(p)$ ,
- $\psi^{-1}([0, 1] \times \{0\} \cup [0, 1] \times \{1\}) \subset W^s(p)$

and

- $\psi\varphi\psi^{-1}$  is linear on the rectangle  $[0, 1] \times [0, c]$ .

We shall make use of this coordinates and identify  $\varphi$  on  $\psi^{-1}([0, 1] \times [0, c])$  with  $\psi\varphi\psi^{-1}$  on points in  $[0, 1] \times [0, c]$ . We can write  $\varphi(x, y) = (\varphi_1(x), \varphi_2(y))$  and we will only perturb  $\varphi_2$ .

In the segment  $\{0\} \times [0, 1]$  we have three segments  $\bar{a}, \bar{b}, \bar{c}$  (a Markov partition see Fig. 2.2) where  $\bar{b}$  is a gap and  $\varphi(\bar{a}) = \{1\} \times [0, 1]$ ,  $\varphi(0, 1) = (1, 0)$ ,  $\varphi(\bar{c}) = \{0\} \times [0, 1]$ ,  $\varphi(0, 0) = (0, 0)$  and  $\varphi(0, c) = (0, 1)$  where  $c$  is the length of  $\bar{c}$  ( $z$  will always mean the length of  $\bar{z}$ ).

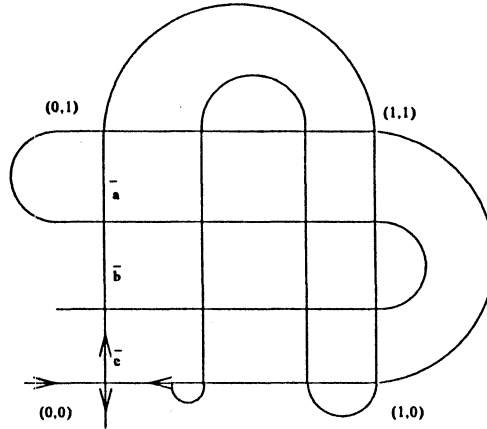


Fig. 2.2.

We define  $\bar{a}_m, \bar{b}_m$  and  $\bar{c}_m$  as  $\varphi^{-m}(\bar{a})$ ,  $\varphi^{-m}(\bar{b})$  and  $\varphi^{-m}(\bar{c})$  respectively and consider a continuous function  $\rho : [0, 1] \rightarrow \mathbb{R}$  such that:

- (1)  $\rho(c) = \rho(0) = 0$ .
- (2)  $\rho|_{[b_1+c_1, c-\varepsilon]} = \frac{1}{n+1}$  with  $\varepsilon > 0$  small enough.
- (3)  $\rho|_{\bar{a}_i} = \frac{1}{n+i} i \geq 2$ .
- (4)  $\int_{c_i}^{c_{i-1}} \rho(t) dt = 0$ ,  $c_0 = c$ .
- (5)  $|\rho(t)| < \frac{C}{n}$  for some constant  $C > 0$ .

$$(6) \int_y^c \rho(t) dt \geq 0, \forall 0 \leq y \leq c.$$

It is clear that this function exists because  $\bar{a}_m = \lambda^{-m} \bar{a}$ ,  $\bar{b}_m = \lambda^{-m} \bar{b}$  and  $\bar{c}_m = \lambda^{-m} \bar{c}$  where  $\lambda > 1$  is the unstable eigenvalue of  $\varphi$  at  $p$ . Then, let  $g_n(x, y) = \left( \varphi_1(x), \varphi_2(y) - \int_y^c \rho(t) dt \right)$  in  $[0, 1] \times [0, c]$ . It is clear that  $g_n$  and  $Dg_n$  are equal to  $\varphi$  and  $D\varphi$  at  $[0, 1] \times \{0\}$  and  $[0, 1] \times \{c\}$ .

The coordinates are defined in an open set that contains  $[0, 1] \times [0, 1]$  and then, we can take a rectangle as in Figure 2.3.

