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BIFURCATION OF CONTRACTING SINGULAR CYCLES *

BY RAFAEL LABARCA

Dedicated to the memory of Professor R. Chuaqui (R.I.P.)

ABSTRACT. – The aim of this work is to continue the analysis of a new mechanism, the singular cycles, through which a vector field, depending on parameter, may evolve when the parameter varies from a vector field exhibiting simple dynamics into one having non-trivial dynamics. Specifically; if we start with a Morse - Smale vector field and move through a generic one - parameter family of vector fields to a contracting singular cycle and beyond, we reach a region filled up mostly with hyperbolic flows. In fact, the Lebesgue measure of parameter values corresponding to non Axiom A flows is zero. Moreover we provide a complete description of the bifurcation set that appear in these families.

1. Introduction

The aim of this work is to continue the analysis of a new mechanism, the singular cycles, introduced in [3] and [1] through which a vector field, depending on parameters, may envolve when the parameter varies from a vector field exhibiting simple dynamics into one having non-trivial dynamics.

Let M be a C^∞, m-dimensional, compact, connected, boundaryless, riemannian manifold. Let $X \in \mathcal{X}^r(M)$ be a C^r-vector field on M.

DEFINITION 1. – *A cycle for the vector field X is a compact, invariant set* $\Gamma \subset M$ *formed by:*

(i) *a finite number of singularities and periodic orbits* $\Gamma_0 = \{\sigma_0, \cdots, \sigma_n\}$;

(ii) *the complement* $\Gamma_1 = (\Gamma \setminus \Gamma_0)$ *is a set of non-periodic regular trajectories of the vector field* X *that satisfies:*

$(CC)_1$ *for any trajectory* $\gamma \subset \Gamma_1$, *there exists* $0 \le i \le n$ *such that* $\omega(\gamma) \subset \sigma_{(i+1) mod (n+1)}$ *and* $\alpha(\gamma) \subset \sigma_i$;

$(CC)_2$ *given* $0 \le i \le n$ *there exists a trayectory* $\gamma \subset \Gamma_1$ *such that* $\omega(\gamma) \subset \sigma_{(i+1) mod (n+1)}$ *and* $\alpha(\gamma) \subset \sigma_i$.

Here $\omega(\gamma)$ (respectively $\alpha(\gamma)$) denotes the ω-limit set (respectively the α-limit set) of the trayectory γ.

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A cycle will be called *singular* if it contains a singularity; *hyperbolic* if all the critical elements in Γ are hyperbolic.

In this article we will deal with a 3-dimensional, hyperbolic, singular cycle, $\Gamma \subset M^3$, that contains a unique singularity, $\sigma_0(X)$, and periodic orbits $\sigma_1(X), \cdots, \sigma_n(X)$, $n \geq 1$ (*Fig.* 1).

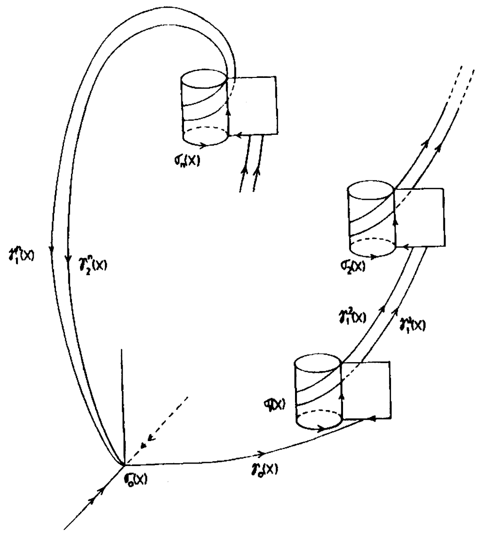

Fig. 1

We will assume the following regularity conditions:

(1) $\Gamma = \{\sigma_0(X), \gamma_0(X), \sigma_1(X), \gamma_1^1(X), \gamma_1^2(X), \cdots, \sigma_n(X), \gamma_n^1(X), \gamma_n^2(X)\}$, where $W_i^u = W_{\sigma_i(X)}^u$ intersects transversally $W_{(i+1)\mathrm{mod}(n+1)}^s$ along the orbits $\gamma_i^1(X) \cup \gamma_i^2(X), i = 1, \cdots, n$.

We let $\sigma_0(Y), \sigma_1(Y), \cdots, \sigma_n(Y)$ denote, respectively, the analytic continuation of $\sigma_0(X), \sigma_1(X), \cdots, \sigma_n(X)$; for any $Y \in \mathcal{U}_X$. Here $\mathcal{U}_X$ denotes a small neighborhood of X in $\mathcal{X}^r(M^3)$ with the usual C^r-topology, $r \geq 3$;

(2) For any $Y \in \mathcal{U}_X$, the eigenvalues of $D_{\sigma_0(Y)}(Y) : T_{\sigma_0(Y)}(M^3) \to T_{\sigma_0(Y)}(M^3)$ are real numbers $-\lambda_3(Y) < -\lambda_1(Y) < 0 < \lambda_2(Y)$ and satisfy a k-Sternberg condition, k big enough to guarantee that we have C^3-linearizing coordinates which depend C^2 on $Y \in \mathcal{U}_X$ in a neighborhood of $\sigma_0(Y)$;

(3) For every $p \in \gamma_0(X)$ and every invariant manifold of X, passing through $\sigma_0(X)$ and $p, W(\sigma_0(X))$, and tangent (at $\sigma_0(X)$) to the space spanned by the eigenvectors associated to $-\lambda_1(X)$ and $\lambda_2(X)$, we have $T_p(W(\sigma_0(X))) + T_p(W_{\sigma_1(X)}^s) = T_pM^3$;

(4) Γ is isolated: that is, there exists an open set $U \supset \Gamma$ such that $\cap_t X_t(U) = \Gamma$; here X_t denotes the flow defined by the vector field X;

(5) Let $Q_i \subset M^3, 1 \leq i \leq n$, be a transversal section at $q_i(Y) \in \sigma_i(Y)$. We let $P_i(Y) : V_i \subset Q_i \to Q_i$ denote the first return map defined in a neighborhood of $q_i(Y)$, any $Y \in \mathcal{U}_X$. We assume the eigenvalues of $D_{q_i}P_i : T_{q_i}(V_i) \to T_{q_i}(Q_i)$ are real numbers

and satisfy a k-Sternberg condition, k big enough to guarantee that we have C^3-linearizing coordinates which depend C^2 on $Y \in \mathcal{U}_X$ in a neighborhood of $q_i(Y)$;

(6) The number $\alpha(Y) = \dfrac{\lambda_1(Y)}{\lambda_2(Y)}$ is greater than one and

$$\beta(Y) = \frac{\lambda_3(Y)}{\lambda_2(Y)} > \alpha(Y) + 2 .$$

A cycle Γ as above is called a *contracting singular cycle.*

We let $\Gamma(Y, U) \subset M$ denote the set $\cap_t Y_t(U)$, for $Y \in \mathcal{U}_X$ (that is, the maximal invariant set in the neighborhood U for the vector field Y).

We let $\gamma_0(Y), \gamma_1^1(Y), \gamma_1^2(Y), \cdots ; \gamma_n^1(Y), \gamma_n^2(Y)$ denote, respectively, the analytic continuation of the trajectories $\gamma_0(X), \cdots, \gamma_n^2(X)$ for any $Y \in \mathcal{U}_X$. These trajectories are included in the unstable manifolds $W^u(\sigma_0(Y)), \cdots, W^u(\sigma_n(Y))$ respectively.

Comment: It is easy to see that there exists a codimension-one submanifold, $\mathcal{N} \subset \mathcal{X}^r(M)$, containing X such that:

(i) $Y \in \mathcal{N}$ implies $\Gamma(Y, U) = \{\sigma_0(Y), \gamma_0(Y), \cdots, \gamma_n^2(Y)\}$;

(ii) $(\mathcal{U}_X \setminus \mathcal{N})$ has two connected components and one of them, which is denoted $\mathcal{U}^-$, is such that $Y \in \mathcal{U}^-$ implies $\Gamma(Y, U) = \{\sigma_0(Y), \sigma_1(Y), \gamma_1^1(Y), \gamma_1^2(Y), \cdots, \sigma_n(Y), \gamma_n^1(Y), \gamma_n^2(Y)\}$; and

(iii) Bifurcations for the maximal invariant set $\Gamma(Y, U)$ may appear only for $Y \in \mathcal{U}^+ = (\mathcal{U}_X \setminus (\mathcal{N} \cup \mathcal{U}^-))$.

$\mathcal{U}_H^+$ is defined to be the set of $Y \in \mathcal{U}^+$ such that $\Gamma(Y, U)$ consists of Γ_0, a transitive hyperbolic set and a denumerable number of isolated hyperbolic periodic orbit, and $\mathcal{U}_A^+$ as the set of $Y \in \mathcal{U}^+$ such that $\Gamma(Y, U)$ consists of $\sigma_0(Y)$, a transitive hyperbolic set, a hyperbolic attracting periodic orbit (which is contained in the closure of the trajectory $\gamma_0(Y)$), and a denumerable number of isolated hyperbolic periodic orbit.

Under the above conditions we have the following :

THEOREM 1. – *a)* $\mathcal{U}^+ \setminus (\mathcal{U}_H^+ \cup \mathcal{U}_A^+)$ *is laminated by codimension-one* C^1*-submanifolds of the following type:*

a_1) *those laminas that present a saddle-node or a flip bifurcation for periodic orbits;*

a_2) *those laminas that present a contracting singular cycle;*

a_3) *those laminas that present a homoclinic behavior for the singularity; and*

a_4) *those laminas that present a recurrent behavior for the analytic continuation of the trayectory* $\gamma_0(X)$.

Moreover all elements in the same lamina have the same dynamics in the neighborhood U (that is, given a lamina $L \subset \mathcal{U}^+ \setminus (\mathcal{U}_H^+ \cup \mathcal{U}_A^+)$ *and* $Y_1, Y_2 \in L$, *there exists a homeomorphism* $h : U \to U$ *that is a topological equivalence between* $Y_1|_U$ *and* $Y_2|_U$).

b) Any $Y \in \mathcal{U}_H^+ \cup \mathcal{U}_A^+$ *is structurally stable.*

c) For any $Y \in (\mathcal{U}^+ \setminus (\mathcal{U}_H^+ \cup \mathcal{U}_A^+))$, $\Gamma(Y, U)$ *decomposed into a chain recurrent expansive set, a denumerable number of isolated hyperbolic periodic orbits plus the closure of the trajectory* $\gamma_0(Y)$.

Now let $\{X_\mu\} \subset \mathcal{U}_X$ be a one-parameter family of vector fields such that $X_{\mu=0} \in \mathcal{N}$ and $\{X_\mu\}$ is transversal to $\mathcal{N}$ at $\mu = 0$.

THEOREM 2. – *There exists* $\nu = \nu(X_\mu) > 0$ *such that :*

$$m(\{\mu; 0 \leq \mu \leq \nu, X_\mu \notin (\mathcal{U}_H^+ \cup \mathcal{U}_A^+)\}) = 0$$

(*here* $m(A)$ *denotes the Lebesgue measure of the set* $A \subset \mathbf{R}$).

Following [3] we may now state a corollary for Theorem 1.

COROLLARY. – *Let* $\{Y_\mu\}$ *be another one-parameter family transversal to* $\mathcal{N}$ *at* $\mu = 0$. *There exists a reparametrization* $\rho : [0, \nu(X_\mu)] \to [0, \nu(Y_\mu)]$ *and, for each* $\mu \in [0, \nu(X_\mu)]$, *a homeomorphism* $h_\mu : U \to U$ *that is a topological equivalence between* $X_\mu|_U$ *and* $Y_{\rho(u)}|_U$.

Remark. – *a*) A particular case of Theorem 2 was proven by Pacifico and Rovella in [2]. In their case, Γ is given by $\{\sigma_0(X), \gamma_0(X), \sigma_1(X), \gamma_1(X)\}$ and the associated first return map preserves orientation. A more general case of the Pacifico-Rovella result was proven by San Martin in [8].

The techniques they use to prove their result do not apply in our case.

b) For the case $\alpha(X) < 1$ (an expanding singular cycle), theorems 1 and 2 and the above Corollary 1 were proven by Bamón, Labarca, Mañé and Pacifico in [1].

c) The main difference between the unfolding of expanding and contracting singular cycles is the following: the unfolding of contracting singular cycles must have saddle-node and flip bifurcations whereas the unfolding of the expanding singular cycles does not.

ACKNOWLEDGEMENTS

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2. Proof of Theorem 1

This Chapter is organized in the following way : In section 2.1 we make the necessary change of coordinates to obtain a simpler form of the First Return Map. Section 2.2 is devoted to give a characterization of the elements in $\mathcal{U}_H^+ \cup \mathcal{U}_A^+$. Sections 2.3 - 2.11 are devoted to the study of the one dimensional dynamics associated to a contracting singular cycle. In particular we obtain the proof of Theorem 1.

2.1. CHANGE OF COORDINATES AND THE FIRST RETURN MAP

Let $X \in \mathcal{X}^r(M^3)$ be a vector field having a contracting singular cycle, Γ, with isolated neighbohood $U \subset M$. For the sake of simplicity we will assume Γ contains a unique periodic orbit, and later on in Section III.5 we will make comments on the general case. Here Γ is the union of a singularity $\sigma_0 = \sigma_0(X)$, a periodic orbit $\sigma_1 = \sigma_1(X)$, an orbit $\gamma_0 = \gamma_0(X) \subset W^u_{\sigma_0}$ of nontransversal intersection between $W^u_{\sigma_0}$ and $W^s_{\sigma_1}$, and two orbits of transversal intersection between $W^u_{\sigma_1}$ and $W^s_{\sigma_0}$, $\gamma_1^1 = \gamma_1^1(X)$ and $\gamma_1^2 = \gamma_1^2(X)$.

Let Q be a cross section to the flow X at $q \in \sigma_1$ parametrized by $\{(x,y)/|x|,|y| \leq 1\}$ and satisfying $W^s_{\sigma_1} \supseteq \{(x,0);|x| \leq 1\}$ and $W^u_{\sigma_1} \supseteq \{(0,y);|y| \leq 1\}$.

Let $p = p(X)$ be the first intersection between γ_0 and Q. Then $p = (x_0, 0) = (x_0(X), 0)$ and we assume $x_0 > 0$. It is clear that a first return map, $F = F(X)$, is defined on a subset of Q. Moreover if $q_1 = (0, y_1) = (0, y_1(X))$ and $q_2 = (0, y_2) = (0, y_2(X))$ are such that their ω-limit set is σ_0, then there are horizontal strips $R_1 = R_1(X)$ and $R_2 = R_2(X)$ such that F is defined on $R_1 \cup R_2$. Here a horizontal strip is a closed set $C \subset Q$ bounded (in Q) by two disjoint continuous curves connecting the vertical sides of $Q, \{(-1,y)/|y| \leq 1\}$, and $\{(1,y)/|y| \leq 1\}$.

Since Γ is isolated, we have that $\Gamma \cap Q \subset \{(x,y)/y \geq 0\}$ and that :

$$F(R_1 \cup R_2) \subset \{(x,y)/y \leq 0\}$$

(*See Fig.* 2).

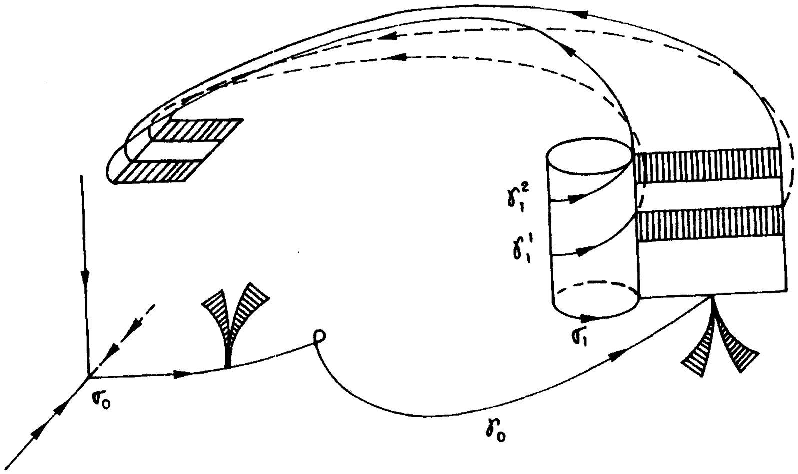

Fig. 2

If $Y \in \mathcal{X}^r$ is near X, then $W^s(\sigma_1(Y))$ intersects Q at a curve $c(Y)$, and the first intersection of $W^u(\sigma_0(Y))$ with Q is a point $p(Y)$. Note that both $c(Y)$ and $p(Y)$ vary smoothly with Y. The implicit function theorem on Banach spaces implies that the condition $p(Y) \in c(Y)$ defines a C^2-codimension one submanifold, $\mathcal{N}$, in a neighborhood of $X, \mathcal{U} \subset \mathcal{X}^r$, such that $(\mathcal{U} \setminus \mathcal{N})$ has two connected components: one of them, which we denote by $\mathcal{U}^-$, is characterized by $p(Y) \in Q$ and lies below $c(Y)$; we let $\mathcal{U}^+$ denote the other component.

Clearly, $Y \in \mathcal{U}^-$ implies $\Gamma(Y,U) = \{\sigma_0(Y), \sigma_1(Y), \gamma_1^1(Y), \gamma_1^2(Y)\}$ and hence the dynamics of the vector field Y in U is simple.

If $Y \in \mathcal{U}^+$, then $\sigma_1(Y)$ has transversal homoclinic orbits and therefore Y does not have simple dynamics in U. As before we note that there exists a first return map F_Y defined on a subset of Q, every $Y \in \mathcal{U}^+$.

Since $\Gamma(Y,U)$ is the saturation of $\Gamma(Y,U) \cap Q$ by the flow Y_t, and $\Gamma(Y,U) \cap Q$ is the maximal invariant set of F_Y, it is necessary to describe the dynamics of F_Y to understand the dynamics of Y on $\Gamma(Y,U)$. For this we choose coordinates (x,y) on Q, that depend C^2 on Y, such that:

(i) $\{(x,0)/|x| \leq 1\} \subset W^s(\sigma_1(Y))$;

(ii) $\{(0,y)/|y| \leq 1\} \subset W^u(\sigma_1(Y))$;

(iii) $\Gamma(Y,U) \cap Q \subset Q^+ = \{(x,y)/x \geq 0, y \geq 0\}$; and

(iv) the analytic continuation of the point $p = p(X) = \gamma_0(X) \cap Q$ is a point $p(Y) = (x(Y), y(Y))$, with $0 < x(Y) < 1$.

Note that $Y \in \mathcal{U}^+$ if and only if $y(Y) > 0$.

Moreover $\Gamma(Y,U) \nsubseteq \{\sigma_0(Y), \sigma_1(Y), \gamma_1^1(Y), \gamma_1^2(Y)\}$ if and only if $y(Y) \geq 0$.

For $Y \in \mathcal{U}$ such that $y(Y) \geq 0$, let $q_1(Y) = (0, y_1(Y))$ (resp., $q_2(Y) = (0, y_2(Y))$) be the analytic continuation of the point q_1 (resp., q_2). Since $\omega(q_i(Y)) = \sigma_0(Y)$ and $\alpha(q_i(Y)) = \sigma_1(Y), i = 1,2$, there are horizontal strips $R_Y^i \ni q_i(Y)$ such that the positive orbits of points at R_Y^i first pass near $\sigma_0(Y)$ and afterwards return to Q. On the other hand, the positive orbits of points at a horizontal strip R_Y containing $W^s(\sigma_1(Y)) \cap Q$ goes around the closed orbit $\sigma_1(Y)$ and then return to Q (*see Fig*. 3).

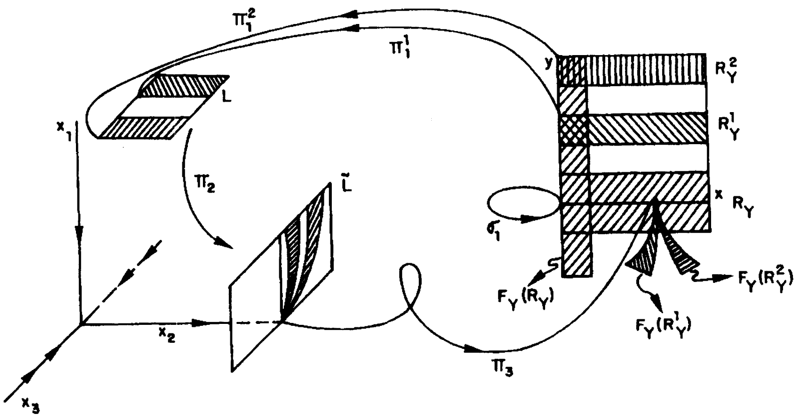

Fig. 3

Therefore F_Y is defined on $R_Y \cup R_Y^1 \cup R_Y^2$, and the restriction of F_Y to R_Y coincides with the Poincaré map, P_Y, associated to $\sigma_1(Y)$. We further assume P_Y is linear on R_Y.

