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DEGENERATIONS FOR REPRESENTATIONS OF TAME QUIVERS

BY KLAUS BONGARTZ

ABSTRACT. – Let A be the path algebra of a quiver with underlying diagram of type $\tilde{A}_n, \tilde{D}_n, \tilde{E}_6, \tilde{E}_7$ or $\tilde{E}_8$. We show that a finite dimensional A-module M degenerates to another A-module N if and only if $\dim_k \mathrm{Hom}_A(M, U) \leq \dim_k \mathrm{Hom}_A(N, U)$ holds for all finite dimensional A-modules U.

1. Introduction

We fix an algebraically closed field k of arbitrary characteristic. If A is a finite dimensional associative k-algebra with basis $a_1 = 1, \ldots, a_\alpha$, we have the corresponding structure constants defined by

$$a_i a_j = \sum a_{ijk} a_k.$$

The affine variety Mod_A^d of d-dimensional unital left A-modules consists in the α-tuples

$$m = (m_1, \ldots, m_\alpha)$$

of $d \times d$-matrices with coefficients in k such that m_1 is the identity and such that

$$m_i m_j = \sum a_{ijk} m_k$$

holds for all indices i and j. The general linear group $\mathrm{Gl}_d(k)$ acts on Mod_A^d by conjugation, and the orbits correspond to the isomorphism classes of d-dimensional modules. We denote by $O(m)$ the orbit of a point m in Mod_A^d and by M the A-module on k^d given by m. By abuse of notation we also write M for the isomorphism class of M. Thus N is a degeneration of M if $O(n)$ belongs to the closure of $O(m)$, and we denote this fact by $M \leq_{\deg} N$ and *not* by $N \leq_{\deg} M$ as one might expect. It is not clear how to characterize

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the partial order $\leq_{\text{deg}}$ on the set of isomorphism classes of d-dimensional modules in terms of representation theory.

However, there are two other partial orders $\leq_{\text{ext}}$ and $\leq$ on the isomorphism classes which have been introduced by S. Abeasis, A. del Fra and C. Riedtmann. They are defined in terms of representation theory as follows (*see* [1], [2], [16], [6]):

• $M \leq_{\text{ext}} N :\Leftrightarrow$ there are modules M_i, U_i, V_i and exact sequences $0 \to U_i \to M_i \to V_i \to 0$ such that $M = M_1, M_{i+1} = U_i \oplus V_i$ and $N = M_{n+1}$ are true for some natural number n .

• $M \leq N :\Leftrightarrow [M, X] \leq [N, X]$ holds for all modules X.

Here and later on we abbreviate $\dim_k \text{Hom}_A(M, X)$ by $[M, X]$ and $\dim_k \text{Ext}_A^i(M, X)$ by $[M, X]^i$.Note that $\leq$ is a partial order on the isomorphism classes by a result of M. Auslander (*see* [3], [4]). Recall that the dimension vector $\underline{\dim} M$ of a module M counts the composition factors with multiplicities. If N and M have the same dimension vectors then M. Auslander and I. Reiten have shown in [3] for all non-injective indecomposable U the remarkable formula $[N, U] - [M, U] = [TrDU, N] - [TrDU, M]$. It follows in particular that $M \leq N$ is also equivalent to the inequalities $[X, M] \leq [X, N]$ for all modules X.

It is easy to see that

$$M \leq_{\text{ext}} N \quad \Rightarrow \quad M \leq_{\text{deg}} N \quad \Rightarrow \quad M \leq N$$

holds for all modules (*see* [6]). Unfortunately, the reverse implications are not true in general, and it is interesting to find out when they are. Our main result in this note asserts that $M \leq N$ implies $M \leq_{\text{deg}} N$ for all modules over path algebras of quivers whose underlying graph is an extended Dynkin diagram of type $\tilde{A}_n, \tilde{D}_n, \tilde{E}_6, \tilde{E}_7$ or $\tilde{E}_8$. The representation theory of these algebras is reasonably well understood and all indecomposables are known ([9], [15], [8]). Therefore these quivers are called tame. As a special case of our main result one also has a complete description of the degeneration behaviour of certain subspace configurations inside a given vector space. For instance the variety of quadruples of subspaces with prescribed dimensions is in a natural way the geometric quotient of certain representations of a $\tilde{D}_4$-quiver.

In Section 2 we review some details of the representation theory of tame quivers, e.g. that the indecomposables are divided into three types, namely into preprojective, preinjective and regular modules. Then we present two types of extensions between indecomposables in different connected components of the Auslander-Reiten quiver.

Section 3 contains the proof that $\leq$ and $\leq_{\text{ext}}$ are equivalent for representations of an oriented cycle. This has been shown by G.Kempken in her thesis [13], but we include a more conceptual proof for the convenience of the reader. The representations of an oriented cycle occur for regular modules over a tame quiver, and it is possible that $\leq$ and $\leq_{\text{ext}}$ are equivalent for all representations of tame quivers as it is shown for the double arrow in [6], Section 5 by a rather technical direct analysis. The main obstacle to carry through such an analysis for the other tame quivers seems to be the combinatorial jungle of extensions between preprojective modules. Therefore we introduce a technique in Section 4 which allows to normalize the preprojective parts of the modules involved. Thereafter, the proof is based on an inductive argument whose most difficult part is the beginning.

In Chapter 5 we extend the main result to all tame concealed algebras. We also show how one can derive geometrically the equivalence of $\leq$ and $\leq_{\text{ext}}$ from the equivalence of

$\leq$ and $\leq_{\text{deg}}$ in certain cases. In particular, this argument proves again the equivalence of $\leq$ and $\leq_{\text{ext}}$ for preprojective modules. This has been shown before in [6] by a somewhat mysterious direct construction. At the end of the article we indicate how to decide by finitely many rational operations whether a given representation over a tame quiver degenerates to another representation.

We conclude this introduction by a general remark on this problem which explains the geometric relation between $\leq$ and $\leq_{\text{deg}}$.

PROPOSITION 1. – *Let M be an A-module of dimension d. Then we have:*

a) For any A-module U the set $C(U) := \{n \in \mathrm{Mod}_A^d \mid [M, U] \leq [N, U]\}$ is closed in Mod_A^d. In particular, $C_M := \{n \in \mathrm{Mod}_A^d \mid [M, U] \leq [N, U]$ for all $U\}$ is closed, and it is a finite intersection of some $C(U_i)$ with indecomposable U_i's.

b) The Zariski-closure of the orbit of m is always an irreducible component of $C(M)$, whence a fortiori *of C_M. Thus C_M is irreducible if and only if the conditions $M \leq N$ and $M \leq_{\text{deg}} N$ are equivalent for all modules N. Furthermore, if m is given explicitely by some matrices , one can write down a finite number of equations defining $C(M)$.*

Proof. – Most of this is well-known ([6], [10]). Suppose U belongs to Mod_A^r. The matrix-equations $fn_i = u_i f$ defining an element of $\mathrm{Hom}(N, U)$ are equivalent to a system of homogeneous linear equations with the entries of f as the unknowns. The matrix $B(n, u)$ of coefficients depends regularly on n and u. If the dimension of $\mathrm{Hom}(M, U)$ equals t, the wanted set $C(U)$ is defined by the vanishing of all subminors of $B(n, u)$ of size $rd - t + 1$. This shows the first part of a) and the last part of b). The remaining assertions of a) follow from Hilberts Basissatz.

By the above argument we see that $\mathcal{S} = \{n \in \mathrm{Mod}_A^d \mid [M, M] = [N, M]\}$ is open in $C(M)$. We look at the vector bundle $\mathcal{V}$ over $\mathcal{S}$ consisting of pairs (n, f) with $n \in \mathcal{S}$, $f \in k^{d \times d}$ such that $fn_i = m_i f$. The open projection maps the open non-empty subset $\mathcal{V}_{detf}$ onto the orbit of m. Therefore, this orbit is open in $\mathcal{S}$, whence in $C(m)$. It is also irreducible because Gl_d is so, and part b) follows.

Thus the geometric meaning of our main result is that for the path-algebra of a tame quiver the varieties C_M are always irreducible. It would be nice to have a description of the finitely many "test modules" in part a) of the proposition. For tame quivers we give this in 5.4.

Let us end with an interesting consequence of the proposition. Namely, for certain fields k, G. Hermann — building on earlier work of L. Kronecker, D. Hilbert, E. Noether and especially K. Hentzelt — has described a finite algorithm using only rational operations which determines for any ideal I with a given finite set of generators in a polynomial ring over k the minimal prime ideals above I by producing finite sets of generators (*see* [14]). For these fields we obtain from the proposition a finite algorithm which decides whether M degenerates to N or not. For instance,the method applies to the algebraic closures of the prime fields provided one interpretes e.g. $\overline{\mathbf{Q}}$ as the union of the fields $\mathbf{Q}_i$ obtained from the rationals by adjoining all roots of polynomials of degree i with integer coefficients of absolute value at most i. Unfortunately this algorithm is quite impractical because the number of involved operations grows exponentially with the dimension of the modules.

2. Representations of tame quivers

2.1. Some known facts

All facts and definitions mentioned in the following section can be found in Ringels book [17] or in the article [8]. The term module always means a finite-dimensional left module. We do not distinguish between modules over the path algebra of a quiver and representations of the quiver because both points of view are algebraically and geometrically equivalent by [5]. The point set of a quiver is denoted by Q_0. Two modules are called disjoint if they have no common direct summand.