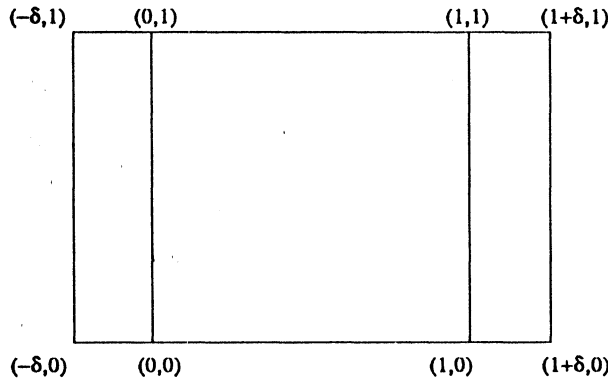


Fig. 2.3.

We take a  $C^1$  function  $\beta : \mathbb{R} \rightarrow [0, 1]$  such that:

- (1)  $\beta|_{[0,1]} = 1$ .
- (2)  $\beta(-\delta) = \beta(1 + \delta) = \beta'(-\delta) = \beta'(1 + \delta) = 0$ .
- (3)  $\beta(x) = \beta'(x) = 0$  for  $x \notin [-\delta, 1 + \delta]$ .

Then we denote  $K = \sup |\beta'|$ .

We define  $g_n$  as  $\varphi$  outside  $[-\delta, 1 + \delta] \times [0, c]$  and  $\left( \varphi_1(x), \varphi_2(y) - \beta(x) \int_y^c \rho(t) dt \right)$  otherwise.

We obtain that:

- (i)  $g_n = \varphi$  on the boundary of the rectangle  $[-\delta, 1 + \delta] \times [0, c]$ .
- (ii)  $Dg_n = D\varphi$  on the same set.

Then,  $g_n$  is a  $C^1$  function. Moreover,  $g_n$  is  $C^1$  near  $\varphi$  for  $n$  large enough:

$$\begin{aligned} \|\varphi - g_n\|_{C_0} &= \left| \beta(x) \int_y^c \rho(t) dt \right| < \frac{C}{n}, \\ \|D\varphi - Dg_n\| &= \left\| \begin{matrix} 0 & 0 \\ -\beta'(x) \int_y^c \rho(t) dt & \beta(x) \rho(y) \end{matrix} \right\| \\ &= \left| \beta'(x) \int_y^c \rho(t) dt \right| + |\beta(x) \rho(y)| < \frac{KC}{n} + \frac{C}{n} < \frac{K_1}{n}. \end{aligned}$$



Now we only need to prove that the hyperbolic set of  $g_n$  corresponding to the continuation of  $\Lambda$  has stable thickness equal to zero.

In order to prove this, first of all we observe that  $g_{n,2}(0, c_i) = (0, c_{i-1})$ ,  $\forall i \geq 1$ ,  $g_{n,2}(0, c) = (0, 1)$ ,  $g_n(\bar{a}_i) \supset \bar{a}_{i-1}$  and, if we call  $\bar{a}'_i = g_n^{-i}(\bar{a})$ ,  $\bar{b}'_i = g_n^{-i}(\bar{b})$  and  $\bar{c}'_i = g_n^{-i}(\bar{c})$ , then

$$(i) \quad a'_i = \frac{a'_1}{\prod_{j=n+2}^{n+i} \left( \lambda + \frac{1}{j} \right)}.$$

$$(ii) \quad c'_i = c_i = \frac{c}{\lambda^i}.$$

$$(iii) \quad b'_i = c'_{i-1} - c'_i - a'_i.$$

CLAIM. -  $a'_i \lambda^i \rightarrow 0$ .

$$\begin{aligned} a'_i \frac{\lambda^i}{\prod_{j=n+2}^{n+i} \left( \lambda + \frac{1}{j} \right)} &= a'_i \lambda^2 \prod_{j=n+2}^{n+i} \frac{\lambda}{\lambda + \frac{1}{j}} \\ &= a'_i \lambda^2 \prod_{j=n+2}^{n+i} \left( 1 - \frac{1}{\lambda j + 1} \right) \xrightarrow{i \rightarrow \infty} 0 \end{aligned}$$

As a consequence  $b'_i \lambda^i \rightarrow \lambda c - c > 0$ . In particular  $b'_i < b'_{i-1}$  if  $i$  is sufficiently large.