Let $\xi_Y > 1$ and $\tau_Y < 1$ be the eigenvalues of $DP_Y(0,0)$. We have $R_Y^1 = \{(x,y)/x \geq 0, \Theta_Y^1(x) \leq y \leq \Theta^1\}, R_Y^2 = \{(x,y)/x \geq 0, \Theta^2 \leq y \leq \Theta_Y^2(x)\}$, where $\Theta_Y^i(x) = \Theta^i(Y,x)$ is a smooth real function satisfying $\{(x, \Theta_Y^i(x)), 0 \leq x \leq 1\} \subseteq W^s(\sigma_0(Y))$ and $(0, \Theta_Y^i(0)) = q_i(Y), i = 1,2$. Moreover if $\delta_Y^i(x) = \delta^i(Y,x)$ is such that $\{(x, \Theta_Y^i(x) + (-1)^{i+1}\delta_Y^i(x)), 0 \leq x \leq 1\} \subset F_Y^{-1}(\{(x,0); 0 \leq x \leq 1\}) \subset F_Y^{-1}(W^s(\sigma_1(Y)))$ $i = 1,2$, then there is $\varepsilon > 0$ such that $\Theta^1 - \varepsilon > \Theta_Y^1(x) + \delta_Y^1(x)$ and $\Theta^2 + \varepsilon < \Theta_Y^2(x) - \delta_Y^2(x)$, every x.

Making a linear change of coordinates we may also assume that

(v) $|(\Theta_Y^i)'(x)| < \dfrac{1}{100}$ and that δ_Y goes to zero uniformly in the C^2-topology when Y approaches $\mathcal{N}$.

Clearly $R_Y = \{(x,y)/x \geq 0, 0 \leq y \leq \xi_Y^{-1}\Theta_Y(x)\}$ and $F_Y(x,y) = (\tau_Y x, \xi_Y y)$, for $(x,y) \in R_Y$.

To obtain the expressions of $F_Y(x,y)$, for $(x,y) \in R_Y^1 \cup R_Y^2$, we proceed as follows:

Let $-\lambda_3(Y) < -\lambda_1(Y) < 0 < \lambda_2(Y)$ be the eigenvalues of $DY(\sigma_0(Y))$. We set $\alpha(Y) = \dfrac{\lambda_1(Y)}{\lambda_2(Y)}$ and $\beta(Y) = \dfrac{\lambda_3(Y)}{\lambda_2(Y)}$.

For $Y \in \mathcal{U}$, let (x_1, x_2, x_3) be C^3-linearizing coordinates, in a neighborhood $U_0 \ni \sigma_0(Y)$, that depend C^2 on Y. We let L and $\tilde{L}$ denote the planes $x_1 = 1$ and $x_2 = 1$, respectively.

For $(x,y) \in R_Y^i$, we have $F_Y(x,y) = \pi_3 \circ \pi_2 \circ \pi_1^i(x,y) = (f_Y^i(x,y), g_Y^i(x,y))$ where:

(*a*) $\pi_1^i : V_i \subset Q^+ \to L$ is a diffeomorphism such that $\pi_1^i(x, \Theta_Y^i(x)) = (x_3, 0)$, for $0 \leq x \leq 1$, and $D\pi_1^i(x,y) = \begin{bmatrix} a_i(x,y) & b_i(x,y) \\ c_i(x,y) & d_i(x,y) \end{bmatrix}$ where $k_1 \leq |a_i(x,y)|, |d_i(x,y)| \leq K_1$, and k_1, K_1 are positive real constants. Up to replacing $\{(x, \Theta_Y^i(x)), x \in [0,1]\}$ with some negative iterate of it (and shrinking $\mathcal{U}$) if necessary; we may assume that there are $0 < \eta << 1$ such that $\dfrac{|c_i(x,y)|}{|d_i(x,y)|} \leq \eta$, every $(x,y) \in R_Y^i$ and $Y \in \mathcal{U}^+$;

(*b*) $\pi_2 : L \to \tilde{L}$ is given by $\pi_2(x_3, x_2) = (\tilde{x}_3 = x_3 x_2^{\beta_Y}, \tilde{x}_1 = x_2^{\alpha_Y})$;

(*c*) $\pi_3 : \tilde{L} \to Q$ is a diffeomorphism such that

$$D\pi_3(\tilde{x}_3, \tilde{x}_1) = \begin{bmatrix} \tilde{a}(\tilde{x}_3, \tilde{x}_1) & \tilde{b}(\tilde{x}_3, \tilde{x}_1) \\ \tilde{c}(\tilde{x}_3, \tilde{x}_1) & \tilde{d}(\tilde{x}_3, \tilde{x}_1) \end{bmatrix}$$

with $k_2 \leq |\tilde{a}(\tilde{x}_3, \tilde{x}_1)|, |\tilde{d}(\tilde{x}_3, \tilde{x}_1)| \leq K_2$, some positive constants k_2, K_2. Moreover, by replacing $p(Y)$ with some positive iterate of it (also contained in $W^u(\sigma_0(Y)) \cap S$), if necessary, we may assume that the quotient $|\tilde{b}|/|\tilde{d}|$ is small enough, and hence that $|\tilde{b}|/|\tilde{d}| \leq \eta$, some small $\eta > 0$.

We now state a very useful lemma that establishes the existence of a C^3-invariant stable foliation for F_Y that depends C^2 on Y. The proof follows from the tecniques in [4]; e.g. as may be found in [1] and [5].

LEMMA 1. – *For every $Y \in \mathcal{U}$, there exists an invariant C^3 stable foliation for F_Y, $\mathcal{F}_Y^s$, that depends C^2 on Y.*

After a C^3 change of coordinates, this lemma implies that $\Theta_Y^i(x), \delta_Y^i(x)$ and $g_Y^i(x,y)$ are maps that do not depend on x.

For the sake of simplicity, we assume that $\Theta_Y^2(x) \equiv 1$ and that $\Theta_Y^1(x) = 1 - \delta$. We also have $c_i(x,y) \equiv 0$. Since $\pi_1^i(x,y)$ is a diffeomorphism, we have that $a_i(x,y) \neq 0$ and that $d_i(x,y) \neq 0$, every (x,y). Thus we conclude that there are real positive constants C and K such that:

(*d*)

$$0 \leq \left| \frac{\partial}{\partial x} f_Y^i(x,y) \right| \leq K x_2^{\beta_Y} + r_1^i(x,y),$$

$$\left| \frac{\partial}{\partial y} f_Y^i(x,y) \right| = K x_2^{\alpha_Y - 1} + r_2^i(x,y)$$

and

$$\left| \frac{\partial}{\partial y} g_Y^i(x) \right| \leq C x_2^{\alpha_Y - 1} + r_3^i(y)$$

where, respectively, $|r_1^i(x,y)| \leq$ (constant) $\cdot\, x_2^{\beta_Y - 1}, |r_2^i(x,y)| \leq$ (constant) $\cdot\, x_2^{\beta_Y}$ and $|r_3^i(y)| \leq$ (constant) $\cdot\, x_2^{\alpha_Y}$. In the above inequalities we replace x_2 with $y - (1-\delta)$ or $1 - y$, according that $i = 1$ or 2.

Moreover,

(*e*) $f_Y^1(x, 1-\delta) = x_Y = f_Y^2(x,1)$, for $x \in [0,1]$, and $g_Y^1(1-\delta) = y_Y = g_Y^2(1)$;

(*f*)

$$f_Y^1(x, 1-\delta+\delta_Y^1) \subset \{(x,0), x \in]0,1[\},$$
$$f_Y^2(x, 1-\delta_Y^2) \subset \{(x,0); x \in]0,1[\}, \text{ any } x \in [0,1],$$

and $g_Y^1(1-\delta+\delta_Y^1) = 0 = g_Y^2(1-\delta_Y^2)$.

Conditions (d), (e) and (f) imply $\delta_Y^i = A_Y^i y_Y^{1/\alpha_Y}$, where A_Y^i is a positive constant for $i = 1,2$.

Finally, by making another C^3-change of coordinates, we obtain $F_Y(x,y) = (f_Y(x,y), g_Y(y))$, with

$$g_Y(y) = \begin{cases} \xi_Y y, & \text{for } y \in [0, \xi_Y^{-1}] \\ y_Y - J(Y,y)(y-(1-\delta))^{\alpha_Y}, & \text{for } y \in [1-\delta, 1-\delta+\delta_Y^1] \\ y_Y - K(Y,y)(1-y)^{\alpha_Y}, & \text{for } y \in [1-\delta_Y^2, 1]. \end{cases}$$

Here $J(Y,y)$ and $K(Y,y)$ are C^2-maps on Y, whereas C^3-maps on y for $y \neq 1, 1-\delta$. Furthermore using (d), (e) and (f), we obtain:

(*g*) $\left|\frac{\partial}{\partial y} g_Y(x)\right| \leq C|1-y|^{\alpha_Y - 1}$ or $\left|\frac{\partial}{\partial y} g_Y(y)\right| \leq C|y-(1-\delta)|^{\alpha_Y-1}$ according, respectively, that $y \in [1-\delta_Y^2, 1]$ or that $y \in [1-\delta, 1-\delta+\delta_Y^1]$.

Also

(i) $\left|\frac{\partial}{\partial y} K(Y,y)\right| \leq K_0$ and $\left\|\frac{\partial}{\partial Y} K(Y,y)\right\|$ is small;

(ii) $\left|\frac{\partial}{\partial y} J(Y,y)\right| \leq K_0$ and $\left\|\frac{\partial}{\partial Y} J(Y,y)\right\|$ is small;

(iii) $J(X, 1-\delta) > 0$ and $K(X,1) > 0$.

(*h*) $0 \leq \left|\frac{\partial}{\partial x} f_y(x,y)\right| \leq K|1-y|^{\beta_Y}$ or $0 \leq \left|\frac{\partial}{\partial x} f_Y(x,y)\right| \leq K|y-(1-\delta)|^{\beta_Y}$, and $\left|\frac{\partial}{\partial y} f_Y(x,y)\right| \leq K|1-y|^{\alpha_Y-1}or\left|\frac{\partial}{\partial y} f_Y(x,y)\right| \leq K|y-(1-\delta)|^{\alpha_Y-1}$; according, respectively, that $y \in [1-\delta_Y^2, 1]$ or that $y \in [1-\delta, 1-\delta+\delta_Y^1]$.

We do not lose generality if, in the sequel, we assume that, for $Y \in \mathcal{U} : \alpha(Y) = \alpha, \beta(Y) = \beta, \xi_Y = \xi$ and $\tau_Y = \tau$.

Furthermore since the map $Y \to y_Y$ is a C^2-submersion, we can find C^2-coordinates (v, μ) in the neighborhood $\mathcal{U}(\mu \in \mathbf{R})$ such that:

(i) $\{(v,\mu)/\mu = 0\} \subset \mathcal{N} \cap \mathcal{U}$;

(ii) $F_{(v,\mu)}(x,y) = (\tau x, \xi y)$ if $0 \leq y \leq \xi^{-1}$,

$$F_{(v,\mu)}(x,y) = (x(\mu,v) + f^2(v,\mu;x,y), \mu - K(v,\mu;y)(1-y)^{\alpha})$$
$$\text{for } 1-\delta^2(v,\mu) \leq y \leq 1,$$
$$F_{(v,\mu)}(x,y) = (x(v,\mu) + f^1(v,\mu;x,y), \mu - J(v,\mu;y)(y-(1-\delta))^{\alpha}), \text{for}$$
$$1-\delta \leq y \leq 1-\delta+\delta^1(v,\mu).$$

Under these conditions we obtain $\delta^i(v,\mu) = A^i(v)\mu^{1/\alpha}$, with $\left\|\frac{\partial A^i}{\partial v}\right\|$ small numbers, for $i = 1, 2$.

We will use the notations $a(v,\mu) = 1 - \delta^2(v,\mu)$ and $b(v,\mu) = 1 - \delta + \delta^1(v,\mu)$.

2.2.

For a proof of Theorem 1 we first give a characterization of the elements in $\mathcal{U}_H^+ \cup \mathcal{U}_A^+$.
Choose $\mu_1 > 0$ and $n_0 \in \mathbf{N}$ such that $\xi^{n_0}\mu_1 = 1,1 > 1$.

LEMMA 2. – *For $(v,\mu) \in \mathcal{U}$ such that $\xi^{-n_0} < \mu \leq \mu_1$, we have that*

$$\Lambda(v,\mu) = \{(x,y)/F^n_{(v,\mu)} \in R(v,\mu) \cup R_1(v,\mu) \cup R_2(v,\mu), n \in \mathbf{Z}\}$$

is a hyperbolic transitive set.

Proof. – *See* Lemma 2 in [1].

We next assume $0 \leq \mu \leq \xi^{-n_0} = \mu_0$.

Set $I_0(v,\mu) = [0,\xi^{-1}], I_{01}(v,\mu) =]\xi^{-1}, 1-\delta[$,

$$I_1(v,\mu) = [1-\delta, b(v,\mu)], I_{12}(v,\mu) =]b(v,\mu), a(v,\mu)[\text{ and}$$
$$I_2(v,\mu) = [a(v,\mu), 1].$$

For $(v,\mu) \in \mathcal{U}$, let $L(v,\mu,\cdot) : \cup_{i=0}^2 I_i(v,\mu) \to [0,1]$ be the map $L(v,\mu;y) = \pi_y \circ F_{(v,\mu)}(x,y) =$ second component of the first return map $F_{(v,\mu)}(x,y)$.

Define $L_1(v,\mu;y) = L(v,\mu;y)$ and $L_{n+1}(v,\mu;y) = L(v,\mu;L_n(v,\mu;y))$ for $n \geq 1$.

Let

$$\Lambda(v,\mu) = \{y \in [0,1]/L_n(v,\mu;y) \in \cup_{i=0}^2 I_i(v,\mu), n \geq 0\}$$
$$\Gamma_0 = \{(v,\mu) \in \mathcal{U} : 1 \notin \Lambda(v,\mu)\}$$

and

$$\Gamma_1 = \{(v,\mu) \in \mathcal{U} : 1 \in \Lambda(v,\mu) \textit{ and there exists a hyperbolic attracting periodic orbit for the map } L(v,\mu;\cdot)\}$$

LEMMA 3. – *For $(v,\mu) \in \Gamma_0$ we have that $\Lambda(v,\mu)$ is a hyperbolic set for the map $L(v,\mu;\cdot)$.*

Proof. – Let $(v,\mu) \in \Gamma_0$ and $n = n(v,\mu)$ be the integer such that $L_n(v,\mu;1) \in I_{01}(v,\mu) \cup I_{12}(v,\mu)$. Due to the continuity of the map $(v,\mu;y) \longmapsto L_n(v,\mu;y)$ we can find neighborhoods $U_{1-\delta} \subset I_1(v,\mu), U_1 \subset I_2(v,\mu)$ of the points $1-\delta$ and 1, respectively, such that $y \in U_{1-\delta} \cup U_1$ implies $L_n(v,\mu;y) \in I_{01}(v,\mu) \cup I_{12}(v,\mu)$. This, in turn, implies that $\Lambda(v,\mu)$ is a compact invariant set with all its periodic points hyperbolic repelling and without critical points. Hence, by applying a result proved by Mañé [6] to the restriction map

$$L_{(v,\mu;\cdot)}\big/_{(I_0(v,\mu)\cup I_1(v,\mu)\cup I_2(v,\mu)\backslash U_{1-\delta}\cup U_1)}$$

the result now follows. ■

DEFINITION 2. – *Let $I \subset J$ be two intervals. We will say $f \in C^k(I,J), k \geq 1$, satisfies Axiom A if:*

(i) *f has a finite number of hyperbolic, attracting periodic orbits and no other attractors,*
(ii) *Let $B(f)$ denote the basin of attraction of the attracting periodic orbits for f. The set $\sum(f) = I \setminus B(f)$ is a hyperbolic set for f.*

LEMMA 4. – *For $(u, \mu) \in \Gamma_1$ we have that $L(v, \mu; \cdot)$ satisfies Axiom A.*

Proof. – We note that $L(v, \mu; \cdot)|_{I_1(v,\mu)\cup I_2(v,\mu)}$ has negative Schwarzian derivative. By Singer's theorem we obtain that the attracting periodic orbit attracts all the critical points (since that all critical points eventually have the same orbit).

Since $L(v, \mu; \cdot)$ has a hyperbolic attracting periodic orbit, we have that it does not have saddle-node or attracting flip bifurcations. Since these are the only non-hyperbolic periodic orbits that appear in our family (*see* sections 2.3 through 2.14), we conclude that $\Lambda(v, \mu)$ does not contain non-hyperbolic periodic orbits. In particular, all the periodic points in $(\Lambda(v, \mu) \setminus B(L(v, \mu, \cdot)))$ are hyperbolic. This implies that $(\Lambda(v, \mu) \setminus B(L(v, \mu; \cdot)))$ is a hyperbolic set (*see* [dM, pg. 128]). ■

Using the techniques of [3] or [1], it is easy to see that $(v, \mu) \in \Gamma_0$ if and only if $(v, \mu) \in \mathcal{U}_H^+$ and $(v, \mu) \in \Gamma_1$ if and only if $(v, \mu) \in \mathcal{U}_A^+$. Part b) of Theorem 1 now follows.

2.3.

Since $X \in \mathcal{U}_X$ we have $X = (v_0, 0)$ some v_0.

In the sequel we will deal with $(v, \mu) \in \mathcal{U}_X$ such that : $-\xi^{-(n_0-1)} \leq \mu \leq \xi^{-(n_0-1)}$; $||v - v_0|| \leq r_0$, some $r_0 > 0$ small, and $n_0 \in \mathbf{N}$ choosen such that the number :

$$Q_0 = \inf\{\alpha((A^1(v))^{-1}\xi^{\frac{n_0}{\alpha}}(1 - \delta - \xi^{-1}), \alpha(A^2(v))^{-1}\xi^{\frac{n_0}{\alpha}}(1 - \delta - \xi^{-1}); v \in V\}$$

satisfies $Q_0 > 2, \dfrac{2}{Q_0(1 - \xi^{-1})} < 1$ and, $\xi^{-1/\alpha}Q_0 > 1$.

Throughout, we will consider $k_0 \in \mathbf{N}$ such that $k_0 \geq n_0$.

Let $B(k_0)$ be the set $\{(v, \mu) \in \mathcal{U}/1 - \delta \leq \xi^{k_0-1}\mu \leq 1; ||v - v_0|| \leq r_0\}$.

