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BOUNDARY COHOMOLOGY OF SHIMURA VARIETIES I. – COHERENT COHOMOLOGY ON TOROIDAL COMPACTIFICATIONS

BY MICHAEL HARRIS (1) AND STEVEN ZUCKER (2)

ABSTRACT. – We study the coherent cohomology of automorphic vector bundles, restricted to the toroidal boundary strata of Shimura varieties associated to maximal rational parabolic subgroups. The cohomology is computed in terms of coherent cohomology of the Shimura varieties attached to the boundary components. The main result concerns the restriction of a global coherent cohomology class to the boundary stratum associated with the maximal parabolic P; it is shown that, in terms of Dolbeault cohomology with growth conditions, this restriction is given by taking the constant term along the unipotent radical of P. This result is used to show that certain non-holomorphic, absolutely convergent Eisenstein series define rational global (coherent) cohomology classes. The main technical construction is a comparison between the (simplicial) Dolbeault complex associated to a complex torus embedding and the (simplicial) de Rham complex associated to its "real part".

Introduction

This paper is the first in a series devoted to analyzing the mixed Hodge structure on the cohomology of a non-compact Shimura variety Sh, with coefficients in a locally homogeneous variation of Hodge structure $\tilde{V}$. The mixed Hodge structure consists of two filtrations, the Hodge filtration and the weight filtration. Of these, the Hodge filtration is easier to understand concretely, since the associated graded object is the direct sum of spaces of coherent cohomology of automorphic vector bundles, canonically extended to toroidal compactifications (*cf.* [H4], and § 3 below). These coherent cohomology spaces have natural restriction maps to the boundary of the toroidal compactifications, and the kernels of these are reasonably well understood: they consist of spaces of square-integrable automorphic forms, usually (but not always) cusp forms [H5]. Furthermore, the kernels of the restrictions themselves comprise the Hodge components of what Harder (*see* [Ha2]) calls the *interior cohomology* of $\tilde{V}$, namely the image of $H_c^*(\mathrm{Sh}, \tilde{V})$ in $H^\bullet(\mathrm{Sh}, \tilde{V})$, where H_c^* denotes cohomology with compact support. The natural mixed Hodge structure of the

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interior cohomology is known to be pure, so for our purposes it is reasonable to concentrate initially on the cohomology supported on the boundary.

The present article proves basic results on the coherent cohomology of automorphic vector bundles, restricted to the toroidal boundary strata of Shimura varieties associated to *maximal* rational parabolic subgroups. We give a formula for the cohomology of these boundary strata, first in classical, then adelic, language (Corollaries 3.7.6, 3.13.6, and 4.1.14). Our main result (Theorems 3.10.3 and 3.12.7) computes the restriction of a global cohomology class on the compactification, given by a differential form (Dolbeault cohomology with growth conditions), to the boundary stratum associated to a maximal parabolic P, and to its closure, and shows that it is obtained by taking the constant term along the unipotent radical of P. In Section 4, we show that these computations are compatible with the rational structures provided by canonical models of the boundary and of the automorphic vector bundles (Theorem 4.8.1).

These considerations correspond roughly to Ch. I, Sections 1 and 2 of Schwermer's book on Eisenstein cohomology [Sch], which covers the analogous material for the topological (or de Rham) cohomology of local systems on arithmetic quotients of (not necessarily hermitian) symmetric spaces. There, the much simpler and nicer Borel-Serre compactification [BS] could be used, which realizes the space as the interior of a manifold-with-corners. As such, the space has the same homotopy type as its compactification. Moreover, all closed faces of the boundary – in natural correspondence with rational parabolic subgroups P of *all* types – are themselves the Borel-Serre compactifications of their interiors, denoted $e'(P)$. For quite general reasons, these faces admit collars (though one must avoid a well-known pitfall when one writes them down in terms of the group theory; *see* our 3.11.3). Thus, any $e'(P)$ can be moved in from the boundary, allowing for an easy description of restriction of a cohomology class to a boundary face, and this plays an important role in the topological theory.

Our treatment of coherent cohomology, although it eventually arrives at similar-looking results (esp. 3.12.7 and 3.13.6), takes much longer to work out. The main reason for this is that the toroidal compactifications, which play the role of the Borel-Serre compactifications in the holomorphic category, are topologically quite different, and are not homotopy equivalent to anything directly related to group theory. The theory of [H5] computes the coherent cohomology of a toroidal compactification of a Shimura variety in terms of C^∞-differential forms having moderate singularities along the boundary. Although coherent cohomology admits an obvious restriction to the toroidal boundary, it was by no means clear how to express it in terms of these singular forms. In fact, most of the techniques in Sections 2 and 3 were developed in order to find a substitute for the deformation retraction of the Borel-Serre boundary into the locally symmetric space. The main construction is an identification of the (simplicial) Dolbeault complexes associated to a complex torus embedding with the (simplicial) de Rham complexes associated to its "real part" (Prop. 2.7.3), which we subsequently see as the "fine resolution" version of something more basic (2.8). This construction, which we have not seen elsewhere, allows us, more or less, to retract coherent cohomology classes from the boundary to the interior of a partially holomorphic, partially C^∞ quotient of a neighborhood of the boundary stratum, which retains the means for computing the coherent cohomology of that stratum (*see* 3.9-3.10).

The technical difficulties do not end there. In the topological case, $e'(\mathrm{P})$ is fibered by compact nilmanifolds associated to the unipotent radical of P, with base an arithmetic quotient of the lower-rank symmetric space of the Levi quotient, and it is integration over the fiber that produces the constant term. In the toroidal setting, (for P maximal) the geometric structure near the boundary comes from the Siegel domain picture associated to P [*see* our (1.2.5)]: a part of the unipotent radical gives rise to a family of abelian varieties, which is stuck as the middle term of a two-step fibration over the corresponding (holomorphic) boundary stratum of the Baily-Borel Satake compactification, and is itself covered by something of a purely topological nature, coming from the "link subgroup" G_l [*see* (1.2.1)]; and an arithmetic subgroup of the link group acts on the whole fibration. The most natural way that emerged to deal with this problem is to use equivariant cohomology (2.9, 3.7, 3.9, 3.12, 4.8), which circumvents some technical issues arising in the hybrid de Rham/Dolbeault calculations suggested above, and simultaneously returns us to the purely algebraic category.

At several points we make reference to Pink's work on mixed Shimura varieties and their canonical models, which allows us to replace growth conditions on non-reductive groups by geometric constructions. We understand that Pink is in the process of working out the theory of automorphic vector bundles on mixed Shimura varieties, a part of which we have developed *ad hoc* in paragraph 4. The availability of a systematic theory along these lines may simplify a number of our constructions; more significantly, it will provide a unified framework for considering the boundary strata of toroidal compactifications corresponding to non-maximal rational parabolic subgroups. We hint at this theory at several points, especially in our treatment of growth conditions along the boundary of the chosen boundary stratum, and the general answer is described during the proof of Lemma 5.3.12. We have decided to defer consideration of non-maximal strata because the paper is already long enough, and because the general result will require a purely geometric proof of our key formula, Corollary 3.13.6, that eliminates the use of differential forms with growth conditions.

Section 5 contains a construction of certain Eisenstein cohomology classes lifting cusp forms on (Levi factors of) maximal parabolic subgroups, along the lines sketched in section 6 of [H4], and a proof of their rationality in the range of absolute convergence (Theorem 5.3.11). That construction requires the embedding of discrete series modules in induced modules, the classification of which in general remains an open problem. J. Schwermer and J. Franke have both advised us to bypass this embedding problem by computing the $\bar{\partial}$-cohomology of the space of Eisenstein series directly; Franke has in fact calculated the $\bar{\partial}$-cohomology of induced representations. We agree in principle, but computing the restriction to the boundary of these cohomology classes requires a substantial understanding of intertwining operators. In the long run, and as in recent work of Harder, we expect that these intertwining operators, which are closely related to special values of L-functions, will be essential to describing the "mixed motives" of which the Eisenstein cohomology classes form a part. Among other things, we consider the present work to be a contribution to the elucidation of these mixed motives, especially their de Rham realizations.

Some of the results of the present paper were announced at the conference on Cohomology of Arithmetic Groups at Luminy in 1989. In Part II of this paper, we will generalize the results of Part I to arbitrary (non-maximal) boundary strata, and will take up the problem of describing the mixed Hodge structure of the boundary cohomology in terms of automorphic forms.

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Notation

By $\mathbf{A}$ (resp. $\mathbf{A}^f$) we mean the ring of rational adeles (resp. of rational finite adeles). If B is any $\mathbb{Q}$-algebra, then $\mathbf{A}_B = \mathbf{A} \otimes_{\mathbb{Q}} B$. By $\bar{\mathbb{Q}}$ we always mean the algebraic closure of $\mathbb{Q}$ in $\mathbb{C}$; $\mathbb{Q}^{ab}$ is the maximal abelian extension of $\mathbb{Q}$ in $\bar{\mathbb{Q}}$.

If V and T are schemes over the scheme S, then V(T) denotes the set of T-valued points of V; $V_T = V \times_S T$. If T is Spec(A) for some ring A, we often write V(A) and V_A in place of V(T) and V_T. If $S = \operatorname{Spec} k'$, where k' is a finite field extension of the field k, then $R_{k'/k} V$ is the scheme over k obtained by Weil's restriction of scalars functor. The structure sheaf of V is denoted $\mathcal{O}_V$.

If G is an algebraic group, then G^{ad}, G^{der}, G^{ab}, and Z_G are the adjoint group, the derived subgroup, the abelianization G/G^{der}, and the center, respectively, of G. The unipotent radical of G is denoted $R_u(G)$. If G is a topological group, then G^0 is its connected component containing the identity; the same notation is used for algebraic groups. The group schemes $GL(n)$ and $\mathbb{G}_m$ are denoted as usual. By X(G) we denote the group $\operatorname{Hom}(G, \mathbb{G}_m)$.

If $f: X \to Y$ is a continuous map of topological spaces and $\mathcal{F}$ is a sheaf on X, we denote by $\mathbf{R} f_* \mathcal{F}$ the total direct image of $\mathcal{F}$ (in the derived category). If $\mathcal{E}$ is a sheaf on Y, then $f^{-1} \mathcal{E}$ denotes the pullback in the category of sheaves; if X and Y are schemes and $\mathcal{E}$ is a sheaf of $\mathcal{O}_Y$-modules then $f^* \mathcal{E}$ denotes $\mathcal{O}_X \otimes_{\mathcal{O}_Y} \mathcal{E}$. If $\mathcal{E}^\bullet$ and $\mathcal{F}^\bullet$ are complexes of abelian sheaves on X then "$\mathcal{E}^\bullet \approx \mathcal{F}^\bullet$" means that $\mathcal{E}^\bullet$ and $\mathcal{F}^\bullet$ are quasi-isomorphic. The case where Y is a point (f is thus the unique map from X to Y) produces $\mathbf{R}\Gamma(X, \mathcal{E})$, whose cohomology is the hypercohomology of $\mathcal{E}^\bullet$, denoted $\mathbb{H}^\bullet(X, \mathcal{E}^\bullet)$.

Unavoidably, an enormous quantity of special notation is used in this paper. Here is a list of the most frequently used notation, preceded by the number of the section or subsection in which it first appears.

1.1. $\underline{S}$, (G, X), Sh(G, X), ${}_K\mathrm{Sh}(G, X)$, E(G, X), $\mathfrak{g}$.

1.2. D, $G(\mathbb{R})^+$, S^+, Γ, M_Γ.

1.2.1. F, P_F, L_F, W_F, $G_l = G_{l,F}$, u_F.

1.2.2. $\tilde{A}$, A, w_F, $F_h^\bullet$, W^F, $\mathfrak{z}$, $\mathfrak{g}^i$, $\mathfrak{g}_l$, $\mathfrak{g}_t$, G_t, $\mathfrak{g}_h(0)$, $G_h(0)$, $G_h = G_{h,F}$, $\mathfrak{g}_h$, $\mathfrak{a}$.

1.2.3-1.2.5. C_F, $G^0_{l,F}$, $\Gamma_l = \Gamma_{l,F}$, Γ_U, $P' = P'_F$, Γ'_F, D_F, Γ'_F, M'_F, T_F, $X(T_F)$, π_2, π_1, A_F, M_F.

1.3. σ, T_σ, T_Σ, ∂T_Σ, Σ^0, Σ^c, $\bar{C}_F$, $(M'_F)_\sigma$, $D_{F,\sigma}$, $(M'_F)_{\{\sigma'\}}$, $D_{F,\{\sigma'\}}$, $(M'_F)_\Sigma$, $D_{F,\Sigma}$, $\pi_{2,\Sigma}$, $\pi_{F,\Sigma}$, $X_*(T)$.

1.4. Σ_F, $M_{\Gamma,\Sigma}$, $\partial M_{\Gamma,\Sigma}$, $\varphi_{F,\sigma}$, $\varphi_{F,\Sigma}$, SNC.

1.5. π_Σ, M^*_Γ, ∂M^*_Γ, $\tilde{M}_F$, $\bar{Z}_\sigma$.

$Z_{F,\Sigma}$, ${}^{>}Z_{F,\Sigma}$, ${}^{<}Z_{F,\Sigma}$, ${}^{0}Z_{F,\Sigma}$, $\partial Z_{F,\Sigma}$, ${}^{>}\partial Z_{F,\Sigma}$, ${}^{<}\partial Z_{F,\Sigma}$, $\bar{Z}_\tau$, ${}^{<}\tilde{Z}_{F,\Sigma}$, $\Phi_{F,\Sigma}$, $\delta_{F,\Sigma}$, Z_τ, ψ_τ.

1.6. $\bar{\pi}_2$, $\bar{\pi}_1$, $M'_{F,\Sigma(\Xi)}$, $A_{F,\Xi}$, $M_{F,\Xi}$, $\tilde{Z}_{F,\Sigma(\Xi)}$, $\partial\tilde{Z}_{F,\Sigma(\Xi)}$, ${}^{>}\partial\tilde{Z}_{F,\Sigma(\Xi)}$, ${}^{<}\partial\tilde{Z}_{F,\Sigma(\Xi)}$.

1.7. ${}_K Sh(G,X)_\Sigma$, $Sh(G,X)^\sim$, $Sh(G,X)^*$, ${}_K Sh(G,X)^*$, $\pi^\sim$, $\mathcal{B}(D)$, $\mathcal{B}(F)$, $(G_{h,F}, X(F))$, $(G_{h,F}, F_P)$, $Sh(\mathbb{G}_m, N)$, $Sh(G,X)^F$, $Sh(G,X)^{P'}$, $Sh(G,X)^{P_F}$, $Sh(G,X)^{\sim,F}$, $Sh(G,X)^{\sim,P'}$, $Sh(G,X)^{\sim,P_F}$.

1.8. K_p, $\mathcal{P}_p$, $\mathfrak{k}_{p,\mathbb{C}}$, $\mathfrak{p}^+$, $\mathfrak{p}^-$, K_h, K_l, $(G^{(2)}, \Delta(P_F))$, $\Delta(P_F)^0$, $\Gamma^{(2)}$, D_t, $K_p^{(2)}$, K_t, N_F, c_F, $\mathfrak{p}_2^+$, $Q_{F,p}$, V_F, $\mathfrak{v}_F$, $\mathfrak{v}_x^+$, $\mathfrak{v}_x^-$, $\mathfrak{p}_1^+$, $\mathfrak{p}_h^+$, $J = J^{P_{F,p}}$.

2.1. $\hat{\sigma} = \sigma/\mathbb{R}_+^\times$, $\hat{\Sigma}$, T^+, T_σ^+, T_Σ^+, T^c, ∂T_Σ^+.

2.2. $Sk^1(\sigma)$, $\hat{\Sigma}^c$, $\partial\hat{\Sigma}$, $\hat{\Sigma}'$, $\Sigma^{(1)}$, $\hat{\Sigma}^{(1),c}$, $X(\Gamma_l)$.

2.3. U_σ, U_Σ, $\underline{\varepsilon}_\sigma$, $\langle\sigma'\rangle$, $U_\sigma(\mathbb{R})$.

2.4. ∂U_σ, Star(σ'), Con(σ'), Y, $(C_F)_\Sigma$, $\partial(C_F)_\Sigma$, $\partial\hat{\Sigma}^c$.

2.6. $\mathcal{A}^\bullet(T_\Sigma^+)$.

2.7. $\mathcal{A}^{0,\bullet}(T_\Sigma)$.

2.9. $H_\Gamma^\bullet$, R_Γ^i.

3.1. $\check{M} = \check{M}(G,X)$, $[\mathcal{V}_\lambda]$.

3.2. $\mathcal{V}'_F$, $\mathcal{V}^A$, $\mathcal{V}'_{F,\Sigma}$, $\mathcal{V}_{\Gamma,\Sigma}$, $[\mathcal{V}]_\Sigma$.

3.3. J_λ.

3.4. q_σ, p_σ.

3.5. A_x, Γ_V, $S_{F,p}$, J', J'_λ, Φ_σ, $\mathfrak{s}_{F,p}$.

3.6. H, $\mathfrak{h}$, R^+, ρ, ρ_c, H_h, H_l, $R^{+,(2)}$, $W^{F,p}$, $l(w)$, $W^{F,p}(q)$, $\lambda(h,w)$, $\lambda(l,w)$.

3.8. $\mathcal{A}^\bullet_{si}$, $\mathcal{A}^\bullet_{rd}$, $\mathcal{A}^\bullet_{sia}$, $\mathcal{A}^\bullet_{rda}$, $\mathfrak{P}_p$.

3.9. p, q, Z_1, Z_2, Z_3, $\tilde{\Phi}$, $\tilde{\Psi}$.

3.10. r_F, η_F, $\mathfrak{p}_p$, $\tilde{\mathfrak{p}}_l$, p_l.

3.12. $\bar{p}$, $\bar{q}$, $\bar{\tilde{\Phi}}$, $\bar{\tilde{\Psi}}$, $\mathcal{V}'_{F,\Sigma}$.

4.1. $\mathcal{M}'_F$, $\mathcal{A}_F$, ${}_K\mathcal{M}'_F$, ${}_K\mathcal{A}_F$, $\pi_{2,K}$, $\pi_{1,K}$, ${}_K\mathcal{M}'_{F,\Sigma}$, $\widetilde{Sh}_\Sigma^{P'}$, $X(G_l)$, $\tilde{H}^\bullet(\mathcal{V}^{can})$, $H^\bullet(\mathcal{V}^{sub})$, $\bar{H}^\bullet$, $\mathcal{V}_\Sigma^P$, $\mathcal{V}_\Sigma^F$, Δ, Δ_0, Δ_1, $\mathcal{H}^\bullet(w)$, I^P.

4.2. $\hat{\mathfrak{K}}_h$, $\hat{\mathfrak{K}}_l$, I_P^G.

4.3. $\tilde{V}$, $[\tilde{V}]$, i_γ, $I(G,X)$, $Per_{(p,g')}$, $E(\mathcal{V},h)$, $\mathfrak{M}(\chi)$, $p(h,\chi)$.

4.4. $I_\Sigma = {}_K I(G,X)_\Sigma$, $I(G,X)_\Gamma$, $W(N)$, $W(N,w)$, $W(N\,\mathrm{rel}\,W^\bullet)$.

4.5. $I^F_{\Sigma,\sigma}$, I^F_Σ.

4.6. $I^F_{\Sigma,j}$, $\pi^F_{\Sigma,j}$, $\tilde{I}^F_{\Sigma,j}$, I'_2, I'_1.

1. Shimura varieties and Toroidal compactifications

1.1. Shimura varieties. – Let $\underline{S}$ be the real algebraic torus $\mathrm{R}_{\mathbb{C}/\mathbb{R}}\,\mathbb{G}_m$. Let (G, X) be a pair consisting of a connected reductive algebraic group G defined over $\mathbb{Q}$ and a $\mathrm{G}(\mathbb{R})$-conjugacy class of homomorphisms h: $\underline{S} \to \mathrm{G}_{\mathbb{R}}$, satisfying the following conditions ([D3]; *cf.* [Mi]):

(1.1.1) The Hodge structure on the Lie algebra $\mathfrak{g}$ of G, given by

$$\mathrm{Ad} \circ h: \quad \underline{S} \to \mathrm{GL}(\mathfrak{g}), \text{ is of type } (1, -1) + (0,0) + (-1,1);$$

(1.1.2) The automorphism $\mathrm{Ad}(h(i))$ of $\mathrm{G}(\mathbb{R})$ induces a Cartan involution on $\mathrm{G}^{\mathrm{der}}(\mathbb{R})^0$;

(1.1.3) Let w: $\mathbb{G}_{m,\mathbb{R}} \to \underline{S}$ be the canonical conorm map. The *weight map* $h \circ w$: $\mathbb{G}_{m,\mathbb{R}} \to \mathrm{G}_{\mathbb{R}}$, whose image is (by (1.1.1)) central in $\mathrm{G}_{\mathbb{R}}$, is defined over $\mathbb{Q}$;

(1.1.4) Let Z'_{G} be the maximal $\mathbb{Q}$-split torus of $\mathrm{Z_G}$. Then $\mathrm{Z_G}(\mathbb{R})/\mathrm{Z}'_{\mathrm{G}}(\mathbb{R})$ is compact.

We call such a (G, X) a *basic pair*. The space X has a natural $\mathrm{G}(\mathbb{R})$-invariant complex structure.

The associated Shimura variety is defined as follows: if $\mathrm{K} \subset \mathrm{G}(\mathbf{A}^f)$ is an open compact subgroup, then

$${}_{\mathrm{K}}\mathrm{Sh}(\mathrm{G}, \mathrm{X})(\mathbb{C}) \underset{\mathrm{def}}{=} \mathrm{G}(\mathbb{Q})\backslash \mathrm{X} \times \mathrm{G}(\mathbf{A}^f)/\mathrm{K}$$

is a (non-connected) quasi-projective complex algebraic variety, as follows from [BB]. Then

$$\mathrm{Sh}(\mathrm{G}, \mathrm{X})(\mathbb{C}) = \underset{\mathrm{K}}{\overset{\lim}{\leftarrow}}\, {}_{\mathrm{K}}\mathrm{Sh}(\mathrm{G}, \mathrm{X})(\mathbb{C})$$

is a pro-algebraic complex variety with continuous $\mathrm{G}(\mathbf{A}^f)$-action. Then Sh(G, X)($\mathbb{C}$) is the set of complex points of the Shimura variety Sh(G, X) associated to (G, X); Sh(G, X) has a *canonical model* over a certain number field E(G, X) (the *reflex field*) *cf.* [D3].

1.1.5. For simplicity, we assume throughout the paper that G^{ad} is $\mathbb{Q}$-simple; however, all substantial assertions are valid as stated for general G. Similarly, (1.1.4) is imposed only for convenience, and the results of the paper have natural generalizations in the absence of this hypothesis, *cf.* ([H2], Remark 4.9.2).

1.2. We recall the construction in [AMRT] of the toroidal compactifications of the connected components of ${}_{\mathrm{K}}\mathrm{Sh}(\mathrm{G}, \mathrm{X})_{\mathbb{C}}$, and return to the adelic setting in 1.7.

Let D be a connected component of X; let $\mathrm{G}(\mathbb{R})^+ \subset \mathrm{G}(\mathbb{R})$ be the group that stabilizes D. For any subgroups $\mathrm{S} \subset \mathrm{G}(\mathbb{R})$ let $\mathrm{S}^+ = \mathrm{S} \cap \mathrm{G}(\mathbb{R})^+$. Let $\mathrm{G}_0 = \mathrm{G}^{\mathrm{der}}(\mathbb{R})^0$. The action of G_0 on D identifies D with the Riemannian symmetric space associated to G_0.

Let $\Gamma \subset \mathrm{G}(\mathbb{Q})^+$ be an arithmetic subgroup, and let $\mathrm{M} = \mathrm{M}_\Gamma$ be the quotient $\Gamma\backslash\mathrm{D}$; every connected component of ${}_{\mathrm{K}}\mathrm{Sh}(\mathrm{G}, \mathrm{X})_{\mathbb{C}}$ is of the form M_Γ for some Γ. We assume for convenience that Γ is neat ([B1], Sec. 17): whenever $\rho: G \to GL(V)$ is a faithful representation, and $\gamma \in \Gamma$, none of the eigenvalues of $\rho(\gamma)$ is a root of unity other than 1. Then Γ is, in particular, torsion-free, so M_Γ is a *smooth* quasi-projective complex variety.

1.2.1. To the pair $(D, G(\mathbb{Q})^+)$ is associated, as in [BB], the collection of *rational boundary components* F. For each F, let $P_F \subset G$ be the maximal $\mathbb{Q}$-parabolic subgroup such that $P_F(\mathbb{R})^+$ stabilizes F. Choose a Levi decomposition $P_F = L_F \cdot W_F$, and let U_F be the center of the unipotent radical W_F. Let $G_l = G_{l,F} \subset L_F$ be the subgroup described in chapter III, paragraph 4 of [AMRT], which can be characterized as the maximal $\mathbb{Q}$-rational connected reductive subgroup of L_F which (i) contains the identity component of Z_G, (ii) acts trivially on F, and (iii) modulo $Z_G \cdot \{\text{finite subgroup}\}$, acts faithfully (by conjugation) on U_F. The logarithm map identifies the abelian U_F with the vector space $\mathfrak{u}_F = \mathrm{Lie}(U_F)$ as group schemes over $\mathbb{Q}$.

1.2.2. Let $\tilde{A}$ be the split component of the center of $G_l \cap G^{der}$, $A = \tilde{A} \cdot Z_G^0$. Then A is a $\mathbb{Q}$-rational torus, and modulo Z_G^0 is one-dimensional and split; $\tilde{A} = A \cap G^{der}$. We let $w_F : \mathbb{G}_m \to A$ be an *admissible Cayley morphism* in the sense of Deligne ([D2], 3.1), *cf.* ([Br2], 4.1; [H2], § 5). It is uniquely determined by the following properties:

(1.2.2.1) $m_F := w_F \cdot (h \circ w)^{-1}$, ($w$ as in 1.1.3), which is actually independent of $h \in X$ ([D3], 2.1.1), maps $\mathbb{G}_m$ to G^{der}.

(1.2.2.2) For any $h \in X$ and any rational representation $\rho: G \to GL(V)$, the pair $(F_h^\bullet, W_\bullet^F)$ defines a mixed Hodge structure on V.

(1.2.2.3) When ρ is the adjoint representation, $W_0^F \mathfrak{g} = \mathrm{Lie}(P_F)$.

In (1.2.2.2-3), $F_h^\bullet$ is the filtration induced by $\rho \circ h$, whereas $W_\bullet^F$ is defined in terms of the weight space decomposition under w_F:

$$W_i^F V = \bigoplus_{j \leq i} V_{j, w_F}, \qquad \text{where} \quad V_{j, w_F} = \{v \in V \mid \rho \circ w_F(t)\, v = t^{-j} v,\ t \in \mathbb{G}_m\}.$$

It follows easily from (1.2.2.2) and (1.1.1) that the weight filtration on $\mathfrak{g}$ is of the form

$$\text{(1.2.2.4)} \qquad \{0\} = W_{-3}^F \mathfrak{g} \subset W_{-2}^F \mathfrak{g} \subset \cdots \subset W_2^F \mathfrak{g} = \mathfrak{g}.$$

We let $\mathfrak{g}^i \subset \mathfrak{g}$ be the t^i-eigenspace of $w_F(t)$, $t \in \mathbb{G}_m$. Then we have

$$\text{(1.2.2.5)} \qquad \mathfrak{g}^{-2} = \mathrm{Lie}(U_F), \qquad \mathfrak{g}^{-2} \oplus \mathfrak{g}^{-1} = \mathrm{Lie}(W_F), \qquad \mathfrak{g}^0 = \mathrm{Lie}(L_F).$$

The following notation will be used throughout. Let $\mathfrak{z} = \mathrm{Lie}(Z_G)$,

$$\mathfrak{g}_l = [\mathfrak{g}^{-2}, \mathfrak{g}^2] \oplus \mathfrak{z} = \mathrm{Lie}(G_l).$$

Let $\mathfrak{g}_t = \mathfrak{g}^{-2} \oplus \mathfrak{g}_l \oplus \mathfrak{g}^2$, and $\mathfrak{g}_h(0)$ = the orthogonal complement of $\mathfrak{g}_l$ in $\mathfrak{g}^0 \cap \mathfrak{g}^{der}$ with respect to the Killing form of $\mathfrak{g}$. Denote by G_t and $G_h(0)$ the corresponding connected subgroups of G; note that $G_h(0)$ contains all compact factors of $L_F(\mathbb{R})$. Let $\mathfrak{g}_h = \mathfrak{g}_h(0) \oplus \mathfrak{a}$, where $\mathfrak{a} = \mathrm{Lie}(A)$, and let G_h be the corresponding connected subgroup of L_F. (Note that, with our notational conventions – which differ a little from those of [AMRT], where "G_h" refers to our $G_h(0)$, modulo its compact factors – G_l and G_h intersect in A; we want G_h to determine the basic pair for a Shimura variety, as in 1.7, below. Note esp. (1.1.3)). When necessary, we write $G_{h,F}$ instead of G_h.