In this article, we call a module preprojective provided all its indecomposable direct summands belong to preprojective components of the Auslander-Reiten quiver Γ . There is a partial order $\preceq$ on the (isomorphism classes of) indecomposable preprojectives with $U \preceq V$ if there is a path in Γ leading from U to V. There is the dual notion of a preinjective module.

For tame quivers, an indecomposable representation is either preprojective or else preinjective or else regular. Accordingly, any module M can be decomposed in an essentially unique way into its preprojective part M_P, its regular part M_R and its preinjective part M_I. A very important fact is that there are no non-zero maps from preinjectives to regular or preprojective modules and from regular modules to preprojectives. The full subcategory $\mathcal{R}$ of all regular modules is an abelian subcategory which breaks up into a direct sum of subcategories $\mathcal{T}(p)$, $p \in \mathbf{P}^1(k)$. Each of these categories is equivalent to the category $\mathcal{N}(n_p)$ of nilpotent representations of an oriented cycle with n_p points. In fact for all but at most three values of p we have $n_p = 1$, *i.e.* $\mathcal{T}(p)$ has only one simple object and these categories are called homogeneous. In the other cases the simples are conjugate under DTr. Given a simple regular module E and a natural number t we denote by $P(E,t)$ resp. $I(E,t)$ the regular indecomposable of regular length t with E as its top resp. socle.

The tame quivers are characterized by the fact that the associated quadratic form

$$q(x) = \sum_{i \varepsilon Q_0} x_i^2 - \sum_{i — j} x_i x_j$$

is positive semi-definite on $\mathbf{R}^{Q_0}$. Its radical admits a generator h in $\mathbf{N}^{Q_0}$ with strictly positive entries one of which is 1. The sum of the dimension vectors of all simple regular modules in a category $\mathcal{T}(p)$ as above equals h. The global dimension of a quiver algebra is at most one so that the map $(M,N) \mapsto [M,N] - [M,N]^1$ induces a bilinear form on the Grothendieck-group $K_0(A)$. Identifying $K_0(A)$ and $\mathbf{Z}^{Q_0}$ by $\underline{\dim}$, its associated quadratic form is q.

There is a linear form ∂ called defect on $K_0(A)$ whose values on the dimension vector of an indecomposable are strictly negative for a preprojective, zero for a regular and strictly positive for a preinjective. Given any module E with dimension vector h, one has $\partial(\underline{\dim} U) = [U,E]^1 - [U,E]$. Another definition of the defect uses the Coxeter-transformation c. This is the unique endomorphism of $\mathbf{R}^{Q_0}$ that sends $\underline{\dim} U$ to $\underline{\dim} DTrU$ for each non-projective indecomposable U. In particular, h is fixed by c which induces on the quotient space an automorphism of finite order $p(Q)$. In contrast to c itself this

Coxeter-number $p(Q)$ depends only on the underlying non-oriented graph of Q and there is for any indecomposable U the formula

$$c^{p(Q)}(\underline{\dim} U) = \underline{\dim} U + \partial(\underline{\dim} U)h.$$

2.2. Two types of short exact sequences

Throughout this section we are dealing with modules over a tame quiver.

LEMMA 1. – *Let P be a simple projective of defect -1, and let $E_1, E_2, \ldots, E_n$ be simple regular modules in different $\mathcal{T}(p)'s$ such that $[P, E_i] \neq 0$. Then there is for any natural number t a short exact sequence*

$$0 \longrightarrow P \longrightarrow M(t) = \bigoplus P(E_i, t) \longrightarrow I(t) \longrightarrow 0,$$

where $I(t)$ is indecomposable of defect 1.

Proof. – For any index i, there is an injection $f_i : P \longrightarrow P(E_i, t)$ that does not factor through the regular radical of $P(E_i, t)$. Define $f : P \longrightarrow M(t)$ by these homomorphisms and look at the cokernel C. Suppose that C is not preinjective. Being a homomorphic image of $M(t)$, C contains some $P(E_i, t')$ with $0 \neq t' \leq t$ as a direct summand. Let π be the projection from $M(t)$ onto C and let π_i be the projection from $P(E_i, t)$ onto $P(E_i, t')$. From $0 = \pi f$ we conclude $0 = \pi_i f_i$ because of $[P(E_j, t), P(E_i, t')] = 0$ for $j \neq i$. This contradiction to the choice of f_i shows that C is preinjective, whence of defect 1 and indecomposable.

LEMMA 2. – *Let $0 \longrightarrow P_1 \longrightarrow P_0 \longrightarrow E \longrightarrow 0$ be the minimal projective resolution of a simple regular module E. Then there exists for all t a short exact sequence*

$$0 \longrightarrow P_1 \longrightarrow P_0 \oplus R \longrightarrow P(E, t) \longrightarrow 0,$$

where R is the regular radical of $P(E, t)$.

Proof. – The pull-back of the minimal projective resolution under the projection $P(E, t) \longrightarrow E$ is the desired exact sequence.

It is easy to see that the orbit of $P \oplus I(t)$ has codimension one in the orbit-closure of $M(t)$, so that $M(t) <_{\deg} P \oplus I(t)$ is a minimal degeneration, *i.e.* there is no module L satisfying $M(t) <_{\deg} L <_{\deg} P \oplus I(t)$. In general, this is not true for the degeneration $P_0 \oplus R \leq_{\deg} P_1 \oplus P(E, t)$.

3. Degenerations for representations of an oriented cycle

3.1. A general strategy to prove the equivalence of $\leq_{\text{ext}}$ and $\leq$

The main result in this section reads as follows:

PROPOSITION. – *The partial orders $\leq$ and $\leq_{\text{ext}}$ coincide for representations of an oriented cycle.*

The proof is based on the following general strategy valid for any finite-dimensional algebra. To a short exact sequence

$$\Sigma : 0 \longrightarrow X \longrightarrow Y \longrightarrow Z \longrightarrow 0$$

we associate the functions $\delta_\Sigma(V) = [X \oplus Z, V] - [Y, V]$ and $\delta'_\Sigma(V) = [V, X \oplus Z] - [V, Y]$ from the set of (isomorphism classes of) indecomposables to the natural numbers. Then S. Abeasis, A. del Fra and C. Riedtmann made the following useful observation whose proof is obvious. Furthermore, there is a dual lemma.

LEMMA 3. – *Let $M \leq N$ be two modules and let $\Sigma : 0 \longrightarrow X \longrightarrow Y \longrightarrow Z \longrightarrow 0$ be a non-split exact sequence such that $\delta'_\Sigma(V) \leq [V, N] - [V, M]$ holds for all indecomposables V.*

a) If $M = M' \oplus Y$ holds we have $M < M' \oplus X \oplus Z \leq N$.

b) If $N = N' \oplus X \oplus Z$ holds we have $M \leq N' \oplus Y < N$.

Thus if one can find for any disjoint pair $M \leq N$ an exact sequence as in the lemma an obvious induction on $[N, N] - [M, M]$ (use [6], Lemma 1.2) proves the equivalence of $\leq$ and $\leq_{\text{ext}}$. In that case there even exists such an exact sequence with indecomposable ends by [7], Theorem 4. But since the description of these extensions between indecomposables can be quite difficult even for representation-finite algebras (*see* Section 5.3) and since it is usually even more difficult to single out an exact sequence satisfying the assumptions of the lemma, the above strategy is hard to apply directly except for some simple cases, e.g. for the representations of an oriented cycle. The proof we give is an adaptation of Riedtmanns nice combinatorial proof in [16] for quivers of type A_n .

3.2. The universal covering of an oriented cycle

Let Z_n be an oriented cycle with point set $\mathbf{Z}/(n)$ and arrows pointing from $\bar{i}$ to $\overline{i+1}$. Here $\bar{i}$ is the residue of i in $\mathbf{Z}/(n)$. By Fittings lemma any finite dimensional representation V of Z_n admits a canonical decomposition $V = \oplus V(\lambda)$. On $V(\lambda)$ the composition of the n arrows from $V_{\bar{0}}$ to $V_{\bar{0}}$ has λ as its only eigenvalue. Thus the category of finite-dimensional representations is the direct sum of categories $C(\lambda)$ which are all equivalent to the category of nilpotent representations of the loop Z_1 except for $C(0)$ which is the category of nilpotent representations of Z_n. To study the degenerations of these nilpotent representations we can clearly fix the order $l \geq 2$ of nilpotence. Thus we are looking at modules over the finite-dimensional algebra A obtained from the path algebra kZ_n by factoring out all the paths of length $\geq l$. The universal covering $\tilde{A}$ of A is the "path category" of the quiver $\tilde{Q}$ with point set $\mathbf{Z}$ and arrows pointing from i to $i+1$ divided by all paths of length ≥ 2. Note that both A and $\tilde{A}$ are self-dual. The covering functor π induces the canonical projection on the point sets. We refer the reader to [11] for all basic facts and notions concerning coverings that are used before and later on. If we denote by (i, j) resp. $(\bar{i}, j)$ the indecomposable $\tilde{A}$- resp. A-module of dimension j with top at i resp. $\bar{i}$, then the Auslander-Reiten quivers $\tilde{\Gamma}$ resp. Γ have $\mathbf{Z} \times \{1, 2, ..., l\}$ resp. $\mathbf{Z}/(n) \times \{1, 2, ..., l\}$ as underlying set. All points with second component l are projective and injective. On the other points $DTr = \tau$ increases the first component by one. The push-down functor induced by π is denoted the same way. Recall the fundamental relation

$\operatorname{Hom}_A(\pi\tilde{M}, N) \simeq \bigoplus \operatorname{Hom}_{\tilde{A}}(\tilde{M}, \tilde{N})$. Here $\tilde{M}$ is an arbitrary fixed $\tilde{A}$-module, N is a fixed A-module and the direct sum goes over all $\tilde{N}$ which project onto N.