For $(v, \mu) \in B(k_0)$ denote by $D\begin{pmatrix}1\\j\end{pmatrix}(v, \mu) \subset I_1(v, \mu)$ $\left(D\begin{pmatrix}2\\j\end{pmatrix}(v, \mu) \subset I_2(v, \mu))\right)$ the interval satisfying :

$$L\left(v, \mu, D\begin{pmatrix}i\\j\end{pmatrix}(v, \mu)\right) = \xi^{-(k_0-1)}\xi^{-j}[1 - \delta, 1], \quad \text{for } j \geq 1, \quad i = 1, 2.$$

$D\begin{pmatrix}i\\0\end{pmatrix}(v, \mu) \subset I_i(v, \mu)$ will denote, the interval satisfying :

$$L\left(v, \mu; D\begin{pmatrix}i\\0\end{pmatrix}(v, \mu)\right) = \xi^{-(k_0-1)}[1 - \delta, \xi^{k_0-1}\mu], \quad i = 1, 2.$$

Note that

$$D\begin{pmatrix}1\\0\end{pmatrix}(v, \xi^{-(k_0-1)}(1 - \delta)) = \{1 - \delta\} \text{ and that } D\begin{pmatrix}2\\0\end{pmatrix}(v, \xi^{-(k_0-1)}(1 - \delta)) = \{1\}.$$

For $j \geq 1$, we let $\left\{z\begin{pmatrix} i \\ j \end{pmatrix}(v,\mu), y\begin{pmatrix} i \\ j \end{pmatrix}(v,\mu)\right\}$ denote the boundary points of the interval $D\begin{pmatrix} i \\ j \end{pmatrix}(v,\mu)$. These two points are defined by the equations

$$L\left(v,\mu; z\begin{pmatrix} i \\ j \end{pmatrix}(v,\mu)\right) = \xi^{-(k_0-1)}\xi^{-j}(1-\delta) \text{ and}$$
$$L\left(v,\mu; y\begin{pmatrix} i \\ j \end{pmatrix}(v,\mu)\right) = \xi^{-(k_0-1)}\xi^{-j}.$$

For $j=0$, we have that $D\begin{pmatrix} 1 \\ 0 \end{pmatrix}(v,\mu) = \left[1-\delta, z\begin{pmatrix} 1 \\ 0 \end{pmatrix}(v,\mu)\right]$ and that $D\begin{pmatrix} 2 \\ 0 \end{pmatrix}(v,\mu) = \left[z\begin{pmatrix} 2 \\ 0 \end{pmatrix}(v,\mu); 1\right]$ where $L\left(v,\mu; z\begin{pmatrix} i \\ 0 \end{pmatrix}(v,\mu)\right) = \xi^{-(k_0-1)}(1-\delta)$, $i=1,2$.
We note that :

$$\lim_{\mu\to\xi^{-(k_0-1)}(1-\delta)} \frac{\partial z\begin{pmatrix} 1 \\ 0 \end{pmatrix}}{\partial\mu}(v\,,\mu) = +\infty \quad \text{and} \quad \lim_{\mu\to\xi^{-(k_0-1)}(1-\delta)} \frac{\partial z\begin{pmatrix} 2 \\ 0 \end{pmatrix}}{\partial\mu}(v\,,\mu) = -\infty$$

The proof of the following lemma is easy and left to the reader.

LEMMA 5. – *Given $\varepsilon > 0$ we can find $j_0 \in \mathbf{N}$ such that*

$$\max\Bigg\{ \sup\left\{\left|b(v,\mu) - z\begin{pmatrix} 1 \\ j \end{pmatrix}(v,\mu)\right|, \left|\frac{\partial b}{\partial\mu}(v,\mu) - \frac{\partial z\begin{pmatrix} 1 \\ j \end{pmatrix}}{\partial\mu}(v,\mu)\right|,\right.$$
$$\left.\left\|\frac{\partial b}{\partial v}(v,\mu) - \frac{\partial z\begin{pmatrix} 1 \\ j \end{pmatrix}}{\partial v}(v,\mu)\right\|\right\},$$
$$\sup\left\{\left|b(v,\mu) - y\begin{pmatrix} 1 \\ j \end{pmatrix}(v,\mu)\right|, \left|\frac{\partial b}{\partial\mu}(v,\mu) - \frac{\partial y\begin{pmatrix} j \\ 1 \end{pmatrix}}{\partial\mu}(v,\mu)\right|,\right.$$
$$\left.\left\|\frac{\partial b}{\partial v}(v,\mu) - \frac{\partial y\begin{pmatrix} 1 \\ j \end{pmatrix}}{\partial v}(v,\mu)\right\|\right\},$$
$$\sup\left\{\left|(a(v,\mu) - z\begin{pmatrix} 2 \\ j \end{pmatrix}(v,\mu)\right|, \left|\frac{\partial a}{\partial\mu}(v,\mu) - \frac{\partial z\begin{pmatrix} 2 \\ j \end{pmatrix}}{\partial\mu}(v,\mu)\right|,\right.$$
$$\left.\left\|\frac{\partial a}{\partial v}(v,\mu) - \frac{\partial z\begin{pmatrix} 2 \\ j \end{pmatrix}}{\partial v}(u,\mu)\right\|\right\},$$

$$\sup\left\{\left|a(v,\mu)-y\binom{2}{j}(v,\mu)\right|,\left|\frac{\partial a}{\partial v}(v,\mu)-\frac{\partial y\binom{2}{j}}{\partial \mu}(v,\mu)\right|,\right.$$
$$\left.\left\|\frac{\partial a}{\partial v}(v,\mu)-\frac{\partial y\binom{2}{j}}{\partial v}(v,\mu)\right\|\right\};\ (v,\mu)\in B(k_0)\Big\}<\varepsilon,$$

for any $j \geq j_0$: that is, the sequences of maps $\left(z\binom{1}{j}\right),\left(y\binom{1}{j}\right)$ $\left(\text{resp. }\left(z\binom{2}{j}\right),\left(y\binom{2}{j}\right)\right)$ converge to $b(v,\mu)$ (resp. $a(v,\mu)$) in the uniform C^1-topology in $B(k_0)$. ■

We also note the following fact: for any $j \geq 1, y \in D\binom{i}{j}(v,\mu)$ and $y' \in D\binom{i}{j+1}(v,\mu)$ we have

$$\left|\frac{\frac{\partial L}{\partial y}(v,\mu,y')}{\frac{\partial L}{\partial y}(v,\mu;y)}\right| \geq \lambda_j > 1,$$

where the sequence (λ_j) satisfies $\lim_{j\to\infty} \lambda_j = 1$

We now have the following result for $(v,\mu) \in B(k_0)$.

LEMMA 6.

$$\min\left\{\left|\frac{\partial L_{k_0}}{\partial y}\left(v,\mu;y\binom{1}{1}(v,\mu)\right|,\left|\frac{\partial L_{k_0}}{\partial y}\left(v,\mu,y\binom{2}{1}(v,\mu)\right|\right\}\geq \xi^{\frac{k_0-n_0-1}{\alpha}}Q_0\right.\right.$$

Proof. – Since $L_{k_0}(v,\mu;y) = \xi^{k_0-1}L(v,\mu;y)$, for $(v,\mu) \in B(k_0), y\binom{1}{1}(v,\mu) \leq y \leq b(v,\mu)$ or $a(v,\mu) \leq y \leq y\binom{2}{1}(v,\mu)$ we have

$$(\star)\quad \begin{aligned}&\frac{\partial L_{k_0}}{\partial y}(v,\mu,y\binom{1}{1}(v,\mu))\\ &= -\xi^{k_0-1}\alpha J\left(v,\mu;y\binom{1}{1}(v,\mu)\right)\left(y\binom{1}{1}(v,\mu)-(1-\delta)\right)^{\alpha-1}\cdot\\ &\left[1+\frac{y\binom{1}{1}(v,\mu)-(1-\delta)}{\alpha J\left(v,\mu,y\binom{1}{1}\right)}\frac{\partial J}{\partial y}\left(v,\mu,y\binom{1}{1}\right)\right]\end{aligned}$$

For $y\binom{1}{1}(v,\mu)$ we have : $\mu - J\left(v,\mu,y\binom{1}{1}\right)\left(y\binom{1}{1}-(1-\delta)\right)^{\alpha} = \xi^{-k_0}$ and $1-\delta < y\binom{1}{1}(v,\mu) < 1-\delta + A^1(v)\mu^{1/\alpha}$.

Since $\xi^{-(k_0-1)}(1-\delta) \leq \mu \leq \xi^{-(k_0-1)}$, we obtain

$$\xi^{-(\frac{k_0-1}{\alpha})}(1-\delta)^{1/\alpha} \leq \mu^{1/\alpha} \leq \xi^{-(\frac{k_0-1}{\alpha})}$$

and hence $(\mu^{1/\alpha})^{-1} \geq \xi^{\frac{k_0-1}{\alpha}}$.

Therefore

$$\left|\alpha\xi^{k_0-1}J\left(v,\mu;y\binom{1}{1}\right)\left(y\binom{1}{1}-(1-\delta)\right)^{\alpha-1}\right| > \alpha(A^1(v))^{-1}\xi^{\frac{k_0-1}{\alpha}}(1-\delta-\xi^{-1}).$$

Using this fact in equation $(*)$ the result follows for $y\binom{1}{1}(v,\mu)$. The proof for $\left|\frac{\partial L_{k_0}}{\partial y}\left(v,\mu,y\binom{2}{1}(v,\mu)\right)\right|$ is analogous. ■

COROLLARY 1. – *For* $(v,\mu) \in B(k_0)$ *and* $y \in D\binom{i}{j}(v,\mu), j \geq 1$, *we have that*

$$\left|\frac{\partial L_{k_0}}{\partial y}(v,\mu,y)\right| \geq \xi^{\frac{k_0-n_0-1}{\alpha}}Q_0, \textit{ for } y \in D\binom{i}{1}(v,\mu),$$

and that

$$\left|\frac{\partial L_{k_0}}{\partial y}(v,\mu;y)\right| \geq \lambda_1\cdots\lambda_{j-1}\xi^{\frac{k_0-n_0-1}{\alpha}}Q_0, \textit{ for } y \in D\binom{i}{j}(v,\mu)$$

and any $j \geq 2$.

2.4. Associated to $\binom{i}{j}$ we next define the one-dimensional map

$$g\binom{i}{j}(v,\mu,\cdot) : D\binom{i}{j}(v,\mu) \to [1-\delta,1] \text{ by } g\binom{i}{j}(v,\mu;y) = L_{k_0+j}(v,\mu;y).$$

Applying Corollary 1 we have that

$$\left|\frac{\partial g\binom{i}{1}}{\partial y}(v,\mu,y)\right| \geq \xi\xi^{\frac{k_0-n_0-1}{\alpha}}Q_0 = P_1, \text{ for } y \in D\binom{i}{1}(v,\mu)$$

and that

$$\left|\frac{\partial g\binom{i}{j}}{\partial y}(v,\mu;y)\right| \geq \xi^j\lambda_1\cdots\lambda_{j-1},\xi^{\frac{k_0-n_0-1}{\alpha}}Q_0 = P_j, \text{for} y \in D\binom{i}{j}(v,\mu)$$

any $j \geq 2$.

From these estimates we get that the maps $g\binom{i}{j}(v,\mu;y)\,, i = 1,2, j \geq 1$, are C^∞-expanding diffeomorphisms onto their images (that are $[1-\delta,1]$). Moreover, for $i=1$ all the maps $g\binom{1}{j}(v,\mu)$ reverse orientation, and for $i=2$ all the maps $g\binom{2}{j}(v,\mu)$ preserve orientation.

Now given any sequence of two symbols, $\left(\binom{i_0}{j_0}, \binom{i_1}{j_1}, \cdots\right)$, let us define a sequence of nested sets and maps:

$$D\binom{i_0}{j_0}(v,\mu) \supset D\left(\binom{i_0}{j_0}, \binom{i_1}{j_1}\right)(v,\mu) \supset \cdots \supset D\left(\binom{i_0}{j_0}, \cdots, \binom{1_r}{j_r}\right)(v,\mu) \supset \cdots$$

and

$$g\left(\binom{i_0}{j_0}, \binom{i_1}{j_1}\right)(v,\mu,\cdot), \cdots, g\left(\binom{i_0}{j_0}, \binom{i_1}{j_1}, \cdots, \binom{1_r}{j_r}\right)(v,\mu;\cdot), \cdots$$

as follows:

$$D\left(\binom{i_0}{j_0}, \binom{i_1}{j_1}\right)(v,\mu) = \left\{ y \in D\binom{i_0}{j_0}(v,\mu) \ : \ g\binom{i_0}{j_0}(v,\mu;y) \in D\binom{i_1}{j_1}(v,\mu)\right\}.$$

For $D\left(\binom{i_0}{j_0}, \binom{i_1}{j_1}\right)(v,\mu) \neq \emptyset$ we associate a map

$$g\left(\binom{i_0}{j_0}, \binom{i_1}{j_1}\right)(v,\mu;\cdot) \ : \ D\left(\binom{i_0}{j_0}, \binom{i_1}{j_1}\right)(v,\mu) \to [1-\delta, 1]$$

defined by

$$g\left(\binom{i_0}{j_0}, \binom{i_1}{j_1}\right)(v,\mu;y) = g\binom{i_1}{j_1}\ \left(v,\mu, g\binom{i_0}{j_0}\right)(v,\mu;y).$$

For $r \geq 2$ and $D\left(\binom{i_0}{j_0}, \cdots, \binom{i_{r-1}}{j_{r-1}}\right)(v,\mu) \neq \emptyset$, we define

$$D\left(\binom{i_0}{j_0}, \binom{i_1}{j_1}, \cdots, \binom{1_r}{j_r}\right)(v,\mu) = \left\{ y \in D\left(\binom{i_0}{j_0}, \cdots, \binom{i_{r-1}}{j_{r-1}}\right)(v,\mu) \ / \ g\left(\binom{i_0}{j_0}, \binom{i_1}{j_1}, \cdots, \binom{i_{r-1}}{j_{r-1}}\right)(v,\mu;y) \in D\binom{i_r}{j_r}\right\}.$$

Associated to those $D\left(\binom{i_0}{j_0}, \cdots, \binom{1_r}{j_r}\right)(v,\mu)$ that are non-empty define the map

$$g\left(\binom{i_0}{j_0}, \cdots, \binom{1_r}{j_r}\right)(v,\mu;\cdot) \ : \ D\left(\binom{i_0}{j_0}, \cdots, \binom{1_r}{j_r}\right)(v,\mu) \to [1-\delta,\, 1]$$

by

$$g\left(\binom{i_0}{j_0}, \cdots, \binom{1_r}{j_r}\right)(v,\mu;y) = g\binom{1_r}{j_r}\left(v,\mu, g\left(\binom{i_0}{j_0}, \cdots \binom{i_{r-1}}{j_{r-1}}\right)\right)(v,\mu;y).$$

Remark 1. – Given any finite set of two symbols, $\left\{\binom{i_0}{j_0}, \cdots, \binom{1_r}{j_r}\right\}$, such that $j_k \geq 1$, for $k = 0, 1, \cdots, r$, by Corollary 1 we have that:

$$\left|\frac{\partial}{\partial y}\left(g\left(\binom{i_0}{j_0}, \cdots, \binom{i_r}{j_r}\right)\right)(v, \mu; y)\right| \geq P_{j_0} \cdots P_{j_r},$$

any $y \in D\left(\binom{i_0}{j_0}, \cdots, \binom{1_r}{j_r}\right)(v, \mu)$. From this inequality we conclude

$$\left|D\left(\binom{i_0}{j_0}, \cdots, \binom{i_r}{j_r}\right)(v, \mu)\right| \leq \left(P_{j_0} \cdots P_{j_{r-1}}\right)^{-1}\left|D\binom{i_r}{j_r}(v, \mu)\right|$$

and hence

$$\sum_{\substack{(i_0, j_0)\\ j_0 \geq 1}}\left(\sum_{\substack{(i_1, j_1)\\ j_1 \geq 1}}\left(\cdots\left(\sum_{\substack{(i_r, j_r)\\ j_r \geq 1}}\left|D\left(\binom{i_0}{j_0}, \cdots, \binom{i_r}{j_r}\right)(v, \mu)\right|\right)\cdots\right)\right) \leq \delta \cdot \left(\frac{2}{P_1(1-\xi^{-1})}\right)^r;$$

that is, for any $(v, \mu) \in B(k_0)$ we have :

COROLLARY 2. – *The set of points*

$$y \in \left(I_1(v, \mu) \setminus D\binom{1}{0}(v, \mu)\right) \cup \left(I_2(v, \mu) \setminus D\binom{2}{0}(v, \mu)\right)$$

that satisfy

(i) *$L_i(v, \mu; y)$ is defined, all $i \geq 1$, and*

(ii) *there is no $i_0 \in \mathbf{N}$ such that $L_{i_0}(v, \mu; y) \in D\binom{1}{0}(v, \mu) \cup D\binom{2}{0}(v, \mu)$.*

is a hyperbolic set of zero Lebesgue measure. ■

Remark 2. – Let denote the set above by $C\left(\binom{1}{1}, \binom{2}{1}\right)(v, \mu)$. As a consequence we obtain that its closure is a Cantor set of zero Lebesgue measure.

2.5. Let us now consider any sequence of two symbols $\left(\binom{i_0}{j_0}, \binom{i_1}{j_1}, \cdots\right)$, where $i_k = 1, 2$ and $j_k \geq 1$, all $k \in \mathbf{N}$.

Let

$$z_r(v, \mu) = z\left(\binom{i_0}{j_0}, \cdots, \binom{i_r}{j_r}\right)(v, \mu),\ y_r(v, \mu) = y\left(\binom{i_0}{j_0}, \cdots, \binom{i_r}{j_r}\right)(v, \mu)$$

denote the boundary points of the interval $D\left(\binom{i_0}{j_0}, \cdots, \binom{i_r}{j_r}\right)(v, \mu)$ defined, respectively, by the conditions

$$\Delta_r(v, \mu, z_r(v, \mu)) = g\left(\binom{i_0}{j_0}, \cdots, \binom{i_r}{j_r}\right)\left(v, \mu; z\left(\binom{i_0}{j_0}, \cdots, \binom{i_r}{j_r}\right)(v, \mu)\right) = 1-\delta$$

and

$$\Delta_r(v,\mu,y_r(v,\mu)) = g\left(\begin{pmatrix} i_0 \\ j_0 \end{pmatrix}, \cdots, \begin{pmatrix} i_r \\ j_r \end{pmatrix}\right)\left(v,\mu; y\left(\begin{pmatrix} i_0 \\ j_0 \end{pmatrix}, \cdots, \begin{pmatrix} i_r \\ j_r \end{pmatrix}\right)(v,\mu)\right) = 1$$

From these relations we obtain

$$\frac{\partial z_r}{\partial v}(v,\mu) = \frac{-\dfrac{\partial \Delta_r}{\partial v}(v,\mu,z_r(v,\mu))}{\dfrac{\partial \Delta_r}{\partial y}(v,\mu,z_r(v,\mu))}$$

$$\frac{\partial z_r}{\partial \mu}(v,\mu) = \frac{-\dfrac{\partial \Delta_r}{\partial \mu}(v,\mu,z_r(v,\mu))}{\dfrac{\partial \Delta_r}{\partial y}(v,\mu,z_r(v,\mu))}$$

Let us compute inductively the derivatives in the right-hand side.

Since $\Delta_r(v,\mu;y) = g\begin{pmatrix} i_r \\ j_r \end{pmatrix}(v,\mu;\Delta_{r-1}(v,\mu;y))$, we have

$$\begin{aligned}
\frac{\partial \Delta_r}{\partial v}(v,\mu;y) &= \frac{\partial g\begin{pmatrix} i_r \\ j_r \end{pmatrix}}{\partial v}(v,\mu,\Delta_{r-1}(v,\mu;y)) \\
&\quad + \frac{\partial g\begin{pmatrix} i_r \\ j_r \end{pmatrix}}{\partial y}(v,\mu;\Delta_{r-1}(v,\mu,y)) \cdot \frac{\partial \Delta_{r-1}}{\partial v}(v,\mu;y) = \\
&= \frac{\partial g\begin{pmatrix} i_r \\ j_r \end{pmatrix}}{\partial v}(y,\mu,\Delta_{r-1}(v,\mu;y)) \\
&\quad + \frac{\partial g\begin{pmatrix} i_r \\ j_r \end{pmatrix}}{\partial v}(v,\mu,\Delta_{r-1}(v,\mu;y)) \cdot \frac{\partial g\begin{pmatrix} i_{r-1} \\ j_{r-1} \end{pmatrix}}{\partial v}(y,\mu,\Delta_{r-2}) \\
&\quad + \frac{\partial g\begin{pmatrix} i_r \\ j_r \end{pmatrix}}{\partial y}(v,\mu,\Delta_{r-1}(v,\mu;y)) \cdot \frac{\partial g\begin{pmatrix} i_{r-1} \\ j_{r-1} \end{pmatrix}}{\partial y}(v,\mu,\Delta_{r-2}) \\
&\quad \cdot \frac{\partial g\begin{pmatrix} i_{r-3} \\ j_{r-3} \end{pmatrix}}{\partial v}(v,\mu;\Delta_{r-3}(v,\mu;y)) \\
&\quad + \cdots + \frac{\partial g\begin{pmatrix} i_r \\ j_r \end{pmatrix}}{\partial y}(v,\mu;\Delta_{r-1}(v,\mu;y)) \\
&\quad \cdots \frac{\partial g\begin{pmatrix} i_1 \\ j_1 \end{pmatrix}}{\partial y}(v,\mu;\Delta_0(v,\mu;y)) \cdot \frac{\partial \Delta_0}{\partial v}(v,\mu;y)
\end{aligned}$$

We have a similar relation for $\dfrac{\partial \Delta_r}{\Delta \mu}(v,\mu;y)$ by replacing $\dfrac{\partial}{\partial \mu}$ for $\dfrac{\partial}{\partial v}$ wherever it corresponds in the above formulas.