Moreover, for  $i$  large enough,  $\frac{a'_i}{b'_i} \xrightarrow{i \rightarrow \infty} 0$ . This proves the proposition.  $\square$

*Proof of Proposition 2.2.* - If the thickness  $\tau(K) = \tau$  we have, for every  $\varepsilon$ , a quotient between a gap  $\bar{b}$  and a bridge  $\bar{a}$  such that  $\left| \tau - \frac{\bar{b}}{\bar{a}} \right| < \varepsilon$  and this remains true for diffeomorphisms  $C^1$  close to  $\varphi$ .  $\square$

### 3. Proof of Theorem B

Consider a  $C^2$  family  $(\tilde{\varphi}_\mu)_{\mu \in (b-\varepsilon, b+\varepsilon)}$  with a first homoclinic bifurcation at  $b$  (as in Remark 1.5). The non-wandering ( $\Omega$ -set) of  $\tilde{\varphi}_b$  consists of the union of  $l$  hyperbolic basic sets  $(\tilde{\Lambda}_i(b), i = 1, \dots, l)$  and the orbit of tangency, say  $\tilde{O}$ . We suppose that the periodic point  $\tilde{p}$  involved with the tangency is in  $\tilde{\Lambda}(b) = \tilde{\Lambda}_i(b)$ .

Without loss of generality, as in Section 2 we will assume that  $\tilde{\Lambda}(b)$  is a "horse-shoe" and  $\tilde{p}_\mu \in \tilde{\Lambda}(\mu)$  is the fixed point with positive eigenvalues.

Using the same technique of Section 2, we will perturb  $(\tilde{\varphi}_\mu)_\mu$ . As in Section 2 we have segments  $\bar{a}_\mu^*$ ,  $\bar{b}_\mu^*$ ,  $\bar{c}_\mu^*$  and a point  $r_\mu^* \in \bar{b}_\mu^*$  such that  $r_b^* = r_b$  is a point of the orbit of tangency and  $r_\mu^*$  depend continuously on  $\mu$  ( $\star = s$  or  $u$ ).

Now we define continuous functions  $\rho_{\mu,n} = \rho_{\mu,n}^s$  (and analogously  $\rho_{\mu,n}^u$ ) such that:

$$(1) \quad \rho_{\mu,n}(c_\mu) = \rho_{\mu,n}(0) = 0.$$

$$(2) \quad \rho_{\mu,n}|_{[\tilde{\varphi}_\mu^{-1}(r_\mu) - \varepsilon, c_\mu - \varepsilon]} = \frac{1}{n+1} \text{ with } \varepsilon > 0 \text{ small enough.}$$

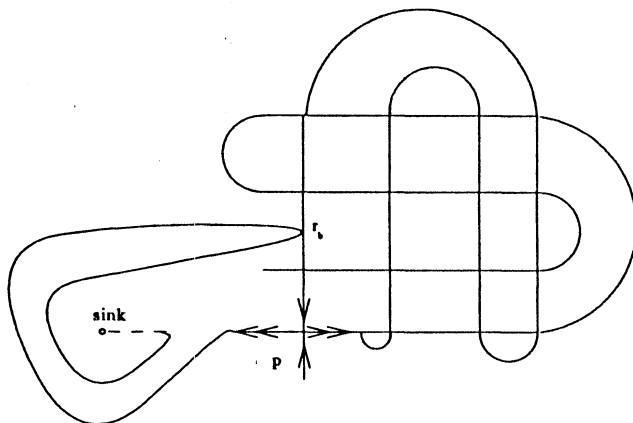


Fig. 3.1

$$(3) \rho_{\mu, n} |_{\tilde{\varphi}_{\mu}^{-i}([\tilde{\varphi}_{\mu}^{-1}(r_{\mu}) - \varepsilon, c_{\mu} + \varepsilon])} = \frac{1}{n + i} \quad i \geq 2.$$

$$(4) \int_{c_{\mu, i}}^{c_{\mu, i-1}} \rho_{\mu, n}(t) dt = 0, \quad c_{\mu, i} = \tilde{\varphi}_{\mu}^{-i}(c_{\mu}) \text{ and } c_{\mu, 0} = c_{\mu}.$$

Then, we approximate the family  $(\tilde{\varphi}_{\mu})_{\mu}$  by

$$\varphi(x, y) = \left( \tilde{\varphi}_{1, \mu}(x) - \int_x^{c_{\mu}^u} \rho_{\mu, n}^u(t) dt, \tilde{\varphi}_{2, \mu}(y) - \int_y^{c_{\mu}^s} \rho_{\mu, n}^s(t) dt \right)$$

and we extend this perturbation as in Section 2.

