ANNALES SCIENTIFIQUES DE L'É.N.S.

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Annales scientifiques de l'É.N.S. 4e série, tome 27, nº 4 (1994), p. 493-527
<http://www.numdam.org/item?id=ASENS_1994_4_27_4_493_0>

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Ann. scient. Éc. Norm. Sup.,
4e série, t. 27, 1994, p. 493 à 527.

EQUIVARIANT CYCLIC HOMOLOGY AND EQUIVARIANT DIFFERENTIAL FORMS

BY JONATHAN BLOCK AND EZRA GETZLER (1)

ABSTRACT. – Let G be a compact Lie group, and let M be a compact manifold on which G acts smoothly. In this paper, we give a description of the equivariant periodic cyclic homology $\mathrm{HP}_k^{\mathrm{G}}(\mathrm{C}^\infty(\mathrm{M}))$ of $\mathrm{C}^\infty(\mathrm{M})$ as the cohomology of global equivariant differential forms on M: these are sections of a sheaf over the group G, whose stalk at $g \in \mathrm{G}$ is the complex of equivariant differential forms on the fixed-point set M^g, with action of the centralizer G^g.

By the isomorphism $\mathrm{HP}_k^{\mathrm{G}}(\mathrm{C}^\infty(\mathrm{M})) \cong \mathrm{K}_{\mathrm{G}}^k(\mathrm{M}) \otimes_{\mathrm{R}(\mathrm{G})} \mathrm{R}^\infty(\mathrm{G})$ with equivariant K-theory [where $\mathrm{R}^\infty(\mathrm{G})$ is the space of smooth functions on G invariant under the adjoint action], we also obtain a de Rham description of equivariant K-theory.

Let G a compact Lie group, and let M be a compact manifold on which G acts smoothly. Let $\mathrm{R}^\infty(\mathrm{G})$ be the ring $\mathrm{C}^\infty(\mathrm{G})^{\mathrm{G}}$ of smooth conjugation invariant functions on the group G; it is an algebra over the representation ring R(G) of G, since R(G) maps into $\mathrm{R}^\infty(\mathrm{G})$ by the character map. Then there is an equivariant Chern character

$$\mathrm{ch}_k^{\mathrm{G}}\colon \quad \mathrm{K}_{\mathrm{G}}^k(\mathrm{M}) = \mathrm{K}_k^{\mathrm{G}}(\mathrm{C}^\infty(\mathrm{M})) \to \mathrm{HP}_k^{\mathrm{G}}(\mathrm{C}^\infty(\mathrm{M}))$$

from the equivariant K-theory of M to the periodic cyclic homology $\mathrm{HP}_\bullet^{\mathrm{G}}(\mathrm{C}^\infty(\mathrm{M}))$ of the algebra $\mathrm{C}^\infty(\mathrm{M})$ of smooth functions on M. This map induces an isomorphism

$$\mathrm{HP}_k^{\mathrm{G}}(\mathrm{C}^\infty(\mathrm{M})) \cong \mathrm{K}_{\mathrm{G}}^k(\mathrm{M}) \otimes_{\mathrm{R}(\mathrm{G})} \mathrm{R}^\infty(\mathrm{G});$$

furthermore, there are graded-commutative products on both $\mathrm{HP}_\bullet^{\mathrm{G}}(\mathrm{C}^\infty(\mathrm{M}))$ and $\mathrm{K}_{\mathrm{G}}^\bullet(\mathrm{M})$ such that the Chern character map is a ring homomorphism. These results are due to Block [3] (although he works with a crossed product involving algebraic functions instead of smooth ones), and Brylinski [5].

In this paper, we will study the equivariant cyclic homology of the algebra $\mathrm{C}^\infty(\mathrm{M})$ in terms of equivariant differential forms on M; this extends the description which Hochschild-Kostant-Rosenberg gave of the Hochschild homology of $\mathrm{C}^\infty(\mathrm{M})$ in terms of differential

(1) This paper is dedicated to the memory of Ellen Block.

forms on M, which was extended by Connes to cyclic homology. Let us give a rough idea of how this works. If

$$c = f_0 \otimes \cdots \otimes f_k \otimes \psi \qquad f_i \in \mathrm{C}^\infty(\mathrm{M}) \quad \text{and} \quad \psi \in \mathrm{C}^\infty(\mathrm{G}),$$

we define a map from the Lie algebra $\mathfrak{g}$ of G to the space of k-forms $\Omega^k(\mathrm{M})$ on M, by the formula

$$\mathrm{X} \mapsto \psi(\exp \mathrm{X}) \int_{\Delta_k} f_0\, d\left(e^{-t_1 \mathrm{X}} \,.\, f_1\right) \wedge \cdots \wedge d\left(e^{-t_k \mathrm{X}} \,.\, f_k\right) dt_1 \cdots dt_k.$$

Here, Δ_k is the k-simplex

$$\{(t_1, \ldots, t_k) | 0 \leq t_1 \leq t_2 \ldots \leq t_k \leq 1\} \subset \mathbb{R}^k.$$

This definition extends to define a map from $\mathrm{C}^\infty(\mathrm{M}^{k+1} \times \mathrm{G})$ to $\mathrm{C}^\infty(\mathfrak{g}, \Omega^k(\mathrm{M}))$, which moreover commutes with the actions of G on these two spaces: these actions are defined as follows: on $\mathrm{C}^\infty(\mathrm{M}^{k+1} \times \mathrm{G})$ by

$$(h.c)(x_0, \ldots, x_k | g) = c(h^{-1} x_0, \ldots, h^{-1} x_k | h^{-1} gh),$$

and on $\mathrm{C}^\infty(\mathfrak{g}, \Omega^k(\mathrm{M}))$ by

$$(h \,.\, \omega)(\mathrm{X}) = \mathrm{L}^*_{h^{-1}}\, \omega(\mathrm{ad}(h)\, \mathrm{X}).$$

Thus, we obtain a map from $\mathcal{C}_k^{\mathrm{G}}(\mathrm{C}^\infty(\mathrm{M})) = \mathrm{C}^\infty(\mathrm{M}^{k+1} \times \mathrm{G})^{\mathrm{G}}$ to

$$\mathrm{C}^\infty(\mathfrak{g}, \Omega^k(\mathrm{M}))^{\mathrm{G}} = (\mathbb{C}[\mathfrak{g}] \otimes \Omega^k(\mathrm{M}))^{\mathrm{G}} \otimes_{\mathbb{C}[\mathfrak{g}]^{\mathrm{G}}} \mathrm{C}^\infty(\mathfrak{g})^{\mathrm{G}}.$$

This map is just one component of our equivariant Hochschild-Kostant-Rosenberg map; the other components correspond to other points of G, and define maps from $\mathcal{C}_k^{\mathrm{G}}(\mathrm{C}^\infty(\mathrm{M}))$ to $\mathrm{C}^\infty(\mathfrak{g}^g, \Omega^k(\mathrm{M}^g))^{\mathrm{G}^g}$, where M^g is the fixed-point set of g acting on M, G^g is the fixed-point set of g acting by conjugation on G (in other words the centralizer of g) and $\mathfrak{g}^g$ is the Lie algebra of G^g. In the above notation, this map is induced by sending $f_0 \otimes \cdots f_k \otimes \psi$ to

$$\mathrm{X} \in \mathfrak{g}^g \mapsto \psi(g \exp \mathrm{X}) \int_{\Delta_k} f_0\, d\left(e^{-t_1 \mathrm{X}}.f_1\right) \wedge \cdots \wedge d\left(e^{-t_k \mathrm{X}}.f_k\right)|_{\mathrm{M}^g}\, dt_1 \cdots dt_k.$$

We call this map α_g.

It turns out that the correct way to describe the situation is by means of sheaves on G, with the topology given by open sets invariant under conjugation; all of our sheaves will be equivariant. In Section 1, we define a sheaf whose stalk at $g \in \mathrm{G}$ is the space of germs at 0 of maps from $\mathfrak{g}^g$ to $\Omega^\bullet(\mathrm{M}^g)$ invariant under the centralizer G^g. In Section 2, we introduce the equivariant cyclic chains; these are just smooth functions on $\mathrm{M}^{k+1} \times \mathrm{G}$ which are invariant under the action of G:

$$c(x_0, \ldots, x_k, g) = c(h^{-1} x_0, \ldots, h^{-1} x_k, h^{-1} gh) \quad \text{for all } h \in \mathrm{G}.$$

It is easy to see how to define the sheaf $\mathcal{C}_\bullet(\mathrm{C}^\infty(\mathrm{M}), \mathrm{G})$ of equivariant k-chains over G: the space of sections $\mathcal{C}_k^{\mathrm{G}}(\mathrm{C}^\infty(\mathrm{M}))$ over the invariant open set U is the space of invariant smooth functions on $\mathrm{M}^{k+1} \times \mathrm{U}$.

The maps $\{\alpha_g | g \in \mathrm{G}\}$ assemble to define a map of sheaves

$$\alpha : \quad \mathcal{C}_\bullet (\mathrm{C}^\infty (\mathrm{M}), \mathrm{G}) \to \Omega^\bullet (\mathrm{M}, \mathrm{G}).$$

The main result of this paper is the following equivariant generalization of the theorems of Hochschild-Kostant-Rosenberg and Connes; in a sense, we are completing the program of Baum-Brylinski-MacPherson.

THEOREM. – *The map α defines a quasi-isomorphism of complexes of sheaves*

$$\alpha : \quad (\mathcal{C}_\bullet (\mathrm{C}^\infty (\mathrm{M}), \mathrm{G}), b + u\,\mathrm{B}) \to (\Omega^\bullet (\mathrm{M}, \mathrm{G}), \iota + ud).$$

Taking the homology of both sides, we see that

$$\mathrm{HP}^{\mathrm{G}}_\bullet (\mathrm{C}^\infty (\mathrm{M})) \cong \mathrm{H}^\bullet (\mathcal{A}^\bullet_{\mathrm{G}} (\mathrm{M}), d + \iota),$$

where $\mathcal{A}^\bullet_{\mathrm{G}} (\mathrm{M}) = \Gamma (\mathrm{G}, \Omega^\bullet (\mathrm{M}, \mathrm{G}))$ is the space of *global equivariant differential forms*. In combination with the result relating equivariant K-theory with equivariant periodic cyclic homology, we obtain the following theorem:

$$\mathrm{K}^\bullet_{\mathrm{G}} (\mathrm{M}) \otimes_{\mathrm{R}(\mathrm{G})} \mathrm{R}^\infty (\mathrm{G}) \cong \mathrm{H}^\bullet (\mathcal{A}^\bullet_{\mathrm{G}} (\mathrm{M}), d + \iota).$$

This work is heavily influenced by the papers of Baum-Brylinski-MacPherson [1], Berline-Vergne [2], and Brylinski [4]. We would like to thank M. Vergne and the referee for a number of helpful suggestions. The paper was written while the first author was at MIT and at the Courant Institute. The second author would like to thank the MSRI and the ENS for their hospitality during the writing of parts of this paper. Both authors are partially funded by the NSF.

Conventions

In this paper, a differential graded algebra is a $\mathbb{Z}/2$-graded algebra (or superalgebra) with odd derivation d such that $d^2 = 0$; commutative in this setting means $\mathbb{Z}/2$-graded commutative. We use the notation $|\mathrm{A}| \in \mathbb{Z}/2$ for the degree of a homogeneous operator A acting on the graded vector space **H**; that is, $|\mathrm{A}| = 0$ if A is even, and $|\mathrm{A}| = 1$ if is odd. In a superalgebra, [A, B] is the supercommutator of the operators A and B, which when A and B are homogeneous equals

$$[\mathrm{A}, \mathrm{B}] = \mathrm{AB} - (-1)^{|\mathrm{A}|.|\mathrm{B}|}\,\mathrm{BA}.$$

A supertrace on a superalgebra is a linear form which vanishes on supercommutators.

1. The sheaf of equivariant differential forms

If G is a Lie group, G acts on the manifold underlying G by conjugation, $g\,.\,h = ghg^{-1}$. Consider G with the topology of invariant open sets:

$$\mathcal{O} = \{\mathrm{U} \subset \mathrm{G} \text{ open} \mid \mathrm{U} = g\,.\,\mathrm{U} \text{ for all } g \in \mathrm{G}\}.$$

In this paper, we will work with sheaves over G with this topology; for example, we have the sheaf of invariant function C^∞_G, defined by

$$\Gamma(U, C^\infty_G) = C^\infty(U)^G.$$

This sheaf is fine, since there exists partitions of unity on G invariant under conjugation, G being a compact Lie group. All of our sheaves on G will be sheaves of modules for C^∞_G, and hence will be fine.

Let M be a compact manifold with a smooth action of a compact Lie group G, which we denote by $(g, x) \in G \times M \mapsto g.x$. The group G acts on the algebra of differential forms on M by the formula

$$g.\omega = L^*_{g^{-1}}\omega,$$

where $L_{g^{-1}} : M \to M$ is the operation of left translation by $g^{-1} \in G$.

If $\omega : \mathfrak{g} \to \Omega^\bullet(M)$ is a map from $\mathfrak{g}$ to $\Omega^\bullet(M)$, the group G acts on ω by the formula

$$(g.\omega)(X) = g.(\omega(\mathrm{Ad}(g^{-1})X)).$$

If V is a finite-dimensional vector space, we will denote by $C^\infty_0(V)$ the algebra of germs at $0 \in V$ of smooth functions on V.

DEFINITION 1.1. – *A (local)* equivariant differential form *on* M *is a smooth germ at* $0 \in \mathfrak{g}$ *of a smooth map from* $\mathfrak{g}$ *to* $\Omega^\bullet(M)$ *invariant under the action of* G:

$$\Omega^\bullet_G(M) = C^\infty_0(\mathfrak{g}, \Omega^\bullet(M))^G.$$

This space is $\mathbb{Z}/2$-graded, and is a module over the algebra $C^\infty_0(\mathfrak{g})^G$ of germs of invariant smooth functions over $\mathfrak{g}$. Since the algebra of invariant polynomials $\mathbb{C}|\mathfrak{g}|^G$ on $\mathfrak{g}$ is a subalgebra of $C^\infty(\mathfrak{g})^G_0$, the space $(\mathbb{C}[\mathfrak{g}] \otimes \Omega^\bullet(M))^G$ is a subspace of $\Omega^\bullet_G(M)$ – this is Cartan's definition of the space of equivariant differential forms, of which our space is a certain completion.

Let us define operators d and ι on $C^\infty(\mathfrak{g}, \Omega^\bullet(M))$ by the formulas

$$(d\omega)(X) = d(\omega(X)),$$
$$(\iota\omega)(X) = \iota(X)(\omega(X)).$$

It is easy to check that d and ι commute with the action of G. By the formula $([d, \iota]\omega)(X) = \mathcal{L}(X)\omega$, we see that d and ι graded commute on elements of $\Omega^\bullet_G(M)$.

In this section, we will define a sheaf $\Omega^\bullet(M, G)$ over G which is an algebra over the sheaf of rings C^∞_G. If $g \in G$, let M^g denote the fixed point set of the diffeomorphism induced by g on M. Let G^g denote the centralizer of g

$$G^g = \{h \in G | gh = hg\},$$

and $\mathfrak{g}^g$ its Lie algebra. The passage from the compact Lie group G acting on a manifold M to the compact Lie group G^g acting on M^g is an example of the procedure of descent.

The stalk of the sheaf $\Omega^\bullet(M, G)$ at $g \in G$ is the space of equivariant differential forms

$$\Omega^\bullet(M, G)_g = \Omega^\bullet_{G^g}(M^g),$$

that is, germs at zero of smooth maps from $\mathfrak{g}^g$ to $\Omega^\bullet(\mathrm{M}^g)$ which are invariant under G^g. If $\omega \in \Omega^\bullet(\mathrm{M}, \mathrm{G})_g = \Omega^\bullet_{\mathrm{G}^g}(g)$, it is easily seen that $k.\omega$ is an element of $\Omega^\bullet(\mathrm{M}, \mathrm{G})_{k.g} = \Omega^\bullet_{k.\mathrm{G}^g}(\mathrm{M}^{k.g})$; thus, the group G acts on the sheaf $\Omega^\bullet(\mathrm{M}, \mathrm{G}) = \bigcup_{g \in \mathrm{G}} \Omega^\bullet(\mathrm{M}, \mathrm{G})_g$ in a way compatible with its conjugation action on G. We will write the differential on $\Omega^\bullet(\mathrm{M}, \mathrm{G})_g$ as d_g; it is easy to see that $k.d_g\,\omega = d_{k.g}\,k.\omega$.

DEFINITION 1.2. – *We say that a point $h = g\exp \mathrm{X} \in \mathrm{G}^g$, where $\mathrm{X} \in \mathfrak{g}^g$, is* near *a point $g \in \mathrm{G}$ if $\mathrm{M}^{g\exp \mathrm{X}} \subset \mathrm{M}^g$ and $\mathrm{G}^{g\exp \mathrm{X}} \subset \mathrm{G}^g$.*

If G is a compact Lie group and M is a compact manifold with smooth G-action, then by a theorem of Mostow and Palais [12], [13], there is a finite-dimensional linear representation V of G and a smooth equivariant embedding $\mathrm{M} \hookrightarrow \mathrm{V}$.

LEMMA 1.3. – *Let $\mathrm{M}(\mathrm{X})$ be the fixed point set of the element $g\exp \mathrm{X}$, where $\mathrm{X} \in \mathfrak{g}^g$. Then for X sufficiently small, $\mathrm{M}(\mathrm{X}) \subset \mathrm{M}(0) = \mathrm{M}^g$. In other words, the set of all points in G^g near g is a neighbourhood of g.*

Proof. – By the above considerations, we may assume that M is a complex vector space on which G acts linearly. We may certainly assume that this action is unitary. In this way, we need only consider the case in which g is a diagonal matrix acting on $\mathbb{C}^N$. Decomposing $\mathbb{C}^N$ according to the eigenvalues of g, we may even assume that g is a multiple of the identity, in which case the results is obvious. □

A section $\omega \in \Gamma(\mathrm{U}, \Omega^\bullet(\mathrm{M}, \mathrm{G}))$ of the sheaf $\Omega^\bullet(\mathrm{M}, \mathrm{G})$ over an invariant open set $\mathrm{U} \subset \mathrm{G}$ is defined by giving, for each point $g \in \mathrm{U}$, an element ω_g of $\Omega^\bullet(\mathrm{M}, \mathrm{G})_g$, such that if h is near g, we have the equality of germs

$$\omega_g|_{\mathfrak{g}^h \times \mathrm{M}^h} = \omega_h \in \Omega^\bullet(\mathrm{M}, \mathrm{G})_h.$$

We see that $\Omega^\bullet(\mathrm{M}, \mathrm{G})$ is an equivariant sheaf of differential graded algebras over G.