1.2.3. (For details and definitions in what follows, *see* [AMRT], III, § 4.) There is a natural $\mathbb{Q}$-rational positive-definite quadratic form $\langle\, ,\, \rangle$ on U_F. Inside $U_F(\mathbb{R})$ is an open convex cone C_F, self-adjoint with respect to $\langle\, ,\, \rangle$, on which $G_l^0 \underset{\text{def}}{=} G_l(\mathbb{R})^0$ acts transitively. Let $\Gamma_l = \Gamma_{l,F} = \Gamma \cap G_l^0$. Then Γ_l acts freely on C_F. Let $\Gamma_U = U_F(\mathbb{Q}) \cap \Gamma$; Γ_U is a lattice in $U_F(\mathbb{R})$.

Let $P' = P'_F \subset P_F$ denote the largest connected subgroup whose adjoint action on U_F is given by homotheties; thus $P' = G_h \cdot W_F$. Let $D_F = U_F(\mathbb{C}) \cdot \beta(D) \subset \check{M}(\mathbb{C})$, where β is the Borel imbedding of D in its compact dual $\check{M}(\mathbb{C})$, *see* 3.1. Then D has the structure of a Siegel domain of the third kind over F:

$$D \subset D_F \cong \{(z, v, t) | z \in U_F(\mathbb{C}),\ v \in \mathbb{C}^a,\ t \in F\} \tag{1.2.4}$$

is defined by a well-known inequality [*cf.* (2.5.2), below]; here $2a = \dim(W_F/U_F)$. The (transitive) action of $P_F(\mathbb{R})^0$ on D extends to an action of $P_F(\mathbb{R})^0 \cdot U_F(\mathbb{C})$ on D_F. By choosing a base-point for D_F lying on the boundary component opposite to F [*see* our (1.8)], one finds G_l in the corresponding isotropy subgroup, which implies that D_F is homogeneous under $P'(\mathbb{R}) \cdot U_F(\mathbb{C})$. Let $\Gamma'_F \underset{\text{def}}{=} \Gamma \cap P'(\mathbb{Q})$, $M'_F = \Gamma'_F \backslash D_F$, and let T_F be the $\mathbb{Q}$-split torus with character group $X(T_F) = \mathrm{Hom}(\Gamma_U, \mathbb{Z})$. Thus $T_F(\mathbb{C}) \cong U_F(\mathbb{C})/\Gamma_U$ acts holomorphically on M'_F, and Γ_l acts on T_F.

It was proved by Brylinski ([Br1]; *cf.* [P], § 10), that M'_F has the structure of an algebraic variety. More precisely, M'_F fits into a tower of fibrations, corresponding to (1.2.4):

$$\begin{array}{ccc} D_F & \twoheadrightarrow & M'_F \\ \downarrow & & \downarrow \pi_2 \\ D_F/U_F(\mathbb{C}) & \twoheadrightarrow & A_F \\ \downarrow & & \downarrow \pi_1 \\ F & \twoheadrightarrow & M_F \end{array} \tag{1.2.5}$$

where

(*a*) M_F is of the form $\Gamma_F \backslash F$, for some neat arithmetic group Γ_F, and in particular is a smooth quasi-projective algebraic variety;

(*b*) π_1 represents A_F as an abelian scheme over M_F;

(*c*) π_2 is a principal algebraic torus fibration over A_F, with structure group T_F.

The discrete group $\Gamma_{l,F}$ acts on both M'_F and A_F over M_F. The action on M'_F is proper and discontinuous in the classical topology, whereas the action on A_F does not admit a nice quotient (unless $A_F = M_F$).

1.3. Let $\sigma \subset \mathrm{U_F}(\mathbb{R})$ be a closed rational polyhedral cone (rpc); *i.e.*, a subset of the form $\left\{\sum_{i=1}^{a} \lambda_i v_i \,\middle|\, \lambda_i \geqq 0\right\}$, where $v_i \in \mathrm{U_F}(\mathbb{Q})$, $i = 1, \ldots, a$. The dual space $\mathrm{X}(\mathrm{T_F}) \otimes \mathbb{R} \cong \mathrm{Hom}(\mathrm{U_F}(\mathbb{R}), \mathbb{R})$ contains the dual cone

$$\check{\sigma} = \{\lambda \in \mathrm{X}(\mathrm{T_F}) \otimes \mathbb{R} | \lambda(v) \geqq 0 \,\forall v \in \sigma\}.$$

Suppose σ contains no non-trivial linear subspace of $\mathrm{U_F}(\mathbb{R})$. As in ([KKMS], I, § 1), let $\mathrm{T}_\sigma = \mathrm{Spec}\,\mathbb{Q}[\mathrm{X}(\mathrm{T_F}) \cap \check{\sigma}]$; then $\mathrm{T_F}$ embeds naturally in T_σ, and the action of T on itself extends to an action on T_σ. In this way one defines a 1-1 correspondence between rpc's $\sigma \subset \mathrm{U_F}(\mathbb{R})$ and normal equivariant affine embeddings $\mathrm{T_F} \hookrightarrow \mathrm{T}_\sigma$. One puts $\partial \mathrm{T}_\sigma = \mathrm{T}_\sigma - \mathrm{T}$. If $\sigma = \bigcup \sigma'$ is a finite simplicial decomposition, then the $\mathrm{T}_{\sigma'}$ patch together to an equivariant torus embedding $\mathrm{T}_{\{\sigma'\}}$, and the natural maps $\mathrm{T}_{\sigma'} \to \mathrm{T}_\sigma$ patch together to a proper surjective T-equivariant morphism $\mathrm{T}_{\{\sigma'\}} \to \mathrm{T}_\sigma$.

More generally, define a *fan* (rppd in [AMRT]) in $\mathrm{U_F}(\mathbb{R})$, as in [O], [Mi]: if Σ is a fan, then each $\sigma \in \Sigma$ is an rpc in the closure $\bar{\mathrm{C}}_\mathrm{F}$ of $\mathrm{C_F}$, and the intersections of the different σ satisfy certain natural axioms. One obtains thereby an equivariant torus embedding T_Σ, constructed by patching together the T_σ for $\sigma \in \Sigma$; we write $\partial \mathrm{T}_\Sigma = \mathrm{T}_\Sigma - \mathrm{T}$.

Given a fan Σ, there are two distinguished subfans:

$$\Sigma^o = \{\sigma \in \Sigma | \sigma \cap \mathrm{C_F} \neq \varnothing\}, \tag{1.3.1}$$

$$\Sigma^c = \{\sigma \in \Sigma | \sigma - \{0\} \subset \mathrm{C_F}\}; \tag{1.3.2}$$

the latter is a subcomplex.

For any rpc $\sigma \subset \mathrm{U_F}(\mathbb{R})$, let

$$(\mathrm{M}'_\mathrm{F})_\sigma = \mathrm{M}'_\mathrm{F} \times {}^{\mathrm{T_F}}\mathrm{T}_\sigma, \tag{1.3.3}$$

and let $\mathrm{D}_{\mathrm{F},\sigma}$ denote the interior of the closure in $(\mathrm{M}'_\mathrm{F})_\sigma$ of $\Gamma'_\mathrm{F}\backslash \mathrm{D}$. We let

$$\pi_{2,\sigma} : \quad \mathrm{D}_{\mathrm{F},\sigma} \to \mathrm{A_F}$$

be the natural projection. If $\sigma = \bigcup \sigma'$ is a decomposition as above, define $(\mathrm{M}'_\mathrm{F})_{\{\sigma'\}}$ and $\mathrm{D}_{\mathrm{F},\{\sigma'\}}$ analogously; then there is a proper morphism $\mathrm{D}_{\mathrm{F},\{\sigma'\}} \to \mathrm{D}_{\mathrm{F},\sigma}$ of analytic spaces over $\mathrm{A_F}$. Finally, if Σ is a fan, then we define $(\mathrm{M}'_\mathrm{F})_\Sigma = \mathrm{M}'_\mathrm{F} \times {}^{\mathrm{T_F}}\mathrm{T}_\Sigma$ and $\mathrm{D}_{\mathrm{F},\Sigma}$ in analogy with the above. The morphism $\pi_2 : \mathrm{M}'_\mathrm{F} \to \mathrm{A_F}$ extends to a morphism $\pi_{2,\Sigma} : (\mathrm{M}'_\mathrm{F})_\Sigma \to \mathrm{A_F}$. Let

$$\pi_{\mathrm{F},\Sigma} = \pi_1 \circ \pi_{2,\Sigma} : (\mathrm{M}'_\mathrm{F})_\Sigma \to \mathrm{M_F}. \tag{1.3.4}$$

The pullbacks to $\mathrm{D}_{\mathrm{F},\Sigma}$ of (the analytic morphisms corresponding to) $\pi_{2,\Sigma}$ and $\pi_{\mathrm{F},\Sigma}$ will be denoted by the same symbols.

It will be useful to consider the following slight generalization of torus embeddings. Let T_1 and T_2 be split algebraic tori over the field k with cocharacter groups

$X_i = X_*(T_i) = \mathrm{Hom}(\mathbb{G}_m, T_i)$, $i=1, 2$. Let $\Sigma_i \subset X_i \otimes \mathbb{R}$ be fans. A *morphism* $h:(T_1, \Sigma_1) \to (T_2, \Sigma_2)$ is a homomorphism $h: T_1 \to T_2$ of algebraic groups such that the induced map $h_*: X_1 \to X_2$ has the property that, for each $\sigma_1 \in \Sigma_1$, there exists $\sigma_2 \in \Sigma_2$ such that $h_*(\sigma_1) \subset \sigma_2$ (*cf.* [O], § 1.5). This condition easily implies that the homomorphism h extends to a T_1-equivariant morphism

$$h_{\sigma_1, \sigma_2}: \quad T_{1, \sigma_1} \to T_{2, \sigma_2}.$$

These maps patch together to define a T_1-equivariant map of torus embeddings (*cf.* [H3], § 3; P, 5.4)

$$h_{\Sigma_1, \Sigma_2}: \quad T_{1, \Sigma_1} \to T_{2, \Sigma_2}.$$

When $T_2 = \{1\}$, any (T_1, Σ_1) gives a morphism trivially. When $T_1 = T_2$, we obtain the notion of *refinement* (*cf.* 1.4, below).

Until the end of 1.3, we assume Σ_1 and Σ_2 to be *finite* complexes; the generalization to locally finite torus embeddings is easy.

1.3.5. Lemma. – *The morphism h_{Σ_1, Σ_2} is proper if and only if, for all $\sigma \in \Sigma_2$, the set $|\Sigma_1| := \{\bigcup \sigma' \in \Sigma_1 | h_*(\sigma') \subset \sigma\}$ equals $h_*^{-1}(\sigma)$.*

Proof. – This is proved in the same way as Theorem 8 of ([KKMS], § 2).

If the hypothesis of the lemma is satisfied, we call the morphism $h:(T_1, \Sigma_1) \to (T_2, \Sigma_2)$ *proper.*

Write $Y_i = T_{i, \Sigma_i} \cdot \mathcal{O}_i = \mathcal{O}_{Y_i} \cdot \partial_i = \partial Y_i$ with reduced scheme structure $i=1, 2$.

(N.B.: one need not have that the support of $(h_{\Sigma_1, \Sigma_2})^* \partial_2$ equals ∂_1, nor, when one does have it, that $(h_{\Sigma_1, \Sigma_2})^* \partial_2$ is reduced, *i.e.* equals ∂_1.)

Assume Y_i smooth, and ∂_i is a divisor with normal crossings on Y_i, $i=1, 2$. Let $\tilde{\partial}_1$ denote the union of the irreducible components of ∂_1 whose image is contained in ∂_2. Let $\mathbb{K}_1 = \mathcal{O}_1(-\tilde{\partial}_1)$, $\mathbb{K}_2 = \mathcal{O}_2(-\partial_2)$.

1.3.6. Lemma. – *Suppose $h:(T_1, \Sigma_1) \to (T_2, \Sigma_2)$ is proper, with the map of tori surjective. Then (a) $R^i h_{\Sigma_1, \Sigma_2, *}(\mathcal{O}_1) = R^i h_{\Sigma_1, \Sigma_2, *}(\mathbb{K}_1) = 0$ for $i>0$; (b) $h_{\Sigma_1, \Sigma_2, *}(\mathcal{O}_1) \cong \mathcal{O}_2$ and $h_{\Sigma_1, \Sigma_2, *}(\mathbb{K}_1) \cong \mathbb{K}_2$, canonically.*

Proof. – The assertions are local on Y_2. Thus we may assume Y_2 affine, *i.e.*, $\Sigma_2 = \sigma$ is a single cone. We do this throughout. There are canonical adjunction morphisms $\mathcal{O}_2 \to h_{\Sigma_1, \Sigma_2, *}(\mathcal{O}_1)$, $\mathbb{K}_2 \to h_{\Sigma_1, \Sigma_2, *}(\mathbb{K}_1)$. The second part of the lemma is the statement that these morphisms are isomorphisms, *i.e.*

(1.3.6.1) The natural adjunction maps

$$H^0(Y_2, \mathcal{O}_2) \to H^0(Y_1, \mathcal{O}_1) \qquad \text{and} \qquad H^0(Y_2, \mathbb{K}_2) \to H^0(Y_1, \mathbb{K}_1)$$

are isomorphisms.

Similarly, the first part of the lemma is the statement that

$$H^1(Y_1, \mathcal{O}_1) = H^1(Y_1, \mathbb{K}_1) = 0 \qquad \text{for} \quad i > 0. \tag{1.3.6.2}$$

The vanishing of $H^i(Y_1, \mathcal{O}_1)$ follows directly from ([KKMS], p. 44, Cor. 2), by the hypothesis of properness, since $h_*^{-1}(\sigma)$ is in particular convex (*cf.* 1.3.5). Let $\mathcal{S}_1 = \mathcal{O}_{\tilde{\partial}_1}$, so that we have the short exact sequence

(1.3.6.3) $$0 \to \mathbb{K}_1 \to \mathcal{O}_1 \to \mathcal{S}_1 \to 0.$$

It is enough to show that

(1.3.6.4) $$H^i(Y_1, \mathcal{S}_1) = 0 \quad \text{for } i > 0;$$

(1.3.6.5) $$H^0(Y_1, \mathcal{O}_1) \to H^0(Y_1, \mathcal{S}_1) \quad \text{is surjective;}$$

The sheaf $\mathcal{S}_1$ is supported on the closed divisor $\tilde{\partial}_1$. Write $\tilde{\partial}_1 = \bigcup \partial_\alpha$ as the union of its irreducible components; then the ∂_α are in one-to one correspondence with the set of one-dimensional cones $\sigma_\alpha \in \Sigma_1$ such that $h_*(\sigma_\alpha) \neq 0$. More generally, for any $\tau \in \Sigma_1$, let ∂_τ be the closure of the corresponding T_1-orbit in Y_1, and let $i_\tau : \partial_\tau \to Y_1$ be its canonical closed immersion. Define

(1.3.6.6) $$\hat{\tau} = \tau - \{0\}//\mathbb{R}_+^\times, \qquad \hat{\Sigma}_1 = \{\hat{\tau} | \tau \in \Sigma_1\}.$$

Then $\hat{\Sigma}_1$ is a polyhedral complex. Let

(1.3.6.7) $${}_h\hat{\Sigma}_1 = \{\hat{\tau} | \tau \in {}_h\Sigma_1\},$$

where ${}_h\Sigma_1 = \{\tau \in \Sigma_1 | \tau \cap \mathrm{Ker}(h_*) = \{0\}\}$. The normal crossings hypothesis means that $\hat{\Sigma}_1$ is actually a *simplicial* complex; *i.e.*, every polyhedron in $\hat{\Sigma}_1$ of dimension k has exactly $k+1$ vertices.

We compute $H^i(Y_1, \mathcal{S}_1) = H^i(\tilde{\partial}_1, \mathcal{S}_1)$ using the spectral sequence for the closed covering by the ∂_α. The nerve $\mathcal{N}$ of this closed covering is evidently PL-homeomorphic to ${}_h\hat{\Sigma}_1$. Now for all $\tau \in {}_h\Sigma_1$,

(1.3.6.8) $$H^i(\partial_\tau, i_\tau^*(\mathcal{S}_1)) \cong H^i(\partial_\tau, \mathcal{O}_{\partial_\tau}) = 0 \qquad \text{for} \quad i > 0$$

([KKMS], p. 44, Cor. 2).

Define systems of coefficients $L(\cdot, \mathcal{S}_1)$ on $\mathcal{N} \cong {}_h\hat{\Sigma}_1$ by the formula

$$L(\hat{\tau}, \mathcal{S}_1) = H^0(\partial_\tau, \mathcal{O}_{\partial_\tau}).$$

By (1.3.6.8), the spectral sequence degenerates at E_1 and yields

(1.3.6.9) $$H^i(Y_1, \mathcal{S}_1) \cong H^i({}_h\hat{\Sigma}_1, L(\cdot, \mathcal{S}_1)), \qquad i = 0, 1, \ldots$$

We compute the terms $L(\hat{\tau}, \mathcal{S}_1)$ by direct calculation, decomposing it into weight spaces $L(\hat{\tau}, \chi)$ for the action of T_1, where χ runs through the character group X_1^* of T_1. For any $\chi \in X_1^*$, let

$$R(\chi) = \{x \in X_1 \otimes \mathbb{R} | \chi(x) \geqq 0\}; \qquad H(\chi) = \operatorname{Ker}(\chi) \subset X_1 \otimes \mathbb{R}.$$

Recalling that ∂_τ is taken to be a closed T_1-invariant subset, it follows from the definitions of affine torus embeddings (*cf.* also 2.4.3, below) that

(1.3.6.10) $H^0(\partial_\tau, \mathcal{O}_{\partial_\tau})^\chi = k \cdot \chi$ (canonically) if $\tau \subset H(\chi)$ and $\tau' \subset R(\chi)$ for every τ' containing τ as a face; otherwise $H^0(\partial_\tau, \mathcal{O}_{\partial_\tau})^\chi \cong 0$.

We determine the support $A(\chi)$ of $L(\cdot, \chi)$. Recall from 1.3.5 that $|\Sigma_1| = h_*^{-1}(\sigma)$. Thus $|\Sigma_1| = \bigcap R(\eta_i)$, where the η_i in the character group X_2^* of T_2, now identified as a subgroup of X_2^* via h^*, define the boundary hyperplanes of σ. It is easy to see that if τ as in (1.3.6.10) exists, *i.e.* with $A(\chi) \neq \emptyset$, then (with additive notation in X_2^*):

(1.3.6.11) If $\chi = \Sigma c_i \cdot \eta_i \in X_2^*$ with $c_i \geqq 0$, then

$$A(\chi) = \{|\Sigma_1| \cap \bigcap_{c_i \text{ positive}} H(\eta_i) - \operatorname{Ker}(h_*)\}/\mathbb{R}_+^\times.$$

In every case $A(\chi)$ is either empty or contractible, and (1.3.6.4) thus follows from (1.3.6.9). The proofs of (1.3.6.5) and (1.3.6.1) are similar and are omitted.

1.3.7. *Remark.* – The above argument also shows that $H^i(Y_1, \mathbb{K}_1)$ does not change (up to canonical isomorphism) if Σ_1 is replaced by a refinement.

1.4. Toroidal compactifications. – A toroidal compactification $M_\Gamma \hookrightarrow M_{\Gamma, \Sigma}$ is associated to a collection $\Sigma = \{\Sigma_F | F \text{ a rational boundary component}\}$ where each Σ_F is a fan in $U_F(\mathbb{R})$, satisfying a list of hypotheses to ensure compatibility (*cf.* [AMRT], p. 252), which we recall as we need them. We refer to such Σ as a Γ-*admissible family of fans*. The Σ's are partially ordered by the relation of *refinement*: Σ' is a refinement of Σ if every $\sigma' \in \Sigma'$ is contained in some $\sigma \in \Sigma$, such that, for each $\sigma \in \Sigma$, the set $\{\sigma' \in \Sigma' | \sigma' \subset \sigma\}$ is a finite simplicial decomposition of σ. The spaces $M_{\Gamma, \Sigma}$ are in general only algebraic spaces over $\mathbb{C}$. Let $\partial M_{\Gamma, \Sigma} = M_{\Gamma, \Sigma} - M_\Gamma$.

For a complete picture of the structure of $M_{\Gamma, \Sigma}$, we refer the reader to ([AMRT], Ch. III, § 5). For our present purposes, it suffices to note the following:

(1.4.1) For each F and each $\sigma \in \Sigma_F$, the natural map $\Gamma'_F \backslash D \to M_\Gamma$ extends to an analytic local isomorphism $\varphi_{F, \sigma} : D_{F, \sigma} \to M_{\Gamma, \Sigma}$; these patch to yield a local isomorphism $\varphi_{F, \Sigma} : D_{F, \Sigma} \to M_{\Gamma, \Sigma}$.

(1.4.2) The union of the images of the $\varphi_{F, \sigma}$ form an open covering of $M_{\Gamma, \Sigma}$.

(1.4.3) Any Σ has a refinement Σ' such that the toroidal compactification $M_{\Gamma, \Sigma'}$ is smooth and projective, and such that $\partial M_{\Gamma, \Sigma'}$ is a divisor with normal crossings, each of whose irreducible components is smooth. Such a compactification will be called *SNC*.

1.4.4. *Remark.* – A toroidal compactification (or torus embedding) with the normal crossings property mentioned in (1.4.3) necessarily has the property that every σ in Σ is *simplicial*, in the sense that the number of its 1-dimensional faces equals its dimension.

If Σ' is a refinement of Σ, then there is a natural proper surjective morphism $\pi_{\Sigma',\Sigma} : M_{\Gamma,\Sigma'} \to M_{\Gamma,\Sigma}$, consistent with (1.4.1) and (1.4.2).

1.5. STRUCTURE OF THE BOUNDARY. – The variety M_Γ also has a natural compactification as a normal projective variety. This compactification, due to Satake and Baily-Borel ([S1], [BB]), will be called the *minimal compactification*, and denoted M_Γ^*. For any Γ-admissible Σ, there is a unique proper morphism of algebraic spaces

$$\pi_\Sigma : \quad M_{\Gamma,\Sigma} \to M_\Gamma^*$$

which restricts to the identity on M_Γ.

The complement $\partial M_\Gamma^* := M_\Gamma^* - M_\Gamma$ decomposes as the disjoint union of locally closed subvarieties $\amalg \tilde{M}_F$, where the union is taken over the set of rational boundary components F modulo the action of Γ. For each F, there is a commutative diagram:

$$\text{(1.5.1)} \qquad \begin{array}{ccccc} D_{F,\Sigma_F} - D & \overset{\varphi_{F,\Sigma}}{\to} & \partial M_{\Gamma,\Sigma} & \subset & M_{\Gamma,\Sigma} \\ \downarrow & & \downarrow & & \downarrow \pi_\Sigma \\ M_F & \overset{\sim}{\to} & \tilde{M}_F \subset \partial M_\Gamma^* & \subset & M_\Gamma^* \end{array}$$

Let $Z_{F,\Sigma}$ denote the closure in $M_{\Gamma,\Sigma}$ of $\pi_\Sigma^{-1}(\tilde{M}_F)$.

By refining Σ, we can arrange that, for each F, (i) $Z_{F,\Sigma}$ is a subdivisor with normal crossings of $\partial M_{\Gamma,\Sigma}$ with

$$\partial M_{\Gamma,\Sigma} = \bigcup_F Z_{F,\Sigma},$$

and (ii) the irreducible components $\bar{Z}_\sigma$ of $Z_{F,\Sigma}$ are smooth and in one-to-one correspondence with the set of 1-dimensional cones σ in Σ_F^c, modulo the action of Γ_l. In this case, we say $M_{\Gamma,\Sigma}$ is *SNC*. The same terminology will be used without comment for torus embeddings (here Γ_l is of course irrelevant) and for partial toroidal compactifications. *We assume henceforward that* $M_{\Gamma,\Sigma}$ *is SNC.*

Whenever F′ is a rational boundary component of containing F in its closure (one writes $F' \geqq F$), then $C_{F'}$ is a boundary component of C_F, and one of the conditions on Σ requires that $\Sigma_F \cap C_{F'} = \Sigma_{F'}$. This provides the means for determining $Z_{F,\Sigma} \cap Z_{F',\Sigma}$ from the data in Σ_F, when $F' > F$. Define

$$\text{(1.5.2)} \qquad {}^{>}Z_{F,\Sigma} = Z_{F,\Sigma} - \bigcup_{F' > F \,(\mathrm{mod}\,\Gamma)} Z_{F',\Sigma}, \quad {}^{>}\partial Z_{F,\Sigma} = Z_{F,\Sigma} - {}^{>}Z_{F,\Sigma};$$

$${}^{<}Z_{F,\Sigma} = Z_{F,\Sigma} - \bigcup_{F' < F \,(\mathrm{mod}\,\Gamma)} Z_{F',\Sigma}, \quad {}^{<}\partial Z_{F,\Sigma} = Z_{F,\Sigma} - {}^{<}Z_{F,\Sigma};$$

$${}^{0}Z_{F,\Sigma} = {}^{>}Z_{F,\Sigma} \cap {}^{<}Z_{F,\Sigma}; \qquad \partial Z_{F,\Sigma} = Z_{F,\Sigma} - {}^{0}Z_{F,\Sigma} = {}^{>}\partial Z_{F,\Sigma} \cup {}^{<}\partial Z_{F,\Sigma}.$$

Then ${}^{>}\partial Z_{F,\Sigma}$, ${}^{<}\partial Z_{F,\Sigma}$, and $\partial Z_{F,\Sigma}$ are all divisors with normal crossings on $Z_{F,\Sigma}$.

Said in words, $Z_{F,\Sigma}$ is the union of the complete divisors corresponding to the one-dimensional cones in Σ_F^c. To get ${}^{<}Z_{F,\Sigma}$ (the *F-stratum* of $\partial M_{\Gamma,\Sigma}$), one removes from $Z_{F,\Sigma}$

all points coming from Σ_F^o but not Σ_F^c, which thus are really associated to smaller boundary components; these are precisely the points that map to the boundary of $\tilde{M}_F$ under π_Σ. To produce ${}^{>}Z_{F,\Sigma}$, one removes from $Z_{F,\Sigma}$ those points that serve to compactify divisors associated to larger boundary components (these still map to $\tilde{M}_F$).

For any cone $\tau \in \Sigma_F$, write

$$\overline{Z}_\tau = \bigcap_{\sigma \subset \tau,\, \dim \sigma = 1} \overline{Z}_\sigma.$$

Via (1.3.3) and (1.4.1), it comes from the (closed) T-orbit in T_τ associated to τ.