REMARK 3.1. – *If we repeat the calculations of Section 2 we obtain that*

$$\frac{l(\tilde{\varphi}_{\mu}^{-i}([\tilde{\varphi}_{\mu}^{-1}(r_{\mu}) - \varepsilon, c_{\mu} + \varepsilon]))}{c_{\mu, i-1}} \xrightarrow{i \rightarrow \infty} 0$$

where  $l(I)$  denotes the length of the interval  $I$ .

REMARK 3.2. – *The foliations  $\mathcal{F}_{\mu}^s$  and  $\mathcal{F}_{\mu}^u$  are  $C^1$  (the tangent vectors depend  $C^1$  on the parameter and the point) and when we make the perturbation above, they remain invariant in the “square”  $Q_{\mu}$ .*

In fact we need  $\int_y^{c_{\mu}} \rho_{\mu, n}(t) dt$  to be a  $C^1$  function on both  $y$  and  $\mu$  (we also want that it has small derivatives but, once we have that it is  $C^1$ , it is enough to multiply it by a small constant). In order to assume that this is indeed possible let us describe the definition of  $\rho_{\mu, n}$  in somewhat more detail. Let us call  $D_{\mu} = (c_{\mu, 1}, c_{\mu}]$  and define functions  $h_{\mu, i} : D_{\mu} \rightarrow \mathbb{R}$  such that:

$$(1) \int_{D_{\mu}} h_{\mu, i}(t) dt = 0.$$

$$(2) h_{\mu, i} |_{[\tilde{\varphi}_{\mu}(r_{\mu}) - \varepsilon, c_{\mu}]} \equiv \frac{1}{i}.$$

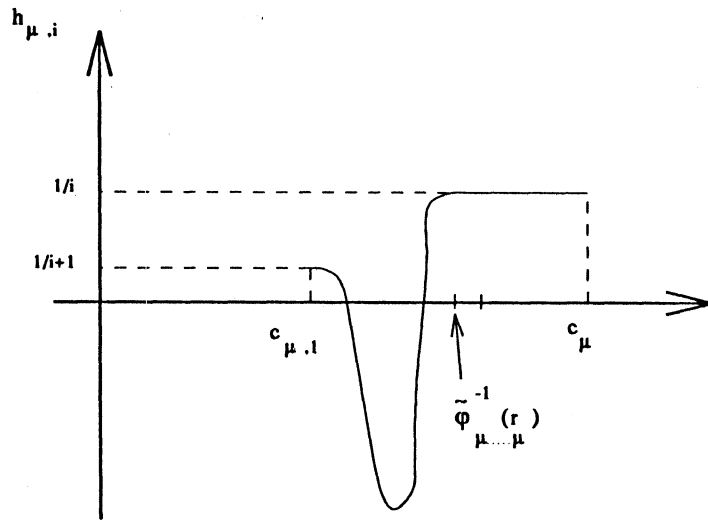


Fig. 3.2.

$$(3) \quad h_{\mu, i} |_{(c_{\mu, i}, c_{\mu, i+1} + \varepsilon]} \equiv \frac{1}{i + 1}.$$

These functions can be taken in such a way that:

- (i)  $\frac{\partial}{\partial \mu} \Big|_{\mu = \mu_0} h_{\mu, i}(y_0)$  is a continuous function on  $(\mu_0, y_0)$ .
- (ii)  $\left| \frac{\partial}{\partial \mu} \Big|_{\mu = \mu_0} h_{\mu, i}(y_0) \right| < K_1$  where  $K_1$  does not depend neither on  $i$  neither on  $(\mu_0, y_0)$ .
- (iii)  $|h'_{\mu_0, i}(y_0)| < K_2$  where  $K_2$  does not depend neither on  $i$  nor on  $(\mu_0, y_0)$ .