DEFINITION 1.4. – *A* global equivariant differential form $\omega \in \mathcal{A}^\bullet_{\mathrm{G}}(\mathrm{M})$ *is a global section* $\omega \in \Gamma(\mathrm{G}, \Omega^\bullet(\mathrm{M}, \mathrm{G}))$.

EXAMPLE 1.5. – The simplest example of the above construction is where M is a point *pt*. Observe that $\mathrm{C}^\infty_0(\mathfrak{g}, \Omega^\bullet(pt))^{\mathrm{G}}$ is equal to $\mathrm{C}^\infty_0(\mathfrak{g})^{\mathrm{G}}$, and hence the stalk of $\Omega^\bullet(pt, \mathrm{G})$ at the identity may be identified with the stalk of $\mathrm{C}^\infty_{\mathrm{G}}$ at the identity. A similar argument at other point of G shows that $\Omega^\bullet(\mathrm{M}, \mathrm{G}) = \mathrm{C}^\infty_{\mathrm{G}}$ is the sheaf of invariant functions on G, concentrated in degree 0, and hence that $\mathcal{A}^\bullet_{\mathrm{G}}(\mathrm{M}) = \mathrm{R}^\infty(\mathrm{G})$.

EXAMPLE 1.6. – As another example, we may consider the case of a manifold P with a free action of the group G (that is, a principal bundle). In this case, the stalks of $\Omega^\bullet(\mathrm{M}, \mathrm{G})$ vanish except at the identity $e \in \mathrm{G}$, where we have $\Omega^\bullet(\mathrm{M}, \mathrm{G})_e = \Omega^\bullet_{\mathrm{G}}(\mathrm{P})$; hence, we see that

$$\mathcal{A}^\bullet_{\mathrm{G}}(\mathrm{P}) = \Omega^\bullet_{\mathrm{G}}(\mathrm{P}).$$

EXAMPLE 1.7. – Let G be a compact connected Lie group with maximal torus T. Consider the action of G on the flag variety $\mathrm{M} = \mathrm{G}/\mathrm{T}$. It suffices to calculate the stalk of the sheaf $\Omega^\bullet(\mathrm{M}, \mathrm{G})_g$ for $g \in \mathrm{T}$, since the conjugates of T cover G. Let H be a connected reductive subgroup of G containing the maximal torus T, and having positive

roots $\Delta^+(\mathrm{H}) \subset \Delta^+(\mathrm{G})$. If $g \in \mathrm{T}$ lies in the centralizer of H, that is, the intersection of the sets

$$\{\alpha(g) = 1 | \alpha \in \Delta^+(\mathrm{H})\},$$

we see that M^g may be identified with N (H)/T, where N (H) is the normalizer of the group H, and that G^g may be identified with H. In this way, we see that $\Omega^\bullet(\mathrm{M}, \mathrm{G})_g = \Omega^\bullet_{\mathrm{H}}(\mathrm{N}(\mathrm{H})/\mathrm{T})$. In particular, the case $\mathrm{H} = \mathrm{T}$ corresponds to the set of regular points $g \in \mathrm{T}$, and we see that for such points,

$$\Omega^\bullet(\mathrm{M}, \mathrm{G})_g = \Omega^\bullet_{\mathrm{T}}(\mathrm{W}(\mathrm{G}, \mathrm{T})),$$

where W (G, T) is the Weyl group of G with respect to T, and that the boundaries d and ι vanish.

2. The equivariant Hochschild complex

Before continuing, we must recall a little of the theory of Hochschild homology for topological algebras; as references, we suggest Taylor [15] or Block [3]. If V_1 and V_2 are two locally convex topological vector spaces, we will denote by $\mathrm{V}_1 \otimes \mathrm{V}_2$ their completed projective tensor product, usually denoted $\mathrm{V}_1 \hat{\otimes} \mathrm{V}_2$ (we will have no cause for considering the algebraic tensor product, so this should not cause any confusion). A topological algebra is a locally convex topological vector space A with associative multiplication given by a continuous linear map from $\mathrm{A} \otimes \mathrm{A}$ to A; in other words, the product is jointly continuous. Given a right module K and a left module L (again, with jointly continuous actions), the tensor product $\mathrm{K} \otimes_{\mathrm{A}} \mathrm{L}$ is defined to be the quotient

$$\mathrm{K} \otimes_{\mathrm{A}} \mathrm{L} = \frac{\mathrm{K} \otimes \mathrm{L}}{\operatorname{span}(ma \otimes n - m \otimes an | m \in \mathrm{K},\ a \in \mathrm{A},\ n \in \mathrm{L})}.$$

Unless otherwise stated, all algebras will have identities.

In the category of modules of a topological algebra A, an exact sequence is a complex of modules

$$\cdots \to \mathrm{L}_{i+1} \to \mathrm{L}_i \to \mathrm{L}_{i-1} \to \cdots$$

which is split exact as a complex of topological vector spaces. We may develop a relative homological algebra using exact resolutions of a module by projective modules, where we define projective to mean with respect to this definition of exact sequence.

The Hochschild homology of the topological algebra A with coefficients in a topological $\mathrm{A} \otimes \mathrm{A}^\circ$-module L is defined to be the sequence of derived functors of the functor

$$\mathrm{H}_0(\mathrm{A}, \mathrm{L}) = \mathrm{A}/\operatorname{span}([a, m] | a \in \mathrm{A},\ m \in \mathrm{L}).$$

If E is a vector bundle over a manifold, denote by $\Gamma(\mathrm{M}, \mathrm{E})$ the space of smooth sections of E; it is a nuclear Fréchet space. If M and N are manifolds with vector bundles E and F respectively, the projective tensor product $\Gamma(\mathrm{M}, \mathrm{E}) \otimes \Gamma(\mathrm{N}, \mathrm{F})$ is isomorphic to $\Gamma(\mathrm{M} \times \mathrm{N}, \mathrm{E} \boxtimes \mathrm{F})$.

By the results of Section 4 of Taylor [15], the Hochschild homology $\mathrm{H}_\bullet(\mathrm{A}, \mathrm{L})$ of a nuclear Fréchet module L over a nuclear Fréchet algebra A may be calculated by taking a resolution

$$\cdots \xrightarrow{\partial_3} \mathrm{F}_2 \xrightarrow{\partial_2} \mathrm{F}_1 \xrightarrow{\partial_1} \mathrm{F}_0 \xrightarrow{\varepsilon} \mathrm{A}$$

of A by projective $\mathrm{A} \otimes \mathrm{A}^\circ$-modules and forming the homology of the complex $(\mathrm{F}_i \otimes_{\mathrm{A}\otimes\mathrm{A}^\circ} \mathrm{L}, \partial_i)$. Thus, at least in this case, relative homological algebra is not too different from ordinary homological algebra.

If A is a topological algebra, the vector spaces $[k] \mapsto \mathrm{A}^{\otimes(k+1)}$ form a cyclic vector space in the sense of Connes [6]; the generators are represented by the formulas

$$\begin{cases} d_i\,(a_0 \otimes \cdots \otimes a_k) = a_0 \otimes \cdots \otimes a_i\, a_{i+1} \otimes \cdots \otimes a_k & \text{for} \quad 0 \leq i < k, \\ d_k\,(a_0 \otimes \cdots \otimes a_k) = a_k\, a_0 \otimes \cdots \otimes a_{k-1} & \\ s_i\,(a_0 \otimes \cdots \otimes a_k) = a_0 \otimes \cdots \otimes a_i \otimes 1 \otimes a_{i+1} \otimes \cdots \otimes a_k & \text{for} \quad 0 \leq i < k, \\ t\,(a_0 \otimes \cdots \otimes a_k) = a_k \otimes a_0 \otimes \cdots \otimes a_{k-1}. & \end{cases}$$

Now let G be a compact Lie group which acts smoothly on A (preserving the identity). The group G acts on $\mathrm{C}^\infty(\mathrm{G}, \mathrm{A}^{\otimes(k+1)}) \cong \mathrm{C}^\infty(\mathrm{G}) \otimes \mathrm{A}^{\otimes(k+1)}$ by the formula

$$h\,.\,(\varphi \otimes a_0 \otimes \cdots \otimes a_k) = h\,.\,\varphi \otimes h\,.\,a_0 \otimes \cdots \otimes h\,.\,a_k,$$

where we recall that $(h\,.\,\varphi)\,(g) = \varphi\,(hgh^{-1})$. Let $\mathrm{C}^\infty(\mathrm{G}, \mathrm{A}^{\otimes(k+1)})^{\mathrm{G}}$ be the subspace of invariant chains. The vector spaces $[k] \mapsto \mathrm{C}^\infty(\mathrm{G}, \mathrm{A}^{\otimes(k+1)})^{\mathrm{G}}$ form a cyclic vector space, with generators represented by the formulas

$$\begin{cases} d_i\,(\varphi \otimes a_0 \otimes \cdots \otimes a_k) = \varphi \otimes a_0 \otimes \cdots \otimes a_i\, a_{i+1} \otimes \cdots \otimes a_k & \text{for} \quad 0 \leq i < k, \\ d_k\,(\varphi \otimes a_0 \otimes \cdots \otimes a_k)\,(h) = \varphi\,(h)\,(h\,.\,a_k)\, a_0 \otimes \cdots \otimes a_{k-1} & \\ s_i\,(\varphi \otimes a_0 \otimes \cdots \otimes a_k) = \varphi \otimes a_0 \otimes \cdots \otimes a_i \otimes 1 \otimes a_{i+1} \otimes \cdots \otimes a_k & \text{for} \quad 0 \leq i < k, \\ t\,(\varphi \otimes a_0 \otimes \cdots \otimes a_k)\,(h) = \varphi\,(h)\, h\,.\,a_k \otimes a_0 \otimes \cdots \otimes a_{k-1}. & \end{cases}$$

Given a cyclic vector space $[k] \mapsto \mathrm{V}_k$, its *normalization* is defined as the complex

$$\mathrm{N}\,(\mathrm{V})_k = \frac{\mathrm{V}_k}{\sum_{i=0}^{k-1} s_i\,[\mathrm{V}_{k-1}]},$$

with boundary

$$b = \sum_{i=0}^{k} (-1)^i\, d_i : \mathrm{N}\,(\mathrm{V})_k \to \mathrm{N}\,(\mathrm{V})_{k-1}.$$

Applying this construction to the cyclic space $[k] \mapsto \mathrm{A}^{\otimes(k+1)}$, we obtain the *Hochschild complex* of A, denoted $\mathcal{C}_\bullet(\mathrm{A})$. The homology of this complex is called the Hochschild homology of A, and denoted $\mathrm{HH}_\bullet(\mathrm{A})$.

If A is an algebra without identity, we must the Hochschild homology a little more careffuly. Let $\mathrm{A}^+ = \mathrm{A} \oplus \mathbb{C}$ be the unital algebra obtained by adjoining an identity to A.

The homomorphism $A \to \mathbb{C}$ which sends A to zero induces a map

$$\mathrm{HH}_k(A^+) \to \mathrm{HH}_k(\mathbb{C}) = \begin{cases} \mathbb{C}, & k = 0, \\ 0, & k > 0, \end{cases}$$

of Hochschild homology groups, and $\mathrm{HH}_k(A)$ is defined to be the kernel of this map. If A already has an identity, this definition agrees with the earlier one.

If M is a manifold on which acts the compct Lie group G, and $A = C^\infty(M)$ we may realize this cyclic vector space as the series of vetor spaces

$$[k] \mapsto C^\infty(M^{k+1} \times G)^G,$$

where G acts on $C^\infty(M^{k+1} \times G)$ by the formula

$$(h \,.\, c)(x_0, \ldots, x_k | g) = c(h^{-1} x_0, \ldots, h^{-1} x_k | h^{-1} \,.\, g).$$

The generators of the cyclic category are represented by the formulas

$$\begin{cases} (d_i c)(x_0, \ldots, x_k | g) = c(x_0, \ldots, x_i, x_i, \ldots, x_k | g) & \text{for} \quad 0 \le i < k, \\ (d_k c)(x_0, \ldots, x_k | g) = c(x_0, \ldots, x_k, g \,.\, x_0 | g) & \text{for} \quad i = k, \\ (s_i c)(x_0, \ldots, x_k | g) = c(x_0, \ldots, x_{i-1}, x_{i+1}, \ldots, x_k | g) & \text{for} \quad 0 \le i \le k, \\ (tc)(x_0, \ldots, x_k | g) = c(x_1, \ldots, x_k, g \,.\, x_0 | g). \end{cases}$$

We may check that t commutes with the action of G:

$$\begin{aligned} h^{-1} \,.\, (tc)(x_0, \ldots, x_k | g) &= (tc)(hx_0, \ldots, hx_k | h \,.\, g) \\ &= c(hx_1, \ldots, hx_k, (hgh^{-1}) hx_0 | h \,.\, g) \\ &= t(h^{-1} \,.\, c)(x_0, \ldots, x_k | g). \end{aligned}$$

The only relation in the cyclic category which is not obvious is $t^{k+1} = 1$ on $C^\infty(M^k \times G)^G$, but we see that

$$(t^{k+1} c)(x_0, \ldots, x_k | g) = c(gx_0, \ldots, gx_k | g),$$

which equals $c(x_0, \ldots, x_k | g)$ by invariance.

Parenthetically, we note that the vector spaces $[k] \mapsto C^\infty(M^{k+1} \times G)^G$ actually form a dihedral vector space, in the sence of Loday, with the action of θ being given by the formula

$$(\theta c)(x_0, \ldots, x_k | g) = \overline{c(x_0, gx_k, \ldots, gx_1 | g^{-1})}.$$

This dihedral structure may be related to equivariant KR-theory.

Normalizing the cyclic space $[k] \mapsto C^\infty(G, A^{\otimes(k+1)})^G$, we obtain the *equivariant Hochschild complex* of A, hich we will denote $C_\bullet^G(A)$. The homology of this complex is called the equivariant Hochschild homology of A, and denoted $\mathrm{HH}_\bullet^G(A)$.

If A is a topological algebra with smooth action of a compact Lie group G, we denote by $A \rtimes G$ the *crossed product algebra* $C^\infty(G, A)$, with multiplication

$$(u * v)(g) = \int_G u(h)\, h.v(h^{-1} g)\, dh.$$

Note that if G is not discrete, this algebra does not have an identity.

The following theorem is due to Brylinski [4], [5].

THEOREM 2.1. – *Let* A *be a topological algebra with identity, and let* G *be a compact Lie group acting on* A. *There is a natural isomomorphism* $\mathrm{HH}_\bullet^{\mathrm{G}}(\mathrm{A}) \cong \mathrm{HH}_\bullet(\mathrm{A} \rtimes \mathrm{G})$.

On the normalization of a cyclic vector space, we also have a differential B of degree minus one, which graded commutes with b; it is given by the formula

$$\mathrm{B} = \sum_{i=0}^{k} (-1)^{ki}\, s\,.\,t^i : \mathrm{N}(\mathrm{V})_k \to \mathrm{N}(\mathrm{V})_{k+1},$$

where $s = ts_0\, t^{-1}$.

If W is a module over the algebra of polynomials $\mathbb{C}[u]$, graded by $\deg(u) = -2$, we may form a complex

$$(\mathcal{C}_\bullet^{\mathrm{G}}(\mathrm{A})[\![u]\!] \otimes_{\mathbb{C}[u]} \mathrm{W},\ b + u\,\mathrm{B});$$

the degree of u is fixed so that the operator $b + u\mathrm{B}$ will have degree –1. We denote the homology of this complex by $\mathrm{HC}_\bullet^{\mathrm{G}}(\mathrm{A};\ \mathrm{W})$, and call it equivariant cyclic homology with coefficients in W. For example,

(1) $\mathrm{W} = \mathbb{C}[u]$ gives the equivariant negative cyclic homology, which is the fundamental theory; this theory is usually denoted by $\mathrm{HC}_\bullet^{-,\,\mathrm{G}}(\mathrm{A})$;

(2) $\mathrm{W} = \mathbb{C}((u))$ (Laurent series) gives periodic cyclic homology, denoted by $\mathrm{HP}_\bullet^{\mathrm{G}}(\mathrm{A})$, which is a $\mathbb{Z}$-graded version of the $\mathbb{Z}/2$-graded theory obtained by taking the homology of $\mathcal{C}_\bullet^{\mathrm{G}}(\mathrm{A})$ with respect to the boundary $b + \mathrm{B}$.

(3) $\mathrm{W} = \mathbb{C}$ with u acting by zero gives Hochschild homology $\mathrm{HH}_\bullet^{\mathrm{G}}(\mathrm{A})$.

In the case of $\mathrm{A} = \mathrm{C}^\infty(\mathrm{M})$, the operators b and B are given on a chain $c \in \mathrm{C}^\infty(\mathrm{M}^{k+1} \times \mathrm{G})$ by the formulas

$$\begin{aligned}(bc)(x_0, \ldots, x_{k-1}|g) = &\sum_{i=0}^{k-1} (-1)^i\, c(x_0, \ldots, x_i, x_i, \ldots, x_{k-1}|g)\\ &+ (-1)^k\, c(x_0, \ldots, x_{k-1}, gx_0|g),\end{aligned}$$

and

$$(\mathrm{B}\,c)(x_0, \ldots, x_{k+1}|g) = \sum_{i=1}^{k+1} (-1)^{(k-i)\,k}\, c(x_i, \ldots, x_{k+1}, gx_1, \ldots, gx_{i-1}|g).$$

There are a number of multilinear operators that may be introduced on the equivariant Hochschild complex of a commutative algebra. We will start with the shuffle product, which defines a graded commutative product on the space of equivariant chains $\mathcal{C}_\bullet^{\mathrm{G}}(\mathrm{C}^\infty(\mathrm{M}))$, since $\mathrm{C}^\infty(\mathrm{M})$ is commutative.