Note that ${}^{<}Z_{F,\Sigma} = \pi_\Sigma^{-1}(\tilde{M}_F)$ and that $Z_{F,\Sigma}$ is the closure in $M_{\Gamma,\Sigma}$ of ${}^{0}Z_{F,\Sigma}$. We note the following description of ${}^{<}Z_{F,\Sigma}$:

1.5.3. LEMMA. – *Let $\overline{\partial T_{F,\Sigma_F^c}}$ denote the closure of $\partial T_{F,\Sigma_F^c}$ in $\partial T_{F,\Sigma_F}$. Let*

$${}^{<}\tilde{Z}_{F,\Sigma} = M'_F \times^{T_F} \overline{\partial T_{F,\Sigma_F^c}}, \quad {}^{0}\tilde{Z}_{F,\Sigma} - M'_F \times^{T_F} \partial T_{F,\Sigma_F^c}.$$

Then

$${}^{<}Z_{F,\Sigma} \cong \Gamma_l \backslash {}^{<}\tilde{Z}_{F,\Sigma}, \qquad {}^{0}Z_{F,\Sigma} \cong \Gamma_l \backslash {}^{0}\tilde{Z}_{F,\Sigma} \quad \textit{as algebraic varieties.}$$

Proof. – Everything is clear but the algebraicity, which follows from ([P], Prop. 9.36).

Let $\delta_{F,\Sigma} : {}^{<}\tilde{Z}_{F,\Sigma} \to {}^{<}Z_{F,\Sigma}$ be the projection determined by the Lemma. For any cone $\tau \in \Sigma_F$ let $\tilde{Z}_\tau \subset {}^{<}\tilde{Z}_{F,\Sigma}$ be the stratum corresponding to τ. Then assuming Σ is sufficiently fine or Γ is sufficiently small, which we do, $\delta_{F,\Sigma}$ restricts to an algebraic isomorphism $\tilde{Z}_\tau \xrightarrow{\sim} Z_\tau \coloneqq \bar{Z}_\tau \cap {}^{<}Z_{F,\Sigma}$. Define

$$\psi_\tau = \pi_{2,\Sigma} \circ \delta_{F,\Sigma}^{-1} : Z_\tau \to A_F.$$

1.5.4. COROLLARY. – For each τ, the morphism ψ_τ defined above is algebraic.

1.6. STRUCTURE OF THE BOUNDARY (CONTINUED). – In order to extend Lemma 1.5.3 to something directly related to the boundary stratum $Z_{F,\Sigma}$ attached to F, we must take a detour through the theory of *mixed Shimura varieties* ([Mi], P). These are attached to pairs (Q, $\mathcal{X}$), where Q is an algebraic group with a three step-filtration by normal subgroups

$$\{1\} \subset W_{-2}Q \subset W_{-1}Q = R_u Q \subset W_0 Q = Q, \tag{1.6.1}$$

and $\mathcal{X}$ is a homogeneous space for $Q(\mathbb{R}) \cdot W_{-2}Q(\mathbb{C})$. It is assumed that $W_{-2}Q$ is commutative, and for any arithmetic subgroup $\Gamma \subset Q$ the quotient

$$T_\Gamma(Q, \mathcal{X}) := \Gamma \cap W_{-2}Q(\mathbb{Q}) \backslash W_{-2}Q(\mathbb{C})$$

is viewed as the set of complex points of the split torus with character group $\mathrm{Hom}(\Gamma \cap W_{-2}Q, \mathbb{Z})$. The remaining axioms satisfied by the pair (Q, $\mathcal{X}$) are listed in ([P], Definition 2.1). The mixed Shimura variety is denoted $\mathrm{Sh}(Q, \mathcal{X})$; its connected components at finite level are of the form $\mathcal{X}_\Gamma = \Gamma \backslash \mathcal{X}^+$, with $\mathcal{X}^+$ a connected component of $\mathcal{X}$. In what follows, by "connected component" we mean "connected component at finite level".

Now, the spaces on the right-hand side of diagram (1.2.5) are all (connected components of) mixed Shimura varieties: M'_F is attached to the pair $(P', X(D_F))$, with $W_{-2}P'=U$, $W_{-1}P'=W$, and $X(D_F)$ is a certain $P'(\mathbb{R})\cdot U_F(\mathbb{C})$-homogeneous space containing D_F as a connected component; A_F is attached to the pair $(P'/U, X(D_F)/U_F(\mathbb{C}))$, with filtration induced from that on P'; and M_F is the (pure) Shimura variety attached to $(G_h, X(F))$, where $X(F)$ will be discussed in 1.7, below.

More generally, Pink defines ([P], § 4) a set of rational boundary components $(Q(\mathcal{R}), \mathcal{X}(\mathcal{R}))$ of $(Q, \mathcal{X})$, consisting of a "canonical subgroup" $Q(\mathcal{R})$ of an "admissible parabolic subgroup" of Q, and a space $\mathcal{X}(\mathcal{R})$, such that $(Q(\mathcal{R}), \mathcal{X}(\mathcal{R}))$ is a pair defining a mixed Shimura variety, and such that

$$(1.6.2)\qquad W_{-2}Q(\mathcal{R})\supset W_{-2}Q,\qquad T_\Gamma(Q,\mathcal{X})\subset T_\Gamma(Q(\mathcal{R}),\mathcal{X}(\mathcal{R})).$$

For each $\mathcal{R}$ there is also an open cone $C(\mathcal{R})\subset W_{-2}Q(\mathcal{R})(\mathbb{R})$, which is invariant under translation by the vector space $(W_{-2}Q(\mathcal{R})\cap W_{-1}Q)(\mathbb{R})$. He then constructs partial toroidal compactifications, associated to families of fans in the various $C(\mathcal{R})$, just as in 1.4. A rational polyhedral cone $\sigma\subset W_{-2}Q(\mathcal{R})(\mathbb{R})$ defines a torus embedding $T_\Gamma(Q(\mathcal{R}), \mathcal{X}(\mathcal{R}))_\sigma$ for any arithmetic subgroup $\Gamma\subset Q$, and the connected component $\Gamma\backslash\mathcal{X}(\mathcal{R})^+$ is a principal $T_\Gamma(Q(\mathcal{R}), \mathcal{X}(\mathcal{R}))$-bundle over a connected component of the mixed Shimura variety attached to $(Q(\mathcal{R})/W_{-2}Q(\mathcal{R}), \mathcal{X}(\mathcal{R})/W_{-2}Q(\mathcal{R})(\mathbb{C}))$; one defines $(\Gamma\backslash\mathcal{X}(\mathcal{R})^+)_\sigma$ as in the pure case.

Now consider the case $(Q, \mathcal{X})=(P', X(D_F))$ of a rational boundary component of (G, X). Pink shows ([P], Cor. 4.20) that each rational boundary component of $(P', X(D_F))$ is of the form $(P'_1, X(D_{F_1}))$, where F_1 is a rational boundary component of F and P_1 is its normalizer in G. Furthermore, he shows ([P], Corollary 7.17) that the closed divisor $\bar{Z}_\sigma\subset Z_1^*$ (notation from 1.5) is the toroidal compactification, attached to an explicitly determined family of fans, of (a connected component of) the mixed Shimura variety attached to the pair $(P'/U(\langle\sigma\rangle), D_F/U(\langle\sigma\rangle))$, where $U(\langle\sigma\rangle)\subset U_F$ is the one-dimensional subgroup whose Lie algebra is spanned by σ.

1.6.3. *Examples.*

(*a*) The rational boundary components $(Q(\mathcal{R}_\alpha), \mathcal{X}(\mathcal{R}_\alpha))$ of M'_F correspond exactly to the rational boundary components F_α of F. Then the torus $T_\Gamma(Q(\mathcal{R}_\alpha), \mathcal{X}(\mathcal{R}_\alpha))$ is the corresponding T_{F_α}, which contains T_F [as in (1.6.2)], and the cone $C(\mathcal{R}_\alpha)=C_{F_\alpha}$.

(*b*) The rational boundary components of A_F, which we denote $(Q^A(\mathcal{R}_\alpha), \mathcal{X}^A(\mathcal{R}_\alpha))$, correspond as in (*a*) to the rational boundary components F_α of F, but now the corresponding torus $T_\Gamma(Q^A(\mathcal{R}_\alpha), \mathcal{X}^A(\mathcal{R}_\alpha))$ equals T_{F_α}/T_F. Denote the corresponding open cone $C^A_\alpha\subset W_{-2}(Q(\mathcal{R}_\alpha))/W_{-2}(Q)$, and let $\varphi_{2,\alpha}: C_{F_\alpha}\to C^A_\alpha$ be the natural map.

(*c*) Finally, we denote by $T_{F_\alpha,F}$ the torus attached to the boundary component F_α of F for the pure Shimura variety $M_F=\Gamma_F\backslash F$, and $C_{F_\alpha,F}$ the corresponding homogeneous cone. Then there are natural homomorphisms $T_\Gamma(Q^A(\mathcal{R}_\alpha), \mathcal{X}^A(\mathcal{R}_\alpha))\to T_{F_\alpha,F}$, inducing maps $\varphi_{1,\alpha}: C^A_\alpha\to C_{F_\alpha,F}$.

Let $M'_{F,\Sigma}$ be the toroidal embedding of M_F, in the sense of [P], associated to the partial cone decomposition inherited from Σ. Then $M'_{F,\Sigma}$ is SNC and contains as an open subset $(M'_F)_{\Sigma_F}$ (notation 1.3). Let $\tilde{Z}_{F,\Sigma}$ denote the closure in $M'_{F,\Sigma}$ of ${}^0Z_{F,\Sigma}$. If Γ is sufficiently small, there is a natural isomorphism, extending those of Lemma 1.5.3 (*cf.* [P], 7.13, 7.17):

$$\Gamma_l \backslash \tilde{Z}_{F,\Sigma} \xrightarrow{\sim} Z_{F,\Sigma}. \tag{1.6.4}$$

We assume henceforward that Γ is chosen sufficiently small so that (1.6.4) holds; this is always possible (*cf.* [*loc. cit.*]; alternatively, we could replace Σ by an appropriate refinement).

According to ([P], Prop. 6.25 (*b*)), after replacing Σ by a suitable refinement, we can construct complete SNC toroidal compactifications $A_{F,\Xi}$ and $M_{F,\Xi}$ of A_F and M_F, respectively, such that the morphism π_1 of (1.2.5) extends to a Γ_l-equivariant morphism $\bar{\pi}_1 : A_{F,\Xi} \to M_{F,\Xi}$. To be more precise, one constructs families of fans $\{\Xi_\alpha \subset C_{F_\alpha, F}\}$ and $\{\Xi^A_\alpha \subset C^A_\alpha\}$, with the property that, for all $\sigma \in \Xi^A_\alpha$, there exists $\xi \in \Xi_\alpha$ such that $\varphi_{1,\alpha}(\sigma) \subset \xi$, and such that $A_{F,\Xi} := A_{F,\Xi^A}$ (for simplicity) and $M_{F,\Xi}$ are SNC.

1.6.5. LEMMA. – *When Σ is sufficiently fine, the morphism π_2 of (1.2.5) extends to a Γ_l-equivariant morphism $\bar{\pi}_2 : M'_{F,\Sigma} \to A_{F,\Xi}$. Furthermore, given $A_{F,\Xi}$ and $M_{F,\Xi}$ as above for each rational boundary component F (modulo the action of Γ), Σ can be chosen to work for all F's simultaneously.*

Proof. – This is essentially due to Pink. We first consider a fixed F, and let $\{\mathcal{S}_{[\Sigma_F],\alpha} \subset C_{F_\alpha}\}$ denote the induced family of fans, defined by analogy with ([P], 7.7):

$$\mathcal{S}_{[\Sigma_F],\alpha} = \{\tau \in \Sigma_{F_\alpha} | \tau \text{ has a face in } \Sigma_F\}. \tag{1.6.6}$$

The one-dimensional cones in $\mathcal{S}_{[\Sigma_F],\alpha}$ correspond to the divisors in $Z_{F_\alpha,\Sigma}$ that have non-empty intersections with $Z_{F,\Sigma}$. Let $\tilde{\Sigma} = \{\tilde{\Sigma}_\alpha\}$ be a refinement of $\{\mathcal{S}_{[\Sigma_F],\alpha}\}$ such that:

(1.6.7) For all $\tau \in \tilde{\Sigma}_\alpha$, there exists $\sigma \in \Xi^A_\alpha$ such that $\varphi_{2,\alpha}(\tau) \subset \sigma$. Then we can define the morphism $\bar{\pi}_2$, as in ([P], 6.25).

Since (1.6.7) is stable under refinement, we can find a global refinement $\tilde{\Sigma}$ of Σ such that $\mathcal{S}_{[\tilde{\Sigma}_F],\alpha}$ satisfies (1.6.7) for all F and all α. Replacing Σ by such a $\tilde{\Sigma}$, we obtain the lemma.

Henceforward, we assume Σ to be sufficiently fine in the sense of the lemma.

Let ${}^>\partial\tilde{Z}_{F,\Sigma(\Xi)}$ denote the inverse image under (1.6.7) of ${}^>\partial Z_{F,\Sigma}$; it is the closed subvariety corresponding to cones in the boundary of C_F. Define ${}^<\partial\tilde{Z}_{F,\Sigma(\Xi)}$ and $\partial\tilde{Z}_{F,\Sigma(\Xi)}$ analogously. For any $\tau \in \Sigma^o_F$, let $\bar{\pi}_{2,\tau} : \bar{Z}_\tau \to A_{F,\Xi}$ be the morphism deduced from $\bar{\pi}_2$ via (1.6.7), $\pi_{2,\tau}$ its restriction to $Z_\tau := \bar{Z}_\tau \cap {}^<Z_{F,\Sigma}$. We record the following facts for future reference:

1.6.8. LEMMA.

(i) *$\partial\tilde{Z}_{F,\Sigma(\Xi)} = {}^<\partial\tilde{Z}_{F,\Sigma(\Xi)} \cup {}^>\partial\tilde{Z}_{F,\Sigma(\Xi)}$ is a divisor with normal crossings on $\tilde{Z}_{F,\Sigma(\Xi)}$. More precisely, for any non-empty intersection of components of $\tilde{Z}_{F,\Sigma(\Xi)}$, $E = \cap\delta_\alpha$, the closure of $[\partial\tilde{Z}_{F,\Sigma(\Xi)} - \cup\delta_\alpha]$ induces on E a divisor with normal crossings.*

(ii) *For any* $\tau \in \Sigma_F^{\circ}$, *the morphism* $\pi_2 : Z_\tau \to A_F$ *is a smooth fibration. The morphism* $\bar{\pi}_{2,\tau} : \bar{Z}_\tau \to A_{F,\Xi}$ *is proper.*

(iii) *For any* $\tau \in \Sigma_F^{\circ}$, *let* ${}^{>}\partial_\tau = \bar{Z}_\tau \cap {}^{>}\partial Z_{F,\Sigma}$, *and let* $\partial_\Xi = A_{F,\Xi} - A_F$. *Then*

$$\mathbf{R}^i \bar{\pi}_{2,*}(\mathcal{O}_{\bar{Z}_\tau}) = \mathbf{R}^i \bar{\pi}_{2,*}(\mathcal{O}_{\bar{Z}_\tau}(-{}^{>}\partial_\tau)) = 0 \qquad for \quad i > 0;$$

and there are canonical isomorphisms

$$\mathcal{O}_{A_{F,\Sigma}} \xrightarrow{\sim} \pi_{2,*}(\mathcal{O}_{\bar{Z}_\tau}), \qquad \mathcal{O}_{A_{F,\Xi}}(-\partial_\Xi) \xrightarrow{\sim} \bar{\pi}_{2,*}(\mathcal{O}_{\bar{Z}_\tau}(-{}^{>}\partial_\tau)).$$

Proof. – Part (i) is just the hereditary property of divisors with normal crossings. The first statement in (ii) is standard, and the second is obvious. Finally, GAGA reduces (iii) to the corresponding analytic statement. Write $T_\alpha^A = T_\Gamma(Q^A(\mathcal{R}_\alpha), \mathcal{X}^A(\mathcal{R}_\alpha)) \cong T_{F_\alpha}/T_F$ [notation (1.6.3) (*b*)]. The assertions being local on $A_{F,\Xi}$, we may restrict our attention to a neighborhood of the stratum on $A_{F,\Xi}$ corresponding to the boundary component $\mathcal{X}^A(\mathcal{R}_\alpha)$. Thus we may replace A_F by a T_α^A-fibration over a certain mixed Shimura variety M_α^A, and $A_{F,\Xi}$ by the partial toroidal compactification $M_\alpha^A \times^{T_\alpha^A} (T_\alpha^A)_\sigma$, for some $\sigma \in \Xi_\alpha^A$. Let $T(\tau) \subset T_F$ be the torus generated by the images of $\{\lambda(\mathbb{G}_m) | \lambda \in X_*(T_F) \cap \mathbb{R}\cdot\tau\}$, and let $T_{\tau,\alpha} = T_{F_\alpha}/T(\tau)$; this is the torus associated to the stratum Z_τ in a partial compactification of the mixed Shimura variety M'_{F_α}, *cf.* ([P], § 7). Let

$$\Sigma(\sigma) = \{\tilde{\sigma} \in \mathcal{S}_{[\Sigma_F],\alpha} | \varphi_2(\tilde{\sigma}) \subset \sigma\}.$$

Localizing further, we can replace the morphism $\bar{\pi}_{2,\tau}$ by

$$h_\sigma \times 1_B : T_{\tau,\alpha,\Sigma(\sigma)} \times B \to (T_\alpha^A)_\sigma \times B$$

[notation (1.6.3) (*b*)]

where B is a complex ball in M_α^A, and h_α is a proper surjective morphism of torus embeddings. In this form, (iii) is an easy consequence of Lemma 1.3.6.

1.6.9. *Remark.* – The condition 1.6.8 (i) for ${}^{>}\partial Z_{F,\Sigma} \subset {}^{<}Z_{F,\Sigma}$ can be written in terms of Σ_F: for all $\tau \in \Sigma_F$,

$$\bar{Z}_\tau \cap \{\text{closure of } ({}^{>}\partial Z_{F,\Sigma} - \bigcup_{\sigma\subset\tau,\, \dim(\sigma)=1} Z_\sigma)\}$$

is a divisor with normal crossings on $\bar{Z}_\tau$.

1.7. We still have to discuss the adelic toroidal compactifications. Choose a minimal rational parabolic $P_0 \subset G$. A rational boundary component F of D will be called *standard* if P_F is standard, *i.e.* contains P_0. Fix an open compact subgroup $K \subset G(\mathbf{A}^f)$, which is *neat* in the *ad hoc* sense of [H5] or the more canonical sense of [P]. The toroidal compactifications ${}_K\mathrm{Sh}(G, X)_\Sigma$ of ${}_K\mathrm{Sh}(G, X)$ are associated to adelic fans $\Sigma = \bigcup_F \Sigma_F$, where F runs through the standard rational boundary components. Each Σ_F is now a fan in

(1.7.1) $$G(\mathbb{Q})^+ \times^{P_F(\mathbb{Q})^+} (\bar{C}_F \times G(\mathbf{A}^f)/K),$$

where $P_F(\mathbb{Q})^+$ acts on the left on the last two factors and on the right on $G(\mathbb{Q})^+$, satisfying the axioms of ([H3], 2.5). It is proved in [*loc. cit.*, Prop. 2.8] that, if Σ is moreover *projective* and *equivariant* (definitions in [*loc. cit.*]), then ${}_K\mathrm{Sh}(\mathrm{G}, \mathrm{X})_\Sigma$ has a model over $\mathrm{E}(\mathrm{G}, \mathrm{X})$ which is compatible with the canonical model of ${}_K\mathrm{Sh}(\mathrm{G}, \mathrm{X})$. Pink [P] has proved a more precise result: he has defined canonical models of toroidal compactifications with good functorial properties, and proved that ${}_K\mathrm{Sh}(\mathrm{G}, \mathrm{X})_\Sigma$, and more generally toroidal compactifications of *mixed Shimura varieties*, (*see* also 3.1.2.2) possess canonical models.

The toroidal compactifications of ${}_K\mathrm{Sh}(\mathrm{G}, \mathrm{X})$ associated to projective equivariant fans will be called *admissible* toroidal compactifications if the restriction to any connected component is an SNC toroidal compactification. As remarked in ([H3], 2.8), the admissible toroidal compactifications are cofinal in the projective system of all toroidal compactifications, ordered by the relation of refinement on fans. Thus the following ugly object:

$$\mathrm{Sh}(\mathrm{G}, \mathrm{X})^\sim = \varprojlim {}_K\mathrm{Sh}(\mathrm{G}, \mathrm{X})_\Sigma,$$

where the limit is taken over all (compact open) K and (fans) Σ, has a canonical model over $\mathrm{E}(\mathrm{G}, \mathrm{X})$, which is respected by the canonical extension of the $\mathrm{G}(\mathbf{A}^f)$-action on $\mathrm{Sh}(\mathrm{G}, \mathrm{X})$.

The *minimal* (Baily-Borel Satake) compactification $\mathrm{Sh}(\mathrm{G}, \mathrm{X})^*$ of $\mathrm{Sh}(\mathrm{G}, \mathrm{X})$ is constructed as $\varprojlim {}_K\mathrm{Sh}(\mathrm{G}, \mathrm{X})^*$, the limit being taken over compact open $\mathrm{K} \subset \mathrm{G}(\mathbf{A}^f)$. Here ${}_K\mathrm{Sh}(\mathrm{G}, \mathrm{X}) = \amalg \mathrm{M}_{\Gamma_\alpha}$ is the disjoint union of a finite set of locally symmetric varieties $\mathrm{M}_{\Gamma_\alpha}$, and ${}_K\mathrm{Sh}(\mathrm{G}, \mathrm{X})^*$ is simply $\amalg \mathrm{M}^*_{\Gamma_\alpha}$. The morphisms $\pi_\Sigma : {}_K\mathrm{Sh}(\mathrm{G}, \mathrm{X})_\Sigma \to {}_K\mathrm{Sh}(\mathrm{G}, \mathrm{X})^*$, defined on connected components in 1.5, define in the limit a canonical $\mathrm{G}(\mathbf{A}^f)$-equivariant morphism

$$\pi^\sim : \quad \mathrm{Sh}(\mathrm{G}, \mathrm{X})^\sim \to \mathrm{Sh}(\mathrm{G}, \mathrm{X})^*. \tag{1.7.2}$$

The Baily-Borel compactification $\mathrm{Sh}(\mathrm{G}, \mathrm{X})^*$ is naturally stratified as follows (*cf.* [P], § 6). A *rational boundary component* of $\mathrm{Sh}(\mathrm{G}, \mathrm{X})$ is a connected component of the set

$$\mathrm{G}(\mathbb{Q})^+\backslash\mathcal{B}(\mathrm{D}) \times \mathrm{G}(\mathbf{A}^f) = \mathrm{G}(\mathbb{Q})\backslash\mathcal{B}(\mathrm{X}) \times \mathrm{G}(\mathbf{A}^f), \tag{1.7.3}$$

where $\mathcal{B}(\mathrm{D})$ (resp. $\mathcal{B}(\mathrm{X})$) is the set of rational boundary components of D (resp. X), and the equality is a consequence of real approximation [D3]. Then $\mathrm{Sh}(\mathrm{G}, \mathrm{X})^*$ can be written as the disjoint union of connected (pro-algebraic) varieties of the form

$$\mathrm{M}(\Phi, g) = \varprojlim g^{-1}\mathrm{K}g \cap \mathrm{G}_{h,\Phi}(\mathbb{Q})\backslash\Phi \cdot g,$$

where $\Phi \in \mathcal{B}(\mathrm{D})$, $\mathrm{G}_{h,\Phi}$ is the corresponding group defined in 1.2, $g \in \mathrm{G}(\mathbf{A}^f)$, and K runs through compact open subgroups of $\mathrm{G}(\mathbf{A}^f)$.

Let F be a standard rational boundary component. The *F-stratum* $\mathrm{Sh}(\mathrm{G}, \mathrm{X})^\mathrm{F}$ of $\mathrm{Sh}(\mathrm{G}, \mathrm{X})^*$ is the set of $\mathrm{M}(\Phi, g)$ with $\Phi \in \mathrm{G}(\mathbb{Q})^+ \cdot \mathrm{F}$. Then the F-stratum of $\mathrm{Sh}(\mathrm{G}, \mathrm{X})^\sim$ is $(\pi^\sim)^{-1}(\mathrm{Sh}(\mathrm{G}, \mathrm{X})^\mathrm{F}) \subset \mathrm{Sh}(\mathrm{G}, \mathrm{X})^\sim$, and, for any compact open $\mathrm{K} \subset \mathrm{G}(\mathbf{A}^f)$ and any admissible fan Σ, the F-stratum of ${}_K\mathrm{Sh}(\mathrm{G}, \mathrm{X})_\Sigma$ is the image in ${}_K\mathrm{Sh}(\mathrm{G}, \mathrm{X})_\Sigma$ of the F-stratum of $\mathrm{Sh}(\mathrm{G}, \mathrm{X})^\sim$.

At this point it is necessary to correct a misunderstanding which has led to incorrect (or at least incomplete) formulations in much of the literature, including a number of papers of the first author. Proposition 5.1.11 of [H2] associates to F a basic pair $(\mathrm{G}_h, \mathrm{F}_\mathrm{P})$

(denoted there $(\mathrm{G_P}, \mathrm{F_{P_1}})$, where F is a connected component of $\mathrm{F_P}$, and the F-stratum $\mathrm{Sh}(\mathrm{G}, \mathrm{X})^{\mathrm{F}}$ is implicitly viewed as the disjoint union of a family of Shimura varieties isomorphic to $\mathrm{Sh}(\mathrm{G}_h, \mathrm{F_P})$. This is done in order to identify the Fourier-Jacobi expansion, and especially the constant term, of an automorphic form along F, in terms of sections of automorphic vector bundles (*cf.* 3.1, below) on $\mathrm{Sh}(\mathrm{G}_h, \mathrm{F_P})$. Unfortunately, $\mathrm{F_P}$ may have fewer connected components than X. The result is that $\mathrm{Sh}(\mathrm{G}, \mathrm{X})^{\mathrm{F}}$ is actually the disjoint union of a family of *finite Galois coverings* Sh′ of $\mathrm{Sh}(\mathrm{G}_h, \mathrm{F_P})$, with Galois group a subquotient π of $\pi_0(\mathrm{G}_{h,\mathrm{F}}(\mathbb{R}))$. These Galois coverings also have canonical models, as we see presently, and admit a theory of automorphic vector bundles, all of which are naturally homogeneous with respect to $\pi \times \mathrm{G}_h(\mathbf{A}^f)$.