Then, we define  $\rho_{\mu, n}$  in the following way:

$$\rho_{\mu, n}(y) = h_{\mu, n+i}(\lambda_{\mu}^i y) \quad \text{if } y \in \varphi_{\mu}^{-i} D_{\mu}, \quad i \geq 1.$$

In  $D_{\mu}$  we define  $\rho_{\mu, n}$  in a similar way with the condition that  $\rho_{\mu, n}(c_{\mu}) = 0$ .

LEMMA 3.3. -  $\frac{\partial}{\partial \mu} \int_y^{c_{\mu}} \rho_{\mu, n}(t) dt$  is a continuous function on  $(\mu, y)$ .

Proof. - Suppose that  $y \in (c_{\mu, i}, c_{\mu, i-1})$ . Then,

$$\begin{aligned} & \left| \frac{\partial}{\partial \mu} \int_y^{c_{\mu}} \rho_{\mu, n}(t) dt \right| \\ &= \left| \frac{\partial}{\partial \mu} \int_y^{c_{\mu, i-1}} \rho_{\mu, n}(t) dt + \frac{\partial}{\partial \mu} \int_{c_{\mu, i-1}}^{c_{\mu}} \rho_{\mu, n}(t) dt \right| \\ &= \left| \frac{\partial}{\partial \mu} \int_y^{c_{\mu, i-1}} \rho_{\mu, n}(t) dt \right| = \left| \frac{\partial}{\partial \mu} \int_y^{\lambda_{\mu}^{-(i-1)}} h_{\mu, n+i-1}(\lambda_{\mu}^{i-1} t) dt \right| \end{aligned}$$

$$\begin{aligned}
 &= \left| h_{\mu, n+i-1} (\lambda_{\mu}^{i-1} t) (-i+1) \lambda_{\mu}^{-i} \frac{\partial \lambda_{\mu}}{\partial \mu} \right. \\
 &\quad \left. + \int_y^{\lambda_{\mu}^{-(i-1)}} \left( \frac{\partial h_{\mu, n+i-1}}{\partial \mu} (\lambda_{\mu}^{i-1} t) + h'_{n+i-1, \mu} (\lambda_{\mu}^{i-1} t) (i-1) \lambda_{\mu}^{i-2} \frac{\partial \lambda_{\mu}}{\partial \mu} t \right) dt \right| \\
 &\leq K_4 i \lambda_{\mu}^{-i} + \int_y^{\lambda_{\mu}^{-(i-1)}} \left| (K_1 + K_2 (i-1) \lambda_{\mu}^{i-2}) \frac{\partial \lambda_{\mu}}{\partial \mu} \lambda_{\mu}^{-(i-1)} \right| dt \\
 &\leq K_4 i \lambda_{\mu}^{-i} + (K_1 + K_3 (i-1) \lambda_{\mu}^{-1}) |\lambda_{\mu}^{-(i-1)} - y| \\
 &\leq K_5 i \lambda_{\mu}^{-(i-1)} \xrightarrow{i \rightarrow \infty} 0.
 \end{aligned}$$

As  $\frac{\partial \rho_{\mu, n}}{\partial \mu} (0) = 0$  the lemma is proved.  $\square$

A main step in the proof of Theorem B is

PROPOSITION 3.4. – *There exists a dense set  $\mathcal{A} \subset \mathcal{B}^1$  such that*

$$\frac{m(H_{\varepsilon}(\{\varphi_{\mu}\}))}{\varepsilon} \xrightarrow{\varepsilon \searrow 0} 1 \quad (m \text{ denotes the Lebesgue measure})$$

for any  $\{\varphi_{\mu}\} \in \mathcal{A}$ .

For the proof of this proposition we just show that all perturbed families  $(\varphi_{n, \mu})_{\mu}$  as above satisfy its conclusion. Before we can do that we have to introduce two auxiliary results.