The other derivative yields

$$\frac{\partial \Delta_r}{\partial y}(v,\mu;y) = \frac{\partial g\binom{i_r}{j_r}}{\partial y}(v,\mu;\Delta_{r-1}(v,\mu;y))\cdots\frac{\partial g\binom{i_0}{j_0}}{\partial y}(v,\mu;y)\,.$$

Denoting by g_r the map $g\binom{i_r}{j_r}$, we have:

$$\frac{\partial z_r}{\partial v}(v,\mu) = \frac{\left\{\begin{array}{l} -\left[\dfrac{\partial g_r}{\partial y}(v,\mu;\Delta_{r-1}(z_r)) + \cdots \right.\\ \qquad \left. + \dfrac{\partial g_r}{\partial y}(v,\mu;\Delta_{r-1}(z_r))\cdots\dfrac{\partial g_1}{\partial y}(v,\mu;\Delta_0(z_r))\cdot\dfrac{\partial \Delta_0}{\partial v}(v,\mu;z_r)\right]\end{array}\right\}}{\dfrac{\partial g_r}{\partial y}(v,\mu;\Delta_{r-1}(z_r))\cdots\dfrac{\partial g_0}{\partial y}(v,\mu;z_r)}$$

and

$$\frac{\partial z_r}{\partial \mu}(v,\mu) = \frac{\left\{\begin{array}{l} -\left[\dfrac{\partial g_r}{\partial \mu}(v,\mu;\Delta_{r-1}(z_r)) + \cdots \right.\\ \qquad \left. + \dfrac{\partial g_r}{\partial y}(v,\mu;\Delta_{r-1}(z_r))\cdots\dfrac{\partial g_1}{\partial y}(v,\mu;\Delta_0(z_r))\dfrac{\partial \Delta_0}{\partial \mu}(v,\mu;z_r)\right]\end{array}\right\}}{\dfrac{\partial g_r}{\partial y}(v,\mu;\Delta_{r-1}(z_r))\cdots\dfrac{\partial g_0}{\partial y}(v,\mu;z_r)}.$$

Now, for any $\binom{i_0}{j_0}$, we have

$$\left|\frac{\dfrac{\partial g\binom{i_0}{j_0}}{\partial v}(v,\mu;y)}{\dfrac{\partial g\binom{i_0}{j_0}}{\partial y}(v,\mu;y)}\right| = \left|\frac{\dfrac{\partial L}{\partial v}(v,\mu;y)}{\dfrac{\partial L}{\partial y}(v,\mu;y)}\right|$$

and

$$\left|\frac{\dfrac{\partial g\binom{i_0}{j_0}}{\partial \mu}(v,\mu;y)}{\dfrac{\partial g}{\partial y}(v,\mu;y)}\right| = \left|\frac{\dfrac{\partial L}{\partial \mu}(v,\mu;y)}{\dfrac{\partial L}{\partial y}(v,\mu;y)}\right|.$$

We note that the sequence $(z_r(v,\mu))$ converges uniformly in the C^0-topology to

$$z_\infty(v,\mu) = z\left(\begin{pmatrix} i_0 \\ j_0 \end{pmatrix}, \begin{pmatrix} i_1 \\ j_1 \end{pmatrix}, \cdots\right)(v,\mu)$$

i.e.,

$$\lim_{r\to\infty} \sup\{|z_\infty(v,\mu) - z_r(v,\mu)|; (v,\mu) \in B(k_0\} = 0.$$

From this fact and the above computation for the derivatives of the maps $z_r(v,\mu)$, and since all the $g\begin{pmatrix} i_r \\ j_r \end{pmatrix}$, $j_r \geq 1$ are C^∞-diffeomorphisms, after a cumbersome computation, we obtain

LEMMA 7. – *The sequence $(z_r(v,\mu))$ satisfies the following property: Given $\varepsilon > 0$ there is an $r_0 \in \mathbf{N}$ such that*

$$\sup\{|z_{r+p}(v,\mu) - z_r(v,\mu)|, \left\|\frac{\partial z_{r+p}}{\partial v}(v,\mu) - \frac{\partial z_r}{\partial v}(v,\mu)\right\|,$$

$$\left\|\frac{\partial z_{r+p}}{\partial \mu}(v,\mu) - \frac{\partial z_r}{\partial \mu}(v,\mu)\right\|; (v,\mu) \in B(k_0)\} < \varepsilon \; for \; r \geq r_0, \quad p \in \mathbf{N};$$

that is, the sequence $(z_r(v,\mu))$ is a Cauchy sequence of maps in the uniform C^1-topology. ■

In particular we have that the map $(v,\mu) \longmapsto z_\infty(v,\mu)$ is a C^1-map on $B(k_0)$.

Let us now denote by

$$G(v,\mu,\cdot) : \cup_{i=1}^2 \left(\cup_{j\geq 1} D\begin{pmatrix} i \\ j \end{pmatrix}(v,\mu)\right) \to [1-\delta, 1]$$

the map defined by $G(v,\mu,y) = g\begin{pmatrix} i \\ j \end{pmatrix}(v,\mu,y)$, for $y \in D\begin{pmatrix} i \\ j \end{pmatrix}(v,\mu)$.

Let

$$C\left(\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \end{pmatrix}\right)(v,\mu)$$

denote the set of points $y \in \left[y\begin{pmatrix} 1 \\ 1 \end{pmatrix}(v,\mu), y\begin{pmatrix} 2 \\ 1 \end{pmatrix}(v,\mu)\right]$ such that it is defined $G_k(v,\mu,y)(G_{k+1}(v,\mu,y) = G(v,\mu,G_k(v,\mu,y)), G_1(v\,\mu\,y) = G(v,\mu,y))$ for all $k \in \mathbf{N}$ and $G_k(v,\mu,y) \in \left[y\begin{pmatrix} 1 \\ 1 \end{pmatrix}(v,\mu), y\begin{pmatrix} 2 \\ 1 \end{pmatrix}(v,\mu)\right]$.

Associated with any point $y \in C\left(\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \end{pmatrix}\right)(v,\mu)$ we may define a sequence $\Gamma(v,\mu) : \mathbf{N} \to \left\{\begin{pmatrix} i \\ j \end{pmatrix}; i = 1,2; j \geq 1\right\}$ by

$$\Gamma(v,\mu)(k) = \begin{pmatrix} i_s \\ j_s \end{pmatrix} \Longleftrightarrow G_k(v,\mu)(y) \in D\begin{pmatrix} i_s \\ j_s \end{pmatrix}(v,\mu).$$

This defines a map $C\left(\binom{1}{1}, \binom{1}{2}\right)(v, \mu) \overset{\Gamma(v,\mu)}{\to} \sum_1$,

$$\Sigma_1 = \left\{\Gamma : \mathbf{N} \to \left\{ \binom{i}{j} ; i = 1, 2; j \geq 1\right\}\right\}$$

which is, as usual, a homeomorphism and satisfies

$$\Gamma(v, \mu) \circ G(v, \mu) = \sigma_1 \circ \Gamma(v, \mu),$$

where $\Sigma_1 \overset{\sigma_1}{\to} \Sigma_1$ denotes the shift map $\sigma_1(\Gamma)(k) = \Gamma(k+1)$.

For $\Gamma \in \sum_1$ we denote $p_\Gamma(v, \mu) = (\Gamma(v, \mu))^{-1}(\Gamma)$. As in Lemma 7 we may prove the following:

COROLLARY 3. – *The map* $B(k_0) \overset{p_\Gamma}{\to} [1-\delta, 1], (v, \mu) \longmapsto p_\Gamma(v, \mu)$ *is* C^1. ■

We observe that the closure of the set $C\left(\binom{1}{1}, \binom{2}{1}\right)(v, \mu)$ contains the points $b(v, \mu), a(v, \mu)$ and all their preimages under the map $G(v, \mu, \cdot)$ which are contained in the interval $\left[y\binom{1}{1}(v, \mu), y\binom{2}{1}(v, \mu)\right]$.

Denoting by $s(v, \mu)$ any of these preimages it is clear that the map $B(k_0) \longrightarrow [1-\delta, 1]$, $(v, \mu) \longmapsto s(v, \mu)$ is a C^1 map and can be approximated, in the C^1-uniform topology, by a sequence of maps $z\left(\binom{i_0}{j_0}, \ldots \binom{i_r}{j_r}\right)(v, \mu)$ (or $y\left(\binom{i_0}{j_0}, \ldots \binom{i_r}{j_r}\right)(v, \mu)$) as in lemma 5.

In this sense we will say that the closure of the set $C\left(\binom{1}{1}, \binom{2}{1}\right)(v, \mu)$ is a C^1-Cantor set of Lebesgue measure zero for any $(v, \mu) \in B(k_0)$.

2.6. Let us now consider the surface

$$S_0 = \{(v, \mu; \xi^{k_0-1}\mu); (v, \mu) \in B(k_0)\} \subset \mathcal{U} \times [1-\delta, 1].$$

Since S_0 is transversal to $Y\binom{i}{j} = \left\{(v, \mu; y\binom{i}{j}(v, \mu); (v, \mu) \in B(k_0)\right\}$, we have that the intersection $S_0 \cap Y\binom{i}{j}$ defines a C^1-surface, $\bar{Y}\binom{i}{j}$, parametrized by

$$\left\{\left(v, C\binom{i}{j}(v), \xi^{k_0-1} C\binom{i}{j}(v)\right); ||v - v_0|| \leq r_0\right\}.$$

This defines a C^1-map $C\binom{i}{j} : V \to [0, \mu_0], v \longmapsto C\binom{i}{j}(v)$ that satisfies

$$G_0\left(v, C\binom{i}{j}(v)\right)\left(y\binom{i}{j}\left(v, C\binom{i}{j}(v)\right)\right) = 1.$$

This implies that the vector field $X\binom{i}{j}(v)$, associated to the point $\left(v, C\binom{i}{j}(v)\right) \in B(k_0) \subset \mathcal{U}$, will satisfy the homoclinic condition

$$\gamma_0\left(\sigma_0\left(X\binom{i}{j}(v)\right)\right) \subset W^s\left(\sigma_1\left(X\binom{i}{j}(v)\right)\right).$$

The same will apply to the intersection $S_0 \cap Z\binom{i}{j}$ where

$$Z\binom{i}{j} = \left\{\left(v, \mu; Z\binom{i}{j}(v,\mu)\right);\ (v,\mu) \in B(k_0)\right\}.$$

Next we consider

$$\begin{aligned} C\left(\binom{1}{1}, \binom{2}{1}\right) &= \left\{\left(v, \mu; C\left(\binom{1}{1}, \binom{2}{1}\right)(v,\mu)\right); (v,\mu) \in B(k_0)\right\} \\ &= \{(v,\mu; p_\Gamma(v,\mu)); (v,\mu) \in B(k_0), \Gamma \in \Sigma_1\}. \end{aligned}$$

For any given C^1-surface $\{(v,\mu;p_\Gamma(v,\mu));(v,\mu) \in B(k_0)\} = P_\Gamma$, we have that P_Γ is transversal to S_0 and hence the intersection $S_0 \cap P_\Gamma$ will define a C^1-surface, C_Γ, parametrized by $\{(v, C_\Gamma(v); P_\Gamma(v, C_\Gamma(v))\,;\ v \in V\}$. We denote by $X_\Gamma(v)$ the vector field associated to $(v, C_\Gamma(v)) \in B(k_0) \subset \mathcal{U}$. This vector field must satisfy one of the following conditions:

(i) the point $p_\Gamma(v, C_\Gamma(v))$ represents a periodic point of the map $G(v, C_\Gamma(v))$. In this case denote by $\sigma(p_\Gamma(v, C_\Gamma(v)))$ the hyperbolic periodic orbit of the vector field $X_\Gamma(v)$ associated to $p_\Gamma(v, C_\Gamma(v))$. Under these conditions we must have $\gamma_0(\sigma_0(X_\Gamma(v))) \subset W^s(\sigma(p_\Gamma(v, C_\Gamma(v))))$, that is, the vector field $X_\Gamma(v)$ presents a contracting singular cycle or

(ii) the point $p_\Gamma(v, C_\Gamma(v))$ has recurrent behavior with respect to the set $C\left(\binom{1}{1}, \binom{1}{2}\right)(v, C_\Gamma(v))$ under the map $G(v, C_\Gamma(v))$. In this case the trajectory $\gamma_0(\sigma_0(X_\Gamma(v)))$ has recurrent behavior in the neighborhhod U; *or*

(iii) the point $p_\Gamma(v, C_\Gamma(v))$is eventually periodic under the map $G(v, C_\Gamma(v), \cdot)$ (that is there is $s \in \mathbf{N}$ such that $G_{s_0}(v, C_\Gamma(v), p_\Gamma(v, C_\Gamma(v)))$ is a periodic point of the map $G(v, C_\Gamma(v), \cdot)$. In this case the situation for the vector field $X_\Gamma(v)$ is analogous to (i) above.

Now take any preimage,$s(v, \mu)$,of the points $b(v, \mu)$ or $a(v, \mu)$,in the closure of the set $C\left(\binom{1}{1}, \binom{1}{2}\right)(v,\mu)$. Since the C^1 surface $S = \{(v, \mu, s(v, \mu)); (v, \mu) \in B(k_0)\}$ is transversal to S_0 then the intersection $S \cap S_0$ define a C^1 surface S_b (resp S_a) parametrized by $\{(v, \bar{b}(v), s(v, \bar{b}(v)); v \in V\}$ (resp. $\{(v, \bar{a}(v), s(v, \bar{a}(v)); v \in V\}$). Let denote by $X_{\bar{b}}(v)$ (resp. $X_{\bar{a}}(v)$)the vector field associated to $(v, \bar{b}) \in B(k_0)$ (resp.$(v, \bar{a}) \in B(k_0)$). This vector field satisfies that :

$$\gamma_0(\sigma_0(X_{\bar{b}}(v))) \subset W^s(\sigma_1(X_{\bar{b}}(v))).$$

(resp.$\gamma_0(\sigma_0(X_{\bar{a}}(v))) \subset W^s(\sigma_1(X_{\bar{a}}(v)))$).

2.7.

In general let us consider the set of bisequences

$$\Sigma_0 = \left\{ \Gamma : \mathbf{N} \to \left\{ \binom{i}{j}, i = 1, 2; j \geq 0 \right\} \right\}$$

and the map

$$G(v, \mu, \cdot) : \bigcup_{i=1}^{2} \left(\cup_{j \geq 0} D\binom{i}{j}(v, \mu) \right) \to [1 - \delta, 1]$$

given by

$$G(v, \mu, y) = g\binom{i}{j}(v, \mu; y), y \in D\binom{i}{j}(v, \mu)$$

and $(v, \mu) \in B(k_0)$.

Denote by $M(v, \mu)$ the set of points $y \in [1 - \delta, 1]$ such that it is defined $G_k(v, \mu, y)$ for all $k \in \mathbf{N}$.

Associated with any $y \in M(v, \mu)$ we can define a bisequence $\Gamma(v, \mu)(y) \in \Sigma_0$ by:

$$(\Gamma(v, \mu)(y))(k) = \binom{i_s}{j_s} \Longleftrightarrow G_k(v, \mu, y) \in D\binom{i_s}{j_s}(v, \mu)$$

Clearly $\Gamma(v, \mu) : M(v, \mu) \to \Sigma_1$ is continuous and satisfies $\Gamma(v, \mu) \circ G(v, \mu) = \sigma_1 \circ \Gamma(v, \mu)$. Here $\sigma_0 : \Sigma_0 \to \Sigma_0$ is the shift map $\sigma_0(\Gamma)(k) = \Gamma(k + 1)$.

DEFINITION 3. – *We will say that the bisequence $\Gamma \in \Sigma_0$ is admissible at the level (v, μ) if $\Gamma(v, \mu)^{-1}(\Gamma) \neq \emptyset$.*

Remark 3. – 1) We note that $\Gamma(v, \xi^{-(k_0-1)})$ is a surjective map, for any $(v, \xi^{-(k_0-1)}) \in B(k_0)$.

2) From 1) we conclude that, given $\Gamma \in \Sigma_0$, we can find a first parameter value $\mu_\Gamma(v); \xi^{-(k_0-1)}(1 - \delta) \leq \mu_\Gamma(v) \leq \xi^{-(k_0-1)}$ such that Γ is admissible at the level (v, μ), any $\mu \geq \mu_\Gamma(v)$ [for instance $\mu_\Gamma(v) = \xi^{-(k_0-1)}(1 - \delta)$, any $\Gamma \in \Sigma_1$].

DEFINITION 4. – *Assume $(v, \mu) \in B(k_0)$ is a parameter value that satisfies $\{1 - \delta, 1\} \subset M(v, \mu)$. In this case we will call the bisequence $\sigma_0(\Gamma(v, \mu))(1) = \sigma_0(\Gamma(v, \mu))(1 - \delta)$ the itinerary of the map $G(v, \mu, \cdot)$, and we will denote it by $\Theta(v, \mu)$. We will say a bisequence $\Gamma \in \Sigma_0$ is realizable if there is a parameter value $(v, \mu) \in B(k_0)$ such that $\Theta(v, \mu) = \Gamma$. We will denote the bisequence $\Gamma(v, \mu)(1)$ (resp. $\Gamma(v, \mu)(1 - \delta)$). by $\Gamma_1(v, \mu)$(resp.$\Gamma_{1-\delta}(v, \mu)$).*

Remark 4. – The only bisequence that satisfies $\Gamma = \left(\begin{pmatrix}1\\0\end{pmatrix}, \cdots\right)$ and is realizable is the bisequence $\left(\begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}1\\0\end{pmatrix}, \cdots\right)$. From here we conclude that there are bisequences which are not realizable.

Denote by $Per(\sigma_0) \subset \Sigma_0$ the set of all periodic bisequences $\Gamma \in \Sigma_0$. It is clear that $Per(\sigma_0)$ is a dense subset of Σ_0. Let $\Sigma_2 \subset Per(\sigma_0)$ be the set of all periodic bisequences $\Gamma \in (Per(\sigma_0) \setminus \Sigma_1)$ such that $\Gamma = \left(\begin{pmatrix}1\\0\end{pmatrix}, \cdots\right)$ or $\Gamma = \left(\begin{pmatrix}2\\0\end{pmatrix}, \cdots\right)$.

