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LIFTING DIFFERENTIAL OPERATORS FROM ORBIT SPACES

BY GERALD W. SCHWARZ

ABSTRACT. – Let X be an affine complex algebraic variety, and let $\mathcal{D}(X)$ denote the (non-commutative) algebra of algebraic differential operators on X . Then $\mathcal{D}(X)$ has a filtration $\{\mathcal{D}^n(X)\}$ by order of differentiation, and the associated graded $\text{gr } \mathcal{D}(X)$ is commutative. Now assume that X is smooth and a G -variety, where G is a reductive complex algebraic group. Let $\pi_X : X \rightarrow X//G$ be the quotient morphism. Then we have a natural map $(\pi_X)_* : (\mathcal{D}^n(X))^G \rightarrow \mathcal{D}^n(X//G)$. We find conditions under which $(\pi_X)_*$ is surjective for all n , in which case $\text{gr } \mathcal{D}(X//G)$ is finitely generated. We conjecture that the latter is always true. We also consider generalizations to algebras of differential operators on sections of G -vector bundles.

0. Introduction

All varieties we consider will be algebraic and defined over our base field \mathbb{C} .

Let Z be an affine variety, and let $\mathcal{D}(Z)$ denote the (non-commutative) algebra of differential operators on Z . Then $\mathcal{D}(Z)$ has a filtration $\{\mathcal{D}^n(Z)\}$ by order of differentiation, and the associated graded $\text{gr } \mathcal{D}(Z)$ is commutative. If Z is smooth, then $\text{gr } \mathcal{D}(Z)$ is finitely generated [Bj], hence $\mathcal{D}(Z)$ is finitely generated, left and right noetherian. If Z is not smooth, all of these properties can fail ([BGG], *see* 3.11). It seems to be difficult to determine the properties of $\mathcal{D}(Z)$ in the singular case, e.g., to determine when $\text{gr } \mathcal{D}(Z)$ is finitely generated.

We will be considering the case where Z is a quotient: Let X be an affine G -variety, where G is reductive. Then canonically there is a quotient variety $X//G$ and a surjection $\pi_X : X \rightarrow X//G$ (*see* 1.1). Work of Kantor, Musson, Levasseur, Stafford and others has shown that quotients of *smooth* varieties, although usually singular, often have well-behaved algebras of differential operators. Their work leads one to formulate the following:

(0.1) CONJECTURE. – *Let X be a smooth affine G -variety, where G is reductive. Then $\text{gr } \mathcal{D}(X//G)$ is finitely generated.*

One can also consider analogous problems and conjectures for differential operators on sections of G -vector bundles. There are some new phenomena in this case (*see* 0.13-0.14).

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For most of this introduction we restrict ourselves to the case of “ordinary” differential operators. We will also assume here that all of our affine G -varieties are irreducible.

(0.2) There is a canonical morphism $(\pi_X)_* : \mathcal{D}(X)^G \rightarrow \mathcal{D}(X//G)$ which respects the filtrations by order. If $(\pi_X)_*(\mathcal{D}^n(X)^G) = \mathcal{D}^n(X//G)$ for every $n \geq 0$ (equivalently, if $\text{gr}(\pi_X)_* : \text{gr}(\mathcal{D}(X)^G) \rightarrow \text{gr} \mathcal{D}(X//G)$ is surjective) we say that $(\pi_X)_*$ is *graded surjective*. If X is smooth we will show that

- (1) $\text{gr}(\mathcal{D}(X)^G) = (\text{gr} \mathcal{D}(X))^G$ is finitely generated (3.19).
- (2) $(\pi_X)_*$ is “usually” graded surjective.

Graded surjectivity clearly implies that the conjecture holds. The main focus of our paper is on properties of $(\pi_X)_*$.

(0.3) Let X be an affine G -variety. Set $k := \min\{\dim G_x : Gx \text{ is closed}\}$ and $l := \min\{\text{no. of components of } G_x : Gx \text{ is closed and } \dim G_x = k\}$. Let X' denote those $x \in X$ such that Gx is closed, $\dim G_x = k$ and G_x has l components. The orbits in X' are called *principal orbits*, and their isotropy groups are all conjugate and are called *principal isotropy groups* (see 1.4). If $k = 0$, then X has finite principal isotropy groups (abbreviation: X has FPIG). Set $X_{\text{pr}} = \pi_X^{-1}(\pi_X(X'))$.

We have a geometric criterion for $(\pi_X)_*$ to be graded surjective, X smooth, consisting of the following three conditions:

- (1) X has FPIG.
- (2) The codimension in X of $X \setminus X_{\text{pr}}$ is at least 2.
- (3) $\text{mod}(X \setminus X_{(0)}, G) \leq \dim X//G - 2$.

Here $X_{(0)}$ denotes the orbits with zero dimensional stabilizer. If Z is a G -variety, then $\text{mod}(Z, G)$ is the *modularity of Z* or the “number of parameters in the orbit space Z/G .” More precisely, it is the maximum of $\dim Z_{(n)} - \dim G + n$ where $Z_{(n)} := \{z \in Z : \dim G_z = n\}$. We say that X is *2-principal* if (2) is satisfied, and we say that X is *2-large* if (1)–(3) are satisfied. Note that X is 2-large if $X = X_{\text{pr}}$, e.g., if G acts freely on X .

Our main results on (graded) surjectivity are the following:

(0.4) THEOREM. – *Let X be a smooth affine G -variety.*

(1) *If X is 2-large, then $(\pi_X)_*$ is graded surjective (9.10).*

(2) *Suppose that G is semisimple. Consider G -modules V such that $\text{Ker}(G \rightarrow \text{GL}(W))$ is finite for each non-zero irreducible G -submodule W of V . Then, up to isomorphism, all but finitely many V are 2-large (11.6).*

Now suppose that $(\pi_X)_$ is surjective.*

(3) *If all G -orbits have the same dimension (e.g., G is finite), or if G^0 is semisimple or a torus, then X is 2-principal (6.5, 5.16 and 7.11, 10.2).*

(4) *If G^0 is a torus, then $(\pi_X)_*$ is graded surjective. If, in addition, X has FPIG, then X is 2-large (10.4).*

(5) *Suppose that $X = V$ is a G -module. If V is coregular, then V is fix pointed (5.5).*

Recall that a G -module V is said to be *coregular* if $V//G$ is smooth, equivalently, if $\mathcal{O}(V)^G$ is a polynomial algebra. We say that V is *fix pointed* if $V^G \simeq V//G$, equivalently, if all closed G -orbits are fixed points (1.5).

Parts (3) and (4) generalize results of Kantor [Ka] on finite groups (see also [Le1]) and of Musson [Mu] on tori. One is led to

(0.5) CONJECTURES. – *Let X be a smooth affine G -variety. Then*

(1) $(\pi_X)_*$ is surjective if and only if it is graded surjective.

(2) If $(\pi_X)_*$ is surjective, then X is 2-principal.

The property of 2-largeness has some interesting consequences for G -modules.

(0.6) PROPOSITION. – *Let V be a G -module.*

(1) If V is faithful and 2-large, then its principal isotropy groups are trivial (7.7).

(2) Let V be coregular with FPIG. If V is 2-large or $V \setminus V_{\text{pr}}$ is of codimension 3 in V , then V is fix pointed (9.12).

(3) ([Po2], [Go], cf. [Kn]) If G is semisimple, then up to isomorphism and the addition of trivial factors, there are only finitely many coregular V (11.7).

Regarding conjecture 0.1 we have the following:

(0.7) THEOREM. – *Let X be a smooth affine G -variety. Then conjecture 0.1 holds in the following cases.*

(1) X is 2-large (9.10).

(2) $X//G$ is smooth.

(3) G is commutative (10.7).

(4) All the G -orbits on X have the same dimension, e.g., G is finite (6.7).

There is an interesting dichotomy in (1) and (2) above. Let us just consider G -modules for the moment. If V is 2-large, then $(\pi_V)_*$ is graded surjective, conjecture 0.1 holds, and V is *not* coregular (unless V is fix pointed). On the other hand, if V is coregular, then we know that $\mathcal{D}(V//G)$ is finitely generated, etc. without needing to know anything about $(\pi_V)_*$. Coregularity indicates that V is a “small” representation. Unfortunately, there are representations in a “gray area” which are neither 2-large nor coregular. In general, we have no tools to determine whether or not conjecture 0.1 holds in these cases. Examples are some of the SL_n -modules of the form $k\mathbb{C}^n \oplus l(\mathbb{C}^n)^*$ (see 11.15).

(0.8) There is another type of “small” representation; those for which the principal isotropy groups are positive dimensional. We are sometimes able (e.g., for tori) to reduce to the case of finite principal isotropy groups by the Luna-Richardson theorem (7.10). In other cases we can show that lifting does not hold (7.13), i.e., $(\pi_V)_*$ is not surjective.

(0.9) The inspiration for this paper was the work of Levasseur and Stafford ([LS], [Le2]). They showed that conjecture 0.1 holds for the actions of the classical groups on sums of their standard representations. More precisely:

(0.10) THEOREM ([LS]). – *Let $(V, G) = (k\mathbb{C}^n \oplus l(\mathbb{C}^n)^*, GL_n), (k\mathbb{C}^n, O_n), (k\mathbb{C}^n, SO_n)$ or $(k\mathbb{C}^{2n}, Sp_{2n})$; $k, l \geq 0, n \geq 1$. Then*

(1) V is coregular, or

(2) $(\pi_V)_*$ is graded surjective.

In either case

(3) $\mathcal{D}(V//G)$ is simple.

The methods of [LS] depend upon results in the theory of enveloping algebras and Howe's theory of reductive dual pairs [Ho], and they essentially only apply in the cases of 0.10. Levasseur [Le2] put the results of [LS] in a more general setting, and provided general criteria for $(\pi_V)_*$ to be graded surjective. His criteria depend upon estimating the homological codimension (depth) of certain algebras, and in the cases of 0.10, he obtains these estimates from [LS].

(0.11) *Definition.* – We say that a certain collection of representations satisfies the *LS-alternative* if for every (V, G) in the collection, (V, G) is coregular or 2-large (so conjecture 0.1 holds).

In this paper, we establish the LS-alternative in the following cases:

- (1) $G = \mathrm{SL}_2$ (11.9)
- (2) For the representations in Theorem 0.10 (11.10–20).
- (3) For the “classical” representations of G_2 and Spin_7 (11.21).

In [S7] we prove:

(0.12) *THEOREM.* – *Let G be simple. Then irreducible representations of G satisfy the LS-alternative.*

Regarding the simplicity of $\mathcal{D}(V//G)$ we have the result 0.10(3) of Levasseur and Stafford. Recently, Van den Bergh [VdB] showed that $\mathcal{D}(V//G)$ is simple when $G = \mathbb{C}^*$ (His techniques also hold for tori, although he does not work out the details). He also considers differential operators on certain $\mathcal{O}(V)^G$ -modules of covariants. It would be interesting to extend his work to more general group actions.

(0.13) We now consider the case of differential operators on (trivial) G -vector bundles: Let V and W be G -modules and let E denote the trivial G -vector bundle $V \times W \xrightarrow{\mathrm{proj}} V$. Then $\Gamma(E)^G$, the G -invariant sections of E , is just the $\mathcal{O}(V)^G$ -module $\mathrm{Mor}(V, W)^G$ of covariants. There is a corresponding sheaf of $\mathcal{O}_{V//G}$ -modules \mathcal{E} on $V//G$, and we let $\mathcal{D}_{\mathcal{E}}(V//G) = \cup_n \mathcal{D}_{\mathcal{E}}^n(V//G)$ denote the algebra of differential operators on \mathcal{E} (see §§2–3). There is a canonical map $\pi_{V,E} : (\mathcal{D}^n(V) \otimes \mathrm{End}(W))^G \rightarrow \mathcal{D}_{\mathcal{E}}^n(V//G)$ and we are able to prove the following results.

(0.14) *THEOREM.* – *Let V , W and E be as above. Then*

(1) $\mathrm{gr}((\mathcal{D}(V) \otimes \mathrm{End}(W))^G) = (\mathrm{gr}(\mathcal{D}(V) \otimes \mathrm{End}(W)))^G$ is a finite $\mathrm{gr}(\mathcal{D}(V)^G)$ -module (3.19).

(2) *Suppose that V is 2-large. Then $(\pi_V)_*$ and $\pi_{V,E}$ are graded surjective, hence $\mathrm{gr} \mathcal{D}_{\mathcal{E}}(V//G)$ is a finite $\mathrm{gr} \mathcal{D}(V//G)$ -module, where $\mathrm{gr} \mathcal{D}(V//G)$ is finitely generated (3.20 and 9.10).*

This result is somewhat surprising. One expects that $\mathcal{D}_{\mathcal{E}}(X)$ is nasty for the general sheaf of \mathcal{O}_X -modules \mathcal{E} on an affine variety X . For quotients of G -modules which are 2-large, there are a huge number of sheaves \mathcal{E} with $\mathcal{D}_{\mathcal{E}}$ finitely generated. Theorem 0.14 generalizes to the case of G -vector bundles E over smooth affine G -varieties X , as do many of our other results. However, not all of theorem 0.7 generalizes. For $G = \mathbb{C}^*$ and $G = \mathrm{SL}_n$ we give examples of V and $E = V \times W$ such that V is coregular and $\mathrm{gr} \mathcal{D}_{\mathcal{E}}(V//G)$ is *not* finitely generated over any finitely generated commutative algebra (3.27–28). Moreover,

some of these examples just “barely” miss being 2-large. Thus, in the case of G -vector bundles, the condition of 2-largeness is close to being necessary as well as sufficient for $\text{gr } \mathcal{D}_\varepsilon(V//G)$ to be finite over a finitely generated commutative algebra.

(0.15) The contents of this paper are as follows. In §1 we recall fundamental properties of quotient spaces and Luna’s slice theorem. In §2 and §3 we discuss G -vector bundles and properties of differential operators on G -vector bundles. In §4 we reduce conjecture 0.1 to the case of representations. More specifically, we show that $(\pi_X)_*$ is (graded) surjective if and only if the analogous property holds for all the slice representations of X . In §6 we discuss the connection between the conjecture for G and G^0 . In particular, we handle the case when G is finite.

In §5 we consider properties of $(\pi_X)_*$ when $X//G$ is smooth, and §7 is devoted to considering representations which have positive dimensional principal isotropy groups. In §8 we develop homological criteria which are sufficient for graded surjectivity and which also enable us to determine the kernel $\mathcal{K}(X)^G$ of $(\pi_X)_*$. These criteria require certain collections of functions to be regular sequences in $\mathcal{O}(T^*X)$. In §9 we show that these criteria hold if X is 2-large. In §10 we consider representations of tori, and in §11 those of SL_2 and of the classical groups. In §12 we apply our results to the Nakai Conjecture.

(0.16) The results of this paper (in the case of “ordinary” differential operators) were announced in [S6]. In [S6] we attached a slightly different meaning to the term “LS-alternative.”

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1. Quotient Spaces and Stratifications

The symbol G will always denote a *reductive* complex algebraic group.

(1.1) Let X be an affine G -variety. (We will sometimes write (X, G) in place of X to emphasize the group involved. We do not assume that X is irreducible, but make a related assumption in 1.3 below.) The algebra $\mathcal{O}(X)^G$ of G -invariant polynomial functions on X is finitely generated ([Kr, II.3.2]). Let $X//G$ denote the corresponding affine variety, and let $\pi_{X,G}$ (or just π_X) denote the morphism $X \rightarrow X//G$ corresponding to the inclusion $\mathcal{O}(X)^G \subseteq \mathcal{O}(X)$.

(1.2) PROPOSITION (see [Kr, II.3.2] or [MumF, Ch. I §2]).

- (1) $\text{Im } \pi_X = X//G$.
- (2) π_X separates disjoint closed G -invariant algebraic subsets of X .
- (3) Every orbit contains a unique closed orbit in its closure, and π_X sets up a bijection between the closed orbits in X and the points of $X//G$.

(1.3) Throughout this paper we will always assume that the group G acts transitively on the irreducible components of the affine G -varieties that we consider. If X is an affine G -variety, then our assumption gives that $X = GX_0$ where X_0 is an irreducible component of X . Let G_0 denote $\{g \in G : gX_0 = X_0\}$. Then $X//G \simeq X_0//G_0$ is irreducible.

(1.4) Let $x \in X$. Then G_x denotes the *isotropy group* of G at x , and (G_x) denotes its *conjugacy class* in G , which we also call an *isotropy class* of X . We say that an isotropy class (and its conjugacy class) are *closed* if the corresponding orbit Gx is closed. Then G_x is reductive (Matsushima's theorem, *see* [Lu1] or [PS]), and there are only finitely many closed isotropy classes of X ([Lu1]). If (H) is a conjugacy class of subgroups of G , then we let $(X//G)_{(H)}$ denote the points in $X//G$ corresponding to closed orbits with isotropy class in (H) , and $X^{(H)}$ denotes its inverse image in X .

The isotropy classes are partially ordered, where $(L) \leq (M)$ if L is conjugate to a subgroup of M . Since $X//G$ is irreducible, there is a unique minimal closed isotropy class (H) , the *principal isotropy class* [Lu1]. We call H a *principal isotropy group*, and closed orbits Gx with $G_x \in (H)$ are called *principal orbits*. The subset $(X//G)_{\text{pr}} \subseteq X//G$ of principal orbits is Zariski open and dense, and $X_{\text{pr}} := (\pi_X)^{-1}((X//G)_{\text{pr}})$ is open and dense in X .

(1.5) We say that X is a *fix pointed G -variety* if all the closed G -orbits are fixed points. Equivalently, the canonical injective morphism $X^G \hookrightarrow X//G$ is an isomorphism. If X is a G -module, then clearly X is fix pointed if and only if $\{0\}$ is a principal orbit.

(1.6) Assume that X is smooth, and let Gx be a closed orbit in X . Let H denote G_x . Then the tangent space $T_x(Gx)$ is isomorphic to $\mathfrak{g}/\mathfrak{h}$, where \mathfrak{g} , \mathfrak{h} denotes the Lie algebra of G , H , respectively. Since H is reductive, we have an H -decomposition

$$T_x(X) \simeq N \oplus \mathfrak{g}/\mathfrak{h}.$$

The representation $\lambda := (N, H)$ is called the *slice representation of H at x* , or the *slice representation at x* . Luna's slice theorem (*see* below) shows that the isomorphism class of the slice representation of H is constant on components of $(X//G)_{(H)}$. We denote by $(X//G)_\lambda$ the union of the components of $(X//G)_{(H)}$ with slice representation λ . Since $X//G$ is irreducible, the principal stratum is connected, so the slice representation of a principal isotropy group is uniquely determined. In fact, the principal orbits are exactly the closed orbits whose associated slice representations are fix pointed.

(1.7) Suppose that $X = V$ is a G -module. Then the H -module N of 1.6 is uniquely determined by H , since $T_x(V) \simeq V$ as H -module. Thus the stratifications of $V//G$ by "isotropy type" (*i.e.*, by the $(V//G)_{(H)}$) and "slice type" (by the $(V//G)_\lambda$) coincide. If (L) and (M) are closed isotropy classes, then $(L) \leq (M)$ if and only if the closure of $(V//G)_{(L)}$ contains $(V//G)_{(M)}$ (*see* [S3, §5]). An important role is played by the *null cone* $\mathcal{N}_G(V) := \pi_V^{-1}(\pi_V(0))$.

(1.8) Let H be a reductive subgroup of G and Y an affine H -variety. We denote by $G *^H Y$ the quotient of $G \times Y$ by the H -action: $h(g, y) = (gh^{-1}, hy)$, $h \in H$, $g \in G$, $y \in Y$. The orbit of (g, y) is denoted $[g, y]$. Now $G *^H Y$ is a G -variety (obvious G -action), and $(G *^H Y)//G \simeq Y//H$. If Y is an H -module, then $G *^H Y$ is a G -vector bundle (*see* §2).

(1.9) Let $p : P \rightarrow Z$ be a surjective morphism of varieties. We say that p is a *fibration with fiber F* if there is an étale surjective map $\varphi : Z' \rightarrow Z$ and an isomorphism $\tilde{\varphi} : Z' \times_Z P \simeq Z' \times F$ preserving the projections to Z' . If G acts on F and on P preserving the fibers of p and $\tilde{\varphi}$ is G -equivariant, then we say that p is a *G -fibration*. If, in addition, G acts freely on the fibers of p , then $p : P \rightarrow Z$ is called a *principal G -bundle* (by

convention, one assumes that G acts on the right in this case). If p is a principal G -bundle and Y is a G -variety, then the quotient $P *^G Y$ is a fiber bundle with fiber Y , where the G -action on $P \times Y$ is given by $g(p, y) = (pg^{-1}, gy)$, $g \in G$, $p \in P$, $y \in Y$.

Suppose that H is a reductive subgroup of G . Then H acts on G on the right in the obvious way, and the quotient $G//H$ is the coset space G/H . One can easily show that $G \rightarrow G/H$ is a principal H -bundle. If Y is an affine H -variety, then the G -variety $G *^H Y$ is a G -fiber bundle over G/H .

(1.10) *Remark.* – Fiber bundles are not necessarily locally trivial in the Zariski topology, although they are in the usual Hausdorff topology. For example, the $\mathbb{Z}/2$ quotient $\mathbb{C}^* \xrightarrow{z^2} \mathbb{C}^*$ is a principal bundle, but is certainly not locally trivial in the Zariski topology. For certain G (*special groups* [Gr1]) any principal G -bundle is automatically locally trivial in the Zariski topology. For example, $SL_n(\mathbb{C})$ and $GL_n(\mathbb{C})$ are special.

(1.11) PROPOSITION ([Lu1]). – Let X be a smooth affine G -variety, and let Z denote $X//G$. Let $\lambda_i = (N_i, H_i)$ represent the slice representations of X , and let Z_i denote Z_{λ_i} , $i = 1, \dots, r$. Write N_i as a direct sum of H_i -modules: $N_i = N_i^{H_i} \oplus N_i'$. Then

(1) The Z_i are locally closed smooth subvarieties of Z .

(2) The map $X_i := \pi_X^{-1}(Z_i) \rightarrow Z_i$ is a G -fibration with fiber $G *^{H_i} \mathcal{N}_{H_i}(N_i')$.

We now present a version of Luna's slice theorem ([Lu1], [SI]).

(1.12) *Definitions.* – Let X and Y be affine G -varieties. A subset Z of X is said to be G -saturated if $Z = \pi_X^{-1}(\pi_X(Z))$. A G -morphism $\varphi : X \rightarrow Y$ is said to be *excellent* if

(1) φ is étale,

(2) the induced morphism $\varphi//G : X//G \rightarrow Y//G$ is étale, and

(3) the morphism $(\varphi, \pi_X) : X \rightarrow Y \times_{Y//G} X//G$ is an isomorphism.

(1.13) *Remark.* – If φ is excellent, then clearly it induces an isomorphism of the fibers over $\pi_X(x)$ and $\pi_Y(\varphi(x))$, $x \in X$. Thus $\varphi//G$ preserves isotropy type, and it preserves slice type if X and Y are smooth.

(1.14) THEOREM (Luna). – Let X be an affine G -variety, Gx a closed orbit, and let H denote G_x .

(1) There is a locally closed affine H -stable and H -saturated subvariety S of X containing x such that $U := G \cdot S$ is a G -saturated affine open subset of X . Moreover, the canonical G -morphism

$$\varphi : G *^H S \rightarrow U \subseteq X$$

$$[g, s] \mapsto gs$$

is excellent.

Now suppose that X is smooth at x , and let (N, H) denote the corresponding slice representation. Then

(2) S is smooth at x and the H -modules $T_x S$ and N are isomorphic. Possibly shrinking S we can arrange:

(3) *There is an excellent surjective H -morphism $\psi : S \rightarrow N_f$ which sends x to 0, inducing an excellent G -morphism*

$$\tau : G *^H S \rightarrow G *^H N_f,$$

where $f \in \mathcal{O}(N)^H$ and $f(0) \neq 0$.

(1.15) *Remark.* – Let X , etc. be as in (2) and (3). Then Luna's slice theorem says that, up to excellent maps, X is locally isomorphic to affine open subsets of G -vector bundles of the form $G *^H N_f$.

(1.16) **COROLLARY** ([Lu1]). – *Let X be an affine G -variety where G acts freely. Then $X \rightarrow X//G$ is a principal G -bundle.*

2. G -Vector Bundles

We assume that the reader is familiar with the notions of algebraic vector bundles and bundle maps. In this section we assume that X is an affine G -variety.

(2.1) *Definition.* – A G -vector bundle over X is a vector bundle E over X such that

(1) E is a G -variety.

(2) The projection $p_E : E \rightarrow X$ is G -equivariant.

(3) The elements of G act on E as vector bundle maps. In other words, for all $g \in G$ and $x \in X$, g maps the fiber E_x at x linearly to the fiber E_{gx} at gx .

We denote the sections of E over $U \subset X$ by $\Gamma(U, E)$ and abbreviate $\Gamma(X, E)$ by $\Gamma(E)$. The functor $U \mapsto \Gamma(U, E)$ is a coherent sheaf of \mathcal{O}_X -modules, which we denote by \underline{E} . There is also the coherent sheaf \mathcal{E} of $\mathcal{O}_{X//G}$ -modules associated to E , where $\Gamma(U, \mathcal{E}) = \Gamma(\pi_X^{-1}(U), E)^G$, $U \subset X//G$ an open set.

Given G -vector bundles E and F over X , we have natural G -vector bundle structures on $\text{Hom}(E, F)$, $E \otimes F$, etc.

(2.2) We call a G -vector bundle E over X *trivial* if it is isomorphic to a G -vector bundle $\Theta_W := X \times W \rightarrow X$ where W is a G -module and the G -action is diagonal. Note that the G -invariant sections of Θ_W are isomorphic to $\text{Mor}(X, W)^G$, the $\mathcal{O}(X)^G$ -module of covariants of type W^* (usually W is assumed irreducible, but we will not require this). We use $\mathbf{1}_X$ (or $\mathbf{1}_{X,G}$) to denote the trivial bundle, *i.e.*, the trivial G -bundle whose fibers are isomorphic to \mathbb{C} with trivial G -action.

While not all G -vector bundles are locally G -isomorphic to trivial G -bundles, they do have a nice local form.