This issue has been addressed by Pink, whose formulation seems to be the most natural possible. Instead of the pair $(\mathrm{G}_h, \mathrm{F_P})$, Pink introduces a slightly more general pair, which we denote $(\mathrm{G}_h, \mathrm{X}(\mathrm{F}))$. Here $\mathrm{X}(\mathrm{F})$ is a homogeneous space for $\mathrm{G}_h(\mathbb{R})$, which admits a finite-to-one equivariant map $\mathrm{X}(\mathrm{F}) \to \mathrm{F_P}$, which makes $\mathrm{X}(\mathrm{F})$ a finite Galois covering of $\mathrm{F_P}$, with Galois group π as above. However, $\mathrm{X}(\mathrm{F})$ is not a conjugacy class of homomorphisms $\underline{S} \to \mathrm{G}_{h,\mathbb{R}}$. The Shimura variety is defined as before:

$$\mathrm{Sh}(\mathrm{G}_h, \mathrm{X}(\mathrm{F})) = \overset{\lim}{\leftarrow} \mathrm{G}_h(\mathbb{Q})\backslash \mathrm{X}(\mathrm{F}) \times \mathrm{G}_h(\mathbf{A}^f)/\mathrm{K_F}, \qquad \mathrm{K_F} \subset \mathrm{G}_h(\mathbf{A}^f)$$

and there is evidently a natural Galois covering

$$\mathrm{Sh}(\mathrm{G}_h, \mathrm{X}(\mathrm{F})) \to \mathrm{Sh}(\mathrm{G}_h, \mathrm{F_P}) \tag{1.7.4}$$

with Galois group $\pi = \mathrm{Stab}(\mathrm{F})/\mathrm{Stab}(\mathrm{X}(\mathrm{F})^0)$, for any connected component $\mathrm{X}(\mathrm{F})^0$ of $\mathrm{X}(\mathrm{F})$ which maps to F. This construction is taken from ([P], 4.11, Prop. 2.9). In Proposition 12.1, Pink shows that the reflex field $\mathrm{E}(\mathrm{G}_h, \mathrm{X}(\mathrm{F}))$ equals $\mathrm{E}(\mathrm{G}, \mathrm{X})$ (cf. [H2], Corollary 6.1.4), and in paragraph 11 Pink defines and constructs the canonical model of $\mathrm{Sh}(\mathrm{G}_h, \mathrm{X}(\mathrm{F}))$ and proves the expected functoriality properties; in particular, the covering (1.7.4) and the action of π are defined over $\mathrm{E}(\mathrm{G}, \mathrm{X})$. Automorphic vector bundles on $\mathrm{Sh}(\mathrm{G}_h, \mathrm{F_P})$ thus pull back to $\pi \times \mathrm{G}_h(\mathbf{A}^f)$-equivariant vector bundles on $\mathrm{Sh}(\mathrm{G}_h, \mathrm{X}(\mathrm{F}))$, which we also call automorphic.

Moreover, let $\mathrm{T} = (\mathrm{G}_h)^{\mathrm{ab}}$, and let $t : \underline{S} \to \mathrm{T}_{\mathbb{R}}$ be the image of any element of $\mathrm{F_P}$; let T′ be the maximal $\mathbb{Q}$-split quotient of T, and let (T', t') be the corresponding basic pair. By our hypothesis (1.1.4), the natural map $\mathrm{T}(\mathbb{R})/\mathrm{T}(\mathbb{R})^0 \to \mathrm{T}'(\mathbb{R})/\mathrm{T}'(\mathbb{R})^0$ is injective. It then follows from Proposition 2.3 of [P] that (1.7.4) is the pullback, via the map $(\mathrm{G}_{h,\mathrm{F}}, \mathrm{F_P}) \to (\mathrm{T}', t')$, of a Galois covering

$$\mathrm{Sh}(\mathrm{T}', \tilde{t}') \to \mathrm{Sh}(\mathrm{T}', t')$$

where $\tilde{t}'$ is some $\mathrm{T}'(\mathbb{R})$-equivariant cover of the point $\{t'\}$. In particular, if one adds the double cover of $\mathrm{Sh}(\mathbb{G}_m, \mathrm{N})$, where $\mathrm{N} : \underline{S} \to \mathbb{G}_{m,\mathbb{R}}$ is the norm map:

$$\begin{aligned} &\mathrm{Sh}(\mathbb{G}_m, \pi_0(\mathbb{R}^\times)) \\ &\quad = \mathbb{Q}^\times\backslash\mathbb{R}^\times \times \mathbf{A}^{f,\times}/(\mathbb{R}^\times)^0 \to \mathbb{Q}^\times\backslash\mathbb{R}^\times \times \mathbf{A}^{f,\times}/\mathbb{R}^\times = \mathrm{Sh}(\mathbb{G}_m, \mathrm{N}) \end{aligned} \tag{1.7.5}$$

then every Shimura variety in Pink's sense can be obtained from the old ones and $\mathrm{Sh}(\mathbb{G}_m, \pi_0(\mathbb{R}^\times))$ by taking fiber products and pullbacks.

Now in the case of the boundary component F, X(F) has the property that, if $X(F)^0$ is a connected component of X(F),

$$\mathrm{Stab}_{G_h(\mathbb{R})}(X(F)^0) = G_h(\mathbb{R}) \cap G(\mathbb{R})^+, \tag{1.7.6}$$

which in general is smaller than $\mathrm{Stab}_{G_h(\mathbb{R})}(F)$. Of course, $X(F)^0$ may be identified with the boundary component F of D, and this identification is equivariant with respect to the actions of $G_h(\mathbb{R}) \cap G(\mathbb{R})^+$. Thus

$$\mathrm{Sh}(G_h, X(F)) \overset{\lim}{\underset{\leftarrow}{\simeq}} G_h(\mathbb{Q})^+ \backslash F \times G_h(\mathbf{A}^f)/K_F. \tag{1.7.7}$$

Now the action on F extends naturally to $\mathrm{Stab}_{G(\mathbb{R})^+}(F) = P(\mathbb{R})^+$. Recall the group P′ introduced in (1.2.3). Then the *P′-stratum* of $\mathrm{Sh}(G, X)^*$, defined by

$$\mathrm{Sh}(G, X)^{P'} \overset{\lim}{\underset{\leftarrow}{=}} P(\mathbb{Q})^+ \backslash F \times P(\mathbb{Q})^+ \cdot P'(\mathbf{A}^f) \cdot K/K, \qquad K \subset G(\mathbf{A}^f), \tag{1.7.8}$$

is the image of $\mathrm{Sh}(G_h, X(F))$ under its natural map to $\mathrm{Sh}(G, X)^*$. Indeed, the map $\mathrm{Sh}(G_h, X(F)) \to \mathrm{Sh}(G, X)^{P'}$, the natural one from the right-hand side of (1.7.7) to the right hand side of (1.7.8), is surjective, by strong approximation in the unipotent group W_F. Whether or not it is an isomorphism appears to depend on whether or not the commutative algebraic group $G_l \cap G_h$ satisfies the Hasse principle. In any case, $\mathrm{Sh}(G, X)^{P'}$ is the quotient of $\mathrm{Sh}(G_h, X(F))$ by a subgroup of $P(\mathbb{Q})^+$, acting by conjugation. This action is given by an automorphism of the data $(G_h, X(F))$, hence respects the canonical model by functoriality (*cf.* [P], Prop. 11.10).

More generally, the F-stratum of $\mathrm{Sh}(G, X)^*$ is given by

$$\begin{aligned}\mathrm{Sh}(G, X)^F &\overset{\lim}{\underset{\leftarrow}{=}} P(\mathbb{Q})^+ \backslash F \times G(\mathbf{A}^f)/K \\ &= \mathrm{Sh}(G_h, X(F)) \times^{\overline{P(\mathbb{Q})^+} \cdot P'(\mathbf{A}^f)} G(\mathbf{A}^f),\end{aligned} \tag{1.7.9}$$

where $\overline{P(\mathbb{Q})^+}$ is the closure of $P(\mathbb{Q})^+$ in $G(\mathbf{A}^f)$.

The equality (1.7.9) imposes a canonical model on $\mathrm{Sh}(G, X)^F$. It is proved in paragraph 12 of [P] that this model is compatible with the canonical model of $\mathrm{Sh}(G, X)^*$ defined (for example) by the universal properties of the latter among normal compactifications of Sh(G, X).

It follows from (1.7.9) that the F-stratum $\mathrm{Sh}(G, X)^{\sim, F}$ of $\mathrm{Sh}(G, X)^{\sim}$ is

$$(\pi^{\sim})^{-1}(\mathrm{Sh}(G, X)^F) = (\pi^{\sim})^{-1}(\mathrm{Sh}(G_{h,F}, X(F))) \times^{\overline{P(\mathbb{Q})^+} \cdot P'(\mathbf{A}^f)} G(\mathbf{A}^f). \tag{1.7.10}$$

We define the *P-stratum* $\mathrm{Sh}(G, X)^P$ of $\mathrm{Sh}(G, X)^*$ to be the set of $M(F, g)$ with $g \in P(\mathbf{A}^f)$. The P- and P′-strata of $\mathrm{Sh}(G, X)^{\sim}$ or ${}_K\mathrm{Sh}(G, X)_{\Sigma}$ are defined analogously. The P′- and P-strata of $\mathrm{Sh}(G, X)^{\sim}$ are given simply by

$$\begin{cases} \mathrm{Sh}(G, X)^{\sim, P'} := (\pi^{\sim})^{-1}(\mathrm{Sh}(G, X)^{P'}) \\ \mathrm{Sh}(G, X)^{\sim, P} := (\pi^{\sim})^{-1}(\mathrm{Sh}(G, X)^{P'}) \times^{\overline{P(\mathbb{Q})^+} \cdot P'(\mathbf{A}^f)} G(\mathbf{A}^f). \end{cases} \tag{1.7.11}$$

The difference between these two notions is that $\mathrm{Sh}(\mathrm{G}, \mathrm{X})^{\sim, \mathrm{P}'}$ maps naturally to (a quotient of) $\mathrm{Sh}(\mathrm{G}_h, \mathrm{X}(\mathrm{F}))$, whereas $\mathrm{Sh}(\mathrm{G}, \mathrm{X})^{\sim, \mathrm{P}}$ is more natural with respect to the adeles of G_l. Here is the picture: (1.7.12)

$$\mathrm{Sh}(\mathrm{G}, \mathrm{X})^{\sim, \mathrm{F}} \supset \mathrm{Sh}(\mathrm{G}, \mathrm{X})^{\sim, \mathrm{P}} \supset \mathrm{Sh}(\mathrm{G}, \mathrm{X})^{\sim, \mathrm{P}'} \xrightarrow{\pi^{\sim}} \mathrm{Sh}(\mathrm{G}, \mathrm{X})^{\mathrm{P}'} \leftarrow \mathrm{Sh}(\mathrm{G}_h, \mathrm{X}(\mathrm{F})).$$

1.8. Cayley Transforms. – We collect here some of the basic facts about Cayley transforms and canonical automorphy factors that will be used in subsequent sections. We introduce the auxiliary basic pair $(\mathrm{G}^{(2)}, \Delta(\mathrm{P}_\mathrm{F}))$ as in [H2], 5.1, where $\mathrm{G}^{(2)}$ was denoted $\mathrm{G}^{\mathrm{even}}$. Here $\mathrm{G}^{(2)}$ is the connected algebraic subgroup of G with Lie algebra $\mathfrak{g}^{-2} \oplus \mathfrak{g}^0 \oplus \mathfrak{g}^2$, in the notation of 1.2.2, and a connected component $\Delta(\mathrm{P}_\mathrm{F})^0 \subset \Delta(\mathrm{P}_\mathrm{F})$ is given, in the coordinates (1.2.4), by the subset of D for which $v = 0$. There is a product decomposition

$$\Delta(\mathrm{P}_\mathrm{F})^0 \cong \mathrm{D}_t \times \mathrm{F},$$

where D_t is the tube domain over the cone C_F, corresponding to an isogeny

$$j: \quad \mathrm{G}_t \times \mathrm{G}_h(0) \to \mathrm{G}^{(2)}$$

(notation in the domain as in 1.2.2). The symmetric domain F is then a boundary component of $\Delta(\mathrm{P}_\mathrm{F})^0$. More precisely, $\mathrm{P}_{\mathrm{F}, t} = \mathrm{G}_l \cdot \mathrm{U}_\mathrm{F}$ is a rational maximal parabolic subgroup of G_t, the stabilizer of a point boundary component $\{q\}$ of the tube domain D_t; then $\{q\} \times \mathrm{F}$ is naturally a boundary component of $\Delta(\mathrm{P}_\mathrm{F})^0$. In particular, $\mathrm{G}_t \supset \mathrm{Z}_\mathrm{G}^0$.

Let $\mathrm{N}_\mathrm{F} = \mathrm{G}_l \cdot \mathrm{W}_\mathrm{F}$; $\mathrm{N}_\mathrm{F}(\mathbb{R})$ acts trivially on F. Fix a point $p \in \mathrm{D}$; $\mathfrak{g} = \mathfrak{k}_p \oplus \mathfrak{p}_p$ be the corresponding Cartan decomposition of the Lie algebra $\mathfrak{g}$ of $\mathrm{G}(\mathbb{R})$, and let

$$\mathfrak{g}_\mathbb{C} = \mathfrak{k}_{p, \mathbb{C}} \oplus \mathfrak{p}^+ \oplus \mathfrak{p}^- = \mathfrak{g}^{(0,0)} \oplus \mathfrak{g}^{(1,-1)} \oplus \mathfrak{g}^{(-1,1)} \tag{1.8.1}$$

be the Hodge decomposition (1.1.1). Let K_p (resp. $\mathcal{P}_p$) $\subset \mathrm{G}_\mathbb{C}$ be the connected subgroups with Lie algebra $\mathfrak{k}_{p,\mathbb{C}}$ (resp. $\mathfrak{P}_p := \mathfrak{k}_{p,\mathbb{C}} \oplus \mathfrak{p}^-$). Then K_p is defined over $\mathbb{R}$, and we frequently write K_p for the subgroup of $\mathrm{G}(\mathbb{R})$ which stabilizes p; K_p contains a maximal compact subgroup of $\mathrm{G}(\mathbb{R})^0$, and also contains $\mathrm{Z}_\mathrm{G}(\mathbb{R})$. We may assume that $p \in \Delta(\mathrm{P}_\mathrm{F})$, so that $\mathrm{K}_h = \mathrm{K}_p \cap \mathrm{G}_h(\mathbb{R})$ contains a maximal compact subgroup of $\mathrm{G}_h(\mathbb{R})^0$, $\mathrm{G}_h(\mathbb{R})^0/\mathrm{A}(\mathbb{R}) \cdot \mathrm{K}_h$ is hermitian symmetric, and $\mathrm{K}_l = \mathrm{K}_p \cap \mathrm{G}_l(\mathbb{R})$ contains a maximal compact subgroup of $\mathrm{G}_l(\mathbb{R})^0$ (*cf.* [H2], § 5). Note that K_h is smaller than the stabilizer in $\mathrm{G}_h(\mathbb{R})$ of a point in X(F), since K_h does not contain $\mathrm{A}(\mathbb{R})$. Let $\mathrm{K}_p^{(2)} = \mathrm{K}_p \cap \mathrm{G}^{(2)}(\mathbb{R})$; it is isogenous modulo $\mathrm{Z}_\mathrm{G}^0(\mathbb{R})$ to $\mathrm{K}_t \times \mathrm{K}_h$, where $\mathrm{K}_t = \mathrm{K}_p^{(2)} \cap j(\mathrm{G}_t)$, with j as above. Again, we frequently use K_*, $* = h, l$, to denote the connected $\mathbb{R}$-algebraic group with Lie algebra $\mathrm{Lie}(\mathrm{K}_*)$.

The point $p \in \mathrm{D}$ is a *CM point* if there is an algebraic torus $\mathrm{H} \subset \mathrm{G}$ such that $p: \underline{S} \to \mathrm{G}_\mathbb{R}$ factors through $\mathrm{H}_\mathbb{R}$; then the basic pair (H, p) is called a *CM pair*, with reflex field $\mathrm{E}(\mathrm{H}, p)$. There are many CM pairs; indeed

$$\begin{cases} \mathrm{E}(\mathrm{G}, \mathrm{X}) = \bigcap \mathrm{E}(\mathrm{H}, p), \\ \text{where } \{(\mathrm{H}, p)\} \text{ runs over CM pairs in} (\mathrm{G}, \mathrm{X}) \quad ([\mathrm{D5}], 5.1). \end{cases} \tag{1.8.2}$$

We assume henceforward that p is a CM point. In ([H2], 5.2.3), we construct an element $c_F \in G_t^{der}(E(H, p))$ with the following properties:

$$(1.8.3)\quad \begin{cases} \mathrm{Ad}(c_F)(\mathfrak{g}_{\mathbb{C}}^{-2}) = \mathfrak{p}_2^+ \subset \mathfrak{p}^+, \qquad \mathrm{Ad}(c_F)(G_l) = K_t, \qquad \mathrm{Ad}(c_F)(K_t) = G_l; \\ c_F \text{ commutes with } K_l \end{cases}$$

This corresponds to the element denoted $(c_p, P)^{-1}$ in [*loc. cit.*], and we call it the inverse Cayley transform. The purely analytic theory of the (inverse) Cayley transform was developed in [WK]. The point $c_F(p)$ is in the $U(\mathbb{C})$-orbit of p and lies on the boundary component *opposite* to F, *i. e.*, the one whose normalizer is the parabolic subgroup opposite to P_F (relative to our choice of Levi factor).

The composite $\mathrm{Ad}(c_F) \circ w_F : \mathbb{G}_m \to K_p$ determines a parabolic subgroup

$$Q_{F,p} \subset K_p : \quad Q_{F,p} = \{x \in K_p | \lim_{t\to\infty} \mathrm{Ad}(\mathrm{Ad}(c_F) \circ w_F(t))(x) \text{ exists in } K_p\}.$$

Let V_F denote the vector group W_F/U_F, $v_F = \mathrm{Lie}(V_F) = \mathrm{Gr}_{-1}^W \mathfrak{g}$, relative to the weight filtration (1.2.2.4). Thus V_F is the unipotent radical of P'/U_F. By (1.2.2.2), v_F has a pure Hodge structure of weight -1, (with Hodge filtration $F_p^\bullet$), and one verifies easily that it is of type $(-1, 0)+(0, -1)$ (*cf.* [Br2], [D2], [P]):

$$(1.8.4)\qquad \mathfrak{v}_{F,\mathbb{C}} \cong (\mathfrak{v}_{F,\mathbb{C}})^{(-1,0)} \oplus (\mathfrak{v}_{F,\mathbb{C}})^{(0,-1)} := \mathfrak{v}_x^+ \oplus \mathfrak{v}_x^-.$$

When D is realized, à la Harish-Chandra, as a bounded domain in $\mathfrak{p}^+$, $c_F \cdot D$ becomes a Siegel domain of the third kind inside the complex vector space $\mathfrak{p}^+$ with respect to a certain K_h-invariant decomposition ([WK]: 7.1):

$$(1.8.5)\qquad \mathfrak{p}^+ = \mathfrak{p}_2^+ \oplus \mathfrak{p}_1^+ \oplus \mathfrak{p}_h^+,$$

with $\mathfrak{p}_2^+$ as in (1.8.3) and $\mathfrak{p}_h^+ = \mathfrak{p}^+ \cap \mathfrak{g}_{h,\mathbb{C}}$.

1.8.6. LEMMA. – $\mathrm{Ad}(c_F)$ *induces a K_h-equivariant identification* $\mathfrak{v}_x^+ \cong \mathfrak{p}_1^+$. *Furthermore,* $Q_{F,p}$ *is a maximal parabolic subgroup of* K_p *with Levi component* $K_p^{(2)}$ *and* $\mathrm{Lie}(R_u Q_{F,p}) \cong \mathfrak{v}_x^-$. *Alternatively,* $Q_{F,p}$ *is the projection on* $K_p = \mathcal{P}_p/R_u\mathcal{P}_p$ *of* $\mathcal{P}_p \cap P$.

Proof. – See ([WK]: 6.3, 7.4) for the first two assertions. Now $\mathcal{P}_p \cap P$ has Lie algebra $\mathfrak{s} := F^0\mathfrak{g} \cap W_0\mathfrak{g}$. Since $W_{-2}\mathfrak{g}$ is purely of type $(-1,-1)$ [D2], $\mathfrak{s}$ is an extension of $F^0\mathfrak{g} \cap \mathrm{Gr}_0^W \mathfrak{g} = F^0(\mathrm{Lie}(L_F))$ by $F^0\mathfrak{g} \cap \mathrm{Gr}_{-1}^W\mathfrak{g} = F^0\mathrm{Gr}_{-1}^W\mathfrak{g} = \mathfrak{v}_x^-$. Similarly,

$$\mathrm{Lie}(P \cap R_u\mathcal{P}_p) \cong F^1\mathfrak{g} \cap \mathrm{Gr}_0^W\mathfrak{g} = \mathfrak{p}_h^- := \mathfrak{p}^- \cap \mathfrak{g}_{h,\mathbb{C}}.$$

The third assertion follows immediately.

1.8.7. A *canonical automorphy factor* for the pair (P_F, p) is a morphism $J : G(\mathbb{R})^0 \times D \to K_p(\mathbb{C})$ satisfying the following seven conditions:

(1.8.7.1) $J(g\cdot g', x) = J(g, g'(x)) \cdot J(g', x)$, $g, g' \in G(\mathbb{R})^0$, $x \in D$;

(1.8.7.2) $J(k, p) = k$, $\forall k \in K_p(\mathbb{R})$;

(1.8.7.3) For any $g \in G(\mathbb{R})^0$, the function $J(g, \cdot) : D \to K_p(\mathbb{C})$ is holomorphic;

(1.8.7.4) The map $J(\cdot, x) : N_F(\mathbb{R}) \cap G(\mathbb{R})^0 \to K_p(\mathbb{C})$ comes from an $E(H, p)$-rational homomorphism $N_F \to K_p$ of algebraic groups that is trivial on U_F, and is independent of $x \in D$.

(1.8.7.5) The restriction $J^{(2)}$ of J to $G^{(2)}(\mathbb{R})^0 \times \Delta(P_F)^0$ takes values in the subgroup $K_p^{(2)}(\mathbb{C}) \subset K_p(\mathbb{C})$. Then the homomorphism

$$J^{(2)}|_{G_l(\mathbb{R})^0} : G_l(\mathbb{R})^0 \to K_p^{(2)}(\mathbb{C})$$

takes values in $K_t(\mathbb{C})$, and comes from the isomorphism (1.8.3) $c_F : G_l \xrightarrow{\sim} K_t$ of algebraic groups.

(1.8.7.6) The restriction of J to $P_F(\mathbb{R})^0 \times D$ takes values in $Q_{F,p}(\mathbb{C})$.

(1.8.7.7) There is an automorphy factor $J_h : G_h(\mathbb{R})^0 \times F \to K_h(\mathbb{C})$, satisfying the analogues of (1.8.7.1-3), such that, for $g \in G_h(\mathbb{R})^0$, $z \in \Delta(P_F)^0$, $J^{(2)}(g, z) = J_h(g, z_F)$, where z_F is the projection of z onto F.

1.8.8. PROPOSITION. – *A canonical automorphy factor* $J = J^{P_F, p}$ *for* (P_F, p) *exists.*

Proof. – J was constructed in ([H2], 5.2) in terms of the Cayley transform c_F. With the exception of (1.8.7.6), which follows easily from Lemma 1.8.6 and the definitions, the above properties were verified there, following [WK]. It is essentially uniquely determined by these properties.

2. Differential forms on torus embeddings and their real quotients

2.1. Let T be the torus $\mathbb{G}_m^n$, and let T_Σ be a non-singular torus embedding, associated to a fan Σ in $\mathbb{R}^n$. Recall that if Σ^0 denotes the set of n-dimensional cones of Σ, and Σ^1 the set of cones of codimension one, we have

(i) $T_\Sigma = \bigcup_{\sigma \in \Sigma^0} T_\sigma$ (with $T_\sigma(\mathbb{C}) \cong \mathbb{C}^n$),

(ii) $T_\sigma \cap T_{\sigma'} \cong T_\tau$ whenever $\sigma \cap \sigma' = \tau \in \Sigma^1$ (with $T_\tau(\mathbb{C}) \cong \mathbb{C}^{n-1} \times \mathbb{C}^\times$),

and so on for intersections of higher dimension (assuming, as of course we always do, that every τ in Σ is a face of some top-dimensional cone). Since each σ in Σ is a cone, one can define $\hat{\sigma} = (\sigma - \{0\})/\mathbb{R}_+^\times$, and $\hat{\Sigma} = \{\hat{\sigma} | \sigma \in \Sigma\}$, as in (1.3.6.6).

In what follows, we use the analytic topology on $T_\Sigma(\mathbb{C})$. The data contained in Σ also gives rise to a real "connected torus embedding" T_Σ^+ as follows. Inside $T(\mathbb{C}) = (\mathbb{C}^\times)^n$ we have $T(\mathbb{R}) \cong (\mathbb{R}^\times)^n$, a group with 2^n connected components. We take its identity component $T(\mathbb{R})^0 \cong (\mathbb{R}_+^\times)^n$, which we denote T^+. The construction of $T_\sigma^+ \subset T_\sigma(\mathbb{R})$ can be described abstractly as the adjoining of the usual boundary (with corners when $n > 1$) to T^+, viz.

$$T_\sigma^+ \cong (\mathbb{R}_{\geq 0})^n \quad \text{if} \quad \sigma \in \Sigma^0.$$

(*See* also 2.3 below.) This is a natural process: if $\sigma \cap \sigma' = \tau$, as above, then

$$T_\sigma^+ \cap T_{\sigma'}^+ \cong T_\tau^+ \cong (\mathbb{R}_{\geq 0})^{n-1} \times \mathbb{R}_+^\times, \qquad \text{if} \quad \sigma, \sigma' \in \Sigma^0,$$

the intersection taking place inside T_Σ, and so on. We put

$$T_\Sigma^+ = \bigcup_{\sigma\in\Sigma^0} T_\sigma^+ .$$

It is a manifold with corners, equipped with a closed embedding in T_Σ, as the closure of T^+ in T_Σ.

One can also regard T_Σ^+ as a *quotient* of T_Σ, as follows (*see* [O], § 1.3). Note first that the pair (T, T^+) acts on the pair (T_Σ, T_Σ^+). Let $T^c \cong (S^1)^n$ denote the maximal compact subgroup of T (this is written $c-T$ in [AMRT]). The following is evident:

2.1.1. PROPOSITION. – *The composite mapping*

$$T_\Sigma^+ \overset{\iota}{\hookrightarrow} T_\Sigma \overset{\pi}{\to} T_\Sigma/T^c$$

is a homeomorphism.

Let $\partial T_\Sigma^+ = T_\Sigma^+ - T^+$. We have:

2.1.2. COROLLARY. – *The composite mapping*

$$\partial T_\Sigma^+ \overset{\iota}{\hookrightarrow} \partial T_\Sigma \overset{\pi}{\to} (\partial T_\Sigma)/T^c$$

is a homeomorphism.

2.1.3. PROPOSITION. – *The space ∂T_Σ^+ is* PL-*isomorphic to the polyhedral complex dual to $\hat\Sigma$.*

Proof. – We have:

$$\partial T_\Sigma^+ = \bigcup_{\sigma\in\Sigma} \partial T_\sigma^+ = \bigcup_{\sigma\in\Sigma} \operatorname{orb}(\sigma),$$

where $\operatorname{orb}(\sigma)$ denotes the unique closed T^+-orbit in T_σ^+, a cell whose dimension is the codimension of σ in $\mathbb{R}^n$. Moreover, one has that $\sigma' \subset \bar\sigma$ if and only if $\overline{\operatorname{orb}(\sigma')} \supset \operatorname{orb}(\sigma)$. These are the data that define the dual of $\hat\Sigma$.

2.2. THE SIMPLICIAL COMPLEXES ASSOCIATED TO THE TORUS EMBEDDINGS. – In the remainder of this section, we fix F and study the topology of the boundary of $M_{\Gamma,\Sigma}$ near ${}^{<}Z_{F,\Sigma}$. Thus we drop the subscript F until section 2.5.

Let $\Sigma = \Sigma_F$ be as in Section 1.3. For $\sigma\in\Sigma$, let $\mathrm{Sk}^1(\sigma)$ denote the set of one-dimensional faces of σ. In order to simplify the exposition, we assume from now on that the elements of Σ are simplicial cones (as can always be arranged by subdivision), viz.

(2.2.1) $$\text{For all } \sigma\in\Sigma, \qquad \dim(\sigma) = \operatorname{Card} \mathrm{Sk}^1(\sigma).$$

The condition (2.2.1) implies that $\hat\Sigma$ is a simplicial complex, whose i-simplices are in natural one-to-one correspondence with the $(i+1)$-dimensional cones of Σ. Also, let $\hat\Sigma^c$ denote the subcomplex of $\hat\Sigma$ given by

(2.2.2) $$\{\hat\sigma\in\hat\Sigma | \sigma\in\Sigma^c\}.$$

Both $\hat\Sigma$ and $\hat\Sigma^c$ are Γ_l-equivariant.