Note first that, up to iterating negatively, resp. positively, the foliations  $\mathcal{F}_{\mu}^s$ , resp.  $\mathcal{F}_{\mu}^u$ , we may suppose that they are defined in a neighbourhood of the first point of tangency. We define  $l_{\mu}$  as the differentiable curve where the leaves of  $\mathcal{F}_{\mu}^s$  and  $\mathcal{F}_{\mu}^u$  are tangent. We call  $\mathcal{F}^s(u)(A) = \{F^s(u) \in \mathcal{F}^s(u); F^s(u) \cap A \neq \emptyset\}$ . Let  $\mathcal{O}_{\mu}^s(u)$  be a  $\varphi_{\mu}$ -orbit of  $r_{\mu}^s(u)$ . Without loss of generality we suppose  $b = 0$ .

PROPOSITION 3.5. – *For each  $\alpha > 0$ , there is a  $\mu_1(\alpha) > 0$ , such that for every  $\mu \in (0, \mu_1(\alpha))$  such that the distance between  $\mathcal{F}^s(\Lambda_{\mu} \cup \mathcal{O}_{\mu}^s) \cap l_{\mu}$  and  $\mathcal{F}^u(\Lambda_{\mu} \cup \mathcal{O}_{\mu}^s) \cap l_{\mu}$  is at least  $\alpha\mu$ ,  $\varphi_{\mu}$  is hyperbolic.*

This proposition follows, essentially, from the arguments in [PT2] and we postpone (a sketch of) its proof until the end of the section. We also use another result from [PT2] but give here a shorter proof.

PROPOSITION 3.6 (see [PT2]). – *Let  $A(\mu) = \{\lambda^i(\mu)\}_{i \geq 0} \cup \{0\}$  with  $0 < \lambda(\mu) < 1$  for any  $\mu \in (-\delta, \delta)$  and  $\lambda$  be a Lipschitz function of  $\mu$ . Then, for each  $c > 0$  there is a  $\mu(c) > 0$  such that for each  $0 < \mu_1 < \mu(c)$*

$$A(0)_{\mu c} \supset A(\mu) \cap [0, \mu_1]$$

for all  $0 < \mu < \mu_1$  ( $A_{\varepsilon}$  means an  $\varepsilon$ -neighbourhood of  $A$ ).

*Proof.* – Note that  $|\lambda^k(\mu) - \lambda^k(0)| \leq ((1 + K\mu)^k - 1) \lambda^k(0) \leq k(1 + K\mu)^{k-1} K\mu \lambda^k(0)$ , where  $K$  is a Lipschitz constant for  $\lambda(\mu)$ . Now, if  $k$  is large enough and  $\mu$  is small enough (depending only on  $\lambda(0)$ ), we have  $Kk(1 + K\mu)^{k-1} \leq \lambda^{-(k/2)}$ .

Given  $c > 0$  take  $\mu(c) = c^2$ . Then for every  $\mu \leq \mu_1 \leq \mu(c)$

$$|\lambda^k(\mu) - \lambda^k(0)| \leq \mu \lambda^{k/2} \leq \mu \sqrt{\mu_1} \leq c\mu$$

whenever  $\lambda^k(0) \in [0, \mu_1]$ .  $\square$

*Proof of Proposition 3.4.* – Now, we consider the sequences  $\{c_{m,\mu}^s\}$  and  $\{c_{m,\mu}^u\}$ . As a consequence of our construction  $c_{m,\mu}^u = \lambda_1^m(\mu)$  and  $c_{m,\mu}^s = \lambda_2^{-m}(\mu)$  where  $\lambda_1$  and  $\lambda_2$  are, respectively, the contracting and expanding eigenvalues.

Given  $\alpha$  and for  $\mu$  small enough,  $\{c_{m,0}^u\} \cap [0, \mu]$  and  $\{c_{m,0}^s\} \cap [0, \mu]$  can be covered by  $N_{\alpha\mu}(\{c_{m,0}^u\})$  and  $N_{\alpha\mu}(\{c_{m,0}^s\})$  intervals of length  $\alpha\mu$  such that

$$N_{\alpha\mu}(\{c_{m,0}^u\}) < \frac{\log \alpha}{\log \lambda_1} + 2,$$

$$N_{\alpha\mu}(\{c_{m,0}^s\}) < \frac{\log \alpha}{\log \lambda_2^{-1}} + 2.$$

If  $\pi_{s,\mu}$  and  $\pi_{u,\mu}$  are the projections onto  $l_\mu$  along the stable and unstable foliations we have, shrinking  $\mu$  if necessary,

$$N_{\alpha\mu}(\pi_{u,\mu}\{c_{m,0}^u\}) < \frac{\log \alpha}{\log \lambda_1} + 2,$$

$$N_{\alpha\mu}(\pi_{s,\mu}\{c_{m,0}^s\}) < \frac{\log \alpha}{\log \lambda_2^{-1}} + 2$$