Now we have $[\tilde{M}, \tilde{N}] \leq 1$ for all $\tilde{A}$-modules. The support of $\operatorname{Hom}_{\tilde{A}}((i,j), -)$ is the possibly degenerated quadrilateral with vertices $(i,j), (i,1), (i+j-l,l)$ and $(i+1-l,l)$ (*see* Fig. 1).

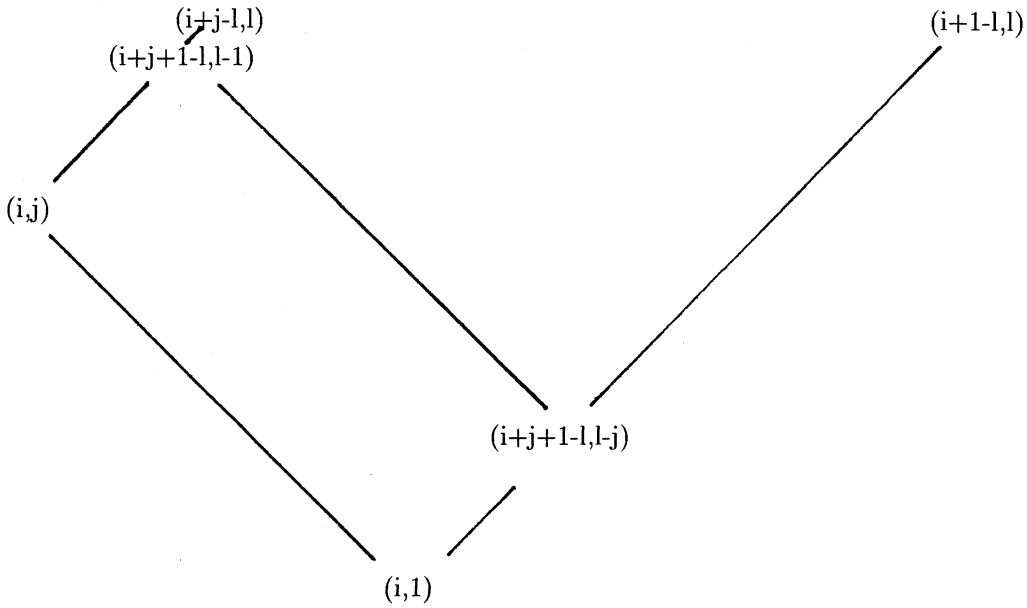

Fig. 1.

Since we are interested in short exact sequences we also want to know the support of $\operatorname{Ext}^1(-, DTr(i,j))$ which coincides by a useful formula of M. Auslander and I. Reiten with the support of $\underline{\operatorname{Hom}}_{\tilde{A}}((i,j), -)$ where $\underline{\operatorname{Hom}}_{\tilde{A}}(\tilde{M}, \tilde{N})$ is the quotient of $\operatorname{Hom}_{\tilde{A}}(\tilde{M}, \tilde{N})$ gotten by annullating maps factoring through projectives. This support is the possibly degenerated rectangle with vertices $(i,j), (i,1), (i+j+1-l, l-1)$ and $(i+j+1-l, l-j)$ (*see* Fig. 1). Note that any two non-split exact sequences with indecomposable $\tilde{A}$-modules at the ends are proportional, whence have isomorphic middle terms.

The following easy result will be used in the next section. Its proof is based on the fact that all indecomposable A- and $\tilde{A}$-modules are uniserial.

LEMMA 4. – *Let V be an indecomposable A-module and $\tilde{V} = (i,j)$ be a lifting. Set $\epsilon_V(X) := [V, X] - [V/\operatorname{soc}V, X]$. Let $X = \oplus U_k^{m_k}$ be a decomposition into pairwise non-isomorphic indecomposables U_k. Then we have $\epsilon_V(X) := \sum m_k$, where the sum runs over all indices k such that V embeds into U_k or equivalently over all k such that $\tilde{V}$ embeds into a lifting of U_k. In particular we have $\epsilon_V(X) = 0$ if and only if no point on the straight line between (i,j) and $(i+j-l,l)$ projects onto a direct summand of X.*

There seems to be an obvious way to prove Proposition 2: Given two A-modules with $M \leq N$ one lifts them to two $\tilde{A}$-modules with $\tilde{M} \leq \tilde{N}$. They live on a finite-dimensional simply connected algebra (in the sense of [12]), whence we have $\tilde{M} \leq_{\text{ext}} \tilde{N}$ by Section 5.2. Since π is exact we obtain also $M \leq_{\text{ext}} N$. This argument is wrong because it might be impossible to lift M and N in such a way that $\tilde{M} \leq \tilde{N}$ holds. In fact the algebra given by an arrow $a \rightarrow b$ and a loop α at b with $\alpha^2 = 0$ as the only relation admits a degeneration $U \leq_{\text{deg}} V$ between indecomposables. This example due to C. Riedtmann shows that there is no lifting $\tilde{U} \leq \tilde{V}$. On the other hand our proof of Proposition 2 consists in showing that for an oriented cycle one can always lift minimal degenerations. It is an interesting problem to find out those representation-finite algebras, where this is possible.

3.3. The proof of proposition 2

We keep all the notations from 3.2. Fix two disjoint A-modules satisfying $M \leq N$. A rectangle is a convex subset $R(a, d)$ of $\tilde{\Gamma}$ with vertices $a = (i, j), b_1 = (i + j - q, q)$, $b_2 = (i, p)$ and $d = (i + j - q, p + q - j)$ such that $1 \leq p \leq j \leq q < l$ holds. Following C.Riedtmann the rectangle $R(a, d)$ is admissible (for M and N) provided we have for all indecomposables U the inequality

$$[U, N] - [U, M] \geq |\pi^{-1}(U) \cap R(a, d)|.$$

Now, choose an indecomposable direct summand U of N of maximal dimension and lift it to an $\tilde{A}$-module (i, j). Then we have by the equality $\pi(i, j)/soc\,\pi(i, j) = \pi(i, j - 1)$ and by Lemma 4

$$[\pi(i, l), N] = [\pi(i, l - 1), N] = \ldots [\pi(i, j), N] > [\pi(i, j - 1), N].$$

From $\underline{\dim}M = \underline{\dim}N$ and from $M \leq N$ we get

$$[\pi(i, l), M] = [\pi(i, l), N] > [\pi(i, j - 1), N] \geq [\pi(i, j - 1), M].$$

Thus by Lemma 4 the triangle T in Figure 2 with lower vertex (i, j) contains a $\tilde{W}$ with $M = M' \oplus \pi(\tilde{W})$.

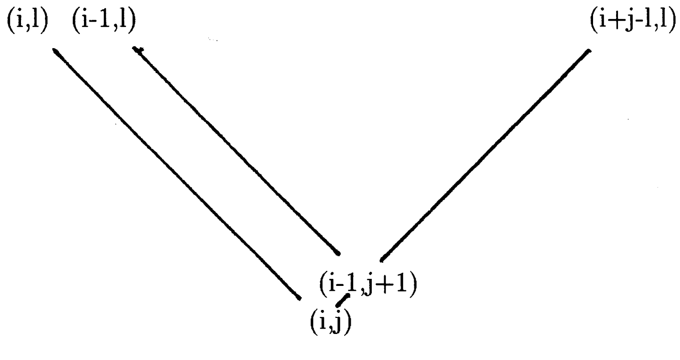

Fig. 2.

Because M and N are disjoint and because A and $\tilde{A}$ are self-dual, we even can assume that $\tilde{W}$ belongs to the triangle T' with lower vertex $(i - 1, j + 1)$. Put $a = (i - 1, j)$.

Descending the straight line with first component $i-1$ and applying at each step Lemma 4, we see that $R(a,a)$ is admissible. By construction $\pi(\tau a)$ is a direct summand of N. We choose j' minimal with the property that the degenerated rectangle $R(a,(i-1,j'))$ is admissible and we set $a' = (i-1,j')$. Then we have $j' = 1$ or else we have

$$[\pi(b_2), N] = [\pi(b_2), M] \tag{1}$$

for the point $b_2 = (i-1, j'-1)$. Finally let $R(a,d)$ be the maximal admissible rectangle that contains $R(a,a')$ (*see* Fig. 3).

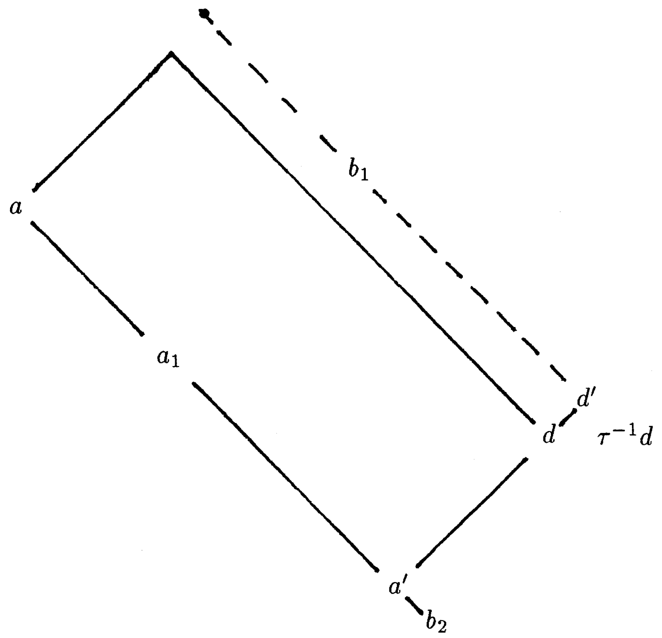

Fig. 3.