(2.3) **LEMMA.** – *Let E be a G -vector bundle over X , and let Gx be a closed orbit. Choose a slice S at x as in 1.14, so that there is an excellent map $\varphi : G *^H S \rightarrow U \subseteq X$, $H = G_x$. Set $W := E_x$, an H -module. Then, after perhaps shrinking S , we have an isomorphism of G -vector bundles $\varphi^*(E) \simeq (G *^H (S \times W) \xrightarrow{\text{proj}} G *^H S)$.*

Proof. – Since $E|_{Gx} \simeq G *^H W$, we have a G -isomorphism Φ of $G *^H (S \times W)$ and $\varphi^*(E)$ defined over the closed G -invariant subset $G *^H \{x\} \subset G *^H S$. Then Φ extends to a morphism (also called Φ) of vector bundles over $G *^H S$, and applying the Reynold's

operator, we may assume that Φ is G -invariant. Since Φ is an isomorphism on Gx , it is a G -isomorphism on a G -neighborhood of Gx , which we can assume to be all of $G *^H S$. \square

If E is a G -vector bundle over a G -module V , then there is an open cover $\{U_\alpha\}$ of $V//G$ such that $E|_{\pi_V^{-1}(U_\alpha)}$ is trivial for all α ([Kr2]). However, E may fail to be trivial ([S5]).

(2.4) There is a 1-1 correspondence between vector bundles over X of fiber dimension n and principal GL_n -bundles over X . If $E \rightarrow X$ is a vector bundle, then the associated principal bundle $P_E \rightarrow X$ has fibers $(P_E)_x = \{\text{bases of } E_x\}$, $x \in X$. Given a principal GL_n -bundle P , then $P *^{GL_n} \mathbb{C}^n$ is the associated vector bundle. If E is a G -vector bundle, then G acts on P_E such that $P_E \rightarrow X$ is equivariant, and the actions of G and GL_n on P_E commute.

Suppose that \tilde{E} is a vector bundle on $X//G$. Then $E := \pi_X^*(\tilde{E})$ is a G -vector bundle on X such that G_x acts trivially on E_x for every closed orbit Gx in X . Conversely, we have

(2.5) PROPOSITION ([Kr2]). – Let E be a G -vector bundle on X such that G_x acts trivially on E_x for every closed orbit Gx in X . Then $E \simeq \pi_X^*(\tilde{E})$ for some vector bundle \tilde{E} on $X//G$.

Proof. – Let $\varphi : P_E \rightarrow X$ be the principal bundle of E . Then $\varphi//G : P_E//G \rightarrow X//G$ is the quotient by GL_n . The condition on E assures that GL_n acts freely, hence $\varphi//G$ is a principal GL_n -bundle, and $\tilde{E} := (P_E//G) *^{GL_n} \mathbb{C}^n$ is the required vector bundle on $X//G$.

(2.6) Definition. – Let E be a G -vector bundle over X . We say that E is *admissible* if G_x acts trivially on E_x whenever Gx is a principal orbit. (Recall that principal orbits are closed.)

(2.7) Remarks.

- (1) 1_X is always admissible.
- (2) If X has trivial principal isotropy groups, then all G -vector bundles on X are admissible.
- (3) E is admissible if and only if $\Gamma(E)$ evaluated at x spans E_x for every principal orbit Gx .
- (4) E is admissible if and only if $E|_{X_{\text{pr}}}$ is the pull-back of a vector bundle on $(X//G)_{\text{pr}}$.
- (5) E is admissible if and only if $E|_{X_{\text{pr}}}$ is locally G -isomorphic to a trivial bundle $X_{\text{pr}} \times \mathbb{C}^r \rightarrow X_{\text{pr}}$, where G acts trivially on \mathbb{C}^r .

Suppose that $X = V$ is a G -module and that $E = \Theta_W$ is a trivial G -bundle. Let H be a principal isotropy group of V . Then

- (6) E is admissible if and only if H acts trivially on W .
- (7) If G is finite, then $H = \text{Ker}(G \rightarrow GL(V))$.
- (8) (Exercise in using the Luna-Richardson Theorem ([LR]), cf. 7.2). If E is admissible, then $\Gamma(E)^G = \text{Mor}(V, W)^G \simeq \text{Mor}(V^H, W)^{N_G(H)/H}$.

“Most” affine G -varieties have trivial principal isotropy groups, so admissibility usually holds. We found it difficult to make meaningful statements about differential operators on sections of nonadmissible G -vector bundles (see 3.23, 5.4). In a very few cases, however, we can work around the problem of nonadmissible bundles (see 5.2–3, 6.7, 7.10). The reader who prefers to consider only differential operators on functions can always assume that $E = 1_X$.

3. Differential Operators

We recall the basic definitions and properties of algebras of differential operators on sheaves on a variety X ([Le1], [Bj], [Gr2], [SmSt] and [Sw] are references for what follows). We then consider some properties of differential operators on G -vector bundles over G -varieties.

(3.1) Let A be a localization of a finitely generated commutative \mathbb{C} -algebra, and let M and N be A -modules. If $P \in \text{Hom}_{\mathbb{C}}(M, N)$ and $a \in A$, then $[P, a]$ denotes the usual commutator: $[P, a](m) = P(am) - a(P(m))$, $m \in M$. Define $D_A^n(M, N) = 0$ for $n < 0$, and for $n \geq 0$ inductively define:

$$D_A^n(M, N) = \{P \in \text{Hom}_{\mathbb{C}}(M, N) : [P, a] \in D_A^{n-1}(M, N) \text{ for all } a \in A\}.$$

Clearly, $D_A^0(M, N) = \text{Hom}_A(M, N)$. An element of $D_A^n(M, N)$ is called a *differential operator from M to N of order at most n* . An element of $D_A^n(M, N) \setminus D_A^{n-1}(M, N)$ is said to have order (exactly) n . Note that $D_A^n(M, N) \subseteq D_A^{n+1}(M, N)$ for all n . We call $D_A(M, N) := \bigcup D_A^n(M, N)$ the *differential operators from M to N* . We set $D_A^n(M) := D_A^n(M, M)$ and $D_A(M) := D_A(M, M)$.

Let $P \in D_A^n(M, N)$ and let $a \in A$. Then aP and Pa are in $D_A^n(M, N)$, where $(aP)(m) := a(P(m))$, $(Pa)(m) := P(am)$, $m \in M$. We call the action $P \mapsto aP$ (resp. $P \mapsto Pa$) the left (resp. right) action of A on $D_A^n(M, N)$, and we speak of the left (resp. right) A -module structures on $D_A^n(M, N)$ and $D_A(M, N)$. We always use the left A -module structure.

(3.2) PROPOSITION. – *Let M, N and R be A -modules.*

- (1) (see 3.4 below). *If M and N are finite A -modules, then so is each $D_A^n(M, N)$.*
- (2) ([Gr2, 16.8.9]). *If $P \in D_A^n(M, N)$ and $Q \in D_A^m(N, R)$, then $Q \circ P \in D_A^{n+m}(M, R)$.*
- (3) ([Le1, I.2]). *If $M = N = R = A$, then $[Q, P] := Q \circ P - P \circ Q \in D_A^{n+m-1}(A)$.*

From 3.2(2) we see that $D_A(M)$ is a \mathbb{C} -algebra, called the *algebra of differential operators on M* .

(3.3) As in the case of smooth manifolds, there are universal differential operators of order n . Set $\tilde{A} := A \otimes_{\mathbb{C}} A$ and $\tilde{M} := A \otimes_{\mathbb{C}} M$. Give \tilde{M} the \tilde{A} -module structure such that $(a \otimes a')(a'' \otimes m) = aa'' \otimes a'm$; $a, a', a'' \in A$, $m \in M$. Let I_A denote the kernel of the multiplication mapping $\tilde{A} \rightarrow A$ sending $a \otimes a' \mapsto aa'$, $a, a' \in A$. Define $P_{A,M}^n := \tilde{M}/I_A^{n+1}\tilde{M}$ and define $j_{A,M}^n : M \rightarrow P_{A,M}^n$ by the formula: $j_{A,M}^n(m) = 1 \otimes m + I_A^{n+1}\tilde{M}$. We give $P_{A,M}^n$ the A -module structure induced by multiplication on the first factor of $\tilde{M} = A \otimes_{\mathbb{C}} M$. We denote $P_{A,A}^n$ by P_A^n and $j_{A,A}^n$ by j_A^n .

(3.4) PROPOSITION. – *Let $j_{A,M}^n : M \rightarrow P_{A,M}^n$ be as above. Then*

- (1) ([Gr2, 16.7.3]). *If M is a finite A -module, then so is $P_{A,M}^n$.*
- (2) ([Gr2, 16.8.2]). $j_{A,M}^n \in D_A^n(M, P_{A,M}^n)$.
- (3) ([Gr2, 16.8.4]). $j_{A,M}^n$ is universal, i.e., *if N is an A -module and $Q \in D_A^n(M, N)$, then there is a unique $q \in \text{Hom}_A(P_{A,M}^n, N)$ such that $Q = q \circ j_{A,M}^n$. In other words, $q \mapsto q \circ j_{A,M}^n$ induces an isomorphism of $\text{Hom}_A(P_{A,M}^n, N)$ with $D_A^n(M, N)$.*

(3.5) PROPOSITION. – Suppose that (A, \mathfrak{M}_A) is local with $A/\mathfrak{M}_A = \mathbb{C}$. Let M be a finite A -module. Then

$$(1) \quad \begin{aligned} M/\mathfrak{M}_A^{n+1}M &\simeq (A/\mathfrak{M}_A) \otimes_{\mathbb{C}} M/\mathfrak{M}_A^{n+1}M \rightarrow P_{A,M}^n/\mathfrak{M}_A P_{A,M}^n \\ m + \mathfrak{M}_A^{n+1}M &\mapsto 1 \otimes (m + \mathfrak{M}_A^{n+1}M) \mapsto (1 \otimes m + \mathfrak{M}_A P_{A,M}^n) \end{aligned}$$

is an isomorphism of vector spaces over $\mathbb{C} = A/\mathfrak{M}_A$.

$$(2) \quad D_A^n(M, A/\mathfrak{M}_A) \simeq (M/\mathfrak{M}_A^{n+1}M)^*.$$

Proof. – As A -module, I_A is generated by elements $a \otimes 1 - 1 \otimes a$, where $a \in \mathfrak{M}_A$. Thus, modulo $\mathfrak{M}_A \otimes_{\mathbb{C}} M$, $I_A^{n+1}M$ is generated by $1 \otimes_{\mathbb{C}} \mathfrak{M}_A^{n+1}M$, and

$$\begin{aligned} P_{A,M}^n/\mathfrak{M}_A P_{A,M}^n &\simeq (A \otimes_{\mathbb{C}} M)/(\mathfrak{M}_A \otimes_{\mathbb{C}} M + A \otimes_{\mathbb{C}} \mathfrak{M}_A^{n+1}M) \\ &\simeq A/\mathfrak{M}_A \otimes_{\mathbb{C}} M/\mathfrak{M}_A^{n+1}M, \end{aligned}$$

giving (1). Part (2) follows from 3.4(3). \square

(3.6) All of the notions above localize nicely: Let S be a multiplicative subset of A , and let M_S denote $S^{-1}M$. Then $j_{A,M}^n : M \rightarrow P_{A,M}^n$ canonically gives rise to a differential operator $(j_{A,M}^n)_S : M_S \rightarrow A_S \otimes_A P_{A,M}^n$ which can be identified with $j_{M_S}^n : M_S \rightarrow P_{A_S, M_S}^n$ (see [Sw, §13]). In particular, the P_{A_S, M_S}^n give rise to a sheaf on $\text{Spec } A$ which is canonically identified with the sheaf corresponding to the A -module $P_{A,M}^n$. Let N be an A -module, $q \in \text{Hom}_A(P_{A,M}^n, N)$ and $Q = q \circ j_{A,M}^n \in D_A^n(M, N)$. Set

$$q_S := \text{id} \otimes q : A_S \otimes_A P_{A,M}^n \simeq P_{A_S, M_S}^n \rightarrow A_S \otimes_A N,$$

and set $Q_S := q_S \circ j_{A_S, M_S}^n \in D_{A_S}^n(M_S, N_S)$. The homomorphism $Q \mapsto Q_S$ gives rise to an isomorphism of $A_S \otimes_A D_A^n(M, N)$ with $D_{A_S}^n(M_S, N_S)$.

(3.7) Because of 3.6, one can define differential operators on varieties; we briefly sketch the definitions. Let X be a variety and let \mathcal{F} be a coherent sheaf of \mathcal{O}_X -modules. Let $\tilde{\mathcal{F}}$ denote the tensor product $\mathcal{O}_X \otimes_{\mathbb{C}} \mathcal{F}$ with the obvious $\mathcal{O}_X \otimes_{\mathbb{C}} \mathcal{O}_X \simeq \mathcal{O}_{X \times X}$ -module structure. Let \mathcal{I}_X denote the sheaf of $\mathcal{O}_{X \times X}$ -ideals of the diagonal $X \rightarrow X \times X$. For each $n \geq 0$, define $\mathcal{P}_{\mathcal{F}}^n$ to be the quotient $\tilde{\mathcal{F}}/(\mathcal{I}_X)^{n+1}\tilde{\mathcal{F}}$. We give $\mathcal{P}_{\mathcal{F}}^n$ the \mathcal{O}_X -module structure induced by multiplication on the first factor of $\mathcal{O}_X \otimes_{\mathbb{C}} \mathcal{F}$. Define $\mathcal{D}_{\mathcal{F}}^n$ to be $\text{Hom}_{\mathcal{O}_X}(\mathcal{P}_{\mathcal{F}}^n, \mathcal{F})$.

Let U be an affine open subset of X . Set $A := \mathcal{O}_X(U)$ and $M := \Gamma(U, \mathcal{F})$. Then there are canonical isomorphisms of $\Gamma(U, \mathcal{D}_{\mathcal{F}}^n)$ with $D_A^n(M)$, $n \geq 0$. Applying the “local” results in 3.1–3.6 we see that $\mathcal{D}_{\mathcal{F}} := \bigcup \mathcal{D}_{\mathcal{F}}^n$ is an \mathcal{O}_X -algebra, the *sheaf of differential operators on \mathcal{F}* . The \mathcal{O}_X -modules $\mathcal{D}_{\mathcal{F}}^n$ are coherent, while $\mathcal{D}_{\mathcal{F}}$ is quasi-coherent. Let $\mathcal{D}_{\mathcal{F}}^n(X) = \Gamma(X, \mathcal{D}_{\mathcal{F}}^n)$ and $\mathcal{D}_{\mathcal{F}}(X) = \bigcup \mathcal{D}_{\mathcal{F}}^n(X)$ denote the global sections. If $\mathcal{F} = \mathcal{O}_X$, then we use the notation $\mathcal{D}^n(X)$, $\mathcal{D}(X)$, etc.

(3.8) EXAMPLE. – Let $A := \mathcal{O}(\mathbb{C}^k) = \mathbb{C}[x_1, \dots, x_k]$. Then $D(A)$ is the *k*th Weyl algebra, i.e., the noncommutative algebra $\mathbb{C}\langle x_1, \dots, x_k, \partial_1, \dots, \partial_k \rangle$ generated by the x_i and the $\partial_j := \partial/\partial x_j$. Note that $\text{gr } D(A) \simeq \mathbb{C}[x_1, \dots, x_k, y_1, \dots, y_k]$ is a polynomial ring. If $\alpha = (\alpha_1, \dots, \alpha_k) \in \mathbb{N}^k$, let $|\alpha|$ denote $\sum_i \alpha_i$, let $\alpha!$ denote $\alpha_1! \cdots \alpha_k!$, let x^α denote

$x_1^{\alpha_1} \cdots x_k^{\alpha_k}$ and let ∂^α denote $\partial_1^{\alpha_1} \cdots \partial_k^{\alpha_k}$. Then every element $Q \in D^n(A)$ is a sum $\sum_{|\alpha| \leq n} a_\alpha \partial^\alpha$ where the a_α are in A . The (n th order) symbol of Q is $\sum_{|\alpha|=n} a_\alpha \partial^\alpha$.

We now describe P_A^n , etc. Let $x \in \mathbb{C}^k$ and let \mathfrak{M}_x denote the corresponding maximal ideal of A . Then, by 3.5–6, we can identify $P_A^n / \mathfrak{M}_x P_A^n$ with $J_x^n := A / \mathfrak{M}_x^{n+1} = \{n$ th order Taylor series at x of elements of $A\}$. Thus P_A^n is the module of sections of the trivial bundle J^n over \mathbb{C}^k whose fibers J_x^n are all isomorphic to $\mathbb{C} \oplus (\mathbb{C}^k)^* \oplus \cdots \oplus S^n(\mathbb{C}^k)^*$. If $f \in A$, then $j_A^n(f)$ is the section of J^n whose value at x is the n th order Taylor series of f at x . If $q \in \text{Hom}_A(P_A^n, A)$, then q is uniquely determined by the values $a_\alpha(v)$, $v \in \mathbb{C}^k$, $|\alpha| \leq n$, where $(\alpha!) a_\alpha(v) = q(j_A^n((x-v)^\alpha))(v)$. The corresponding differential operator $Q = q \circ j_A^n$ is, as expected, $\sum_{\alpha \leq n} a_\alpha \partial^\alpha$.

(3.9) *Remarks.* (1) Suppose that $B = A/I$ where I is an ideal in $A := \mathcal{O}(\mathbb{C}^k)$. Let $P \in D^n(B)$. Then the composition $A \rightarrow B \xrightarrow{P} B$ is a differential operator $P' = \sum_{|\alpha| \leq n} a_\alpha \partial^\alpha$ where $P'(I) = 0$ and the a_α are in B . Lifting the a_α to A we obtain $Q \in D^n(A)$ which induces P . It follows that $D^n(B)$ is the quotient of $\{Q \in D^n(A) : Q(I) \subset I\}$ by $D_A^n(A, I) = I \cdot D^n(A)$.

(2) Let $F = A^n$ be a free A -module and N a submodule. Then, as above, one shows that $D_A^n(F/N)$ is the quotient of $\{Q \in D_A^n(F) : Q(N) \subset N\}$ by $D_A^n(F, N) = \text{Hom}_A(F, N) D_A^n(F)$.

(3.10) *EXAMPLE (T. Stafford).* – Let $A = \mathbb{C}[x, y]$ and let $M = xA + yA$, the homogeneous maximal ideal. We show that $D_A(M)$ is not left noetherian: Let $Q \in D_A(M)$. Since $M_x = A_x$ (localization), Q extends to a differential operator $Q_x \in D(A_x)$ which preserves $M \subset A_x$. Similarly Q gives rise to $Q_y \in D(A_y)$ preserving M . Write $Q_x = \sum_{\alpha} a_\alpha \partial^\alpha$, $a_\alpha \in A_x$ and $Q_y = \sum_{\alpha} b_\alpha \partial^\alpha$, $b_\alpha \in A_y$. Since $Q_x = Q_y$ on $M_{xy} = A_{xy}$, we have $a_\alpha = b_\alpha \in A$ for all α . Thus Q is simply an element of $D(A)$ which preserves $M \subset A$. Hence $D_A(M) = \mathbb{C} + xD(A) + yD(A) \subset D(A)$. By Resco [Re], $D_A(M)$ is finitely generated and right noetherian, but not left noetherian. In fact, $D_A(M)$ is generated by $D_A^3(M)$ and the left ideals I_k generated by $x(\partial/\partial y)^i$, $0 \leq i \leq k$, form an increasing sequence which does not stabilize.

(3.11) We now consider the BGG example: Let X be $\{x^3 + y^3 + z^3 = 0\} \subseteq \mathbb{C}^3$. Then $\mathcal{O}(X) =: A = \sum_{n=0}^{\infty} A_n$ is a graded algebra. Let $\mathcal{D}_j^k(X)$ denote the elements of $\mathcal{D}^k(X)$ which send elements of A_n to A_{n+j} for all n . Then in [BGG] one finds a proof that:

- (1) $\mathcal{D}_j^k(X) = 0$ for $j < 0$.
- (2) $\mathcal{D}(X)$ has an infinite ascending chain of two-sided ideals.
- (3) $\mathcal{D}(X)$ is not generated by $\mathcal{D}^k(X)$ for any k .

(3.12) *Remarks.* – (1) If $Y = V//G$ is a quotient of a G -module V , then for every $0 \neq f \in \mathcal{O}(Y)$, there is a $Q \in \mathcal{D}(Y)$ such that $Q(f) = 1$: Since $f \in S^\bullet(V^*)^G$, we may

choose a dual element $P \in S^\bullet(V)^G \subset \{\text{constant coefficient differential operators}\}$ so that $P(f) = 1$. Set $Q = (\pi_V)_*(P)$.

(2) (T. Bloom) The BGG variety X fails to have the property in (1), because of 3.11(1). Thus X cannot be a quotient. It is known that X does not have rational singularities, so it cannot be a quotient for this reason. However, one can modify the BGG example to have rational singularities ([LS, 0.13]).

We now consider differential operators on sections of vector bundles.

(3.13) Let E be a vector bundle on X . We have the associated coherent sheaf \underline{E} of \mathcal{O}_X -modules. We will usually use the notation \mathcal{D}_E^n for $\mathcal{D}_{\underline{E}}^n$ and \mathcal{D}_E for $\mathcal{D}_{\underline{E}}$. Note that the zeroth order differential operators on E are just $\Gamma(\text{End}(\underline{E}))$.

(3.14) Assume that X is smooth, let $x \in X$ and let $x_1, \dots, x_n \in \mathcal{O}(X)$ generate \mathfrak{M}_x , the maximal ideal of $\mathcal{O}_{X,x}$, where $n = \dim X$. Then \mathcal{D}_x^1 (the germs at x of differential operators of order at most 1) is freely generated over $\mathcal{O}_{X,x}$ by the function 1 and the vector fields $\partial/\partial x_1, \dots, \partial/\partial x_n$, and \mathcal{D}_x^n is freely generated by all monomials in the $\partial/\partial x_i$ of degree at most n ([Gr2]). Thus $\mathcal{D}_x^n/\mathcal{D}_x^{n-1} \simeq \Gamma(X, S^n(TX))_x$.

This generalizes to the case of differential operators on a vector bundle E on X . We have the exact *symbol sequence*

$$0 \rightarrow \mathcal{D}_E^{n-1}(X) \rightarrow \mathcal{D}_E^n(X) \xrightarrow{\sigma_E^n} \mathcal{O}(T^*X)_n \otimes_{\mathcal{O}(X)} \Gamma(\text{End}(E)) \rightarrow 0,$$

where $\mathcal{O}(T^*X)_n \simeq \Gamma(X, S^n(TX))$ denotes the elements of $\mathcal{O}(T^*X)$ homogeneous of degree n with respect to the scalar action of \mathbb{C}^* on T^*X . The homomorphisms σ_E^n are called the *symbol maps*. If $P \in \mathcal{D}_E^n(X)$, $e \in E_x$ and $\xi \in T_x^*X$, the value of $\sigma_E^n(P)(\xi)(e)$ can be computed as follows: Choose $s \in \Gamma(E)$ such that $s(x) = e$, choose $f \in \mathcal{O}(X)$ such that $f(x) = 0$ and $df(x) = \xi$. Then $\sigma_E^n(P)(\xi)(e) = P(f^n s)(x)$. The symbol map gives an isomorphism of $\text{gr } \mathcal{D}_E(X)$ with $\mathcal{O}(T^*X) \otimes_{\mathcal{O}(X)} \Gamma(\text{End}(E))$.

(3.15) Let Y and X be affine varieties, and let E be a vector bundle on $Y \times X$. Then $\mathcal{D}_E(Y \times X)$ has a bifiltration $\{\mathcal{D}_E^{n,m}(Y \times X)\}$, where the (n, m) th subspace consists of the elements $P \in \mathcal{D}_E^{n+m}(Y \times X)$ such that any $(n + 1)$ -fold (resp. $(m + 1)$ -fold) commutator of P with elements of $\mathcal{O}(Y)$ (resp. $\mathcal{O}(X)$) is zero.

(3.16) LEMMA. – Let Y and X be affine, let E be a vector bundle on X , and let p_1 (resp. p_2) denote projection onto the first (resp. second) factor of $Y \times X$. Then

(1) $\mathcal{O}(Y) \otimes_{\mathbb{C}} \mathcal{D}_E^n(X) \simeq \mathcal{D}_{p_2^*(E)}^{0,n}(Y \times X)$.

(2) There is a projection of $\mathcal{O}(Y \times X)$ -modules, $\rho : \mathcal{D}_{p_2^*(E)}^n(Y \times X) \rightarrow \mathcal{D}_{p_2^*(E)}^{0,n}(Y \times X)$, where $\rho(P)(f \otimes s) = p_1^*(f) \cdot P(p_2^*(s))$, $P \in \mathcal{D}_{p_2^*(E)}^n(Y \times X)$, $s \in \Gamma(X, E)$, $f \in \mathcal{O}(Y)$.

(3) ρ is a left inverse to the natural inclusion $\mathcal{D}_{p_2^*(E)}^{0,n}(Y \times X) \subset \mathcal{D}_{p_2^*(E)}^n(Y \times X)$.

Proof. – Set $A := \mathcal{O}(Y)$, $B := \mathcal{O}(X)$, $M' := \Gamma(X, E)$, $R := A \otimes_{\mathbb{C}} B$ and $M := A \otimes_{\mathbb{C}} M' \simeq R \otimes_B M'$. Set $D_R^{0,n}(M) := \{Q \in D_R^n(M) : [Q, a] = 0 \text{ for all } a \in A\}$. Set $\tilde{M} := R \otimes_{\mathbb{C}} M$ (resp. $\tilde{M}' = B \otimes_{\mathbb{C}} M'$) with $(R \otimes_{\mathbb{C}} R)$ -module (resp. $(B \otimes_{\mathbb{C}} B)$ -module) structure as in 3.3. There is a canonical R -module mapping

$$R \otimes_B P_{B,M'}^n \xrightarrow{\alpha} P_{R,M}^n$$

$$(a \otimes b) \otimes (b' \otimes m' + (I_B^{n+1} \tilde{M}')) \mapsto (a \otimes bb') \otimes (1 \otimes m') + I_R^{n+1} \tilde{M}.$$

Composition with α gives an R -module morphism

$$\begin{aligned} \tilde{\rho} : D_R^n(M) &\simeq \text{Hom}_R(P_{R,M}^n, M) \rightarrow \text{Hom}_R(R \otimes_B P_{B,M'}^n, R \otimes_B M') \\ &\simeq A \otimes_{\mathbb{C}} \text{Hom}_B(P_{B,M'}^n, M') \simeq A \otimes_{\mathbb{C}} D_B^n(M'). \end{aligned}$$

The projection ρ of (2) is just $\tilde{\rho}$ followed by the inclusion

$$\mathcal{O}(Y) \otimes_{\mathbb{C}} \mathcal{D}_E^n(X) \simeq A \otimes_{\mathbb{C}} D_B^n(M') \hookrightarrow D_R^{0,n}(M) \simeq \mathcal{D}_{p_2^*(E)}^{0,n}(Y \times X).$$

Since ρ is the identity on $\mathcal{D}_{p_2^*(E)}^{0,n}(Y \times X)$, we have (1), (2) and (3). \square

(3.17) Suppose that $\Phi : E \rightarrow E$ is a vector bundle isomorphism over $\varphi : X \rightarrow X$. Then Φ induces an isomorphism Φ_* on $\Gamma(E)$, where $\Phi_*(fs) = (f \circ \varphi^{-1})(\Phi \circ s \circ \varphi^{-1})$, $f \in \mathcal{O}(X)$, $s \in \Gamma(E)$. On $\mathcal{D}_E^n(X)$ we have the isomorphism $\Phi_{\#}$, where $\Phi_{\#}(f \cdot Q) = (f \circ \varphi^{-1}) \cdot (\Phi_* \circ Q \circ \Phi_*^{-1})$, $f \in \mathcal{O}(X)$, $Q \in \mathcal{D}_E^n(X)$.