A (k, l)-*shuffle* is a permutation $\chi \in \mathrm{S}_{k+l}$ with the property that $\chi(i) < \chi(j)$ if $1 \leq i < j \leq k$, or if $k+1 \leq i < j \leq k+l$. The shuffle product on $\mathcal{C}_\bullet^{\mathrm{G}}(\mathrm{C}^\infty(\mathrm{M}))$ is defined as the sum over all (k, l)-shuffles

$$\begin{aligned}&(c * \tilde{c})(x_0, \ldots, x_{k+l}|g)\\ &\quad = \sum_\chi (-1)^{\varepsilon(\chi)} c(x_0, x_{\chi(1)}, \ldots, x_{\chi(k)}|g)\, \tilde{c}(x_0, x_{\chi(k+1)}, \ldots, x_{\chi(k+l)}|g).\end{aligned}$$

The following proposition summarizes the properties of this product.

PROPOSITION 2.2. – (1) *The shuffle product on* $\mathcal{C}_\bullet^{\mathrm{G}}(\mathrm{C}^\infty(\mathrm{M}))$ *is associative and graded commutative with identity* $1 \in \mathcal{C}_0^{\mathrm{G}}(\mathrm{C}^\infty(\mathrm{M}))$.

(2) *The differential* b *on* $\mathcal{C}_\bullet^{\mathrm{G}}(\mathrm{C}^\infty(\mathrm{M}))$ *satisfies Leibniz's rule with respect to the shuffle product, so that* $\mathcal{C}_\bullet^{\mathrm{G}}(\mathrm{C}^\infty(\mathrm{M}))$ *is made into a commutative differential graded algebra.*

It is possible to generalize the results of [10] to the equivariant setting, and define an A_∞-structure on $\mathcal{C}_\bullet^{\mathrm{G}}(\mathrm{C}^\infty(\mathrm{M}))[\![u]\!]$ with differential $b + u\mathrm{B}$ and product a deformation of the shuffle product.

In order to define the higher maps B_n, we need a little combinatorial machinery. Given numbers $i_1, \ldots, i_n$, order the set

$$\mathrm{C}(i_1, \ldots, i_n) = \{(1, 0), \ldots, (1, i_1), \ldots, (n, 0), \ldots, (n, i_n)\}$$

lexicographically, that is $(k_1, l_1) < (k_2, l_2)$ if and only if $k_1 < k_2$ or $k_1 = k_2$ and $l_1 < l_2$. A *cyclic shuffle* χ is a permutation of the set $\mathrm{C}(i_1, \ldots, i_n)$ which satisfies the following two conditions:

(1) $\chi(i, 0) < \chi(j, 0)$ if $i < j$, and

(2) for each $1 \leq m \leq n$, there is a number $0 \leq j_m \leq i_m$ such that

$$\chi(m, j_m) < \cdots < \chi(m, i_m) < \chi(m, 0) < \cdots < \chi(m, j_m - 1).$$

We will denote the set of cyclic shuffles by $S(i_1, \cdots, i_n)$.

Given chains c_k, $1 \leq k \leq n$, in $\mathcal{C}_\bullet^{\mathrm{G}}(\mathrm{C}^\infty(\mathrm{M}))$, we define the result of the operation $\mathrm{B}_n(c_1, \ldots, c_n) \in \mathcal{C}_\bullet^{\mathrm{G}}(\mathrm{C}^\infty(\mathrm{M}))$ by the formula

$$\begin{aligned}&\mathrm{B}_n(c_1, \ldots, c_n)(x_0, x_{(1,0)}, \ldots, x_{(1,i_1)}, \ldots, x_{(n,0)}, \ldots, x_{(n,i_n)}|g)\\ &\quad = \sum_{\chi \in S(i_0, \ldots, i_n)} (-1)^{\varepsilon(\chi)} \prod_{k=1}^n c_k(g^{\eta(k,0)} . x_{\chi(k,0)}, \ldots, g^{\eta(k,i_k)} . x_{\chi(k,i_k)}|g)\end{aligned}$$

Here, $\eta_\chi(i, j)$ equals 0 if $\chi(i, 0) \leq \chi(i, j)$, and 1 if $\chi(i, j) < \chi(i, 0)$.

Using the operators B_n, we may define a series of multilinear products m_n on $\mathcal{C}_\bullet^{\mathrm{G}}(\mathrm{C}^\infty(\mathrm{M}))[\![u]\!]$ by the formula

$$m_n(c_1, \ldots, c_n) = \begin{cases} (b + u\mathrm{B})\, c_1, & n = 1,\\ c_1 * c_2 + (-1)^{|c_1|} u\mathrm{B}_2(c_1, c_2), & n = 2,\\ (-1)^{(n-1)(|c_1|-1)+\cdots+(|c_{n-1}|-1)} u\mathrm{B}_n(c_1, \ldots, c_n), & \text{otherwise.}\end{cases}$$

It may be proved, in much the same way as in Getzler-Jones [10], that the operators m_n define an A_∞-structure on $\mathcal{C}_\bullet^G(C^\infty(M))[\![u]\!]$. Thus, $\mathcal{C}_\bullet^G(C^\infty(M))$ is a sort of homotopy-associative algebra: the operator $m_1 = b + uB$ is a differential and m_2 is a (non-associative) product on $\mathcal{C}_\bullet^G(C^\infty(M))$ for which Leibniz's rule holds:

$$m_1(m_2(c_1, c_2)) - m_2(m_1(c_1), c_2) - (-1)^{|c_1|} m_2(c_1, m_1(c_2)) = 0.$$

Finally, m_3 defines a homotopy which corrects the non-associativity of m_2:

$$\begin{aligned} &- m_1(m_3(a, b, c)) + m_2(m_2(a, b, c)) - (-1)^{|a|} m_2(a, m_2(b, c)) \\ &\quad - m_3(m_1(a), b, c) - (-1)^{|a|} m_3(a, m_1(b), c) - (-1)^{|a|+|b|} m_3(a, b, m_1(c)) = 0 \end{aligned}$$

In particular, m_2 defines an associative (graded) commutative product on the homology theories $HC_\bullet^{-,G}(C^\infty(M))$ and $HP_\bullet^G(C^\infty(M))$.

We will now give a construction which is basic to the study of cyclic homology of algebras of smooth functions. If U is an invariant open subset of M, let $\mathcal{C}_\bullet(C^\infty(M), C^\infty(U \times G))$ be the complex such that

$$\mathcal{C}_k(C^\infty(M), C^\infty(U \times G)) = C^\infty(U \times M^k \times G),$$

with boundary

$$\begin{aligned} (bc)(y, x_1, \ldots, x_{k-1}|g) = &c(y, y, x_1, \ldots, x_{k-1}|g) \\ &+ \sum_{i=1}^{k-1} (-1)^i c(y, x_1, \ldots, x_i, x_i, \ldots, x_{k-1}|g) \\ &+ (-1)^k c(y, x_1, \ldots, x_{k-1}, gy|g). \end{aligned}$$

This may be identified with the (unnormalized) chain complex for the algebra $C^\infty(M)$, with coefficients in the bimodule $C^\infty(U \times G)$, with respect to the two actions

$$\begin{aligned} (f.m)(x, g) &= f(gx)\, m(x, g), \\ (m.f)(x, g) &= m(x, g)\, f(x), \end{aligned}$$

for $m \in C^\infty(U \times G)$ and $f \in C^\infty(M)$.

Let $\mathcal{C}_\bullet(C^\infty(M), C^\infty(U \times G))^G$ denote the subspace of G-invariant elements of $\mathcal{C}_\bullet(C^\infty(M), C^\infty(U \times G))$. By restriction from $U \times M^k \times G$ to $U^{k+1} \times G$, we obtain a map

$$\mathcal{C}_\bullet(C^\infty(M), C^\infty(U \times G))^G \xrightarrow{\beta} \mathcal{C}_\bullet^G(C^\infty(U)).$$

PROPOSITION 2.3. – *For any invariant open subset* $U \subset M$, *the map* β *is a quasi-isomorphism of complexes.*

Proof. – We use the following abstract result.

LEMMA 2.4. – *Suppose* $\varphi : A \to B$ *is a continuous homomorphism of nuclear Fréchet algebras such that*

(1) $B \otimes_A B \cong B$, *and*

(2) $H_i(A, B \otimes B) = 0$ *for* $i > 0$.

Then for any B*-bimodule* L, *the natural map*

$$\mathrm{H}_\bullet(\mathrm{A}, \mathrm{L}) \to \mathrm{H}_\bullet(\mathrm{B}, \mathrm{L})$$

is an isomorphism, where L *is considered as an* A*-bimodule in the obvious way.*

Proof. – Consider the following resolution of B by free $\mathrm{B} \otimes \mathrm{B}^{\mathrm{o}}$ modules:

$$\cdots \to \mathrm{B} \otimes \mathrm{A}^{\otimes i} \otimes \mathrm{B} \to \mathrm{B} \otimes \mathrm{A} \otimes \mathrm{B} \to \mathrm{B} \otimes \mathrm{B} \to \mathrm{B}$$

with the boundaries

$$\begin{aligned}\partial(b_0 \otimes a_1 \otimes \cdots \otimes a_n \otimes b_{n+1}) &= b_0\, \varphi(a_1) \otimes a_2 \otimes \cdots \otimes a_n \otimes b_{n+1} \\ &\quad + \sum_{i=1}^{n-1} (-1)^i\, b_0 \otimes \cdots \otimes a_i\, a_{i+1} \otimes \cdots \otimes a_n \otimes b_{n+1} \\ &= (-1)^n\, b_0 \otimes \cdots \otimes a_{n-1} \otimes \varphi(a_n)\, b_{n+1}.\end{aligned}$$

This complex is the standard complex computing $\mathrm{H}_i(\mathrm{A}, \mathrm{B} \otimes \mathrm{B})$ which is exact by hypothesis so provides a resolution of B by free $\mathrm{B} \otimes \mathrm{B}^{\mathrm{o}}$ modules. So by tensoring the above resolution by L over $\mathrm{B} \otimes \mathrm{B}^{\mathrm{o}}$ we find that $\mathrm{H}_\bullet(\mathrm{B}, \mathrm{L})$ can be computed from

$$\cdots \to \mathrm{L} \otimes \mathrm{A}^{\otimes 2} \to \mathrm{L} \otimes \mathrm{A} \to \mathrm{L}$$

which also computes $\mathrm{H}_\bullet(\mathrm{A}, \mathrm{L})$. □

In order to apply this result, we use the following lemma; this is Lemme 6.1 of Tougeron [16].

LEMMA 2.5. – *Let* U *be an open subset of a manifold* M. *If* $\{f_\alpha\}$ *is a countable collection of functions in* $\mathrm{C}^\infty(\mathrm{U})$, *then there exists a function* $\varphi \in \mathrm{C}^\infty(\mathrm{M})$ *with the following properties:*

(1) φ *is nowhere vanishing on* U;

(2) φ *and all of its derivatives vanish on* M\U;

(3) *For each* α, φf_α *extends to* M, *and* φf_α *and all of its derivatives vanish on* M\U.

We use this to verify that the hypotheses of Lemma 2.4 are satisfied, with $\mathrm{A} = \mathrm{C}^\infty(\mathrm{M})$ and $\mathrm{B} = \mathrm{C}^\infty(\mathrm{U})$:

(1) $\mathrm{C}^\infty(\mathrm{U}) \otimes_{\mathrm{C}^\infty(\mathrm{M})} \mathrm{C}^\infty(\mathrm{U}) \cong \mathrm{C}^\infty(\mathrm{U})$, and

(2) for each $i > 0$, $\mathrm{H}_i(\mathrm{C}^\infty(\mathrm{M}), \mathrm{C}^\infty(\mathrm{U}) \otimes \mathrm{C}^\infty(\mathrm{U})) = 0$.

This shows that the map

$$\mathcal{C}_\bullet(\mathrm{C}^\infty(\mathrm{M}), \mathrm{C}^\infty(\mathrm{U} \times \mathrm{G})) \to \mathcal{C}_\bullet(\mathrm{C}^\infty(\mathrm{U}), \mathrm{C}^\infty(\mathrm{U} \times \mathrm{G}))$$

is a quasi-isomorphism, and Proposition 2.3 follows by taking invariants.

The statements (1) and (2) are both contained in the exactness of the following complex:

$$\cdots \to \mathrm{C}^\infty(\mathrm{U} \times \mathrm{M} \times \mathrm{U}) \to \mathrm{C}^\infty(\mathrm{U} \times \mathrm{U}) \to \mathrm{C}^\infty(\mathrm{U}) \to 0.$$

An element $h \in C^\infty(U \times M^k \times U)$ such that $\partial_k h = 0$ may be written as a countable sum

$$h(y, x_1, \ldots, x_k, z) = \sum_{j=1}^{\infty} f_j(y)\, g_j(x_1, \ldots, x_k, z),$$

where $f_j \in C^\infty(U)$ and $g_j \in C^\infty(M^k \times U)$; this follows from the isomorphism $C^\infty(U \times M^k \times U) \cong C^\infty(U) \otimes C^\infty(M^k \times U)$. Let $\varphi \in C^\infty(M)$ be the function whose existence is guaranteed by Lemma 2.5 applied to the countable set of functions $\{f_j\} \subset C^\infty(U)$. Since $\partial_k h = 0$, it follows that $s_\varphi(h) \in C^\infty(U \times M^{k+1} \times U)$, defined by the formula

$$s_\varphi(h)(y, x_0, \ldots, x_k, z) = \sum_{j=1}^{\infty} \varphi(y)^{-1}\, \varphi(0)\, f_j(y_0)\, g(x_1, \ldots, x_k, z)$$

satisfies $\partial_{k+1} s_\varphi(h) = h$. Hence the above complex is exact. $\square$

3. The equivariant Hochschild-Kostant-Rosenberg map

There is a sheaf $\mathcal{C}_\bullet(C^\infty(M), G)$ of cyclic vector spaces associated to the space of equivariant Hochschild chains over the topological space G (with the quotient topology of the last section). Over an invariant open set $\mathcal{O}$ of G, we associate the graded space of chains

$$\Gamma(\mathcal{O}, C_\bullet(C^\infty(M), G)) = C^\infty(\mathcal{O}, C^\infty(M^{\bullet+1}))^G.$$

The sheaf $\mathcal{C}_\bullet(C^\infty(M), G)$ is easily seen to be a module for the sheaf of ring C^∞_G, and $\Gamma(G, \mathcal{C}_\bullet(C^\infty(M), G)) = \mathcal{C}_\bullet(C^\infty(M))$. We also see that the differentials b and B act on the stalks of the sheaf $\mathcal{C}_\bullet(C^\infty(M), G)$.

We will now construct a map α between the sheaves $\mathcal{C}_\bullet(C^\infty(M), G)$ and $\Omega^\bullet(M, G)$. In fact, since both sheaves are fine, it suffices to construct a map from the space of global sections $\mathcal{C}^G_\bullet(C^\infty(M))$ to $\Gamma(G, \Omega^\bullet(M, G))$, in other words, for each $g \in G$, a map α_g from $\mathcal{C}^G_\bullet(C^\infty(M))$ to $\Omega^\bullet_{G^g}(M^g)$ such that for h near g,

$$\alpha_g\, c|_{M^h} = \alpha_h\, c,$$

and also which is equivariant in the sense that

$$k \,.\, (\alpha_g\, c) = \alpha_{k.g}\, (k \,.\, c).$$

We will actually construct the map α_g from $C^\infty(M^{k+1} \times G)$ to $C^\infty_0(\mathfrak{g}^g, \Omega^k(M^g))$. On chains of the form

$$c(x_0, \ldots, x_k | g) = f_0(x_0) \cdots f_k(x_k) \psi(g),$$

where $f_i \in C^\infty(M)$ and $\psi \in C^\infty(G)$, and for $X \in \mathfrak{g}^g$, $(\alpha_g\, c)(X) \in \Omega^\bullet(M^g)$ is defined by the integral over the k-simplex

$$\alpha_g(c)(X) = \psi(g \exp X) \int_{\Delta_k} f_0\, d(e^{-t_1 X} \,.\, f_1) \wedge \cdots \wedge d(e^{-t_k X} \,.\, f_k)|_{M^g} dt_1 \cdots dt_k.$$

It is easy to see that this map extends to all of $C^\infty(M^{k+1} \times G)$.

The maps α_g generalize the map from the Hochschild chain complex $\mathcal{C}_\bullet(\mathrm{C}^\infty(\mathrm{M}))$ to $\Omega^\bullet(\mathrm{M})$ defined by the formula

$$\alpha(f_0 \otimes \cdots \otimes f_k) \to \frac{1}{k!} f_0\, df_1 \cdots df_k.$$

This map sends the Hochschild differential b to zero, and Conne's differential B to the exterior differential d. Our goal is to generalize this result.

The next result shows that the maps α_g combine to give a map of sheaves $\alpha : \mathcal{C}_\bullet(\mathrm{M}, \mathrm{G}) \to \Omega^\bullet(\mathrm{M}, \mathrm{G})$.