Given $\Gamma \in \Sigma_2$ we let Γ_0 denote its period (*i.e.*, $\Gamma = (\Gamma_0, \Gamma_0, \Gamma_0, \cdots)$.) We have the following proposition:

PROPOSITION 1. – *For those $\Gamma \in \Sigma_2$ which satisfy that $\sigma_0(\Gamma)$ is realizable and the number of $\begin{pmatrix}1\\j\end{pmatrix}$ that appears in Γ_0 is odd, we can find values of the parameter $\mu_{\Gamma_0}(v) < \mu^f_{\Gamma_0}(v) < \mu_{2\Gamma_0}(v)$ such that:*

i) *for any $(v, \mu) \in B(k_0)$, $\mu_{\Gamma_0}(v) < \mu < \mu^f_{\Gamma_0}(v)$, the associated one-dimensional map $G(v, \mu, \cdot)$ has an attracting, hyperbolic, periodic orbit whose period is $\sharp(\Gamma_0)$. Moreover, one point of this orbit is contained in $D(\sigma_0^k(\Gamma_0))(v, \mu)$, any $0 \le k \le \sharp(\Gamma_0) - 1$.*

ii) *for any $(v, \mu) \in B(k_0)$, $\mu^f_{\Gamma_0}(v) < \mu < \mu_{2\Gamma_0}(v)$, the associated one-dimensional map $G(v, \mu, \cdot)$ has an attracting, hyperbolic, periodic orbit whose period is $2\sharp(\Gamma_0)$. Moreover, two points of this orbit are contained in $D(\sigma_0^k(\Gamma_0))(v, \mu)$, any $0 \le k \le \sharp(\Gamma_0) - 1$.*

iii) *for $(v, \mu_{\Gamma_0}(v)) \in B(k_0)$ we have that $D(\sigma_1^k(\Gamma_0))(v, \mu)$ is a single point, and the associated one-dimensional map $G(v, \mu, \cdot)$ satisfies*

$$G_{\sharp(\Gamma_0)}(v, \mu)(D(\sigma_0^k(\Gamma_0))(v, \mu)) = D(\sigma_0^k(\Gamma_0)(v, \mu)),$$

any $0 \le k \le \sharp(\Gamma_0) - 1$, $\mu = \mu_{\Gamma_0}(v)$.

iv) *for $(v, \mu^f_{\Gamma_0}) \in B(k_0)$ the associated one-dimensional map $G(v, \mu, \cdot)$ has a flip bifurcation of the attracting periodic orbit. Moreover, one point of this orbit is contained in the interior of $D(\sigma_0^k(\Gamma_0))(v, \mu^f_{\Gamma_0})$, any $0 \le k \le \sharp(\Gamma_0) - 1$.*

v) *for $(v, \mu_{2\Gamma_0}(v)) \in B(k_0)$ the associated one-dimensional map $G(v, \mu_{2\Gamma_0}(v), \cdot)$ satisfies*

$$G_{\sharp(\Gamma_0)}(\partial D(\sigma_0^k(\Gamma_0))(v, \mu_{2\Gamma_0})) = \partial D(\sigma_0^k(\Gamma_0)(v, \mu_{2\Gamma_0})$$

and interchanges the points in $\partial D(\sigma_0^k(\Gamma_0))(v, \mu_{2\Gamma_0})$, any $0 \le k \le \sharp(\Gamma_0) - 1$.[in particular for $\Gamma_0 = \left(\begin{pmatrix}1\\0\end{pmatrix}, \cdots\right)$. $\left(\text{resp. } \Gamma_0 = \left(\begin{pmatrix}2\\0\end{pmatrix}, \cdots\right)\right)$ we have that $G_{2\sharp(\Gamma_0)}(v, \mu, 1 - \delta) = 1 - \delta$ (resp. $G_{2\sharp(\Gamma_0)}(v, \mu, 1 - \delta) = 1$), $\mu = \mu_{2\Gamma_0}$].

vi) *for $\mu_{\Gamma_0}(v) \le \mu \le \mu_{2\Gamma_0}(v)$, the pre-image $\Gamma(v, \mu)^{-1}(\sigma_0^k(\Gamma))$ is the interval $D(\sigma_0^k(\Gamma))(v, \mu)$.*

vii) *for any $(v, \mu) \in B(k_0)$ such that $\mu > \mu_{2\Gamma_0}(v)$, we have that $\Gamma(v, \mu)^{-1}(\sigma_0^k(\Gamma))$ is a hyperbolic repelling fixed point of the map $G_{\sharp(\Gamma_0)}(v, \mu, \cdot)$. Moreover $D(\sigma_0^k(\Gamma))(v, \mu)$ is exactly this repelling fixed point and*

viii) *all the maps $v \to \mu_{\Gamma_0}(v), v \longmapsto \mu^f_{\Gamma_0}(v)$, and $v \longmapsto \mu_{2\Gamma_0}(v)$ are C^1.*

Proof. – Without loss assume $\Gamma = (\Gamma_0, \Gamma_0, \ldots)$ where $\Gamma_0 = \left(\begin{pmatrix}1\\0\end{pmatrix}\right)$. Later we will make some comments on the general case.

In this situation $\mu_{\Gamma_0} = \xi^{-(k_0-1)}(1-\delta)$. For $(v, \mu) \in B(k_0)$ and $y \in D\begin{pmatrix}1\\0\end{pmatrix}(v, \mu)$ define :

$$E(v, \mu; y) = G(v, \mu, y) - y.$$

We have ; $E(v, \xi^{-(k_0-1)}(1-\delta), 1-\delta) = 0$ and

$$\frac{\partial E}{\partial y}(v, \mu; y)\Big|_{\mu=\xi^{-(k_0-1)}(1-\delta)}^{y=1-\delta} = -1$$

By applying the implicit function theorem we can find a C^2-map $y = y(v, \mu) \in D\begin{pmatrix}1\\0\end{pmatrix}$ such that $E(v, \mu; y(v, \mu)) = 0$.

That is, $G(v, \mu, y(v, \mu)) = y(v, \mu)$.

For fixed v such that $(v, \mu) \in B(k_0)$ we have

$$\frac{\partial y}{\partial \mu}(v, \mu) = \frac{\dfrac{\partial G}{\partial \mu}(v, \mu; y(v, \mu))}{1 - \dfrac{\partial G}{\partial y}(v, \mu; y(v, \mu))}.$$

Since $\frac{\partial G}{\partial \mu}(v, \mu; y) > 0; (v, \mu) \in B(k_0), y \in D\begin{pmatrix}1\\0\end{pmatrix}(v, \mu)$, and $\frac{\partial G}{\partial y}(v, \mu; y) \leq 0$, for $(v, \mu) \in B(k_0), y \in D\begin{pmatrix}1\\0\end{pmatrix}(v, \mu)$, we conclude that $\frac{\partial y}{\partial \mu}(v, \mu) > 0, (v, \mu) \in B(k_0)$ and

$$\frac{\partial y}{\partial \mu}(v, \mu) \leq \frac{\partial y}{\partial \mu}(v, \mu)\Big|_{\mu=\xi^{-(k_0-1)}(1-\delta)} = \xi^{k_0-1}$$

Since

$$\frac{\partial G}{\partial y}(v, \mu, y(v, \mu))\Big|_{\mu=\xi^{-(k_0-1)}(1-\delta)} = 0$$

we conclude , for μ near $\xi^{-(k_0-1)}(1-\delta)$ such that $(v, \mu) \in B(k_0)$ that $y = y(v, \mu)$ is an attracting fixed point for the map $G(v, \mu, \cdot)$.

Now a cumbersone computation will show that

$$\frac{\partial}{\partial \mu}\left(\frac{\partial}{\partial y}(G(v, \mu, y))\right)\Big|_{y=y(v,\mu)} \leq 0.$$

Moreover, for $\mu > \xi^{-(k_0-1)}(1-\delta)$ we have :

$$\frac{\partial G}{\partial \mu}(v, \mu, y(v, \mu)) = -\frac{\xi^{k_0-1}\mu - y(v, \mu)}{J(v, \mu, y(v, \mu))}\left[\frac{\partial J}{\partial y}(v, \mu, y) + \frac{\alpha J^{1+\frac{1}{\alpha}}\xi^{\frac{k_0-1}{\alpha}}}{\xi^{k_0-1}\mu - y^{\frac{1}{\alpha}}}\right].$$

So, there exist a unique value $\mu = \mu^f_{\Gamma_0}(v)$ such that

$$\frac{\partial G}{\partial y}(v\,,\mu\,,y)\Big|_{\mu=\mu^f_{\Gamma_0}(v)} = -1.$$

Now it is not hard to see that :

$$\frac{\partial^3}{\partial y^3}(G(v,\mu\,,y(v\,,\mu)))\Bigg|_{\substack{y=y(v,\mu^f_{\Gamma_0}(v))\\ \mu=\mu^f_{\Gamma_0}(v)}} < 0.$$

Under these circumstances we may consider the C^2-map

$$H(v,\mu;y) = \begin{cases} \dfrac{G_2(v\,,\mu\,,y) - y}{y - y(v,\mu)}\,, & y \neq y(v,\mu) \\ \dfrac{\partial}{\partial y}(G_2(v\,,\mu\,y)) - 1\,, & y = y(v,\mu) \end{cases} .$$

Clearly $H(v,\mu^f_{\Gamma_0}(v)\,,y(v\,,\mu^f_{\Gamma_0}(v))) = 0$ and

$$\frac{\partial H}{\partial \mu}(v,\mu;y)\Bigg|_{\substack{\mu=\mu^f_{\Gamma_0}(v)\\ y=y(v;\mu^f_{\Gamma_0}(v))}} = \frac{\partial}{\partial\mu}\left(\frac{\partial}{\partial y}(G_2(v\,,\mu\,,y)\right)\Bigg|_{\substack{\mu=\mu^f_{\Gamma_0}(v)\\ y=y(v,\mu^f_{\Gamma_0}(v))}} \neq 0.$$

In this case there is a smooth map $\mu = \mu(v,y)$ such that $H(v,\mu(v,y),y) = 0$.

For $y \neq y(v,\mu)$ we have $G_2(v\,,\mu\,,y) = y$ which is a period two point for the map $G(v,\mu\,,\cdot)$.

It is easy to see that

$$\frac{\partial\mu}{\partial y}(v,y)\Bigg|_{\substack{y=y(v\,,\mu)\\ \mu=\mu^f_{\Gamma_0}(v)}} = 0$$

and that

$$\frac{\partial^2\mu}{\partial y^2}\Bigg|_{\substack{y=y(v\,,\mu)\\ \mu=\mu^f_{\Gamma_0}(v)}} > 0\ .$$

We note that, whenever defined, the interval $\{(v,\mu)\} \times [0,1]$ intersects the graph of the map $\mu = \mu(v,y)$ into two points: $(v,\mu;y_1),(v,\mu;y_2)$. These two points satisfy $G(v\,,\mu(v\,,y_1)\,,y_1) = y_2, G(v,\mu(v\,,y_2)\,,y_2) = y_1$, and $y_1 \leq y(v,\mu) \leq y_2$. Since

$$\left|\frac{\partial G_2}{\partial y}(v\,,\mu\,,y(v\,,\mu))\right| \geq 1,$$

for $\mu \geq \mu^f_{\Gamma_0}(v)$, and since this absolute value is equal to one only for $\mu = \mu^f_{\Gamma_0}(v)$, we have that

$$\left|\frac{\partial G_2}{\partial y}(v\,,\mu(v\,,y_2)\,,y_2)\right| < 1,$$

any $\mu > \mu^f_{\Gamma_0}(v)$ wherever y_2 is defined.

Since the graph of the map $\mu = \mu(v,y)$ intersects transversally the graph of the map $(v,\mu) \longmapsto G(v,\mu 1-\delta)$, their intersection defines a C^1-map $\mu = \mu_{2\Gamma_0}(v)$ and thus the proof of Proposition 1 is now complete in the case $\Gamma_0 = \left(\begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}1\\0\end{pmatrix}, \cdots\right)$. ■

In the general case we can proceed as follows:

Let $\Gamma_0 = \left(\begin{pmatrix}1\\0\end{pmatrix}\begin{pmatrix}i_1\\j_1\end{pmatrix}\cdots,\begin{pmatrix}i_r\\j_r\end{pmatrix}\right)$ here $r = \sharp(\Gamma_0) - 1$, and consider

$$D\left(\begin{pmatrix}1\\0\end{pmatrix}\begin{pmatrix}i_1\\j_1\end{pmatrix}\cdots,\begin{pmatrix}i_r\\j_r\end{pmatrix}\right)(v,\mu)$$
$$= D(\Gamma_0)(v,\mu) \subset D\left(\begin{pmatrix}1\\0\end{pmatrix}\begin{pmatrix}i_1\\j_1\end{pmatrix}\cdots,\begin{pmatrix}i_{r-1}\\j_{r-1}\end{pmatrix}\right)(v,\mu) \subset \cdots \subset D\left(\begin{pmatrix}1\\0\end{pmatrix}\right)(v,\mu).$$

Clearly we have $G_{\sharp(\Gamma_0)}(v,\mu,1-\delta) \in D(\Gamma_0)(v,\mu)$.

Let $\mu_{\Gamma_0}(v) = \inf\{\mu; (v,\mu) \in B(k_0), \Theta(v,\mu) = \sigma_0(\Gamma)\}$. For $\mu = \mu_{\Gamma 0}(v)$ we must have $G_{\sharp(\Gamma_0)}(v,\mu,1-\delta) = 1-\delta($ and therefore $D(\Gamma_0)(v,\mu_{\Gamma 0}(v)) = 1-\delta)$.

Now we define the map $E(v,\mu,y), y \in D(\Gamma_0)(v,\mu), (v,\mu) \in B(k_0)$ such that $\mu \geq \mu_{\Gamma_0}(v)$ by:

$$E(v,\mu,y) = G_{\sharp(\Gamma_0)}(v,\mu,y) - y$$

Now the proof of the proposition 1 follows as in the previous case.

2.8.

Let $\Gamma \in \Sigma_2$ and denote by Γ_0 its period.

PROPOSITION 2. – *For those $\Gamma \in \Sigma_2$ such that $\sigma_0(\Gamma)$ is realizable and the number of $\begin{pmatrix}1\\j\end{pmatrix}$ that appears in Γ_0 is even, we can find values of the parameter $\mu_\Gamma(v) = \mu^{sn}_{\Gamma_0}(v) < \mu_{\Gamma_0}(v)$ such that:*

i) *for $(v,\mu^{sn}_{\Gamma_0}(v)) \in B(k_0)$, the associated one-dimensional map $G(v,\mu^{sn}_{\Gamma_0}(v),\cdot)$ has a saddle-node bifurcation whose period is $\sharp(\Gamma_0)$. Moreover, one point of this orbit is contained in the boundary of the interval $D(\sigma_0^k(\Gamma))(v,\mu)$, any $0 \leq k \leq \sharp(\Gamma_0) - 1$.*

ii) *for $(v,\mu) \in B(k_0)$; $\mu^{sn}_{\Gamma_0}(v) < \mu < \mu_{\Gamma_0}(v)$, the associated one-dimensional map $G(v,\mu,\cdot)$ has an attracting, hyperbolic, periodic orbit and a repelling, hyperbolic, periodic orbit contained in the interior of $D(\Gamma)(v,\mu) \cup D(\sigma_0(\Gamma))(v,\mu) \cup \cdots \cup D(\sigma_0^{\sharp(\Gamma_0)-1}(\Gamma))(v,\mu)$.*

Moreover one point, of any of the two periodic orbits, is contained in $D(\sigma_0^k(\Gamma))(v,\mu)$, any $0 \leq k \leq (\sharp(\Gamma_0) - 1)$.

iii) *for $(v,\mu = \mu_{\Gamma_0}(v)) \in B(k_0)$, the associated one-dimensional map satisfies*

$$G_{\sharp(\Gamma_0)}(v,\mu,\partial D(\sigma_0^k(\Gamma))(v,\mu)) = \partial D(\sigma_0^k(\Gamma))(v,\mu).$$

Under these circumstances the points in the boundary are fixed points for the map $G_{\sharp(\Gamma_0)}$. Note that the boundary $\partial D(\Gamma)(v,\mu)$ contains $1-\delta$ or 1 depending on $\Gamma_0 = \left(\begin{pmatrix}1\\0\end{pmatrix}, \cdots\right)$ or $\left(\begin{pmatrix}2\\0\end{pmatrix}, \cdots\right)$, respectively.

iv) *for* $(v\,,\mu)\in B(k_0);\mu^{sn}_{\Gamma_0}(v)\le\mu\le\mu_{2\Gamma_0}(v)$ *the pre-image* $\Gamma(v,\mu)^{-1}(\sigma_0^k(\Gamma))$ *is the interval* $D(\sigma_0^k(\Gamma))(v,\mu)$.

v) *for any* $(v,\mu)\in B(k_0)$ *such that* $\mu>\mu_{\Gamma_0}(v)$ *we have that* $\Gamma(v,\mu)^{-1}(\sigma_0^k(\Gamma))$ *is a hyperbolic, repelling fixed point of the map* $G^{\sharp(\Gamma_0)}(v,\mu)(\cdot)$. *Moreover* $D(\sigma_0^k(\Gamma))(v,\mu)$ *is exactly this repelling fixed point.*

vi) *The maps* $V\longrightarrow[1-\delta\,,1];v\longmapsto\mu^{sn}_{\Gamma_0}(v)$, *and* $v\longmapsto\mu_{\Gamma_0}(v)$ *are* C^1.

Proof. – Assume $\Gamma_0=\left(\begin{pmatrix}2\\0\end{pmatrix},\begin{pmatrix}2\\0\end{pmatrix},\cdots\right)$. Later we will comments on the general case. In this situation $\mu_{\Gamma_0}(v)=\xi^{-(k_0-1)}$.

For $(v,\mu)\in B(k_0)$ and $y\in D\begin{pmatrix}2\\0\end{pmatrix}(v,\mu)$ define the map: $E(v,\mu;y)=G(v,\mu;y)-y$.

We have :

$$E(v,\mu;y)=\xi^{k_0-1}[\mu-K(v,\mu;y)(1-y)^\alpha]-y\,.$$

and, hence, $\dfrac{\partial E}{\partial\mu}(v,\mu;y)|_{y=1}=\xi^{k_0-1}\neq0$, for any $(v,\mu)\in B(k_0)$. Therefore, by the implicit function theorem we obtain a C^1-map, twice differentiable in the y-variable $\mu=\mu(v,y)$ such that: We solve the equation $E(v,\mu;y)=0$ for $(v,\mu)\in B(k_0),y\in D\begin{pmatrix}2\\0\end{pmatrix}(v,\mu)$ if and only if $\mu=\mu(v,y)$.

From the relation $E(v,\mu(v,y);y)=0$ we obtain

$$\frac{\partial\mu}{\partial y}(v,y)=\frac{\xi^{k_0-1}\left[-\dfrac{\partial K}{\partial y}(v,\mu;y)(1-y)^\alpha-\alpha K(v,\mu;y)(1-y)^{\alpha-1}\right]-1}{\xi^{k_0}-\dfrac{\partial K}{\partial\mu}(v,\mu;y)(1-y)^\alpha}\,,$$

and from this relation we have that: $\dfrac{\partial\mu}{\partial y}(v,y)=0$ if and only if

$$H(v,y)=-\frac{\partial K}{\partial y}(v,\mu(v,y);y)(1-y)^\alpha+\alpha K(v,\mu(v,y);y)(1-y)^{\alpha-1}-\xi^{-(k_0-1)}=0\,.$$

Since $|1-y|$ is small, $K(v,\mu;y)\neq0$ and

$$\begin{aligned}\frac{\partial H}{\partial y}(v,y)=&(1-y)^{\alpha-2}\left[\frac{\partial^2K}{\partial y^2}(v,\mu(v,y);y)(1-y)^2+\right.\\&\left.+2\alpha\frac{\partial K}{\partial y}(v,\mu(v,y);y)(1-y)-\alpha(\alpha-1)K(v,\mu(v,y),y)\right],\end{aligned}$$

we have $\dfrac{\partial H}{\partial y}(v,y)\neq0$, any (v,y) such that $H(v,y)=0$.

Hence by the implicit function theorem we find a C^1-map, $y=y(v)$, that simultaneously satisfies equations $E(v,\mu(v,y(v));y(v))=0$ and $\dfrac{\partial\mu}{\partial y}(v,y(v))=0$.

Figure 4 shows the above relations obtained for the maps $\mu(v,y)$ and $y(v)$.

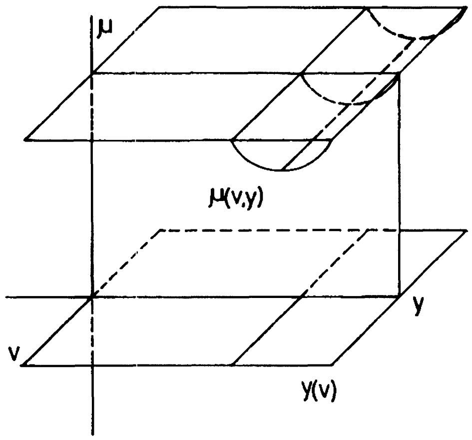

Fig. 4

Denote by $\mu^{sn}_{\Gamma_0} = \mu(v\,,y(v))$. For this map we have:

$$G(v\,,\mu^{sn}_{\Gamma_0}\,,y(v)) = y(v)\,;\frac{\partial G}{\partial y}(v\,,\mu^{sn}_{\Gamma_0}\,,y(v)) \equiv 1$$

and

$$\frac{\partial^2 G}{\partial y^2}(v\,,\mu^{sn}_{\Gamma_0}\,,y) \neq 0$$

That is ; the one dimensional map $G(v\,,\mu^{sn}_{\Gamma_0}\,,\cdot)$, has a saddle-node at the point $y = y(v) \in D\binom{2}{0}(v\,,\mu^{sn}_{\Gamma_0})$.