(we are assuming that the projectons have derivative 1, if this is not the case we have the same bounds but for  $N_{K\alpha\mu}$ , where  $K$  is the derivative of the projection). So, an  $\alpha\mu$ -neighborhood of  $\pi_{u,\mu}\{c_{m,0}^u\}$  can be covered with no more than  $3N_{\alpha\mu}(\pi_{u,\mu}\{c_{m,0}^u\})$  intervals of length  $\alpha\mu$ .

This means that

$$\tilde{B}^{(\mu,\alpha)} = \{\mu' \in (0, \mu); \text{distance between } \pi_{s,\mu'}\{c_{m,0}^s\} \\ \text{and } \pi_{u,\mu'}\{c_{m,0}^u\} + \mu' \text{ is less than } \alpha\mu\}$$

can be covered with no more than  $3N_{\alpha\mu}(\pi_{u,\mu}\{c_{m,0}^u\})N_{\alpha\mu}(\pi_{s,\mu}\{c_{m,0}^s\})$  intervals of length  $\alpha\mu$ .

Then

$$\frac{m(\tilde{B}^{(\mu,\alpha)})}{\mu} < 3 \left( \frac{\log \alpha}{\log \lambda_1} + 2 \right) \left( \frac{\log \alpha}{\log \lambda_2^{-1}} + 2 \right) \alpha$$

which goes to 0 as  $\alpha$  goes to 0.

We know (see Proposition 3.6) that for each  $\alpha > 0$  there exists  $\mu_1(\alpha)$  such that for any  $\mu \in (0, \mu_1(\alpha))$ ,  $\{c_{m,\mu}^u\} \cap [0, \mu]$  and  $\{c_{m,\mu}^s\} \cap [0, \mu]$  are contained, respectively, in a  $\frac{1}{4}\alpha\mu$ -neighbourhood of  $\{c_{m,0}^u\}$  and  $\{c_{m,0}^s\}$ .

In  $W^u(p_\mu)$  we define the set  $A(\mu)$  as  $(W^u(p_\mu) \cap \Lambda_\mu) \cup \mathcal{O}_\mu^s$ . In the same way we define  $B(\mu)$  in  $W^s(p_\mu)$ .

We observe that, as a consequence of Remark 3.1, given  $\alpha > 0$  there exists  $\bar{\mu} > 0$  such that for  $0 < \mu < \bar{\mu}$

$$A(\mu) \cap [0, \mu] \subset_{(1/4)\alpha\mu} \{c_{m,\mu}^s\} \cap [0, \mu],$$

where  $\beta\{c_n\} = \{\alpha \in [0, 1]; \exists c_i \text{ such that } |c_i - x| < \beta c_i\}$ . Analogous reasonings and conclusions hold for  $B(\mu)$  and  $\{c_m^u\}$ .

This, using Proposition 3.5, ends the proof.  $\square$

*Proof of Proposition 3.5.* – As we said before, this result is, essentially, a consequence of the arguments in [PT2] and so here we only sketch the main points.

The proof is based on the fact that the “new” nonwandering orbits are all contained in a small neighbourhood of  $\mathcal{F}^s(\Lambda_\mu \cup \mathcal{O}^s) \cap \mathcal{F}^u(\Lambda_\mu \cup \mathcal{O}^u)$ . An argument involving filtrations gives that the “new” nonwandering orbits have to pass through a small neighbourhood  $U_\mu$  of  $r_\mu$  and, as the distance of  $\mathcal{F}^s(\Lambda_\mu \cup \mathcal{O}^s) \cap l_\mu$  and  $\mathcal{F}^u(\Lambda_\mu \cup \mathcal{O}^u) \cap l_\mu$  is at least  $c\mu$  we can obtain a lower bound for the angle between leaves of order  $(c\mu)^{1/2}$ . As these orbits must stay a long time near the hyperbolic set  $\Lambda_\mu$  before returning to a neighbourhood of  $l_\mu$ , we can prove the hyperbolicity constructing cone fields (see [PT2], Section 4.4). A filtration argument also gives that  $\varphi_\mu$ , for these parameter values, has the nocycle property.  $\square$