We need to study more closely the topology of the simplicial complexes

$$\Gamma_l\backslash\hat{\Sigma} \supset \Gamma_l\backslash\hat{\Sigma}^c, \tag{2.2.3}$$

and their relation to the boundary divisor $Z_{F,\Sigma}$ via 2.1.3. It turns out that, under mild additional hypotheses, these spaces actually contain the most essential information about the topology of the boundary; to be precise, it is possible to construct a homotopy equivalence from the "real quotient" [*cf.* (2.1)] of ${}^{<}Z_{F,\Sigma}$ to fiber bundles with fiber the simplicial complexes in (2.2.3), in terms of the coordinates of the cone C_F. We now set:

$$(2.2.4)\quad \begin{cases} \text{(i)} & \partial\Sigma = \{\sigma \in \Sigma | \sigma \subset \partial C_F\} \\ \text{(ii)} & \partial\hat{\Sigma} = \{\hat{\sigma} | \sigma \in \partial\Sigma\} \\ \text{(iii)} & \hat{\Sigma}' = \hat{\Sigma} - \partial\hat{\Sigma} \end{cases}$$

(N.B. – $\hat{\Sigma}'$ is not a simplicial subcomplex of $\hat{\Sigma}$), and note that

$$\hat{\Sigma}^o \subset \hat{\Sigma}' \subset \hat{\Sigma}. \tag{2.2.5}$$

2.2.6. Definition. – Σ *is said to have* full *boundary if every cone in* Σ*, all of whose edges are in* $\partial\Sigma$*, is itself in* $\partial\Sigma$ (cf. [ES], II, 9.2).

When Σ has full boundary, one can describe Σ^o as the set of cones $\sigma \subset \bar{C}_F$ for which some edge (*i.e.*, one-dimensional face) lies in the interior. It is always the case that Σ^c is the subfan of cones in Σ such that *every* edge lies in the interior.

It is clear that (2.2.6) induces a corresponding notion for $\hat{\Sigma}$, expressible in terms of its simplices and vertices, and conversely, so the two are equivalent. We observe that given any Σ, the process of barycentric subdivision produces a Γ_l-equivariant refinement $\Sigma^{(1)}$ of Σ having full boundary (*cf.* [ES], II, 9.4). Thus:

(2.2.7) *One can always assume without loss that* Σ *has full boundary.*

2.2.8. Proposition. – *If* (2.2.7) *holds, the simplicial complex* $\Gamma_l\backslash\hat{\Sigma}$ *is a (piecewise-linear) deformation retract of* $\Gamma_l\backslash\hat{\Sigma}'$.

Proof. – Since Σ has full boundary, it follows that $\partial\hat{\Sigma}$ has a regular neighborhood in $\hat{\Sigma}^{(1)}$ (see [*loc. cit.*], Ch. II, § 9). We claim that our assertion is a variant of that; indeed it is a case of the following construction.

First, let σ be a simplex, and τ any face of σ. Then one has canonically that $\sigma = \tau \star \alpha$, where α is the face opposite to τ, and $\star$ denotes join (by convention, $\tau\star\varnothing = \tau$). Whenever $\alpha \neq \varnothing$, this displays σ as the quotient of $\tau \times I \times \alpha$, where I is the closed interval [0, 1], in which $\tau \times \{0\} \times \alpha$ projects onto τ in σ and $\tau \times \{1\} \times \alpha$ projects onto α.

Next, let K be a simplicial complex, and L a full subcomplex ([ES], II, 9.2). Let

$$N = \bigcup_{\tau\in L} \mathrm{Star}(\tau) \quad \text{(notation recalled in 2.4, below).}$$

For any simplex σ of N, put $\tau = \sigma \cap L$, which is non-empty by hypothesis, and a face of σ by fullness; then write $\sigma = \tau \star \alpha(\sigma, \tau)$ as above. By mild abuse of notation, we put

$$\partial N = \bigcup_{\tau \in L} \alpha(\sigma, \tau), \qquad \mathring{N} = N - \partial N.$$

Compare the two assertions:

(i) The linear retraction of I onto $\{0\}$ (for simplices not contained in L) generates a deformation retraction of N onto L (regular neighborhood property);

(ii) The linear retraction of I onto $\{1\}$ generates a deformation retraction of $N-L$ onto ∂N (hence of $K-L$ onto $K-\mathring{N}$). Then (ii) always holds (whereas (i) requires that the retraction be well-defined at the boundary, i.e.

$$\alpha(\sigma, \tau) = \alpha(\sigma', \tau') \Rightarrow \tau' = \tau$$

and this can be arranged by barycentric subdivision using the "I" of the *original* triangulation). For Proposition 2.2.8, we take

$$K = \Gamma_l \backslash \hat{\Sigma}, \qquad L = \Gamma_l \backslash \partial \hat{\Sigma}, \qquad K - \mathring{N} = \Gamma_l \backslash \hat{\Sigma}^c,$$

and apply (ii).

Thus, if we impose (2.2.7), we may assume without loss that Σ has the property:

(2.2.9) $\Gamma_l \backslash \hat{\Sigma}^c$ *is a deformation retract of* $\Gamma_l \backslash \hat{\Sigma}'$.

Since we are assuming that $C_F = \bigcup(\sigma \cap C_F)$ ([AMRT], p. 117, Hyp. 5), we have that $\Sigma' - \{0\}$ is homeomorphic to $C_F - \{0\}$, hence $\Gamma_l \backslash \hat{\Sigma}' \cong \Gamma_l \backslash \hat{\Sigma}^{(1),\prime}$ (non-canonical homeomorphism) is topologically the quotient $X(\Gamma_l)$ of the symmetric space of $(G_{l,F} \cap G^{der})(\mathbb{R})^0/\tilde{A}(\mathbb{R})$ by the arithmetic group Γ_l. Moreover, it is a consequence of reduction theory (see *loc. cit.*, Hyp. 4) that $\Gamma_l \backslash \Sigma^c$ is a *compact* simplicial complex (with boundary), and by (2.2.8) a deformation retract of $\Gamma_l \backslash \hat{\Sigma}'$. We thereby get

2.2.10. COROLLARY. – *The simplicial complexes* $\Gamma_l \backslash \hat{\Sigma}^c$ *and* $\Gamma_l \backslash \hat{\Sigma}^{(1),c}$ *are triangulations of compact topological deformation retracts of the locally symmetric space* $X(\Gamma_l)$.

2.2.11. *Remark.* – $\hat{\Sigma}^{(1),c}$ can be seen as a triangulation of the complex dual to $\hat{\Sigma}$ (*cf.* 2.1.1).

2.3. We recall some of the terminology from ([AMRT], I). We have

(2.3.1) $$T = \Gamma_U \backslash U(\mathbb{C}) = T^c \times i\,U(\mathbb{R}),$$

where T^c is now the compact torus $\Gamma_U \backslash U(\mathbb{R})$. Identifying $i\,U(\mathbb{R})$ with $U(\mathbb{R})$ in the obvious way, let

(2.3.2) $$\text{ord}: \quad T \to U(\mathbb{R})$$

be the projection onto the second factor. This induces a homeomorphism

(2.3.3) $$T^0_{\mathbb{R}} \cong T/T^c \cong U(\mathbb{R}).$$

This permits one to realize the real torus embedding T_Σ/T^c in terms of the ambient space for Σ, as follows.

For $\sigma \in \Sigma$, one has a continuous extension of ord to a mapping

$$\mathrm{ord}_\sigma : \quad T_\sigma \to U_\sigma,$$

where U_σ is the partial compactification of $U(\mathbb{R})$ to be described below, which factors through a homeomorphism $T_\sigma/T^c \cong U_\sigma$. These mappings patch together to define a homeomorphism

$$\mathrm{ord}_\Sigma : \ T_\Sigma/T^c \cong U_\Sigma. \tag{2.3.4}$$

We use this to provide explicit coordinate charts. Let $Q(\sigma)$ be the (unique) minimal set of generators of $\sigma \cap \Gamma_U$. With our running hypothesis that $\partial M_{\Gamma,\Sigma}$ is a divisor with normal crossings, $Q(\sigma)$ is a subset of a basis of Γ_U (*see* [KKMS], p. 14, Thm. 4). First, suppose that σ is a cone of top dimension. Let $\check{Q}(\check{\sigma})$ be the basis dual to $Q(\sigma)$. For $q \in \check{Q}(\check{\sigma})$, put $\varepsilon_q(x) = e^{-2\pi q(x)}$; these define *canonical coordinates*:

$$\underline{\varepsilon}_\sigma : \quad U(\mathbb{R}) \to (\mathbb{R}_{>0})^{\check{Q}(\check{\sigma})}, \tag{2.3.5}$$

a real analytic isomorphism. One obtains U_σ as the partial compactification of $U(\mathbb{R})$ corresponding under $\underline{\varepsilon}_\sigma$ to $(\mathbb{R}_{\geqq 0})^{\check{Q}(\check{\sigma})}$.

2.3.6. *Remark.* – One should keep in mind the elementary fact that $\varepsilon_q(x) \leqq 1$ if and only if $q(x) \geqq 0$. Thus, the closure of $\underline{\varepsilon}_\sigma(\sigma)$ is the closed unit hypercube.

If σ' is a lower-dimensional cone in Σ, let σ be any top-dimensional cone having σ' as a face. Then σ' is defined in σ as the locus of zeros of some subset $Q'(\sigma)$ of $\check{Q}(\check{\sigma})$. Using $\underline{\varepsilon}_\sigma$ again, let $U_{\sigma,\sigma'}$ be the (dense) subset of U_σ corresponding to

$$(\mathbb{R}_{\geqq 0})^{\check{Q}(\check{\sigma})-Q'(\sigma)} \times (\mathbb{R}_{>0})^{Q'(\sigma)}. \tag{2.3.7}$$

One checks that (2.3.7) is independent of σ in the following sense:

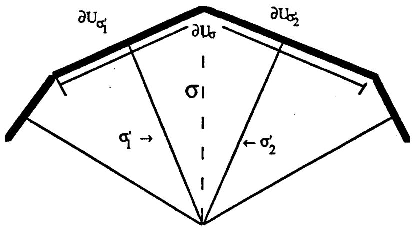
2.3.8. LEMMA. – *If σ_1 and σ_2 are top-dimensional cones containing σ' as a face, then the identity mapping of* $U(\mathbb{R})$ *extends to a diffeomorphism* $U_{\sigma_1,\sigma'} \cong U_{\sigma_2,\sigma'}$.

Proof. – Let $\check{Q}_{\check{\sigma}}(\sigma')$ denote the set of restrictions of $\check{Q}(\check{\sigma})-Q'(\sigma)$ to the linear span $\langle\sigma'\rangle$ of σ'. These give generators of the non-negative $\mathbb{Z}$-valued functionals on $\langle\sigma'\rangle \cap U(\mathbb{Z})$, so this set is independent of σ. It follows that for each $q_1 \in \check{Q}_{\check{\sigma}_1}(\sigma')$, there are a unique $q_2 \in \check{Q}_{\check{\sigma}_2}(\sigma')$ and q' in the $\mathbb{Z}$-span of $Q'(\sigma_2)$ with $q_1 = q_2 + q'$. With this said, the composite diffeomorphism $\underline{\varepsilon}_{\sigma_1}\underline{\varepsilon}_{\sigma_2}^{-1}$ is seen to extend to the boundaries in (2.3.7), as desired.

(2.3.9) *Remark.* – By the above, we have canonical coordinates $\underline{\varepsilon}_{\sigma'}$ on $\langle\sigma'\rangle$, determined by an intrinsic set $\check{Q}(\sigma')$.

We can now write simply $U_{\sigma'}$, and patch together the $U_{\sigma'}$ ($\sigma' \in \Sigma$) to obtain U_Σ. Furthermore, we have the canonical homeomorphisms $T_\sigma/T^c \cong U_\sigma$, which likewise patch together to define (2.3.4). Note that we can apply the same construction to Σ^c, and we get $T_{\Sigma^c}/T^c \cong U_{\Sigma^c}$ by restricting ord_Σ. *See* the figure below for a picture in dimension 2.

(2.3.10)



2.4. We write ∂U_σ for the complement of $U(\mathbb{R})$ in U_σ; by a mild abuse of notation, we can say that it is defined as the locus where at least one $\underline{\varepsilon}_\sigma$-coordinate vanishes. Let $\bar{\sigma}$ denote the closure of σ in U_σ. Note that

$$\sigma = \{x \in \bar{\sigma} | \varepsilon_q(x) > 0 \text{ for all } q \in \check{Q}(\check{\sigma})\}. \tag{2.4.1}$$

If σ' is a face of σ – say then that $\sigma \in \mathrm{Star}(\sigma')$, – put

$$\partial U_{\sigma'}(\sigma) = \partial U_{\sigma'} \cap \bar{\sigma}. \tag{2.4.2}$$

It is useful to keep in mind the following hereditary feature of torus embeddings, which also passes naturally to their real quotients:

2.4.3. PROPOSITION (see [O], 1.3). – *Let Σ be a fan in $\mathbb{R}^n$. The* T*-orbit in* T_Σ *associated to* σ (*the unique closed* T*-orbit in* T_σ) *is itself a torus. Its dimension equals the codimension of* σ *in* $\mathbb{R}^n$, *and its closure in* T_Σ *is the torus embedding determined by the fan in* $\mathbb{R}^n/\langle\sigma\rangle$ *consisting of the projections of the elements of* $\mathrm{Star}(\sigma)$.

2.4.4. COROLLARY. – *If* $\dim\sigma' = 1$, *the closure of* $\partial U_{\sigma'}$ *in* U_Σ *is the real torus embedding determined by the projection of* $\mathrm{Star}(\sigma')$ *in* $\mathbb{R}^n/\langle\sigma'\rangle$.

It is also convenient to introduce the *constellation* of σ':

$$\mathrm{Con}(\sigma') = \{\sigma \in \Sigma | \sigma \cap \sigma' \neq 0\} = \bigcup \{\mathrm{Star}(\tau) | \tau \text{ is an edge of } \sigma'\}.$$

2.4.5. LEMMA.

(i) $\partial U_{\sigma'}(\sigma_1) \cap \partial U_{\sigma'}(\sigma_2) = \partial U_{\sigma'}(\sigma_1 \cap \sigma_2)$.

(ii) *For any* $\sigma' \in \Sigma^c$, *we have* $\bigcup \partial U_{\sigma'}(\sigma) = \partial U_{\sigma'}$, *where* σ *runs over the (finite) set of top-dimensional cones in* $\mathrm{Con}(\sigma')$. *Thus,* $\partial U_{\Sigma^c} \subset \bigcup_{\sigma\in\Sigma} \bar{\sigma}$.

Proof. – It is easy to see that whenever σ is a face of τ, $\bar{\sigma}$ [defined by (2.4.1)] is closed in U_τ. Then (i) follows directly from (2.4.2). To prove (ii), one must verify that $\partial U_{\sigma'} \subset \bigcup \bar{\sigma}$, where the union is as above. Since $\sigma' \in \Sigma^c$, this union is finite. Also, the union of $\{\partial U_\tau | \tau$ a 1-dimensional face of $\sigma'\}$ is dense in $\partial U_{\sigma'}$. It thus suffices to consider the case where $\dim\sigma' = 1$; then $\mathrm{Con}(\sigma') = \mathrm{Star}(\sigma')$ is just the union of the closed top-dimensional cones of which σ' is an edge. Our assumptions on σ' imply that in $U(\mathbb{R})$,

σ' is in the interior of its star, *i.e.*, $\hat{\sigma}' \subset \operatorname{Int}(\bigcup \hat{\sigma})$. The desired assertion now follows from 2.4.4.

For each top-dimensional $\sigma \in \Sigma$, we use the corresponding canonical coordinates to define the linear homotopy:

$$(2.4.6) \qquad \begin{cases} h_\sigma : \quad [0,1] \times \bar{\sigma} \to \bar{\sigma} \\ \varepsilon_q(h_\sigma(t,x)) = t\,\varepsilon_q(0) + (1-t)\,\varepsilon_q(x). \end{cases}$$

(N.B., $\varepsilon_q(0) = 1$ for all q.) For $t<1$, its restriction $h_{\sigma,t}$ to $\{t\} \times \bar{\sigma}$ is one-to-one; for $t>0$, the image of $h_{\sigma,t}$ is contained in $U(\mathbb{R})$. Note that if q vanishes on a face of σ, then $\varepsilon_q \circ h_\sigma \equiv 1$ on that face, hence $q \circ h_\sigma \equiv 0$. It follows that

(2.4.7) *The homotopy respects the simplicial structure of* σ.

Moreover, to each proper subset $I \subset \check{Q}(\check{\sigma})$, let $p_I \in \partial U_\sigma$ be the vertex of the hypercube such that $\varepsilon_q(p_I) = 1$ for $q \in I$, $\varepsilon_q(p_I) = 0$ for $q \notin I$, and let $\sigma(I)$ be the face of σ defined by the vanishing of all $q \in I$. Then it is clear that

(2.4.8) *For each* I, $h_\sigma(t, p_I)$ *traces the barycentric ray in* $\sigma(I)$ *for* $t \in (0,1)$.

Let $(C_F)_\Sigma$ denote $\bigcup_{\sigma \in \Sigma} \bar{\sigma}$, endowed with the weak topology, and do likewise for other fans; let $|\Sigma|$ denote the support of Σ. Then $(C_F)_\Sigma$ admits a continuous injection into U_Σ. However, it is usually not an embedding, for the induced topology on

$$(C_F)_{\Sigma,f} - \{0\} = \bigcup_{\sigma \in \Sigma} (\sigma - \{0\}) = |\Sigma| - \{0\}$$

is, in fact, a Satake topology (this fact, to be proved in a forthcoming paper of the two authors, will not be used here), generally finer than the one induced from $U(\mathbb{R})$. We repeat that the topology on the union above is the weak topology; this is also the topology imposed on $\hat{\Sigma}$ in 2.4.11, below. On the other hand, things are nicer at the boundary:

$$(2.4.9) \qquad \partial(C_F)_\Sigma := (C_F)_\Sigma - |\Sigma| \quad \textit{is embedded in } \partial U_\Sigma.$$

To be precise, after reminding ourselves that

$$\partial\bar{\sigma} := \partial U_\sigma \cap \bar{\sigma} \subsetneq \partial U_\sigma,$$

we have [*cf.* (2.4.5, (ii)]

$$\overline{\partial U_{\Sigma^c}} \subseteq \partial(C_F)_\Sigma \subseteq \partial U_\Sigma,$$

and the inclusions are proper unless $\Sigma^c = \Sigma$. Moreover, the inclusions are embeddings, for the topology of ∂U_Σ *is* the weak topology. From 2.1.3, we have that

$$\overline{\partial U_{\Sigma^c}} = \bigcup_{\tau \in \Sigma^c} \overline{\partial U_\tau}$$

is the dual complex of $\hat{\Sigma}^c$, and is realized inside $\hat{\Sigma}$ as $\hat{\Sigma}^{(1),c}$ [cf. (2.3.10)]. Finally, it is clear that the Γ_l-action on C_F extends continuously to $(C_F)_\Sigma$.

The argument used in the proof of Lemma 2.3.8 shows that the h_σ's patch together to define a continuous homotopy

$$h_\Sigma: \quad [0, 1] \times (\mathrm{C_F})_\Sigma \to (\mathrm{C_F})_\Sigma,$$

which tautologically respects the cone decomposition.

From (2.4.6), we see at once:

2.4.10. PROPOSITION. – *The homotopy* h_Σ *is piecewise linear and* Γ_l*-equivariant. For* $0<t<1$, $h_{\Sigma, t}$ *moves* $\partial(\mathrm{C_F})_\Sigma$ *homeomorphically into* $(\mathrm{C_F})_{\Sigma, f} - \{0\}$.

Henceforth, we write $\partial(\mathrm{C_F})_\Sigma(t)$ for $h_{\Sigma, t}(\partial(\mathrm{C_F})_\Sigma)$, etc. Let $\partial\hat{\Sigma}^c$ be the boundary of $\hat{\Sigma}^c$ as a PL manifold with boundary.

2.4.11. PROPOSITION. – *The projection of* $\partial(\mathrm{C_F})_\Sigma(t)$ *for* $0<t<1$, *onto* $((\mathrm{C_F})_{\Sigma, f} - \{0\})/\mathbb{R}_+^\times$ *defines a* Γ_l*-equivariant homeomorphism of the triples* $(\partial(\mathrm{C_F})_\Sigma, \overline{\partial \mathrm{U}_{\Sigma^c}}, \partial(\mathrm{C_F})_{\Sigma^c})$ *and* $(\hat{\Sigma}, \hat{\Sigma}^{(1), c}, \hat{\Sigma}^c)$, *with* $\partial(\mathrm{C_F})_{\partial\Sigma}$ *going to* $\partial\hat{\Sigma}$, *etc.*

Proof. – We first verify that $\partial(\mathrm{C_F})_\Sigma(t)$ is transverse to the lines of dilation in $\mathrm{C_F}$. In terms of canonical coordinates, we need to check that on any face of the boundary, the equations

$$[t + (1-t)\,\varepsilon_q(x)] = [t + (1-t)\,\varepsilon_q(y)]^r \quad q \in \check{\mathrm{Q}}(\check{\sigma})$$

imply that $r=1$. But there is some q for which $\varepsilon_q \equiv 0$ on that face, which gives $t=t^r$, thus yields the desired conclusion for $\partial(\mathrm{C_F})_\Sigma$, $\partial(\mathrm{C_F})_{\Sigma^c}$, and $\partial(\mathrm{C_F})_{\partial\Sigma}$.

Now for any 1-dimensional $\sigma \in \Sigma^c$ and any top-dimensional $\sigma' \in \mathrm{Star}(\sigma)$, let $\mathrm{B}(\sigma, \sigma') \subset \hat{\Sigma}^{(1), c}$ be the hypercube whose vertices are the barycenters of all $\hat{\sigma}'' \in \hat{\Sigma}$ such that $\hat{\sigma} \subset \hat{\sigma}'' \subset \hat{\sigma}'$. It follows from (2.4.8) that for $0<t<1$, the projection of $h_\Sigma(t \times \partial\mathrm{U}_{\sigma'}(\sigma))$ onto $((\mathrm{C_F})_{\Sigma f} - \{0\})/\mathbb{R}_+^\times$ is just $\mathrm{B}(\sigma, \sigma')$. The assertion regarding $\overline{\partial\mathrm{U}_{\Sigma^c}}$ now follows from Lemma 2.4.5.

The following is immediate:

2.4.12. COROLLARY. – *Let* Y *denote any* Γ_l*-invariant cross-section to the dilations of* $\mathrm{C_F}$. *Then there exists a* Γ_l*-equivariant homeomorphism from* $\partial(\mathrm{C_F})_\Sigma - \partial(\mathrm{C_F})_{\partial\Sigma}$ *onto* Y. *If* $\bar{\mathrm{Y}}$ *is such a cross-section for* $(\mathrm{C_F})_\Sigma$, *then there is a* Γ_l*-equivariant homeomorphism from* $\partial(\mathrm{C_F})_\Sigma$ *onto* $\bar{\mathrm{Y}}$.

By 2.2.10, we also have:

2.4.13. COROLLARY. – $\Gamma_l\backslash\overline{\partial\mathrm{U}_{\Sigma^c}}$ *has the homotopy type of* $\mathrm{X}(\Gamma_l)$.

2.5. We restore the subscript F in our notation. Recall the spaces $\mathrm{D}_{\mathrm{F}, \sigma}$ and $\mathrm{D}_{\mathrm{F}, \Sigma}$ from 1.3. Let Y be as in 2.4.12, and put

$$\mathrm{C_F}(\mathrm{Y}) = \{ry \in \mathrm{C_F} \mid r > 1,\ y \in \mathrm{Y}\}. \tag{2.5.1}$$

For $t \in \mathrm{F}$, let $h_t(\cdot, \cdot)$ be the real-bilinear form on $\mathbb{C}^a \cong \mathrm{W_F}(\mathbb{R})/\mathrm{U_F}(\mathbb{R})$ given by the theory of Siegel domains of the third kind (coordinates as in (1.2.4); *cf.* [AMRT], p. 239), and let

$$\mathrm{D_F}(\mathrm{Y}) = \{(z, v, t) \in \mathrm{D} \mid \mathrm{Im}(z) - h_t(v, v) \in \mathrm{C_F}(\mathrm{Y})\}, \tag{2.5.2}$$

Let $D_{F,\Sigma}(Y) \subset D_{F,\Sigma}$ denote the interior of the closure of $\Gamma'_F \backslash D_F(Y)$ in $(M'_F)_\Sigma$.

The morphism $\pi_{F,\Sigma} : (M'_F)_\Sigma \to M_F$ of (1.3.2) restricts to a mapping

$$\pi_Y : \quad D_{F,\Sigma}(Y) \to M_F.$$

When Y is understood, we omit the "Y", writing $D_{F,\Sigma}(\alpha)$ for $D_{F,\Sigma}(\alpha Y)$, etc. The following is well-known, deduced in ([AMRT]: Ch. 2, § 5) from the theory of cores:

2.5.3. LEMMA. – *Let* $\varphi_{F,\Sigma} : D_{F,\Sigma} \to M_{\Gamma,\Sigma}$ *be as in* (1.4.1). *There exists* Y *as above, such that the set of interiors of the closures in* $M_{\Gamma,\Sigma}$ *of*

$$W(\alpha) = \varphi_{F,\Sigma}(\Gamma_{l,F} \backslash D_{F,\Sigma}(\alpha Y))$$

for $\alpha > 1$, *is a fundamental system of neighborhoods of* Z_{F,Σ_F} *in* $M_{\Gamma,\Sigma}$.

2.6. We next return to the setting of 2.1, and discuss the C^∞ de Rham sheaves for T^+_Σ and ∂T^+_Σ, from all angles suggested therein. We begin with T^+_Σ. Let $\mathcal{A}^\bullet(T^+_\Sigma)$ denote its intrinsic de Rham complex as a manifold with corners, and $\mathcal{A}^\bullet(T_\Sigma/T^c)$ denote its de Rham complex as a quotient, in the sense of Koszul [Ko] (*see* also [Sj]). The latter is, by definition, the subcomplex of $\pi_* \mathcal{A}^\bullet(T_\Sigma)$ consisting of forms which are both invariant under the action of T^c and annihilated by contraction with vectors tangent to the T^c-action. Clearly, the diagram in 2.1.1 defines a diagram

$$\mathcal{A}^\bullet(T_\Sigma/T^c) \hookrightarrow \pi_* \mathcal{A}^\bullet(T_\Sigma) \xrightarrow{\iota^*} \mathcal{A}^\bullet(T^+_\Sigma). \tag{2.6.1}$$

Since we shall be making local statements, we can restrict our attention to a single $\sigma \in \Sigma^0$:

$$T^+_\Sigma \hookrightarrow T_\sigma \to T_\sigma/T^c,$$

is just

$$(\mathbb{R}_{\geqq 0})^n \hookrightarrow \mathbb{C}^n \to \mathbb{C}^n/T^c.$$

Let H denote the subgroup $(\mathbb{Z}/2\mathbb{Z})^n$ of T^c. Then H acts on $\mathbb{R}^n$ by the sign ± 1 on each factor.

2.6.2. PROPOSITION. – $\mathcal{A}^\bullet(\mathbb{C}^n/T^c) \cong \mathcal{A}^\bullet(\mathbb{R}^n/H) = (\pi_* \mathcal{A}^\bullet(\mathbb{R}^n))^H$.

Proof. – It is enough to consider the case $n=1$, where the assertion can be checked using the usual polar coordinates. Indeed, $\mathcal{A}^\bullet(\mathbb{R}/\{\pm 1\})$ is generated at $r=0$ by 1 and rdr over the even functions of r.