Now assume $(v,\mu) \in B(k_0)$ satisfies $\mu^{sn}_{\Gamma_0} < \mu < \mu_{\Gamma_0}(v)$. In this case the interval $\{(v,\mu)\} \times [1-\delta,1]$ intersects the graph of the map $\mu(v\,,y)$ into two points $(v,\mu;y_1)$ and $(v,\mu;y_2)$. These two points satisfy $G(v,\mu;y_1) = y_1$ and $G(v,\mu;y_2) = y_2$ with $y_1 < y_2$. Again, an easy computation shows $\frac{\partial G}{\partial y}(v,\mu;y_1) > 1 > \frac{\partial G}{\partial y}(v,\mu;y_2)$: that is the map $G(v,\mu;\cdot)$ has a hyperbolic, attracting periodic orbit whose period is k_0, at $y = y_2$; and a hyperbolic repelling , fixed point at $y = y_1$.

Observe that , for $(v\,,\mu) \in B(k_0)\,,\mu \leq \mu^{sn}_{\Gamma_0}$, the one dimensional map $G(v\,,\mu\,\cdot)$;does not have fixed points in $D\binom{2}{0}$. This complete the proof of proposition 2 in this particular case.

In the general case we can proceed as follows :

Let $\Gamma_0 = \left(\begin{pmatrix}2\\0\end{pmatrix},\begin{pmatrix}i_1\\j_1\end{pmatrix},\ldots,\begin{pmatrix}i_r\\j_r\end{pmatrix}\right)$, here $r = \#(\Gamma_0) - 1$. Let us consider $D\left(\begin{pmatrix}2\\0\end{pmatrix},\begin{pmatrix}i_1\\j_1\end{pmatrix},\ldots,\begin{pmatrix}i_r\\j_r\end{pmatrix}\right)(v,\mu) \subset D\left(\begin{pmatrix}2\\0\end{pmatrix},\begin{pmatrix}i_1\\j_1\end{pmatrix}\ldots,\begin{pmatrix}i_{r-1}\\j_{r-1}\end{pmatrix}\right)(v,\mu) \subset \ldots \subset D\begin{pmatrix}2\\0\end{pmatrix}(v,\mu)$. Clearly we have $G_{\#(\Gamma_0)}(v,\mu,1) \in D(\Gamma)(v,\mu)$. Let $\mu_{\Gamma_0}(v) = \sup\{\mu\,;\,(v,\mu) \in B(k_0)\,,\Theta(v,\mu) = \sigma_0(\Gamma)\}$. For $\mu = \mu_{\Gamma_0}(v)$ we must have

$G_{\#\Gamma_0}(v,\mu;1) \equiv 1$. Now we define the map $E(v,\mu;y)$, $y \in D(\Gamma_0)(v,\mu)$, $(v,\mu) \in B(k_0)$ such that $\mu \leq \mu_{\Gamma_0}(v)$ by:

$$E(v,\mu;y) = G_{\#(\Gamma_0)}(v,\mu;y) - y.$$

Now the proof follows as in the previous case. ■

As a consequence of proposition 1 and 2 we get the following :

Remark 5. – Asumme $\Gamma_1(v,\mu)$ or $\Gamma_{1-\delta}(v,\mu)$ is a periodic itinerary. In this situation the associated one dimensional map $G(v,\mu,\cdot)$ satisfies one of the following:

(i) $D(\Gamma_1(v,\mu))(v,\mu)$ (or $D(\Gamma_{1-\delta}(v,\mu))(v,\mu)$) is an interval which contains, in its interior, a hyperbolic, attracting periodic orbit or

(ii) $D(\Gamma_1(v,\mu))(v,\mu)$ (or $D(\Gamma_{1-\delta}(v,\mu))(v,\mu)$) is an interval which contains a flip or a saddle-node periodic orbit or

(iii) $D(\Gamma_1(v,\mu))(v,\mu)$ (or $D(\Gamma_{1-\delta}(v,\mu))(v,\mu)$) is an interval and $y = 1$ (or $y = 1-\delta$) is an attracting periodic orbit or

(iv) $D(\Gamma_1(v,\mu))(v,\mu) = \{1\}$ (or $D(\Gamma_{1-\delta}(v,\mu))(v,\mu) = \{1-\delta\}$).

2.9.

Let us now define an order relation among the elements of Σ_0.

We initially define

$$\binom{1}{0} < \binom{1}{1} < \cdots < \binom{1}{n} < \binom{1}{n+1} < \cdots < \binom{2}{n+1} < \binom{2}{n} < \cdots < \binom{2}{0}.$$

Let $\Gamma_1 \neq \Gamma_2$ be any two bisequences. Assume that

$$\left(\binom{i_0^1}{j_0^1}, \ldots, \binom{i_k^1}{j_k^1}\right) = \left(\binom{i_0^2}{j_0^2}, \ldots, \binom{i_k^2}{j_k^2}\right) \text{and that} \binom{i_{k+1}^1}{j_{k+1}^1} \neq \binom{i_{k+1}^2}{j_{k+1}^2}.$$

– If there is an even number of $\binom{1}{j}$ among $\binom{i_0^1}{j_0^1}, \cdots, \binom{i_k^1}{j_k^1}$ and $\binom{i_{k+1}^1}{j_{k+1}^1} > \binom{i_{k+1}^2}{j_{k+1}^2}$, we will say Γ_1 is greater than Γ_2 and we will denote $\Gamma_1 > \Gamma_2$.

– If there is an odd number of $\binom{1}{j}$ among $\binom{i_0^1}{j_0^1}, \cdots, \binom{i_k^1}{j_k^1}$ and $\binom{i_{k+1}^1}{j_{k+1}^1} < \binom{i_{k+1}^2}{j_{k+1}^2}$, we will say Γ_1 is greater than Γ_2 and we will denote $\Gamma_1 > \Gamma_2$.

Lemma 8. – *The map* $\Gamma(v,\mu) : M(v,\mu) \to \Sigma_0$ *is order-preserving.*

Proof. – Let $x_1, x_2 \in M(v,\mu)$ be two points such that $x_1 \leq x_2$. If $x_1 \in D\binom{i_0}{j_0}(v,\mu)$ and $x_2 \in D\binom{i_1}{j_1}$ with $\binom{i_0}{j_0} \neq \binom{i_1}{j_1}$, the result follows.

Assume $\Gamma(v,\mu)(x_1) = \Gamma_1$, and $\Gamma(v,\mu)(x_2) = \Gamma_2$ are such that

$$\left(\binom{i_0^1}{j_0^1}, \ldots, \binom{i_k^1}{j_k^1}\right) = \left(\binom{i_0^2}{j_0^2}, \ldots, \binom{i_k^2}{j_k^2}\right) \quad \text{and} \quad \binom{i_{k+1}^1}{j_{k+1}^1} \neq \binom{i_{k+1}^2}{j_{k+1}^2}.$$

If there is an even number of $\binom{i}{j}$'s among the $\left(\binom{i_0^1}{j_0^1}, \ldots, \binom{i_k^1}{j_k^1}\right)$, then the restriction of the map $G_k(v,\mu)$ to the interval that contains $[x_1, x_2]$ preserves orientation. This implies that $G_k(v,\mu)(x_1) \leq G_k(v,\mu)(x_2)$ and therefore $\binom{i_{k+1}^1}{j_{k+1}^1} < \binom{i_{k+1}^2}{j_{k+1}^2}$. By the definition of the order relation in Σ_0 this implies $\Gamma_1 < \Gamma_2$.

If there is an odd number of $\binom{i}{j}$ among the $\binom{i_0^1}{j_0^1}, \cdots, \binom{i_k^1}{j_k^1}$, then the restriction map $G_k(v,\mu)(\cdot)$ to the interval $D\left(\binom{i_0^1}{j_0^1}, \ldots, \binom{i_k^1}{j_k^1}\right)(v,\mu)$, which contains $[x_1, x_2]$, reverses orientation. This implies that $G_k(v,\mu)(x_1) > G_k(v,\mu)(x_2)$ and therefore $\binom{i_{k+1}^1}{j_{k+1}^1} > \binom{i_{k+1}^2}{j_{k+1}^2}$. By the definition of the order relation in Σ_0 we obtain $\Gamma_1 < \Gamma_2$. ■

Let us now consider two bisequences $\Gamma_1, \Gamma_2 \in \Sigma_0$ such that $\Gamma(v,\mu)(x_1) = \Gamma_1, \Gamma(v,\mu)(x_2) = \Gamma_2$, some $x_1, x_2 \in M(v,\mu)$.

LEMMA 9. – *If* $\Gamma_1 < \Gamma_2$, *then* $x_1 < x_2$.

Proof. – The proof is easy and left to the reader. ■

Let $\Gamma \in \Sigma_0$ be any realizable sequence and denote by $\mu_\Gamma = \inf\{\mu; \Theta(v,\mu) = \Gamma\}$. Let $\Gamma_2 \in \Sigma_0$ be any admissible bisequence at the level $(v, \mu_\Gamma(v))$ such that $\Gamma_2 > \Gamma$.

LEMMA 10. – Γ_2 *is realizable.*

Proof. – Denote by $x_1(v,\mu) \in M(v,\mu), x_2(v,\mu) \in M(v,\mu)$ two points which satisfy $\Gamma(v,\mu)(x_1(v,\mu)) = \Gamma$ and $\Gamma(v,\mu)(x_2(v,\mu)) = \Gamma_2$. We have $x_1(v,\mu) < x_2(v,\mu)$ and $x_1(v,\mu_\Gamma(v)) = \xi^{k_0-1}\mu_\Gamma(v)$. Since $\mu \longmapsto \xi^{k_0-1}\mu$ is an increasing map we can find a parameter value $\mu_2 > \mu_\Gamma(v)$ such that $x_2(v,\mu_2) = \xi^{k_0-1}\mu_2$. This implies $x_2(v,\mu_2) = \Gamma(v,\mu)(G(v,\mu_2, 1-\delta)) = \sigma_0 \circ (\Gamma(v,\mu_2)(1-\delta)) = \Theta(v,\mu_2)$. That is Γ_2 is realizable. ■

Remark 6. – 1) Let $\Gamma \in \Sigma_0$ be any realizable sequence and $\mu_\Gamma(v) = \inf\{\mu, \Theta(v,\mu) = \Gamma\}$. Let $\Gamma_2 \in \Sigma_0; \Gamma_2 \leq \Gamma$ be any bisequence which is not realizable for $\xi^{-(k_0-1)}(1-\delta) \leq \mu \leq \mu_\Gamma(v)$ then Γ_2 is not realizable at all, that is there no exists $\xi^{-(k_0-1)}(1-\delta) \leq \mu \leq \xi^{-(k_0-1)}$ such that $\Theta(v,\mu) = \Gamma_2$.

2) Assume $(v,\mu_1), (v,\mu_2) \in B(k_0)$ satisfy $\xi^{k_0-1}\mu_1 \in M(v,\mu_1)$, $\xi^{k_0-1}\mu_2 \in M(v,\mu_2)$. If $\mu_1 < \mu_2$ then we have $\Theta(v,\mu_1) = \Gamma(v,\mu_1)(\xi^{k_0-1}\mu_1) \leq \Theta(v,\mu_2) = \Gamma(v,\mu_2)(\xi^{k_0-1}\mu_2)$

3) Assume $(v,\mu_1), (v,\mu_2) \in B(k_0)$ satisfy $\xi^{k_0-1}\mu_1 \in M(v,\mu_1)$, $\xi^{k_0-1}\mu_2 \in M(v,\mu_2)$ and $\Theta(v,\mu_1) < \Theta(v,\mu_2)$ then we have $\mu_1 < \mu_2$.

2.10.

Let $\Gamma \in \Sigma_2$ be any periodic bisequence which is realizable.

Assume $\mu_\Gamma(v) = \inf\{\mu; \Theta(v,\mu) = \Gamma\}$.

(A) Let $\Gamma_k = \sigma_0^k(\Gamma)$, for $1 \leq k \leq \sharp(\Gamma_0)-1$. Suppose $\Gamma_j > \Gamma$, for some j. By Lemma 21 we have that Γ_j is realizable. In fact denote by $x_j(v,\mu) \in M(v,\mu)$ a point which satisfies $\Gamma(v,\mu)(x_j(v,\mu)) = \Gamma_j$. By (2.11) we know that $D(\Gamma_j)(v,\mu)$ is a hyperbolic, repelling, fixed point of the map $G_{\sharp(\Gamma_0)}(v,\mu)$, for $\mu > \mu_{2\Gamma_0}(v)$ or $\mu > \mu_{\Gamma_0}(v)$. Since the C^1-surface $C_{\Gamma_j} = \{(v,\mu;x_j(v,\mu))/\mu \geq \mu_{\Gamma_0}(v)$ or $\mu \geq \mu_{2\Gamma_0}(v), (v,\mu) \in B(k_0)\}$ is transversal to $S_0 = \{(v,\mu;\xi^{k_0-1}\mu)/(v,\mu \in B(k_0)\}$ we have that $S_0 \cap C_{\Gamma_j}$, define a C^1 surface contained in $\mathcal{U} \times [1-\delta, 1]$ and parametrized by $\{(v, C_{\Gamma_j}(v), x_j(v, C_{\Gamma_j}(v))); v \in V\}$.

Let us denote by $X_{\Gamma_j}(v)$ the vector field associated to $(v, C_{\Gamma_j}(v)) \in B(k_0)$.

Let $\sigma(x_j(v, C_{\Gamma_j}(v))) \subset U$ be the hyperbolic, periodic orbit associated to the point $x_j(v, C_{\Gamma_j}(v))$. We have

$$\gamma_0(\sigma_0(X_{\Gamma_j}(v))) \subset W^s(\sigma(x_j(v, C_{\Gamma_j}(v)))),$$

that is, the associated vector field $X_{\Gamma_j}(v)$ represents a contracting singular cycle.

(B) Let $\mathcal{X} \in \sum_0, \mathcal{X} > \Gamma$ be any admissible bisequence, at the level $(v, \mu_\Gamma(v))$, such that $\sigma_0^k(\mathcal{X}) = \Gamma$, some $k \in \mathbf{N}$.

Let us denote by $x_{\mathcal{X}}(v,\mu) \in M(v,\mu)$ a point which satisfies $\Gamma(v,\mu)(x_{\mathcal{X}}(v,\mu)) = \mathcal{X}$. We have: $\sigma_0^k \circ \Gamma(v,\mu)(x_{\mathcal{X}}(v,\mu)) = \sigma_0^k(\mathcal{X}) = \Gamma$. That is: $\Gamma(v,\mu)G_k(v,\mu)(x_{\mathcal{X}}(v,\mu)) = \Gamma(v,\mu)(p_\Gamma(v,\mu))$ (here $p_\Gamma(v,\mu)$ denotes the fixed point of the map $G_{\sharp\Gamma_0}(v,\mu)$ which satisfies $p_\Gamma(v,\mu) \in D(\Gamma)(v,\mu)$. In particular, $G_k(v,\mu)(x_{\mathcal{X}}(v,\mu)) \in D(\Gamma)(v,\mu)$. That is $x_{\mathcal{X}}(v,\mu) \in G^{-k}(v,\mu)(D(\Gamma)(v,\mu))$. From here we conclude that, for $\mu > \mu_{2\Gamma_0}(v)$ or $\mu > \mu_{\Gamma_0}(v)$, the point $x_{\mathcal{X}}(v,\mu)$ is a pre-image of the hyperbolic, repelling, fixed point $p_\Gamma(v,\mu)$. So in particular

$$C_{\mathcal{X}} = \{(v,\mu; x_{\mathcal{X}}(v,\mu)); (v,\mu) \in B(k_0), \mu > \mu_{2\Gamma_0}(v), \mu > \mu_{\Gamma_0}(v)\}$$

is a C^1-surface transversal to S_0. Therefore the intersection $S_0 \cap C_{\mathcal{X}}$ defines a C^1-surface, $C^0_{\mathcal{X}}$, contained in $\mathcal{U} \times [1-\delta, 1]$ and parametrized by

$$\{(v, C^0_{\mathcal{X}}(v), \mathcal{X}_{\mathcal{X}}(v, C^0_{\mathcal{X}}(v))); v \in V\}.$$

Denote by $X_{\mathcal{X}}(v)$ the vector field associated to $(v, C^0_{\mathcal{X}}(v)) \in B(k_0)$.

Let $\sigma(p_\Gamma(v, C^0_{\mathcal{X}}(v))) \subset U$ be the hyperbolic, periodic orbit associated to the point $p_\Gamma(v, C^0_{\mathcal{X}}(v)) \in M(v, C^0_{\mathcal{X}}(v))$. We have

$$\gamma_0(\sigma_0(X_{\mathcal{X}}(v))) \subset W^s(\sigma(p_\Gamma(v, C^0_{\mathcal{X}}(v)))),$$

that is, the vector field $X_{\mathcal{X}}(v)$ has a contracting singular cycle.

2.11.

Let $\Gamma \in \Sigma_0$ be any realizable bisequence. Assume $\mu_\Gamma = \mu_\Gamma(v)$ is the parameter value which satisfies $\Theta(v, \mu_\Gamma(v)) = \Gamma$ and $x_\Gamma = x_\Gamma(v,\mu) \in M(v,\mu)$ be a point which satisfies

$$\Gamma(v,\mu)(x_\Gamma(v,\mu)) = \Gamma.$$

(A) Assume $\Gamma \in Per(\sigma)$. In this case we have $\Gamma \in \Sigma_1$ or $\Gamma \in \Sigma_2$ or there is $k \in \mathbf{N}$ such that $\sigma_0^k(\Gamma) \in \Sigma_2$. In all the cases, as we have seen in (2.6), (2.7) (2.8) and (2.10), we known that associated to Γ we can find a C^1-surface $C^0_\Gamma = \{(v, C_\Gamma(v)); v \in V\} \subset B(k_0)$ such that: the vector field $X_\Gamma(v)$, which represents the point $(v, C_\Gamma(v)) \in C^0_\Gamma$, presents a contracting singular cycle or a homoclinic orbit for the singularity $\sigma_0(X_\Gamma(v))$ or a saddle-node or a flip bifurcation.

(B) Suppose that $\Gamma \notin Per(\sigma)$ and that there is $k \in \mathbf{N}$ such that $\sigma_0^k(\Gamma) \in Per(\sigma)$. In this situation, as we have seen in (2.6) and (2.10), we know that associated to Γ, we can find a C^1-surface $C^0_\Gamma = \{(v, C_\Gamma(v)); v \in V\} \subset B(k_0)$ such that: the vector field $X_\Gamma(v)$, which represents the point $(v, C_\Gamma(v)) \in C^0_\Gamma$, presents a contracting singular cycle.

(C) Suppose $\Gamma \notin Per(\sigma)$ and $\sigma_0^k(\Gamma) \notin Per(\sigma)$, for any $k \in \mathbf{N}$. In this case we can find a sequence of realizable sequences $\Gamma_k \in Per(\sigma_0)$, $\Gamma_k < \Gamma$, such that

(i) $\lim_{k\to\infty} \Gamma_k = \Gamma$

(ii) $\mu_{\Gamma_i}(v) \to \mu_\Gamma(v), \mu_{\Gamma_i}(v) < \mu_\Gamma(v)$ and

(iii) $(\mu_{\Gamma_i}(\cdot))$ is a Cauchy sequence of maps in the C^1-uniform topology (this can be proved as in (2.9)).

In this case, associated to Γ, we find a C^1-surface $\{(v, C_\Gamma(v)); v \in V\}$ such that the vector field which represents the point $(v, C_\Gamma(v)) \in C^0_\Gamma$ satisfies that the trajectory $\gamma_0(X_\Gamma(v))$ has recurrent behavior in the neighborhood U.

(D) Let now $s(v,\mu)$ be any pre image of the points $b(v,\mu)$ or $a(v,\mu)$ in the closure of the set $M(v,\mu)$, such that $s(v,\mu) \geq \xi^{k_0-1}\mu$ for some $\xi^{-(k_0-1)}(1-\delta) \leq \mu \leq \xi^{-(k_0-1}$. In this situation the C^1-surface $\{(v,\mu,s(v,\mu))\} = S$ is transversal to S_0 and, therefore, the intersection $S \cap S_0$ define a C^1-surface S_b (resp S_a) parametrized by $\{(v, \bar{b}(v), S(v, \bar{b}(v))); v \in V\}$ (resp. $\{(v, \bar{a}(v), S(v, \bar{a}(v))); v \in V\}$). Let $X_{\bar{b}}(v)$ (resp. $X_{\bar{a}}(v)$) denote the vector field associated to $(v, \bar{b}(v)) \in B(k_0)$ (resp. $v, \bar{a}(v)) \in B(k_0)$). This vector field satisfies that

$$\gamma_0(\sigma_0(X_{\bar{b}}(v))) \subset W^s(\sigma_1(X_{\bar{b}}(v)))$$

(resp. $\gamma_0(\sigma_0(X_{\bar{a}}(v))) \subset W^s(\sigma_1(X_{\bar{a}}(v))))$. That is presents a contracting singular cycles.