Now we claim:

a) $R(a,d)$ contains a point e with $N = N' \oplus \pi(e)$

b) $R(a,d)$ even contains a point e with $N = N' \oplus \pi(e) \oplus \pi(\tau a)$

If this is true we can choose a point e as in b) and we look at the exact sequence

$$0 \longrightarrow \tau a \longrightarrow c := c_1 \oplus c_2 \longrightarrow e \longrightarrow 0.$$

Here c_2 is missing for $e = (i-1,1)$ (*see* Fig. 4). Part b) of Lemma 3 applies to the push-down Σ of this exact sequence. Namely for any indecomposable V we have

$\delta'_{\Sigma}(V) = |\pi^{-1}(V) \cap R(a, e)|$ because of the shape of the supports of $\mathrm{Hom}(-, \tilde{M})$ (*see* 3.2). Since $R(a, d)$ is admissible we find $\delta'_{\Sigma}(V) \leq [V, N] - [V, M]$.

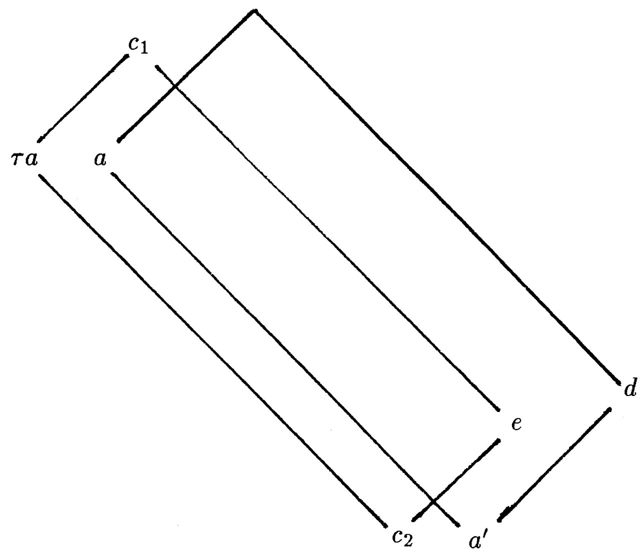

Fig. 4.

Suppose now that claim a) does not hold. Then we construct on the dashed line in figure 3 a point b_1 satisfying $[\pi(b_1), N] = [\pi(b_1), M]$. If the dashed line hits the upper border of $\tilde{\Gamma}$ we can take that point. In the other case the rectangle $R(a, d')$ is not admissible. Therefore there is a point b_1 on the dashed line with

$$0 \leq [\pi(b_1), N] - [\pi(b_1), M] < |\pi^{-1}(\pi(b_1)) \cap R(a, d')|.$$

From $0 < [\pi(b_1), N] - [\pi(b_1), M]$ it would follow $2 \leq |\pi^{-1}(\pi(b_1)) \cap R(a, d')|$. But then $R(a, d)$ would contain a preimage of $\pi(\tau a)$. Thus we have

$$(2) \qquad 0 = [\pi(b_1), N] - [\pi(b_1), M].$$

Now we consider (*see* Fig. 3) the exact sequence

$$E : 0 \longrightarrow a_1 \longrightarrow b := b_1 \oplus b_2 \longrightarrow \tau^{-1} d \longrightarrow 0.$$

Here b_2 is missing for $j' = 1$. Arguments dual to the ones used above imply now for any indecomposable V the relation

$$[\pi(a_1) \oplus \pi(\tau^{-1} d), V] - [\pi(b), V] = |\pi^{-1}(V) \cap R(a_1, d)|.$$

Thus we find

$$[\pi(\tau^{-1} d), N] - [\pi(b), N] + [\pi(a_1), N] = 0,$$

because we assume that $|\pi^{-1}(V) \cap R(a,d)| = 0$ holds for all direct summands of N and also

$$[\pi(\tau^{-1}d), M] - [\pi(b), M] + [\pi(a_1), M] - ([\pi(a_1) \oplus \pi(\tau^{-1}d), M] - [\pi(b), M]) = 0.$$

Subtraction gives

$$[\pi(a_1), N] - [\pi(a_1), M] = ([\pi(b), N] - [\pi(b), M]) - ([\pi(\tau^{-1}d), N] - [\pi(\tau^{-1}d), M]) - ([\pi(a_1) \oplus \pi(\tau^{-1}d), M] - [\pi(b), M]).$$

The first term on the right hand side is zero by the equations (1) and (2) and the other two are not positive so that we have $[\pi(a_1), N] - [\pi(a_1), M] \leq 0$. On the other hand this term is strictly positive because $R(a_1, d)$ is admissible. This contradiction shows that claim a) is true.

The proof of the second claim is again by contradiction. If b) is not true we have $\pi e = \pi\tau a$ for all e as in claim a). Furthermore, $U = \pi\tau a$ has multiplicity one in N. The situation is illustrated in Figure 5. Let ρ be the translation by n. The whole triangle with $(i-1, j') = a'$ as the low vertex contains only points— say u — in the ρ-orbit of τa as liftings of direct summands of N and at least the points in the ρ-orbit of $\tilde{W}$ as liftings of direct summands of M — say v. Figure 5 shows $v > u$. Descending step by step the straight line through $(i-1, l)$, using Lemma 4 and equation (1) we infer from this $j' = 1$. It follows the final contradiction

$$[\pi(i-1,l), N] = |\{r \varepsilon \mathbf{Z} \mid \operatorname{Hom}_{\tilde{A}}((i-1,l), \rho^r \tau a) \neq 0\}| < |\{r \varepsilon \mathbf{Z} \mid \operatorname{Hom}_{\tilde{A}}((i-1,l), \rho^r(\tilde{W})) \neq 0\}| \leq [\pi(i-1,l), M].$$

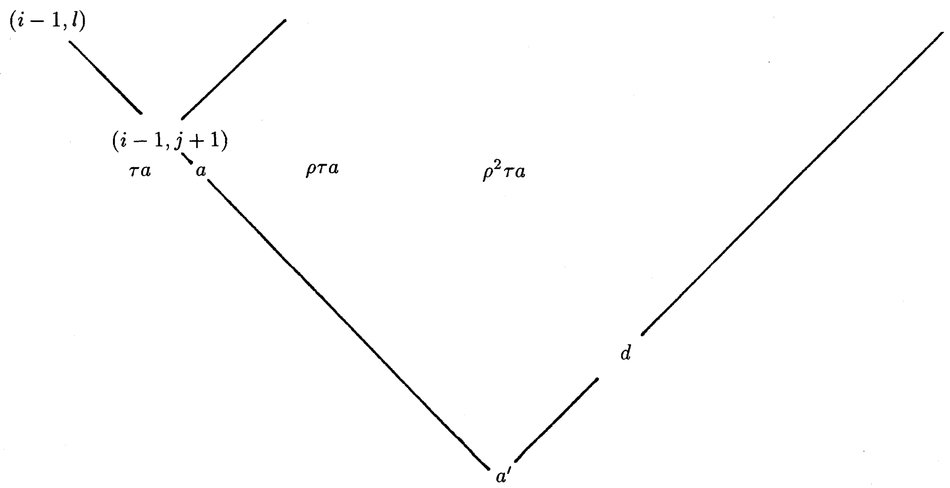

Fig. 5.

4. The proof of the main result

We will now prove:

THEOREM 1. – *The partial orders $\leq$ and $\leq_{deg}$ coincide for modules over path algebras of tame quivers.*

Given two disjoint modules with $M \leq N$ we decompose them into their preprojective, regular and preinjective parts. If both modules are regular we are done by Section 3. So we can assume up to duality that the preprojective parts are non-zero. Then we apply the technique introduced already in [6] to "normalize" the preprojective parts. After that we make an induction on the defect of $\underline{\dim} M_P - \underline{\dim} N_P$ which is easily seen to be non-negative. If the defect is zero, the dual of Lemma 3 applies to an appropriate sequence described in Lemma 2 . In the other case one can use an extension as in Lemma 1 to decrease the defect.

4.1. Translation of the preprojective parts to the right

LEMMA 5. – *Let $M \leq N$ be two modules without common direct summands. If U is a $\preceq$ – minimal preprojective direct summand of $M \oplus N$, the following is true.*

a) U occurs with multiplicity $a := [N,U] - [M,U] \neq 0$ as a direct summand in N.

b) U is not injective, whence there is an almost split sequence $0 \longrightarrow U \longrightarrow X \longrightarrow TrDU \longrightarrow 0$.

c) Writing $N = N' \oplus U^a$ we have $M \oplus TrDU^a \leq N' \oplus X^a$ and $[N,V] - [M,V] = [N' \oplus X^a, V] - [M \oplus TrDU^a] + a\delta(V)$ for all indecomposables V. Here $\delta(V)$ is one for V isomorphic to U and zero otherwise.

d) If $M \oplus TrDU^a$ degenerates to $N' \oplus X^a$, then M degenerates to N.