2.6.3. *Remark.* – Explicitly, this gives the following. For any subset S of $\{1, 2, \ldots, n\}$, put

$$H^S = \{(t_1, \ldots, t_n) \in H : t_j = 1 \text{ if } j \notin S\}$$
$$\check{F}_S = \{(x_1, \ldots, x_n) \in \mathbb{R}^n : x_j = 0 \text{ if and only if } j \in S\},$$

and the open faces of the corner,

$$F_S = \check{F}_S \cap (\mathbb{R}_{\geqq 0})^n.$$

Then $H_S \underset{\text{def}}{=} H/H^S$ acts on $\check{F}_S$, hence on $\mathcal{A}^\bullet(\check{F}_S)$, and the restriction to F_S of

$$\operatorname{coker}\{\mathcal{A}^\bullet(\mathbb{R}^n/H) \to \mathcal{A}^\bullet((\mathbb{R}_{\geqq 0})^n)\},$$

which is supported at the boundary, is isomorphic to

$$\operatorname{im}\{\oplus(\pi_* \mathcal{A}^\bullet(\check{F}_S))^\chi \to \mathcal{A}^\bullet(F_S)\},$$

where the sum runs over the non-trivial characters of H^S.

2.6.4. Corollary. – *The inclusion*

$$\mathcal{A}^\bullet(\mathbb{C}^n/T^c) \to \mathcal{A}^\bullet((\mathbb{R}_{\geqq 0})^n)$$

is a quasi-isomorphism.

2.6.5. *Remark.* – When $n=1$, this is asserting that at $r=0$, the germs of forms on the r-line decompose into the complexes of odd and even forms (note that if $f(r)$ is an even function, then $f(r)\,dr$ is odd, as $(-1)^*\,dr = -dr$), and that the little cohomology that there is, is carried by the even summand.

2.7. We now turn our attention to the boundary, and consider

$$\text{(2.7.1)}\qquad \begin{cases} \mathcal{A}^\bullet(T_\Sigma/T^c)|_{(\partial T_\Sigma/T^c)} & \to \mathcal{A}^\bullet(T_\Sigma^+)|_{\partial T_\Sigma^+} \\ \qquad\downarrow & \qquad\downarrow \\ \mathcal{A}^\bullet(\partial T_\Sigma/T^c) & \overset{\alpha}{\to}\ \mathcal{A}^\bullet(\partial T_\Sigma^+) \end{cases}$$

For the bottom line, we can use the simplicial de Rham complex for a divisor with normal crossings or a polyhedral complex (defined in terms of those of its components or faces) and the corresponding Koszul construction. (Alternatively, the vertical arrows are seen to be surjective.) That α is a quasi-isomorphism is a consequence of the following standard spectral sequence argument (*cf.* [D6], § 5):

2.7.2. Proposition. – *Let* $\{\mathcal{A}_\sigma^\bullet\}_{\sigma\in\Sigma}$ *and* $\{\mathcal{B}_\sigma^\bullet\}_{\sigma\in\Sigma}$ *define simplicial sheaves* $\underline{\mathcal{A}}^\bullet$ *and* $\underline{\mathcal{B}}^\bullet$, *and let* $\underline{\Phi}: \underline{\mathcal{A}}^\bullet \to \underline{\mathcal{B}}^\bullet$ *be a morphism. Suppose that for all* σ, Φ_σ *is a quasi-isomorphism. Then so is* $\underline{\Phi}$.

Now let $\mathcal{A}^{0,\bullet}(T_\Sigma)$ be the Dolbeault resolution of $\mathcal{O}_{T_\Sigma}$, and define $\mathcal{A}^{0,\bullet}(\partial T_\Sigma)$ likewise.

2.7.3. Proposition. – *The projection*

$$\mathcal{A}^\bullet(T_\Sigma/T^c) \to \pi_* \mathcal{A}^{0,\bullet}(T_\Sigma)^{T^c}$$

is an isomorphism of complexes. Moreover, the same is true of

$$\mathcal{A}^\bullet(\partial T_\Sigma/T^c) \to \pi_* \mathcal{A}^{0,\bullet}(\partial T_\Sigma)^{T^c}.$$

Proof. – We consider only the case of T_Σ, for the assertion for ∂T_Σ is just the simplicial version. In terms of the usual coordinates of $T_\sigma \cong \mathbb{C}^n$, the differentials

$$z_j\,d\bar{z}_j = r_j\,dr_j - ir_j^2\,d\theta_j \quad (1 \leqq j \leqq n),$$

the (0, 1)-component of $d|z_j|^2$, are (T^c)-invariant, hence generate $\pi_* \mathcal{A}^{0,\bullet}(T_\Sigma)^{T^c}$, as an exterior algebra, over $\pi_* \mathcal{A}^0(T_\Sigma)^{T^c}$. By what we said earlier, along F_S the latter is given by smooth functions of the r_j's that are even with respect to the variable r_j whenever $j \in S$ (hence are functions of r_j^2 if $j \in S$). The isomorphism is effected simply by replacing dr_j by $1/2\,(dr_j - ir_j\,d\theta_j)$.

2.8. We will need a relative and equivariant version of 2.7.3. It is useful to introduce the following abstract setting. Let $\pi : E \xrightarrow{\pi} B$ be a holomorphic fiber bundle, on which the torus T acts vertically. We assume that the fiber is a SNC torus embedding for T. Let Z be a T^c-invariant subset of E that satisfies the conditions:

$$(2.8.1)\quad \begin{cases} \text{(i)} & \pi' = \pi|_Z \text{ is a fiber bundle,} \\ \text{(ii)} & Z \text{ is open in the closure of a union of T-orbits,} \end{cases}$$

We decompose π' as

$$(2.8.2)\qquad Z \xrightarrow{p} Z/T^c \xrightarrow{q} B.$$

By hypothesis, we have locally on B that

$$(2.8.3)\qquad E \cong T_\Sigma \times B,$$

and then p is deduced from $T_\Sigma \to T_\Sigma/T^c$. Since we will be working locally on B, we assume that (2.8.3) holds. For simplicity, assume that Z is open in $T_\Sigma \times B$ (otherwise, we must argue simplicially, as in 2.7). Again, p is induced by the product of n factors of $\mathbb{C} \to \mathbb{R}_{\geq 0}$. But then for any (relatively) open cube K in standard position in $T/T^c \cong (\mathbb{R}_{\geq 0})^n$, $p^{-1}(K)$ is a product of discs, punctured discs and annuli, hence is Stein. Thus we deduce:

2.8.4. Proposition. – *For all* $i>0$, $R^i p_* \mathcal{O}_Z = 0$.

Let $\mathcal{E}$ be a locally-free sheaf on B, and put $\mathcal{V} = \pi^* \mathcal{E}$. Then:

2.8.5. Corollary. – *For all* $i>0$, $R^i p_* \mathcal{V} = 0$. *Thus the morphism* $p_* \mathcal{V} \to \mathbf{R} p_* \mathcal{V}$ *is a quasi-isomorphism.*

If we represent $\mathbf{R} p_* \mathcal{V}$ by the direct image of the Dolbeault complex for $\mathcal{V}$, and take invariants for the compact group T^c, we obtain:

2.8.6. Corollary. – *The natural mapping* $(p_* \mathcal{V})^{T^c} \to [p_* (\mathcal{A}^{0,\bullet}(Z) \otimes \mathcal{V})]^{T^c}$ *is a quasi-isomorphism.*

There is one more elementary ingredient:

2.8.7. Proposition. – *The natural inclusion*

$$q^{-1}\,\mathcal{E} \to (p_* \mathcal{V})^{T^c}$$

is an isomorphism.

Proof. – We may assume that $\mathcal{E} = \mathcal{O}_B$. The discussion preceding 2.8.4 shows that we can reduce to issues about functions of one complex variable. The assertion here comes down to the fact that a holomorphic function which is independent of θ is necessarily constant.

2.8.8. COROLLARY. – *The mapping* $q^{-1}(\mathcal{A}^{0,\bullet}(\mathrm{B}) \otimes \mathcal{E}) \to [p_*(\mathcal{A}^{0,\bullet}(\mathrm{Z}) \otimes \mathcal{V})]^{\mathrm{T}^c}$ *is a quasi-isomorphism.*

2.8.9. *Remark.* – If we write $q^{-1}\mathcal{E} = p_* \pi^{-1}\mathcal{E}$, we see that 2.8.8 can be viewed as a generalization of 2.7.3. Note that $\pi^{-1}\mathcal{E}$ is the sheaf of relatively horizontal sections of the natural relative connection (*see* [D]: I, 2.20) on $\mathcal{V}$.

We will also have to permit $\pi : \mathrm{E} \to \mathrm{B}$ to degenerate. The kind of local structure that occurs at the boundary is a surjective morphism of fiber bundles of torus embeddings:

$$\begin{array}{ccc} \mathrm{Z}^* \subset \mathrm{E}^* & \to & \mathrm{B}^* \\ \searrow & & \swarrow \\ & \mathrm{S} & \end{array}$$

(*i. e.*, locally the product of a morphism of torus embeddings with a parameter space). If h: $\mathrm{T}_1 \to \mathrm{T}_2$ is the associated morphism of tori, then the torus in the preceding corresponds to $\mathrm{T} = (\mathrm{Ker}\, h)^0$ here, and we assume Z^* to satisfy (2.8.1) (ii) relative to T_1.

Let $\bar{p} : \mathrm{E}^* \to \mathrm{E}^*/\mathrm{T}^c$ be the quotient map, and $\bar{\pi} = \bar{q} \circ \bar{p}$ the factorization of $\bar{\pi}$ [*cf.* (2.8.2)].

2.8.10. PROPOSITION. – *For all* $i > 0$, $\mathrm{R}^i \bar{p}_* \mathcal{O}_{\mathrm{Z}^*} = 0$.

Proof. – The argument is similar to the one for 2.8.4. Since the question is local on $\mathrm{E}^*/\mathrm{T}^c$, we can quickly reduce to the case where S is a point, and then to an affine torus embedding $\mathrm{E}^* \cong \mathbb{C}^m$. It is always possible to choose a basis $\{t_j \,(1 \leqq j \leqq n),\ s_i \,(1 \leqq i \leqq m-n)\}$ of the characters of T_1 such that:

(2.8.10.1) $\begin{cases} \text{(i)} & \text{each } s_i \text{ is pulled back from } \mathrm{T}_2, \\ \text{(ii)} & s_i \text{ defines a regular function on } \mathrm{E}^*. \end{cases}$

Then $\mathrm{T} = \bigcap \mathrm{Ker}\, s_i$, and the t_j's give a basis for the characters of T. The coordinates on E^* are of the form

(2.8.10.2) $$z_k = \mu_k(t)\, v_k(s) = \prod (t_j)^{a_{j,k}} \cdot \prod (s_i)^{b_{i,k}},$$

with $a_{j,k}$, $b_{i,k} \in \mathbb{Z}$. This can be inverted, yielding

(2.8.10.3) $$t_j = \prod (z_k)^{c_{j,k}}, \qquad s_i = \prod (z_k)^{d_{i,k}},$$

with $c_{j,k}$, $d_{i,k} \in \mathbb{Z}$, and $d_{i,k} \geqq 0$.

Now, a set of the form $\bar{p}^{-1}(\mathrm{U})$ is the same as a T^c-invariant neighborhood of a single T^c-orbit. The T^c-orbit of $\mathbf{z}^0 \in \mathrm{E}^*$ is given parametrically by:

(2.8.10.4) $$z_k(\boldsymbol{\theta}) = z_k^0\, e^{i\langle \mathbf{a}_k, \boldsymbol{\theta}\rangle} \quad (\boldsymbol{\theta} \in \mathbb{R}^n),$$

where $\mathbf{a}_k$ is the vector with components $a_{j,k}$, and is thus defined by the equations:

$$\text{(2.8.10.5)} \qquad \begin{cases} |z_k| = |z_k^0| \quad (1 \leqq k \leqq m), \\ \prod (z_k)^{d_{i,k}} = \prod (z_k^0)^{d_{i,k}} \quad (1 \leqq i \leqq m-n). \end{cases}$$

It follows from (2.8.10.5) that a set $\bar{p}^{-1}$ (U), with U "small", is given by specifying that each $r_k = |z_k|$ belong to a small interval, and in addition, the values of the holomorphic function $f_i(\mathbf{z}) = \prod (z_k)^{d_{i,k}}$ belongs to a small disc (for each i). Each condition separately would define a domain of holomorphy; since the intersection of domains of holomorphy is again a domain of holomorphy (*see* [Kr]: (3.4.5)), and domains of holomorphy have no higher cohomology for coherent sheaves, we are done.

Let $\bar{\mathcal{E}}$ be a locally-free sheaf on B*, and put $\bar{\mathcal{V}} = \bar{\pi}^* \bar{\mathcal{E}}$. Generalizing 2.8.5, we have:

2.8.11. COROLLARY. – *For all* $i>0$, $\mathrm{R}^i \bar{p}_* \bar{\mathcal{V}} = 0$. *Thus,* $\bar{p}_* \bar{\mathcal{V}} \to \mathbf{R}\bar{p}_* \bar{\mathcal{V}}$ *is a quasi-isomorphism.*

We next verify that 2.8.7 carries over to the present setting:

2.8.12. PROPOSITION. – *Consider the natural inclusion*

$$\iota : \quad \bar{q}^{-1} \bar{\mathcal{E}} \to (\bar{p}_* \bar{\mathcal{V}})^{\mathrm{T}^c}.$$

Suppose that for some complex subvariety V *of* B*, *the map* ι *is an isomorphism outside* $\bar{q}^{-1}$ (V). *Then* ι *is an isomorphism.*

Proof. – We use the notation of 2.8.7 for the restrictions over B = B* – V; there is no loss in taking V to be a hypersurface. Let j : B → B* denote the inclusion. It is clear that every element of $(\bar{p}_* \bar{\mathcal{V}})^{\mathrm{T}^c}$ comes from an element of $\bar{q}^{-1}(j_* \mathcal{E})$. If the latter had singularities along V (relative to $\bar{\mathcal{E}}$), there would also be singularities along $\bar{\pi}^{-1}$ (V) ⊂ Z*. This shows that ι is surjective, as required.

2.9. EQUIVARIANT SHEAF COHOMOLOGY. – Although equivariant cohomology (in algebraic topology) and sheaf cohomology have been around for a long time, we failed to find a systematic treatment of equivariant sheaf cohomology in the literature. (A very brief discussion can be found in [J].) We give a terse account here. We begin by bringing in some customary terminology. Let B be a Hausdorff topological space on which the discrete group Γ acts, and $\mathcal{S}$ a sheaf on B admitting an equivariant action of Γ.

2.9.1. DEFINITION. – *The* Γ-equivariant cohomology *of* $\mathcal{S}$ *is the cohomology of the complex* $\mathrm{RHom}_{\mathbb{Z}[\Gamma]}(\mathbb{Z}, \mathbf{R}\Gamma(\mathrm{B}, \mathcal{S}))$. *It is denoted* $\mathrm{H}^{\bullet}_{\Gamma}(\mathrm{B}, \mathcal{S})$.

There is a straightforward generalization of this notion to equivariant hypercohomology for complexes of sheaves with equivariant Γ-action.

There are some choices to be made in the above definition. For $\mathbf{R}\Gamma$, one can simply take the sections of the canonical resolution of Godement ([G], p. 167). However, because "Hom" is a bi-functor, there are two alternatives for "RHom" that can be selected according to purpose: use either

(i) a $\mathbb{Z}[\Gamma]$-injective resolution of $\mathbf{R}\Gamma(\mathrm{B}, \mathcal{S})$, or

(ii) a $\mathbb{Z}[\Gamma]$-projective resolution of $\mathbb{Z}$.

A typical choice for (ii) is a suitable chain complex with $\mathbb{Z}$-coefficients on $E\Gamma$ (a contractile space on which Γ acts freely).

Equivariant cohomology is a cohomological functor with the following basic property:

(2.9.2) If the action of Γ on B is free, then $H^\bullet_\Gamma(B, \mathcal{S}) \cong H^\bullet(\Gamma\backslash B, \mathcal{S}_\Gamma)$, where $\mathcal{S}_\Gamma$ denotes the sheaf $[\psi_* \mathcal{S}]^\Gamma$ of Γ-invariants in $\psi_* \mathcal{S}$ and $\psi : B \to \Gamma\backslash B$ is the quotient mapping.

As with ordinary sheaf cohomology, the definition gives rise to the spectral sequence for the equivariant cohomology of a filtered complex. In particular, if $\pi : Q \to B$ is Γ-equivariant, and $\mathcal{G}$ is an equivariant sheaf on Q, the canonical filtration of $R\pi_* \mathcal{G}$ determines the Leray spectral sequence:

(2.9.3) $$E_2^{p,q} = H^p_\Gamma(B, R^q\pi_* \mathcal{G}) \Rightarrow H^{p+q}_\Gamma(Q, \mathcal{G}).$$

Thus we have:

2.9.4. Proposition. – *Let $\pi : Q \to B$ be a Γ-equivariant mapping of topological spaces, with Γ acting freely on Q. Then there is a mapping*

$$H^\bullet_\Gamma(B, \mathcal{S}) \to H^\bullet(\Gamma\backslash Q, (\pi^{-1}\mathcal{S})_\Gamma).$$

It is an isomorphism whenever $\mathcal{S} \to R\pi_*(\pi^{-1}\mathcal{S})$ is a quasi-isomorphism.

2.9.5. *Remarks.*

(i) One often takes $Q = B \times E\Gamma$ in 2.9.3 (the *Borel construction*); one then has isomorphism in 2.9.4. Indeed, one can use it to *define* equivariant cohomology (*cf.* [J]).

(ii) From 2.9.1, one obtains a spectral sequence

$$E_2^{p,q} = H^p(\Gamma, H^q(B, \mathcal{S})) \Rightarrow H^{p+q}_\Gamma(B, \mathcal{S}).$$

Consider next the case where Γ acts trivially on a space that we now call M. Then for any Γ-equivariant sheaf $\mathcal{F}$ on M, we have

$$\mathrm{Hom}_{\mathbb{Z}[\Gamma]}(\mathbb{Z}, \boldsymbol{\Gamma}(M, \mathcal{F})) = \Gamma(M, \mathcal{F})^\Gamma \cong \Gamma(M, \mathcal{F}^\Gamma) = \Gamma(M, \mathcal{H}om_{\mathbb{Z}[\Gamma]}(\mathbb{Z}_M, \mathcal{F})).$$

It follows that the Γ-equivariant hypercohomology of a complex of sheaves $\mathcal{F}^\bullet$ can be expressed as the cohomology of

$$R\Gamma(M, R\mathcal{H}om_{\mathbb{Z}[\Gamma]}(\mathbb{Z}_M, \mathcal{F}^\bullet)).$$

This yields a variant of the canonical spectral sequence

(2.9.6) $$E_2^{p,q} = H^p(M, \mathcal{H}^q_\Gamma(\mathcal{F}^\bullet)) \Rightarrow \mathbb{H}^{p+q}_\Gamma(M, \mathcal{F}^\bullet).$$

Suppose now that, in the situation of (2.8), we impose compatible actions of a discrete group Γ, and assume further that Γ acts relative to a fibration $\rho : B \to M$ (*i.e.*, ρ is equivariant for the trivial Γ-action on M). From (2.9.6), we get another version of the Leray spectral sequence:

(2.9.7) $$E_2^{p,q} = H^p(M, R^q_\Gamma \mathcal{F}^\bullet) \Rightarrow \mathbb{H}^{p+q}_\Gamma(B, \mathcal{F}^\bullet).$$

The actions on Z and Z/T^c will be assumed free. Let

$$\tilde{\Phi} = \rho \circ \pi, \qquad \tilde{\Psi} = \rho \circ q,$$

and let Φ and Ψ denote the induced mappings on $\Gamma\backslash \mathrm{Z}$ and $\Gamma\backslash(\mathrm{Z}/\mathrm{T}^c)$ respectively. Assume, finally, that $\mathcal{V}$ admits an equivariant Γ-action. We compute:

$$\mathrm{R}^i\,\Phi_*\,\mathcal{V}_\Gamma \cong \mathrm{R}^i_\Gamma\,\tilde{\Phi}_*\,\mathcal{V} \cong \mathrm{R}^i_\Gamma\,\tilde{\Psi}_*\,(p_*\,\mathcal{V}). \tag{2.9.8}$$

Then,

$$\mathrm{R}^i_\Gamma\,\tilde{\Psi}_*\,(p_*\,\mathcal{V})^{\mathrm{T}^c} \cong \mathrm{R}^i_\Gamma\,(\rho\circ q)_*\,(q^{-1}\,\mathcal{E}), \tag{2.9.9}$$

which equals $\mathrm{R}^i_\Gamma\,\rho_*\,\mathcal{E}$ whenever the fibers of q are contractible, or more generally whenever $\mathcal{E} \to \mathbf{R}\,q_*\,(q^{-1}\,\mathcal{E})$ is a quasi-isomorphism. It is to be noted that the last expression depends only on B and $\mathcal{E}$, and the action of Γ thereon, and is thus independent of Z. We will apply this in Section 3 to $\mathrm{B}=\mathrm{A}_\mathrm{F}$ and $\mathrm{M}=\mathrm{M}_\mathrm{F}$ (from 1.2.5), and $\mathrm{Z} \subset (\mathrm{M}'_\mathrm{F})_\Sigma$ (*see* 1.3.4), and give cases where (2.9.8) and (2.9.9) are isomorphic.

2.10. The following will be needed for "adelization" in paragraph 4. Suppose that the group Γ acts (on the left) on the space X, and acts transitively on the space Y. Choose a point $y \in \mathrm{Y}$, and let Γ_y denote the isotropy subgroup of Γ at y.

We first recall a simple fact:

2.10.1. LEMMA. – *In the above situation, the maps*

$$\mathrm{X}\times\mathrm{Y} \underset{\beta}{\overset{\alpha}{\rightleftarrows}} \mathrm{X};$$

$$\alpha\,(x,\,\gamma\,y) = \gamma^{-1}\,x, \qquad \beta\,(x) = (x,\,y)$$

induce mutually inverse homeomorphisms

$$\Gamma\backslash(\mathrm{X}\times\mathrm{Y}) \cong \Gamma_y\backslash\mathrm{X}.$$

This has an analogue for equivariant cohomology. Let $\pi_\mathrm{X} : \mathrm{X}\times\mathrm{Y} \to \mathrm{X}$ be the projection, and let $\mathcal{F}^\bullet$ be a complex of sheaves on X with equivariant Γ-action.

2.10.2. PROPOSITION. – *Assume that* Y *is discrete. Then there is a canonical isomorphism*

$$\mathbb{H}^\bullet_\Gamma\,(\mathrm{X}\times\mathrm{Y},\,\pi_\mathrm{X}^{-1}\,(\mathcal{F}^\bullet)) \cong \mathbb{H}^\bullet_{\Gamma_y}\,(\mathrm{X},\,\mathcal{F}^\bullet).$$

Proof. – It suffices to check this when $\mathcal{F}^\bullet$ is a single sheaf $\mathcal{S}$; we must compare the cohomology of the complexes

$$\mathrm{RHom}_{\mathbb{Z}[\Gamma]}\,(\mathbb{Z},\,\mathbf{R}\,\mathbf{\Gamma}\,(\mathrm{X}\times\mathrm{Y},\,\pi_\mathrm{X}^{-1}(\mathcal{S}))) \qquad \text{and} \qquad \mathrm{RHom}_{\mathbb{Z}[\Gamma_y]}\,(\mathbb{Z},\,\mathbf{R}\,\mathbf{\Gamma}\,(\mathrm{X},\,\mathcal{S}))$$

Because these come by derived functor constructions, underlying this are the functors on Γ-equivariant sheaves $\mathcal{F}$ on X:

$$[\Gamma\,(\mathrm{X},\,\mathcal{F})\otimes\Gamma\,(\mathrm{Y},\,\mathbb{Z})]^\Gamma \qquad \text{and} \qquad \Gamma\,(\mathrm{X},\,\mathcal{F})^{\Gamma_y}.$$

We show that these two functors coincide. Indeed, this is a consequence of the same for the following constructions on $\mathbb{Z}$-modules A:

$$(A \otimes \mathbb{Z}^Y)^\Gamma \qquad \text{and} \qquad A^{\Gamma_y};$$

this can be checked directly.

It is easy to eliminate the condition that Γ act transitively on Y by decomposing Y into Γ-orbits. One obtains:

2.10.3. Corollary. – *Let* Γ *act on the spaces* X *and* Y, *with* Y *discrete; let* $\mathcal{F}^\bullet$ *be a* Γ*-equivariant complex of sheaves on* X. *Then*

$$H^\bullet_\Gamma(X \times Y, \pi_X^{-1}(\mathcal{F}^\bullet)) \cong \prod_{y \in \Gamma \backslash Y} H^\bullet_{\Gamma_y}(X, \mathcal{F}^\bullet).$$

N.B.: If $y' = \gamma y$, then $\Gamma_{y'}$ is a conjugate of Γ_y, and hence $H^\bullet_{\Gamma_{y'}}$ and $H^\bullet_{\Gamma_y}$ are canonically isomorphic.

3. Boundary cohomology of automorphic vector bundles

3.1. Automorphic vector bundles. – For what follows, a reference is [H2], especially its paragraph 3. We define p, K_p, $\mathcal{P}_p$, etc. as in 1.8.

Let $\check{M}(\mathbb{C})$ be the compact dual symmetric space of X. We may define $\check{M}(\mathbb{C})$ as the set of complex points of the flag variety $G/\mathcal{P}_p$, which has a natural rational structure $\check{M} = \check{M}(G, X)$ over the reflex field E(G, X), described in ([H2], § 3). Let $\lambda : K_p \to GL(V_\lambda)$ be a finite-dimensional algebraic representation, and extend λ trivially to a representation of $\mathcal{P}_p$. This defines, by the usual procedure, a G-homogeneous vector bundle $\mathcal{V}_\lambda$ on $\check{M}$, rational over some number field. More generally, let $\mathcal{V}$ be a G-homogeneous vector bundle on $\check{M}$. Let $\beta : X \hookrightarrow \check{M}(\mathbb{C})$ be the Borel embedding, defined as in ([H2], 3.1); it is the unique $G(\mathbb{R})$-equivariant mapping whose restriction to D is the open immersion defined above. For any compact open subgroup $K \subset G(\mathbf{A}^f)$,

$$[\mathcal{V}]_K = G(\mathbb{Q}) \backslash \beta^*(\mathcal{V}) \times G(\mathbf{A}^f)/K \tag{3.1.1}$$

is an algebraic vector bundle over ${}_K M_{\mathbb{C}}$ ([BB], § 10), and $[\mathcal{V}] = \varprojlim_K [\mathcal{V}]_K$ is a $G(\mathbf{A}^f)$-homogeneous algebraic vector bundle over $M_{\mathbb{C}}$. The restriction of $[\mathcal{V}]$ to the connected component $\Gamma \backslash D$ of ${}_K M_{\mathbb{C}}$ will be denoted $\mathcal{V}_\Gamma$.

3.1.2. Definition. – *A bundle of the form* $[\mathcal{V}]$, *with* $\mathcal{V}$ *a* G*-homogeneous vector bundle on* $\check{M}$, *is called an* automorphic vector bundle *on* M. *When* $\mathcal{V}$ *is of the form* $\mathcal{V}_\lambda$, *the automorphic vector bundle* $[\mathcal{V}_\lambda]$ *is called* fully decomposed.

One of the main theorems of [H2] is the following:

3.1.3. Theorem. – *The functor* $\mathcal{V} \to [\mathcal{V}]$, *from* G*-homogeneous vector bundles on* $\check{M}$ *to* $G(\mathbf{A}^f)$*-homogeneous vector bundles over* Sh(G, X), *is rational over* E(G, X).

The algebraic construction underlying this theorem is recalled in paragraph 4.3.

3.2. CANONICAL EXTENSIONS. – Let M_Σ be an admissible toroidal compactification of $M_{\mathbb{C}}$. In this section we construct a "best possible" extension of $[\mathcal{V}_\lambda]$ to a vector bundle $[\mathcal{V}_\lambda]_\Sigma$ over M_Σ, following Mumford [Mu2].