This completes the proof of Theorem 1. ∎

An easy consequence of the results in (2.7) through (2.11) is

COROLLARY 4. – $\Gamma_0 \cup \Gamma_1$ *is a dense subset of* $B(k_0)$, *any* $k_0 \geq n_0$. ∎

3. Proof of Theorem 2

Without loss of generality, we may assume that the family $\{X_\mu\}$ such that $X_{\mu=0} \in \mathcal{N}$ is given by $\{(\bar{v}, \mu); -\varepsilon_0 < \mu < \varepsilon_0\}$ for some $\bar{v} \in V$ and $\varepsilon_0 > 0$ small.

We let $L(\mu; y)$ denote the map $L(\bar{v}, \mu; y)$ given by

$$L(\mu;y) = \begin{cases} \xi y, & 0 \leq y \leq \xi^{-1} \\ \mu - J(\mu;y)(y-(1-\delta))^\alpha, & 1-\delta \leq y \leq b(\mu) \\ \mu - K(\mu;y)(1-y)^\alpha, & a(\mu) \leq y \leq 1; \end{cases}$$

where $a(\mu) = 1 - \delta^2(\mu), b(\mu) = 1 - \delta + \delta^1(\mu), \delta^i(\mu) = A^i(\mu)\mu^{1/\alpha}, i = 1,2; J$ and K are C^2-map in the μ-variable, C^3 in the y-variable for $y \neq 1-\delta, 1$ and whose derivatives are small with μ small.

Also $J(\mu, y) > 0$ and $K(\mu; y) > 0$ for any $(\mu; y), 0 \leq \mu \leq \mu_0 = \xi^{-n_0}; y \in I_1(\mu) \cup I_2(\mu)$..

Given $0 \leq \mu \leq \mu_0$ we define $\Lambda(\mu) = \{y \in [0,1]/L^n_\mu(y) \in \cup_{i=0}^2 I_i(\mu)\,,$ for all $n \geq 0\}$. Let $\Gamma_0 = \{\mu \in [0, \mu_0]/1 \notin \Lambda(\mu)\}$ and $\Gamma_1 = \{\mu \in [0, \mu_0]/1 \in \Lambda(\mu)$ and there exists an hyperbolic attracting periodic orbit for the map $L_\mu(\cdot)\}$. Here $L_\mu(y) = L(\mu; y)$.

As we have seen in Chapter II, $\mu \in \Gamma_0 \cup \Gamma_1$ implies that the associated vector field $X(\bar{v}, \mu)$ is structurally stable in U. Let $H = \Gamma_0 \cup \Gamma_1$ and $B = [0, \mu_0] \setminus H$.

Theorem 2 will follow from the following

THEOREM 2′. – $m(H \cap [0, \mu_0]) = \mu_0$. *(Here m denotes the Lebesgue measure.)*

Using the Lebesgue density theorem it is enough to prove that given $0 \leq \mu \leq \mu_0$ we have

$$(*) \lim_{\varepsilon \to 0} \frac{m(B \cap [\mu - \varepsilon, \mu + \varepsilon])}{2\varepsilon} < 1\,.$$

3.1.

For $\mu \in [0, \mu_0]\,,$ define $L_1(\mu) = L(\mu; 1)$ and $L_{n+1}(\mu) = L(\mu; L_n(\mu))$.

We have $L_{i+1}(\mu) = \xi L_i(\mu)\,,$ for any $1 \leq i \leq n_0$ and $L_{n_0+1}(\mu) = \xi^{n_0}\mu$. Hence these maps satisfy:

a) $L'_i(\mu) > 0$ and $L''_i(\mu) = 0, \mu \in [0, \mu_0], 1 \leq i \leq n_0 + 1\,,$

b) $L'_i(\mu) \leq L'_i(0), 0 \leq \mu \leq \mu_0, 1 \leq i \leq n_0 + 1$.

For any $k \geq n_0 + 2\,,$ let $I_k = I^1_k \cup \cdots \cup I^{m_k}_k$ be the domain of definition of the map L_k.

Let $I^j_k = [\nu_0, \nu_1]$ be a component of the domain I_k that satisfies $L'_i(\mu) \neq 0\,,$ for $1 \leq i \leq k - 1$ and any $\mu \in I^j_k$.

LEMMA 11. – *The map L_k satisfies one and only one of the following possibilities:*

(i) *there exists a unique $\bar{\nu} \in I^j_k$ such that $L'_k(\bar{\nu}) = 0$ and $L''_k(\bar{\nu}) < 0$ or*

(ii) *$L'_k(\mu) \neq 0$ and $L''_k(\mu) = 0$ for any $\mu \in I^j_k$ or*

(iii) *$L'_k(\mu) \neq 0$ and $L''_k(\mu) < 0$ for any $\mu \in I^j_k$.*

Proof. – *See* the appendix. ■

COROLLARY 5. – *Let $I = [\nu_0, \nu_1] \subset I^j_k$ be an interval and assume $L'_i(\mu) \neq 0$ for $\mu \in I, 1 \leq i \leq k$. Then for any $\alpha, \beta, \nu_0 \leq \alpha \leq \beta \leq \nu_1$ we have $L'_k(\alpha) \geq L'_k(\beta)$.*

Proof. – Let $\mathcal{X}(\mu) = \dfrac{L'_k(\mu)}{L'_k(\nu_0)}, \mu \in I$. We have $\mathcal{X}(\nu_0) = 1$.

If $L'_k(\mu) < 0\,,$ then $\mathcal{X}'(\mu) = \dfrac{L''_k(\mu)}{L'_k(\nu_0)} > 0$ and $\mathcal{X}$ is an increasing map. So $\mathcal{X}(\alpha) \leq \mathcal{X}(\beta)$ and hence $L'_k(\alpha) \geq L'_k(\beta)$.

If $L'_k(\mu) > 0\,,$ then $\mathcal{X}'(\mu) = \dfrac{L''_k(\mu)}{L'_k(\nu_0)} \leq 0$ and $\mathcal{X}$ is a decreasing map. In particular, $\mathcal{X}(\alpha) \leq \mathcal{X}(\beta)$ and hence $L'_k(\alpha) \geq L'_k(\beta)$. ■

3.2.

We note that $[0, \mu_0] = \{0\} \cup \cup_{k=n_0}^\infty \xi^{-k}]\xi^{-1}, 1]$.

Let $k \geq n_0$ be a given number and $I_k = \xi^{-k}\,]\,\xi^{-1}\,, 1\,]$. For any given $\mu \in I_k$ we have $\xi^{-1} < \xi^k\mu \leq 1$. Clearly that it is enough to prove that $m(B \cap I_k) = 0\,,$ for any $k \geq n_0$.

Given $\mu \in I_k$ let $D\binom{i}{j}(\mu)$ and $G_\mu(\cdot)$ denote the interval $D\binom{i}{j}(\bar{v},\mu)$ and the map $G(\bar{v},\mu)$ as defined in (2.11).

Let $J_0 = \xi^{-k}[1-\delta,1]$ and $g_0 : J_0 \to [1-\delta,1]$ be the map $g_0(\mu) = \xi^k\mu$.

Let us define, inductively,

$$J_r = \left\{\mu \in J_{r-1}/g_{r-1}(\mu) \in \cup_{j=0}^{\infty} \cup_{i=1,2} D\binom{i}{j}(\mu)\right\}$$

and $g_r : J_r \to [1-\delta,1]$ by $g_r(\mu) = G_\mu(g_{r-1}(\mu)), r \geq 1$.

Let $J_r^t = [\nu_0,\nu_1]$ be a component of the domain J_r such that $g_i'(\mu) \neq 0$, for $0 \leq i \leq r-1$ and any $\mu \in J_r^t$.

COROLLARY 6. – *For the map $g_r|J_r^t$ we have one and only one of the following possibilities:*
(i) *there exists a unique $\bar{\nu} \in J_r^t$ such that $g_r'(\bar{\nu}) = 0$ and $g_r''(\mu) < 0$, any $\mu \in J_r^t$ or*
(ii) $g_r'(\mu) \neq 0$ *and* $g_r''(\mu) < 0$ *for any* $\mu \in J_r^t$.

Proof. – The proof follows from Lemma 11. ■

COROLLARY 7. – *Let $J = [\nu_0,\nu_1] \subset J_r^t \subset J_r$ be an interval such that $g_i'(\mu) \neq 0$, for $0 \leq i \leq r$ and $\mu \in J_r^t$. Let α,β be the parameter values such that $\nu_0 \leq \alpha \leq \beta \leq \nu_1$ we have $g_r'(\alpha) \geq g_r'(\beta)$.*

Proof. – Similar to Corollary 5. ■

3.3.

Let us now consider a parameter value $\mu \in J_r$ that satisfies: there is an interval $[\alpha,\beta] \subset J_r$ such that $\mu \in]\alpha,\beta[$ and $g_i'(\nu) \neq 0, 0 \leq i \leq r, \nu \in [\alpha,\beta]$.

(A_1) Let us assume $g_r'(\nu) > 0, \nu \in [\alpha,\beta]$;

$$[b(\beta),a(\beta)] \subset]g_r(\alpha),g_r(\beta)[\text{ and } g_r(\mu) \in I_1(\mu)$$

PROPOSITION 3. – *There exists $\bar{\mu} \in]\mu,\beta[$ such that $\dfrac{m(B\cap[\mu,\bar{\mu}])}{\bar{\mu}-\mu} \leq 1/3$, for k big enough.*

Proof. – Denote by $\mu \leq \mu_1 \leq \mu_2 \leq \beta$ the parameter values which satisfy $g_r(\mu_1) = b(\beta)$, and $g_r(\mu_2) = a(\beta)$. We have $g_r(\mu_2) - g_r(\mu_1) = \int_{\mu_1}^{\mu_2} g'(\nu)d\nu \leq g_r'(\mu_1)(\mu_2-\mu_1)$ and $g_r(\mu_1) - g_r(\mu) = \int_{\mu}^{\mu_1} g'(\nu)d\nu \geq g_r'(\mu_1)(m_1-\mu)$.

Since $g_r(\mu_1) - g_r(\mu) \leq b(\mu_1) - (1-\delta)$ we have

$$\frac{m(B\cap[\mu,\mu_2])}{\mu_2-\mu} \leq \frac{\mu_1-\mu}{\mu_2-\mu_1} \leq \frac{b(\mu_1)-(1-\delta)}{a(\beta)-b(\beta)},$$

which can be taken smaller or equal to $1/3$ for k big. ■

(A_2) Assume $g_r'(\nu) < 0, \nu \in [\alpha,\beta]; [b(\beta),a(\beta)] \subset]g_r(\beta),g_r(\alpha)[$ and $g_r(\mu) \in I_1(\mu)$.

PROPOSITION 4. – *There exists $\bar{\mu} \in [\alpha,\mu]$ such that $\dfrac{m(B\cap[\bar{\mu},\mu])}{\mu-\bar{\mu}} \leq 1/3$, for k big enough.*

Proof. – The proof is similar to that of Proposition 3. ■

(A_3) Assume there is $\begin{pmatrix} i \\ j \end{pmatrix}$, $j \neq 0$, such that $D\begin{pmatrix} i \\ j \end{pmatrix}(\nu) \subset [g_r(\alpha), g_r(\beta)]$.

Given $\nu \in [\alpha, \beta]$ denote by $I_1\begin{pmatrix} i \\ j \end{pmatrix}(\nu)$ the interval contained in $D\begin{pmatrix} i \\ j \end{pmatrix}(\nu)$ such that $G\left(\nu, I_t\begin{pmatrix} i \\ j \end{pmatrix}(\nu)\right) = I_t(\nu)$, for $t = 1, 2$.

(A_{31}) Assume that $g_r(\mu) \in I_1\begin{pmatrix} i \\ j \end{pmatrix}(\mu); i = 2$ and $g_r'(\nu) > 0$, for $\nu \in [\alpha, \beta]$. Denote by $\mu < \mu_1 < \mu_2 < \beta$ the parameter values which satisfy $G(\mu_1, g_r(\mu_1)) = b(\beta)$ and $G(\mu_2, g_r(\mu_2)) = a(\beta)$, respectively. We have

PROPOSITION 5. – $\dfrac{m(B \cap [\mu, \mu_2])}{\mu_2 - \mu} \leq \dfrac{1}{3}$, *for k big enough.*

Proof. – The proof is similar to that of Proposition 3. ■

(A_{32}) Assume that $g_r(\mu) \in I_1\begin{pmatrix} i \\ j \end{pmatrix}(\mu); i = 2$ and that $g_r'(\nu) < 0$, for $\nu \in [\alpha, \beta]$. Let denote by $\alpha < \mu_2 < \mu_1 < \mu$ the parameter values which satisfy $G(\mu_1, g_r(\mu_1)) = b(\beta), G(\mu_2, g_r(\mu_2)) = a(\beta)$, respectively.

We have:

PROPOSITION 6. – $\dfrac{m(B \cap [\mu_2, \mu])}{\mu - \mu_2} \leq 1/3$ *for k big enough.*

Proof. – The proof is similar to that of Proposition 3. ■

(A_{33}) Assume that $g_r(\mu) \in I_2\begin{pmatrix} i \\ j \end{pmatrix}(\mu), i = 1$ and that $g_r'(\nu) > 0$, for $\nu \in [\alpha, \beta]$. Denote by $\mu < \mu_1 < \mu_2 < \beta$ the parameter values which satisfy $G(\mu_1, g_r(\mu_1)) = a(\beta)$ and $G(\mu_2, g_r(\mu_2)) = b(\beta)$, respectively. We have

PROPOSITION 7. – $\dfrac{m(B \cap [\mu, \mu_2])}{\mu_2 - \mu} \leq 1/3$, *for k big enough.*

Proof. – The proof is similar to that of Proposition 3. ■

(A_{34}) Assume that $g_r(\mu) \in I_2\begin{pmatrix} i \\ j \end{pmatrix}(\mu), i = 1$ and $g_r'(\nu) < 0$ for $\nu \in [\alpha, \beta]$. Let denote by $\alpha < \mu_2 < \mu_1 < \mu$ the parameter values which satisfy $G(\mu_2, g_r(\mu_2)) = b(\beta)$ and $G(\mu_1, g_r(\mu_1)) = a(\beta)$, respectively.

We have:

PROPOSITION 8. – $\dfrac{m(B \cap [\mu_2, \mu])}{\mu - \mu_2} \leq 1/3$, *for k big enough.*

Proof. – The proof is similar to that of Proposition 3. ■

(A_{35}) Assume that $g_r(\mu) \in I_2\begin{pmatrix} i \\ j \end{pmatrix}(\mu)$, $i = 2$ and that $g_r'(\nu) > 0$, for $\nu \in [\alpha, \beta]$ and, additionally, $\left[y\begin{pmatrix} 2 \\ j \end{pmatrix}(\beta), z\begin{pmatrix} 2 \\ j-1 \end{pmatrix}(\beta)\right] \subset]g_r(\alpha), g_r(\beta)[$.

Denote by $\mu < \mu_1 < \mu_2 < \beta$ the parameter values which satisfy $g_r(\mu_1) = y\binom{2}{j}(\beta)$, $g_r(\mu_1) = z\binom{2}{j-1}(\beta)$, respectively.

We have

PROPOSITION 9. – $\dfrac{m(B \cap [\mu, \mu_2])}{\mu_2 - \mu} \leq 1/3$, *for k big enough.*

Proof. – The proof is similar to that of Proposition 3. ■

(A_{36}) Assume that $i = 2; g_r(\mu) \in I_2\binom{i}{j}(\mu)$ and that $g_r'(\nu) < 0$, for $\nu \in [\alpha, \beta]$ and

$$\left[y\binom{2}{j}(\beta), z\binom{2}{j-1}(\beta)\right] \subset]\, g_r(\alpha), g_r(\beta)\, [.$$

Denote by $\alpha < \mu_2 < \mu_1 < \mu$ the parameter values which satisfy $g_r(\mu_2) = z\binom{2}{j-1}(\beta)$, $g_r(\mu_1) = y\binom{2}{j}(\beta)$, respectively.

We have

PROPOSITION 10. – $\dfrac{m(B \cap [\mu_2, \mu])}{\mu - \mu_2} \leq 1/3$, *for k big enough.*

Proof. – The proof is similar to that of Proposition 3. ■

(A_{37}) Assume that $i = 1; g_r(\mu) \in I_1\binom{i}{j}(\mu); g_r'(\nu) > 0$, for $\nu \in [\alpha, \beta]$ and $\left[z\binom{1}{j}(\beta), y\binom{1}{j+1}(\beta)\right] \subset]\, g_r(\alpha), g_r(\beta)\, [$.

Denote by $\mu < \mu_1 < \mu_2 < \beta$ the parameter values which satisfy $g_r(\mu_1) = z\binom{1}{j}(\beta)$ and $g_r(\mu_2) = y\binom{1}{j+1}(\beta)$, respectively.

We have

PROPOSITION 11. – $\dfrac{m(B \cap [\mu, \mu_2])}{\mu_2 - \mu} \leq 1/3$, *for k big enough.* ■

(A_{38}) Assume that $i = 1; g_r(\mu) \in I_1\binom{i}{j}(\mu); g_r'(\nu) < 0$ for $\nu \in [\alpha, \beta]$ and $\left[z\binom{1}{j}(\beta), y\binom{1}{j+1}(\beta)\right] \subset]g_r(\alpha), g_r(\beta)[$.

Let denote by $\alpha < \mu_2 < \mu_1 < \mu$ the parameter values which satisfy $g_r(\mu_2) = y\binom{1}{j+1}\beta)$ and $g_r(\mu_1) = z\binom{1}{j}(\beta)$, respectively.

We have

PROPOSITION 12. – $\dfrac{m(B \cap [\mu_2, \mu])}{\mu - \mu_2} \leq 1/3$ *for k big enough.* ■

(A_4) Assume that $\left[z\binom{1}{0}(\beta), y\binom{1}{1}(\beta)\right] \subset]\, g_r(\alpha), g_r(\beta)\, [$ and $g_r(\mu) \in D\binom{1}{0}(\mu)$.

(A_{41}) Assume that $g'_r(\nu) > 0$ for $\nu \in [\alpha, \beta]$.

Let denote by $\mu < \mu_1 < \mu_2 < \beta$ the parameter values which satisfy $g_r(\mu_1) = z\begin{pmatrix}1\\0\end{pmatrix}(\beta)$ and $g_r(\mu_2) = y\begin{pmatrix}1\\1\end{pmatrix}(\beta)$, respectively.

We have

PROPOSITION 13. – $\dfrac{m(B \cap [\mu, \mu_2])}{\mu_2 - \mu} \leq 1/3$, *for k big enough.*

Proof. – The proof is similar to that of Proposition 3. ∎

(A_{42}) Assume that $g'_r(\nu) < 0$, for $\nu \in [\alpha, \beta]$.

Denote by $\alpha < \mu_2 < \mu_1 < \mu$ the parameter values that satisfy $g_r(\mu_2) = y\begin{pmatrix}1\\1\end{pmatrix}(\beta)$ and $g_r(\mu_2) = z\begin{pmatrix}1\\0\end{pmatrix}(\beta)$, respectively.

We have

PROPOSITION 14. – $\dfrac{m(B \cap [\mu_2, \mu])}{\mu - \mu_2} \leq 1/3$, *for k big enough.*

Proof. – The proof is similar to that of Proposition 3. ∎

(3.4). Consider a parameter value $\mu \in J_0$ which satisfies: there exists $r_0 \in \mathbf{N}$ that

$$G_\mu^{r_0}(\xi^k \mu) \in \left([1 - \delta, 1] \setminus \bigcup_{j=0}^{\infty} \bigcup_{i=1,2} D\begin{pmatrix}i\\j\end{pmatrix}(\mu)\right)$$

In this case we have $\mu \in \Gamma_0$ or $G_\mu^{r_0}(\xi^k \mu) = b(\mu)$ or $G_\mu^{r_0}(\xi^k \mu) = a(\mu)$. It is clear that assertion $(*)$ is true in any of the cases above. Let

$$T = \left\{\mu \in J_0 / g_r(\mu) \in \cup_{j=0}^{\infty} \cup_{i=1,2} D\begin{pmatrix}i\\j\end{pmatrix}(\mu), \text{ for any } r \geq 0\right\}$$

For a given $\mu \in T$ we have three possibilities for the itinerary Γ_μ :

(1) Γ_μ is a periodic itinerary;

(2) Γ_μ is an itinerary which is eventually periodic and

(3) Γ_μ do not satisfies (1) and (2) above.