Proof. – a) Because of $M \leq N$ and the $\preceq$ – minimality, U occurs in N. From $[U,U] = 1$ we see that a is the multiplicity.

b) Since M and N have the same dimension vector we have $[M,I] = [N,I]$ for all injectives. Thus U is not injective.

c) The definition of almost split sequences implies $[TrDU,V] - [X,V] + [U,V] - \delta(V) = 0$ for all indecomposables V. Part c) follows.

d) We have $M \oplus X^a \leq_{\text{deg}} M \oplus (U \oplus TrDU)^a$ for obvious reasons and $M \oplus (U \oplus TrDU)^a <_{\text{deg}} N \oplus X^a$ by assumption, whence $M \oplus X^a$ degenerates to $N \oplus X^a$. The $\preceq$ – minimality of U implies $[N, DTrX] = [M, DTrX]$ and therefore also $[X,N] = [X,M]$ by the formula cited in the introduction. The cancellation theorem for degenerations in [6], 2.5 gives what we want.

Removing from $M \oplus TrDU^a$ and $N' \oplus X^a$ all common direct summands we obtain two new modules M_1 and N_1 with $M_1 \leq N_1$ whose preprojective parts do no more contain U and are situated more to the right.

4.2. Reduction to radical square zero algebras and projective preprojective parts

We have to recall some facts about admissible changes of orientations. Given a sink s in a quiver Q one obtains a new quiver Q' by reversing all arrows pointing to s. There is a dual construction for sources and the transition from Q to Q' is called an admissible change of orientation. Two quivers Q_1 and Q_2 are in the same orientation class if Q_1 is obtained from Q_2 by some admissible changes of orientation. For instance there lives only one orientation class on a tree, but $\tilde{A}_n$ admits many different classes which are characterized by the number of arrows having clockwise orientation.

If Γ is a preprojective component, a slice S is a full connected subquiver of Γ which meets each TrD-orbit once. For instance, the projective indecomposables of a finite dimensional quiver algebra kQ always form a slice isomorphic to Q^{op} in the preprojective component Γ of kQ, and all quivers in the orientation class of Q^{op} occur as slices.

We consider first the case $A = kQ$, where the underlying graph of Q is a tree. Let $M \leq N$ be two kQ-modules with non-zero preprojective part $M_P \oplus N_P$. Applying Lemma 4.1 several times and removing in each step the common direct summands we arrive in a situation where the preprojective parts live on a section S with zig-zag orientation, *i.e.* all points of S are sources or sinks. By the condition $M \leq N$, sources occur only in N. One more application of 4.1 to all these sources leads us to a situation where all summands of N_P are sources and all summands of M_P are sinks. Clearly, the direct sum T of the indecomposables lying on S is a classical tilting module with endomorphism algebra B isomorphic to kS^{op}. Furthermore, M and N are generated by T so that FM and FN have the same dimension vector. Here $F = \mathrm{Hom}_A({}_AT_B, -)$ denotes the usual tilting functor from A-modules to B-modules. Since F has an adjoint functor we still have $FM \leq FN$. By the invariance of degenerations of torsion resp. torsionfree modules under tilting (*see* [7], Theorem 3) we are reduced now to the case where the quiver has only sinks or sources and where N_P is semisimple projective and M_P is projective without simple direct summands.

Next, let Q be a quiver of type $\tilde{A}_n$ with p arrows in clockwise orientation and $q \leq p$ arrows in anti-clockwise orientation. If $q = p$ holds we can argue as above and arrive in the analogous radical square zero case. For $q \neq p$ we reduce to the quiver in the same orientation class with only one source x_1 and one sink x_{p+1}. Let $x_1 \to x_2 \to \ldots x_{p+1}$ be the long path in Q. Then we replace $x_1 \to x_2$ by a hook $x_1 \leftarrow z \to x_2$ and obtain a new quiver Q' of type $\tilde{A}_{n+1}$ where $p - q$ is strictly smaller. There is an obvious embedding F of the kQ-modules to the full subcategory C of those kQ'-modules where $x_1 \leftarrow z$ is represented by a bijection. The functor F has a left and a right adjoint which both induce on C the obvious contraction functor. It follows that $M \leq N$ implies $FM \leq FN$. Since degenerations behave well under contraction of bijectively represented arrows by [7], $FM \leq_{\deg} FN$ also implies $M \leq_{\deg} N$. Repeating this argument often enough we end up with the case $p = q$.

4.3. The behaviour of the preprojective parts under translations to the right

Let Q be a tame quiver consisting only of sinks a_i and sources b_i. Since the double arrow has already been dealt with in [6] we assume that no double arrows occur. This is not essential for our reasoning, but it simplifies the notations a little bit. We denote by

$P(x)$ the indecomposable projective to the point x. We consider two modules $M \leq N$ with preprojective parts given by $N_P = \bigoplus P(a_i)^{n_i}$ and $M_P = \bigoplus P(b_i)^{m_i}$. The multiplicities m_i and n_i are the non-zero coefficients of two vectors m and n in $\mathbf{N}^{Q_0}$. We write c_a for the product in any order of all simple reflections corresponding to sinks. Since these reflections commute, c_a is independent of the order and an involution. Similar remarks hold for the product c_b of the simple reflections to the sources. The following observations are crucial later on:

LEMMA 6. – *We keep all the notations and assumptions of 4.3.*

a) We have $c = c_b c_a$ *and* $c^{-1} = c_a c_b$.

b) $[N_P, TrD^i P(x)] - [M_P, TrD^i P(x)] = c^i c_b(n+m)(x)$ *holds for all* i *and* $x \in Q_0$. *In particular,* $c_b(n+m)$ *has non-negative defect.*

c) If we apply the right translation of 4.1 to all $P(a_i)$ *and then to all* $P(b_i)$ *and if we cancel in each step the common direct summands we get new preprojective parts* $N'_P = \bigoplus TrDP(a_i)^{n'_i}$ *and* $M'_P = \bigoplus TrDP(b_i)^{m'_i}$ *with* $n' + m' = c^{-1}(n+m)$.

d) $\underline{\dim} M_P - \underline{\dim} N_P$ *equals* $c_a(n+m)$. *This vector has the same defect as* $c_b(n+m)$, *and we have* $\underline{\dim} M'_P - \underline{\dim} N'_P = c(\underline{\dim} M_P - \underline{\dim} N_P)$.

Proof. – Part a) is well-known (*see* [8]). The case $i = 0$ of part b) is easily verified. To derive from it the general case we write down the almost split sequences involving projectives, namely

$$0 \longrightarrow P(a_i) \longrightarrow \bigoplus_{b_j - a_i} P(b_j) \longrightarrow TrDP(a_i) \longrightarrow 0$$

and

$$0 \longrightarrow P(b_j) \longrightarrow \bigoplus_{b_j - a_i} TrDP(a_i) \longrightarrow TrDP(b_j) \longrightarrow 0.$$

The other almost split sequences in the preprojective component are obtained from these by applying the functor TrD. Since M_P and N_P are projective the functors $\mathrm{Hom}(M_P, -)$ and $\mathrm{Hom}(N_P, -)$ are exact and the formula of part b) follows. The defect is positive because of $M \leq N$.

Again using the shape of the almost split sequences involving projectives one derives c) and d).

4.4. Preprojective parts with the same defect

We keep all the notations and let Q, M, N, M_P and N_P be as in Section 4.3, but we suppose in addition that M_P and N_P have the same defect.

LEMMA 7. – *Under the above assumptions,* $\underline{\dim} M_P - \underline{\dim} N_P$ *is the dimension vector of a regular module. Any regular module* R *with that dimension vector admits a projective cover* $0 \longrightarrow N_P \longrightarrow M_P \longrightarrow R \longrightarrow 0$.

Proof. – By part d) of Lemma 6 and by our additional assumption we have that $\underline{\dim} M_P - \underline{\dim} N_P = c_a(n+m)$ is a vector of defect zero. Let p be the Coxeter number of the quiver Q. Then we get $c_a(n+m) = c^p c_a(n+m) = c^{p-1} c_b(n+m)$, which has non-negative entries by part b) of Lemma 6. This means that the semi-simple module N_P embeds into M_P. Let C be a $\leq_{deg}$-minimal quotient. Suppose C_P is not zero. Then we choose a $\preceq$-minimal indecomposable preprojective direct summand U of C. The given exact sequence induces the exact sequence $0 \longrightarrow \mathrm{Hom}(C,U) \longrightarrow \mathrm{Hom}(M_P,U) \longrightarrow \mathrm{Hom}(N_P,U) \longrightarrow \mathrm{Ext}(C,U) \longrightarrow 0$. This shows $0 \leq [N_P,U] - [M_P,U] = [C,U]^1 - [C,U]$.

Because of $[C,U] \neq 0$ we also have $[C,U]^1 \neq 0$. Since indecomposable preprojectives have no proper self-extensions there exists a non-split exact sequence $0 \longrightarrow U \longrightarrow D \longrightarrow C' \longrightarrow 0$ with $C = U \oplus C'$. Now M_P is the projective cover of C with semisimple kernel N_P and the same is true for D, which has C as a proper degeneration. This contradiction shows that C_P vanishes. Now C has defect zero and it is therefore regular. Thus $\underline{\dim} M_P - \underline{\dim} N_P$ is the dimension vector of a regular module. Since the top of any regular module R with that dimension vector is the part that lives on the sources, the last assertion is clear.