We now suppose that E is a G -vector bundle over the affine G -variety X . The action of each $g \in G$ gives a bundle isomorphism τ_g over $g : X \rightarrow X$, and we define gQ for $Q \in \mathcal{D}_E^m(X)$ to be $(\tau_g)_{\#}Q$. Then $(gh)Q = g(hQ)$, hence $\mathcal{D}_E^m(X)$ and $\mathcal{D}_E(X)$ are G -modules.

(3.18) PROPOSITION. – Let E be as above. Then $\mathcal{D}_E^m(X)$, $m \geq 0$, is a locally finite G -module. That is, $\mathcal{D}_E^m(X)$ is the union of finite dimensional rational (i.e. algebraic) G -modules.

Proof. – Let $p_2 : G \times X \rightarrow X$ be projection. Consider the isomorphism $\varphi : G \times X \rightarrow G \times X$, $(g, x) \mapsto (g, gx)$. Then there is a vector bundle isomorphism $\Phi : p_2^*E \rightarrow p_2^*E$ over φ , sending $\{g\} \times E_x$ to $\{g\} \times E_{gx}$ via the action of g . Let $Q \in \mathcal{D}_E^m(X)$. Then $1 \otimes Q \in \mathcal{O}(G) \otimes \mathcal{D}_E^m(X) \simeq \mathcal{D}_{p_2^*(E)}^{0,m}(G \times X)$, and $\rho(\Phi_{\#}(1 \otimes Q)) \in \mathcal{O}(G) \otimes_{\mathbb{C}} \mathcal{D}_E^m(X)$. Unwinding the definitions, we see that there are $f_i \in \mathcal{O}(G)$ and $Q_i \in \mathcal{D}_E^m(X)$, $i = 1, \dots, r$, such that $gQ = \sum_{i=1}^r f_i(g)Q_i$. Hence $\mathcal{D}_E^m(X)$ is a locally finite G -module. \square

(3.19) THEOREM (cf. [Bj, Ch. 3], [MR, §15]). – Let X be a smooth affine G -variety and E a G -vector bundle on X . Then

- (1) $\text{gr}(\mathcal{D}(X)^G) = (\text{gr} \mathcal{D}(X))^G$ is a finitely generated commutative \mathbb{C} -algebra.
- (2) $\text{gr}(\mathcal{D}_E(X)^G) = (\text{gr} \mathcal{D}_E(X))^G$ is a finitely generated $\text{gr}(\mathcal{D}(X)^G)$ -module.

Proof. – Since X is smooth, $\text{gr} \mathcal{D}_E(X) \simeq \mathcal{O}(T^*X) \otimes_{\mathcal{O}(X)} \Gamma(\text{End}(E))$ is a finite $\mathcal{O}(T^*X)$ -module, where $\mathcal{O}(T^*X) \simeq \text{gr} \mathcal{D}(X)$ is finitely generated commutative. We get the analogous result for $\text{gr}(\mathcal{D}_E(X)^G)$: Reductivity of G shows that

- (a) $\text{gr}(\mathcal{D}_E(X)^G) \simeq (\text{gr} \mathcal{D}_E(X))^G \simeq (\mathcal{O}(T^*X) \otimes_{\mathcal{O}(X)} \Gamma(\text{End}(E)))^G$.
- (b) $(\mathcal{O}(T^*X) \otimes_{\mathcal{O}(X)} \Gamma(\text{End}(E)))^G$ is a finite $\mathcal{O}(T^*X)^G \simeq (\text{gr} \mathcal{D}(X))^G$ -module. \square

From now on we use $\text{gr} \mathcal{D}(X)^G$ as shorthand for $\text{gr}(\mathcal{D}(X)^G) = (\text{gr} \mathcal{D}(X))^G$, and similarly for $\text{gr} \mathcal{D}_E(X)^G$.

(3.20) PROPOSITION. – Let X be an affine G -variety and E a G -vector bundle on X .

(1) *The restriction map*

$$\mathcal{D}_E(X)^G \ni Q \mapsto Q|_{\Gamma(E)^G = \Gamma(\mathcal{E})}$$

induces a homomorphism $\pi_{X,E} : \mathcal{D}_E(X)^G \rightarrow \mathcal{D}_\mathcal{E}(X//G)$ which preserves the filtrations by order of differentiation.

(2) If X is smooth and $\pi_{X,E}(\mathcal{D}_E^n(X)^G) = \mathcal{D}_\mathcal{E}^n(X//G)$ for every n (we say that $\pi_{X,E}$ is graded surjective), then $\text{gr } \mathcal{D}_\mathcal{E}(X//G)$ is finite over $\text{gr } \mathcal{D}(X)^G$.

(3) If X is smooth and both $\pi_{X,E}$ and $(\pi_X)_* = \pi_{X,1_X}$ are graded surjective, then $\text{gr } \mathcal{D}_\mathcal{E}(X//G)$ is in a natural way a finite $\text{gr } \mathcal{D}(X//G)$ -module.

Proof. – Parts (1) and (2) are obvious. Let $P \in \mathcal{D}^n(X)^G$ and let $P_E \in \mathcal{D}_E^n(X)^G$ have symbol $\sigma^n(P) \otimes \text{id}_E$. Suppose that $(\pi_X)_*P \in \mathcal{D}^{n-1}(X//G)$. To establish (3), we must show that $\pi_{X,E}P_E \in \mathcal{D}_\mathcal{E}^{n-1}(X//G)$. This is obvious if $n = 0$. If $n > 0$, let $f \in \mathcal{O}(X)^G$. Then $\sigma_E^{n-1}([P_E, f]) = \sigma^{n-1}([P, f]) \otimes \text{id}_E$. Since $(\pi_X)_*[P, f] = [(\pi_X)_*P, f] \in \mathcal{D}^{n-2}(X//G)$, induction gives that $\pi_{X,E}[P_E, f] = [\pi_{X,E}P_E, f] \in \mathcal{D}_\mathcal{E}^{n-2}(X//G)$. Since f is arbitrary, $\pi_{X,E}P_E \in \mathcal{D}_\mathcal{E}^{n-1}(X//G)$. \square

(3.21) Let E, \mathcal{E} , etc. be as above, and let $\mathcal{K}_E^n(X)$ denote the elements in $\mathcal{D}_E^n(X)$ which act trivially on $\Gamma(E)^G$. Set $\mathcal{K}_E(X) = \bigcup \mathcal{K}_E^n(X)$. Then $\mathcal{K}_E(X)^G$ is the kernel of $\pi_{X,E} : \mathcal{D}_E(X)^G \rightarrow \mathcal{D}_\mathcal{E}(X//G)$, and similarly for $\mathcal{K}_E^n(X)^G$. In order to show that $\pi_{X,E} : \mathcal{D}_E^n(X)^G \rightarrow \mathcal{D}_\mathcal{E}^n(X//G)$ is surjective, it is obviously very useful to have a good description of $\mathcal{K}_E^n(X)$. Assume that G acts almost faithfully on E . Then we may consider \mathfrak{g} as a subspace of $\mathcal{K}_E^1(X)$ (by differentiating the action of G on $\Gamma(E)$), and clearly $\mathcal{D}_E^{n-1}(X)\mathfrak{g} \subseteq \mathcal{K}_E^n(X)$. In §8 and §9 we find conditions guaranteeing equality when X is smooth.

(3.22) *Remarks.* – Let $A \in \mathfrak{g}$.

(1) An easy calculation shows that $\sigma_E^1(A) = \sigma^1(A) \otimes \text{id} \in \Gamma(TX \otimes \text{End}(E))$, where $\sigma^1(A)$ is the symbol of A as a differential operator on functions.

(2) If x lies on a principal orbit and E is admissible, then we may choose $s_1, \dots, s_r \in \Gamma(E)^G$ such that they give a basis of sections of E near x . If s is a section of E near x , then $s = \sum_{i=1}^r f_i s_i$ for some functions f_i , and $A(s) = \sum_{i=1}^r A(f_i) s_i$. In other words, near x , A can be identified with its symbol.

(3.23) *Remarks.* – Let E be a G -vector bundle on X .

(1) Let Gx be a principal orbit, and write $E_x = E_x^{Gx} \oplus E'_x$ as G_x -module. Then $\text{End}(E_x)^{G_x} = \text{End}(E_x^{G_x}) \oplus \text{End}(E'_x)^{G_x}$, and $\mathcal{K}_E^0(X)^G$ evaluated at x is isomorphic to $\text{End}(E'_x)^{G_x}$. The latter is nonzero if and only if $E' \neq 0$. It follows that $\mathcal{K}_E^0(X)^G = 0$ if and only if E is admissible. Hence, if E is not admissible, one cannot hope to have $(\mathcal{D}_E^{n-1}(X)\mathfrak{g})^G = \mathcal{K}_E^n(X)^G$ for all n .

(2) In example 5.4 we will show that if E is not admissible, then $\pi_{X,E}$ can be surjective without being graded surjective.

(3.24) Let X, E , etc. be as above. We say that $\pi_{X,E}$ is *surjective* (resp. *n-surjective*) if $\pi_{X,E}(\mathcal{D}_E(X)^G) = \mathcal{D}_\mathcal{E}(X//G)$ (resp. $\pi_{X,E}(\mathcal{D}_E^n(X)^G) = \mathcal{D}_\mathcal{E}^n(X//G)$). Recall that $\pi_{X,E}$

is *graded surjective* if $\pi_{X,E}$ is n -surjective for all n . We say that E is *good* if $\pi_{X,E}$ is *surjective*, *n -good* if $\pi_{X,E}$ is n -surjective and $\mathcal{K}_E^n(X) = \mathcal{D}_E^{n-1}(X)\mathfrak{g}$, and *very good* if it is n -good for all n . The notation “ (n) -surjective” will denote “ n -surjective (resp. surjective),” and similarly for “ (n) -good.”

In light of remark 3.23 we say that X is *good* (resp. *n -good*, resp. *very good*) if E is good (resp. n -good, resp. very good) for every *admissible* G -vector bundle on X . Note that all these definitions make sense even if X is not smooth. However, we will be mostly interested in the smooth case.

The following generalization of 0.5(1) seems reasonable.

(3.25) CONJECTURE. – *Let E be an admissible G -vector bundle over the smooth affine G -variety X . Then $\pi_{X,E}$ is surjective if and only if it is graded surjective.*

(3.26) Remark. – The most natural generalization of conjecture 0.1 would say that $\text{gr } \mathcal{D}_E(X//G)$ is finite over some finitely generated commutative algebra whenever E is an admissible G -vector bundle over X and X is smooth. But this is false (see 3.27–28 below). Even the LS-alternative ($X//G$ is smooth or X is 2-large) is not sufficient. One needs to add to smoothness of $X//G$ the condition that π_X is equidimensional (see 5.2–3). If we are considering a G -module V , then the condition is that either V is cofree (see 5.1) or V is 2-large. This stronger version of the LS-alternative follows from the usual one whenever coregular representations of G are automatically cofree. For example, this holds for irreducible representations of the simple algebraic groups.

(3.27) EXAMPLE. – Let $G = \mathbb{C}^*$ act on $V = \mathbb{C}^3$ where there are coordinate functions s, t and u transforming by weights 1, 1 and -1 , respectively. Let W be the one-dimensional G -module of weight 1 and set $E := \Theta_W$. Then $A := \mathcal{O}(V)^G = \mathbb{C}[x, y]$ where $x = su$ and $y = tu$, and $M := \Gamma(E)^G$ is generated over A by s and t with the relation $xt = ys$. Note that V is coregular (so V satisfies the LS-alternative) and that M is A -isomorphic to the homogeneous maximal ideal $xA + yA$. By example 3.10, $D_A(M)$ is not left noetherian. Hence $\text{gr } D_A(M)$ (which is commutative) is not finitely generated. Of course, (V, G) is not 2-large, but it is 1-large, that is, the conditions of 0.3 hold with the twos in 0.3(2–3) replaced by ones.

(3.28) EXAMPLE. – Let $(V, G) = ((n+1)\mathbb{C}^n, \text{SL}_n)$ and $W = (\mathbb{C}^n, \text{SL}_n)$, $n \geq 2$. Then it follows from classical invariant theory ([Weyl]) that $A := \mathcal{O}(V)^G$ is a polynomial algebra generated by the $n+1$ determinant functions x_i , where

$$x_i(v_1, \dots, v_{n+1}) = \det(v_1, \dots, \hat{v}_i, \dots, v_{n+1}),$$

$$1 \leq i \leq n+1, \quad (v_1, \dots, v_{n+1}) \in (n+1)\mathbb{C}^n.$$

Moreover, $M := \Gamma(E)^G \simeq \text{Mor}(V, W)^G$ is generated over $\mathcal{O}(V)^G$ by the projections m_i , $1 \leq i \leq n+1$, where $m_i(v_1, \dots, v_{n+1}) = v_i \in \mathbb{C}^n = W$. The relations are generated by $\sum_{i=1}^{n+1} (-1)^i x_i m_i = 0$, so that (after a slight change of basis) we have $M \simeq A^{n+1}/A(x_1, \dots, x_{n+1})$. The proposition below shows that $\text{gr } D_A(M)$ is not finitely generated over any finitely generated commutative algebra. The case $n = 2$ is especially interesting, since $(V, G) = (3\mathbb{C}^2, \text{SL}_2)$ is 1-large, but, of course, not 2-large (9.11).

(3.29) PROPOSITION. – Let $A = \mathbb{C}[x_1, \dots, x_n]$, $F = A^n$ and $N = A(x_1, \dots, x_n)$, $n \geq 2$. Then $D_A(F/N)$ is not left noetherian, hence $\text{gr } D_A(F/N)$ is not finite over any finitely generated commutative algebra.

If $n = 2$, define $\varphi : F = A^2 \rightarrow A$ by $\varphi(a, b) = x_2a - x_1b$. Then φ is an isomorphism of F/N onto $Ax_1 + Ax_2$, hence 3.10 is the case $n = 2$ of the proposition. Our proof of proposition 3.29 is an elaboration on the ideas in 3.10.

We give a proof in the case $n = 3$ and indicate the necessary changes for the general case in 3.32 below. We rename the variables to be x, y and z and we think of $F = A^3$ as consisting of column vectors. Then $A = \mathbb{C}[x, y, z]$, elements of $D_A(F)$ are 3×3 matrices of elements of $D(A)$ and $N = \left\{ \begin{pmatrix} xf \\ yf \\ zf \end{pmatrix}, f \in A \right\}$. Let \mathcal{B} denote $\{\mathcal{P} \in D_A(F) : \mathcal{P}(N) \subset N\}$, and let $\text{Ann}(N) \subset \mathcal{B}$ denote the two-sided ideal of elements annihilating N .

(3.30) LEMMA. – Let $P \in D(A)$.

(1) $P = xQ + Rx$ for some $Q, R \in D(A)$.

(2) $P = xQ + yR + zS + T$ for some Q, R, S and T , where T has constant coefficients, and T is unique. Moreover, we may assume that $R = R(y, z)$ has coefficients which are polynomials only in y and z , and that $S = S(z)$ has coefficients which are polynomials only in z , in which case Q, R and S are also unique.

(3) Part (2) remains true if one replaces xQ by Qx and/or yR by Ry and/or zS by Sz .

Proof. – Part (1) holds since $\frac{1}{(k+1)}[x, U\partial_x^{k+1}] = U\partial_x^k$ for any differential operator U which does not involve ∂_x . Part (2) is obvious.

In (3), consider the case of expressing P uniquely in the form $Qx + yR(y, z) + zS(z) + T$. We may reduce to the case that $P = xQ_1$ for some Q_1 . Then $P = Q_1x + [x, Q_1]$, where, by induction on order, $[x, Q_1]$ has a unique representation $[x, Q_1] = Q_2x + yR(y, z) + zS(z) + T$. Thus $P = (Q_1 + Q_2)x + yR(y, z) + zS(z) + T$ is a unique representation of the required form. The other possibilities in (3) are handled similarly. \square

(3.31) LEMMA. – $\mathcal{B} = \text{Ann}(N) + D_A(F, N) + \mathbb{C} \cdot \text{id}_F$ where

(1) $\text{Ann}(N)$ consists of the operators whose rows are of the form

$$(Qy + Rz \quad -Qx + Sz \quad -Rx - Sy), \quad Q, R, S \in D(A).$$

(2) $D_A(F, N)$ consists of the operators whose columns are of the form

$$\begin{pmatrix} xQ \\ yQ \\ zQ \end{pmatrix}, \quad Q \in D(A).$$

Proof. – Let $\mathcal{P} \in \text{Ann}(N)$, and let $(Q \quad R \quad S)$ be a row of \mathcal{P} . Then $Qx + Ry + Sz = 0$. Since $\text{gr } D(A) \simeq \mathbb{C}[x, y, z, \partial_x, \partial_y, \partial_z]$ is a polynomial algebra, one can establish (1) by a downward induction on order. Part (2) is obvious.

We now show that \mathcal{B} is as claimed. Let

$$\mathcal{P} = \begin{pmatrix} Q & R & S \\ T & U & V \\ W & X & Y \end{pmatrix}$$

be an element of \mathcal{B} . By 3.30(1) we may write R in the form $xR_1 + R_2x$ for some R_1 and R_2 . Subtracting

$$\begin{pmatrix} -R_2y & R_2x & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & xR_1 & 0 \\ 0 & yR_1 & 0 \\ 0 & zR_1 & 0 \end{pmatrix} \in \text{Ann}(N) + D_A(F, N)$$

from \mathcal{P} we may reduce to the case that $R = 0$. Similarly we can assume that $S = 0$.

By 3.30(2–3) we may write Q uniquely in the form $xQ_1 + Q_2(y, z)y + Q_3(z)z + Q_4$, where Q_4 has constant coefficients, etc. Since \mathcal{P} preserves N , Q preserves xA , so x must divide $xQ_1x + Q_2(y, z)yx + Q_3(z)zx + Q_4x$. Thus $xT + [Q_2(y, z), x]y + [Q_3(z), x]z + [Q_4, x] = 0$ for some T . By uniqueness, we have $[Q_2(y, z), x] = [Q_3(z), x] = [Q_4, x] = 0$. Hence $Q_2 = Q_2(y, z, \partial_y, \partial_z)$, $Q_3 = Q_3(z, \partial_y, \partial_z)$ and $Q_4 = Q_4(\partial_y, \partial_z)$ do not involve ∂_x . We have

$$Q = xQ_1 + Q_2(y, z, \partial_y, \partial_z)y + Q_3(z, \partial_y, \partial_z)z + Q_4(\partial_y, \partial_z).$$

Subtracting the element

$$\begin{pmatrix} Q_2y + Q_3z & -Q_2x & -Q_3x \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} xQ_1 & xQ_2 & xQ_3 \\ yQ_1 & yQ_2 & yQ_3 \\ zQ_1 & zQ_2 & zQ_3 \end{pmatrix} \in \text{Ann}(N) + D_A(F, N)$$

we can reduce to the case that $Q = Q_4(\partial_y, \partial_z)$.

Since \mathcal{P} preserves N , we must have that $yQ_4(\partial_y, \partial_z)x = x(Tx + Uy + Vz)$. We may divide by x to get that

$$yQ_4(\partial_y, \partial_z) = Tx + Uy + Vz,$$

hence $[y, Q_4(\partial_y, \partial_z)] \in D(A)x + D(A)y + D(A)z$. By uniqueness again, $[y, Q_4] = 0$, and $Q_4 = Q_4(\partial_z)$ does not involve ∂_y . Similarly, Q_4 does not involve ∂_z , hence $Q_4 =: \lambda \in \mathbb{C}$. Adding $-\lambda \cdot \text{id}_F$ to \mathcal{P} , we can reduce to the case that the first row of \mathcal{P} is zero. It follows that the second and third rows of \mathcal{P} annihilate N , so $\mathcal{P} \in \text{Ann}(N)$. \square

Proof of 3.29. – We construct left ideals I_k of \mathcal{B} such that

$$I_k + D_A(F, N) \subsetneq I_{k+1} + D_A(F, N) \subsetneq \dots$$

Then $\{I_k + D_A(F, N)\}$ is an increasing sequence of left ideals in $\mathcal{B}/D_A(F, N) \simeq D_A(F/N)$ which does not stabilize, establishing the proposition.

Set

$$Q_i = \begin{pmatrix} 0 & 0 & 0 \\ -\partial_y^i y & \partial_y^i x & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

and let I_j denote the left \mathcal{B} -ideal generated by Q_0, \dots, Q_j . Suppose that $Q_{k+1} \in I_k + D_A(F, N)$. Then $\mathcal{P} + \sum_{i=0}^k \mathcal{R}_i Q_i = Q_{k+1}$ for some $\mathcal{P} \in D_A(F, N)$ and $\mathcal{R}_i \in \mathcal{B}$. Just considering the middle $(= (2, 2))$ terms of the matrices involved we obtain an equation

$$yT + \sum_{i=0}^k (\lambda_i + R_i x + S_i z) \partial_y^i x = \partial_y^{k+1} x,$$

where T and the R_i and S_i are in $D(A)$. Since all terms are multiplied on the right by x , except for yT , we must have $T = T'x$ for some T' . Dividing out the factor of x we obtain an equation

$$\partial_y^{k+1} - \sum_{i=0}^k \lambda_i \partial_y^i = yT' + \sum_{i=0}^k R_i \partial_y^i x + S_i \partial_y^i z.$$

By the uniqueness part of 3.30(2–3), we must have that $\partial_y^{k+1} - \sum_{i=0}^k \lambda_i \partial_y^i = 0$, a contradiction. \square

(3.32) *Remark.* – Suppose that $n > 3$. As above, let \mathcal{B} denote $\{\mathcal{P} \in D_A(F) : \mathcal{P}(N) \subset N\}$, and let $\text{Ann}(N) \subset \mathcal{B}$ denote the two-sided ideal of elements annihilating N . The analogues of 3.30 and 3.31 go through. In particular, $\mathcal{B} = \text{Ann}(N) + D_A(F, N) + \mathbb{C} \cdot \text{id}_F$, where $\text{Ann}(N)$ is generated by matrices whose rows are the “Koszul relations” of x_1, \dots, x_n . For the proof of 3.29, set

$$\mathcal{Q}_i = \begin{pmatrix} 0 & 0 & 0_{1,n-2} \\ -\partial_{x_2}^i x_2 & \partial_{x_2}^i x_1 & 0_{1,n-2} \\ 0_{n-2,1} & 0_{n-2,1} & 0_{n-2,n-2} \end{pmatrix},$$

where $0_{i,j}$ indicates an $i \times j$ matrix of zeroes, and proceed as before.

4. Reduction to representations

We establish some results on behavior of differential operators under morphisms. We show that a smooth affine G -variety X is (n) -good if and only if all of its slice representations are (n) -good, and we establish analogous results for (n) -surjectivity.

We use the following:

(4.1) *Remarks.* – Let E be a G -vector bundle over the affine G -variety X .

(1) Let $\{U_\alpha\}$ be an affine open cover of $X//G$, and let E_α denote the restriction of E to $\tilde{U}_\alpha := \pi_X^{-1}(U_\alpha)$. Then E is (n) -good (resp. $\pi_{X,E}$ is (n) -surjective) if and only if each E_α is (n) -good (resp. each $\pi_{\tilde{U}_\alpha, E_\alpha}$ is (n) -surjective).

(2) (see 3.6) Let $f \in \mathcal{O}(X)^G$, and let E_f denote the G -vector bundle $E|_{X_f} \rightarrow X_f$. Then $\mathcal{D}_{E_f}^n(X_f)^G \simeq \mathcal{O}(X)_f^G \otimes_{\mathcal{O}(X)^G} \mathcal{D}_E^n(X)^G$. In particular, for every $Q \in \mathcal{D}_{E_f}^n(X_f)^G$ there is a $k \geq 0$ and $P \in \mathcal{D}_E^n(X)^G$ such that $P = f^k Q$.

(4.2) **THEOREM.** – Let $\varphi : X \rightarrow Y$ be an étale morphism of varieties, let \mathcal{G} be a coherent sheaf of \mathcal{O}_Y -modules and set $\mathcal{F} := \varphi^* \mathcal{G}$. Then φ induces isomorphisms:

- (1) $\tilde{\varphi} : \varphi^* \mathcal{P}_{\mathcal{G}}^n \rightarrow \mathcal{P}_{\mathcal{F}}^n$, and
- (2) $(\tilde{\varphi})^\vee : \varphi^* \mathcal{D}_{\mathcal{G}}^n \rightarrow \mathcal{D}_{\mathcal{F}}^n$.

Let $x \in X$, $y = \varphi(x) \in Y$, and consider $B := \mathcal{O}_{X,x}$ as an $A := \mathcal{O}_{Y,y}$ -algebra via φ_x^* . Let M (resp. N) denote the stalk of \mathcal{G} at y (resp. of \mathcal{F} at x), so that $B \otimes_A M = N$. Let R be an A -module. Then φ_x^* induces canonical isomorphisms

- (3) $\tilde{\varphi}_x : B \otimes_A P_{A,M}^n \xrightarrow{\sim} P_{B,N}^n$, and
- (4) $(\tilde{\varphi}_x)^\vee : B \otimes_A D_A^n(M, R) \xrightarrow{\sim} D_B^n(N, B \otimes_A R)$.

Proof. – It is enough to establish (3) and (4): Let $\lambda : M \rightarrow N$ denote the canonical A -homomorphism. There is a commutative diagram where μ is the A -module morphism

$$\begin{array}{ccc} P_{A,M}^n & \xrightarrow{\mu} & P_{B,N}^n \\ \uparrow j_{A,M}^n & & \uparrow j_{B,N}^n \\ M & \xrightarrow{\lambda} & N \end{array}$$

which sends $P_{A,M}^n \ni (a \otimes m + I_A^{n+1}(A \otimes M))$ to $\varphi_x^*(a) \otimes \lambda(m) + I_B^{n+1}(B \otimes N) \in P_{B,N}^n$. From μ we obtain in a canonical way a B -module morphism $\tilde{\varphi}_x : B \otimes_A P_{A,M}^n \rightarrow P_{B,N}^n$. Reducing mod \mathfrak{M}_B and using 3.5 and the étaleness of φ_x^* we see that

$$B/\mathfrak{M}_B \otimes_A P_{A,M}^n \simeq A/\mathfrak{M}_A \otimes P_{A,M}^n \simeq M/\mathfrak{M}_A^{n+1}M \simeq N/\mathfrak{M}_B^{n+1}N \simeq B/\mathfrak{M}_B \otimes_B P_{B,N}^n.$$

Thus, by Nakayama, $\tilde{\varphi}_x$ is an isomorphism. We have obtained (3).