PROPOSITION 3.1. – (1) *If $h = g \exp \mathrm{X}$ is near g, the map α_g satisfies the formula*

$$(\alpha_h c)(\mathrm{Y}) = (\alpha_g c)(\mathrm{X} + \mathrm{Y})|_{\mathrm{M}^h} \qquad \text{for} \quad \mathrm{Y} \in \mathfrak{g}^h.$$

(2) *The map α_g is equivariant with respect to the action of G^g, and hence sends invariant chains $c \in \mathrm{C}^\infty(\mathrm{M}^{k+1} \times \mathrm{G})^{\mathrm{G}}$ to elements of $\Omega^k_{\mathrm{G}^g}(\mathrm{M}^g)$.*

(3) *The map α_g descends to the normalized spaces $\mathcal{C}^{\mathrm{G}}_\bullet(\mathrm{C}^\infty(\mathrm{M}))$.*

Proof. – First, observe that $[\mathrm{X}, \mathrm{Y}] = 0$, and hence that

$$\psi(ge^{\mathrm{X}+\mathrm{Y}}) = \psi((ge^{\mathrm{X}})\, e^{\mathrm{Y}}).$$

Taking c of the form $f_0 \otimes \cdots \otimes f_k \otimes \psi$, we see that to prove Part (1), it suffices to show that

$$\begin{aligned}\int_{\Delta_k} & f_0\, d(e^{-t_1(\mathrm{X}+\mathrm{Y})} . f_1) \cdots d(e^{-t_k(\mathrm{X}+\mathrm{Y})} . f_k)|_{\mathrm{M}^h}\, dt_1 \cdots dt_k \\ &= \int_{\Delta_k} f_0\, d(e^{-t_1 \mathrm{Y}} . f_1) \cdots d(e^{-t_k \mathrm{Y}} . f_k)|_{\mathrm{M}^h}\, dt_1 \cdots dt_k,\end{aligned}$$

which is clear since $[\mathrm{X}, \mathrm{Y}] = 0$ and $\mathrm{X}|_{\mathrm{M}^h} = 0$.

Part (2) is clear, since the operators d and $e^{-t\mathrm{X}}$, $\mathrm{X} \in \mathfrak{g}^g$, used to define α_g commute with g. Part (3) reflects the fact that the differential form 1 is closed. □

We will now study the compatibility of the maps α_g with the differential b and B and the products on $\mathcal{C}^{\mathrm{G}}_\bullet(\mathrm{C}^\infty(\mathrm{M}))$ and $\Omega^\bullet(\mathrm{M}, \mathrm{G})_g$.

THEOREM 3.2. – *Consider the map α_g: $\mathcal{C}^{\mathrm{G}}_\bullet(\mathrm{C}^\infty(\mathrm{M})) \to \Omega^\bullet_{\mathrm{G}^g}(\mathrm{M}^g)$.*

(1) *The Hochschild boundary on $\mathcal{C}^{\mathrm{G}}_\bullet(\mathrm{C}^\infty(\mathrm{M}))$ is carried into the differential ι on $\Omega^\bullet(\mathrm{M}, \mathrm{G})_g$:*

$$\alpha_g(bc) = \iota\alpha_g(c).$$

(2) *The shuffle product on $\mathcal{C}^{\mathrm{G}}_\bullet(\mathrm{C}^\infty(\mathrm{M}))$ is carried into the wedge product on $\Omega^\bullet(\mathrm{M}, \mathrm{G})_g$:*

$$\alpha_g(c_1 * c_2) = \alpha_g(c_1) \wedge \alpha_g(c_2).$$

(3) *The B_n-operators and transformed as follows:*

$$\alpha_g(\mathrm{B}_n(c_1, \ldots, c_n)) = \frac{1}{n!} d\alpha_g(c_1) \wedge \cdots \wedge d\alpha_g(c_n).$$

In particular, $\alpha_g(\mathrm{B}c) = d\alpha_g(c)$.

Proof. – (1) On a chain of the form $c(x_0, \ldots, x_k|g) = f_0(x_0) \cdots f_k(x_k)\psi(g)$ where $f_i \in C^\infty(M)$ and $\psi \in C^\infty(G)$, we have

$$\begin{aligned}\iota(X)\alpha_g(c)(X) &= \psi(ge^X)\sum_{i=1}^{k}(-1)^{i-1}\\ &\quad\times\int_{\Delta_k} f_0\, d(e^{-t_1 X}.f_1)\cdots \mathcal{L}(X)(e^{-t_i X}.f_i)\cdots d(e^{-t_k X}.f_k)|_{M^g}\\ &= \psi(ge^X)\sum_{i=1}^{k}(-1)^i\int_{\Delta_k} f_0\, d(e^{-t_1 X}.f_1)\cdots\frac{\partial(e^{-t_i X}.f_i)}{\partial t_i}\cdots d(e^{-t_k X}.f_k)|_{M^g}.\end{aligned}$$

Using integration by parts, we see that

$$\begin{aligned}&\int_{\Delta_k} f_0\, d(e^{-t_1 X}.f_1)\cdots\frac{\partial(e^{-t_i X}.f_i)}{\partial t_i}\cdots d(e^{-t_k X}.f_k)|_{M^g}\\ &= \int_{\{t_1\le\cdots\le t_i=t_{i+1}\le\cdots\le t_k\}} f_0\, d(e^{-t_1 X}.f_1)\cdots e^{-t_{i+1}X}.(f_i\, df_{i+1})\cdots d(e^{-t_k X}.f_k)|_{M^g}\\ &\quad-\int_{\{t_1\le\cdots\le t_{i-1}=t_i\le\cdots\le t_k\}} f_0\, d(e^{-t_1 X}.f_1)\cdots e^{-t_{i+1}X}.(df_{i-1}\, f_i)\cdots d(e^{-t_k X}.f_k)|_{M^g}\end{aligned}$$

Adding up all of these terms, we see that

$$\begin{aligned}\iota(X)\alpha_g(c)(X) &= \psi(ge^X)\int_{\{t_2\le\cdots\le t_k\}} f_0 f_1\, d(e^{-t_2 X}.f_2)\cdots d(e^{-t_k X}.f_k)|_{M^g}\\ &\quad+\psi(ge^X)\sum_{i=1}^{k-1}(-1)^i\int_{\{t_1\le\cdots\le t_{i-1}\le t_{i+1}\le\cdots\le t_k\}} f_0\, d(e^{-t_1 X}.f_1)\, d(e^{-t_k X}.f_k)|_{M^g}\\ &\quad+(-1)^k\psi(ge^X)\int_{\{t_1\le\cdots\le t_{k-1}\}} f_0\, d(e^{-t_1 X}.f_1)\, d(e^{-t_{k-1}X}.f_{k-1})(e^{-X}.f_k)|_{M^g}\\ &= \alpha_g(bc)(X).\end{aligned}$$

(2) Suppose $c_1 = f_0 \otimes \cdots \otimes f_k\psi$ and $c_2 = h_0 \otimes \cdots \otimes h_l \otimes \varphi$. The formula for $\alpha_g(c_1) \wedge \alpha_g(c_2)$ is

$$(\psi\varphi)(ge^X)\int_{\Delta_k\times\Delta_l} f_0\, d(e^{-t_1 X}.f_1)\cdots d(e^{-t_k X}.f_k)\, h_0\, d(e^{-s_1 X}.h_1)\cdots d(e^{-s_l X}.h_l)|_{M^g}$$

Now let χ be a shuffle of the ordered sets $(t_1, \ldots, t_k) \in \Delta_k$, $(s_1, \ldots, s_l) \in \Delta_l$, and let $\Delta(\chi)$ be the subset of $\Delta_k \times \Delta_l$ consisting of those points $(t_1, \ldots t_k, s_1, \ldots, s_l)$ such that the $(k+l)$-tuple $\chi(t_1, \ldots, t_k, s_1, \ldots, s_l)$ is monotonically increasing. It is clear

that each set $\Delta(\chi)$ is a $(k+l)$-simplex and that $\Delta_k \times \Delta_l$ is the union of the $\Delta(\chi)$; this is the shuffle product triangulation of $\Delta_k \times \Delta_l$. It is straightforward to check that

$$\int_{\Delta(\chi)} f_0\, d\,(e^{-t_1 \mathrm{X}}.\, f_1) \cdots d\,(e^{-t_k \mathrm{X}}.f_k)\, h_0\, d\,(e^{-s_1 \mathrm{X}}.\, h_1) \cdots d\,(e^{-s_1 \mathrm{X}}.\, h_l)|_{\mathrm{M}^g}$$
$$= \alpha_g\,(f_0\, h_0 \otimes \mathrm{S}_\chi\,(f_1 \otimes \cdots \otimes f_k \otimes h_1 \otimes \cdots \otimes h_l) \otimes \psi\varphi),$$

and the formula $\alpha_g\,(c_1 * c_2) = \alpha_g\,(c_1) \wedge \alpha_g\,(c_2)$ follows easily.

(3) We will prove the formula for $\alpha_g \mathrm{B}_n$ first for g equal to the identity, since the general case is obtained by replacing M and G by M^g and G^g. Take n chains

$$c_k = f_{(k,\,0)} \otimes \cdots \otimes f_{(k,\,i_k)} \otimes \psi_k, \qquad 1 \leq k \leq n,$$

and form the product

$$\alpha = \int_{\Delta_n} d\,(e^{-s_1 \mathrm{X}}.\, \alpha_1) \wedge \cdots \wedge d\,(e^{-s_n \mathrm{X}}.\, \alpha_n)\, ds_1 \cdots ds_n,$$

where α_k is the element of $\mathrm{C}_0^\infty\,(\mathfrak{g},\, \Omega^{i_k}\,(\mathrm{M}))$ given by the formula

$$\alpha_k\,(\mathrm{X}) = \psi_k\,(\exp \mathrm{X}) \int_{\Delta_{i_k}} f_{(k,\,0)}\, d\,(e^{-t_1 \mathrm{X}}.\, f_{(k,\,1)}) \wedge \cdots \wedge d\,(e^{-t_k \mathrm{X}}.\, f_{(k,\,i_k)})\, dt_1 \cdots dt_{i_k}.$$

Thus, α is given by an integral over product of simplices

$$\Delta_n \times \Delta_{i_1} \times \cdots \times \Delta_{i_n}.$$

If $(t_1, \ldots,\, t_n;\, s_{(1,\,0)}, \ldots,\, s_{(1,\,i_1)}; \ldots;\, s_{(n,\,0)}, \ldots,\, s_{(n,\,i_n)})$ lies in this product, form the $(n + i_1 + \cdots + i_n)$-tuple of numbers

$$(t_1,\, t_1 + \ldots,\, s_{(1,\,i_1)}, \ldots,\, t_1 + s_{(1,\,i_1)}, \ldots,\, t_n,\, t_n + s_{(n,\,0)}, \ldots,\, t_n + s_{(n,\,i_n)}),$$

where each of the real numbers in this expression is taken modulo 1. The permutation needed to reorder these numbers in [0, 1] into increasing order is a cyclic shuffle. Furthermore if we define $\Delta(\chi)$ to be the subset of the product of simplices such that the cyclic shuffle χ puts the above set of points in increasing order then the $\Delta(\chi)$ is a simplex and the collection of simplices $\Delta(\chi)$ gives a triangulation of the product of simplices.

Thus $\Delta_n \times \Delta_{i_1} \times \cdots \times \Delta_{i_n}$ may be partitioned into a disjoint union of simplices of dimension $n + i_1 + \cdots\, i_n$, each one labeled by a cyclic shuffle. A short calculation shows that the integral over the simplex labeled by the cyclic shuffle χ is equal to the image under α_g of the corresponding term in the sum defining $\mathrm{B}_n\,(c_1, \ldots,\, c_n)$. Thus, we see that

$$\alpha = \alpha_g\, \mathrm{B}_n\,(c_1, \cdots,\, c_n).$$

If we now replace the chains c_k by chains invariant under G, then the differential forms α_k are invariant, and hence

$$\alpha = \frac{1}{n!}\, d\,\alpha_1 \wedge \cdots \wedge d\,\alpha_n.$$

This completes the proof of the formula for $\alpha_g\, \mathrm{B}_n$. $\square$

The above proposition allows us to introduce an A_∞-structure on $\Omega^\bullet(M, G)[\![u]\!]$ by the following formulas:

$$m_n(\omega_1, \cdots, \omega_2) = \begin{cases} d_g\,\omega_1 = \iota\omega_1 + u d\omega_1, & n = 1, \\ \omega_1 \circ \omega_2 = \omega_1 \wedge \omega_2 + (-1)^{|\omega_1|}\dfrac{u}{2}\, d\omega_1 \wedge d\omega_2, & n = 2, \\ (-1)^{(n-1)(|\omega_1|-1)+\cdots+(|\omega_{n-1}|-1)}\dfrac{u}{n!}\, d\omega_1 \wedge \cdots \wedge d\omega_2, & n > 2. \end{cases}$$

Strangely enough, the product $\omega_1 \circ \omega_2$ introduced above is associative, and so by throwing away the higher m_n-operators, we obtain a differential graded algebra deforming the usual one. This product was used by Fedosov in his proof of the Atiyah-Singer index theorem for Euclidean space [9], although he obtained it without having any knowledge of cyclic homology theory. Note that the correction term to Fedesov's product is exact, and hence the product induced on the cohomology of the complex

$$(\Omega^\bullet_G(M)[\![u]\!],\ \iota + ud)$$

by the product $\omega_1 \circ \omega_2$ is the same as the product induced by the exterior product $\omega_1 \wedge \omega_2$.

We can now state the main result of this paper.

THEOREM 3.3. – *Let* M *be a compact* G*-manifold, where* G *is a compact Lie group. Let* W *be a module over* $\mathbb{C}[\![u]\!]$ *of finite projective dimension. Then the equivariant Hochschild-Kostant-Rosenberg map*

$$\alpha: \quad (\mathcal{C}^G_\bullet(C^\infty(M), G)[\![u]\!] \otimes_{\mathbb{C}[\![u]\!]} W,\ b + uB) \to (\Omega^\bullet(M, G)[\![u]\!] \otimes_{\mathbb{C}[\![u]\!]} W,\ \iota + ud)$$

is a quasi-isomorphism of complexes of sheaves.

For us, the most important application of the above theorem is where W is the module $\mathbb{C}$ with u acting by the identity; the theorem implies that the Hochschild-Kostant-Rosenberg map induces as isomorphism between $HP^G_\bullet(C^\infty(M))$ and the cohomology of the complex of global equivariant differential $\mathcal{A}^\bullet_G(M)$ with boundary $d + \iota$. By the identification of $K^\bullet_G(M) \otimes_{R(G)} R^\infty(G)$ with $HP^G_\bullet(C^\infty(M))$, this give us a de Rham model for equivariant K-theory of M.

The proof of Theorem 3.3 will be obtained by a sequence of reductions, each of which is straightforward. The first step is an application of the following lemma (*see* [10]). Recall that a *mixed complex* is a graded vector space $C_\bullet$ and two operators $b: C_\bullet \to C_{\bullet-1}$ and $B: C_\bullet \to C_{\bullet+1}$, such that $b^2 = 0$, $B^2 = 0$, and $bB + Bb = 0$.

LEMMA 3.4. – *Let* $f: (C_1, b_1, B_1) \to (C_2, b_2, B_2)$ *be a map of mixed complexes such that* f *induces as isomorphism* $H(C_1, b_1) \to H(C_2, b_2)$. *Then for any coefficients* W *of finite projective dimension over* $\mathbb{C}[\![u]\!]$,

$$f: \quad H_\bullet(C_1[\![u]\!] \otimes_{\mathbb{C}[u]} W,\ b_1 + uB_1) \to H_\bullet(C_2[\![u]\!] \otimes_{\mathbb{C}[u]} W,\ b_2 + uB_2)$$

is an isomorphism.

By this lemma, we see that it suffices to prove Theorem 3.3 in the case in which W equals the module $\mathbb{C}$ with u acting by zero; we must show that the equivariant Hochschild-Kostant-Rosenberg map

$$\alpha : \quad (\mathcal{C}_\bullet (\mathrm{C}^\infty (\mathrm{M}), \mathrm{G}), b) \to (\Omega^\bullet (\mathrm{M}, \mathrm{G}), \iota)$$

is a quasi-isomorphism of complexes of sheaves under the hypotheses of Theorem 3.3.

We can now explain the Mayer-Vietoris short exact sequences for equivariant cyclic homology and for equivariant differential forms. Let U_1 and U_2 be two invariant open subsets of M, and choose an invariant partition of unity $\{\varphi_1, \varphi_2\}$ for the covering $\{\mathrm{U}_1, \mathrm{U}_2\}$ of $\mathrm{U}_1 \cup \mathrm{U}_2$. That is, $\varphi_i \in \mathrm{C}^\infty (\mathrm{U}_1 \cup \mathrm{U}_2)^{\mathrm{G}}$ are such that $\varphi_1 + \varphi_2 = 1$ and $\mathrm{supp}(\varphi_i) \subset \mathrm{U}_i$. Using this partition of unity, we may show that the following sequence of $\mathrm{C}^\infty (\mathrm{U}_1 \cup \mathrm{U}_2)$-bimodules is split exact,

$$0 \to \mathrm{C}^\infty((\mathrm{U}_1 \cup \mathrm{U}_2) \times \mathrm{G}) \to \mathrm{C}^\infty (\mathrm{U}_1 \times \mathrm{G}) \oplus \mathrm{C}^\infty (\mathrm{U}_2 \times \mathrm{G}) \to \mathrm{C}^\infty ((\mathrm{U}_1 \cap \mathrm{U}_2) \times \mathrm{G}) \to 0,$$

where the first map sends f to $f|_{\mathrm{U}_1} \oplus f|_{\mathrm{U}_2}$, and the second map sends $f_1 \oplus f_2$ to $f_1|_{\mathrm{U}_1 \cap \mathrm{U}_2} - f_2|_{\mathrm{U}_1 \cap \mathrm{U}_2}$. Indeed, the splitting sends $f \in \mathrm{C}^\infty ((\mathrm{U}_1 \cap \mathrm{U}_2) \times \mathrm{G})$ to

$$(\varphi_1 f, -\varphi_2 f) \in \mathrm{C}^\infty (\mathrm{U}_1 \times \mathrm{G}) \oplus \mathrm{C}^\infty (\mathrm{U}_2 \times \mathrm{G}).$$

It is important that this splitting is G-equivariant.