Assume Γ_μ is periodic. In this case we know (*see* (2.11)) that there is an interval $[\alpha, \beta] \subset T$ such that $\Gamma_\nu = \Gamma_\mu$, for any $\nu \in [\alpha, \beta]; \mu \in [\alpha, \beta]$ and $B \cap [\alpha, \beta]$ is a finite number of points. So for these parameter values assertion $(*)$ is true.

Assume Γ_μ is eventually periodic. Under these circumstances it is easy to see that we can find an interval $[\alpha, \beta] \subset J_0$ and an index $r \in \mathbf{N}$ such that

(i) $\mu \in]\alpha, \beta[$;

(ii) $g'_r(\nu) \neq 0, 0 \leq i \leq r$ for any $\nu \in [\alpha, \beta]$ and

(iii) $g_r/[\alpha, \beta]$ satisfies the conditions of one of the Propositions specified in (3.3) above.

It is clear that we can find a sequence of intervals $[\alpha_n, \beta_n] \subset]\alpha_{n-1}, \beta_{n-1}[$ and a sequence of indexes $r_n > r_{n-1}$ such that (i), (ii) and (iii) hold for any of the given $n \in \mathbf{N}$.

Therefore we can conclude the following

LEMMA 12. – *There exists a sequence $\mu_n \to \mu$ such that*

$$\frac{m(B \cap [\mu, \mu_n])}{\mu_n - \mu} \leq \frac{1}{3} \text{ or } \frac{m(B \cap [\mu_n, \mu])}{\mu - \mu_n} \leq \frac{1}{3},$$

for k big enough. ■

In particular, for any of these parameter values assertion $(*)$ is true.

Assume Γ_μ satisfies (3) above. In this case we can find a sequence $\mu_n \to \mu$ such that Γ_{μ_n} satisfies (1) or (2) above. For these parameter values assertion $(*)$ holds, therefore we conclude that it $(*)$ is true for μ.

This completes the proof of Theorem 2. ■

(3.5) Comments on the general case

Let us now consider the general case for contracting singular cycles. In his paper San Martin [8] introduces a nice idea with which to work in this case. Let us consider the periodic orbits $\sigma_1(X), \cdots, \sigma_r(X)$ that belong to the singular cycle Γ. Let $q_i(X) \in \sigma_i(X)$ be a point and $Q_i \subset M$ be a transversal section associated to $q_i(X), i = 1, \cdots, n$. Assume this cross section is parametrized by $\{(x_i, y_i); |x_i|, |y_i| \leq 1\}$ satisfying $W^s_{\sigma_i} \supseteq \{(x_i, 0); |x_i| \leq 1\}$ and $W^u_{\sigma_i} \supseteq \{(0, y_i); |y_i| \leq 1\}$.

Let $p_i^j = p_i^j(X)$ be the first intersection between $\gamma_i^j(X)$ and $Q_{i+1}, i = 1, 2, \cdots, n-1; j = 1, 2$. We have $p_i^j = (x_{i+1}^j(X), 0)$ and assume $x_{i+1}^j > 0$. Denote by $q_i^j = q_i^j(X) = (0, y_i^j(X))$ the first intersection of the backward orbit of p_i^j with Q_i.

We will assume $y_i^j(X) > 0, i = 1, \cdots, n-1; j = 1, 2$.

Since p_i^j and q_i^j are in the same orbit we can find horizontal strips $R_j^i(X) \ni q_i^j$ and neighborhoods $U_i^j \ni p_i^j$, so that the positive orbits of points at R_j^i intersect U_i^j. This procedure define Poincaré maps $P_i^j : R_j^i \to U_i^j; i = 1, 2, \cdots, n-1; j = 1, 2$.

On the other hand, the positive orbit of points at a horizontal strip $R_i(X)$, containing $W^s(\sigma_i(X)) \cap Q_i$, turns around the closed orbit $\sigma_i(X)$ and then returns to Q_i. This define a return map $P_i : R_i \to Q_i, i = 1, \cdots, n$.

Denote by $q_n^j = q_n^j(X)$ the last intersection of the orbit $\gamma_n^j(X)$ with $Q_n, j = 1, 2$. Since $w(q_n^j) = \sigma_0(X)$ and $\alpha(q_n^j) = \sigma_n(X)$, there are horizontal strips $R_j^n(X) \ni q_r^j$ such that the positive orbit of points at R_j^n pass first near $\sigma_0(X)$ and afterwards intersect Q_1. This define maps $P_j^n : R_j^n \to Q_1, j = 1, 2$.

Therefore the first return map F_X is defined on $\cup_{i=1}^n (R_i \cup R_i^1 \cup R_i^2)$ with values on $\cup_{i=1}^n Q_i$ and its restriction to R_i coincides with the Poincaré map associated to $\sigma_i(X)$.

The same construction applies to vector field Y, near enough to X in the C^r-topology, $r \geq 3$.

From now and on the proof follows as in chapters II and III (3.1)-(3.4), that is: Give an explicit formula to the map F_Y; show that there is an invariant stable foliation for F_Y; change coordinates in the neighborhhod $\mathcal{U}$ and prove the result for the one-dimensional map associated to F_Y.

4. Appendix

In this paragraph we prove Lemma 13. Let $L(\mu;y)$ denote the map given by

$$L(\mu;y)=\begin{cases}\xi y, & 0\le y\le \xi^{-1}\\ \mu-J(\mu;y)(y-(1-\delta))^{\alpha}, & 1-\delta\le y\le b(\mu)\\ \mu-K(\mu;y)(1-y)^{\alpha}, & a(\mu)\le y\le 1,\end{cases}$$

where $a(\mu)=1-\delta^2(\mu), b(\mu)=1-\delta+\delta^1(\mu); \delta^i(\mu)=A^i\mu^{1/\alpha}, A^i>0$, for $i=1,2; J$ and K are C^2-maps in the μ-variable, C^3 in the y-variable $y\neq 1-\delta, 1$ and whose derivatives $\frac{\partial J}{\partial y},\frac{\partial J}{\partial \mu},\frac{\partial^2 J}{\partial\mu\partial y},\frac{\partial^2 J}{\partial y^2},\frac{\partial^2 J}{\partial \mu^2},\frac{\partial K}{\partial \mu},\frac{\partial K}{\partial y},\frac{\partial^2 K}{\partial\mu\partial y},\frac{\partial^2 K}{\partial y^2},\frac{\partial^2 K}{\partial \mu^2}$ are small numbers, with μ small. Moreover $J(\mu;y)>0$ and $K(\mu;y)>0$, any $(\mu;y), 0\le\mu\le\mu_0=\xi^{-n_0}$.

Define $L_1(\mu)=L(\mu;1)=\mu$ and $L_{n+1}(\mu)=L(\mu;L_n(\mu)), n\ge 1$.

We have $L_{i+1}(\mu)=\xi L_i(\mu), 1\le i\le n_0$ and $L_{n+1}(\mu)=\xi^{n_0}\mu$. Hence these maps satisfy:
(a) $L_i'(\mu)>0$ and $L_i''(\mu)=0, \mu\in[0,\mu_0], 0\le i\le n_0+1$ and
(b) $L_i'(\mu)\le L_i'(0), 0\le\mu\le\mu_0$.

For any $k\ge n_0+2$, let $I_k=I_k'\cup I_k^2\cup\cdots\cup I_k^{n_k}$ be the domain of definition of the map L_k.

Let $I_k^j=[\nu_0,\nu_1]$ be a component of the domain I_k that satisfies $L_i'(\mu)\neq 0$, for $0\le i\le k-1$ and $\mu\in I_k^j$.

LEMMA 13. – *For the map L_k we have one and only one of the following:*
(i) *there exists only one $\bar\nu\in I_k^j$ such that $L_k'(\bar\nu)=0$ and $L_k''(\bar\nu)<0$ or*
(ii) *$L_k'(\mu)\neq 0$ and $L_k''(\mu)=0$ for $\mu\in I_k^j$, or*
(iii) *$L_k'(\mu)\neq 0$ and $L_k''(\mu)<0$ for $\mu\in I_k^j$.*

Proof. – For $L_{k-1}(\mu)\le\xi^{-1}, \mu\in I_k^j$, we have $L_k(\mu)=\xi L_{k-1}(\mu)$ and the result follows by the induction hypothesis. Otherwise let us consider $A=\bigcup_{\mu\in[0,\mu_0]}[\{\mu\}\times I_1(\mu)]$ and $B=\bigcup_{\mu\in[0,\mu_0]}(\{\mu\}\times I_2(\mu))$.

We must have $A\cap(Graph(L_k/I_k^j))\neq\emptyset$ or $B\cap(Graph(L_k/I_k^j))\neq\emptyset$ (only one of these intersections in non-empty).

I) Assume $L_{k-1}'(\mu)<0$ for $\mu\in I_k^j$.
(i) We have $L_{k-1}(\nu_0)=1$ and $L_{k-1}(\nu_1)=a(\nu_1)$.
Under these conditions $L_k(\mu)=L(\mu;L_{k-1}(\mu))=\mu-K(\mu;L_{k-1}(\mu))(1-L_{k-1}(\mu))^{\alpha}$.
So

$$\begin{aligned}L_k'(\mu)=&1-\frac{\partial K}{\partial\mu}(\mu;L_{k-1}(\mu))(1-L_{k-1}(\mu))^{\alpha}\\&+\Big[-\frac{\partial K}{\partial y}(\mu;L_{k-1}(\mu))(1-L_{k-1}(\mu))^{\alpha}\\&+\alpha K(\mu;L_{k-1}(\mu))(1-L_{k-1}(\mu))^{\alpha-1}\Big]\cdot L_{k-1}'(\mu)\end{aligned}$$

and

$$\begin{aligned}L''_k(\mu) =&(1 - L_{k-1}(\mu))^{\alpha-2}[-\alpha(\alpha-1)K(\cdot,\cdot)(L'_{k-1}(\mu))^2\\ &- K_{\mu\mu}(\cdot,\cdot)(1-L_{k-1}(\cdot))^2 + 2\alpha K_\mu(\cdot,\cdot)L'_{k-1}(\cdot)(1-L_{k-1}(\cdot))\\ &- K_{yy}(\cdot,\cdot)(1-L_{k-1}(\cdot))^2(L'_{k-1}(\cdot))^2\\ &+ \alpha K(\cdot,\cdot)(1-L_{k-1}(\cdot)) \cdot L"_{k-1}(\cdot)\\ &- 2K_{\mu y}(\cdot,\cdot)(1-L_{k-1}(\cdot))^2 \cdot L'_{k-1}(\cdot)\\ &+ 2\alpha K_y(\cdot,\cdot)(1-L_{k-1}(\cdot))(L'_{k-1}(\cdot))^2\\ &- K_y(\cdot,\cdot)(1-L_{k-1}(\cdot))^2 \cdot L"_{k-1}(\cdot)].\end{aligned}$$

Since

$$\begin{aligned}L_{k-1}(\mu) &= \xi L_{k-2}(\mu) = \cdots = \xi^{j-1}L_{k-j}(\mu)\\ &= \xi^{j-1}[\mu - K(\mu; L_{k-j-1}(\cdot))(1 - L_{k-j-1}(\mu))^\alpha]\\ &\quad \text{if } a(\mu) \le L_{k-j-1}(\mu) \le 1\\ &= \xi^{j-1}[\mu - J(\mu; L_{j-k-1}(\cdot))(L_{k-j-1}(\mu) - 1 - \delta))^\alpha]\\ &\quad \text{if } 1-\delta \le L_{k-j-1}(\mu) \le b(\mu).\end{aligned}$$

Therefore we have

$$\begin{aligned}L'_{k-1}(\mu) =&\xi^{j-1}[1 - J_\mu(\cdot,\cdot)(L_{k-j-1}(\mu) - (1-\delta))^\alpha\\ &- J_y(\cdot,\cdot)(L_{k-j-1}(\mu) - (1-\delta))^\alpha \cdot L'_{k-j-1}(\mu)\\ &- \alpha J(,)(L_{k-j-1}(\mu) - (1-\delta))^{\alpha-1}L'_{k-j-1}(\cdot)]\end{aligned}$$

or

$$\begin{aligned}L'_{k-1}(\mu) =&\xi^{j-1}[1 - K_\mu(,)(1 - L_{k-j-1}(\cdot))^\alpha\\ &- K_y(,)(1 - L_{k-j-1}())^\alpha L'_{k-j-1}(\cdot)\\ &+ \alpha K(,)(1 - L_{k-j-1}())^{\alpha-1}L'_{k-j-1}(\cdot)],\end{aligned}$$

depending on $1-\delta \le L_{k-j-1}(\mu) \le b(\mu)$ or $a(\mu) \le L_{k-j-1}(\mu) \le 1$, respectively. Since $L'_{k-1}(\mu) < 0$ we have

$$L'_{k-j-1}(\mu) > \frac{1 - J_\mu(,)(L_{k-j-1}(\mu) - (1-\delta))^\alpha}{[L_{k-j-1}(\mu) - (1-\delta)]^{\alpha-1}[\alpha J(,) + J_y(,)(L_{k-j-1}(\mu) - (1-\delta))]}$$

or

$$-L'_{k-j-1}(\mu) > \frac{1 - K_\mu(,)(1 - L_{k-j-1}(\mu))^\alpha}{(1 - L_{k-j-1}(\mu))^\alpha[\alpha K(,) - K_y(,)(1 - L_{k-j-1}(\mu))]}$$

depending on $1-\delta \le L_{k-j-1}(\mu) \le b(\mu)$ or $a(\mu) \le L_{k-j-1}(\mu) \le 1$, respectively. In any case we get $|L'_{k-j-1}(\mu)| \gg 20$, for $\mu \in I^j_k$.

Now consider the map $\rho(\mu)$ given by

$$\begin{aligned}\rho(\mu) =&J_\mu(\mu; L_{k-j-1}(\mu))(L_{k-j-1}(\mu) - (1-\delta))^\alpha\\ &+ [J_y(\mu; L_{k-j-1}(\mu))(L_{k-j-1}(\mu) - (1-\delta))^\alpha + \alpha J(\cdot,\cdot)(L_{k-j-1}(\mu)\\ &- (1-\delta))^{\alpha-1}] \times L'_{k-j-1}(\mu)\end{aligned}$$

or

$$\begin{aligned}\rho(\mu) =&K_\mu(\mu;L_{k-j-1}(\mu))(1-L_{k-j-1}(\mu))^\alpha\\&+[K_y(\mu;L_{k-j-1}(\mu))(1-L_{k-j-1}(\mu))^\alpha\\&-\alpha K(\mu;L_{k-j-1}(\mu))(1-L_{k-j-1}(\mu))^{\alpha-1}]\times L'_{k-j-1}(\mu),\end{aligned}$$

depending on whether $1-\delta \leq L_{k-j-1}(\mu) \leq b(\mu)$ or $a(\mu) \leq L_{k-j-1}(\mu) \leq 1$, respectively. In the first case an easy computation, using the facts that $L'_{k-j-1}(\mu) \gg 20$; $L''_{k-j-1}(\mu) < 0$ and $L_{k-j-1}(\mu) - (1-\delta) > 0$ gives $\rho'(\mu) > 0$, for $\nu_0 \leq \mu \leq \nu_1$.

Similarly in the second case we get $\rho'(\mu) > 0$.

Since $L'_{k-1}(\mu) = \xi^{j-1}[1-\rho(\mu)]$, we have:

$$\begin{aligned}L''_k(\mu) =&[1-L_{k-1}(\mu)]^{\alpha-2}[-\alpha(\alpha-1)(K(\mu;L_{k-1}(\mu))[\xi^{j-1}(1-\rho(\mu))]^2\\&-K_{\mu\mu}(\mu;L_{k-1}(\mu))(1-L_{k-1}(\mu))^2\\&+2\alpha K_\mu(\mu;L_{k-1}(\mu))\xi^{j-1}(1-\rho(\mu))(1-L_{k-1}(\mu))\\&-K_{yy}(\mu;L_{k-1}(\mu))(1-L_{k-1}(\mu))^2-(\xi^{j-1}(1-\rho(\mu)))^2\\&-\alpha K(,)(1-L_{k-1}(\mu))\xi^{j-1}\rho'(\mu)\\&-2K_{\mu y}(\cdot,\cdot)(1-L_{k-1}(\mu))^2\xi^{j-1}(1-\rho(\mu))\\&+2\alpha K_y(\cdot,\cdot)(1-L_{k-1}(\mu))(\xi^{j-1}(1-\rho(\mu)))^2\\&+K_y(\cdot,\cdot)(1-L_{k-1}(\mu))^2\xi^{j-1}\rho'(\mu)];\end{aligned}$$

which is clearly a negative number.

We note that $L'_k(\nu_0) = 1$. Let us compute $L'_k(\nu_1)$.

We have

$$L'_k(\nu_1) = 1 + \nu_1^{1-1/\alpha}\left[\alpha K^{1/\alpha}L'_{k-1}(\nu_1) - \frac{K_y}{K}L'_{k-1}(\nu_1)\nu_1^{1/\alpha} - \frac{K_\mu}{K}\nu_1^{1/\alpha}\right].$$

Since $L'_{k-1}(\nu_1) < 0$ and $L_{k-1}(\nu_1) = a(\nu_1)$, we get $L'_k(\nu_1) < 0$.

Since $L''_k(\mu) < 0$, we find only one $\bar{\nu} \in [\nu_0, \nu_1]$ such that $L'_k(\bar{\nu}) = 0$.

(ii) Assume $L_{k-1}(\nu_0) < 1$ and $L_{k-1}(\nu_1) = a(\nu_1)$

Similarly, as in (i) of above, we obtain $L''_k(\mu) < 0$ for $\mu \in I^j_k$. If $L'_k(\nu_1) \geq 0$ then there exists only one $\bar{\nu} \in I^j_k$ such that $L'_k(\bar{\nu}) = 0$. If $L'_k(\nu_1) < 0$, we have $L'_k(\mu) < 0$ for $\mu \in I^j_k$.

(iii) Assume $L_{k-1}(\nu_0) = 1$ and $L_{k-1}(\nu_1) > a(\nu_1)$.

As before we get $L''_k(\mu) < 0$ for $\mu \in I^j_k$. If $L'_k(\nu_1) > 0$ then $L'_k(\mu) > 0$ for $\mu \in I^j_k$. If $L'_k(\nu_1) \leq 0$ then there is only one $\bar{\nu} \in I^j_k$ such that $L'_k(\bar{\nu}) = 0$.

(iv) Assume $L_{k-1}(\nu_0) < 1$ and $L_{k-1}(\nu_1) > a(\nu_1)$.

As before we prove that $L'_k(\mu)$ is a decreasing map and we get the result.

(v) Assume $L_{k-1}(\nu_0) = b(\nu_0)$ and $L_{k-1}(\nu_1) = 1-\delta$.

We proceed as in (i) to prove $L''_k(\mu) < 0$ and hence we obtain $L'_k(\mu) \geq L'_k(\nu_1) = 1$, any $\mu \in I^j_k$.

(vi) Assume $L_{k-1}(\nu_0) < b(\nu_0)$ and $L_{k-1}(\nu_1) = 1-\delta$.

In a similar way as in (i) we get $L''_k(\mu) < 0$ and then $L'_k(\mu) \geq L'_k(\nu_1) = 1$, any $\mu \in I^j_k$.

(vii) Assume $L_{k-1}(\nu_0) < b(\nu_0)$ and $L_{k-1}(\nu_1) > 1-\delta$.

As before we get $L''_k(\mu) < 0$ and $L'_k(\mu) \geq 1$, any $\mu \in I^j_k$.

(viii) Assume $L_{k-1}(\nu_0) = b(\nu_0)$ and $L_{k-1}(\nu_1) > 1 - \delta$.

As before we get the result.

II) Assume $L'_{k-1}(\mu) > 0$ (non-constant) for $\mu \in I_k^j$.

As in Case (I) we have eight possibilities. We proceed as in (I)(i) to get the result in all of the cases.

III) The case $L'_{k-1}(\mu) =$ constant, *i.e.*, $\nu_0 = 0 \in I_k^j$ satisfies $L'_{k-1}(\mu) > 0$ and $L''_k(\mu) = 0$, for $\mu \in I_k^j$. ■

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