Now $A \rightarrow B$ is flat and $P_{A,M}^n$ is a finite A -module. Thus the canonical homomorphism $B \otimes_A \text{Hom}_A(P_{A,M}^n, R) \rightarrow \text{Hom}_B(P_{B,N}^n \simeq B \otimes_A P_{A,M}^n, B \otimes_A R)$ is an isomorphism, and we have (4). \square

(4.3) *Remark.* – In [Le1] Levasseur proves the theorem above in the case that X and Y are smooth.

(4.4) *COROLLARY.* – Let $\varphi : X \rightarrow Y$ be an excellent morphism of affine G -varieties. Let F be a G -vector bundle on Y and $E = \varphi^*F$ the pull-back. Then E is (n) -good (resp. $\pi_{X,E}$ is (n) -surjective) if F is (n) -good (resp. $\pi_{Y,F}$ is (n) -surjective), and conversely if φ is surjective.

Proof. – Set $A := \mathcal{O}(Y)$, $B := \mathcal{O}(X)$, $M = \Gamma(Y, F)$ and $N = \Gamma(X, E) \simeq B \otimes_A M$. Since φ is excellent, we have $B^G \otimes_{A^G} A \xrightarrow{\sim} B$ and $B^G \otimes_{A^G} M \xrightarrow{\sim} N$. There is a commutative diagram of B^G -modules where the horizontal maps are isomorphisms by 4.2 and the

$$\begin{array}{ccc} B^G \otimes_{A^G} D_A^n(M) & \xrightarrow{\sim} & D_B^n(N) \\ \downarrow \text{id} \otimes \text{res}_{A^G} & & \downarrow \text{res}_{B^G} \\ B^G \otimes_{A^G} D_{A^G}^n(M^G, M) & \xrightarrow{\sim} & D_{B^G}^n(N^G, N) \end{array}$$

vertical maps are induced by restriction of domain. Since B^G is A^G -flat, we have $\text{Ker}(\text{res}_{B^G}) \simeq B^G \otimes_{A^G} \text{Ker}(\text{res}_{A^G})$. If $\text{Ker}(\text{res}_{A^G})$ is the product of $D_A^{n-1}(M)$ and the image of \mathfrak{g} in $D_A^1(M)$, then the analogous result holds for $\text{Ker}(\text{res}_{B^G})$. Taking G -invariants in the diagram we obtain that $\text{res}_{B^G} D_B^n(N)^G \simeq B^G \otimes_{A^G} \text{res}_{A^G} D_A^n(M)^G$. Thus E is n -good (resp. $\pi_{X,E}$ is n -surjective) if F (resp. $\pi_{Y,F}$) is. If φ is surjective, then B^G is faithfully flat over A^G , and one sees that F is n -good if E is, and similarly for n -surjectivity.

Suppose that E is good, and let $n \in \mathbb{N}$. Then there is an $m \geq n$ such that

$$(*) \quad \mathcal{D}_E^n(X//G) \subseteq \pi_{X,E} \mathcal{D}_E^m(X)^G \subseteq \mathcal{D}_E^m(X//G).$$

Arguments as above show that $(*)$ holds if it holds with $F, Y//G$, etc. in place of $E, X//G$, etc., and conversely if φ is surjective. \square

(4.5) COROLLARY. – Let X be an affine G -variety where G acts freely. Let $E = (\pi_X)^*(\tilde{E})$ where \tilde{E} is a vector bundle on $X//G$ (see 2.5). Then E is a G -vector bundle and

(1) For all $n \geq 0$, $\mathcal{D}_E^n(X//G) = \mathcal{D}_E^n(X)^G / (\mathcal{D}_E^{n-1}(X)\mathfrak{g})^G$.

(2) X is very good.

Proof. – We handle the case where $\tilde{E} = 1_{X//G}$ (hence $E = 1_X$), and leave the general case to the reader. Working locally on $X//G$ we may reduce to the case that $X = G \times Y$, where Y is an affine variety on which G acts trivially and where G acts on itself by left multiplication. Then π_X is just projection $G \times Y \rightarrow Y$. We can consider the elements of $\mathfrak{g} \subset \Gamma(G, TG)$ as vector fields on $G \times Y$. Clearly the kernel of $\rho : \mathcal{D}^n(G \times Y) \rightarrow \mathcal{D}^{0,n}(G \times Y)$ is $\mathcal{D}^{n-1}(G \times Y)\mathfrak{g}$ (see 3.16), hence

$$\mathcal{D}^n(X)^G / (\mathcal{D}^{n-1}(X)\mathfrak{g})^G \simeq \mathcal{D}^{0,n}(G \times Y)^G \simeq (\mathcal{O}(G) \otimes_{\mathbb{C}} \mathcal{D}^n(Y))^G \simeq \mathcal{D}^n(Y).$$

The composition of the isomorphisms is induced by $(\pi_X)_*$. \square

(4.6) COROLLARY. – Let H be a reductive subgroup of G , let Y be an affine H -variety and let E be an H -vector bundle on Y . Set $\tilde{E} = G *^H E \rightarrow G *^H Y =: X$. Then

(1) $\pi_{X,\tilde{E}}$ is (n) -surjective if and only if $\pi_{Y,E}$ is (n) -surjective

(2) \tilde{E} is (n) -good if and only if E is (n) -good

Proof. – We will use the symbol E to also denote the $(G \times H)$ -bundle $G \times E \rightarrow G \times Y$. Consider the commutative diagram below: The vertical sequence is exact, by 4.5, and

$$\begin{array}{ccccc}
 & & 0 & & \\
 & & \uparrow & & \\
 & & \mathcal{D}_E^n(G *^H Y) & \xleftarrow{\alpha} & \mathcal{D}_E^{n-1}(G *^H Y)\mathfrak{g} \\
 & & \uparrow \gamma & & \uparrow \beta \\
 \mathcal{D}_E^{0,n}(G \times Y)^H & \xrightarrow{\delta} & \mathcal{D}_E^n(G \times Y)^H & \xleftarrow{\quad} & \mathcal{D}_E^{n-1}(G \times Y)^H \mathfrak{g} \\
 & & \uparrow & & \\
 & & \mathcal{D}_E^{n-1}(G \times Y)^H \mathfrak{h} & & \\
 & & \uparrow & & \\
 & & 0 & &
 \end{array}$$

$\mathcal{D}_E^n(G \times Y)^H$ is clearly the direct sum of the terms to its right and left.

We now show that $\pi_{X, \tilde{E}}$ is (n) -surjective if and only if $\pi_{Y, E}$ is (n) -surjective. Let $Q \in \mathcal{D}_E^n(G *^H Y)^G$. Then $Q = (\gamma \circ \delta)(Q')$ modulo $(\alpha \circ \beta)(\mathcal{D}_E^{n-1}(G \times Y)\mathfrak{g})^{G \times H} \subset \mathcal{K}_E^n(G *^H Y)^G$, where $Q' \in \mathcal{D}_E^{0,n}(G \times Y)^{G \times H} \simeq \mathcal{D}_E^n(Y)^H$. Under the isomorphisms $\Gamma(X, \tilde{E})^G \simeq \Gamma(G \times Y, E)^{G \times H} \simeq \Gamma(Y, E)^H$, the action of Q goes over into that of Q' . Hence we have (1).

We now consider $\mathcal{K}_E^n(X)$ and $\mathcal{K}_E^n(Y)$. Assume that $\mathcal{K}_E^n(Y) = \mathcal{D}_E^{n-1}(Y)\mathfrak{h}$, and suppose that $Q \in \mathcal{K}_E^n(G *^H Y)$. As above, modulo $\text{Im}(\alpha \circ \beta)$, Q equals $(\gamma \circ \delta)(Q')$ for some $Q' \in \mathcal{D}_E^{0,n}(G \times Y)^H \simeq (\mathcal{O}(G) \otimes_{\mathbb{C}} \mathcal{D}_E^n(Y))^H$. Write $Q' = \sum_{i=1}^r f_i \otimes P_i$ where the $f_i \in \mathcal{O}(G)$ are linearly independent and the $P_i \in \mathcal{D}_E^n(Y)$. Since Q kills $\Gamma(X, \tilde{E})^G$, Q' kills $\Gamma(Y, E)^H$ and the P_i must lie in $\mathcal{K}_E^n(Y) = \mathcal{D}_E^{n-1}(Y)\mathfrak{h}$. Hence $\delta(Q') \in \text{Ker } \gamma$, i.e., $Q \in \text{Im } \alpha$.

Conversely, suppose that $\mathcal{K}_E^n(X) = \mathcal{D}_E^{n-1}(X)\mathfrak{g}$. Let U be a finite dimensional irreducible H -module in $\mathcal{K}_E^n(Y)$. Since $\mathcal{K}_E^n(Y)$ is a locally finite H -module, it suffices to show that some non-zero element of U is in $\mathcal{D}_E^{n-1}(Y)\mathfrak{h}$. Since H is a subgroup of G , there is certainly a copy of U^* in the H -module $\mathcal{O}(G)$ (H acting on G on the right). Thus there is a non-zero element Q' of the form $\sum_{i=1}^r f_i \otimes P_i$ in $\mathcal{D}_E^{0,n}(G \times Y)^H \simeq (\mathcal{O}(G) \otimes_{\mathbb{C}} \mathcal{D}_E^n(Y))^H$, where the P_i are in U . Then Q' annihilates $\Gamma(G \times Y, E)^{G \times H} \simeq \Gamma(X, \tilde{E})^G$, and our hypotheses imply that $(\gamma \circ \delta)(Q')$ lies in $\text{Im}(\alpha \circ \beta)$. It follows that Q' lies in the projection of $\mathcal{D}_E^{n-1}(G \times Y)^H \mathfrak{h}$ to $(\mathcal{O}(G) \otimes_{\mathbb{C}} \mathcal{D}_E^n(Y))^H$. But this projection is clearly $(\mathcal{O}(G) \otimes_{\mathbb{C}} \mathcal{D}_E^{n-1}(Y)\mathfrak{h})^H$. Thus each P_i lies in $\mathcal{D}_E^{n-1}(Y)\mathfrak{h}$. \square

(4.7) Let V be a G -module, and let E be a trivial G -vector bundle over V . Then $\Gamma(E)$ is a graded $\mathcal{O}(V)^G$ -module and, as in 3.11, we let $\mathcal{D}_{d,E}^n(V) \subset \mathcal{D}_E^n(V)$ and $\mathcal{D}_{d,E}(V) \subset \mathcal{D}_E(V)$ denote the differential operators sending $\Gamma(E)_m$ to $\Gamma(E)_{m+d}$ for all m . We have induced gradings on $\mathcal{D}_E(V)^G$ and $\mathcal{D}_{\mathcal{E}}(V//G)$, where \mathcal{E} is the sheaf corresponding to $\Gamma(E)^G$.

(4.8) PROPOSITION. – Let V and W be G -modules, let E denote Θ_W , and let $f \in \mathcal{O}(V)^G$, $f(0) \neq 0$. Let E_f denote $E|_{V_f}$. Then

- (1) $\pi_{V,E}$ is (n) -surjective if and only if π_{V_f, E_f} is (n) -surjective
- (2) E is (n) -good if and only if E_f is (n) -good

Proof. – Applying 4.4 to the inclusion $V_f \subseteq V$, we see that if $\pi_{V,E}$ is (n) -surjective, then π_{V_f, E_f} is (n) -surjective. Conversely, suppose that π_{V_f, E_f} is (n) -surjective. Let $Q \in \mathcal{D}_E^n(V//G)$ be homogeneous of degree d . By 4.1(2) there is an $l \geq 0$ and a $P \in \mathcal{D}_E^m(V)^G$, $m \geq n$ ($m = n$ if π_{V_f, E_f} is n -surjective), such that $\pi_{V,E}P = f^l Q$. Write $P = \sum P_i$ and $f = \sum f_i$ where the P_i and f_i are homogeneous of degree i . Then $\pi_{V,E}P_d = (f_0)^l Q$, where f_0 is a non-zero constant. Thus $\pi_{V,E}$ is (n) -surjective, and we have (1). Part (2) is similar. \square

The following result is now immediate from Luna's slice theorem (see 1.15), 2.3, 4.1, 4.6 and 4.8.

(4.9) THEOREM. – Let X be a smooth affine G -variety and E a G -vector bundle on X . Let $x \in X$ where Gx is closed, let $(N, H := G_x)$ be the slice representation at x and let E' denote $\Theta_{E_x} = (E_x \times N \rightarrow N)$.

(1) E is (n) -good (resp. $\pi_{X,E}$ is (n) -surjective) in a G -neighborhood of x if and only if E' is (n) -good (resp. $\pi_{N,E'}$ is (n) -surjective).

(2) X is (n) -good if and only if each slice representation of X is (n) -good

5. Coregular and Cofree Representations

(5.1) Definitions. – Let V be a G -module. We say that V is *cofree* if $\mathcal{O}(V)$ is a free $\mathcal{O}(V)^G$ -module. Equivalently ([S3, 17.29]),

(1) V is coregular.

(2) $\pi_V : V \rightarrow V//G$ is equidimensional.

Consequently, if G is finite, then V is coregular if and only if it is cofree.

Coregular and cofree modules are “small” actions (see discussion following 0.7). We begin with some positive results for such actions. Then we establish the main negative result: If V is coregular and not fix pointed, then V is not good. In fact, we have: $\mathcal{D}(V//G)$ cannot be finitely generated as both a left and right $\mathcal{D}(V)^G$ -module.

(5.2) PROPOSITION. – Let \tilde{G} be reductive, where G is a normal reductive subgroup of \tilde{G} and X is a smooth affine \tilde{G} -variety. Assume that $X//G$ is smooth, and let E be a \tilde{G} -vector bundle on X . Suppose that

(1) $E = \mathbf{1}_X$, or

(2) $\pi_X : X \rightarrow X//G$ is equidimensional.

Then \mathcal{E} is the sheaf of germs of sections of a \tilde{G}/G -vector bundle \tilde{E} on $X//G$. In particular, $\text{gr } \mathcal{D}_{\mathcal{E}}(X//G)$ is a finite $\text{gr } \mathcal{D}(X//G)$ -module, where $\text{gr } \mathcal{D}(X//G)$ is finitely generated.

Proof. – Clearly \tilde{G}/G acts on the sections $\Gamma(E)^G = \Gamma(\mathcal{E})$ compatibly with its action on $\mathcal{O}(X//G)$, so we have our desired result if we can show that \mathcal{E} is locally free. This is clear in case (1). In case (2), [Ma, Thm. 81] shows that π_X is flat. It follows that the push forward of the sheaf of sections of E is a flat sheaf of $\mathcal{O}_{X//G}$ -modules, hence so is its (coherent) direct summand \mathcal{E} . Thus \mathcal{E} is locally free. The finite generation claims follow from 3.19 (with $G = \{e\}$!). \square

(5.3) PROPOSITION. – Let \tilde{G} , etc. be as above where all the closed G -orbits are principal. Let E be a \tilde{G} -vector bundle on X . Then

(1) $X \rightarrow X//G$ is equidimensional.

(2) \mathcal{E} is the sheaf of sections of a \tilde{G}/G -vector bundle \tilde{E} on $X//G$.

(3) $\pi_{X,E} : \mathcal{D}_E(X)^G \rightarrow \mathcal{D}_{\mathcal{E}}(X//G)$ is surjective.

(4) Set $E' := (\pi_X)^*(\tilde{E})$. Then E' is admissible, and $E = E'$ if and only if E is admissible.

(5) $\pi_{X,E'} : \mathcal{D}_{E'}(X)^G \rightarrow \mathcal{D}_{\mathcal{E}}(X//G)$ is graded surjective, hence $\pi_{X,E}$ is graded surjective if E is admissible.

Proof. – By 1.11, $X \rightarrow X//G$ is a fibration, so π_X is equidimensional and we have (1). Part (2) follows from 5.2. Using 4.9 we can reduce (3) and (4) to the case of slice representations. So we may assume that $X = V$ and $E = \Theta_W$, where V and W are G -modules and $V = V_{\text{pr}}$.

Clearly V is fix pointed, so $V = V^G \oplus V'$ as G -module where $\mathcal{O}(V')^G = \mathbb{C}$. Set $W' := (\mathcal{O}(V') \otimes W)^G$. Then W' is finite dimensional, and \tilde{E} is the trivial bundle $V//G \times W' \rightarrow V//G$. Hence $E' := (\pi_V)^*(\tilde{E}) = \Theta_{W'}$. We let E and E' also denote their restrictions to V^G and V' . Note that $E = E'$ if and only if $W' = 0$, i.e., if and only if E is admissible. Thus (4) holds, and (5) holds since the restriction of $\pi_{V,E'}$ to $\mathcal{D}_{E'}(V^G) \subset \mathcal{D}_{E'}(V)^G$ is an isomorphism.

We now show that $\mathcal{D}_E(V')^G \rightarrow \mathcal{D}_E(V//G)$ is surjective. It then follows that

$$\pi_{V,E} : \mathcal{D}_E(V)^G \simeq \mathcal{D}(V^G) \otimes \mathcal{D}_E(V')^G \rightarrow \mathcal{D}(V^G) \otimes \mathcal{D}_E(V//G) \simeq \mathcal{D}_E(V//G)$$

is surjective, giving (3). Choose a basis $\{s_1, \dots, s_r\}$ of W' where each $s_i : V' \rightarrow W$ is homogeneous of degree a_i and $a_1 \geq \dots \geq a_r$. Suppose that $a_1 = a_m$ where $m \leq r$ is maximal. We view elements of $\text{Mor}(V'^*, W^*)$ as constant coefficient differential operators from $\Gamma(V', W)$ to $\Gamma(V', \Theta_C)$. It is clear that there are elements $P_i \in \text{Mor}(V'^*, W^*)^G$, homogeneous of degree a_i , such that $P_i(s_j) = \delta_{ij} \in \mathbb{C} = \text{Mor}(V', \Theta_C)^G$, $1 \leq i, j \leq m$. Moreover, the P_i annihilate the s_j for $j > m$. Multiplying the P_i by the s_j we obtain differential operators P_{ij} such that

$$P_{ij}(s_k) = \delta_{ik} s_j, \quad 1 \leq i \leq m, \quad 1 \leq j, k \leq r.$$

Now consider the covariants of the next lower degree $a_{m+1} = \dots = a_{m+t}$. Then, as above, we can construct differential operators Q_{ij} such that

$$Q_{ij}(s_k) = \delta_{ik} s_j, \quad m+1 \leq i \leq m+t, \quad 1 \leq j \leq r, \quad m+1 \leq k \leq r.$$

We would like that $Q_{ij}(s_k) = 0$ for $k \leq m$, and this can be accomplished by modifying the Q_{ij} by the P_{ki} . Thus, inductively, we obtain elements of $\mathcal{D}_E(V')^G$ which map onto a basis of $\text{End}(W') = \mathcal{D}_E(V//G)$. \square

(5.4) *Example.* – Let $G = \mathbb{C}^*$, $V = \mathbb{C}$ and $W = \mathbb{C}^2$ where G acts with weight 1 on V and with weights 1 and 2 on W . Set $E = \Theta_W$. We leave it to the reader to show that $\pi_{V,E} \mathcal{D}_E^0(V)^G \neq \mathcal{D}_E^0(V//G)$ but that $\pi_{V,E} \mathcal{D}_E^n(V)^G = \mathcal{D}_E^0(V//G) = \mathcal{D}_E(V//G)$ for all $n \geq 1$.

The following shows that there are cases where one can definitely *not* lift differential operators.

(5.5) **THEOREM.** – *Let G be reductive, and let V be a coregular G -module such that $V^G = (0)$ and $m := \dim V//G > 0$. Then 1_V is not good, i.e., $(\pi_V)_* \mathcal{D}(V)^G \neq \mathcal{D}(V//G)$.*

(5.6) *Remark* ([Le2]). – If V is as above, then $(\pi_V)_*$ cannot be 1-surjective: Let f_1, \dots, f_m be a minimal homogeneous generating set of $\mathcal{O}(V)^G$, where $\deg f_1 = d > 1$. There is a derivation of $\mathcal{O}(V)^G \simeq \mathcal{O}(\mathbb{C}^m)$ which sends f_1 to 1. If $A \in \mathcal{D}^1(V)^G$ and $A(f_1) = 1$, then the degree $-d$ part of A is nonzero. But the degree of any $A \in \mathcal{D}^1(V)^G$ is at worst -1 .

(5.7) *Example.* – We illustrate the proof of theorem 5.5 in the case that $\dim V//G = 1$.

(1) Suppose that $G = \{\pm 1\}$ and $V = \mathbb{C}$ where G acts on V by multiplication. Let x be a coordinate function on V and let D_x denote differentiation with respect to x . Then $\mathcal{O}(V)^G$ is generated by x^2 . Let y denote the coordinate function on $V//G = \mathbb{C}$ such that $(\pi_V)^*(y) = x^2$, and let D denote differentiation with respect to y . Let P denote $D_x^2 \in \mathcal{D}(V)^G$. Now $\mathcal{D}(V)$ is a Lie algebra, where $(\text{ad}Q)R = [Q, R]$, $Q, R \in \mathcal{D}(V)$. Then P , being a constant coefficient operator, is locally ad-nilpotent, *i.e.*, given an element of $\mathcal{D}(V)$, some power of $\text{ad}P$ annihilates it. If $(\pi_V)_*$ were surjective, then $\tilde{P} := (\pi_V)_*P$ would be locally ad-nilpotent in $\mathcal{D}(\mathbb{C})$. This is not the case:

It is easy to see that $\tilde{P} = 4yD^2 + 2D$. Set $Q_j = (\text{ad}\tilde{P})^j D$, $j \geq 0$. Then, up to nonzero constants, $Q_j = D^{j+1} + \text{lower order terms}$. Hence \tilde{P} is not locally ad-nilpotent, and consequently $(\pi_V)_*$ is not surjective.

(2) In general, $\mathcal{O}(V)^G = \mathbb{C}[h]$ where h is a homogeneous polynomial of degree $n \geq 2$. The constant coefficient differential operator $P \in S^n(V)^G \subset \mathcal{D}(V)^G$ dual to h gives rise to $\tilde{P} := (\pi_V)_*P \in \mathcal{D}(\mathbb{C})$, where the symbol of \tilde{P} is a multiple of $y^{n-1}D^n$. The following sequence of facts shows that $Q_j := (\text{ad}\tilde{P})^j D$, $j \geq 0$, has symbol a non-zero multiple of $y^{j(n-2)}D^{j(n-1)+1}$, proving that $(\pi_V)_*$ is not surjective.

$$(i) D^k y^l = y^l D^k + (kl)y^{l-1}D^{k-1} + \text{lower order terms}, \quad k, l \in \mathbb{N}.$$

$$(ii) [y^i D^j, y^k D^l] = (jk - il)y^{i+k-1}D^{j+l-1} + \text{lower order terms}, \quad k, l \in \mathbb{N}.$$

$$(iii) [y^{n-1}D^n, y^{j(n-2)}D^{j(n-1)+1}] = (1 - j - n)y^{(j+1)(n-2)}D^{(j+1)(n-1)+1} + \text{lower order terms}.$$

(5.8) LEMMA. – Let V be as in 5.5 and let f_1, \dots, f_m be a minimal homogeneous generating set of $\mathcal{O}(V)^G$. Then there are coordinates x_1, \dots, x_n on V such that $f_1 \in x_1^a + I^2$ and $f_i \in I$, $i \geq 2$, where $a := \deg f_1$ and I denotes the ideal in $\mathcal{O}(V)$ generated by x_2, \dots, x_n .

Proof. – Since V is coregular, we may find a point $z \in V//G$ such that $f_1(z) = 1$ and $f_i(z) = 0$ for $i \geq 2$. Let $v \in (\pi_V)^{-1}(z)$. Choose a basis v, v_2, \dots, v_n of V , and let x_1, \dots, x_n be the corresponding dual basis. Then it follows from our construction that the f_i have the desired form, $i \geq 2$. Replacing x_1 by x_1 plus a linear combination of the x_i , $i \geq 2$, we may arrange that $f_1 \in x_1^a + I^2$. \square

(5.9) LEMMA. – Let V be a complex vector space. Let $a \geq 2$ and let $f \in S^a(V^*)$. Choose coordinates x_1, \dots, x_n on V so that $f \in x_1^a + I^2$ where I is the ideal generated by x_2, \dots, x_n . Let $P \in S^a(V)$ be dual to f under the isomorphism of $S^a(V)$ and $S^a(V^*)$ given by the choice of coordinates. Set $\partial_i = \partial/\partial x_i$, $i = 1, \dots, n$, so that, up to a constant, $P \in \partial_1^a + \mathcal{D}^{a-2}(V)(\delta I)^2$ where δI is the span of $\partial_2, \dots, \partial_n$. Inductively define differential operators P_j by: $P_a = P$ and $P_j = [P_{j+1}, f]$; $0 \leq j < a$. Then, up to nonzero constants,

$$P_1 \in (x_1^{(a-1)(a-1)} + I^2)\partial_1 + I\delta I + \mathcal{O}(V), \quad a \geq 3,$$

$$P_1 \in x_1^{(a-1)(a-1)}\partial_1 + I\delta I + \mathcal{O}(V), \quad \text{if } a = 2,$$

$$P_0 \in x_1^{a(a-1)} + I^2.$$

Proof. – By a straightforward induction one can show that

$$P_j \in a^{a-j}(a!/j!)x_1^{(a-1)(a-j)}\partial_1^j + \mathcal{D}^{j-2}(V)(\delta I)^2 + I\mathcal{D}^{j-1}(V)\delta I + I^2\mathcal{D}^j(V) + \mathcal{D}^{j-1}(V),$$

where the term $I^2\mathcal{D}^j(V)$ only appears for $j \leq a - 2$. Thus P_0 and P_1 have the desired form. \square

(5.10) COROLLARY. – Let V be a G -module, and let $P \in \mathcal{S}^d(V)^G$ be an invariant constant coefficient differential operator. Then $(\pi_V)_*(P)$ has order exactly d .