From this, we obtain a commutative diagram of short exact sequences of complexes,

$$\begin{array}{ccc}
0 & & 0 \\
\downarrow & & \downarrow \\
\mathcal{C}_\bullet (\mathrm{C}^\infty (\mathrm{M}), \mathrm{C}^\infty (\mathrm{U}_1 \cup \mathrm{U}_2))^{\mathrm{G}} & \xrightarrow{\beta \circ \alpha} & \Omega^\bullet_{\mathrm{G}} (\mathrm{U}_1 \cup \mathrm{U}_2) \\
\downarrow & & \downarrow \\
\mathcal{C}_\bullet (\mathrm{C}^\infty (\mathrm{M}), \mathrm{C}^\infty (\mathrm{U}_1 \times \mathrm{G}) \oplus \mathrm{C}^\infty (\mathrm{U}_2 \times \mathrm{G}))^{\mathrm{G}} & \xrightarrow{\beta \circ \alpha} & \Omega^\bullet_{\mathrm{G}} (\mathrm{U}_1) \oplus \Omega^\bullet_{\mathrm{G}} (\mathrm{U}_2) \\
\downarrow & & \downarrow \\
\mathcal{C}_\bullet (\mathrm{C}^\infty (\mathrm{M}), \mathrm{C}^\infty (\mathrm{U}_1 \cap \mathrm{U}_2 \times \mathrm{G}))^{\mathrm{G}} & \xrightarrow{\beta \circ \alpha} & \Omega^\bullet_{\mathrm{G}} (\mathrm{U}_1 \cap \mathrm{U}_2) \\
\downarrow & & \downarrow \\
0 & & 0
\end{array}$$

Applying Proposition 2.3 to each complex in the left-hand column, we see that in the diagram of complexes of sheaves

$$\begin{array}{ccc}
0 & & 0 \\
\downarrow & & \downarrow \\
\mathcal{C}_\bullet (\mathrm{C}^\infty (\mathrm{U}_1 \cup \mathrm{U}_2), \mathrm{G}) & \xrightarrow{\alpha} & \Omega^\bullet (\mathrm{U}_1 \cup \mathrm{U}_2, \mathrm{G}) \\
\downarrow & & \downarrow \\
\mathcal{C}_\bullet (\mathrm{C}^\infty (\mathrm{U}_1), \mathrm{G}) \oplus \mathcal{C}_\bullet(\mathrm{C}^\infty (\mathrm{U}_2), \mathrm{G}) & \xrightarrow{\alpha} & \Omega^\bullet (\mathrm{U}_1, \mathrm{G}) \oplus \Omega^\bullet (\mathrm{U}_2, \mathrm{G}) \\
\downarrow & & \downarrow \\
\mathcal{C}_\bullet (\mathrm{C}^\infty (\mathrm{U}_1 \cap \mathrm{U}_2), \mathrm{G}) & \xrightarrow{\alpha} & \Omega^\bullet (\mathrm{U}_1 \cap \mathrm{U}_2, \mathrm{G}) \\
\downarrow & & \downarrow \\
0 & & 0
\end{array}$$

the stalks of the left column are quasi-isomorphic to an exact sequence of complexes. Replacing M by M^g, U_i by U_i^g, and G by G^g, we obtain the same result at all of the

stalks. In this way, we see that if Theorem 3.3 is shown to hold for M equal to the two invariant open sets U_1 and U_2, and to their intersection $U_1 \cap U_2$, then it holds for their union $U_1 \cup U_2$.

Suppose H acts freely on a $G \times H$-manifold M. We may form a commutative diagram

$$\begin{array}{ccc} \mathcal{C}_\bullet(C^\infty(M/H), G) & \xrightarrow{\alpha} & \Omega^\bullet(M/H, G) \\ \downarrow & & \downarrow \\ \mathcal{C}_\bullet(C^\infty(M), G \times H) & \xrightarrow{\alpha} & \Omega^\bullet(M, G \times H) \end{array}$$

where the vertical arrows are induced by the quotient map $M \to M/H$, while the horizontal arrows are the Hochschild-Kostant-Rosenberg maps. The next two lemmas show that the vertical arrows are quasi-isomorphisms.

LEMMA 3.6. – *If* H *acts freely on a* $G \times H$*-manifold* M, *the map*

$$\mathcal{C}_\bullet(C^\infty(M/H), G) \to \mathcal{C}_\bullet(C^\infty(M), G \times H)$$

is a quasi-isomorphism.

Proof. – Suppose that A and B are topological algebras and P and Q are flat modules for, respectively, $A \otimes B^o$ and $B \otimes A^o$, such that

$$P \otimes_B Q \cong A \qquad \text{and} \qquad Q \otimes_A P \cong B.$$

Under the extra assumption of H-unitality, which is explained in the Appendix, it follows that if L is an A-bimodule,

$$H_\bullet(A, L) \cong H_\bullet(B, Q \otimes_A L \otimes_B P);$$

this is called Morita invariance of Hochschild homology. The proposition is an application of this result with $A = C^\infty(M/H)$, $B = C^\infty(M) \rtimes H$, $P = C^\infty(M)$, $Q = C^\infty(M)$, $L = C^\infty((M/H) \times G)$, and $Q \otimes_A L \otimes_B P = C^\infty(M \times G)$. Then Theorem A.3 shows that

$$H_\bullet(\mathcal{C}_\bullet(C^\infty(M/H), G)) \cong H_\bullet(\mathcal{C}_\bullet(C^\infty(M) \rtimes H, G)),$$

and this last homology is isomorphic to $H_\bullet(\mathcal{C}_\bullet(C^\infty(M), G \times H))$ by Theorem 2.1. □

We have a similar result on the right-hand side.

LEMMA 3.7. – *Suppose* H *acts freely on a* $G \times H$*-manifold* M. *Then the map from* $\Omega^\bullet(M/H, G)$ *to* $\Omega^\bullet(M, G \times H)$ *induced by the quotient map* $M \to M/H$ *is a quasi-isomorphism.*

Proof. – Since H acts freely on M, we see that the stalk $\Omega^\bullet(M, G \times H)_{(g, h)}$ equals zero unless $h \in H$ is the identity $e \in H$. The stalk at $(g, e) \in G \times H$ equals

$$\Omega^\bullet(M, G \times H)_{(g, e)} = C_0^\infty(\mathfrak{g}, \Omega_H^\bullet(M))^{G^g},$$

with boundary $\iota_G + \iota_H$, in an evident notation. The complex $(\Omega_H^\bullet(M), \iota_H)$ is quasi-isomorphic to $\Omega^\bullet(M/H)$, and the result follows by a spectral sequence argument. □

If H is a closed subgroup of G and M is a manifold on which H acts smoothly, we may think of $G \times M$ as a $G \times H$-manifold, with the action of $(g, h) \in G \times H$ on $(\gamma, x) \in G \times M$ given by the formula

$$(g, h).(\gamma, x) = (g\gamma h^{-1}, h.x).$$

The actions of G and H on $G \times M$ are free, and the quotient of the action by H is $G \times_H M$. To prove the quasi-isomorphism of complexes

$$\alpha : \quad \mathcal{C}_\bullet (C^\infty (G \times_H M), G) \to \Omega^\bullet (G \times_H M, G)$$

it suffices to prove the quasi-isomorphism

$$\alpha : \quad \mathcal{C}_\bullet (C^\infty (M), H) \to \Omega^\bullet (M, H);$$

this reduction follows from Lemmas 3.6 ande 3.7 which show that the vertical arrows in the following commutative diagram are quasi-isomorphisms:

$$\begin{array}{ccc} \mathcal{C}_\bullet (C^\infty (G \times_H M), G) & \xrightarrow{\alpha} & \Omega^\bullet (G \times_H M, G) \\ \downarrow & & \downarrow \\ \mathcal{C}_\bullet (C^\infty (G \times M), G \times H) & \xrightarrow{\alpha} & \Omega^\bullet (G \times M, G \times H) \\ \uparrow & & \uparrow \\ \mathcal{C}_\bullet (C^\infty (M), H) & \xrightarrow{\alpha} & \Omega^\bullet (M, H) \end{array}$$

We now apply the following result, which shows that any equivariant manifold is built up by a sequence of equivariant surgeries.

LEMMA 3.8. – *If* M *is a compact manifold with a smooth action of a Lie group* G, *there is a covering* $\{Z_i | 0 \leq i \leq k\}$ *of* M *by invariant open sets such that*

(1) $Z_i \subset Z_{i+1}$, $Z_0 = \emptyset$ *and* $Z_k = M$;

(2) *for each i, there is an open cover of* Z_{i+1} *by* Z_i *and* Y_{i+1} *where* Y_{i+1} *is equivariantly diffeomorphic to a set of the form* $G \times_H (B(V) \times B(W))$ *where* H *is a closed subgroup of* G, *and* B (V) *and* B (W) *are the open unit balls in unitary representations* V *and* W *of* H;

(3) $Z_i \cap Y_{i+1} \subset Y_{i+1}$ *is equivariantly diffeomorphic to*

$$G \times_H (B(V) \times S(W)) \subset G \times_H (B(V) \times B(W)),$$

where $S(W) = \{x \in B(W) \mid 1/2 < |x| < 1\}$.

(The set Y_i *is called an equivariant handle-body, and the set* Z_{i+1} *is the result of performing an equivariant surgery on* Z_i.*)*

Proof. – This is simple consequence of equivariant Morse Theory, as developed by Wasserman [17]. Every manifold M with a smooth G-action, where G is a compact Lie group, has an invariant Morse function f, that is, an element of $C^\infty (M)^G$ such that each critical set of f is a single orbit N_c labelled by $c \in \mathbb{R}$ such that $f(N_c) = c$, and the function f is non-degenerate in directions normal of N_c. Order the critical values of f, $c_1 < \cdots < c_k$, and write N_i instead of N_{c_i}.

Wasserman proves that if $N_i \cong G/H$ is a critical orbit, then there are two equivariant vector bundles $\mathcal{V} \cong G \times_H V$ and $\mathcal{W} \cong G \times_H W$ over N_i (the stable and unstable parts of the normal bundle to N_i) and an equivariant neighbourhood Y_i of N_i equivariantly diffeomorphic to the equivariant handle-body

$$B(\mathcal{V}) \times_{N_i} B(\mathcal{W}) \cong G \times_H (B(V) \times B(W)).$$

Write $Z_i = Y_1 \cup \cdots \cup Y_i$. Then the sets Y_i may be chosen in such a way that $Z_k = M$, and that $Z_i \cap Y_i$ is equivariant diffeomorphic to $B(\mathcal{V}) \times_{N_i} S(\mathcal{W}) \cong G \times_H (B(V) \times_{N_i} S(W))$. $\square$

In this way, we have reduced the proof of Theorem 3.3 to the following lemma.

LEMMA 3.9. – *Suppose that the equivariant Hochschild-Kostant-Rosenberg map*

$$(\mathrm{C}^\infty(\mathrm{U}^{\bullet+1} \times g \exp \mathcal{O}),\, b) \xrightarrow{\alpha} (\mathrm{C}^\infty(\mathcal{O},\, \Omega^\bullet(\mathrm{U}^g)),\, \iota)$$

is a quasi-isomorphism of complexes for all convex invariant neighbourhoods U *of zero in a representation of* G, *for all* $g \in \mathrm{G}$, *and for all small balls* $\mathcal{O}$ *around zero in the Lie algebra* $\mathfrak{g}^g$. *Then Theorem 3.3 follows.*

Proof. – Lemma 3.8, combined with an iterative application of the Meyer-Vietoris sequences for the two theories shows that it suffices to prove the quasi-isomorphism of sheaves for equivariant handle-bodies $\mathrm{G} \times_{\mathrm{H}} (\mathrm{B}(\mathrm{V}) \times \mathrm{B}(\mathrm{W}))$ and for their sub-spaces $\mathrm{G} \times_{\mathrm{H}} (\mathrm{B}(\mathrm{V}) \times \mathrm{S}(\mathrm{W}))$. By Lemmas 3.6 and 3.7, we see that is suffices to prove the quasi-isomorphism of sheaves for the H-manifolds $\mathrm{B}(\mathrm{V}) \times \mathrm{B}(\mathrm{W})$ and for their subspaces $\mathrm{B}(\mathrm{V}) \times \mathrm{S}(\mathrm{W})$.

By hypothesis, the quasi-isomorphism holds for $\mathrm{B}(\mathrm{V}) \times \mathrm{B}(\mathrm{W})$. To prove it for the H-manifold $\mathrm{B}(\mathrm{V}) \times \mathrm{S}(\mathrm{W})$, we apply Lemma 3.8 once more, obtaining an equivariant handlebody decomposition of $\mathrm{B}(\mathrm{V}) \times \mathrm{S}(\mathrm{W})$. It is easy to see that this inductive procedure must terminate, since the closed subgroups of G form a partially ordered set under inclusion satisfying the finite chain condition. □

Now, we prove that the hypotheses of Lemma 3.9 hold. Let U be a convex invariant neighbourhood of zero in the linear representation $(\mathrm{V},\, \rho)$ of G, and let $\mathcal{O}$ be a neighbourhood of zero in $\mathfrak{g}^g$. We may think of $\mathrm{C}^\infty(\mathrm{U} \times \mathcal{O})$ as a module over the algebra $\mathrm{C}^\infty(\mathrm{U}^2)$, with action

$$(f.m)(x,\, \mathrm{X}) = f(x,\, e^{\mathrm{X}} x)\, m(x,\, \mathrm{X}),$$

for $f \in (\mathrm{U}^2)$ and $m \in \mathrm{C}^\infty(\mathrm{U} \times \mathcal{O})$. Consider its bar resolution

$$\xrightarrow{\partial} \mathrm{C}^\infty(\mathrm{U}^4 \times \mathcal{O}) \xrightarrow{\partial} \mathrm{C}^\infty(\mathrm{U}^3 \times \mathcal{O}) \xrightarrow{\partial} \mathrm{C}^\infty(\mathrm{U}^2 \times \mathcal{O}) \xrightarrow{\Delta^\bullet} \mathrm{C}^\infty(\mathrm{U} \times \mathcal{O})$$

where $f \in \mathrm{C}^\infty(\mathrm{U}^2)$ acts on $m \in \mathrm{C}^\infty(\mathrm{U}^{k+2} \times \mathcal{O})$ by

$$(f\,.\,m)(x_0, \ldots,\, x_{k+1}|\mathrm{X}) = f(x_0,\, e^{\mathrm{X}} x_{k+1})\, m(x_0, \ldots,\, x_{k+1}|\mathrm{X});$$

although this is not quite the standard action of $\mathrm{C}^\infty(\mathrm{U}^2)$ on $\mathrm{C}^\infty(\mathrm{U}^{k+2} \times \mathcal{O})$, this is nevertheless a free resolution, by an evident isomorphism of the $\mathrm{C}^\infty(\mathrm{U}^2)$-module $\mathrm{C}^\infty(\mathrm{U}^{k+2} \times \mathcal{O})$ with

$$\mathrm{C}^\infty(\mathrm{U}^2) \otimes_{\mathbb{C}} \mathrm{C}^\infty(\mathrm{U}^k \times \mathcal{O}).$$

The boundary ∂ is given by the formula

$$(\partial c)(x_0, \ldots,\, x_{k+1}|\mathrm{X}) = \sum_{i=0}^{k} (-1)^i\, c(x_0, \cdots,\, x_i,\, x_i, \cdots,\, x_{k+1}|\mathrm{X}),$$

and the augmentation $\mathrm{C}^\infty(\mathrm{U}^2) \to \mathrm{C}^\infty(\mathrm{U})$ is pull-back by the diagonal map

$$\Delta: \quad \mathrm{U} \times \mathcal{O} \ni (x,\, \mathrm{X}) \to (x,\, x,\, \mathrm{X}) \in \mathrm{U}^2 \times \mathcal{O}.$$

The bar resolution is exact, with contracting homotopy

$$s: \quad C^\infty(U^{k+2} \times \mathcal{O}) \to C^\infty(U^{k+3} \times \mathcal{O})$$

given by pull-back by the map

$$(x_{-1}, \ldots, x_{k+1}, X) \to (x_0, \ldots, x_{k+1}, X).$$

It may be checked that the complex $(C^\infty(U^{\bullet+1} \times g \exp \mathcal{O}), b)$ of Lemma 3.9 is isomorphic to

$$(C^\infty(U^{\bullet+2} \times \mathcal{O}) \otimes_{C^\infty(U^2)} C^\infty_{(g)}(U), \partial \otimes \mathrm{id}).$$

Thus, we see that the complex on the left-hand side in Lemma 3.9 has cohomology

$$\mathrm{Tor}_\bullet^{C^\infty(U^2)}(C^\infty(U \times \mathcal{O}), C^\infty_{(g)}(U)),$$

where $C^\infty_{(g)}(U)$ is the module, isomorphic to $C^\infty(U)$ as a vector space, with action of $f \in C^\infty(U^2)$ given by the formula

$$(f.m)(x, g) = f(x, gx)\, m(x).$$

Since U is an open subset of the linear representation V of G, there is another resolution of the module $C^\infty(U \times \mathcal{O})$ over $C^\infty(U^2)$, called the Koszul resolution:

$$\to C^\infty(U^2 \times \mathcal{O}, \Lambda^2 V^*) \xrightarrow{\iota(R+X)} C^\infty(U^2 \times \mathcal{O}, V^*) \xrightarrow{\iota(R+X)} C^\infty(U^2 \times \mathcal{O}) \xrightarrow{\Gamma^*} C^\infty(U \times \mathcal{O}).$$

Here, $f \in C^\infty(U^2)$ acts on $\omega \in C^\infty(U^2 \times \mathcal{O}, \Lambda^\bullet V^*)$ by the formula

$$(f.\omega)(x, y, X) = f(x, y)\, \omega(x, y, X),$$

and Γ^* is pull-back by the map

$$\Gamma: \quad U \times \mathcal{O} \ni (x, X) \mapsto (x, e^X x, X) \in U^2 \times \mathcal{O}.$$

It is clear that the spaces $C^\infty(U^2 \times \mathcal{O}, \Lambda^k V^*)$ are free $C^\infty(U^2)$-modules.