Of course the Lemmata 5,6 and 7 dualize. Thus we can reduce to the case that the preinjective parts M_I and N_I lie on a zig-zag slice such that the direct summands of N_I are situated on sinks and those of M_I on sources. Because of

$$0 = \underline{\dim} N - \underline{\dim} M = \underline{\dim} N_P - \underline{\dim} M_P + \underline{\dim} N_R - \underline{\dim} M_R + \underline{\dim} N_I - \underline{\dim} M_I$$

we see that M_P and N_P have the same defect if and only if M_I and N_I do so. By an argument dual to the one used in the beginning of the proof of lemma 7 we get that $\underline{\dim} N_I - \underline{\dim} M_I$ has non-positive entries. Therefore the entries of $\underline{\dim} N_R - \underline{\dim} M_R$ are non-negative. The proof of the theorem for preprojective parts with the same defect proceeds now by induction on $\underline{\dim} N_R - \underline{\dim} M_R$ with respect to the product order on $\mathbf{N}^{Q_0}$. Thus the start of the induction $\underline{\dim} N_R = \underline{\dim} M_R$ is trivial and the induction step is formalized in the next lemma.

LEMMA 8. – *Let T be a tube with DTr-orbit of simple regular modules $E_i, i \in \mathbf{Z}/(e)$, that are numbered in such a way that DTr increases the index by one ($e = 1$ is allowed). Suppose that the regular height H of the T-part of $M_R \oplus N_R$ is not zero, that we have an index k with $0 < [N, P(E_{k+1}, l)] - [M, P(E_{k+1}, l)]$ for all $l \geq H$ and that $\underline{\dim} M_P - \underline{\dim} N_P = \sum \epsilon_i \underline{\dim} E_i + \underline{\dim} Y$ with natural numbers ϵ_i, a regular module Y and $\epsilon_k > 0$. Then the theorem follows for M and N from the induction hypothesis.*

Proof. – Let $0 \longrightarrow P_1 \longrightarrow P_0 \longrightarrow E_k \longrightarrow 0$ be a minimal projective cover and let l be any natural number. We look at the exact sequence

$$\Sigma : 0 \longrightarrow P_1 \longrightarrow P_0 \oplus R \longrightarrow P(E_k, l) \longrightarrow 0$$

given in Lemma 2 and we determine the value of δ_Σ (*see* 3.1) at an indecomposable V. For preprojective V we get

$$\delta_\Sigma(V) = [P_1, V] - [P_0, V] = [E_k, V]^1 - [E_k, V] = 0.$$

If V is regular, we apply $\mathrm{Hom}(-,V)$ to the above projective cover and to $0 \longrightarrow R \longrightarrow P(E_k,l) \longrightarrow E_k \longrightarrow 0$ in order to find exact sequences

$$0 \longrightarrow \mathrm{Hom}(E_k,V) \longrightarrow \mathrm{Hom}(P_0,V) \longrightarrow \mathrm{Hom}(P_1,V) \longrightarrow \mathrm{Ext}(E_k,V) \longrightarrow 0$$

and

$$\begin{aligned}0 \longrightarrow \mathrm{Hom}(E_k,V) \longrightarrow \mathrm{Hom}(P(E_k,l),V) \longrightarrow \mathrm{Hom}(R,V)\\ \xrightarrow{f} \mathrm{Ext}(E_k,V) \longrightarrow \mathrm{Ext}(P(E_k,l),V) \longrightarrow \mathrm{Ext}(R,V) \longrightarrow 0.\end{aligned}$$

A straightforward calculation shows $\delta_\Sigma(V) = [P(E_k,l),V]^1 - [R,V]^1 \leq [E_k,V]^1 \leq 1$, whence $\delta_\Sigma(V) \neq 0$ if and only if f is not epi. Thus $\delta_\Sigma(V) \neq 0$ implies $[E_k,V]^1 \neq 0$ and therefore $V = P(E_{k+1},l')$. Then the non-split extension $0 \longrightarrow P(E_{k+1},l') \longrightarrow P(E_k,l'+1) \xrightarrow{\pi} E_k \longrightarrow 0$ is a push out of $0 \longrightarrow R \longrightarrow P(E_k,l) \longrightarrow E_k \longrightarrow 0$ if and only if $P(E_k,l) \longrightarrow E_k \longrightarrow 0$ factors through π which happens exactly for $l'+1 \leq l$. Thus δ_Σ vanishes at all regular indecomposables except for the $V = P(E_{k+1},l')$ with $l' \geq l$. On the preinjectives δ_Σ is zero because of $\delta_\Sigma(V) = [TrDV, P_1 \oplus P(E_k,l)] - [TrDV, P_0 \oplus R]$.

Now let $P(E_j,H)$ be an indecomposable of maximal height occurring in the T-part of $M \oplus N$. We treat first the case that this module belongs to N. When we translate the preprojective parts of M and N one slice more to the right we obtain by Lemma 6 new preprojective parts with

$$\underline{\dim} M'_P - \underline{\dim} N'_P = \sum \epsilon_i \underline{\dim} E_{i+1} + \underline{\dim} DTrY.$$

This does not change the regular parts, and repeating this translation often enough we can assume $j = k$. Then we consider the exact sequence

$$0 \longrightarrow P_1 \longrightarrow P_0 \oplus R \longrightarrow P(E_k,H) \longrightarrow 0$$

given in Lemma 2 . Recall that

$$0 \longrightarrow N_P \longrightarrow M_P \longrightarrow \bigoplus E_i^{\epsilon_i} \oplus Y \longrightarrow 0$$

is a projective cover by Lemma 7. Thus we have $N = N' \oplus P_1 \oplus P(E_k,H)$. For a preprojective indecomposable V we obtain

$$\begin{aligned}[N,V] - [M,V] = [N_P,V] - [M_P,V] = \sum \epsilon_j([E_j,V]^1 - [E_j,V]) + [Y,V]^1 - [Y,V]\\ \geq ([E_k,V]^1 - [E_k,V]) = \delta_\Sigma(V).\end{aligned}$$

For a regular indecomposable the corresponding inequality holds by the calculation above and by the assumption $0 < [N, P(E_{k+1},l)] - [M, P(E_{k+1},l)]$ for $l \geq H$. Finally, δ_Σ vanishes on preinjectives. From the dual of Lemma 3 we obtain $M \leq L := N' \oplus P_0 \oplus R \leq_{deg} N$. Induction applies to $M \leq L$.

In the case where $P(E_j,H)$ belongs to M, we reduce as above to the case $j = k+1$. Then we look at the sequence

$$0 \longrightarrow P_1 \longrightarrow P_0 \oplus R \longrightarrow P(E_k,H+1) \longrightarrow 0$$

given in Lemma 2 . Since R is isomorphic to $P(E_{k+1}, H)$ we obtain now by the arguments above $M = M' \oplus P_0 \oplus P(E_{k+1}, H) \leq_{deg} L := M' \oplus P_1 \oplus P(E_k, H+1) \leq N$. This time induction applies to $L \leq N$.

It remains to be shown that the additional assumptions of the lemma are always satisfied for some tube. To fix the notations suppose now that we are dealing with three exceptional tubes whose simple regular modules are denoted by A_i, $i \in \mathbf{Z}/(a)$, B_i, $i \in \mathbf{Z}/(b)$ and C_i, $i \in \mathbf{Z}/(c)$. They are numbered in such a way that DTr increases the index by one. The remaining case $\tilde{A}_n$ with two exceptional tubes can be treated in the same way.

Now one knows (*see* [18]) that the kernel of the defect ∂ in $\mathbf{Q}^{Q_0}$ is generated by the $\underline{\dim} A_i, \underline{\dim} B_i$ and $\underline{\dim} C_i$ with the only relations $\sum \underline{\dim} A_i = \sum \underline{\dim} B_i = \sum \underline{\dim} C_i = h$. Therefore the dimension vector d of each regular module admits a canonical decomposition

$$d = \sum \alpha_i \underline{\dim} A_i + \sum \beta_i \underline{\dim} B_i + \sum \gamma_i \underline{\dim} C_i + \eta h$$

with natural numbers $\alpha_i, \beta_i, \gamma_i$ and η such that at least one of the α_i, one of the β_i and one of the γ_i is zero.

First we treat the case that $\underline{\dim} M_P - \underline{\dim} N_P$ is non-homogeneous. Thus we can assume $\alpha_k > 0 = \alpha_{k-1}$ for some index k in the canonical decomposition of $\underline{\dim} M_P - \underline{\dim} N_P$. Then we have for all non-preinjective indecomposables V the inequality

$$\begin{aligned} 0 &\leq [N, V] - [M, V] = [N_P, V] - [M_P, V] + [N_R, V] - [M_R, V] \\ &= [N_R, V] - [M_R, V] + \sum \alpha_i ([A_i, V]^1 - [A_i, V]) + \sum \beta_i ([B_i, V]^1 - [B_i, V]) \\ &+ \sum \gamma_i ([C_i, V]^1 - [C_i, V]). \end{aligned}$$

Let T be the tube containing the A_i, and let H be the regular height of the T-part of $M_R \oplus N_R$. Then we obtain for all j and all $l \geq H$ the relation

$$0 \leq [N, I(A_j, l)] - [M, I(A_j, l)] = \alpha_{j-\bar{l}} - \alpha_j + m(A_j, N_R) - m(A_j, M_R).$$

Here $\bar{l}$ is the residue of l, and $m(E, R)$ is the multiplicity of the regular simple module E inside a regular composition series of the regular module R. From the special case $j - \bar{l} = k - 1$ we get

$$0 \leq -\alpha_j + m(A_j, N_R) - m(A_j, M_R).$$

In particular, A_k has to occur in N_R and we get for $k = j - \bar{l}$ from $I(A_j, l) = P(A_{k+1}, l)$

$$0 < \alpha_k - \alpha_j + m(A_j, N_R) - m(A_j, M_R) = [N, P(A_{k+1}, l)] - [M, P(A_{k+1}, l)].$$

Therefore the tube T satisfies the assumptions of Lemma 8.