Proof of 5.5. – Let $f_1, \dots, f_m, x_1, \dots, x_n, I, \delta I$ and $a = \deg f_1$ be as above. Let P be dual to f_1 as in 5.9. Let \mathfrak{M} denote the ideal $(f_1, \dots, f_m) \subset \mathcal{O}(V)^G \subset \mathcal{O}(V)$. We have a decreasing filtration \mathcal{F}^k of $\mathfrak{M}/\mathfrak{M}^2$, where $\mathcal{F}^k = (I^k \cap \mathfrak{M} + \mathfrak{M}^2)/\mathfrak{M}^2, k \geq 0$. Note that f_1 projects to a \mathbb{C} -basis of $(\mathcal{F}^0 = \mathfrak{M})/\mathcal{F}^1$. We may assume that there is an $r \geq 1$ such that f_2, \dots, f_r project to a basis of $\mathcal{F}^1/\mathcal{F}^2$ and that f_{r+1}, \dots, f_m lie in \mathcal{F}^2 . Let $D_i, 1 \leq i \leq m$ denote the differential operators on $\mathcal{O}(V)^G \simeq \mathbb{C}[f_1, \dots, f_m]$ dual to the f_i . Let J (resp. K) denote the ideal generated by the $f_i, 2 \leq i \leq r$ (resp. $f_i, i > r$), and let δJ (resp. δK) denote the linear span of the $D_i, 2 \leq i \leq r$ (resp. $i > r$). By our choice of the f_i ,

$$I \cap \mathcal{O}(V)^G \subset J + K, \quad \text{and} \quad I^2 \cap \mathcal{O}(V)^G \subset J^2 + K.$$

Set $\mathcal{D}^j := \mathcal{D}^j(V//G), j \geq 0$, and for $s \geq 0$ set

$$M^s := J^2\mathcal{D}^s + J\mathcal{D}^{s-1}\delta J + K\mathcal{D}^s + \mathcal{D}^{s-2}(\delta J)^2 + \mathcal{D}^{s-1}.$$

For the moment assume that

$$(*) \quad (\tilde{P} := (\pi_V)_*(P)) \in f_1^{a-1}D_1^a + M^a.$$

Define $Q_j := (\text{ad } \tilde{P})^j D_1, j \geq 0$. A calculation shows that, up to a nonzero constant, Q_j lies in $f_1^{j(a-2)}D_1^{j(a-1)+1} + M^{j(a-1)+1}$ (see 5.7). Thus \tilde{P} is not locally ad-nilpotent, hence $(\pi_V)_*$ cannot be surjective.

We now establish (*). The symbol of \tilde{P} has the form $\sum_{|\alpha|=m} a_\alpha D^\alpha$, where $\alpha = (\alpha_1, \dots, \alpha_m) \in \mathbb{N}^m, D^\alpha = D_1^{\alpha_1} \dots D_m^{\alpha_m}$, etc. as in 3.8. Since $P \in \partial_1^a + \mathcal{D}^{a-2}(V)(\delta I)^2, [[P, f_i], f_j]$ has coefficients in I^2 if $i, j > r$. Thus $a_\alpha \in J^2 + K$ whenever $D^\alpha \in \mathcal{D}^{a-2}(\delta K)^2$. Similarly, $a_\alpha \in J + K$ if $D^\alpha \in \mathcal{D}^{a-2}\delta J\delta K$. Reinterpreted in terms of the $f_i, 5.9$ shows that the coefficient of D_1^a lies in $\mathbb{C}^* f_1^{a-1} + (J^2 + K)$. Similarly, the coefficient of $D_1^{a-1}D_i$ lies in $J + K$ if $i > 1$ and in $J^2 + K$ if $i > r$. These results establish (*). \square

Our techniques can also be used to establish non surjectivity of $(\pi_V)_*$ in certain (non-coregular) classical cases.

(5.11) PROPOSITION. – Let $(V, G) = ((n+k)\mathbb{C}^n, \text{SL}_n)$ where $n \geq 3$ and $2 \leq k \leq n - 1$. Then $(\pi_V)_*$ is not surjective.

Proof. – Let $x_i : (n+k)\mathbb{C}^n \rightarrow \mathbb{C}^n$ and $\xi_i : (n+k)(\mathbb{C}^n)^* \rightarrow (\mathbb{C}^n)^*$ denote the projections onto the i th factor, $1 \leq i \leq n+k$. Now $\text{gr } \mathcal{D}(V)^G \simeq \mathcal{O}(V \oplus V^*)^G$, and classical invariant theory ([Weyl], [S1, §2]) gives us the following generators:

- (1) Determinants $[x_{i_1}, \dots, x_{i_n}], 1 \leq i_1 < \dots < i_n \leq k+n$.
- (2) Contractions $\langle x_i, \xi_j \rangle, 1 \leq i, j \leq n+k$.
- (3) Determinants $[\xi_{i_1}, \dots, \xi_{i_n}], 1 \leq i_1 < \dots < i_n \leq k+n$.

The elements in (1) generate $\mathcal{O}(V)^G$, and those in (2) correspond to vector fields. The invariants in (3) correspond to constant coefficient differential operators, which, via $(\pi_V)_*$, give elements in $\mathcal{D}(V//G)$ of order exactly n (corollary 5.10). The quotient $V//G$ lies in \mathbb{C}^N , $N = \binom{n+k}{n}$, and we let f_1, \dots, f_N be coordinates corresponding to the generators in (1), where f_1 corresponds to $[x_1, \dots, x_n]$. Let D_1, \dots, D_N be the partial derivatives dual to the f_i and let I denote the ideal generated by the f_j , $j \geq 2$. The ideal J of $V//G \subset \mathbb{C}^N$ is generated by quadratic polynomials in the f_i , and $J \subset I$. We consider $\mathcal{D}^n(V//G)$ as the quotient of $\{P \in \mathcal{D}^n(\mathbb{C}^N) : P(J) \subset J\}$ by $J\mathcal{D}^n(\mathbb{C}^N)$ (see 3.9). Let \mathcal{D}^m be shorthand for $\mathcal{D}^m(\mathbb{C}^N)$.

Set $P_1 = [\xi_1, \dots, \xi_n] \in \mathcal{D}(V)^G$. Then, up to a nonzero constant,

$$\begin{aligned} \tilde{P} &:= (\pi_V)_* P_1 \in f_1^{n-1} D_1^n + I\mathcal{D}^n + \mathcal{D}^{n-1} + J\mathcal{D}^n \\ &= f_1^{n-1} D_1^n + I\mathcal{D}^n + \mathcal{D}^{n-1}. \end{aligned}$$

It follows from classical invariant theory that $\mathcal{O}(V)^G$ is isomorphic to $\mathcal{O}(V')^{G'}$, where $(V', G') = ((n+k)\mathbb{C}^k, \text{SL}_k)$. Let $x'_i : (n+k)\mathbb{C}^k \rightarrow \mathbb{C}^k$, $1 \leq i \leq n+k$, denote projection onto the i th factor. Then, under the isomorphism, $[x_1, \dots, x_n] \in \mathcal{O}(V)^G$ goes to $[x'_{n+1}, \dots, x'_{n+k}] \in \mathcal{O}(V')^{G'}$, etc. It follows that there is a constant coefficient differential operator in $\mathcal{D}(V')^{G'}$ which gives an operator Q on $V//G$ lying in

$$f_1^{k-1} D_1^k + I\mathcal{D}^k + \mathcal{D}^{k-1}.$$

Now, reasoning as in 5.5 and 5.7, we see that, up to nonzero constants,

$$(\text{ad } \tilde{P})^j Q \in f_1^{k-1+j(n-2)} D_1^{k+j(n-1)} + I\mathcal{D}^{k+j(n-1)} + \mathcal{D}^{k+j(n-1)-1}.$$

Hence \tilde{P} is not locally ad-nilpotent in $\mathcal{D}(V//G)$, and $(\pi_V)_*$ is not surjective. \square

The next results show that $(\pi_V)_*$ is rather far from being surjective when (V, G) is coregular.

(5.12) LEMMA ([LS, IV 1.3]). – *Suppose that $\mathcal{D}(V//G)$ is a finitely generated left and right $\mathcal{D}(V)^G$ -module. If $\mathcal{D}(V//G)$ is simple, then V is good.*

Proof. – Let $A := (\pi_V)_* \mathcal{D}(V)^G \subseteq B := \mathcal{D}(V//G)$. We give B its natural filtration $\{B_n\}$. Let $0 \neq f \in \mathcal{O}(V)^G$ such that f vanishes on the nonprincipal strata. By 4.1, 4.9 and 5.3, given $Q \in B$ there is an $l \in \mathbb{N}$ and $P \in A$ such that $f^l Q = P$. Since B is finitely generated as a right A -module, $B = B_n A$ for some n , where B_n is a finitely generated $\mathcal{O}(V//G)$ -module. Thus there is an l such that $f^l B_n \subseteq A$, hence $f^l B \subseteq A$.

Similarly, there are m and k such that $B = AB_m$ and $f^k B_m \subseteq A$. An easy induction shows that $B f^{k+m} = AB_m f^{k+m} \subseteq A f^k B_m \subseteq A$. Thus there is a non-trivial left (resp. right) B -ideal I (resp. J) contained in A . Since B is a simple domain, $IJ = B$, hence $A = B$. \square

(5.13) COROLLARY. – *Let V be as in theorem 5.5. Then $\mathcal{D}(V//G)$ cannot be finitely generated as both a left and right $\mathcal{D}(V)^G$ -module.*

Proof. – By assumption, $\mathcal{D}(V//G)$ is the m th Weyl algebra, which is simple [Bj, Ch. I]. \square

We now give more global versions of the results above.

(5.14) Let X be a smooth affine G -variety, and let $(X//G)_{\text{sm}}$ denote the smooth points of $X//G$. Recall that $(X//G)_{\text{pr}} \subseteq (X//G)_{\text{sm}}$.

(5.15) THEOREM. – *Let X be a smooth affine G -variety such that $(X//G)_{\text{pr}} \neq (X//G)_{\text{sm}}$. Then*

- (1) 1_X (hence X) is not good, in fact,
- (2) $\mathcal{D}(X//G)$ cannot be finitely generated as both a left and right $\mathcal{D}(X)^G$ -module.

Proof. – There has to be a slice representation (V, H) of X , where (H) is a non-principal isotropy class such that $(X//G)_{(H)} \cap (X//G)_{\text{sm}} \neq \emptyset$. Apply 5.13. \square

(5.16). COROLLARY. – *Let X be a smooth affine G -variety. If X is good, then $X//G$ has no codimension 1 strata.*

Proof. – Since $X//G$ is normal, it is smooth in codimension 1. \square

6. Extensions by Finite Groups

Let X be an affine G -variety. Then X is also a G^0 -variety, and we investigate relationships between various lifting properties for (X, G) and (X, G^0) . In particular, we investigate the case where G is finite.

We will need the following.

(6.1) LEMMA (see [Ka], [Le1]). – *Let X be normal and Y a subvariety of codimension at least 2. Let E be a vector bundle on X . Then every element of $\mathcal{D}_E^m(X \setminus Y)$ extends uniquely to an element of $\mathcal{D}_E^m(X)$.*

Proof. – We may assume that $E \simeq X \times \mathbb{C}^r$ is trivial. Then the restriction map $\Gamma(X, E) \simeq \mathcal{O}(X)^r \rightarrow \mathcal{O}(X \setminus Y)^r \simeq \Gamma(X \setminus Y, E)$ is an isomorphism, and the result follows.

(6.2) By abuse of language, we say that X has no codimension 1 strata if $X//G \setminus (X//G)_{\text{pr}}$ has codimension 2 in $X//G$. If G is finite, then X is good if and only if it has no codimension 1 strata, a result which goes back to J.-M. Kantor [Ka] (see also [Le1]).

(6.3) THEOREM. – *Let X be normal and G finite. Let E be an admissible G -vector bundle on X . Then*

- (1) $\pi_{X, E}$ is injective.

Let X' denote the smooth points of X and set $X'_{\text{pr}} = X' \cap X_{\text{pr}}$. The following are equivalent:

- (2) $\text{codim} X \setminus X'_{\text{pr}} \geq 2$.
- (3) X has no codimension 1 strata.
- (4) The slice representations at points of X' contain no pseudoreflections.

(5) X is very good.

(6) 1_X is good.

Proof. – Any $P \in \mathcal{D}_E(X)^G$ is determined by its restriction to X'_{pr} . Since $\pi_X|_{X'_{\text{pr}}}$ is a covering and E is the pull-back of a vector bundle on $(X//G)_{\text{pr}}$, (1) follows from 4.2.

If $X = V$ is a G -module, then clearly $V//G$ has a codimension 1 stratum if and only if there is a $g \in G$, $g \neq e$, such that V^g is a hyperplane, i.e., if and only if G contains a pseudoreflection. Since $\text{codim}(Y := X \setminus X') \geq 2$ and $\text{codim} Y//G \geq 2$, Luna's slice theorem shows that (2), (3) and (4) are equivalent.

Let $Q \in \mathcal{D}_E^n(X//G)$. By 4.5, for any affine open G -stable subvariety $Z \subseteq X'_{\text{pr}}$, $Q|_{\pi_X(Z)}$ is the image of some $P_Z \in \mathcal{D}_E^n(Z)$. By (1), the P_Z patch together to give $P \in \mathcal{D}_E(X'_{\text{pr}})^G$ covering $Q|_{\pi_X(X'_{\text{pr}})}$. If (3) holds, then P is the restriction of an element of $\mathcal{D}_E^n(X)^G$, and clearly $\pi_{X,E}P = Q$. Thus (3) implies (5), which in turn implies (6). If (6) holds, then we may apply 5.16 to the affine open G -stable subvarieties of X'_{pr} to obtain (3). \square

(6.4) *Example.* – Let $G = \{\pm 1\}$ act on $V = \mathbb{C}^2$ by multiplication. Let s and t be coordinate functions on V . Then $\mathcal{O}(V)^G$ is generated by $x := s^2 + t^2$, $y := s^2 - t^2$ and $z := 2st$. Thus $V//G$ is the cone $C := \{x^2 = y^2 + z^2\} \subset \mathbb{C}^3$. Since $\text{gr } \mathcal{D}(V)^G \simeq \mathcal{O}(V \oplus V^*)^G$, the generators of $\mathcal{D}(V)^G$ are

(1) The invariants x , y and z .

(2) The vector fields $s\partial/\partial s$, $t\partial/\partial s$, $s\partial/\partial t$ and $t\partial/\partial t$.

(3) The order 2 differential operators $\partial^2/\partial s^2$, $\partial^2/\partial s\partial t$ and $\partial^2/\partial t^2$.

Since G contains no pseudoreflections, $(\pi_V)_*$ is graded surjective, and we see that $\text{gr } \mathcal{D}(C)$ is generated by $\mathcal{D}^2(C)$.

(6.5) **COROLLARY.** – Let X be a normal affine G -variety such that all G -orbits have the same dimension. Then 6.3(2)-6.3(6) are equivalent.

Proof. – Since all G -orbits have the same dimension, all G -orbits are closed. Let S be a slice at $x \in X$ (see 1.9). Then S is normal since X is, and the G_x -orbits on S all have the same dimension. Thus G_x acts on S via a finite quotient and we may apply 6.3. \square

(6.6) **COROLLARY.** – Let V and W be G -modules, G finite. Let E denote Θ_W , let $Q \in \mathcal{D}^n(V//G)$ and let $P \in \mathcal{D}^n(V_{\text{pr}})^G$ be the unique lift of $Q|_{(V//G)_{\text{pr}}}$. Then there is a $P_E \in \mathcal{D}_E^n(V_{\text{pr}})^G$ such that

(1) $\sigma_E^n(P_E) = \sigma^n(P) \otimes \text{id}_E$, where σ_E^n and σ^n are symbol maps (see 3.14).

(2) There is a $Q_E \in \mathcal{D}_E^n(V//G)$ such that P_E is a lift of $Q_E|_{(V//G)_{\text{pr}}}$.

(3) If \tilde{P}_E and \tilde{Q}_E satisfy (1) and (2), then $Q_E - \tilde{Q}_E \in \mathcal{D}_E^{n-1}(V//G)$.

Consider $\text{gr } \mathcal{D}(V//G)$ as a subalgebra of $\text{gr } \mathcal{D}_E(V//G)$ by sending $Q + \mathcal{D}^{n-1}(V//G)$ to $Q_E + \mathcal{D}_E^{n-1}(V//G)$, $Q \in \mathcal{D}^n(V//G)$. Then

(4) $\text{gr } \mathcal{D}_E(V//G)$ is a finite $\text{gr } \mathcal{D}(V//G)$ -module, where $\text{gr } \mathcal{D}(V//G)$ is finitely generated.

Proof. – Let $K = \text{Ker}(G \rightarrow \text{GL}(V))$, and set $(V', G') = (V, G/K)$, $E' = \Theta_{W^k}$. Then $\Gamma(E')^{G'} \simeq \Gamma(E)^G$, and if we can prove (1), etc. for E' , then it follows for E . Hence we may assume that G acts effectively on V so that E is admissible.

Let H be the (normal) subgroup of G generated by its pseudoreflections. Then it is easy to see that $V' := V//H$ has a vector space structure such that $G' := G/H$ acts linearly and effectively and contains no pseudoreflections (see [S3, 8.1]). Moreover, since (V, H) is cofree, $(\mathcal{O}(V) \otimes W)^H \simeq \mathcal{O}(V)^H \otimes W'$ is a free $(\mathcal{O}(V)^H = \mathcal{O}(V'))$ -module where we can choose W' to be a G/H -module (compare 5.2). Set $E' := \Theta_{W'}$. Now (V', G') is very good, $V'//G' \simeq V//G$ and $\Gamma(E')^{G'} \simeq \Gamma(E)^G$. There is a unique lift $P' \in \mathcal{D}^n(V')^{G'}$ of Q , and $P_{E'} := P' \otimes \text{id}_{E'} \in \mathcal{D}_{E'}^n(V')^{G'}$ induces an operator $Q_\varepsilon \in \mathcal{D}_\varepsilon^n(V//G)$. If P_E is the lift of Q_ε to $\mathcal{D}_E^n(V_{\text{pr}})^G$, then (1) and (2) hold. If \tilde{P}_E and \tilde{Q}_ε are as in (3), then the element $\tilde{P}_{E'} \in \mathcal{D}_{E'}^n(V')^{G'}$ covering \tilde{Q}_ε has the same symbol as $P_{E'}$ on V'_{pr} , hence on all of V' . Thus $P_{E'} - \tilde{P}_{E'} \in \mathcal{D}_{E'}^{n-1}(V')^{G'}$, and $Q_\varepsilon - \tilde{Q}_\varepsilon \in \mathcal{D}_\varepsilon^{n-1}(V//G)$, giving (3). Part (4) follows from the fact that (V', G') is very good (see 3.20). \square

(6.7) COROLLARY. – *Let X be a smooth affine G -variety such that all the G -orbits have the same dimension. Let E be a G -vector bundle on X . Then $\text{gr } \mathcal{D}_\varepsilon(X//G)$ is a finite module over $\text{gr } \mathcal{D}(X//G)$, where $\text{gr } \mathcal{D}(X//G)$ is finitely generated.*

Proof. – The slice representations of X are all of the form (W, H) where $H \rightarrow \text{GL}(W)$ has finite image. Let $Q \in \mathcal{D}^n(X//G)$. Using the slice theorem and 6.6, we can find an open cover $\{U_\alpha\}$ of $X//G$ such that $\text{gr } \mathcal{D}_\varepsilon(U_\alpha)$ is a finite $\text{gr } \mathcal{D}(U_\alpha)$ -module for all α . In particular, there are elements $Q_{\varepsilon, \alpha} \in \mathcal{D}_\varepsilon^n(U_\alpha)$ such that $Q|_{U_\alpha} + \mathcal{D}^{n-1}(U_\alpha)$ has image $Q_{\varepsilon, \alpha} + \mathcal{D}_\varepsilon^{n-1}(U_\alpha)$ and such that $Q_{\varepsilon, \alpha} - Q_{\varepsilon, \beta} \in \mathcal{D}_\varepsilon^{n-1}(U_\alpha \cap U_\beta)$ for any α and β . Since $X//G$ is affine and $\mathcal{D}_\varepsilon^{n-1}$ coherent (3.7), we can find $Q_\varepsilon \in \mathcal{D}_\varepsilon^n(X//G)$ such that $(Q_\varepsilon)|_{U_\alpha} \in Q_{\varepsilon, \alpha} + \mathcal{D}_\varepsilon^{n-1}(U_\alpha)$. If $Q' \in \mathcal{D}^m(X//G)$, then the uniqueness part of 6.6 shows that $(QQ')_\varepsilon = Q_\varepsilon \cdot Q'_\varepsilon$ modulo $\mathcal{D}_\varepsilon^{m+n-1}(X//G)$. Thus $\text{gr } \mathcal{D}_\varepsilon(V//G)$ is a $\text{gr } \mathcal{D}(X//G)$ -module. Locally (hence globally) on $X//G$, $\text{gr } \mathcal{D}_\varepsilon(X//G)$ is finite over $\text{gr } \mathcal{D}(X//G)$, and $\text{gr } \mathcal{D}(X//G)$ is finitely generated. \square

We now investigate connections between goodness of $\mathbf{1}_{(X, G)}$ and $\mathbf{1}_{(X, G^0)}$.

(6.8) PROPOSITION. – *Let X be a smooth affine G -variety, let H be a normal subgroup of G of finite index, and let Z denote $X//H$. Then*

- (1) (X, G) has no codimension 1 strata if and only if (X, H) and $(Z, G/H)$ have no codimension 1 strata.
- (2) If (X, G) has no codimension 1 strata and $\mathbf{1}_{(X, H)}$ is (n) -good, then $\mathbf{1}_{(X, G)}$ is (n) -good
- (3) If $X//H$ is smooth and $\mathbf{1}_{(X, G)}$ is good, then $X//H = (X//H)_{\text{pr}}$, i.e., every closed orbit is principal.

Proof. – We leave the proof of (1) and (2) to the reader. In case (3) we may assume that we are in the case where $X = V$ is a G -module. Then $\mathcal{D}(V//H) \simeq \mathcal{D}(\mathbb{C}^d)$, $d = \dim V//H$, where $\Gamma := G/H$ acts linearly on \mathbb{C}^d [S3, 8.1]. Since (V, G) is good, Γ contains no pseudoreflections, and $\mathcal{D}(\mathbb{C}^d)^\Gamma \rightarrow \mathcal{D}(\mathbb{C}^d//\Gamma) \simeq \mathcal{D}(V//G)$ is an isomorphism. In particular, $\mathcal{D}(V//H)$ is a finitely generated left and right $\mathcal{D}(V//G)$ -module, hence $\mathcal{D}(V//H)$ is a finitely generated left and right $\mathcal{D}(V)^H$ -module. Now 5.13 shows that (V, H) is fix pointed. \square

(6.9) COROLLARY. – *Suppose that V is a G -module, where G^0 is semisimple, (V, G^0) is not fix pointed and $\dim V//G^0 = 2$. Then (V, G) is not good.*

Proof. – A theorem of Kempf [Ke] shows that (V, G^0) is coregular. \square

7. Finite Principal Isotropy Groups

It is technically easier to work with G -modules V (resp. G -varieties X) with finite principal isotropy groups (FPIG). We find conditions which allow us to reduce to this case (7.10). However, for G -modules V , the failure of FPIG often implies that V is not good (7.13).

(7.1) For the rest of this section X will denote a *normal* affine G -variety and H will denote a principal isotropy group of X . Let N denote $N_G(H)/H$. Let $X^{(H)}$ denote the points in X^H which lie on principal G -orbits. Then $X^{(H)}$ is open in X^H , and its closure $\overline{X}^{(H)}$ is a union of components of X^H .

(7.2) *Remark.* – The inclusion $\overline{X}^{(H)} \hookrightarrow X$ induces an isomorphism $\varphi : \overline{X}^{(H)} // N \simeq X // G$ where $\varphi((\overline{X}^{(H)} // N)_{\text{pr}}) = (X // G)_{\text{pr}}$ (Luna-Richardson theorem [Lu2], [LR]). In case X is smooth, there is a 1-1 correspondence between the Luna (*i.e.* slice type) stratifications of $X // G$ and $\overline{X}^{(H)} // N$ (see [S3, 11.3]).

(7.3) We say that X is *stable* if there is an open dense subset of X consisting of closed orbits. We let $X_{(0)}$ denote $\{x \in X : \dim G_x = 0\}$.

(7.4) *Remarks.* – From Luna's slice theorem we have

- (1) X has FPIG if and only if $X_{\text{pr}} \subseteq X_{(0)}$.
- (2) If X is smooth, then X is stable if and only if the slice representations of principal isotropy groups are trivial.

(7.5) *THEOREM.* – Let X, H , etc. be as in 7.1. Suppose that

- (1) X is 2-principal.
- (2) X is stable.

Then there is a canonical isomorphism

$$\varphi : G *^{N_G(H)} \overline{X}^{(H)} \rightarrow X, \quad [g, x] \mapsto gx.$$

In particular, $\text{codim } \overline{X}^{(H)} \setminus \overline{X}_{\text{pr}}^{(H)} = \text{codim } X \setminus X_{\text{pr}}$.

Proof. – Set $Y := G *^{N_G(H)} \overline{X}^{(H)}$. By construction, $\varphi|_{Y_{\text{pr}}} : Y_{\text{pr}} \rightarrow X_{\text{pr}}$ is an isomorphism. Thus φ is birational. Since $\text{codim } X \setminus \varphi(Y) \geq 2$, Richardson's Lemma ([Kr, II.3.5]) gives that φ is an isomorphism. \square

(7.6) *PROPOSITION.* – Let X, H , etc. be as in 7.1. If H is normal in G , then

- (1) $\overline{X}^{(H)} = X^H$.
- (2) X is a fix pointed H -variety.
- (3) If X is stable, then H is the ineffective part of the G -action on X .
- (4) If X is smooth, then $\pi_{X,H} : X \rightarrow X^H$ is equidimensional.

Proof. – There is an inclusion $\overline{X}^{(H)} \subset X // H$ where G/H acts on $\overline{X}^{(H)}$ and $X // H$ with FPIG. The morphisms $\overline{X}^{(H)} // (G/H) \rightarrow (X // H) // (G/H) \rightarrow X // G$ are isomorphisms.

Thus $(X//H) \setminus \overline{X}^{(H)}$ is an open subset of $X//H$ containing no principal orbits, hence it must be empty, i.e., $\overline{X}^{(H)} = X//H$. It follows that $\overline{X}^{(H)} = X^H = X//H$, giving (1) and (2). If (X, G) is stable, then $X^H = G \cdot X^H$ is dense in X , hence $X = X^H$, and we have (3). Part (4) follows from 5.3. \square

(7.7) COROLLARY. – Let X, H , etc. satisfy 7.5(1) and 7.5(2).

(1) Let $Gx \subset X$ be a closed orbit with $x \in \overline{X}^{(H)}$, and let S be a slice at x . Then H is a normal subgroup of G_x and H acts trivially on S .

(2) If $X^G \neq \emptyset$, then H is normal in G and is the ineffective part of the G -action on X . In particular, if X is a G -module, then H is the ineffective part of the G -action.