The boundary in the Koszul complex is $\iota(R + L)$, where R and L are the elements of $C^\infty(U^2 \times \mathcal{O}, V)$ given at a point $(x, y, X) \in U^2 \times \mathcal{O}$ by the formulas $R(x, y, L) = x - y$ and $L(x, y, X) = d\rho(X)\, x$. To show that the Koszul complex is exact, we introduce a contracting homotopy

$$s: \quad C^\infty(U^2 \times \mathcal{O}, \Lambda^\bullet V^*) \to C^\infty(U^2 \times \mathcal{O}, \Lambda^{\bullet+1} V^*)$$

which is related to the homotopy used to prove the Poincaré lemma. We will abbreviate $d\rho(X)\, x$ to $X\, x$ in this discussion, and similarly write gx for $\rho(g)\, x$.

Consider the flow $\Phi_{(t, x, X)}$ on U given by the formula

$$\Phi_{(t, x, X)}(y) = e^{tX}.(tx + (1 - t)\, y);$$

this is the integral of the vector field $\mathrm{R}+\mathrm{L}$, since the vector fields R and L have vanishing Lie bracket. In defining the contracting homotopy s, we identify the space $\mathrm{C}^\infty(\mathrm{U}^2 \times \mathcal{O}, \Lambda^\bullet \mathrm{V}^*)$ with $\mathrm{C}^\infty(\mathrm{U} \times \mathcal{O}, \Omega^\bullet(\mathrm{U}))$ by composing the two identifications

$$\mathrm{C}^\infty(\mathrm{U}^2 \times \mathcal{O}, \Lambda^\bullet \mathrm{V}^*) \cong \mathrm{C}^\infty(\mathrm{U} \times \mathcal{O}, \mathrm{C}^\infty(\mathrm{U}, \Lambda^\bullet \mathrm{V}^*)) \cong \mathrm{C}^\infty(\mathrm{U} \times \mathcal{O}, \Omega^\bullet(\mathrm{U})),$$

where the first identification sends a function $f(x, y, \mathrm{X})$ to the map $(x, \mathrm{X}) \mapsto f(x, ., \mathrm{X})$. We now define s by the formula

$$(s\omega)(x, \mathrm{X}) = \int_0^1 \Phi^*_{(t, x, \mathrm{X})}(d\omega)(x, \mathrm{X}) \frac{dt}{t}.$$

As in the proof of the Poincaré lemma, we see that

$$[\iota(\mathrm{R}+\mathrm{L}), s]\,\omega(x, \mathrm{X}) = \Phi^*_{(1, x, \mathrm{X})}\,\omega(x, \mathrm{X}) - \Phi^*_{(0, x, \mathrm{X})}\,\omega(x, \mathrm{X}).$$

Since $\Phi_{(0, x, \mathrm{X})}$ is the identity, while $\Phi_{(1, x, \mathrm{X})}$ is the map

$$y \mapsto e^{\mathrm{X}} x,$$

we see that the Koszul complex is indeed a resolution of $\mathrm{C}^\infty(\mathrm{U} \times \mathcal{O})$.

Our proof of the hypothesis of Lemma 3.9 will come from considering the map between our two resolutions of the bimodule $\mathrm{C}^\infty(\mathrm{U} \times \mathcal{O})$:

$$\begin{array}{ccccccc}
\xrightarrow{\partial} & \mathrm{C}^\infty(\mathrm{U}^3 \times \mathcal{O})^{\cdot} & \xrightarrow{\partial} & \mathrm{C}^\infty(\mathrm{U}^2 \times \mathcal{O}) & \xrightarrow{\Delta^*} & \mathrm{C}^\infty(\mathrm{U} \times \mathcal{O}) \\
 & \alpha \downarrow & & \alpha \downarrow & & \| \\
\xrightarrow{\iota(\mathrm{R}+\mathrm{L})} & \mathrm{C}^\infty(\mathrm{U}^2 \times \mathcal{O}, \mathrm{V}^*) & \xrightarrow{\iota(\mathrm{R}+\mathrm{L})} & \mathrm{C}^\infty(\mathrm{U}^2 \times \mathcal{O}) & \xrightarrow{\Gamma^*} & \mathrm{C}^\infty(\mathrm{U} \times \mathcal{O})
\end{array}$$

The map of complexes α is defined by a formula analogous to the Hochschild-Kostant-Rosenberg map α for the equivariant cyclic bar complex: if

$$c = f_0 \otimes \cdots \otimes f_{k+1} \otimes \psi \in \mathrm{C}^\infty(\mathrm{U}^{k+2} \otimes \mathcal{O}),$$

then $\alpha c \in \mathrm{C}^\infty(\mathrm{U}^2 \times \mathcal{O}, \Lambda^k \mathrm{V}^*)$ is given by the formula

$$(\alpha c)(x, y, \mathrm{X}) = \psi(\mathrm{X}) \int_{\Delta_k} f_0(x)\, df_1(x_{t_1}) \wedge \cdots \wedge df_k(x_{t_k})\, f(y)\, dt_1 \cdots dt_k$$

where $x_t = e^{t(\mathrm{R}+\mathrm{L})} x = e^{t\mathrm{X}}((1-t)x + ty)$. It is easily checked, by a proof analogous to that of Theorem 3.2, that this is a map of complexes. Also, α is a map of $\mathrm{C}^\infty(\mathrm{U}^2)$-modules.

Thus, we may calculate $\mathrm{Tor}_\bullet^{\mathrm{C}^\infty(\mathrm{U}^2)}(\mathrm{C}^\infty(\mathrm{U} \times \mathcal{O}), \mathrm{C}^\infty_{(g)}(\mathrm{U}))$ equally well from the complex

$$(\mathrm{C}^\infty(\mathrm{U}^2 \times \mathcal{O}) \otimes \Lambda^\bullet \mathrm{V}^*) \otimes_{\mathrm{C}^\infty(\mathrm{U}^2)} \mathrm{C}^\infty_{(g)}(\mathrm{U}) \cong \mathrm{C}^\infty(\mathcal{O}, \Omega^k(\mathrm{U})).$$

It is easy to identity the boundary in this complex as $\iota(gx - x) + \iota$, where $gx - x \in \mathrm{C}^\infty(\mathrm{U}, \mathrm{V})$ is the vector field which at the point $x \in \mathrm{U}$ equals $gx - x \in \mathrm{V}$. Thus, to verify the hypothesis of Lemma 3.9, we must show that the restriction map from U to U^g induces a quasi-isomorphism

$$(\mathrm{C}^\infty(\mathcal{O}, \Omega^\bullet(\mathrm{U})), \iota(gx - x) + \iota) \to (\mathrm{C}^\infty(\mathcal{O}, \Omega^\bullet(\mathrm{U}^g)), \iota).$$

To do this, we choose an invariant metric on V, and decompose V into the orthogonal direct sum

$$V = V_0 \oplus V_1,$$

where $V_0 = \ker(g-1)$; clearly, $U^g = U \cap V_0$. Denote by P and Q the orthogonal projection from V to V_0 and V_1 respectively, and let $\varphi(v) = |Qv|^2$. Denote by $d^\perp = Qd$ the exterior differential operator acting along V_1. If $X \in \mathfrak{g}^g$ is in the centralizer of g, the corresponding vector field X on U satisfies $QX = XQ$; we will denote $QX = XQ$ by $X^\perp$.

Consider the flow $\Psi_X(t)$ on U obtained by integrating the vector field $gx - x + X^\perp$.

LEMMA 3.10. – *The flow* $\Psi_X(t)$ *commutes with* P *and* Q, *preserves* U, *and satisfies the inequality*

$$\frac{\partial}{\partial t}\Psi_X(t)^*\varphi = 2(Qv, (g-1)Qv) \leq -2\lambda\varphi,$$

where λ *is the distance from the spectrum of* g *to the line* $\operatorname{Re} z = 1$.

Proof. – The vector field $gx - x + X^\perp$ may be written $Q(gx - x + X)Q$, and hence the associated flow will commute with P and Q.

From the convexity and invariance of U, we see that the flow $\Psi_X(t)$ preserves U, since the vector field $gx - x$ points inwards, and $X^\perp \in \mathfrak{g}$.

The calculation of $\partial(\Psi_X(t)^*\varphi)/\partial t$ is elementary:

$$\begin{aligned}\frac{\partial}{\partial t}\Psi_X(t)^*\varphi &= 2(Qv, Q(g-1)v) + 2(Qv, QX^\perp v)\\ &= 2(Qv, (g-1)Qv) + 2(Qv, XQv),\end{aligned}$$

and $(Qv, XQv) = 0$ due to the invariance of the metric by the action of G. □

Define the map

$$p\omega = \int_1^\infty \Psi_X(t)^* d^\perp \omega\, dt;$$

the integral converges absolutely, since $\Psi_X(t)^* d^\perp \omega$ decays exponentially. From the lemma, we see that

$$[p, \iota(gx - x) + \iota]\,\omega = \omega - P^*\omega,$$

where $P^*\omega$ is the pull-back of ω by the projection $P : U_g \to U$. Thus, we see that restriction from the Koszul complex on U to the fixed point set U^g of g induces a quasi-isomorphism of complexes: this completes the proof of Theorem 3.3.

4. The equivariant Chern character

In this section, we will compare the equivariant Chern character of Berline and Vergne with the one which emerges from our theory: these two equivariant differential forms agree with each other (and with the Chern character of Chern-Weil theory) when there is no group action but are different in general. We will construct an explicit homotopy between the two differential forms.

An *equivariant vector bundle* over a G-manifold M is a vector bundle over M on which G acts by bundle maps such that the projection map $\pi : \mathrm{E} \to \mathrm{M}$ is equivariant. Recall that the equivariant K-theory of M, denoted $\mathrm{K_G}(\mathrm{M})$, is the Grothendieck group of the exact category of equivariant vector bundles over M. Any equivariant vector bundle may be realized as the image of an idempotent p in an algebra $\mathrm{C}^\infty(\mathrm{M}, \mathrm{End}(\mathrm{V}))^{\mathrm{G}}$, where (V, ρ) is a finite-dimensional representation of the group G. By this, we mean that there is an equivariant bundle $\mathrm{E}^\perp$ and an isomorphism of $\mathrm{E} \oplus \mathrm{E}^\perp$ with the trivial bundle $\mathrm{M} \times \mathrm{V}$, with G acting diagonally on M and V; p is then the projection onto E with kernel $\mathrm{E}^\perp$.

If (V, ρ) is a finite-dimensional representation of the group G, there is a map of cyclic chain complexes

$$\mathrm{Tr}: \quad \mathcal{C}_\bullet^{\mathrm{G}}(\mathrm{C}^\infty(\mathrm{M}) \otimes \mathrm{End}(\mathrm{V})) \to \mathcal{C}_\bullet^{\mathrm{G}}(\mathrm{C}^\infty(\mathrm{M})),$$

defined by the formula

$$\mathrm{Tr}(c)(x_0, \ldots, x_k|g) = \sum_{i_1 \cdots i_k} c(x_0, \ldots, x_k|g)_{i_0 i_1, i_1 i_2, \ldots, i_k i_0}.$$

Since this is a map of cyclic vector spaces, it intertwines the operators b and B. It follows that we obtain maps of cyclic homology theories

$$\mathrm{Tr}: \quad \mathrm{HC}_\bullet^{\mathrm{G}}(\mathrm{C}^\infty(\mathrm{M}) \otimes \mathrm{End}(\mathrm{V}); \mathrm{W}) \to \mathrm{HC}_\bullet^{\mathrm{G}}(\mathrm{C}^\infty(\mathrm{M}); \mathrm{W}),$$

for all coefficients W. By equivariant Morita invariance, this map is an isomorphism; however, we do not make use of this fact.

There is a map form $\mathrm{K_G}(\mathrm{M})$ to $\mathrm{HP}_0^{\mathrm{G}}(\mathrm{C}^\infty(\mathrm{M}))$, known as the equivariant Chern character. It is defined at the chain level by sending the equivariant vector bundle $\mathrm{im}(p)$ to the closed chain defined as follows.

DEFINITION 4.1. – *Let p be an idempotent in the algebra* $\mathrm{C}^\infty(\mathrm{M}, \mathrm{End}(\mathrm{V}))^{\mathrm{G}}$, *where* (V, p) *is a finite-dimensional representation of the group* G. *The* equivariant Chern character $\mathrm{ch}^{\mathrm{G}}(p) \in \mathcal{C}_\bullet^{\mathrm{G}}(\mathrm{C}^\infty(\mathrm{M}))[\![u]\!]$ *is defined by the formula*

$$\mathrm{ch}^{\mathrm{G}}(p)(g) = \mathrm{Tr}(\rho(g)p) + \sum_{l=1}^{\infty} (-u)^l \frac{(2l)!}{l!} \mathrm{Tr}\left(\rho(g)\left(p - \frac{1}{2}\right) \otimes p^{\otimes 2l}\right).$$

We will use the following properties of the equivariant Chern character $\mathrm{ch}^{\mathrm{G}}(p)$; these were proved by Brylinski [5], and may be proved easily using the methods of [11].

(1) It is closed, that is, $(b + u\mathrm{B})\,\mathrm{ch}^{\mathrm{G}}(p) = 0$.

(2) (homotopy invariance) If $p_\tau : [0, 1] \to \mathrm{C}^\infty(\mathrm{M}, \mathrm{End}(\mathrm{V}))^{\mathrm{G}}$ is a differentiable one-parameter family of idempotents, $\tilde{\mathrm{ch}}^{\mathrm{G}}(p, q)$ denote the odd cyclic chain $\iota(q(2p-1)).\mathrm{ch}^{\mathrm{G}}(p)$, where for $a \in \mathrm{C}^\infty(\mathrm{M}, \mathrm{End}(\mathrm{V}))$, we let

$$\iota(a): \quad \mathcal{C}_\bullet^{G}(\mathrm{C}^\infty(\mathrm{M}, \mathrm{End}(\mathrm{V}))) \to \mathcal{C}_{\bullet+1}^{\mathrm{G}}(\mathrm{C}^\infty(\mathrm{M}, \mathrm{End}(\mathrm{V})))$$

be the map defined by the formula

$$(\iota(a).f)(x_0, \ldots, x_k|g) = \sum_{i=0}^{k} (-1)^i a(x_{i+1}) f(x_0, \ldots, x_i, x_{i+2}, \ldots, x_k|g).$$

Then

$$\frac{d\,\mathrm{ch}^{\mathrm{G}}(p_\tau)}{d\tau} = (b + u\mathrm{B})\,\iota\,(\dot{p}_\tau\,(2p_\tau - 1))\,\mathrm{ch}^{\mathrm{G}}(p_\tau). \tag{4.2}$$

Combining these results, we see that there is a map

$$\mathrm{ch}^{\mathrm{G}}: \quad \mathrm{K}_{\mathrm{G}}(\mathrm{M}) \to \mathrm{HP}_0^{\mathrm{G}}(\mathrm{C}^\infty(\mathrm{M})). \tag{4.2}$$

Let G be a compact Lie group with Lie algebra $\mathfrak{g}$, and let M be a compact manifold with a smooth action of G. If E is an equivariant vector bundle over M, let $\mathrm{X} \mapsto \mathcal{L}^E(\mathrm{X})$ be the infinitesimal action of $X \in \mathfrak{g}$ on sections of E; it is a Lie algebra homomorphism from $\mathfrak{g}$ to the first-order differential operators on the bundle E.

The *moment* of an invariant connection ∇ on E is the differential operator, linearly dependent on $\mathrm{X} \in \mathfrak{g}$, defined by

$$\mu(\mathrm{X}) = \nabla_{\mathrm{X}} - \mathcal{L}^{\mathrm{E}}(\mathrm{X}).$$

It is easily seen that $[\mu(\mathrm{X}), f] = 0$ for all $f \in \mathrm{C}^\infty(\mathrm{M})$, and hence that

$$\mu(\mathrm{X}) \in \Gamma(\mathrm{M}, \mathrm{Hom}(\mathfrak{g}, \mathrm{End}(\mathrm{E})))^{\mathrm{G}}.$$

The space $\mathrm{C}^\infty(\mathfrak{g}, \Omega^\bullet(\mathrm{M}, \mathrm{E}))^{\mathrm{G}}[\![u]\!]$ is a module over the algebra $\mathrm{C}^\infty(\mathfrak{g}, \Omega^\bullet(\mathrm{M}))^{\mathrm{G}}[\![u]\!]$ of equivariant differential forms. The *equivariant connection* $\nabla_{\mathfrak{g}}$ associated to ∇ is the operator on $\mathrm{C}^\infty(\mathfrak{g}, \Omega^\bullet(\mathrm{M}, \mathrm{E}))^{\mathrm{G}}[\![u]\!]$ defined by the formula

$$(\nabla_{\mathfrak{g}}\,\omega)(\mathrm{X}) = \iota(\mathrm{X})(\omega(\mathrm{X})) + u\nabla(\omega(\mathrm{X})).$$

Letting $d_{\mathfrak{g}} = \iota + ud$, we see that

$$\nabla_{\mathfrak{g}}\,(\alpha \wedge \omega) = (d_{\mathfrak{g}}\alpha) \wedge \omega + (-1)^{|\alpha|}\,\alpha \wedge (\nabla_{\mathfrak{g}}\,\omega)$$

for $\alpha \in \mathrm{C}^\infty(\mathfrak{g}, \Omega^\bullet(\mathrm{M}))^{\mathrm{G}}[\![u]\!]$ and $\omega \in \mathrm{C}^\infty(\mathfrak{g}, \Omega^\bullet(\mathrm{M}, \mathrm{E}))^{\mathrm{G}}[\![u]\!]$.

If $\mathrm{F} \in \Omega^2(\mathrm{M}, \mathrm{End}(\mathrm{E}))$ is the curvature of the connection ∇, the *equivariant curvature* $\mathrm{F}_{\mathfrak{g}} \in C^\infty(\mathfrak{g}, \Omega^\bullet(\mathrm{M}, \mathrm{End}(\mathrm{E})))^{\mathrm{G}}[u]$ is defined by the formula

$$\mathrm{F}_{\mathfrak{g}}(\mathrm{X}) = u\mathrm{F} + \mu(\mathrm{X}).$$

This definition is motivated by the equation

$$\nabla_{\mathfrak{g}}^2\,\omega = u\varepsilon(\mathrm{F}_{\mathfrak{g}})\,\omega,$$

which holds for all $\omega \in \mathrm{C}^\infty(\mathfrak{g}, \Omega^\bullet(\mathrm{M}, \mathrm{E}))^{\mathrm{G}}$.