If $\underline{\dim} M_I - \underline{\dim} N_I$ is non-homogeneous we use the dual arguments. So we are left with the case where $\underline{\dim} M_P - \underline{\dim} N_P$ and $\underline{\dim} M_I - \underline{\dim} N_I$ are homogeneous, whence also $\underline{\dim} N_R - \underline{\dim} M_R$. Then there exists a simple regular module E such that $m(E, N_R) > m(E, M_R)$ holds. Let T be the tube containing E and let H be the regular height of the T-part of $M \oplus N$. Then Lemma 8 applies to this tube always. This is clear in case T is homogeneous. In the other case we can assume $E = A_j$ for some j. Then we have $0 < m(A_j, N_R) - m(A_j, M_R) = m(A_i, N_R) - m(A_i, M_R)$ for all i because $\underline{\dim} N_R - \underline{\dim} M_R$ is homogeneous. So we get $0 < [N, I(A_i, l)] - [M, I(A_i, l)] = [N, P(A_{i-\bar{l}+1}, l] - [M, P(A_{i-\bar{l}+1}, l)]$ for all $l \geq H$ and all i. The relation $h = \sum \underline{\dim} A_i$ implies that the other condition is satisfied.

4.5. Preprojective parts with different defect

Again we work in the situation of 4.3, but now we assume in addition that the defect d of $\underline{\dim} M_P - \underline{\dim} N_P$ is strictly positive. Since one component of h is one there exists a projective $P(s)$ of defect -1. If s is not yet a sink in Q we apply 4.1 to all $P(a_i)$ and we tilt to a situation where it is. Then we translate the slice carrying M_P and N_P often enough to the right — say j times — such that the following holds for the translated preprojective parts $\overline{M}_P$ and $\overline{N}_P$:

a) $TrD^j P(s)$ occurs in $\overline{N}_P$ as a direct summand.

b) $[TrD^j P(s), U] \leq [\overline{N}_P, U] - [\overline{M}_P, U]$ holds for all preprojective U.

To see that this is possible let p be the Coxeter-number of Q. Then the multiplicity vectors $\overline{m}$ and $\overline{n}$ of $\overline{M}_P$ and $\overline{N}_P$ are given by

$$\overline{m} + \overline{n} = c^{-j}(m+n) = c_b c^j c_b (m+n)$$

because of $c^{-1} = c_b c c_b$ (*see* Lemma 6). Since the defect of $c_b(m+n)$ is strictly positive all entries of the vector $c^{-j}(m+n)$ tend to infinity with increasing j. Thus condition a) holds for all large indices j. Similarily, one can use part b) of Lemma 6 in the special case $N_P = P(s)$ and $M_P = 0$ to obtain $[P(s), TrD^i P(x)] = c^i c_b e_s(x)$ for all x and i. Here $c_b e_s$ has defect one. Now let q be the maximum of the finitely many numbers $[P(s), TrD^i P(x)]$ with x arbitrary but $0 \leq i < p$. Then we can choose j large enough such that

$$\begin{aligned}&[\overline{N}_P, TrD^{i+j} P(x)] - [\overline{M}_P, TrD^{i+j} P(x)]\\ &= c^i c_b(\overline{m} + \overline{n})(x) \geq q \geq [TrD^j P(s), TrD^{i+j} P(x)]\end{aligned}$$

holds for all x and $0 \leq i < p$. But then this inequality is valid for all i because the defect d of the left hand side is at least as big as the defect of the right hand side which is one.

Next we tilt the whole situation to get N_P semi-simple projective containing a simple projective P of defect -1. All this does not change the defect of $\underline{\dim} M_P - \underline{\dim} N_P$. Now for any t we set $M(t) := \oplus P(E,t)$ the sum running over all homogeneous simples occurring in M_R or N_R and over all non-homogeneous simples with $[P,E] \neq 0$. Note that we always have $[P,E] = 1$ so that $[P,X]$ is the multiplicity of E as a composition factor of a regular module X belonging to the same tube as E.. Let

$$\Sigma : 0 \longrightarrow P \longrightarrow M(t) \longrightarrow I(t) \longrightarrow 0,$$

be the corresponding exact sequence given by Lemma 1 . We want to choose t in such a way that $\delta_\Sigma(V) \leq [N,V] - [M,V]$ holds for all preprojective and regular indecomposables V. On the preprojectives this is true by our choices. So let $V = I(F,t')$ be a regular indecomposable. First let F be homogeneous. If F does not occur in $M \oplus N$ we have

$$\delta_\Sigma(V) = [P,V] = t' \leq t'd = [N_P, V] - [M_P, V] = [N,V] - [M,V]$$

because of $\partial X = [X,F]^1 - [X,F]$ for any X (*see* Section 2.1). If F occurs in $M \oplus N$ we find

$$\delta_\Sigma(V) = \max(t'-t, 0) \leq [N,V] - [M,V] = t'd + [N_R, V] - [M_R, V]$$

as soon as $t \geq \dim M_R \oplus N_R$ holds .

Finally, if F and E belong to a non-homogeneous tube with a simple regular modules we have $\delta_\Sigma(V) = m(E, I(F,t')) - m(E, I(F, \min(t',t)))$, which is zero for $t' \leq t$. For $t' > t$ we write $t' = s'a + r'$ and $t = sa + r$ with natural numbers r, r', s, s' satisfying $0 \leq r, r' < a$. Then we obtain

$$\begin{aligned}\delta_\Sigma(V) &= m(E, I(F,t')) - m(E, I(F,t))\\ &\leq m(E, I(F,(s'+1)a)) - m(E, I(F, sa)) = s' + 1 - s\end{aligned}$$

because the dimension vector of $I(F,a)$ is h, and

$$\begin{aligned}&[N,V] - [M,V] = [N_P,V] - [M_P,V] + [N_R,V] - [M_R,V] \geq [N_P, I(F,s'a)]\\ &- [M_P, I(F,(s'+1)a)] - \dim(M_R \oplus N_R)\\ &= s'd + \partial(\underline{\dim} M_P) - \dim(M_R \oplus N_R).\end{aligned}$$

Thus the wanted inequality holds for $s \geq 1 - \partial(\underline{\dim} M_P) + \dim(M_R \oplus N_R)$, *i.e.* $t \geq a(1 - \partial(\underline{\dim} M_P) + \dim(M_R \oplus N_R))$. Since a is bounded by p (*see* [8]), we conclude that $\delta_\Sigma(V) \leq [N,V] - [M,V]$ holds for all regular indecomposables V as soon as t is larger than $p(1 - \partial(\underline{\dim} M_P) + \dim(M_R \oplus N_R))$.

At the end we adjust the preinjective parts M_I and N_I using the dual of Lemma 6. This does not influence the preprojective or regular parts and the inequalities already achieved. First we can assume that M_I and N_I live on a zig-zag slice S. Because of

$$0 = \underline{\dim} N - \underline{\dim} M = \underline{\dim} N_P - \underline{\dim} M_P + \underline{\dim} N_R - \underline{\dim} M_R + \underline{\dim} N_I - \underline{\dim} M_I$$

the defect of $\underline{\dim} N_I - \underline{\dim} M_I$ is $d > 0$. Arguments dual to the ones used in the beginning of 4.5 show that there is a constant j_0 such that the new preinjective parts $\overline{M}_I$ and $\overline{N}_I$ obtained by translating $j \geq j_0$ times to the left satisfy the following two conditions:

a) Any module in $DTr^j S$ occurs in $\overline{M}_I \oplus \overline{N}_I$.

b) If V is a direct summand of $\overline{N}_I$ of defect one, we have $[U,V] \leq [U, \overline{N}_I] - [U, \overline{M}_I]$ for all preinjective indecomposable U.

We translate j_0 times to the left and still call the new preinjective parts M_I and N_I. Then we choose $t \geq p(1 - \partial(\underline{\dim} M_P) + \dim(M_R \oplus N_R))$ so large that $I(t)$ is a $\preceq$-predecessor of some module in N_I. After that we translate again to the left until the indecomposable $I(t)$ of defect one becomes a direct summand of the new preinjective parts. If $I(t)$ does not yet belong to N_I, we move a half step more to the left. Thus we have reached the case where $N = N' \oplus P \oplus I(t)$ holds. By property b) above we have for all preinjective indecomposable V the relations

$$[N,V] - [M,V] = [TrDV, N] - [TrDV, M] = [TrDV, N_I] - [TrDV, M_I]$$

$$\geq [TrDV, I(t)] = [P \oplus I(t), V] - [M(t), V] = \delta_\Sigma(V).$$

Thus we obtain $M \leq N' \oplus M(t) \leq_{\deg} N = N' \oplus P \oplus I(t)$. Now $\partial(\underline{\dim} M_P - \underline{\dim}(N' \oplus M(t))_P) < \partial(\underline{\dim} M_P - \underline{\dim} N_P)$ ends our inductive proof.