Proof. – In (1), G_x is a subgroup of $N_G(H)$ by 7.5, hence it normalizes H . Now H is the principal isotropy group of (S, G_x) , hence it acts trivially by 7.6. If $x \in X^G$, then S is a G -neighborhood of x in X on which H acts trivially. Hence H acts trivially on X . \square

(7.8) Example. – Let $(V, G) = (2 \wedge^2(\mathbb{C}^5) + (\mathbb{C}^5)^*, \text{SL}_5)$. Then $H := \text{SL}_2$ is a principal isotropy group, where $(\mathbb{C}^5, H) = 3\mathbb{C} + R_1$ with H acting trivially on \mathbb{C} and as usual on $R_1 \simeq \mathbb{C}^2$. The slice representation of H is $R_1 + 2\mathbb{C}$, and $V//G \simeq \mathbb{C}^2$ with principal stratum $\mathbb{C}^2 \setminus \{0\}$. One can compute that

$$(V^H, N_G(H)/H) \simeq (2\nu_{-3} + 2\mathbb{C}^3 \otimes \nu_2 + \mathbb{C}^3 \otimes \nu_{-1}, (\text{SL}_3 \times \mathbb{C}^*)/(\mathbb{Z}/3))$$

where ν_j denotes the one dimensional representation of \mathbb{C}^* of weight j . Thus we have:

- (1) (V, G) is 2-principal (see 7.11(3)).
- (2) $(V^H, N_G(H)/H)$ is not 2-principal.
- (3) H acts effectively on V .

Hence stability of (V, G) is necessary in 7.5 and 7.7.

If G^0 is a torus, then one can sometimes apply a mixture of 7.5 and 7.6 to reduce to the case of FPIG.

(7.9) PROPOSITION. – Suppose that G^0 is a torus and that X is smooth. Then

- (1) The action of H^0 is fix pointed.
- (2) $\pi_{X, H^0} : X \rightarrow X^{H^0}$ is equidimensional.

If, in addition, X is 2-principal, then

- (3) $X^{H^0} \simeq G *^{N_G(H)} \overline{X}^{(H)}$.
- (4) $\text{codim } \overline{X}^{(H)} \setminus \overline{X}_{\text{pr}}^{(H)} = \text{codim } X \setminus X_{\text{pr}}$.

Proof. – Proposition 7.6 shows that the action of the principal isotropy group H' of (X, G^0) is fix pointed, that $\overline{X}^{(H')} = X^{H'}$ and that $\pi_{X, H'}$ is equidimensional. Clearly $(H')^0 = H^0$ is normal in G and an orbit $H'x$ is closed if and only if H^0x is closed. Thus the action of H^0 is fix pointed with $X^{H^0} = X^{H'}$, and we have (1) and (2).

Set $\tilde{G} := G/H^0$, $\tilde{H} := H/H^0$ and $\tilde{X} := X^{H^0}$. Using (2) and our hypotheses, we see that $\text{codim } \tilde{X} \setminus \tilde{X}_{\text{pr}} \geq 2$. Since (\tilde{X}, \tilde{G}) has FPIG, 7.5 shows that $\tilde{X} \simeq \tilde{G} *^{N_{\tilde{G}}(\tilde{H})} \overline{X}^{(H)}$, which gives (3). Part (4) follows from (2) and (3). \square

(7.10) THEOREM. – Let X, H , etc. be as in 7.1. Let E be a G -vector bundle on X and set $\tilde{X} := \overline{X}^{(H)}$, $\tilde{G} := N_G(H)/H$. Assume that X is 2-principal and that

- (1) X is stable, or
- (2) X is smooth and H is normal in G , or
- (3) X is smooth and G^0 is a torus.

Then

(4) There is an admissible G -vector bundle E' on X such that $\Gamma(E)^G \simeq \Gamma(E')^G \simeq \Gamma(\tilde{E})^{\tilde{G}}$, where $\tilde{E} := E'|_{\tilde{X}}$. If E is admissible, then $E' = E$. If X is stable, then $E' \subset E$ is a G -subbundle.

(5) $\pi_{X, E'}$ is (n) -surjective if and only if $\pi_{\tilde{X}, \tilde{E}}$ is (n) -surjective

(6) $\pi_{X, E}$ is surjective if $\pi_{\tilde{X}, \tilde{E}}$ is surjective.

(7) X is good if and only if \tilde{X} is good.

(8) If X is smooth and $\pi_{\tilde{X}, \tilde{E}}$ is graded surjective, then $\text{gr } \mathcal{D}_\varepsilon(X//G)$ is a finite $\text{gr } \mathcal{D}(X)^G$ -module. If $(\pi_{\tilde{X}})_*$ is also graded surjective, then $\text{gr } \mathcal{D}_\varepsilon(X//G)$ is a finite $\text{gr } \mathcal{D}(X//G)$ -module.

Proof. – Clearly (4) and (5) imply (7) and (8). In case (1), set $\tilde{E} := (E|_{\overline{X}^{(H)}})^H$ and $E' := G *^{N_G(H)} \tilde{E}$. Then $E' \subset E$ is an admissible subbundle with the same G -invariant sections, so (4) is obvious, and (5) and (6) follow from 4.6. In case (2), 7.6 shows that $X \rightarrow X//H = X^H$ is fix pointed, and we can apply 5.3. In case (3), 7.9 shows that $X \rightarrow X^{H^0}$ is equidimensional and that $X^{H^0} \simeq G *^{N_G(H)} \overline{X}^{(H)}$, so we can just combine the techniques used above. \square

Here are some useful criteria for showing that a G -module has trivial principal isotropy groups.

(7.11) LEMMA. – Let V be a G -module.

(1) If (V, G^0) is orthogonal (i.e. carries a non-degenerate symmetric G^0 -invariant bilinear form), then V is stable ([Lu1]).

(2) If G^0 is semisimple and $V_{(0)} \neq \emptyset$, then (V, G) has FPIG ([Po1]). In particular, (V, G) is stable.

(3) Let Σ be a stratum of $V//G$ of codimension 2. If (V, G^0) is orthogonal or G^0 is semisimple, then $\text{codim } \pi_V^{-1}(\Sigma) \geq 2$ ([S3, 7.4]).

(7.12) COROLLARY. – Suppose that V is a G -module such that

- (1) (V, G^0) is orthogonal or G^0 is semisimple and $V_{(0)} \neq \emptyset$.
- (2) G acts effectively on V .
- (3) V has no codimension one strata.

Then the principal isotropy groups of V are trivial.

In many cases, the failure of V to satisfy FPIG implies that V is not good.

(7.13) THEOREM. – Let G be connected and let V be a G -module without FPIG such that $V^G = (0)$ and $\dim V//G > 0$. If

- (1) G is simple, or
 - (2) G is semisimple and V is irreducible,
- then $\mathbf{1}_V$ (hence V) is not good.

Proof. – For part (1) it suffices to compare the tables in [E11] and [S1]. One finds that V has a non-principal coregular slice representation, hence V is not good. Now let V be as in (2). We may assume that V is not coregular and “castling reduced” (see [Li, §1]). From [E12] and [Li] one sees that the possibilities are the following:

- (1) $(\mathbb{C}^n \otimes W, \mathrm{SO}_n \times H)$, $\dim W \leq n - 2$.
- (2) $(\mathbb{C}^{2n} \otimes W, \mathrm{Sp}_{2n} \times H)$, $\dim W \leq 2n - 1$.
- (3) $(\mathbb{C}^n \otimes \widetilde{\mathbb{C}^n} \otimes \mathbb{C}^2, \mathrm{SL}_n \times \widetilde{\mathrm{SL}_n} \times \mathrm{SL}_2)$, $n \geq 5$.
- (4) $(\Lambda^2 \mathbb{C}^{2n} \otimes \mathbb{C}^2, \mathrm{SL}_{2n} \times \mathrm{SL}_2)$, $n \geq 5$.
- (5) $(\mathbb{C}^{26} \otimes \mathbb{C}^2, F_4 \times \mathrm{SL}_2)$.
- (6) $(\mathbb{C}^{27} \otimes \mathbb{C}^3, E_6 \times \mathrm{SO}_3)$.

Here (W, H) denotes an irreducible representation of the semisimple group H , and the rest of the notation above is, hopefully, self explanatory.

In cases (3)–(6), let H denote the last simple factor of G and let G' denote the product of all the other simple factors. Then in all cases we have $(V, G) = (V' \otimes W, G' \times H)$ for an appropriate G' and V' . In every case except (2), when $\dim W$ is odd, the quotient $V//G'$ is smooth and has codimension 1 strata. Since H is semisimple, the images of the codimension 1 strata of $V//G'$ in $V//G$ have codimension 1. Hence V has codimension 1 strata and is not good.

There remains the case $(V, G) = (\mathbb{C}^{2n} \otimes W, \mathrm{Sp}_{2n} \times H)$, $\dim W := 2k + 1 < 2n$. Set $K = \mathrm{Sp}_{2n-2k+2}$. Then (K) is the isotropy class corresponding to the codimension 3 stratum of $((2k+1)\mathbb{C}^{2n})//\mathrm{Sp}_{2n}$. The classification of [E12] shows that if (V, G) is not coregular, then $(V^K, N_G(K)/K) = (\mathbb{C}^{2k-2} \otimes W, \mathrm{Sp}_{2k-2} \times H)$ has FPIG. Thus there is a closed orbit Gx , $x \in V^K$, such that $N_G(K)_x$ is a finite extension \tilde{K} of K ([Lu2]). From the special form of (V, G) one can see that $G_x \subset N_G(K)$, hence $G_x = \tilde{K}$. The slice representation of K in (V, Sp_{2n}) is $(3\mathbb{C}^{2n-2k+2}, K)$, modulo trivial representations, and it follows that the slice representation of \tilde{K} in (V, G) , when restricted to K , is of the same form (with $\dim H$ fewer trivial factors). Since $(3\mathbb{C}^{2n-2k+2}, K)$ is coregular, 6.8 shows that the slice representation of \tilde{K} is not good. \square

8. Regular Sequences in $\mathcal{O}(T^*X)$

We find sufficient conditions for a smooth affine G -variety X to be very good.

We begin with some homological preliminaries; see [BE] for a more general treatment.

(8.1) Let A be a noetherian commutative ring with identity and M an A -module. Let f_1, \dots, f_s be a sequence of elements of A , and let (f_1, \dots, f_s) denote the ideal they generate. The f_i are an M -sequence (or M -regular sequence) if multiplication by f_{i+1} is injective on $M/(f_1, \dots, f_i)M$, $0 \leq i < s$. If I is an ideal of A we write $\mathrm{depth}_I M \geq s$ if there is an M -sequence of length s in I . Suppose now that M is finite. Then $\mathrm{depth}_I M \geq s$ if and only if $\mathrm{Ext}_A^i(A/I, M) = 0$ for $i < s$ (see [Ma, Th. 28]). If $M = IM$, then $\mathrm{Ext}_A^i(A/I, M) = 0$ for all i , and $\mathrm{depth}_I M = \infty$.

(8.2) In the following, X denotes an affine variety, Y is a closed subvariety, and $I \subset A := \mathcal{O}(X)$ denotes the corresponding ideal. If M is an A -module, we will confuse M with the corresponding sheaf \tilde{M} of \mathcal{O}_X -modules. For example, we say that $m \in M$ is zero on $X \setminus Y$ if the corresponding section of \tilde{M} vanishes on $X \setminus Y \subset \text{Spec } A$.

Suppose that X is smooth. Then f_1, \dots, f_s are A -regular if and only if their zero set $Z(f_1, \dots, f_s) \subseteq X$ has codimension s ([D], [Se], [Ma]). In particular, $\text{depth}_I A = \text{codim } Y$ (where $\text{codim } \emptyset = \infty$).

Suppose that X is a smooth G -variety and that $Y = \pi_X^{-1}(\pi_X(Y))$. An induction argument produces $h_1, \dots, h_m \in I^G$ which are an $\mathcal{O}(X)$ -regular sequence (cf. [S3, 10.5]), where $m = \text{codim } Y$.

(8.3) LEMMA. – *Let*

$$0 \rightarrow M \xrightarrow{\varphi} N \rightarrow 0$$

be a complex of finite A -modules which is exact when localized at any point of $X \setminus Y$.

(1) *If $\text{depth}_I M \geq 1$, then φ is injective.*

(2) *If $\text{depth}_I M \geq 2$ and $\text{depth}_I N \geq 1$, then φ is an isomorphism.*

(3) *Suppose that M and N are projective. Then φ is injective if $\text{codim } Y \geq 1$ and φ is an isomorphism if $\text{codim } Y \geq 2$.*

Proof. – If $\text{depth}_I M \geq 1$, there is an $f \in I$ which is not a zero divisor on M . Let $m \in \text{Ker } \varphi$. Since $X_f \subset X \setminus Y$, $\varphi : M_f \rightarrow N_f$ is injective and $f^l m = 0$ for some $l \geq 0$. This implies that $m = 0$, and we have (1). In (2), suppose that $n \in N$. Let $f_1, f_2 \in I$ be M -regular. Since φ is an isomorphism over X_{f_1} and X_{f_2} , we can find $k, l \geq 0$ and $m_1, m_2 \in M$ such that $\varphi(m_1) = f_1^k n$ and $\varphi(m_2) = f_2^l n$. Since φ is injective, we must have $f_1^k m_2 = f_2^l m_1$. The f_i are M -regular, hence we must have $m_2 = f_2^l m$ and $m_1 = f_1^k m$ for some $m \in M$. Then $n - \varphi(m)$ vanishes on $X \setminus Y$, and $\text{depth}_I N \geq 1$ implies that $n = \varphi(m)$, so we have (2). Part (3) follows from the remarks in 8.2 \square

(8.4) COROLLARY. – *Let*

$$(*) \quad 0 \rightarrow M_k \xrightarrow{\varphi_k} M_{k-1} \xrightarrow{\varphi_{k-1}} \dots \rightarrow M_1 \xrightarrow{\varphi_1} M_0$$

be a complex of finite A -modules which is exact on $X \setminus Y$. Suppose that $r \geq 0$ and either

(1) *$\text{depth}_I(M_j) \geq j + r$, $j = 0, \dots, k$, or*

(2) *each M_j is projective, X is smooth and $\text{codim } Y \geq k + r$.*

Then () is exact and $\text{depth}_I M_j / \varphi_{j+1}(M_{j+1}) \geq j + r$, $j = 0, \dots, k$.*

Proof. – We may assume (1) since it is implied by (2). If $k = 0$ there is nothing to prove. If $k = 1$, then 8.3(1) and the exact sequence of Ext give the result. Suppose that $k \geq 2$. Then $0 \rightarrow M_k \rightarrow \text{Ker}(\varphi_{k-1}) \rightarrow 0$ is exact by 8.3, and

$$0 \rightarrow M_{k-1}/M_k \rightarrow M_{k-2} \rightarrow \dots$$

is exact over $X \setminus Y$. The exact sequence of Ext shows that $\text{depth}_I(M_{k-1}/M_k) \geq k - 1 + r$, hence by induction (*) is exact and $\text{depth}_I M_j / \varphi_{j+1}(M_{j+1}) \geq j + r$. \square

(8.5) From now on X will always denote a smooth affine G -variety. We construct a Koszul-like complex from the \mathfrak{g} -action on X .

Since \mathfrak{g} consists of sections of TX , we may consider \mathfrak{g} as functions on T^*X . Explicitly, if $A \in \mathfrak{g}$, we obtain a function f_A whose value at $\xi_x \in T_x^*X$ is the contraction $\langle A(x), \xi_x \rangle$. The functions f_A are closely related to the moment mapping $\mu : T^*X \rightarrow \mathfrak{g}^*$. In fact, $\mu(\xi_x)(A) = f_A(\xi_x)$; $\xi_x \in T^*X$, $A \in \mathfrak{g}$.

Let A_1, \dots, A_l be a basis of \mathfrak{g} . Then $Z := Z(f_{A_1}, \dots, f_{A_l}) \subseteq T^*X$ is independent of the basis, and we say that \mathfrak{g} is *regular in* $\mathcal{O}(T^*X)$ if Z has codimension l . Equivalently, the f_{A_i} form an $\mathcal{O}(T^*X)$ -regular sequence. Note that $Z = \mu^{-1}(0)$.

In case $X = V$ is a G -module, then $T^*V \simeq V \oplus V^*$, and the f_A are quadratic elements of $\mathcal{O}(V \oplus V^*)$.

(8.6) *Remarks.* – (1) If \mathfrak{g} is regular in $\mathcal{O}(T^*X)$, then the general G -orbit in X has finite isotropy, *i.e.*, $X_{(0)} \neq \emptyset$. The converse is false (*see* 9.3–4). However, if $X = X_{(0)}$, then \mathfrak{g} is regular in $\mathcal{O}(T^*X)$.

(2) \mathfrak{g} is regular in $\mathcal{O}(T^*X)$ if and only if \mathfrak{h} is regular in $\mathcal{O}(W \oplus W^*)$ for every slice representation (W, H) of X (*see* 9.4).

(3) The moment map $\mu : T^*X \rightarrow \mathfrak{g}^*$ is equivariant with respect to the scalar \mathbb{C}^* -actions on T^*X and \mathfrak{g}^* . Thus μ is equidimensional and dominant if and only if $\text{codim } \mu^{-1}(0) = \dim G$, *i.e.*, if and only if \mathfrak{g} is regular in $\mathcal{O}(T^*X)$.

(8.7) Let E be an admissible G -vector bundle over X . We show how to use the regularity of \mathfrak{g} in $\mathcal{O}(T^*X)$ to obtain information about $\mathcal{K}_E(X)$: Let $\mathcal{U} := U(\mathfrak{g})$ denote the universal enveloping algebra of \mathfrak{g} , and let $\{\mathcal{U}^k\}$ denote the usual filtration. There is a free resolution \mathcal{B}_\bullet of the trivial \mathcal{U} -module \mathbb{C} :

$$0 \rightarrow \mathcal{B}_l \xrightarrow{d_l} \mathcal{B}_{l-1} \xrightarrow{d_{l-1}} \dots \rightarrow \mathcal{B}_0 \rightarrow \mathbb{C} \rightarrow 0$$

where $\mathcal{B}_p = \mathcal{U} \otimes_{\mathbb{C}} \Lambda^p \mathfrak{g}$, as follows (*see* [HS, Ch. VII §4]): Let A_1, \dots, A_l be a basis of $\mathfrak{g} \subset \mathcal{U}^1$. Set

$$\begin{aligned} d_p(A_{k_1} \wedge \dots \wedge A_{k_p}) &= \sum_{i=1}^p (-1)^{i+1} A_{k_i} \otimes (A_{k_1} \wedge \dots \wedge \widehat{A}_{k_i} \wedge \dots \wedge A_{k_p}) \\ &+ \sum_{1 \leq i < j \leq p} (-1)^{i+j} [A_{k_i}, A_{k_j}] \wedge A_{k_1} \wedge \dots \wedge \widehat{A}_{k_i} \wedge \dots \wedge \widehat{A}_{k_j} \wedge \dots \wedge A_{k_p}. \end{aligned}$$

Clearly the d_p preserve the filtration of \mathcal{B}_\bullet by the subcomplexes $F^m \mathcal{B}_\bullet$, where $F^m \mathcal{B}_p := \mathcal{U}^{m-p} \otimes_{\mathbb{C}} \Lambda^p \mathfrak{g}$. The associated graded complex is just the Koszul complex of the elements A_1, \dots, A_l of $S^\bullet(\mathfrak{g})$, and this complex is exact.

We map the $A_i \in \mathfrak{g} \subset \mathcal{U}^1$ to the corresponding elements (also denoted A_i) in $\mathcal{D}_E^1(X)$, and we thus obtain a homomorphism $\mathcal{U} \rightarrow \mathcal{D}_E(X)$. From the complex \mathcal{B}_\bullet we obtain a complex \mathcal{C}_\bullet where

$$\mathcal{C}_p = \mathcal{D}_E(X) \otimes_{\mathcal{U}} \mathcal{B}_p \simeq \mathcal{D}_E(X) \otimes_{\mathbb{C}} \Lambda^p \mathfrak{g},$$

and there is a natural surjection $\mathcal{C}_\bullet \rightarrow \mathcal{D}_E(X)/\mathcal{D}_E(X)\mathfrak{g}$. There is a filtration $\{F^m \mathcal{C}_\bullet\}$ of \mathcal{C}_\bullet by subcomplexes, where $F^m \mathcal{C}_p = \mathcal{D}_E^{m-p}(X) \otimes_{\mathbb{C}} \Lambda^p \mathfrak{g}$, and there are natural surjections $F^m \mathcal{C}_\bullet \rightarrow \mathcal{D}_E^m(X)/\mathcal{D}_E^{m-1}(X)\mathfrak{g}$. The $F^m \mathcal{C}_\bullet$ and \mathcal{C}_\bullet are complexes of $\mathcal{O}(X)$ -modules.

(8.8) LEMMA. – *Let E be an admissible G -vector bundle on X , where X has FPIG. Then*

- (1) $F^m \mathcal{C}_\bullet$ is exact when localized at any point of $X_{(0)}$.
- (2) Locally on X_{pr} , $\mathcal{K}_E^m(X) = \mathcal{D}_E^{m-1}(X)\mathfrak{g}$.
- (3) Locally on $(X//G)_{\text{pr}}$, $\pi_{X,E} \mathcal{D}_E^m(X)^G = \mathcal{D}_E^m(X//G)$.

Proof. – The associated graded to $\mathcal{D}_E(X)$ is $\mathcal{O}(T^*X) \otimes_{\mathcal{O}(X)} \Gamma(\text{End}(E))$, and the image of $A \in \mathfrak{g}$ is $f_A \otimes \text{id}_E$ (see 3.14 and 3.22). Thus the associated graded to the filtration $\{F^m \mathcal{C}_\bullet\}$ of \mathcal{C}_\bullet is the Koszul complex of the f_{A_i} tensored with $\Gamma(\text{End}(E))$. Since the f_{A_i} form a regular sequence on any affine open subvariety Z of $T^*X|_{X_{(0)}}$, the associated graded complex is exact on Z . By induction on m , the complexes $F^m \mathcal{C}_\bullet$ are also exact on Z , and (1) follows. Proposition 5.3 gives (3). By 4.9, we may reduce (2) (and (3)) to the (obvious) case where $X = V$ is a G -module, G is finite, G acts trivially on V , $E = \Theta_W$ and G acts trivially on W (by admissibility). \square

The following result is an extension of the methods of [S3], which dealt only with vector fields, to the case of differential operators.

(8.9) THEOREM. – *Suppose that X is smooth and has FPIG and that $\text{codim}(X \setminus X_{\text{pr}}) \geq m \geq 2$. Let E be an admissible G -vector bundle on X . Then*

- (1) $\mathcal{K}_E^p(X) = \mathcal{D}_E^{p-1}(X)\mathfrak{g}$, $1 \leq p \leq m-1$.
- (2) If $m \geq \dim G + 1$, then $\mathcal{K}_E^p(X) = \mathcal{D}_E^{p-1}(X)\mathfrak{g}$ for all p .
- (3) X is $(m-2)$ -good.
- (4) If $m \geq \dim G + 2$, then X is very good.

Proof. – Let Y denote $X \setminus X_{\text{pr}}$, and let I denote the corresponding ideal in $\mathcal{O}(X)$. If $p \leq \min\{\dim G, m-1\}$, consider the complex $F^p \mathcal{C}_\bullet$:

$$0 \rightarrow \mathcal{D}_E^0(X) \otimes_{\mathbb{C}} \Lambda^p(\mathfrak{g}) \rightarrow \cdots \rightarrow \mathcal{D}_E^{p-1}(X) \otimes_{\mathbb{C}} \mathfrak{g} \rightarrow \mathcal{D}_E^p(X).$$

The image of $\mathcal{D}_E^{p-1}(X) \otimes_{\mathbb{C}} \mathfrak{g} \rightarrow \mathcal{D}_E^p(X)$ lies in $\mathcal{K}_E^p(X)$, so we also have a complex

$$(\#) \quad 0 \rightarrow M := (\mathcal{D}_E^{p-1}(X) \otimes_{\mathbb{C}} \mathfrak{g}) / \text{Im}(\mathcal{D}_E^{p-2}(X) \otimes_{\mathbb{C}} \Lambda^2 \mathfrak{g}) \rightarrow \mathcal{K}_E^p(X) \rightarrow 0.$$

All the $\mathcal{O}(X)$ -modules in $F^p \mathcal{C}_\bullet$ are locally isomorphic to copies of $\mathcal{D}^q(X)$ for some q , and these are, in turn, projective over $\mathcal{O}(X)$. Then 8.4 shows that $F^p \mathcal{C}_\bullet$ is exact and that $\text{depth}_I M \geq 2$. Since $\mathcal{K}_E^p(X) \subset \mathcal{D}_E^p(X)$ is torsion free, $\text{depth}_I \mathcal{K}_E^p(X) \geq 1$. Applying 8.3 and 8.8(2) to $(\#)$, we see that $M \simeq \mathcal{K}_E^p(X)$. If $m \geq \dim G + 1$, our argument applies to $\mathcal{K}_E^p(X)$ for all p , hence we have (1) and (2).

To prove (3) and (4) we may reduce to the case that $X = V$ and $E = \Theta_W$, where V and W are G -modules. Set $J := I^G \subset \mathcal{O}(V)^G$, and suppose that $p \leq \min\{\dim G, m-2\}$. From 8.2 we have that $\text{depth}_J(\mathcal{O}(V)) \geq m$. The complex $(F^p \mathcal{C}_\bullet)^G$ has terms which are sums of $\mathcal{O}(V)^G$ -modules of covariants, i.e., the modules are direct summands of several copies of $\mathcal{O}(V)$ (cf. [S3, 10.5]). Each such module has J -depth at least m . Applying 8.4 to $(F^p \mathcal{C}_\bullet)^G$ we see that

$$\text{depth}_J(\mathcal{D}_E^p(V) / \mathcal{D}_E^{p-1}(V)\mathfrak{g})^G = \text{depth}_J(\mathcal{D}_E^p(V) / \mathcal{K}_E^p(V))^G \geq m - p \geq 2.$$

Since $\mathcal{D}_E^p(X//G)$ has no $\mathcal{O}(V)^G$ -torsion, its J -depth is at least 1. Applying 8.3 and 8.8(3) we see that the sequence

$$0 \rightarrow (\mathcal{D}_E^p(V)/\mathcal{K}_E^p(V))^G \rightarrow \mathcal{D}_E^p(V//G) \rightarrow 0$$

is exact. If $m \geq \dim G + 2$, then we can carry out the same argument for any $\mathcal{D}_E^p(V//G)$, hence (3) and (4) hold. \square

We now establish versions of 8.9 which rely on hypotheses which include regularity of \mathfrak{g} in $\mathcal{O}(T^*X)$.