It is simplest to construct the canonical extension over the connected components of $M_{\mathbb{C}}$. Let $\Gamma\backslash D = M_\Gamma$ be such a connected component. For a rational boundary component F of D, define D_F as in 1.2.3. Let Σ_F be a $\Gamma_{l,F}$-admissible fan in $\bar{C}_F$, as in 1.3, and recall the local isomorphism

$$\varphi_{F,\Sigma}: \quad D_{F,\Sigma} \to M_{F,\Sigma}$$

from 1.4.1. One defines the canonical extension $\mathcal{V}_{\lambda,\Gamma,\Sigma}$ of $\mathcal{V}_{\lambda,\Gamma}$ over $M_{\Gamma,\Sigma}$ by first defining it for $D_{F,\Sigma}$ for each F, and then patching. For simplicity we write $\mathcal{V}_\Sigma = \mathcal{V}_{\lambda,\Sigma}$, $\mathcal{V} = \mathcal{V}_\lambda$, etc.; denote by j_Σ the embedding $M_\Gamma \hookrightarrow M_{\Gamma,\Sigma}$.

We have $D_F = U_F(\mathbb{C}) \cdot D \subset \check{M}(\mathbb{C})$. Let $\mathcal{V}_F$ be the restriction of $\mathcal{V}$ to D_F, $\mathcal{V}'_F$ be the vector bundle $\Gamma'_F\backslash\mathcal{V}_F$ over the space M'_F (from 1.2). Define $\mathcal{V}^A_F$ to be the sheaf $(\pi_{2,*}\mathcal{V}'_F)^{T_F}$ of T_F-invariant sections in $\pi_{2,*}\mathcal{V}'_F$ over A_F. Then

$$\mathcal{V}'_F \cong \pi_2^*(\mathcal{V}^A_F) \tag{3.2.1}$$

(*cf.* [Mu2]). Let $\pi_{2,\Sigma}: (M'_F)_\Sigma \to A_F$ be the natural mapping. We let $\mathcal{V}'_{F,\Sigma} = \pi^*_{2,\Sigma}(\mathcal{V}^A_F)$. The *canonical extension* of $\mathcal{V}_\Gamma$ is the unique subsheaf $\mathcal{V}_{\Gamma,\Sigma}$ of $(j_\Sigma)_*\mathcal{V}$ on $M_{\Gamma,\Sigma}$ for which there exist isomorphisms:

$$f_\Sigma: \quad \varphi^*_{F,\Sigma}(\mathcal{V}_{\Gamma,\Sigma}) \xrightarrow{\sim} \mathcal{V}'_{F,\Sigma} \tag{3.2.2}$$

extending the given isomorphism over M'_F. The canonical extension is characterized up to canonical isomorphism by (3.2.2). It is also characterized by a growth condition when $M_{\Gamma,\Sigma}$ is SNC (*see* 3.8.2, below).

More generally, if $\mathcal{V}$ is a G-homogeneous vector bundle over $\check{M}$, $[\mathcal{V}]$ the corresponding automorphic vector bundle over $M_{\mathbb{C}} = {}_K\mathrm{Sh}(G,X)_{\mathbb{C}}$, and M_Σ is an admissible toroidal compactification of $M_{\mathbb{C}}$, then a canonical extension $[\mathcal{V}]_\Sigma$ of $[\mathcal{V}]$ over $M_{\mathbb{C}}$ is a vector bundle whose restriction to every connected component of M_Σ satisfies conditions (3.2.2). A purely algebraic construction of the canonical extension is given in [H3], and recalled in Lemma 4.4.2.

We note the following special cases:

3.2.3. *Examples* (Mumford, [Mu2]).

(i) Let λ be the trivial representation of K_p, so that $\mathcal{V}_\lambda \cong \mathcal{O}_{\check{M}}$. Then $[\mathcal{O}_{\check{M}}]_\Sigma \cong \mathcal{O}_{M_\Sigma}$.

(ii) Let λ be $\Lambda^r(\mathrm{ad}|\mathfrak{p}^+)^*$, so that $\mathcal{V}_\lambda \cong \Omega^r_{\check{M}}$. If M_Σ is SNC, then $[\Omega^r_{\check{M}}]_\Sigma \cong \Omega^r_{M_\Sigma}(\log Z_\Gamma)$, where $\Omega^\bullet_{M_\Sigma}(\log Z_\Gamma)$ is the logarithmic de Rham complex of Deligne [De1].

The following theorem is proved in [H3]; parts (i) and (ii) are due to Mumford.

3.2.4. THEOREM. – *Assume* $K \subset G(\mathbf{A}^f)$ *is neat, and let* M_Σ *be an admissible toroidal compactification of* $M_{\mathbb{C}} = {}_K\mathrm{Sh}(G,X)_{\mathbb{C}}$.

(i) Any automorphic vector bundle $[\mathcal{V}]$ over $M_{\mathbb{C}}$ has a canonical extension $[\mathcal{V}]_\Sigma$ over M_Σ.

(ii) The functor $[\mathcal{V}] \to [\mathcal{V}]_\Sigma$ is exact and commutes with tensor products and Hom.

(iii) Suppose M_Σ is admissible (1.7), so that M_Σ is also defined over E(G, X). Then the functor $[\mathcal{V}] \to [\mathcal{V}]_\Sigma$ preserves fields of definition. In other words, the functor $\mathcal{V} \to [\mathcal{V}]_\Sigma$, taking G-homogeneous vector bundles over $\check{M}$ to vector bundles over M_Σ, is rational over E(G, X).

3.3. Description via canonical automorphy factors. – We now choose a rational boundary component F, which will remain fixed until the end of paragraph 3. We drop F from the notation whenever this is feasible; thus $P=P_F$, $U=U_F$, $W=W_F$, $\mathcal{V}^A = \mathcal{V}^A_F$, and so on; however, we continue to use A_F, M_F, Γ_F, D_F, and other symbols from which F cannot be dropped.

In a neighborhood of $Z_{F,\Sigma}$, we may construct $\mathcal{V}_{\Gamma,\Sigma}$ explicitly, using canonical automorphy factors. Let $J = J^{P,p}$ be the canonical automorphy factor of Proposition 1.8.8, and define

$$J_\lambda = J_\lambda^{P,p} = \lambda \circ J : G(\mathbb{R})^0 \times D \to GL(V_\lambda(\mathbb{C})). \tag{3.3.1}$$

Then J_λ defines a holomorphic action:

$$j: \quad G(\mathbb{R})^0 \to \mathrm{Aut}(D \times V_\lambda(\mathbb{C})); \qquad j(g)\cdot(x, v) = (g(x), J_\lambda(g, x)v), \tag{3.3.2}$$

and it follows easily from (3.3.2) that there is a canonical isomorphism of vector bundles over $\Gamma\backslash D$ (*see* [H2], 5.3):

$$\mathcal{V}_{\lambda,\Gamma} \xrightarrow{\sim} j(\Gamma)\backslash(D \times V_\lambda(\mathbb{C})), \tag{3.3.3}$$

given by $(g, v) \to (g, J_\lambda(g, p)v)$ when both are trivialized on $G(\mathbb{R})^0$.

Thus, the pullback of $\mathcal{V}_{\lambda,\Gamma}$ to $\Gamma'_F\backslash D$ is isomorphic to $j(\Gamma'_F)\backslash(D \times V_\lambda(\mathbb{C}))$. Furthermore, j extends to an action of $P(\mathbb{R})^0$ on $D_F \times V_\lambda(\mathbb{C})$, and the action (3.3.2) gives rise to a vector bundle $j(\Gamma'_F)\backslash D_F \times V_\lambda(\mathbb{C})$ on M'_F which restricts to the preceding on $\Gamma'_F\backslash D$, and is likewise isomorphic to $\mathcal{V}'_F$.

Now, it follows from (1.8.7.4) that the restriction of j to $P(\mathbb{R})^0$ extends, trivially on $U(\mathbb{C})$, to $P(\mathbb{R})^0 \cdot U(\mathbb{C})$; there are actions

$$\begin{cases} j': \quad (U(\mathbb{R})\backslash P'(\mathbb{R})^0) \to \mathrm{Aut}((U(\mathbb{C})\backslash D_F) \times V_\lambda(\mathbb{C})) \\ \qquad j'_Z: \quad (\Gamma_U\backslash\Gamma'_F) \to \mathrm{Aut}((\Gamma_U\backslash D_F) \times V_\lambda(\mathbb{C})) \end{cases} \tag{3.3.4}$$

which lift to the corresponding restrictions of j, denoted simply j and j_Z, respectively. Let $\tilde{\mathcal{V}}^A$ be the vector bundle $j'(\Gamma'_F)\backslash((U(\mathbb{C})\backslash D_F) \times V_\lambda(\mathbb{C}))$ on $\Gamma'_F\backslash(U(\mathbb{C})\backslash D_F) = A_F$. It follows from the above that there is a natural isomorphism $\tilde{\mathcal{V}}^A \cong \mathcal{V}^A$, and

$$\pi_2^*(\tilde{\mathcal{V}}^A) \xrightarrow{\sim} \mathcal{V}'_F, \tag{3.3.5}$$

recovering (3.2.1). In fact, we have more:

3.3.6. PROPOSITION. – *For any sufficiently small open set* $\mathcal{U} \subset \mathrm{M_F}$, *the restriction to* $\pi^{-1}(\mathcal{U}) \subset \mathrm{M'_F}$ *of the automorphic vector bundle* $\mathcal{V}'_{\mathrm{F}}$ *is flat* (*i.e., is the vector bundle attached to a local system*), *and is the pullback of such on* $\mathrm{A_F}$.

Proof. – From (3.3.2) and (3.3.5), we see that if $\mathcal{U}$ is contractible, the restriction of $\mathcal{V}^{\mathrm{A}}$ to $\pi_1^{-1}(\mathcal{U})$ comes from the local system associated to the representation $\mathrm{J}_\lambda(\cdot, x)$ [*cf.* (1.8.7.4)] of its fundamental group $\Gamma \cap \mathrm{W}(\mathbb{Q})/\Gamma_{\mathrm{U}}$.

By (3.3.4), the action of Γ'_{F} on $(\Gamma_{\mathrm{U}}\backslash \mathrm{D_F}) \times \mathrm{V}_\lambda(\mathbb{C})$ extends to an action, again denoted $j_{\mathbf{z}}$, on $(\Gamma_{\mathrm{U}}\backslash \mathrm{D_F} \times^{\mathrm{T_F}} \mathrm{T}_\sigma) \times \mathrm{V}_\lambda(\mathbb{C})$, for any $\sigma \in \Sigma_{\mathrm{F}}$. Then (3.2.2) and (3.3.5) imply that there are isomorphisms

$$j_\sigma : \quad \varphi^*_{\mathrm{F},\sigma}(\mathcal{V}_{\Gamma,\Sigma}) \xrightarrow{\sim} j_{\mathbf{z}}(\Gamma'_{\mathrm{F}})\backslash((\Gamma_{\mathrm{U}}\backslash \mathrm{D_F} \times^{\mathrm{T_F}} \mathrm{T}_\sigma) \times \mathrm{V}_\lambda(\mathbb{C}))|_{\mathrm{D}_{\mathrm{F},\sigma}}, \tag{3.3.7}$$

compatible with inclusions of simplices. Finally, for $\gamma \in \Gamma_l$, $\sigma \in \Sigma_{\mathrm{F}}$ there is a commutative diagram

$$\begin{array}{ccc} \varphi^*_{\mathrm{F},\sigma}(\mathcal{V}_{\Gamma,\Sigma}) & \xrightarrow{\sim} & j_{\mathbf{z}}(\Gamma'_{\mathrm{F}})\backslash((\Gamma_{\mathrm{U}}\backslash \mathrm{D_F} \times^{\mathrm{T_F}} \mathrm{T}_\sigma) \times \mathrm{V}_\lambda(\mathbb{C}))|_{\mathrm{D}_{\mathrm{F},\sigma}} \\ \downarrow j(\gamma) & & \downarrow j(\gamma) \\ \varphi^*_{\mathrm{F},\gamma(\sigma)}(\mathcal{V}_{\Gamma,\Sigma}) & \xrightarrow{\sim} & j_{\mathbf{z}}(\Gamma'_{\mathrm{F}})\backslash((\Gamma_{\mathrm{U}}\backslash \mathrm{D_F} \times^{\mathrm{T_F}} \mathrm{T}_{\gamma(\sigma)}) \times \mathrm{V}_\lambda(\mathbb{C}))|_{\mathrm{D}_{\mathrm{F},\gamma(\sigma)}} \end{array} \tag{3.3.8}$$

Thus we have the following extension of 3.3.6:

3.3.9. PROPOSITION. – *Over any sufficiently small open subset of* $\mathrm{M_F}$, *the restriction of the bundle* $\mathcal{V}'_{\mathrm{F},\Sigma}$ *on* $(\mathrm{M'_F})_\Sigma$ *is a flat vector bundle, and is the pullback of such on* $\mathrm{A_F}$.

3.4. CALCULATION OF BOUNDARY COHOMOLOGY (BEGINNING). – Fix $\sigma \in \Sigma_{\mathrm{F}}$. Let $\psi_\sigma : \mathrm{Z}_\sigma \to \mathrm{A_F}$ be as in 1.5. Then ψ_σ is a proper, smooth morphism. Let $\mathcal{V}_\sigma = i^*_\sigma(\mathcal{V}_{\Gamma,\Sigma})$, where $i_\sigma : \mathrm{Z}_\sigma \to \mathrm{M}_{\Gamma,\Sigma}$ is the natural imbedding. Via (3.3.5), there is a canonical morphism

$$b_\sigma : \quad \mathcal{V}^{\mathrm{A}} \cong (\psi_\sigma)_*(\psi^*_\sigma \mathcal{V}^{\mathrm{A}}) \to \mathbf{R}\psi_{\sigma,*}(\psi^*_\sigma \mathcal{V}^{\mathrm{A}}) = \mathbf{R}\psi_{\sigma,*}\mathcal{V}_\sigma.$$

3.4.1. PROPOSITION. – *The morphism* b_σ *is a quasi-isomorphism. Moreover, this depends naturally on* σ:

(i) *If* σ' *is a face of* σ, *let* $c(\sigma, \sigma') : \mathbf{R}\psi_{\sigma,*}\mathcal{V}_\sigma \to \mathbf{R}\psi_{\sigma',*}\mathcal{V}_{\sigma'}$ *be the natural mapping; then* $c(\sigma, \sigma') \circ b_\sigma = b_{\sigma'}$.

(ii) *If* $\gamma \in \Gamma_{l,\mathrm{F}}$, *let* $c(\gamma) : \mathbf{R}\psi_{\sigma,*}\mathcal{V}_\sigma \to \mathbf{R}\psi_{\gamma(\sigma),*}\mathcal{V}_{\gamma(\sigma)}$ *be the isomorphism induced by* $j(\gamma)$; *then* $c(\gamma) \circ b_\sigma = b_{\gamma(\sigma)}$.

Proof. – We have to check that the map $\mathcal{V}^{\mathrm{A}} \to \psi_{\sigma,*}\mathcal{V}_\sigma$ is an isomorphism, and that $\mathrm{R}^i\psi_{\sigma,*}\mathcal{V}_\sigma = 0$ for $i>0$; the remaining statements are obvious. The mapping ψ_σ is a proper, smooth fibration, whose fiber is a rational variety (torus embedding) which we denote by Y_σ. The desired assertion comes down to the fact that $\mathrm{H}^\bullet(\mathrm{Y}_\sigma, \mathcal{O}_{\mathrm{Y}_\sigma}) = \mathbb{C}$.

3.4.2. *Remark.* – We can view the preceding as a case of the projection formula: $\mathbf{R}\psi_{\sigma,*}\mathcal{O}_{\mathrm{Z}_\sigma} \otimes \mathcal{V}^{\mathrm{A}} \to \mathbf{R}\psi_{\sigma,*}(\psi^*_\sigma \mathcal{V}^{\mathrm{A}})$ is a quasi-isomorphism.

3.4.3. COROLLARY. – $H^\bullet(Z_\sigma, \mathcal{V}_\sigma) \cong H^\bullet(A_F, \mathcal{V}^A)$.

At this point, it is convenient to make use of the decomposition of ψ_σ as $q_\sigma \circ p_\sigma$, as in (2.8.2). Consider

$$\mathcal{V}^A \approx \mathbf{R}\,\psi_{\sigma,*}\,\mathcal{V}_\sigma \approx \mathbf{R}\,q_{\sigma,*}\,(p_{\sigma,*}\,\mathcal{V}_\sigma),$$

where the last quasi-isomorphism is by (2.8.5); and from (2.8.7):

$$\mathbf{R}\,q_{\sigma,*}\,(p_{\sigma,*}\,\mathcal{V}_\sigma)^{T^c} \cong \mathbf{R}\,q_{\sigma,*}\,(q_\sigma^{-1}\,\mathcal{V}^A),$$

which is quasi-isomorphic to $\mathcal{V}^A$, as the fibers of q_σ are contractible. We thus further obtain:

3.4.4. PROPOSITION. – *For any $\sigma \in \Sigma_F$, the inclusion $(p_{\sigma,*}\,\mathcal{V}_\sigma)^{T^c} \hookrightarrow p_{\sigma,*}\,\mathcal{V}_\sigma$ induces an isomorphism on sheaf cohomology.*

3.5. In this section, we fix a point $p \in D$. Let $\pi_F(p)$ be the F-coordinate of p in the Siegel domain realization (1.2.4), and let x be the image of $\pi_F(p)$ in M_F. Write A_x for $\pi_1^{-1}(x)$, and let $\mathcal{V}_x^A$ denote the restriction of $\mathcal{V}^A$ to A_x; by (3.3.5), this bundle has an underlying flat structure.

Define $V = V_F$ as in 1.8, and put $\Gamma_V = (W(\mathbb{Q}) \cap \Gamma)/\Gamma_U$. The point $x \in M_F$ determines a complex structure c_x on $V(\mathbb{R})$, as in 1.8, and A_x is, by construction, isomorphic to the complex torus $\Gamma_V\backslash V(\mathbb{R})$, with this complex structure (*see* [Br1]). Then

$$\mathcal{V}_x^A \cong j_{\mathbb{Z}}(\Gamma_V)\backslash(V(\mathbb{R})\times V_\lambda(\mathbb{C})). \tag{3.5.1}$$

To be explicit, let $a = 1/2\,(\dim V)$, as in 1.2, and let S_p denote the unipotent radical of $Q_p = Q_{F,p}$ [from (1.8.7.6)]. Then $\dim S_p = a$. The homomorphism

$$J' : \quad U\backslash N \to K_p, \tag{3.5.2}$$

deduced from (1.8.7.4), takes $V(\mathbb{C})$ onto $S_p(\mathbb{C})$; moreover, if we decompose

$$V(\mathbb{C}) = \mathfrak{v}_x^+ \oplus \mathfrak{v}_x^-. \tag{3.5.3}$$

as in (1.8.4), then $\mathfrak{v}_x^-$ is mapped by J' onto $S_p(\mathbb{C})$ (1.8.6).

Let $J'_\lambda : U\backslash N \to GL(V_\lambda)$ be the homomorphism induced [*cf.* (3.3.1) by (3.5.2)]. Then:

3.5.4. *Observation.* – The restriction of J'_λ to $V(\mathbb{R})$ is a unipotent representation; *i.e.*, there is a $V(\mathbb{R})$-invariant filtration ω of V_λ such that $V_F(\mathbb{R})$ acts trivially on each $\mathrm{Gr}_\omega^i V_\lambda$. This induces a filtration ω of the flat vector bundle $\mathcal{V}_x^A$ such that each $\mathrm{Gr}_\omega^i \mathcal{V}_x^A$ is flatly trivial.

Denote by $\mathcal{A}^{0,\bullet}(\mathcal{V}_x^A)$ the Dolbeault complex of $\mathcal{V}_x^A$ on A_x. Then we have canonical isomorphisms for all q

$$\mathcal{A}^{0,q}(\mathcal{V}_x^A) \xrightarrow{\sim} \mathrm{Hom}(\Lambda^q(\mathfrak{v}_x^-), C^\infty(V(\mathbb{R}), V_\lambda)^{\Gamma_V}). \tag{3.5.5}$$

We define an isomorphism as in ([BW], VII, § 2):

(3.5.6) $$C^{\infty}(V(\mathbb{R}), V_{\lambda})^{\Gamma_V} \xrightarrow{\sim} C^{\infty}(\Gamma_V \backslash V(\mathbb{R}), V_{\lambda}) \cong C^{\infty}(\Gamma_V \backslash V(\mathbb{R})) \otimes V_{\lambda},$$

by sending $\varphi \in C^{\infty}(V(\mathbb{R}), V_{\lambda})^{\Gamma_V}$ to $\varphi^0(g) = J'_{\lambda}(g)^{-1}\varphi(g)$. Then, just as in ([BW], Ch. VII; [O-O]), (3.5.5) and (3.5.6) define an isomorphism of complexes

(3.5.7) $$\mathcal{A}^{0,\bullet}(\mathcal{V}_x^{A}) \xrightarrow{\sim} C^{\bullet}(\mathfrak{v}_x^{-}, C^{\infty}(\Gamma_V \backslash V(\mathbb{R})) \otimes V_{\lambda})$$

where the right-hand side is the usual complex for computing the cohomology of the (abelian) Lie algebra $\mathfrak{v}_x^{-}$ with the indicated coefficients. The following is quite standard:

3.5.8. Proposition. – *The imbedding* $V_{\lambda} = \mathbb{C} \otimes V_{\lambda} \hookrightarrow C^{\infty}(\Gamma_V \backslash V(\mathbb{R})) \otimes V_{\lambda}$, *as the space of constant functions on* $\Gamma_V \backslash V(\mathbb{R})$, *induces a quasi-isomorphism of complexes*

(3.5.8.1) $$C^{\bullet}(\mathfrak{v}_x^{-}, V_{\lambda}) \hookrightarrow C^{\bullet}(\mathfrak{v}_x^{-}, C^{\infty}(\Gamma_V \backslash V(\mathbb{R})) \otimes V_{\lambda}).$$

Proof. – The morphism of complexes (3.5.8.1) is compatible with the filtrations induced by ω (3.5.4) on both sides. Applying the comparison theorem for spectral sequences, it thus suffices to verify (3.5.8.1) for $V_{\lambda} = \mathbb{C}$. In this case, the assertion can be found in ([Mu1]: p. 8).

3.5.9. Corollary. – *There is a natural isomorphism*

$$H^{\bullet}(\mathfrak{v}_x^{-}, V_{\lambda}) \to H^{\bullet}(A_x, \mathcal{V}_x^{A}).$$

3.5.10. Proposition. – *Let* $\Phi_{\sigma} : Z_{\sigma} \to M_F$ *be the restriction to* Z_{σ} *of the morphism* π_{Σ} (notation 1.5).

(i) *For each* $x \in M_F$, *there are natural isomorphisms*

(3.5.10.1) $$(R^q \Phi_{\sigma *} \mathcal{V}_{\sigma})_x \xrightarrow{\sim} H^q(\mathfrak{s}_p, V_{\lambda}),$$

where $\mathfrak{s}_p = Lie(S_p) \subset \mathfrak{k}_p$ *acts on* V_{λ} *via the differential of* λ.

(ii) *The Leray spectral sequence:*

(3.5.10.2) $$E_2^{p,q} = H^p(M_F, R^q \Phi_{\sigma *} \mathcal{V}_{\sigma}) \Rightarrow H^{p+q}(Z_{\sigma}, V_{\sigma})$$

is independent of σ (*up to canonical isomorphism*).

Proof. – Since $\Phi_{\sigma} = \pi_1 \circ \psi_{\sigma}$, we obtain from 3.4.1 that

(3.5.10.3) $$R^q \Phi_{\sigma *} \mathcal{V}_{\sigma} \cong R^q \pi_{1*} \mathcal{V}^{A} \quad (cf. 3.4.3).$$

Now, the homomorphism J′ identifies $\mathfrak{s}_p$ isomorphically with $\mathfrak{v}_x^{-}$. Thus, (i) is a direct consequence of 3.5.9. To see (ii), we use (3.5.10.3) to identify the spectral sequence (3.5.10.2) with the one for $\pi_1 : A_F \to M_F$ and the sheaf $\mathcal{V}^{A}$:

(3.5.10.4) $$E_2^{p,q} = H^p(M_F, R^q \pi_{1*} \mathcal{V}^{A}) \Rightarrow H^{p+q}(A_F, \mathcal{V}^{A}).$$

We can make a sharper statement. The Levi factor $K_p^{(2)}$ of Q_q acts naturally on the right-hand side of (3.5.10.1). Consider the action μ^q of $K_h \times G_l$ on $(R^q \Phi_{\sigma *} \mathcal{V}_{\sigma})_x$, defined by (3.5.10.1) and the isogeny $K_h \times G_l \to K_p^{(2)}$ induced by $c_F : G_l \xrightarrow{\sim} K_t$ of (1.8.7.5). First we have:

3.5.11. LEMMA.

(i) $R^q \Phi_{\sigma *} \mathcal{V}_\sigma$ *is the automorphic vector bundle on* M_F *defined by the restriction* μ_h^q *of* μ^q *to* $K_h \cdot A$*;*

(ii) *For every* $\gamma \in \Gamma_l$*, we have a commutative diagram*

$$\begin{array}{ccc} (R^q \Phi_{\sigma *} \mathcal{V}_\sigma) & \xrightarrow{\sim} & H^q(\mathfrak{s}_p, V_\lambda) \\ \downarrow j(\gamma) & & \downarrow c_F(\gamma) \\ (R^q \Phi_{\gamma(\sigma) *} \mathcal{V}_{\gamma(\sigma)}) & \xrightarrow{\sim} & H^q(\mathfrak{s}_p, V_\lambda). \end{array}$$

Proof. – These assertions follow immediately from the properties of the automorphy factor and from (3.3.8).

We next prove a sort of relative analogue of 3.5.8:

3.5.12. PROPOSITION. – *The spectral sequence* (3.5.10.4) *degenerates at* E_2.

Proof. – We follow a line of reasoning from ([Sch]: Thm. 2.7) that is attributed to Borel. As in [*loc. cit.*], we can express the cohomology groups appearing in (3.5.10.4), via Dolbeault cohomology, as relative Lie algebra cohomology, coming from complexes:

(3.5.12.1)
$$C^\bullet(\mathfrak{P}_h, K_h; C^\infty(\Gamma_F \backslash G_h) \otimes H^q(\mathfrak{s}_p, V_\lambda)) \cong [\Lambda^\bullet(\mathfrak{p}_h^-)^* \otimes C^\infty(\Gamma_F \backslash G_h) \otimes H^q(\mathfrak{s}_p, V_\lambda)]^{K_h}$$

and

(3.5.12.2)
$$C^\bullet(\mathfrak{P}_h \oplus \mathfrak{s}_p, K_h; C^\infty(\Gamma'_F \backslash (P'/U)(\mathbb{R})) \otimes V_\lambda) \cong [\Lambda^\bullet(\mathfrak{p}_h^- \oplus \mathfrak{s}_p)^* \otimes C^\infty((\Gamma'_F \backslash (P'/U)(\mathbb{R})) \otimes V_\lambda]^{K_h},$$

where $\mathfrak{p}_h^- = \mathfrak{p}^- \cap \mathfrak{g}_h(0)$, and $\mathfrak{P}_h = \mathfrak{p}_h^- \oplus \mathfrak{k}_{h,\mathbb{C}}$. Via a suitable K_h-equivariant embedding $H^q(\mathfrak{s}_p, V_\lambda) \hookrightarrow \Lambda^q(\mathfrak{s}_p)^* \otimes V_\lambda$, we regard the direct sum of the complexes (3.5.12.1) for all q as a subcomplex of (3.5.12.2), inducing the identity on cohomology. This splits the spectral sequence, from which the asserted degeneration follows.