5. Concluding remarks

5.1. Extension of the main result to all tame concealed algebras

PROPOSITION 3. – *The partial orders $\leq$ and $\leq_{deg}$ coincide for all tame concealed algebras.*

Proof. – Recall that a tame concealed algebra has a preprojective component which contains a slice S which is a tame quiver. Of course, the direct sum T of all indecomposables lying on S is a tilting module. Given $M \leq N$, we can translate the preprojective parts of M and N far enough to the right such that M and N are generated by a power of T. Then we can tilt the problem to a tame quiver and the proposition follows from the main result by the arguments of 4.2.

5.2. The equivalence of $\leq$ and $\leq_{\text{ext}}$ for preprojective modules

One knows that the equivalence of $\leq$ and $\leq_{\text{deg}}$ fails for representations of wild quivers ([6], Section 7). However both partial orders coincide for preprojective modules as shown in [6], Section 4. Here I will give another more transparent proof which is based on the following simple geometric observation whose dual is contained in [6], Section 2.

Let $M \leq_{\text{deg}} N$ be a minimal disjoint degeneration with $N = N' \oplus U$ and $[N, U] = [M, U]$. Then U is a homomorphic image of M. If a generic kernel K exists, then $U \oplus K$ degenerates to N, whence we have $M <_{\text{ext}} U \oplus K \leq N$ and $N = U \oplus K$ by minimality.

PROPOSITION 4. – *Let M and N be preprojective modules. Then the following statements are equivalent:*

a) $M \leq N$

b) $M \leq_{\text{deg}} N$

c) $M \leq_{\text{ext}} N$

Proof. – a)$\Rightarrow$ b): Let $M \leq N$ be two disjoint preprojective modules. Consider the function $V \mapsto [N, V] - [M, V] = [TrDV, N] - [TrDV, M]$. Since M and N are preprojective this function has finite support S and we can argue by induction on the cardinality of S. If S is empty, M and N are isomorphic by Auslanders theorem. In the other case we apply the procedure of 4.1 to make the support smaller.

b)$\Rightarrow$ c): Let $M \leq_{\text{deg}} N$ be a minimal degeneration. By the equivalence of $\leq$ and $\leq_{\text{deg}}$ already shown we can assume that M and N are disjoint. Let U be a $\preceq$-maximal direct summand of N. Then we have $[N, U] - [M, U] = [TrDU, N] - [TrDU, M] = 0$. All kernels of surjections $M \to U$ are preprojective. Therefore, the irreducible constructible subset of all kernels consists only of finitely many orbits ([7], Section 2), and there has to be a generic kernel. The remark made before the proposition ends the proof.

This remark resp. its dual apply also to other situations where one knows already the equivalence of $\leq$ and $\leq_{\text{deg}}$. For instance, the only difficult case in my proof for the coincidence of $\leq$ and $\leq_{\text{ext}}$ for Kronecker-modules in [6], Section 5 is the following one. We have $M \leq N$ with $M_P = 0$ and $N_P \neq 0$. Taking a $\preceq$-minimal preprojective direct summand U of N we have $[U, N] = [U, M]$ and it is clear that there are only finitely many cokernels so that a generic one exists. Since we know already that $M \leq_{\deg} N$ holds by our theorem 1, we obtain $M \leq_{\mathrm{ext}} N$ from the above remark.

5.3. The jungle of extensions between preprojective indecomposables

What prevents us from showing the coincidence of $\leq$ and $\leq_{\mathrm{ext}}$ for representations of all tame quivers seems to be the combinatorial complexity of minimal degenerations induced by extensions of indecomposables.

Here we present for any natural number n a simply connected algebra $A(n)$ (in the sense of [12]) and two indecomposable modules U and V such that $[V, U]^1 = n + 1$. The number of codimension one extensions is at least n. Furthermore, there is such an extension with $2n + 3$ direct summands and no non-zero maps between them, but also one with only three direct summands W_1, W_2 and W_3 with $[W_1, W_2] = 2n$. The verification of these facts is a lengthy matter of routine once the algebra and the two modules are given. Now the Gabriel-quiver of $A(n)$ has points $a_1, a_2, \ldots, a_{2n+1}, b_1, b_2, \ldots, b_{2n+2}$ and $c_1, c_2, \ldots, c_{2n+1}$ and arrows $a_i \to b_i$, $a_i \to b_{i+1}$, $b_i \to c_i$ and $b_{i+1} \to c_i$ for $1 \leq i \leq 2n + 1$. The imposed relations are all commutativity relations and the following zero-relations: $a_i \to b_i \to c_{i-1}$ for $2 \leq i \leq 2n + 1$ and $a_{2i+1} \to b_{2i+2} \to c_{2i+2}$ for $0 \leq i \leq n - 1$. Then U is the unique indecomposable vanishing on the a_i and V the one vanishing on the c_i.

5.4. Test-sets for representations of tame quivers

As mentioned in the introduction, for any module M there is a finite set of indecomposables U_i (depending on M) such $M \leq N$ is equivalent to $[M, U_i] \leq [N, U_i]$ for all modules U_i. Here we describe such a test-set for any module M over the path algebra of a tame quiver Q. For the sake of simplicity we assume that Q consists only of sinks and sources.

Thus let M be given of dimension d. Then there is a zig-zag slice S in the preprojective component such that all modules to the right of S have dimension at least d. Let p be the Coxeter-number of Q. As preprojective test-modules we take all indecomposables on the left of TrD^pS. Dually, there is a slice S' in the preinjective component such that all preinjectives on the left of S' have dimension at least d. We pick all the modules on the right of $DTr^{p+1}S'$ as test-modules. Finally, in the at most three non-homogeneous tubes, in all tubes occurring in M and in one fixed homogeneous tube we take all indecomposables of regular length at most $d + p$. Note that only the regular test-modules depend on M. The verification that we have given now a test-set is easy but lengthy. It uses lemma 6 and its dual in an essential way. Furthermore we have not tried to make the test-set as small as possible.

The above recipe is not constructive because one has to know the decomposition of M into indecomposables. Now, by finitely many rational operations, one can—at least in principle—find out all indecomposable direct summands of M which are preprojective, preinjective or regular non-homogeneous. But to know the homogeneous indecomposables means to know the eigenvalues of a certain matrix. For $k[X]$-modules one can overcome this difficulty by looking at the rational canonical form. A similar way-out exists for the other tame quivers, but we leave the somewhat lengthy details to the reader.

REFERENCES

[1] S. ABEASIS and A. del FRA, *Degenerations for the representations of a quiver of type* A_m (*J. of Algebra*, Vol. 93, 1985, pp. 376–412).

[2] S. ABEASIS and A. del FRA, *Degenerations for the representations of an equioriented quiver of type* D_m (*Adv. Math.*, Vol. 52, 1984, pp. 81–172).

[3] M. AUSLANDER and I. REITEN, *Modules determined by their composition factors* (*Illinois J.Math.*, Vol. 29, 1985, pp. 289–301).

[4] K. BONGARTZ, *A generalization of a theorem of M.Auslander* (*Bull. London Math. Soc.*, Vol. 21, 1989, pp. 255–256).

[5] K. BONGARTZ, *A geometric version of the Morita equivalence* (*J.Algebra*, Vol. 139, 1991, pp. 159–179).

[6] K. BONGARTZ, *On degenerations and extensions of finite dimensional modules* (*accepted 1990 for publication in Adv.Math.*, but not yet appeared, preprint of 53 pages).

[7] K. BONGARTZ, *Minimal singularities for representations of Dynkin quivers* (*to appear in Comm. Math. Helv.*, preprint of 24 pages, 1993).

[8] V. DLAB and C. M. RINGEL, *Indecomposable representations of graphs and algebras* (*Memoirs Amer. Math. Soc.*, Vol. 173, 1976).

[9] P. DONOVAN and M. R. FREISLICH, *The representation theory of finite graphs and associated algebras* (*Carleton Lecture notes 5*, Ottawa, 1973).

[10] S. FRIEDLAND, *Simultaneous similarity of matrices* (*Adv. Math.*, Vol. 50, 1983, pp. 189–265).

[11] P. GABRIEL, *The universal cover of a representation-finite algebra*, (*Springer Lecture Notes*, Vol. 903, 1980, pp. 68–105).

[12] P. GABRIEL, B. KELLER and A. V. ROITER, *Representations of finite-dimensional algebras* (*Encycl. Math. Sc.*, Vol. 73, 1992, pp. 1–176).

[13] G. KEMPKEN, *Eine Darstellung des Kchers* $\tilde{A}_k$ (*Bonner Math. Schriften*, Vol. 137, 1982, pp. 1–159).

[14] G. HERMANN, *Die Frage der endlich vielen Schritte in der Theorie der Polynomideale* (*Math. Ann. Bd.*, Vol. 95, 1926, pp. 736–788).

[15] L. A. NAZAROVA, *Representations of quivers of infinite type* (*Izvest. Akad. Nauk. SSR, Ser. Mat.*, Vol. 37, 1973, pp. 752–791).

[16] C. RIEDTMANN, *Degenerations for representations of quivers with relations* (*Ann. Sci. Ecole Norm. Sup.*, Vol. 4, 1986, pp. 275–301).

[17] C. M. RINGEL, *Tame algebras and quadratic forms* (*Lecture Notes in Math. 1099*, Springer Verlag, 1984).

[18] C. M. RINGEL, *The rational invariants of tame quivers* (*Invent. Math.*, Vol. 58, 1980, pp. 217–239).

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