(8.10) Let A denote $\mathcal{O}(X)$ and let $\mathcal{G}_E^m(X)$ denote $\mathcal{O}(T^*X) \otimes_A \Gamma(\text{End}(E))$. Then $\mathcal{G}_E^m(X)$ is a projective A -module, and the symbol sequence is a (split) exact sequence of A -modules:

$$0 \rightarrow \mathcal{D}_E^{m-1}(X) \rightarrow \mathcal{D}_E^m(X) \xrightarrow{\sigma_E^m} \mathcal{G}_E^m(X) \rightarrow 0.$$

Denote by $\mathcal{G}_E^m(X)\mathfrak{g}$ the product of $\mathcal{G}_E^m(X)$ and $\text{Im}(\mathfrak{g} \rightarrow \mathcal{O}(T^*X))$ in $\mathcal{O}(T^*X) \otimes_A \Gamma(\text{End}(E))$.

(8.11) PROPOSITION. – *Suppose that \mathfrak{g} is regular in $\mathcal{O}(T^*X)$. Then*

(1) *The sequence*

$$0 \rightarrow \frac{\mathcal{D}_E^m(X)}{\mathcal{D}_E^{m-1}(X)\mathfrak{g}} \rightarrow \frac{\mathcal{D}_E^{m+1}(X)}{\mathcal{D}_E^m(X)\mathfrak{g}} \rightarrow \frac{\mathcal{G}_E^{m+1}(X)}{\mathcal{G}_E^m(X)\mathfrak{g}} \rightarrow 0$$

is exact, $m \geq 0$.

(2) *For $k \geq m \geq 0$, we have $\mathcal{D}_E^k(X)\mathfrak{g} \cap \mathcal{D}_E^m(X) = \mathcal{D}_E^{m-1}(X)\mathfrak{g}$.*

(3) *$\text{gr } \mathcal{D}_E(X)/\text{gr}(\mathcal{D}_E(X))\mathfrak{g} \simeq \text{gr}(\mathcal{D}_E(X)/\mathcal{D}_E(X)\mathfrak{g})$.*

Proof. – The complexes $F^{m+1}\mathcal{C}_\bullet/F^m\mathcal{C}_\bullet$ have trivial homology in positive degree since their direct sum is the Koszul complex of $\mathfrak{g} \subset \mathcal{O}(T^*X)$ tensored with $\Gamma(\text{End}(E))$. Using induction on m and the exact sequences

$$0 \rightarrow F^m\mathcal{C}_\bullet \rightarrow F^{m+1}\mathcal{C}_\bullet \rightarrow F^{m+1}\mathcal{C}_\bullet/F^m\mathcal{C}_\bullet \rightarrow 0$$

one establishes the vanishing of higher homology for each $F^m\mathcal{C}_\bullet$. Taking homology in the exact sequences we obtain (1). Part (2) follows from (1) by downward induction on k , and (3) follows from (2). \square

(8.12) PROPOSITION. – *Let $h_1, \dots, h_s \in \mathcal{O}(X)$ and let A_1, \dots, A_l be a basis of \mathfrak{g} . Suppose that $h_1, \dots, h_s, f_{A_1}, \dots, f_{A_l}$ is a regular sequence in $\mathcal{O}(T^*X)$. Then h_1, \dots, h_s is a regular sequence for $\mathcal{D}_E^m(X)/\mathcal{D}_E^{m-1}(X)\mathfrak{g}$, $m \geq 0$.*

Proof. – Let \mathcal{K}_\bullet denote the Koszul complex of the $h_i \in \mathcal{O}(X)$. Let

$$\mathcal{J}_\bullet^m = \mathcal{K}_\bullet \otimes_{\mathcal{O}(X)} \frac{\mathcal{D}_E^m(X)}{\mathcal{D}_E^{m-1}(X)\mathfrak{g}}.$$

Since the \mathcal{K}_p are projective $\mathcal{O}(X)$ -modules, from 8.11(1) we obtain a short exact sequence of chain complexes

$$0 \rightarrow \mathcal{J}_\bullet^m \rightarrow \mathcal{J}_\bullet^{m+1} \rightarrow \mathcal{Q}_\bullet^{m+1} := \mathcal{K}_\bullet \otimes_{\mathcal{O}(X)} \frac{\mathcal{G}_E^{m+1}(X)}{\mathcal{G}_E^m(X)\mathfrak{g}} \rightarrow 0.$$

The higher homology of $\mathcal{Q}_{\bullet}^{m+1}$ is trivial, since the f_{A_i} and h_j are $\mathcal{O}(T^*X)$ -regular. By induction, each \mathcal{J}_{\bullet}^m has trivial higher homology, i.e., the h_j are a regular sequence on each $\mathcal{D}_E^m(X)/\mathcal{D}_E^{m-1}(X)\mathfrak{g}$. \square

(8.13) LEMMA. – Let E be an admissible G -vector bundle on X , where X has FPIG. Let A_1, \dots, A_l be a basis of \mathfrak{g} such that h and f_{A_1}, \dots, f_{A_l} form a regular sequence in $\mathcal{O}(T^*X)$, where $0 \neq h \in \mathcal{O}(X)$ vanishes on $X \setminus X_{\text{pr}}$. Then $\mathcal{K}_E^m(X) = \mathcal{D}_E^{m-1}(X)\mathfrak{g}$ for all m .

Proof. – By 8.12, h is not a zero divisor on $\mathcal{D}_E^m(X)/\mathcal{D}_E^{m-1}(X)\mathfrak{g}$, hence it is not a zero divisor on $M := \mathcal{K}_E^m(X)/\mathcal{D}_E^{m-1}(X)\mathfrak{g}$. Since M vanishes on $X_h \subset X_{\text{pr}}$, $M = 0$. \square

(8.14) COROLLARY. – Let X , h , etc. be as above where we now assume that $h \in \mathcal{O}(X)^G$. Let τ denote the injection $\mathcal{D}_E(X)^G/\mathcal{K}_E(X)^G \hookrightarrow \mathcal{D}_{\mathcal{E}}(X//G)$. Then

- (1) $\text{gr } \tau : \text{gr } (\mathcal{D}_E(X)^G/\mathcal{K}_E(X)^G) \rightarrow \text{gr } \mathcal{D}_{\mathcal{E}}(X//G)$ is injective.
- (2) E is good if and only if it is very good.

Proof. – By 8.11(1), (the proof of) 8.12, and 8.13, h is not a zero divisor on

$$(\mathcal{D}_E^m(X)/(\mathcal{D}_E^{m-1}(X) + \mathcal{K}_E^{m-1}(X)))^G \simeq \left(\frac{\mathcal{G}_E^m(X)}{\mathcal{G}_E^{m-1}(X)\mathfrak{g}} \right)^G.$$

Since

$$0 \rightarrow (\mathcal{D}_E^m(X)/(\mathcal{D}_E^{m-1}(X) + \mathcal{K}_E^{m-1}(X)))^G \rightarrow \mathcal{D}_{\mathcal{E}}^m(X//G)/\mathcal{D}_{\mathcal{E}}^{m-1}(X//G)$$

is injective locally over $(X//G)_{\text{pr}}$, it is injective. Hence $\text{gr } \tau$ is injective, and X is very good if it is good. \square

(8.15) PROPOSITION. – Let A_1, \dots, A_l be a basis of \mathfrak{g} , and let h_1, h_2 be elements of $\mathcal{O}(X)^G$. Suppose that:

- (1) $h_1, h_2, f_{A_1}, \dots, f_{A_l}$ is a regular sequence in $\mathcal{O}(T^*X)$.
- (2) h_1 and h_2 vanish on $X \setminus X_{\text{pr}}$.

Then X is very good.

Proof. – Apply 8.3 to

$$0 \rightarrow \mathcal{D}_E^m(X)^G/\mathcal{K}_E^m(X)^G \rightarrow \mathcal{D}_{\mathcal{E}}^m(X//G) \rightarrow 0.$$

(8.16) Remark. – Let Y denote $X \setminus X_{\text{pr}}$, and let I denote the corresponding ideal of $\mathcal{O}(X)$. Suppose that $\text{codim } Y := m \geq \dim G + 2 = l + 2$. Then there are $h_1, h_2 \in I^G$ which are $\mathcal{O}(X)$ -regular (see 8.2). Now the vector fields A_1, \dots, A_l are linearly independent on $X_{(0)}$, hence the zero set $Z(f_{A_1}, \dots, f_{A_l})$ has codimension l in the restriction of T^*X to $Z(h_1, h_2) \setminus Y$. Since $\text{codim } Y = m \geq l + 2$, it follows that $h_1, h_2, f_{A_1}, \dots, f_{A_l}$ is a regular sequence in $\mathcal{O}(T^*X)$. Thus both 8.15 and 8.9(4) show that X is very good. However, for $(V, G) = (2n\mathbb{C}^n, \text{SL}_n)$, the hypotheses of 8.15 hold (see 11.15), while $\text{codim } V \setminus V_{\text{pr}} = n + 1 < n^2 + 1 = \dim G + 2$, so that 8.9(4) does not apply.

9. Modularity

We reformulate the results of §8 in more geometric terms. In this section, X denotes an affine G -variety.

(9.1) Recall that $X_{(n)} = \{x \in X : \dim G_x = n\}$. Define $\text{mod}(X, G)$, the *modularity* of (X, G) , to be $\sup_n \{\dim X_{(n)} - \dim G + n\}$ (see [Vi]). Our standing assumption on X is that $X = GX_0$ where X_0 is an irreducible component of X . Define $d(X, G)$ to be the transcendence degree of $\mathcal{Q}(X_0)^{G_0}$, where $\mathcal{Q}(X_0)$ denotes the field of rational functions on X_0 and G_0 is the stabilizer of X_0 (see 1.3).

(9.2) *Remarks.* – (1) By a theorem of Rosenlicht, $d(X, G) = \dim X - \sup_x \dim Gx$. Thus $d(X, G) = \text{mod}(X_{(k)}, G)$ where $k \in \mathbb{N}$ is minimal such that $X_{(k)} \neq \emptyset$.

(2) $d(X, G) \leq \text{mod}(X, G)$.

(3) If the G -action is stable, then $d(X, G) = \dim X // G$. In particular, if X has FPIG, then $d(X, G) = \dim X - \dim G$.

(4) Both $\text{mod}(X, G)$ and $d(X, G)$ only depend on (X, G^0) .

(5) If G is a torus, then $\text{mod}(X, G) = d(X, G)$ (Vinberg [Vi]).

(9.3) *Example.* – Let $G = \text{SL}_n$, $V = (k+n)\mathbb{C}^n$, $n \geq 3$, $0 \leq k \leq n-3$. Then $d(V, G) = kn + 1 < [(n+k)^2/4] = \text{mod}(\mathcal{N}_G(V), G) = \text{mod}(V, G)$ (see 11.13).

(9.4) **PROPOSITION.** – *Let X be a smooth affine G -variety. The following are equivalent.*

(1) \mathfrak{g} is regular in $\mathcal{O}(T^*X)$.

(2) $X_{(0)} \neq \emptyset$ and $\text{mod}(X, G) = d(X, G)$.

(3) $\text{codim } X_{(n)} \geq n$ for all $n \in \mathbb{N}$.

(4) $\text{codim } W_{(n)} \geq n$ for all $n \in \mathbb{N}$ for every slice representation (W, H) of X .

(5) \mathfrak{h} is regular in $\mathcal{O}(W \oplus W^*)$ for every slice representation (W, H) of X .

Proof. – If (W, H) is a slice representation of X , then $(G *^H W)_{(n)} = G *^H (W_{(n)})$, so clearly (3) and (4) are equivalent. Thus we need only show that (1), (2) and (3) are equivalent. We may assume that $X_{(0)} \neq \emptyset$ since this is implied by (1) (see 8.6(1)) and (3). Then $d(X, G) = \dim X - \dim G$ by 9.2(1), and the equivalence of (2) and (3) is clear. The zero set of \mathfrak{g} inside $\mathcal{O}(T^*X)|_{X_{(n)}}$ has codimension $\dim G - n$, since $\dim \mathfrak{g}(x) = \dim G - n$ for every $x \in X_{(n)}$. Thus (1) and (3) are equivalent. \square

(9.5) *Definitions.* – Let $k \geq 0$ and let X be a smooth affine G -variety with FPIG. We say that X is

(1) *k-modular* if $\text{mod}(X \setminus X_{(0)}, G) + k \leq \dim X // G$,

(2) *k-principal* if $\text{codim } X \setminus X_{\text{pr}} \geq k$, and

(3) *k-large* if it is k -modular and k -principal.

(9.6) *Remarks.* – Let X be as above.

(1) Suppose that G is finite. Then X is automatically k -modular for all k . In addition, if G acts trivially, then X is automatically k -large for all k .

(2) X is k -modular (resp. k -principal, resp. k -large) if and only if all of its slice representations are also.

- (3) X is k -modular if and only if $\text{codim } X_{(n)} \geq n + k$; $n = 1, 2, \dots, \dim G$.
 (4) If X is 2-principal, then X has no codimension one strata.
 (5) X is k -large if and only if $\text{mod}(X \setminus X_{\text{pr}}, G) + k \leq \dim X // G$.

(9.7) LEMMA. – Let X be smooth and k -modular and let A_1, \dots, A_l be a basis of \mathfrak{g} . Suppose that $h_1, \dots, h_k \in \mathcal{O}(X)^G$ are $\mathcal{O}(X)$ -regular. Then $h_1, \dots, h_k, f_{A_1}, \dots, f_{A_l}$ are $\mathcal{O}(T^*X)$ -regular.

Proof. – Let Y denote an irreducible component of the zero set of the h_i . Then Y has codimension k in X . By hypothesis, $X_{(j)}$ has codimension at least $j + k$ in X , hence $Y \cap X_{(j)}$ has codimension at least j in Y . As in 9.4, this implies that f_{A_1}, \dots, f_{A_l} is a regular sequence on $T^*X|_Y$, and the lemma follows. \square

Let X be smooth. Using 7.10, 8.2, 8.9, 8.13–8.15 and 9.7 we obtain the following three results.

(9.8) THEOREM. – Suppose that X has FPIG and is m -principal, $m \geq 2$. Then X is $(m - 2)$ -good. If $m \geq \dim G + 2$, then X is 2-large and is very good.

(9.9) THEOREM. – Suppose that X is 1-large, and let E be an admissible G -vector bundle on X . Then

- (1) $\mathcal{K}_E^m(X) = \mathcal{D}_E^{m-1}(X)\mathfrak{g}$, $m \geq 0$.
 (2) The canonical morphism $\text{gr}(\mathcal{D}_E(X)^G / \mathcal{K}_E(X)^G) \rightarrow \text{gr } \mathcal{D}_E(X // G)$ is injective.
 (3) E is good if and only if it is very good.

Hence X is good if and only if it is very good.

(9.10) THEOREM. – Suppose that X is 2-large. Then

- (1) X is very good.
 (2) If E is a G -vector bundle on X , then $\pi_{X,E}$ is graded surjective. Hence $\text{gr } \mathcal{D}_E(X // G)$ is finite over the finitely generated algebra $\text{gr } \mathcal{D}(X // G)$.

(9.11) Example. – Let $(V, G) = (k\mathbb{C}^2, \text{SL}_2)$, $k \geq 2$. We will see that V is $(k - 2)$ -large. When $k = 2$, $\mathcal{K}(V)^G \neq (\mathcal{D}(V)\mathfrak{g})^G$ ([S3, 9.2]). When $k = 3$, V is 1-large but not good, since it is coregular. For $k \geq 4$, V is very good.

As a consequence of 5.6, 5.15, 9.8 and 9.10 we have the following, somewhat mysterious, result.

(9.12) THEOREM. – If X is a smooth affine G -variety with FPIG which is 2-large or 3-principal, then $(X // G)_{\text{sm}} = (X // G)_{\text{pr}}$. In particular, if V is a non-trivial G -module which is 2-large or 3-principal, then V cannot be coregular.

One can find related results about the smoothness and flatness of π_X in [Br, 4.3 Cor. 2] and [Kn].

10. Tori

Throughout this section, X denotes a smooth affine G -variety. We also assume that G^0 is a torus T . We find necessary and sufficient conditions for X to be (very) good. We also

prove that conjecture 0.1 is true for actions of tori, a result essentially due to Musson [Mu].

(10.1) PROPOSITION. – *Suppose that X has FPIG. If X is k -principal, then X is k -modular, $k \geq 0$. Hence X is k -large if and only if X is k -principal.*

Proof. – By 9.6(2) we may reduce to the case that X is a G -module V with FPIG. We are given that V is k -principal, and we must show that it is k -modular. We may assume that

(1) $V^G = (0)$.

Now principal G -orbits are finite unions of principal G^0 -orbits. Hence if (V, G) is k -principal, then so is (V, G^0) , and we may add

(2) G is connected, i.e., a torus.

Since the principal isotropy groups are the ineffective part of the action, we may assume

(3) (V, G) has trivial principal isotropy groups.

By induction over slice representations we may reduce to the following problem:

(4) Show that $\text{mod}(\mathcal{N}_G(V), G) \leq \dim V // G - 2$.

Let $n = \dim V$, $l = \dim G$. Then $\dim V // G = n - l$. Choose coordinates x_1, \dots, x_n on V so that the action of G is diagonal. Then $\mathcal{N}_G(V)$ is a finite union of coordinate subspaces.

Let $W \subset \mathcal{N}_G(V)$ be a coordinate subspace, where W has codimension s in V . Let $G' = \text{Ker}(G \rightarrow \text{GL}(W))$. If $\dim G' \leq s - k$, then by Vinberg ([Vi], see 9.2(5)) $\text{mod}(W, G) = \text{mod}(W, G/G') = (n-s) - l + \dim G' = (n-l) - s + \dim G' \leq \dim V // G - k$, so we get the desired estimate.

Suppose that $\dim G' \geq s - k + 1$. Consider the G' -action on the s -dimensional coordinate subspace W' complementary to W . Then (W', G') has trivial principal isotropy groups and $\dim W' // G' \leq k - 1$. Thus $\mathcal{N}_{G'}(W')$, a fiber of $\pi_{W', G'}$, has codimension at most $k - 1$ in W' . Choose nonconstant monomial generators $f_1, \dots, f_t \in \mathcal{O}(W')^{G'}$. Then $\mathcal{O}(V)^G$ is generated by (nonconstant) monomials in the coordinates of W and the f_j . Since $W \subset \mathcal{N}_G(V)$, each such monomial involves some of the f_j . Hence $W \times \mathcal{N}_{G'}(W') \subset \mathcal{N}_G(V)$, where $W \times \mathcal{N}_{G'}(W')$ has codimension at most $k - 1$ in V , contradiction. \square

(10.2) PROPOSITION. – *If X is not 2-principal, then 1_X is not good, hence X is not good.*

Proof. – We assume that 1_X is good and derive a contradiction. There must be a nonprincipal closed isotropy class (H) such that $X^{(H)}$ has codimension 1 in X . Choosing a slice representation at a closed orbit in $X^{(H)}$ and using 4.9 we may reduce to the case that X is a G -module V such that $\text{codim } V^{(G)} = 1$. We may further reduce to the case that $V^G = (0)$, so that $\mathcal{N}_G(V)$ has codimension 1 in V . We now show that we may assume that V has FPIG:

Let H denote a principal isotropy group of (V, G) . Then H^0 is normal in G and we may write $V = V' \oplus V_0$ as G -module, where $V_0 := V^{H^0}$. The action of H^0 is fix pointed (7.9), hence $\mathcal{O}(V')^{H^0} = \mathbb{C}$, $V_{\text{pr}} = V' \times (V_0)_{\text{pr}}$ and (V_0, G) is not 2-principal. If $P \in \mathcal{D}^n(V' \oplus V_0)^G$, then $\rho(P) \in \mathcal{D}^{0,n}(V' \oplus V_0)^G = (\mathcal{O}(V')^{H^0} \otimes \mathcal{D}^n(V_0))^G \simeq \mathcal{D}^n(V_0)^G$ (see 3.16). In other words, if $f \in \mathcal{O}(V)^G$, then $P(f)|_{V_0} = \rho(P)(f|_{V_0})$. Thus 1_{V_0} is good since 1_V is. Since $(V_0, G/H^0)$ has FPIG, we may assume that (V, G) has FPIG.

Since (V, G) is 1-principal with FPIG, it is 1-large by 10.1, and 9.9 shows that $(\pi_V)_*$ is graded surjective. If $Q \in \mathcal{D}^n(V // G)$, then $Q = (\pi_V)_* P$ where $P \in \mathcal{D}^n(V)^G$. Now the degree (see 4.7) of P is at worst $-n$, hence $\text{order } Q + \text{deg } Q \geq 0$ for all $Q \in \mathcal{D}(V // G)$.

We now construct a Q where this inequality fails.

Choose coordinates x_1, \dots, x_n on V so that the x_i transform by characters χ_i of $T = G^0$. Clearly $n > 1$, else 1_V is not good. Thus $\mathcal{N}_G(V) = \mathcal{N}_T(V)$ has dimension $n - 1 > 0$, and there is an x_i , say x_1 , which divides every nonconstant monomial in $\mathcal{O}(V)^T$. Let $\{x^\alpha\}_{\alpha \in A \subset \mathbb{N}^n}$ be monomial generators of $\mathcal{O}(V)^T$. Let

$$a = \min\{\alpha_1 : \alpha = (\alpha_1, \dots, \alpha_n) \in A\}.$$

Then $a > 0$, and there is a monomial $f = x_1^a f_2(x_2, \dots, x_n) \in \mathcal{O}(V)^T$ where $\deg f = d > a$. Let P_2 be the constant coefficient differential operator in $\partial/\partial x_2, \dots, \partial/\partial x_n$ dual to f_2 . Then $P := x_1^{-a} P_2$ is T -invariant and preserves $\mathcal{O}(V)^T$. Clearly $Q := (\pi_{V,T})_* P$ has order $d - a$ and degree $-d$. Set $Q' = \prod_{g \in G/T} gQ$. Then $Q' \in \mathcal{D}(V//T)^{G/T}$ induces

$\tilde{Q} \in \mathcal{D}(V//G)$ such that $\deg \tilde{Q} + \text{order } \tilde{Q} < 0$, contradiction. \square

(10.3) EXAMPLE [Mu, 2.9] Let $G = \mathbb{C}^*$ act on $V = \mathbb{C}^3$ so that there are coordinate functions s, t and u transforming by weights 1, 1 and -2 , respectively. Then $\mathcal{O}(V)^G$ is generated by $x := s^2 u + t^2 u, y := s^2 u - t^2 u$ and $z := 2stu$. Note that these generators satisfy the same relations as those in example 6.4, so that $V//G = C = \{x^2 = y^2 + z^2\} \subset \mathbb{C}^3$. Using the description of the generators of $\mathcal{D}(C)$ given in 6.4, one can see that $(\pi_V)_*$ is 1-surjective but not 2-surjective. The missing order 2 differential operators come from the images of $u^{-1} \partial^2 / \partial s^2, u^{-1} \partial^2 / \partial s \partial t$ and $u^{-1} \partial^2 / \partial t^2$.

(10.4) THEOREM. – Let H denote a principal isotropy group of X . Then the following are equivalent:

- (1) $\pi_{X,E}$ is surjective for every G -vector bundle E on X .
- (2) X is very good.
- (3) X is good.
- (4) 1_X is good.
- (5) X is 2-principal.
- (6) $(\overline{X}^{(H)}, G/H)$ is 2-principal.
- (7) $(\overline{X}^{(H)}, G/H)$ is 2-large.
- (8) $1_{\overline{X}^{(H)}}$ is good.

Proof. – Using 9.10, 10.1 and 10.2 we see that (6), (7) and (8) are equivalent and that (4) implies (5). Clearly, (1) (or (2)) implies (3) which implies (4). By proposition 7.9, (5) implies (6), and by 9.10 and 7.10, (7) implies (1) and (2). \square

We have the following “toral analogue” of 6.9:

(10.5) PROPOSITION. – Suppose that $\dim V//G = 2$, where (V, G^0) is not fix pointed. Then V is not good.

Proof. – We may assume that V is 2-principal. Then (V, G^0) is equidimensional, hence coregular [We], and we may apply 6.8(3). \square

The following results establish finite generation for differential operators on quotients by commutative groups, but only for the case of the trivial line bundle.

(10.6) THEOREM (cf. [Mu]). – *Let V be a G -module, where G is commutative. Then $V//G \simeq X'//G'$ and $\mathcal{D}(V//G) \simeq \mathcal{D}(X'//G')$ where X' is a very good G' -variety. Hence $\text{gr } \mathcal{D}(V//G)$ is finitely generated.*

Proof. – By 7.10 (Luna-Richardson), we may reduce to the case that V has trivial principal isotropy groups. Suppose that $(V//G)_{(H)}$ is a codimension 1 stratum. Then the slice representation of H is of the form $(W \oplus \mathbb{C}^p, H)$, where $\dim W//H = 1$ and H acts trivially on \mathbb{C}^p . Since G is commutative, we have that $V \simeq W \oplus \mathbb{C}^{p'}$ as H -module, where H acts trivially on $\mathbb{C}^{p'}$. Quotienting V by H we obtain a representation V' of $G' := G/H$ such that $V//G \simeq V'//G'$, and $V'//G'$ has fewer codimension 1 strata than $V//G$.

By induction, we may reduce to the case that $V//G$ has no codimension 1 strata. Fix coordinate functions x_1, \dots, x_n on V which transform by characters of G . Suppose that $\text{codim}(V^{(G)} \simeq V^G \times \mathcal{N}_G(V)) = 1$, and let U_1, \dots, U_r be the hyperplanes contained in $V^{(G)}$. We may assume that U_i is the zero set of x_i , $i = 1, \dots, r$. For $1 \leq i \leq r$ there are 1-parameter subgroups λ_i of G such that x_i transforms under λ_i by a negative weight, while all other x_j , $1 \leq j \leq n$, transform by non-negative weights. Thus $\mathcal{O}(V)[(x_1 \cdots x_r)^{-1}]^G = \mathcal{O}(V)^G$ and $\mathcal{O}(V)^G = \mathcal{O}(X)^G$ where $X := V \setminus \{\text{zeroes of } x_1 \cdots x_r\}$. Moreover, $X^{(G)} = V^{(G)} \cap X$ has codimension at least 2 in X .

Now suppose that (H) is a closed isotropy class with $\text{codim } V^{(H)} = 1$, and let (W, H) denote the corresponding slice representation. Note that $(V, H) = (W, H)$ modulo trivial representations. As above, there are coordinate functions on V , say x_1, \dots, x_r , such that $X := V \setminus \{\text{zeroes of } x_1 \cdots x_r\}$ has the same G -invariant functions as V , and $\text{codim}_X X^{(H)} \geq 2$. We may perform this procedure for all (H) such that $\text{codim } V^{(H)} = 1$. The resulting affine G -variety X is 2-principal with quotient $V//G$. \square

(10.7) COROLLARY. – *Let X be a smooth affine G -variety, where G is commutative. Then $\text{gr } \mathcal{D}(X//G)$ is finitely generated.*

Example 3.27 shows that we cannot generalize 10.7 to the case of vector bundles.