Berline and Vergne define the *equivariant Chern character* of E by the formula

$$\mathrm{ch}_{\mathfrak{g}}(\mathrm{E}) = \mathrm{Tr}(e^{-\mathrm{F}_{\mathfrak{g}}}) \in \mathrm{C}^\infty(\mathfrak{g}, \Omega^\bullet(\mathrm{M}))^{\mathrm{G}}[\![u]\!].$$

The following theorem is proved by the same method as the corresponding formulas in the Chern-Weil theory [2].

PROPOSITION 4.3. – (1) *The differential form* $\mathrm{ch}_{\mathfrak{g}}(\mathrm{E})$ *is equivariantly closed:*

$$d_{\mathfrak{g}}\,\mathrm{ch}_{\mathfrak{g}}(\mathrm{E}) = 0.$$

(2) *Let* $\nabla(t)$ *be a one-parameter family of invariant connections on the bundle* E, *with equivariant curvature* $F_{\mathfrak{g}}(t)$ *and equivariant Chern character* $\mathrm{ch}_{\mathfrak{g}}(E, t)$. *Then we have the transgression formula*

$$\frac{d}{dt}\,\mathrm{ch}_{\mathfrak{g}}(E, t) = -d_{\mathfrak{g}}\,\mathrm{Tr}\left(\frac{d\nabla(t)}{dt}\,e^{-F_{\mathfrak{g}}(t)}\right).$$

The above construction may be globalized, to define a section of the sheaf $\Omega^{\bullet}(M, G)$. If $g \in G$, we define $\mathrm{Ch}_G(E)_g \in C^{\infty}(\mathfrak{g}^g, \Omega^{\bullet}(M^g))^{G^g}[\![u]\!]$ by the formula

$$\mathrm{Ch}_G(E)_g(X) = \mathrm{Tr}\left(\rho(g)\,e^{-F_{\mathfrak{g}}(X)}\right)|_{M^g}.$$

It is easily seen that these sections piece together to give a global section of $\Omega^{\bullet}(M, G)[\![u]\!]$, which we will denote by $\mathrm{Ch}_G(E)$.

If we divide the equivariant curvature into the two pieces $\mu(X)$ and uF, we may reexpress the exponential in the definition of the $\mathrm{Ch}(E)_g$ by a perturbative expansion as

$$\mathrm{Ch}_G(E)_g(X) = \sum_{t=0}^{\infty}(-u)^l \int_{\Delta_l} \mathrm{Tr}\left(\rho(g)pe^{-t_1\mu(X)}Fe^{(t_2-t_1)\mu(X)}F\cdots Fe^{(1-t_l)\mu(X)}\right)\Big|_{M^g}.$$

We will show how our theory also leads to the definition of an equivariant Chern character, which differs from that of Berline and Vergne. Our definition proceeds via the equivariant cyclic chain complex. Suppose that E is given as the image of an idempotent p in the algebra $C^{\infty}(M, \mathrm{End}(V))^G$, where (V, ρ) finite-dimensional representation of the group G.

We now apply the equivariant Hochschil-Kostant-Rosenberg map to the equivariant cyclic chain $\mathrm{ch}^G(p)$ of Section 2, obtaining a closed section of $\Omega^{\bullet}(M, G)$, which we denote by $\mathrm{ch}_G(p)$.

PROPOSITION 4.2. – *Let p be an idempotent in the algebra* $C^{\infty}(M, \mathrm{End}(V))^G$, *where* (V, ρ) *is a finite-dimensional representation of* G. *Define* $\mathrm{ch}_G(p) \in \Gamma(G, \Omega^{\bullet}(M, G))$ *by applying the Hochschild-Kostant-Rosenberg map* α *to the cyclic homology chain* $\mathrm{ch}^G(p)$. *Then the germ of* $\mathrm{ch}^G(p)$ *at* $g \in G$ *is given, as a function of* $X \in \mathfrak{g}^g$, *by the formula*

$$\mathrm{Tr}(\rho(g)p)\Big|_{M^g} + \sum_{l=1}^{\infty}(-u)^l\frac{(2l)!}{l!}$$
$$\times\int_{\Delta_{2l}}\mathrm{Tr}\left(\rho(g)\left(p-\frac{1}{2}\right)e^{-t_1\rho(X)}\,dp\,e^{(t_2-t_1)\rho(X)}\,dp\cdots dp\,e^{(1-t_{2l})\rho(X)}\right)\Big|_{M^g}$$

Proof. – The Lie derivative of sections of the trivial bundle over M with fibre V, denoted $\mathcal{L}^V(X)$, is equal to $[d, \iota(X)] + \rho(X)$. Since p is invariant under the action of G, we see that

$$\mathcal{L}^V(X)\,p = [dp, \iota(X)] + [\rho(X), p] = 0. \tag{4.3}$$

Exponentiating this relation, we see that

$$e^{-\mathcal{L}^V(X)}.\,dp = e^{\rho(X)}\,dp\,e^{-\rho(X)}. \tag{4.4}$$

The image under the Hochschild-Kostant-Rosenberg map of $\mathrm{ch}^{\mathrm{G}}(p)$ is given by the formula

$$\mathrm{Tr}\,(\rho\,(g)\,p)\Big|_{\mathrm{M}^g} + \sum_{l=0}^{\infty} (-u)^l \frac{(2l)!}{l!}$$
$$\times \int_{\Delta_{2l}} \mathrm{Tr}\left(e^{\rho\,(\mathrm{X})}\,\rho\,(g)\left(p-\frac{1}{2}\right)(e^{-t_1\,\mathcal{L}\,(\mathrm{X})}.\,dp)\cdots(e^{-t_{2l}\,\mathcal{L}\,(\mathrm{X})}.dp)\right)\Big|_{\mathrm{M}^g}$$

If we insert (4.4) into this formula and use the fact that Tr is a trace to bring the operator $e^{\rho\,(\mathrm{X})}$ to the right end, we obtain the proposition. □

From (4.2), we deduce that there is a theory of secondary characteristic classes associated to the equivariant Chern class $\mathrm{ch}_{\mathrm{G}}(p)$; the variation of the equivariant Chern class as the idempotent $p \in \mathrm{C}^{\infty}(\mathrm{M}, \mathrm{End}\,(\mathrm{V}))^{\mathrm{G}}$ varies differentiably is an explicit equivariantly exact differential form. However, we will leave working out the details to the reader.

It is interesting to compare the two equivariant Chern characters thus defined:

(1) our equivariant Chern character $\mathrm{ch}_{\mathrm{G}}(p)$ is only defined when the bundle E is presented as the image of an idempotent $p \in \mathrm{C}^{\infty}(\mathrm{M}, \mathrm{End}\,(\mathrm{V}))^{\mathrm{G}}$;

(2) the $2l$-form component of the Berline-Vergne Chern character $\mathrm{Ch}_{\mathrm{G}}(\mathrm{E})$ may be expressed as an integral over an l-simple, while for $\mathrm{ch}_{\mathrm{G}}(p)$, it is an integral over a $2l$-simplex;

(3) the zero-form component of $\mathrm{Ch}_{\mathrm{G}}(\mathrm{E})$ may be rewritten using the formula $\mu\,(\mathrm{X}) = p\,.\,\rho\,(\mathrm{X})\,.\,p$ as

$$\mathrm{Tr}\,(e^{p.\rho\,(\mathrm{X}).p}),$$

while for $\mathrm{ch}_{\mathrm{G}}(p)$, the corresponding zero-form component is

$$\mathrm{Tr}\,(p.e^{\rho\,(\mathrm{X})}.p).$$

We will now derive an explicit formula showing the relationship between these two equivariant Chern classes.

If $p \in \mathrm{C}^{\infty}(\mathrm{M}, \mathrm{End}\,(\mathrm{V}))^{\mathrm{G}}$ is an idempotent over M as above, let $p^{\perp} = 1 - p$ be the complementary idempotent, and let ∇ be the Grassmannian connection on the trivial bundle over M with fibre V associated to the idempotent p, given by the formula

$$\nabla = p\,.\,d\,.\,p + p^{\perp}.\,d\,.\,p^{\perp} = d + (2p-1)\,dp.$$

LEMMA 4.5. – *This connection has curvature* $\mathrm{F} = (dp)^2$, and moment $\mu\,(\mathrm{X}) = -p\rho\,(\mathrm{X})\,p - p^{\perp}\,\rho\,(\mathrm{X})\,p^{\perp}$.

Proof. – The calculation of the curvature is standard, using the formulas $p\,(dp) = (dp)\,p^{\perp}$, $p^{\perp}\,(dp) = (dp)\,p$, and $(2p-1)^2 = 1$:

$$\begin{aligned}(d + (2p-1)\,dp)^2 &= d^2 + d\,(2p-1)\,dp + (2p-1)\,dp\,(2p-1)\,dp \\ &= 2\,(dp)^2 - (2p-1)^2\,(dp)^2 = (dp)^2.\end{aligned}$$

The calculation of the moment uses the fact that $\mathcal{L}^{\mathrm{V}}(\mathrm{X}) = [d, \iota(\mathrm{X})] + \rho(\mathrm{X})$. Since p is invariant under the action of G, we see that

$$\begin{aligned}\mu(\mathrm{X}) &= [\nabla, \iota(\mathrm{X})] - \mathcal{L}^{\mathrm{V}}(\mathrm{X}) \\ &= [d + (2p-1)\,dp, \iota(\mathrm{X})] - [d, \iota(\mathrm{X})] - \rho(\mathrm{X}) \\ &= (2p-1)\,[dp, \iota(\mathrm{X})] - \rho(\mathrm{X}).\end{aligned}$$

The formula for $\mu(\mathrm{X})$ now follows by inserting (4.3):

$$\begin{aligned}\mu(\mathrm{X}) &= -(2p-1)\,[\rho(\mathrm{X}), p] - \rho(\mathrm{X}) \\ &= -2p\rho(\mathrm{X})\,p + \rho(\mathrm{X})\,p + p\rho(\mathrm{X}) - \rho(\mathrm{X}) \\ &= -p\rho(\mathrm{X})\,p - p^{\perp}\,\rho(\mathrm{X})\,p^{\perp}. \quad \square\end{aligned}$$

Introduce a formal odd variable ε with $\varepsilon^2 = -u$, and define a supertrace

$$\mathrm{Str}: \quad \Omega^{\bullet}(\mathrm{M}, \mathrm{End}(\mathrm{V}))[\varepsilon] \to \Omega^{\bullet}(\mathrm{M})$$

by setting $\mathrm{Str}(a + b\varepsilon) = \mathrm{Tr}(b)$. The following lemma is basic to our treatment of equivariant characteristic classes.

LEMMA 4.6. – (1) *If* $\alpha \in \mathrm{C}^{\infty}(\mathfrak{g}, \Omega^{\bullet}(\mathrm{M}, \mathrm{End}(\mathrm{V})))^{\mathrm{G}}[\varepsilon]$, *then*

$$d_{\mathfrak{g}}\,\mathrm{Str}(\alpha) = \mathrm{Str}([\nabla_g + \lambda\varepsilon p, \alpha]).$$

(2) *Let* α_0 *and* α_1 *be the elements of* $\mathrm{C}^{\infty}(\mathfrak{g}, \Omega^{\bullet}(\mathrm{M}, \mathrm{End}(\mathrm{V})))^{\mathrm{G}}$ *given by the formulas*

$$\begin{aligned}\alpha_0 &= \rho(X) + \lambda\varepsilon dp, \\ \alpha_1 &= p\rho(X)\,p + p^{\perp}\,\rho(X)\,p^{\perp} - u\,(dp)^2.\end{aligned}$$

Then $[\nabla_{\mathfrak{g}} + \lambda\varepsilon p, \alpha_i] = 0$ *for* $i = 0, 1$.

(3) *If* $\alpha_t = (1-t)\,\alpha_0 + t\alpha_1$, *then*

$$\frac{d\alpha_t}{dt} = -[\nabla_{\mathfrak{g}} + \lambda\varepsilon p, (2p-1)\,dp].$$

Proof. – To prove part (1), we observe that

$$\mathrm{Str}([\nabla_{\mathfrak{g}} + \lambda\varepsilon p, \alpha]) = \mathrm{Str}([d_{\mathfrak{g}}, \alpha]) + \mathrm{Str}([u\,(2p-1)\,dp + \lambda\varepsilon p, \alpha]).$$

The second term on the right-hand side, being the supertrace of a supercommutator, vanishes, while the first term is easily seen to equal $d_{\mathfrak{g}}\,\mathrm{Str}(\alpha)$.

The rest of the lemma is a straightforward calculation, obtained by combining the following easily verified formulas:

$$\begin{gathered}[p, dp] = (2p-1)\,dp, \qquad [p, (2p-1)\,dp] = dp, \qquad [p, (dp)^2] = 0; \\ [\nabla, p] = [\nabla, p^{\perp}] = 0, \qquad [\nabla, dp] = -(2p-1)\,(dp)^2, \qquad [\nabla, (dp)^2] = 0; \\ [\iota(X), dp] = -[\rho(X), p], \qquad [\iota(X), (dp)^2] = -[\rho(X), (2p-1)\,dp].\end{gathered}$$

From these formulas, it is easy to check that $[\nabla_{\mathfrak{g}} + \lambda\varepsilon p, \alpha_i] = 0$ for $i = 0, 1$.

Also from these formulas, we see that

$$\begin{aligned}[\nabla_{\mathfrak{g}} + \lambda\varepsilon p,\ (2p-1)\,dp] &= -(2p-1)\,[p\,(\mathrm{X}),\ p] - u\,(dp)^2 + \lambda\varepsilon dp \\ &= \rho\,(\mathrm{X}) - (p\rho\,(\mathrm{X})\,p + p^{\perp}\,\rho\,(\mathrm{X})\,p^{\perp}) - u(dp)^2 + \lambda\varepsilon dp \\ &= \alpha_0 - \alpha_1 = \frac{\partial\alpha_t}{dt},\end{aligned}$$

which is Part (3) of the lemma. □

Using this lemma, we can now present the main result of this section. In the following theorem, $d\mu$ is the Gaussian measure on the real line, given by the formula

$$d\mu = \frac{1}{\sqrt{4\pi}}\,e^{-\lambda^2/4}\,d\lambda.$$

THEOREM 4.7. – (1) *The equivariant Chern character* $\mathrm{ch}_{\mathrm{G}}\,(p)$ *has germ at* $g \in \mathrm{G}$ *given by the formula*

$$\mathrm{ch}_{\mathrm{G}}\,(p)_g = \int_{-\infty}^{\infty} \mathrm{Str}\left(\varepsilon\rho\,(g)\left(p - \frac{1}{2}\right)e^{\alpha_0}\right)\Big|_{M^g} d\mu + \frac{1}{2}\,\mathrm{Tr}\,(\rho\,(g)).$$

(2) *Let* $\mathrm{Ch}_{\mathrm{G}}\,(p)$ *denote the equivariant Chern character of Berline and Vergne for the trivial bundle over* M *with fibre* V, *thought of as a* $\mathbb{Z}/2$*-graded bundle with grading operator* $2p-1$, *and with connection equal to the Grassmannian connection with respect to p. Then the germ of* $\mathrm{Ch}_{\mathrm{G}}\,(p)$ *at* $g \in \mathrm{G}$ *is given by the formula*

$$\mathrm{Ch}_{\mathrm{G}}\,(p)_g = \int_{-\infty}^{\infty} \mathrm{Str}\left(\varepsilon\rho\,(g)\left(p - \frac{1}{2}\right)e^{\alpha_1}\right)\Big|_{M^g} d\mu.$$

(3) *The following transgression formula links* $\mathrm{Ch}_{\mathrm{G}}\,(p)$ *to* $\mathrm{ch}_{\mathrm{G}}\,(p)$:

$$\begin{aligned}\mathrm{ch}_G\,(p) &= \mathrm{Ch}_G\,(p) + \frac{1}{2}\,\mathrm{Tr}\,(\rho\,(g)) \\ &\quad - d_{\mathfrak{g}} \int_0^1 \left(\int_{-\infty}^{\infty} \mathrm{Str}\left(\varepsilon\rho\,(g)\left(p - \frac{1}{2}\right)e^{\alpha_t + (2p-1)\,dp\,dt}\right)\Big|_{M^g} d\mu\right).\end{aligned}$$

Here, dt is the volume one-form on the unit interval [0, 1].

Proof. – Observe that

$$\int_{-\infty}^{\infty} \lambda^k\,d\mu = \begin{cases} \dfrac{(2l)!}{l!}, & k = 2l, \\ 0, & k = 2l+1. \end{cases}$$

The exponential e^{α_0} is polynomial in λ, as we see from the expansion

$$e^{\alpha_0} = \sum_{k=0}^{\dim(\mathrm{M})} \lambda^k\,\varepsilon^k \int_{\Delta_k} e^{t_1\,\rho\,(\mathrm{X})}\,dp \cdots dp\,e^{(1-t_k)\,\rho\,(\mathrm{X})}\,dt_1 \cdots dt_k.$$

Part (1) now follows by integrating this sum term by term and comparing with the formula for $\mathrm{ch}_{\mathrm{G}}\,(p)$ in Proposition 4.2.

Part (2) is simpler, since the integrand does not depend on λ; using the fact that $d\mu$ has mass 1, we are reduced to showing that

$$\mathrm{Ch}_{\mathrm{G}}(p)_g = \mathrm{Tr}\left(\rho(g)\left(p-\frac{1}{2}\right)e^{p\rho(\mathrm{X})\,p+p^{\perp}\,\rho(\mathrm{X})\,p^{\perp}-u\,(dp)^2}\right)\Big|_{\mathrm{M}^g},$$

which follows from the definition of $\mathrm{Ch}_{\mathrm{G}}(p)$ combined with the formulas of Lemma 4.5 for $\mathrm{F}_{\mathfrak{g}}$.