Finally, we prove Theorem B. Let

$$C_n = \left\{ \{\varphi_\mu\} \in \mathcal{B}^1; \exists \delta = \delta(\{\varphi_\mu\}) > 0 \text{ such that } \frac{m(H_\delta(\{\varphi_\mu\}))}{\delta} > \left(1 - \frac{1}{n}\right) \right\}.$$

By Proposition 3.4  $C_n$  contains an open and dense set. Then,  $C = \bigcap_{n \geq 0} C_n$  is a residual set in  $\mathcal{B}^1$  such that for any  $\{\varphi_\mu\} \in C$ ,

$$\limsup_{\delta \rightarrow 0} \frac{m(H(\{\varphi_\mu\}) \cap [b, b + \delta])}{\delta} = 1.$$

REFERENCES

[AS] R. ABRAHAM and S. SMALE, *Non-genericity of  $\Omega$ -stability* (Proc. A.M.S. Symp. Pure Math., Vol. 14, 1970).  
 [AM] A. ARAÚJO and R. MAÑÉ, *On the existence of hyperbolic attractors and homoclinic tangencies for surfaces diffeomorphisms* (to appear).  
 [M1] R. MAÑÉ, *Contribution to the stability conjecture* (Topology, Vol. 17, 1978, pp. 383-396).  
 [M2] R. MAÑÉ, *A proof of the  $C^1$  stability conjecture* (Publ. I.H.E.S., Vol. 66, 1988, pp. 161-210).  
 [MM] A. MANNING and H. McCLUSKEY, *Hausdorff dimension for horseshoes* (Erg. Th. and Dyn. Syst., Vol. 3, 1983, pp. 251-261).  
 [N1] S. NEWHOUSE, *Nondensity of Axiom A (a) on  $S^2$*  (Proc. A.M.S. Symp. Pure Math., Vol. 14, 1970, pp. 191-202).  
 [N2] S. NEWHOUSE, *Diffeomorphisms with infinitely many sinks* (Topology, Vol. 13, 1974, pp. 9-18).

- [N3] S. NEWHOUSE, *The abundance of wild hyperbolic sets and non smooth stable sets for diffeomorphisms* (*Publ. I.H.E.S.*, Vol. 50, 1979, pp. 101-151).
- [P] J. PALIS, *On the  $C^1$   $\Omega$ -stability conjecture* (*Publ. I.H.E.S.*, Vol. 66, 1988, pp. 211-215).
- [PT1] J. PALIS and F. TAKENS, *Hyperbolicity and sensitive-chaotic dynamics at homoclinic bifurcations*, Cambridge University Press, 1993.
- [PT2] J. PALIS and F. TAKENS, *Hyperbolicity and the creation of homoclinic orbits* (*Ann. of Math.*, Vol. 125, 1987, pp. 337-374).
- [PV] J. PALIS and M. VIANA, *Continuity of Hausdorff dimension and limit capacity for horseshoes*, *Dynamical Systems (Lecture Notes in Math.*, Vol. 1331, 1988, pp. 150-160).
- [PY] J. PALIS and J. C. YOCCOZ, *Homoclinic bifurcations: Large Hausdorff dimension and non-hyperbolic behaviour* (*Acta Math.*, to appear).
- [S] C. SIMON, *Instability in  $\text{Diff}^1(\mathbb{T}^3)$*  (*Lecture Notes in Math.*, Vol. 206, 1971, pp. 94-97).
- [W] R. WILLIAMS, *The "DA" maps of Smale and structural stability* (*Proc. A.M.S. Symp. Pure Math.*, Vol. 14, 1970, pp. 329-334).

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R. URES  
IMERL,  
Facultad de Ingeniería,  
C.C. 30, Montevideo, Uruguay.