11. Classical Groups

We develop techniques for establishing that representations are k -large, and we show that 2-largeness holds generically. We then consider the standard representations of the classical groups.

(11.1) DEFINITIONS. – Let X be an affine G -variety, and let (H) be a closed isotropy class of X . Define:

- (1) $\kappa(X, G) = \dim X//G - \text{mod}(X \setminus X_{\text{pr}}, G)$.
- (2) $\kappa^{(H)}(X, G) = \dim X//G - \text{mod}(X^{(H)}, G)$.
- (3) $\text{comod}(X, G) = \dim X - \text{mod}(X, G)$, the *comodularity* of (X, G) .

If $X_{(0)} \neq \emptyset$, then we define

- (4) $\kappa_m(X, G) = \dim X//G - \text{mod}(X \setminus X_{(0)}, G)$.

We use notation $\kappa(X)$, etc. if the group involved is clear.

(11.2) *Remarks* (cf. 9.4).

(1) $\dim G - \text{comod}(X) = \sup_{n \geq 0} \{n - \text{codim } X_{(n)}\} \geq 0$.

(2) $\text{comod}(X) = \dim G$ if and only if \mathfrak{g} is regular in $\mathcal{O}(T^*X)$ if and only if $X_{(0)} \neq \emptyset$ and $\kappa_m(X) \geq 0$.

(3) Let V be a G -module, and write $V = V^G \oplus V'$ where V' is G -stable. Then $V^{(G)} \simeq V^G \times \mathcal{N}_G(V')$ and $V//G \simeq V^G \times V'//G$. Thus

$$\dim V//G - \text{mod}(V^{(G)}) = \dim V'//G - \text{mod}(\mathcal{N}_G(V'), G),$$

so that $\kappa^{(G)}(V) = \kappa^{(G)}(V')$.

Assume that X has FPIG.

(4) X is k -modular (resp. k -large) if and only if $k \leq \kappa_m(X)$ (resp. $k \leq \kappa(X)$).

(5) $\kappa(X) \leq \kappa_m(X)$.

(11.3) LEMMA. – Let X be a smooth affine G -variety such that G acts non-trivially with FPIG. Let $(W_1, H_1), \dots, (W_r, H_r)$ represent the isomorphism classes of non-principal slice representations. Then $\kappa^{(H_i)}(X) = \kappa^{(H_i)}(W_i)$ for each i and

$$\kappa(X) = \inf_i \{\kappa^{(H_i)}(W_i)\}.$$

Proof. – $X \setminus X_{\text{pr}} = \bigcup_i X^{(H_i)}$. By the slice theorem,

$$\text{mod}(X^{(H_i)}, G) = \text{mod}(G *^{H_i} W_i^{(H_i)}, G) = \text{mod}(W_i^{(H_i)}, H_i),$$

and the lemma follows. \square

(11.4) PROPOSITION. – Let V be a G -module, where G^0 is semisimple or (V, G^0) is orthogonal. Suppose that V is 2-modular, i.e., $V_{(0)} \neq \emptyset$ and $\kappa_m(V) \geq 2$. Then V is 2-large if

- (1) V has no codimension 1 strata, or
- (2) G is connected.

Proof. – Part (1) is immediate from 7.11. If (2) holds, we need to show that there is not a codimension 1 stratum $(V//G)_{(H)}$. If $\dim H > 0$, then $\text{mod}(V^{(H)}, G) \geq \dim V//G - 1$, a contradiction. Thus H is finite. By 1.6, $V \simeq W \oplus \mathfrak{g}$ as H -module, where (W, H) is the slice representation of H and the H actions on V and \mathfrak{g} are the restrictions of the actions $G \rightarrow \text{SL}(V)$ and $G \rightarrow \text{SL}(\mathfrak{g})$. Hence $H \rightarrow \text{GL}(W)$ has image in $\text{SL}(W)$ and $\text{Im}(H \rightarrow \text{SL}(W))$ contains no pseudoreflections. Thus $(V//G)_{(H)}$ cannot be a codimension 1 stratum. \square

(11.5) PROPOSITION. – Let V and W be G -modules where W is almost faithful.

- (1) If $\text{comod}(V) < \dim G$, then $\text{comod}(V \oplus W) > \text{comod}(V)$.
- (2) If $V_{(0)} \neq \emptyset$, then $\kappa_m(V \oplus W) > \kappa_m(V)$.
- (3) If V has FPIG, then so does $V \oplus W$.

Proof. – Let $(v, w) \in (V \oplus W)_{(n)}$, $n > 0$. If $v \in V_{(n)}$, then $w \in W^{G_v^0} \neq W$, else $v \in V_{(m)}$ for some $m > n$. It follows that

$$\text{codim}_{V \oplus W}(V \oplus W)_{(n)} - n > \sup_{m \geq n} \{\text{codim}_V V_{(m)} - m\}.$$

Hence (1) and (2) follow from 11.2(1) and 9.6(3), respectively. If V has FPIG, then the slice representation at points $(v, 0) \in V_{\text{pr}} \times W$ is of the form (U, H) where H is finite. Since (U, H) has FPIG, so does $V \oplus W$. \square

(11.6) COROLLARY. – *Let G be connected.*

(1) *If G is simple, then, up to isomorphism, all but finitely many G -modules V with $V^G = (0)$ are 2-large.*

(2) *Let G be semisimple, and consider G -modules V where $V^G = (0)$ and each irreducible component of V is almost faithful. Then, up to isomorphism, all but finitely many V are 2-large.*

Proof. – In both (1) and (2), the numerical criteria and estimates of [AP], [AVE] or [Go] show that, up to isomorphism, only finitely many irreducible G -modules V_1, \dots, V_r fail to satisfy: V has FPIG and $\kappa_m(V) \geq 2$. By 11.5, any (possibly reducible) V failing these conditions is isomorphic to a direct sum of the V_i , and only finitely many sums can fail the conditions. Now apply 11.4. \square

(11.7) COROLLARY ([Po2], [Go]; cf. [Kn]). – *Let G be connected semisimple and consider G -modules V such that $V^G = 0$. Then, up to isomorphism, there are only finitely many V which are coregular.*

Proof. – It follows from 11.6 and 9.12 that there are, up to isomorphism, only finitely many irreducible (not necessarily faithful) coregular representations V_1, \dots, V_r of G to consider. Since a subrepresentation of a coregular representation is coregular [S1, 1.1], any coregular V must be isomorphic to a sum $\sum_{i=1}^r m_i V_i$. Let G_i denote $\text{Im}(G \rightarrow \text{GL}(V_i))$, $i = 1, \dots, r$. Then G_i is semisimple, and by 11.5 there is an n_i such that $(n_i V_i, G_i)$ is 2-modular, hence 2-large. It follows that $m_i < n_i$ for each i . \square

We now consider some representations satisfying the LS-alternative (see 0.11). This property holds for irreducible representations of simple groups [S7]. It does not hold if one drops the irreducibility assumption.

(11.8) Example (see also 5.11 and 11.15). – Let $(V, G) = (\text{S}^2(\mathbb{C}^n) \oplus 2\mathbb{C}^n, \text{SL}_n)$, $n \geq 3$. Then V is not coregular ([S1]), yet it has a non-principal coregular slice representation (namely, $(\mathbb{C} \oplus 2\mathbb{C}^n, \text{SO}_n)$, where SO_n acts trivially on \mathbb{C}). Thus V is neither coregular nor good.

Let R_i denote the (irreducible) representation of SL_2 on $\text{S}^i(\mathbb{C}^2)$.

(11.9) THEOREM. – *Let $G = \text{SL}_2$ and $V = \bigoplus_{i \geq 1} m_i R_i$. Then V is 2-large, except in the*

following cases:

- (1) kR_1 , $0 \leq k \leq 3$.
- (2) R_2 , $2R_2$, $R_2 \oplus R_1$.
- (3) R_3 , R_4 .

Each of the representations listed is coregular, hence all representations of SL_2 satisfy the LS-alternative.

Proof. – By 11.4, it is enough to show that (1)–(3) list all the representations which are not 2-modular. Let $Y := V \setminus V_{(0)}$. If $0 \neq v \in Y$, then G_v^0 is a copy of \mathbb{C}^+ (the additive group) or \mathbb{C}^* . Up to conjugacy, there is only one subgroup of each type in G , and each such subgroup fixes at most a one-dimensional subspace of any R_i . Thus $\dim Y \leq 2 + \sum_i m_i$ and $\text{mod}(Y, G) \leq \sum_i m_i$. Now V has FPIG if $\dim Y < \dim V = \sum_i (i+1)m_i$, and V is 2-modular if, in addition, $\text{mod}(Y, G) + 2 \leq \dim V//G = \dim V - 3$. Hence V is 2-modular if $5 \leq \sum_i im_i$. The cases where $\sum_i im_i < 5$ not listed in (1), (2) and (3) are:

- (4) $4R_1$.
- (5) $R_2 \oplus 2R_1$.
- (6) $R_3 \oplus R_1$.

Consider case (5). The fixed points of any copy of \mathbb{C}^* have dimension 1. The normalizer of any copy of \mathbb{C}^+ contains a copy of \mathbb{C}^* , so that our estimate for $\text{mod}(Y, G)$ can be improved to $\dim V^{\mathbb{C}^+} - 1 = 2$. Since $\dim V//G = 4$, V is 2-modular. Cases (4) and (6) are similar. \square

We now consider the classical representations of GL_n , SL_n , O_n , SO_n and Sp_{2n} .

(11.10) LEMMA. – Let $k \geq 0$, $p > 0$. Then

$$\text{mod}(k\mathbb{C}^p, \text{GL}_p) = \begin{cases} [k^2/4], & k \leq 2p + 1 \\ pk - p^2, & k \geq 2p - 1 \end{cases}$$

Proof. – Let W_r denote $\{(v_1, \dots, v_k) \in k\mathbb{C}^p : \dim \text{span}\{v_1, \dots, v_k\} = r\}$. Set $U_r := \{(e_1, \dots, e_r)\} \times (k-r)\mathbb{C}^r \subseteq k\mathbb{C}^p$, where e_1, \dots, e_p is the standard basis of \mathbb{C}^p and \mathbb{C}^r denotes $\text{span}\{e_1, \dots, e_r\} \subseteq \mathbb{C}^p$. Let the symmetric group S_k act on $k\mathbb{C}^p$ in the usual way. If $(v_1, \dots, v_k) \in W_r$, then modulo the action of $S_k \times \text{GL}_p$, we can assume that $v_i = e_i$, $i \leq r$. Then v_{r+1}, \dots, v_k lie in \mathbb{C}^r , i.e., $W_r = (S_k \times \text{GL}_p)U_r$. Clearly, if $g \in \text{GL}_p$ and $gU_r \cap U_r \neq \emptyset$, then g acts trivially on U_r . Hence $\text{mod}(W_r, S_k \times \text{GL}_p) = \text{mod}(W_r, \text{GL}_p) = \dim U_r = r(k-r)$. Now $k\mathbb{C}^p$ is the disjoint union of the W_r for $0 \leq r \leq \max\{k, p\}$, and the maximum value of $\text{mod}(W_r, \text{GL}_p)$ occurs when $r = \min\{[k/2], p\}$. \square

(11.11) PROPOSITION. – Let $k \geq l \geq 0$ and $n \geq 1$, and let $(V, G) = (k\mathbb{C}^n \oplus l(\mathbb{C}^n)^*, \text{GL}_n)$. Then $\text{mod}(\mathcal{N}_G(V), G) =$

$$= \begin{cases} [(k^2 + l^2)/4], & k + l \leq 2n + 1 \\ \left[\frac{1}{2}(k + l - n)n + \frac{1}{8}(k - l)^2 \right], & 2n - 1 \leq k + l, k \leq l + 2n + 2 \\ nk - n^2, & 2n - 1 \leq k + l, k \geq l + 2n - 2 \end{cases}$$

If $k \geq l \geq n$, then $\kappa^{(G)}(V) \geq nl$.

Proof. – Let e_1, \dots, e_n denote the standard basis on \mathbb{C}^n with dual basis e_1^*, \dots, e_n^* . Let T denote the standard maximal torus of G . If $\lambda : \mathbb{C}^* \rightarrow T$ is a 1-parameter subgroup, define $Z_\lambda := \{v \in V : \lim_{t \rightarrow 0} \lambda(t)v = 0\}$. Then, by the Hilbert-Mumford criterion,

$\mathcal{N}_G(V) = G \cdot \mathcal{N}_T(V) = G(\cup_\lambda Z_\lambda)$. Clearly, up to the obvious action of $S_k \times S_l$, any Z_λ is determined solely by the number p (resp. q) of positive (resp. negative) weights of the λ -action. Thus we may assume that

$$Z_\lambda = k \operatorname{span}\{e_1, \dots, e_p\} \oplus l \operatorname{span}\{e_{p+1}^*, \dots, e_{p+q}^*\} \simeq k\mathbb{C}^p \oplus l(\mathbb{C}^q)^*.$$

Let $W_{r,s}$ denote the points $(v_1, \dots, v_k, \xi_1, \dots, \xi_l) \in Z_\lambda$ such that the span of $\{v_1, \dots, v_k\}$ has dimension r and the span of $\{\xi_1, \dots, \xi_l\}$ has dimension s . Modulo the action of $S_k \times S_l \times G$, any point in $W_{r,s}$ has a representative in

$$U_{r,s} := \{(e_1, \dots, e_r)\} \times (k-r)\mathbb{C}^r \times \{(e_{p+1}^*, \dots, e_{p+s}^*)\} \times (l-s)(\mathbb{C}^s)^*.$$

Moreover, if $g \in G$ and $gU_{r,s} \cap U_{r,s} \neq \emptyset$, then g acts trivially on $U_{r,s}$. Reasoning as in 11.10 we see that

$$\begin{aligned} \operatorname{mod}(\mathcal{N}_G(V), G) &= \sup_{\lambda} \operatorname{mod}(GZ_\lambda, G) \\ &= \sup\{r(k-r) + s(l-s) : r+s \leq n, r \leq k, s \leq l\} \end{aligned}$$

and it follows that $\operatorname{mod}(\mathcal{N}_G(V), G)$ is as claimed.

Suppose that $k \geq l \geq n$ (which implies that V has FPIG). If $k \leq l + 2n$, then $\kappa^{(G)}(V) \geq 1/2(k+l-n)n - 1/8(k-l)^2$. Fixing $k+l$, the least value occurs if $k = l + 2n$ or $k = l + 2n - 1$, yielding the estimate $\kappa^{(G)}(V) \geq nl$. One has $\kappa^{(G)}(V) = nl$ if $k \geq l + 2n$. \square

(11.12) THEOREM. – *Let $(V, G) = (k\mathbb{C}^n \oplus l(\mathbb{C}^n)^*, \operatorname{GL}_n)$. Then V satisfies the LS-alternative.*

Proof. – We may suppose that $k \geq l$. Classical invariant theory shows that V is coregular if and only if $l \leq n$. Suppose that $l > n$. Then V has FPIG and the non-trivial part of every non-principal slice representation is isomorphic to a representation of the form $(V^{(r)}, G^{(r)}) := ((k-r)\mathbb{C}^{n-r} \oplus (l-r)(\mathbb{C}^{n-r})^*, \operatorname{GL}_{n-r})$, $0 \leq r < n$. Then 11.11 shows that $\kappa^{(G^{(r)})}(V^{(r)}) \geq (l-r)(n-r) \geq 2$. By lemma 11.3, V is 2-large. \square

We now consider representations of SL_n . The techniques we use are the same as for GL_n . One only has to notice that for any 1-parameter subgroup of SL_n , the number of positive (or negative) weights of the action on \mathbb{C}^n cannot be zero. Also, the only closed isotropy groups occurring in $(k\mathbb{C}^n, \operatorname{SL}_n)$ are the trivial group and SL_n itself.

(11.13) PROPOSITION. – *Let $(V, G) = (k\mathbb{C}^n, \operatorname{SL}_n)$, $k \geq 0$, $n \geq 2$. Then*

$$\operatorname{mod}(\mathcal{N}_G(V), G) = \begin{cases} [k^2/4], & k \leq 2n-1 \\ (n-1)(k-n+1), & k \geq 2n-3. \end{cases}$$

Suppose that $k \geq 2n-2$. Then V has FPIG and $\kappa(V) = \kappa^{(G)}(V) = k-2n+2$. Hence V is $(k-2n+2)$ -large.

(11.14) PROPOSITION. – Let $(V, G) = (k\mathbb{C}^n \oplus l(\mathbb{C}^n)^*, \mathrm{SL}_n)$; $k \geq l \geq 0$, $n \geq 2$. Then $\mathrm{mod}(\mathcal{N}_G(V), G) =$

$$= \begin{cases} [(k^2 + l^2)/4], & k + l \leq 2n - 3 \\ [(k^2 + l^2)/4], & k + l = 2n - 2, kl \text{ even} \\ [(k^2 + l^2)/4] + 1, & k + l = 2n - 2, kl \text{ odd} \\ \left[\frac{1}{2}(k + l - n)n + \frac{1}{8}(k - l)^2 \right] + 1, & 2n - 3 \leq k + l, k \leq l + 2n - 2 \\ (n - 1)(k - n + 1) + l, & 2n - 3 \leq k + l, k \geq l + 2n - 6 \end{cases}$$

If $k + l \geq 2n - 2$, then V has FPIG and $\kappa^{(G)}(V) \geq k + l - 2n + 2$.

(11.15) THEOREM. – Let $(V, G) = (k\mathbb{C}^n \oplus l(\mathbb{C}^n)^*, \mathrm{SL}_n)$, $n \geq 2$. Then

(1) V is 2-large if and only if $k + l \geq 2n$.

(2) V is good if and only if V is 2-large or $V//G = \{\mathrm{pt}\}$.

In particular, when $n \geq 3$, there are representations which are neither good nor coregular (e.g., $((n+2)\mathbb{C}^n, \mathrm{SL}_n)$). Hence the representations (V, G) do not satisfy the LS-alternative.

Proof. – Arguing as in 11.12, we obtain (1) from 11.14. Now suppose that $k + l < 2n$. We may assume that $k \geq l$. If $k \leq n$, then V is coregular [S1]. If $k > n$, then there is a slice representation whose non-trivial part is $(W, H) := ((p+m)\mathbb{C}^p, \mathrm{SL}_p)$, where $p = n - l \geq 2$, $m = k - n$ and $1 \leq m < p$. Either (W, H) is coregular or proposition 5.11 shows that 1_W is not good. Hence 1_V is not good. \square

(11.16) Remark. – If $(V, G) = (k\mathbb{C}^n, \mathrm{SL}_n)$ does not satisfy the LS-alternative, then $k = n + p$ where $2 \leq p \leq n - 1$. However, this implies that $V//G \simeq V'//G'$ where $(V', G') = (k\mathbb{C}^p, \mathrm{SL}_p)$ is 2-large. Hence conjecture 0.1 holds for representations of the form $(k\mathbb{C}^n, \mathrm{SL}_n)$.

(11.17) PROPOSITION. – Let $(V, G) = (k\mathbb{C}^n, \mathrm{O}_n)$ or $(k\mathbb{C}^n, \mathrm{SO}_n)$, $n \geq 2$. Then

$$\mathrm{mod}(\mathcal{N}_G(V), G) = \begin{cases} [k^2/4], & k \leq n \\ [n/2](k - [n/2]), & k \geq n - 1, \end{cases}$$

and

$$\kappa^{(G)}(V) = \begin{cases} [(k+1)^2/4], & 1 \leq k \leq n \\ [(n+1)/2](k - [(n+1)/2] + 1), & k \geq n - 1 \end{cases}$$

Proof. – We first consider the case that $n = 2m$ is even. We may choose coordinates e_1, \dots, e_{2m} on \mathbb{C}^{2m} so that $\langle e_i, e_j \rangle = \delta_{i+m, j}$ for $i \leq m$, where $\langle \cdot, \cdot \rangle$ is an SO_{2m} -invariant bilinear form on \mathbb{C}^{2m} . Then $\mathrm{GL}_m \subset \mathrm{SO}_{2m}$ where the GL_m -actions on $V_m := \mathrm{span}\{e_1, \dots, e_m\}$ and $V_m^* := \mathrm{span}\{e_{m+1}, \dots, e_{2m}\}$ are dual. The diagonal maximal torus T of GL_m is a maximal torus of SO_{2m} . Thus if λ is a 1-parameter subgroup of SO_{2m} , up to conjugation by an element of G , we can assume that λ has image in T . Then, modulo a rearrangement of the order of our basis, we can assume that $Z_\lambda = \mathbb{C}^p \subset V_m$. Applying the techniques in 11.10 we see that $\sup_\lambda \mathrm{mod}(GZ_\lambda, G) = \sup\{r(k-r) : 0 \leq r \leq m\}$. One gets the same answer in case $n = 2m + 1$ is odd, hence one obtains the given values for $\mathrm{mod}(\mathcal{N}_G(V), G)$ and $\kappa^{(G)}(V)$. \square

(11.18) THEOREM. – Let $V = k\mathbb{C}^n$ with the standard action of SO_n or O_n , $n \geq 2$. Then V has FPIG for $k \geq n - 1$ and

- (1) $\kappa(V, SO_n) = 1$, $1 \leq k < n$.
- (2) $\kappa(V, SO_n) = k - n + 2$, $k \geq n - 1$
- (3) $\kappa(V, O_n) = 1$, $1 \leq k \leq n$.
- (4) $\kappa(V, O_n) = k - n + 1$, $k \geq n$.

In particular, (V, SO_n) and (V, O_n) satisfy the LS-alternative.

Proof. – Let $G = SO_n$. The nontrivial parts of the non-principal slice representations of (V, G) are of the form $((k - r)\mathbb{C}^{n-r}, SO_{n-r})$, $1 \leq r \leq \max\{n - 2, k - 1\}$. From 11.17 and 11.3 we see that $\kappa^{(SO_{n-r})}(V, G) \geq 1$. If $k \geq n$, then

$$\kappa^{(SO_{n-r})}(V, G) = \kappa^{(SO_{n-r})}((k - r)\mathbb{C}^{n-r})$$

has its minimum value when $r = n - 2$, giving $\kappa(V, SO_n) = k - n + 2$. We have established (1) and (2). Now (V, SO_n) is coregular if and only if $k \leq n - 1$, hence (V, SO_n) satisfies the LS-alternative.

Now suppose that $G = O_n$. If $k \leq n - 1$, then (V, SO_n) and (V, O_n) have isomorphic (stratified) quotients and quotient mappings, so (3) holds for $k < n$. When $k \geq n$ the principal stratum (relative to SO_n) breaks up into the principal stratum for O_n and the stratum corresponding to the slice representation whose nontrivial part is $((k - n - 1)\mathbb{C}, O_1 \simeq \mathbb{Z}/2\mathbb{Z})$. Thus (V, O_n) is $(k - n - 1)$ -principal, and one obtains (3) and (4). Moreover, (V, O_n) is coregular if and only if $k \leq n$, hence (V, O_n) satisfies the LS-alternative. \square

The calculations for the symplectic group are similar to those for the orthogonal group. We omit the details.

(11.19) PROPOSITION. – Let $(V, G) = (k\mathbb{C}^{2n}, Sp_{2n})$, $n \geq 1$. Then

$$\text{mod}(\mathcal{N}_G(V), G) = \begin{cases} [k^2/4], & k \leq 2n + 1 \\ nk - n^2, & k \geq 2n - 1, \end{cases}$$

and

$$\kappa^{(G)}(V) = \begin{cases} [(k - 1)^2/4], & 1 \leq k \leq 2n + 1 \\ kn - n^2 - n, & k \geq 2n \end{cases}$$

(11.20) THEOREM. – Let $(V, G) = (k\mathbb{C}^{2n}, Sp_{2n})$. Then V has FPIG for $k \geq 2n$ and is coregular if and only if $k \leq 2n + 1$. Moreover,

- (1) $\kappa(V) = 0$, $2 \leq k \leq 2n$, k even.
- (2) $\kappa(V) = 1$, $3 \leq k \leq 2n + 1$, k odd.
- (3) $\kappa(V) = k - 2n$, $k \geq 2n$.
- (4) V satisfies the LS-alternative.

Finally, we have the “neoclassical” cases involving G_2 and $B_3 = \text{Spin}_7$ (see [S4]). Here the irreducible representation (φ_1, G_2) (resp. (φ_3, B_3)) we consider has dimension seven (resp. eight). We again omit the details.

(11.21) THEOREM. – Let (V, G) denote $(k\varphi_1, G_2)$ or $(k\varphi_3, B_3)$ and let m denote 3 or 4, respectively. Then

- (1) V is coregular if and only if $k \leq m$.
- (2) V has FPIG if and only if $k \geq m$.
- (3) $\kappa(V) = 1$, $1 \leq k \leq m - 1$.
- (4) $\kappa(V) = 2(k - m)$, $k \geq m$.
- (5) V satisfies the LS-alternative.

12. Nakai's conjecture

The version of Nakai's conjecture ([Ish]) that we consider is the following.

(12.1) CONJECTURE. – *Let Y be a complex affine variety. If $\mathcal{D}(Y)$ is generated by $\mathcal{D}^1(Y)$, then Y is smooth.*

We are able to say something in the case that Y is a quotient variety.

(12.2) PROPOSITION. – *Suppose that $Y = X//G$ where X is smooth and affine and $(\pi_X)_*$ is 1-surjective. Then Nakai's conjecture holds for Y . In particular, Nakai's conjecture holds if X is 2-large or 3-principal.*

Proof. – We may reduce to the case of a G -module V such that $V^G = (0)$ and $V \neq (0)$. Then every element of $\mathcal{D}^1(V)^G$ has degree at least 0, so if $\mathcal{D}(V//G)$ is generated by $\mathcal{D}^1(V//G)$, then every element of $\mathcal{D}(V//G)$ has non-negative degree. If $\mathcal{O}(V)^G \neq \mathbb{C}$, then there are elements of $\mathcal{D}(V)^G$ whose images in $\mathcal{D}(V//G)$ have strictly negative degree (constant coefficient operators of order at least two, see 5.10). It follows that $\mathcal{O}(V)^G = \mathbb{C}$, hence $Y = \{\text{pt}\}$ is smooth. \square

(12.3) COROLLARY. – *Let $Y = X//G$. Then the Nakai conjecture holds for Y if*

- (1) G is commutative, or
- (2) G is finite (see [Ish]) or, more generally, if
- (3) all G -orbits on X have the same dimension.

Proof. – In the cases given, locally we have $Y = X'//G'$ where X' is very good (6.6, 10.6). \square

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