Combining Parts (1) and (2), we see that

$$\mathrm{ch}_{\mathrm{G}}(\mathrm{p}) - \mathrm{Ch}_{\mathrm{G}}(p) - \frac{1}{2}\mathrm{Tr}(\rho(g)) = \int_0^1 \frac{d}{dt}\int_{-\infty}^{\infty} \mathrm{Str}\left(\varepsilon\rho(g)\left(p-\frac{1}{2}\right)e^{\alpha_t}\right)\Big|_{M^g} d\mu\, dt.$$

Lemma 4.6, coupled with the fact that over M^g, the operator $\nabla_{\mathfrak{g}} + \lambda\varepsilon p$ supercommutes with $\varepsilon\rho(g)(2p-1)$, shows that

$$\begin{aligned}&\mathrm{ch}_{\mathrm{G}}(p) - \mathrm{Ch}_{\mathrm{G}}(p) - \frac{1}{2}\mathrm{Tr}(\rho(g))\\ &= \int_0^1\int_{-\infty}^{\infty} \mathrm{Str}\left(\left[\nabla_{\mathfrak{g}} + \lambda\varepsilon p,\ \varepsilon\rho(g)\left(p-\frac{1}{2}\right)e^{\alpha_t+(2p-1)\,dp\,dt}\right]\right)\Big|_{M^g} d\mu.\end{aligned}$$

The trangression formula now follows by application of Part (1) of Lemma 4.6. □

Our use of the formal odd variable ε such that $\varepsilon^2 = -u$, and our representation of the Chern character $\mathrm{ch}_G(p)$ as a Gaussian integral with respect to a parameter $\lambda \in \mathbb{R}$, find parallels in the work of Connes and Quillen; *see* especially [7].

Appendix

Morita equivalence for Hochschild homology

In this appendix, we recall those parts of Wodzicki's theory of H-unitality which are needed to prove Morita invariance for equivariant Hochschild homology [18].

If A is an algebra with right module K and left module L, we may define a complex $\mathrm{B}_n(\mathrm{K}, \mathrm{A}, \mathrm{L}) = \mathrm{K}\otimes\mathrm{A}^{\otimes n}\otimes\mathrm{L}$, with differential

$$\begin{aligned}b(k\otimes a_1\otimes\cdots a_n\otimes l) =& ka_1\otimes a_2\otimes\cdots\otimes a_n\otimes l\\ &+\sum_{i=1}^{n-1}(-1)^i\, k\otimes a_1\otimes\cdots\otimes a_i\, a_{i+1}\otimes\cdots\otimes a_n\otimes l\\ &+(-1)^n\, k\otimes a_1\otimes\cdots\otimes a_{n-1}\otimes a_n\, l.\end{aligned}$$

If A is unital, the complex $\mathrm{B}_\bullet(\mathrm{A}, \mathrm{A}, \mathrm{L})$ has homology

$$\mathrm{H}_n(\mathrm{B}_\bullet(\mathrm{A}, \mathrm{A}, \mathrm{L})) = \begin{cases}\mathrm{L}, & n=0,\\ 0, & n>0.\end{cases}$$

This is conveniently proved by extending the definition of $\mathrm{B}_i(\mathrm{K}, \mathrm{A}, \mathrm{L})$ to $i = -1$, by setting $\mathrm{B}_{-1}(\mathrm{K}, \mathrm{A}, \mathrm{L}) = \mathrm{K} \otimes_{\mathrm{A}} \mathrm{L}$ and defining $b : \mathrm{B}_0(\mathrm{K}, \mathrm{A}, \mathrm{L}) \to \mathrm{B}_{-1}(\mathrm{K}, \mathrm{A}, \mathrm{L})$ given by the formula

$$k \otimes l \to k \otimes_A l.$$

We must now show that the augmented complex $\mathrm{B}_\bullet(\mathrm{A}, \mathrm{A}, \mathrm{L})$, $n \geq -1$, is exact: this follows from the existence of a contracting homotopy $s : B_n(A, A, L) \to B_{n+1}(A, A, L)$, given by the formula

$$s(a_0 \otimes \cdots a_n \otimes l) = 1 \otimes a_0 \otimes \cdots \otimes a_n \otimes l,$$

for $n \geq 0$, while for $n = -1$, we map l to $1 \otimes l$. We then verify the formula $sb + bs = \mathrm{id}$.

Motivated by this, and in imitation of Wodzicki, we say that an algebra A is H-*unital* of

$$\mathrm{H}_n(\mathrm{B}_\bullet(\mathrm{A}, \mathrm{A}, \mathrm{A})) = \begin{cases} \mathrm{A}, & n = 0, \\ 0, & n > 0. \end{cases}$$

while flat right and left A-modules K and L are H-*unitary* if $\mathrm{K} \otimes_{\mathrm{A}} \mathrm{A} = \mathrm{K}$ and $\mathrm{A} \otimes_{\mathrm{A}} \mathrm{L} = \mathrm{L}$.

PROPOSITION A.1. – *If* A *is* H-*unital, and* K *and* L *are flat* H-*unitary right, respectively left,* A-*modules, then*

$$\mathrm{H}_n(\mathrm{B}_\bullet(\mathrm{K}, \mathrm{A}, \mathrm{L})) = \begin{cases} \mathrm{K} \otimes_{\mathrm{A}} \mathrm{L}, & n = 0, \\ 0, & n > 0. \end{cases}$$

Proof. – By H-unitary of K and L, the bar complex $\mathrm{B}_\bullet(\mathrm{K}, \mathrm{A}, \mathrm{L})$ may be written

$$\mathrm{B}_\bullet(\mathrm{K}, \mathrm{A}, \mathrm{L}) \cong \mathrm{K} \otimes_{\mathrm{A}} \mathrm{B}(\mathrm{A}, \mathrm{A}, \mathrm{A}) \otimes_{\mathrm{A}} \mathrm{L}.$$

Since the modules K and L are flat, we see that

$$\begin{aligned} \mathrm{H}_n(\mathrm{B}_\bullet(\mathrm{K}, \mathrm{A}, \mathrm{L})) &\cong \mathrm{K} \otimes_{\mathrm{A}} \mathrm{H}_n(\mathrm{B}_\bullet(\mathrm{A}, \mathrm{A}, \mathrm{A})) \otimes_{\mathrm{A}} \mathrm{L} \\ &= \begin{cases} \mathrm{K} \otimes_{\mathrm{A}} \mathrm{L}, & n = 0, \\ 0, & n > 0. \end{cases} \quad \square \end{aligned}$$

In our discussion of Morita equivalence, we follow Dennis and Igusa [8] (who follow Waldhausen), except that we take advantage of Wodzicki's notion of H-unitality to simplify their account.

DEFINITION A.2. – *Let* A *and* B *be* H-*unital algebras and let* ${}_{\mathrm{A}}\mathrm{P}_{\mathrm{B}}$ *and* ${}_{\mathrm{B}}\mathrm{Q}_{\mathrm{A}}$ *be flat* H-*unitary modules for, respectively,* $\mathrm{A} \otimes \mathrm{B}^{\circ}$ *and* $\mathrm{B} \otimes \mathrm{A}^{\circ}$. *We say that* A *and* B *are Morita equivalent if*

$$\mathrm{P} \otimes_{\mathrm{B}} \mathrm{Q} \cong \mathrm{A} \quad \textit{as an } \mathrm{A}\textit{-bimodule},$$

and

$$\mathrm{Q} \otimes_{\mathrm{A}} \mathrm{P} \cong \mathrm{B} \quad \textit{as a } \mathrm{B}\textit{-bimodule}.$$

THEOREM A.3. – *Let* A *and* B *be Morita equivalent algebras and let* L *be a bimodule over* A. *Then*

$$\mathrm{H}_\bullet(\mathrm{A}, \mathrm{L}) \cong \mathrm{H}_\bullet(\mathrm{B}, \mathrm{Q} \otimes_{\mathrm{A}} \mathrm{L} \otimes_{\mathrm{A}} \mathrm{P}).$$

Proof. – If $(M_\bullet, \delta)$ is a differantial graded $A \otimes A^o$-module, the Hochschild complex of $M_\bullet$ is the double complex $C_p(A, M_q)$ with differentials

$$b: \ C_p(A, M_q) \to C_{p-1}(A, M_q),$$

and

$$\delta: \quad C_p(A, M_q) \to C_p(A, M_{q-1}).$$

The homology of the total differential of this complex is called the Hochschild hyperhomology of the differential graded module $M_\bullet$, and there are two first quadrant homology spectral sequences converging to this hyperhomology, with E^1-terms

$${}^{\mathrm{I}}E^1_{pq} = C_p(A, H_q(M_\bullet, \delta)),$$

and

$${}^{\mathrm{II}}E^1_{pq} = H_p(A, M_q).$$

We will prove the theorem by considering these spectral sequences in the case where

$$M_\bullet = B_\bullet(P, B, Q \otimes_A L),$$

considered as a differential graded $A \otimes A^o$-module. The first spectral sequence has E^1-term

$$\begin{aligned} {}^{\mathrm{I}}E^1_{pq} &= C_p(A, H_q(B_\bullet(P, B, Q \otimes_A L))) \\ &= \begin{cases} C_p(A, P \otimes_B Q \otimes_A L \cong C_p(A, L), & q = 0, \\ 0, & q > 0. \end{cases} \end{aligned}$$

From this, we see that the spectral sequence degenerates, and that the Hochschild hyperhomology of $B_\bullet(P, B, Q \otimes_A L)$ is equal to $H_\bullet(A, L)$.

On the other hand, we see that the second spectral sequence has E^1-term

$$\begin{aligned} {}^{\mathrm{II}}E^1_{pq} &= H_p(A, B_q(P, B, Q \otimes_A L)) \\ &= \begin{cases} C_p(B, Q \otimes_A L \otimes_A P), & p = 0, \\ 0, & p > 0. \end{cases} \end{aligned}$$

since the double complexes $C_p(A, B_q(P, B, Q \otimes_A L))$ and $C_q(B, B_p(Q \otimes_A L, A, P))$ are naturally isomorphic, in such a way as to identify the second spectral sequence of the first double complex with the first spectral sequence of the second. The theorem follows from the degeneration of both spectral sequences at the E^2-term. □

Let G be a Lie group, and let A be a unital topological algebra with smooth action of G.

DEFINITION A.4. – *The crossed product algebra* $A \rtimes G$ *is* H-*unital.*

Proof. – Fix an element $\varphi \in C_c^\infty(G)$ such that $\int_G \varphi \, dg = 1$, where dg is the left Haar measure on G. We may think of φ as defining an element of $A \rtimes G$ whose value at any element of G is a scalar multiple of the identity.

Define $s : \mathrm{B}_n(\mathrm{A} \rtimes \mathrm{G}, \mathrm{A} \rtimes \mathrm{G}, \mathrm{A} \rtimes \mathrm{G}) \to \mathrm{B}_{n+1}(\mathrm{A} \rtimes \mathrm{G}, \mathrm{A} \rtimes \mathrm{G}, \mathrm{A} \rtimes \mathrm{G})$, $n \geq -1$, by the formula

$$(sf)(g_0, \ldots, g_{n+2}) = \varphi(g_0) \otimes (g_0 \otimes 1 \otimes \cdots \otimes 1) f(g_0 g_1, \ldots, g_{n+2}).$$

Then s satisfies the formula $sb + bs = \mathrm{id}$. $\square$

In Section 3, we need that if M is a compact manifold on which the compact Lie group H acts freely, then the two algebras $\mathrm{C}^\infty(\mathrm{M}) \rtimes \mathrm{H}$ and $\mathrm{C}^\infty(\mathrm{M}/\mathrm{H})$ are Morita equivalent. Take the bimodule $\mathrm{P} = \mathrm{C}^\infty(\mathrm{M})$, with the evident left action of $\mathrm{C}^\infty(\mathrm{M}) \rtimes \mathrm{H}$ and right action of $\mathrm{C}^\infty(\mathrm{M}/\mathrm{H})$. The algebra $\mathrm{C}^\infty(\mathrm{M}) \rtimes \mathrm{H}$ is isomorphic to its opposite by the map

$$f(x, g) \mapsto f(x, g^{-1}).$$

Take Q to be the opposite module of P; it is a $(\mathrm{C}^\infty(\mathrm{M}/\mathrm{H}), \mathrm{C}^\infty(\mathrm{M}) \rtimes \mathrm{H})$-bimodule, by means of the identifications of the algebras $\mathrm{C}^\infty(\mathrm{M}) \rtimes \mathrm{H}$ and $\mathrm{C}^\infty(\mathrm{M}/\mathrm{H})$ with their opposites.

Let us show that P and Q implement the Morita equivalence between $\mathrm{C}^\infty(\mathrm{M}) \rtimes \mathrm{H}$ and $\mathrm{C}^\infty(\mathrm{M}/\mathrm{H})$, that is,

$$\mathrm{C}^\infty(\mathrm{M}) \otimes_{\mathrm{C}^\infty(\mathrm{M}) \rtimes \mathrm{H}} \mathrm{C}^\infty(\mathrm{M}) \cong \mathrm{C}^\infty(\mathrm{M}/\mathrm{H}),$$

and

$$\mathrm{C}^\infty(\mathrm{M}) \otimes_{\mathrm{C}^\infty(\mathrm{M}/\mathrm{H})} \mathrm{C}^\infty(\mathrm{M}) \cong \mathrm{C}^\infty(\mathrm{M}) \rtimes \mathrm{H}.$$

Let A be an algebra with an action of a Lie group H, and let K and L be equivariant right and left A-modules. The spaces $\mathrm{K} \otimes_{\mathrm{A} \rtimes \mathrm{H}} \mathrm{A}$ and $\mathrm{A} \otimes_{\mathrm{A} \rtimes \mathrm{H}} \mathrm{L}$ may be identified with $\mathrm{K} \otimes_{\mathbb{C} \rtimes \mathrm{H}} \mathbb{C}$ and $\mathbb{C} \otimes_{\mathbb{C} \rtimes \mathrm{H}} \mathrm{L}$, by identifying $k \otimes_{\mathrm{A} \rtimes \mathrm{H}} a$ and $a \otimes_{\mathrm{A} \rtimes \mathrm{H}} l$ with $ka \otimes_{\mathbb{C} \rtimes \mathrm{H}} 1$ and $1 \otimes_{\mathbb{C} \otimes \mathrm{H}} al$. But it is clear that $\mathrm{K} \otimes_{\mathbb{C} \rtimes \mathrm{H}} \mathbb{C}$ and $\mathbb{C} \otimes_{\mathbb{C} \rtimes \mathrm{H}} \mathrm{L}$ are equal to the coinvariants $\mathrm{H}_0(\mathrm{H}, \mathrm{K})$ and $\mathrm{H}_0(\mathrm{H}, \mathrm{L})$ of the actions of H on K and L.

Now consider the special case in which $\mathrm{A} = \mathrm{C}^\infty(\mathrm{M})$ and H is compact. Since $\mathrm{H}_0(\mathrm{H}, .)$ is an exact functor, we see that P and Q are flat as left, respectively right, $\mathrm{C}^\infty(\mathrm{M}) \rtimes \mathrm{H}$-modules. Similarly, we see that

$$\mathrm{C}^\infty(\mathrm{M}) \rtimes \mathrm{H} \otimes_{\mathrm{C}^\infty(\mathrm{M}) \rtimes \mathrm{H}} \mathrm{C}^\infty(\mathrm{M}) \cong \mathrm{C}^\infty(\mathrm{M}),$$

and

$$\mathrm{C}^\infty(\mathrm{M}) \otimes_{\mathrm{C}^\infty(\mathrm{M}) \rtimes \mathrm{H}} \mathrm{C}^\infty(\mathrm{M}) \rtimes \mathrm{H} \cong \mathrm{C}^\infty(\mathrm{M}),$$

and hence P and Q are H-unitary as $\mathrm{C}^\infty(\mathrm{M}) \rtimes \mathrm{H}$-modules.

Also, since we may identify $\mathrm{H}_0(\mathrm{H}, \mathrm{C}^\infty(\mathrm{M}))$ with $\mathrm{C}^\infty(\mathrm{M}/\mathrm{H})$, we see that

$$\mathrm{C}^\infty(\mathrm{M}) \otimes_{\mathrm{C}^\infty(\mathrm{M}) \rtimes \mathrm{H}} \mathrm{C}^\infty(\mathrm{M}) \cong \mathrm{C}^\infty(\mathrm{M}/\mathrm{H}),$$

which is one of the isomorphisms needed for P and Q to define a Morita equivalence.

Since $\mathrm{M} \times_{\mathrm{M}/\mathrm{H}} \mathrm{M} \cong \mathrm{M} \times \mathrm{H}$, we see that

$$\mathrm{C}^\infty(\mathrm{M}) \otimes_{\mathrm{C}^\infty(\mathrm{M}/\mathrm{H})} \mathrm{C}^\infty(\mathrm{M}) \cong \mathrm{C}^\infty(\mathrm{M}) \otimes \mathrm{C}^\infty(\mathrm{H}),$$

where $\mathrm{C}^\infty(\mathrm{M}) \otimes \mathrm{C}^\infty(\mathrm{H}) \cong \mathrm{C}^\infty(\mathrm{M}) \rtimes \mathrm{H}$ as a $\mathrm{C}^\infty(\mathrm{M}) \rtimes \mathrm{H}$-bimodule. Thus, this gives the other isomorphism needed to prove P and Q implement a Morita equivalence. Since M is locally a product, and flatness is a local condition, $\mathrm{C}^\infty(\mathrm{M})$ is flat over $\mathrm{C}^\infty(\mathrm{M}/\mathrm{H})$. Thus, we see that P and Q define a Morita equivalence between $\mathrm{C}^\infty(\mathrm{M}) \rtimes \mathrm{H}$ and $\mathrm{C}^\infty(\mathrm{M}/\mathrm{H})$.

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(Manuscript received June 27, 1992.)

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