3.6. The structure of $H^q(\mathfrak{s}_p, V_\lambda)$ as a module over $K_p^{(2)}$ is determined by a well-known theorem of Kostant, and goes as follows. As before, we use c_F to identify $K_p^{(2)}$, up to isogeny, with $K_h \cdot G_l$. Let H be a maximal torus of K_p, $\mathfrak{h} = \mathrm{Lie}(H)$, and choose a set R^+ of positive roots for $(\mathfrak{g}_{\mathbb{C}}, \mathfrak{h}_{\mathbb{C}})$ such that the holomorphic tangent space $\mathfrak{p}^+$ of D at the base point p is the sum of the root spaces corresponding to the subset R_n^+ of positive non-compact roots; let R_c^+ be the set of positive compact roots in R^+. Let ρ (resp. ρ_c) be the corresponding half-sum of positive (resp. positive compact) roots. We assume that

$H_h = H \cap K_h$ is a maximal torus of K_h, and that $H_l = c_F^{-1}(H \cap K_t)$ is a maximal torus of G_l. Then $\mathfrak{h}_{\mathbb{C}} = \mathfrak{h}_{h,\mathbb{C}} + c_F(\mathfrak{h}_{l,\mathbb{C}}) \cong \mathfrak{h}_{h,\mathbb{C}} + \mathfrak{h}_{l,\mathbb{C}}$, where $\mathfrak{h}_h = \mathrm{Lie}(H_h)$, $\mathfrak{h}_l = \mathrm{Lie}(H_l)$. If $\xi \in (\mathfrak{h}_{\mathbb{C}})^*$ let $[\xi]_h$ and $[\xi]_l$ denote its restrictions to $\mathfrak{h}_{h,\mathbb{C}}$ and $\mathfrak{h}_{l,\mathbb{C}}$, respectively. Of course $\mathfrak{h}_{h,\mathbb{C}} \cap \mathfrak{h}_{l,\mathbb{C}} = \mathrm{Lie}(Z_G)_{\mathbb{C}}$.

Let $R^{+,(2)}$ be the set of positive roots of $\mathfrak{h}_{\mathbb{C}}$ in $\mathfrak{k}_p^{(2)} = \mathrm{Lie}(K_p^{(2)})$, and define

$$W^{F,p} = \{w \in W(\mathfrak{k}_{p,\mathbb{C}}, \mathfrak{h}_{\mathbb{C}}) | w^{-1}(\alpha) > 0, \forall \alpha \in R^{+,(2)}\}.$$

Then $W^{F,p}$ is a set of representatives of shortest length for the left cosets $W(\mathfrak{k}_p^{(2)}, \mathfrak{h}_{\mathbb{C}}) \backslash W(\mathfrak{k}_{p,\mathbb{C}}, \mathfrak{h}_{\mathbb{C}})$. Denote by l the length function on $W(\mathfrak{k}_{p,\mathbb{C}}, \mathfrak{h}_{\mathbb{C}})$ (N.B.-*not* $W(\mathfrak{g}_{\mathbb{C}}, \mathfrak{h}_{\mathbb{C}})$), and let $W^{F,p}(q) \subset W^{F,p}$ be the set of elements of length q, for $q=0, 1, \ldots$.

Let $\Lambda \in (\mathfrak{h}_{\mathbb{C}})^*$ be the highest weight of the representation λ of K_p, relative to R_c^+. For $w \in W^{F,p}$, let

$$\text{(3.6.1)} \qquad \mu_h(w) = [w(\Lambda + \rho_c) - \rho_c]_h, \qquad \mu_l(w) = [w(\Lambda + \rho_c) - \rho_c]_l,$$

Then $\mu_h(w)$ and $\mu_l(w)$ are the respective highest weights of finite-dimensional representations of K_h and G_l, denoted $(\lambda(h, w), V_{\lambda(h,w)})$ and $(\lambda(l, w), V_{\lambda(l,w)})$, respectively.

3.6.2. Proposition (Kostant [K]). – *For every* $q \leqq \dim(\mathfrak{s}_p)$, *there is an isomorphism of* $K_h \times G_l$ *modules*

$$H^q(\mathfrak{s}_p, V_\lambda) \xrightarrow{\sim} \bigoplus_{w \in W^{F,p}(q)} V_{\lambda(h,w)} \otimes V_{\lambda(l,w)}.$$

The representation of $K_h \times G_l$ on the right-hand side factors through its quotient $K_h \cdot G_l \subset G$. Combining this result with Proposition 3.5.10 gives:

3.6.3. Corollary.

(*i*) *for every* $\sigma \in \Sigma_F$ *and all* q, *we have isomorphisms*

$$r^q(\sigma): \ R^q \Phi_{\sigma *} \mathcal{V}_\sigma \cong R^q \pi_{1*} \mathcal{V}^A \xrightarrow{\sim} \bigoplus_{w \in W^{F,p}(q)} \mathcal{V}_{\lambda(h,w)} \otimes V_{\lambda(l,w)}$$

of automorphic vector bundles on M_F, *where* $\mathcal{V}_{\lambda(h,w)}$ *is the automorphic vector bundle associated to the representation* $\lambda(h, w)$ *of* K_h (*cf. Remark* 3.6.3.1)

(*ii*) *For all* σ *and* q *as above, and* $\gamma \in \Gamma_l$, *there is a commutative diagram*

$$\begin{array}{ccc} (R^q \Phi_{\sigma *} \mathcal{V}_\sigma) & \overset{r^q(\gamma(\sigma))}{\longrightarrow} & \bigoplus_{w \in W^{F,p}(q)} \mathcal{V}_{\lambda(h,w)} \otimes V_{\lambda(l,w)} \\ \downarrow j(\gamma) & & \downarrow \oplus 1 \otimes \lambda(l,w)(\gamma) \\ (R^q \Phi_{\gamma(\sigma) *} \mathcal{V}_{\gamma(\sigma)}) & \overset{r^q(\gamma(\sigma))}{\longrightarrow} & \bigoplus_{w \in W^{F,p}(q)} \mathcal{V}_{\lambda(h,w)} \otimes V_{\lambda(l,w)} \end{array}$$

3.6.3.1. *Remark.* – The use of the terminology "automorphic vector bundle" with regard to $\mathcal{V}_{\lambda(h,w)}$ does not quite conform to the definition in paragraph 3.1, because with our conventions K_h is not the stabilizer in $G_h(\mathbb{R})$ of a point in X(F). This abuse of language, which is only of importance for the arithmetic theory of paragraph 4, will be corrected in paragraph 4.1.

If we now feed this into 3.5.12, we obtain:

3.6.4. COROLLARY. – *One can decompose* $H^s(Z_\sigma, \mathcal{V}_\sigma)$ *as:*

$$H^s(Z_\sigma, \mathcal{V}_\sigma) \cong H^s(A_F, \mathcal{V}^A) \cong \bigoplus_{w\in W^{F,p}} H^{s-l(w)}(M_F, \mathcal{V}_{\lambda(h,w)}) \otimes V_{\lambda(l,w)},$$

compatibly with all restriction mappings, and with the action of Γ_l.

3.7. CALCULATION OF BOUNDARY COHOMOLOGY (END). – Denote by $\mathrm{Div}_F(\Sigma)$ the set of divisors at infinity in $(M'_F)_{\Sigma_F^c}$ (notation as in 1.3), and let $\mathrm{Div}_F(M_{\Gamma,\Sigma})$ be the set of irreducible components of ${}^{<}Z_{F,\Sigma}$, or equivalently $Z_{F,\Sigma}$ (notation as in 1.5). Then

$$\mathrm{Div}_F(M_{\Gamma,\Sigma}) \cong \Gamma_l\backslash \mathrm{Div}_F(\Sigma)$$

is the set of vertices of the simplicial complex $\Gamma_l\backslash\hat{\Sigma}_F^c$ (from 2.2).

Consider the closed covering of ${}^{<}Z_{F,\Sigma}$ given by

$$\mathcal{Z}_{\Sigma_F} = \{Z \in \mathrm{Div}_F(M_\Sigma)\}. \tag{3.7.1}$$

The following is evident:

3.7.2. LEMMA. – *The nerve* $\mathfrak{N}(\mathcal{Z}_{\Sigma_F})$ *of the closed covering* $\mathcal{Z}_{\Sigma_F}$ *is isomorphic to* $\Gamma_l\backslash\hat{\Sigma}_F^c$.

Our hypothesis that Γ is neat implies that Γ_l is torsion-free, hence $\Gamma_l\backslash\hat{\Sigma}_F^c$ is a PL-manifold (a similar observation appears already in the work of Looijenga [L]). Under the mild additional hypothesis (2.2.9) on the simplicial complex Σ_F, we saw (recall 2.2.10) that $\Gamma_l\backslash\hat{\Sigma}_F^c$ has the homotopy type of the locally symmetric space $X(\Gamma_l)$.

Let $i_F : Z_{F,\Sigma} \to M_{\Gamma,\Sigma}$ denote the inclusion; $i_F^* \mathcal{V}_{\Gamma,\Sigma}$ restricts to a coherent sheaf on ${}^{<}Z_{F,\Sigma}$, also denoted $i_F^* \mathcal{V}_{\Gamma,\Sigma}$. We can now compute $H^*({}^{<}Z_{F,\Sigma}, i_F^*\mathcal{V}_{\Gamma,\Sigma})$. Define systems of coefficients $\mathbf{L}^s(\cdot, \mathcal{V})$ on $\mathfrak{N}(\mathcal{Z}_{\Sigma_F})$ by the formula $\mathbf{L}^s(\hat{\sigma}, \mathcal{V}) = H^s(Z_\sigma, \mathcal{V}_\sigma)$. Then there is the spectral sequence

$$E_2^{r,s} = H^r(\mathfrak{N}(\mathcal{Z}_{\Sigma_F}), \mathbf{L}^s(\cdot, \mathcal{V})) \Rightarrow H^{r+s}({}^{<}Z_{F,\Sigma}, i_F^*\mathcal{V}_{\Gamma,\Sigma}). \tag{3.7.3}$$

It follows from Corollary 3.6.4 that $\mathbf{L}^s(\cdot, \mathcal{V})$ defines a locally constant sheaf on $\mathfrak{N}(\mathcal{Z}_{\Sigma_F}) \cong \Gamma_l\backslash\hat{\Sigma}_F^c$, the quotient by Γ_l of the constant sheaf on $\hat{\Sigma}_F^c$ with coefficients

$$H^s(A_F, \mathcal{V}^A) \cong \bigoplus_{w\in W^{F,p}} H^{s-l(w)}(M_F, \mathcal{V}_{\lambda(h,w)}) \otimes V_{\lambda(l,w)}, \tag{3.7.4}$$

where Γ_l acts only on the second factor of the tensor product.

3.7.5. LEMMA. – *For all* t, $H^t({}^{<}Z_{F,\Sigma}, i_F^*\mathcal{V}_{\Gamma,\Sigma}) \cong H^t_{\Gamma_l}(A_F, \mathcal{V}^A)$.

Proof. – We have that ${}^{<}Z_{F,\Sigma} \cong \Gamma_l\backslash{}^{<}\tilde{Z}_{F,\Sigma}$, by Lemma 1.5.3, and Γ_l acts freely on ${}^{<}\tilde{Z}_{F,\Sigma}$ [here we are using hypothesis (1.1.4)]. Let $\tilde{i}_F : {}^{<}\tilde{Z}_{F,\Sigma} \to (M'_F)_\Sigma$ be the natural embedding. It follows from (2.9.2) that

$$H^t({}^{<}Z_{F,\Sigma}, i_F^*\mathcal{V}_{\Gamma,\Sigma}) \cong H^t_{\Gamma_l}({}^{<}\tilde{Z}_{F,\Sigma}, \tilde{i}_F^*\mathcal{V}'_{F,\Sigma}). \tag{3.7.5.1}$$

It will be shown [*see* (3.9.4)], as part of something more general, that the projection of $^{<}\tilde{Z}_{F,\Sigma}$ onto A_F induces an isomorphism between the right-hand side of (3.7.5.1) and $H^t_{\Gamma_l}(A_F, \mathcal{V}^A)$.

3.7.6. *Remark.* – Note that we can rewrite the spectral sequence (3.7.3) as

$$E_2^{r,s} = H^r(\Gamma_l, H^s(A_F, \mathcal{V}^A)) \Rightarrow H^{r+s}_{\Gamma_l}(A_F, \mathcal{V}^A),$$

which is just an instance of 2.9.4 (iii).

3.7.7. PROPOSITION. – *The spectral sequence* (3.7.3) *degenerates at* E_2.

Proof. – By 3.5.12, the spectral sequence

$$E_2^{p,q} = H^p(M_F, R^q\pi_{1*}\mathcal{V}^A) \Rightarrow H^{p+q}(A_F, \mathcal{V}^A)$$

degenerates at E_2. After noting that the embedding of (3.5.12.1) in (3.5.12.2) is Γ_l-equivariant, we see that

(3.7.7.1) $$E_2^{p,q} = H^p_{\Gamma_l}(M_F, R^q\pi_{1*}\mathcal{V}^A) \Rightarrow H^{p+q}_{\Gamma_l}(A_F, \mathcal{V}^A)$$

likewise degenerates at E_2. It now follows that the degeneration of (3.7.3) is equivalent to that of

$$E_2^{r,t} = H^r(\Gamma_l, H^t(M_F, R^q\pi_{1*}\mathcal{V}^A)) \Rightarrow H^{r+t}_{\Gamma_l}(M_F, R^q\pi_{1*}\mathcal{V}^A)$$

for all q. But here, Γ_l acts trivially on M_F, and on $R^q\pi_{1*}\mathcal{V}^A$ the action factors off (recall 3.6.3 and 3.6.4), so the above spectral sequence is a direct sum of ones of the form

$$E_2^{r,t} = H^r(\Gamma_l, V_l) \otimes S^t \Rightarrow H^{r+t}_{\Gamma_l}(M_F, \mathcal{V} \otimes V_l)$$

with $S^t = H^t(M_F, \mathcal{V})$, and Γ_l acts trivially on $\mathcal{V}$. The E_2-degeneration of the latter spectral sequence is just the Künneth theorem.

3.7.8. COROLLARY. – *With hypotheses as above, for each integer t, there is a natural isomorphism*

$$H^t(^{<}Z_{F,\Sigma}, i_F^*\mathcal{V}_{\Gamma,\Sigma}) \xrightarrow{\sim} \bigoplus_r \bigoplus_{w \in W^{F,p}} H^{t-r-l(w)}(M_F, \mathcal{V}_{\lambda(h,w)}) \otimes H^r(X(\Gamma_l), \tilde{\mathbf{V}}_{\lambda(l,w)}),$$

where $\tilde{\mathbf{V}}_{\lambda(l,w)}$ *is the local system on* $X(\Gamma_l)$ *defined by the representation* $\lambda(l, w)|_{\Gamma_l}$.

3.7.9. *Remark.* – Of course, the right-hand factor in the above is isomorphic to $H^r(\Gamma_l, V_{\lambda(l,w)})$ (*i.e.*, group cohomology).

3.8. CONDITIONS OF GROWTH AND DECAY. – Except in the case where F is a minimal boundary component, the space M_F is non-compact, not to mention anything lying over it, such as $^{<}Z_{F,\Sigma}$. It would therefore be useful to extend the results of 3.5-3.7 to some compactification of M_F; or, failing that, to prove versions of those results for forms with growth conditions.

We recall three types of growth conditions.

3.8.1. For any locally G-homogeneous vector bundle (not necessarily holomorphic) associated to a finite-dimensional representation (τ, V) of K_p, on an arithmetic quotient

$M_\Gamma = \Gamma\backslash D$, its sections can be described as functions $f : \Gamma\backslash G(\mathbb{R})^+ \to V$ that are invariant under the action of K_p; *i. e.*, these are Γ-invariant functions on $G(\mathbb{R})^+$ with

$$f(gk^{-1}) = \tau(k) f(g).$$

Choose any $K_p \cap G(\mathbb{R})^{der}$-invariant norm on V (which determines a metric in the bundle); let $\|\cdot\|$ denote the measure of size on $G(\mathbb{R})^+$ induced by matrix norm under some essentially-faithful finite-dimensional representation of G. The function f is said to have *moderate growth* (or, is *slowly increasing*) if

$$|f(g)| \leqq C\|g\|^m \quad \text{for some } C > 0, \qquad m \in \mathbb{Z}.$$

We rewrite this as: for some $m \in \mathbb{Z}$,

$$\text{(3.8.1.1)} \qquad |f(g)| \prec \|g\|^m.$$

Analogously, one says that f is *rapidly decreasing* if (3.8.1.1) holds for every $m \in \mathbb{Z}$. More generally, we say that f is *slowly increasing* (resp., *rapidly decreasing*) *to all orders* if Θf satisfies (3.8.1.1) for every invariant differential operator Θ in the enveloping algebra $U(\mathfrak{g}_\mathbb{C})$ and some (resp. all) $m \in \mathbb{Z}$. If $\Gamma \subset G(\mathbb{R})^+$ is an arithmetic subgroup, we let $C^\infty(\Gamma\backslash G(\mathbb{R})^+)_{si}$, (resp. $C^\infty(\Gamma\backslash G(\mathbb{R})^+)_{rd}$, resp. $C^\infty(\Gamma\backslash G(\mathbb{R})^+)_{sia}$ resp. $C^\infty(\Gamma\backslash G(\mathbb{R})^+)_{rda}$) denote the space of smooth functions on $\Gamma\backslash G(\mathbb{R})^+$ which are slowly increasing (resp. rapidly decreasing, resp. slowly increasing to all orders, resp. rapidly decreasing to all orders).

3.8.2. Let M be a complex manifold, Z a divisor with normal crossings on M, and $(\mathcal{E}, h)$ an Hermitian vector bundle on $M - Z$. Let $\Delta = \{z \in \mathbb{C} \mid |z| < 1/2\}$, $\Delta^* = \Delta - \{0\}$. Locally on M, the inclusion $i : M - Z \to M$ is $(\Delta^*)^k \times \Delta^{n-k} \hookrightarrow \Delta^n$, with Z defined by the equation

$$\underline{m}(z) = \prod_{j=1}^{k} z_j = 0.$$

One says that a section s of $\mathcal{E}$ on $(\Delta^*)^n$ has *logarithmic growth* if for some $m \in \mathbb{Z}$,

$$\text{(3.8.2.1)} \qquad h(s, s) \prec |\log \underline{m}(z)|^m.$$

In case $M = M_{\Gamma,\Sigma}$ and $Z = Z_{\Gamma,\Sigma}$, so that $M - Z = M_\Gamma$, and $\mathcal{E}$ is locally homogeneous with a standard metric, the argument in [Mu2]: Prop. 3.3 (which does not require the holomorphy of the bundle) shows that (3.8.1.1) and (3.8.2.1) are equivalent. It follows that the notions of logarithmic growth, defined for different toroidal compactifications of M_Γ, all actually coincide. Furthermore, it was shown in [Mu2]: Thm. 3.1 that the subsheaf of $i_*\mathcal{E}$ consisting of those local sections with logarithmic growth is locally-free and gives the canonical extension $\mathcal{E}_{\Gamma,\Sigma}$.

3.8.3. We return to the general situation of the beginning of 3.8.2. Start with the sheaf of functions of logarithmic growth in $i_* \mathcal{A}^0_{M-Z}$, *i. e.* those for which there is some $m \in \mathbb{Z}$ such that the condition (3.8.2.1) is satisfied, and form the tensor product over the anti-holomorphic functions on M with the conjugate of the holomorphic log-complex. The largest subcomplex of this under $\bar{\partial}$ is denoted $\mathcal{A}^{\bullet}_{\mathrm{si}}$ ($\mathcal{K}^{\bullet}_{\log}$ in [HP]); explicitly, $\mathcal{A}^{\bullet}_{\mathrm{si}}$ consists of forms φ such that (i) the coefficient functions, in the sense of 3.8.1, are slowly increasing, and (ii) the same holds for $\bar{\partial}\varphi$. In [HP], $\mathcal{A}^{\bullet}_{\mathrm{si}}$ is shown to be a resolution of $\mathcal{O}_{\mathrm{M}}$. If one starts instead with functions satisfying (3.8.2.1) for *all* $m \in \mathbb{Z}$, and continues as before, one obtains a resolution $\mathcal{A}^{\bullet}_{\mathrm{rd}}$ of $\mathcal{O}_{\mathrm{M}}(-\mathrm{Z})$ [H5], (3.1.4). Thus:

3.8.3.1. PROPOSITION. – *For any locally-free sheaf $\bar{\mathcal{E}}$ on* M,

$$\mathcal{A}^{\bullet}_{\mathrm{si}}(\mathrm{M}, \bar{\mathcal{E}}) = \mathcal{A}^{\bullet}_{\mathrm{si}} \otimes \bar{\mathcal{E}}$$

(*tensor product here over $\mathcal{O}_{\mathrm{M}}$*) *is a fine resolution of $\bar{\mathcal{E}}$. Likewise, the subcomplex $\mathcal{A}^{\bullet}_{\mathrm{rd}} \otimes \bar{\mathcal{E}}$ is a fine resolution of $\bar{\mathcal{E}}(-\mathrm{Z})$.*

This gets applied on $\mathrm{M} = \mathrm{M}_{\Gamma,\Sigma}$ to $\bar{\mathcal{E}} = [\mathcal{V}]_{\Sigma}$, where the growth conditions can be seen to coincide with those of 3.8.1 and 3.8.2 (*see* [H5], (3.3.4); note 3.2.3 (ii), and be willing to take the complex conjugate). In order to make comparisons with Lie algebra cohomology, it is more convenient to replace the complexes $\mathcal{A}^{\bullet}_{\mathrm{si}}$ and $\mathcal{A}^{\bullet}_{\mathrm{rd}}$ with subcomplexes $\mathcal{A}^{\bullet}_{\mathrm{sia}}$ and $\mathcal{A}^{\bullet}_{\mathrm{rda}}$, corresponding to forms which are slowly increasing (resp. rapidly decreasing) to all orders in the sense defined above. If $\mathfrak{P}_p = \mathrm{Lie}(\mathcal{P}_p)$ and K_p are as in 1.8, there are natural isomorphisms of complexes

$$\text{(3.8.3.2)} \qquad \mathrm{C}^{\bullet}(\mathfrak{P}_p, \mathrm{K}_p; \mathrm{C}^{\infty}(\Gamma\backslash \mathrm{G}(\mathbb{R})^{+})_{*} \otimes \mathrm{V}_{\lambda}) \xrightarrow{\sim} \mathcal{A}^{\bullet}_{*} \otimes \mathcal{V}_{\lambda,\Gamma}, \qquad * = \mathrm{sia\ or\ rda},$$

where the left-hand side is the relative Lie algebra complex.

For emphasis, we state the result from [H4]: 2.4.1:

3.8.3.3. PROPOSITION. – *$\mathcal{A}^{\bullet}_{\mathrm{sia}}(\mathrm{M}_{\Gamma,\Sigma}, [\mathcal{V}]_{\Sigma})$ is a fine resolution of the canonical extension bundle $[\mathcal{V}]_{\Sigma}$. Likewise, $\mathcal{A}^{\bullet}_{\mathrm{rda}}(\mathrm{M}_{\Gamma,\Sigma}, [\mathcal{V}]_{\Sigma})$ is a fine resolution of the subcanonical extension bundle $[\mathcal{V}]_{\Sigma}(-\mathrm{Z}_{\Gamma,\Sigma})$.*

3.8.3.4. *Remark.* – The notions *sia* and *rda* have purely local geometric definitions, independent of the group-theoretic context, *cf.* [H4], p. 54.

3.9. We now apply the results of (2.8) and (2.9) to subsets of the partial toroidal compactification $(\mathrm{M}'_{\mathrm{F}})_{\Sigma}$ of M'_{F} [*see* (1.2.5) and (1.3.4) for notation]. We take in (2.9)

$$\mathrm{B} = \mathrm{A}_{\mathrm{F}}, \quad \mathrm{M} = \mathrm{M}_{\mathrm{F}}, \quad \rho = \pi_1; \qquad \mathcal{E} = \mathcal{V}^{\mathrm{A}}, \quad \mathcal{V} = (\mathcal{V}'_{\mathrm{F},\Sigma})|_{\mathrm{Z}}$$

(in the last one, we mean *bundle*, not sheaf, restriction). Finally, we take for Z in (2.8.1) any of the following:

(i) $\mathrm{Z}_1 = {}^{<}\tilde{\mathrm{Z}}_{\mathrm{F},\Sigma}$, as in Lemma 1.5.3,

(3.9.1) (ii) Z_2 = a $\mathrm{T}^{c}_{\mathrm{F}}$-invariant neighborhood of Z_1 in $(\mathrm{M}'_{\mathrm{F}})_{\Sigma}$ admitting a deformation retract onto Z_1 over A_{F},

(iii) $\mathrm{Z}_3 = \mathrm{Z}_2 - \mathrm{Z}_1$, a deleted neighborhood of Z_1.

We have a tower of spaces

$$(3.9.2)\qquad \tilde{\Phi}\left[\begin{array}{c} \mathrm{Z} \\ \downarrow p \\ \mathrm{Z}/\mathrm{T}^c \\ \downarrow q \\ \mathrm{A}_\mathrm{F} \\ \downarrow \pi_1 \\ \mathrm{M}_\mathrm{F} \end{array}\right.\quad \left.\begin{array}{c} \\ \\ \mathrm{Z}/\mathrm{T}^c \\ \\ \mathrm{A}_\mathrm{F} \\ \\ \mathrm{M}_\mathrm{F} \end{array}\right]\tilde{\Psi}$$

on which Γ_l acts.

Recall, from 2.8, that we have in all three cases:

$$(3.9.3)\qquad \mathrm{R}\,(q \circ p)_* \,\mathcal{V} \overset{\simeq}{\leftarrow} \mathrm{R}\,q_*\,(p_*\,\mathcal{V}) \overset{inv}{\rightarrow} \mathrm{R}\,q_*\,(p_*\,\mathcal{V})^{\mathrm{T}^c} = \mathrm{R}\,q_*\,(q^{-1}\,\mathcal{V}^{\mathrm{A}}) \cong \mathcal{V}^{\mathrm{A}};$$

it is not hard to see that these mappings are all Γ_l-equivariant. In the case of Z_1, *inv* induces an isomorphism on sheaf cohomology (3.4.4), which yields

$$(3.9.4)\qquad \mathrm{H}^t_{\Gamma_l}(\mathrm{Z}_1, \mathcal{V}) \cong \mathrm{H}^t_{\Gamma_l}(\mathrm{A}_\mathrm{F}, \mathcal{V}^{\mathrm{A}})$$

(recall Lemma 3.7.5 and its proof).

From (2.9.8) and (2.9.9) we also have in all three cases:

$$(3.9.5)\qquad \mathrm{R}^i\,\Phi_*\,(\mathcal{V}'_{\mathrm{F},\Sigma})_{\Gamma_l} \cong \mathrm{R}^i_{\Gamma_l}\,\tilde{\Psi}_*\,(p_*\,\mathcal{V}'_{\mathrm{F},\Sigma}) \overset{inv}{\rightarrow} \mathrm{R}^i_{\Gamma_l}\,\tilde{\Psi}_*\,(p_*\,\mathcal{V}'_{\mathrm{F},\Sigma})^{\mathrm{T}^c} \cong \mathrm{R}^i_{\Gamma_l}\,\pi_{1*}\,\mathcal{V}^{\mathrm{A}},$$

where the mapping is the one induced by the projection

$$p_*\,\mathcal{V}'_{\mathrm{F},\Sigma} \rightarrow (p_*\,\mathcal{V}'_{\mathrm{F},\Sigma})^{\mathrm{T}^c}$$

and the restriction to Z is understood. In the case of Z_1, where $((\mathcal{V}'_{\mathrm{F},\Sigma})|_\mathrm{Z})_{\Gamma_l} = i^*_\mathrm{F}\,\mathcal{V}_{\Gamma,\Sigma}$, we can see that *inv* is an isomorphism again as follows. Recall from (3.5.10.3) that for any $\sigma \in \Sigma_\mathrm{F}$, $\mathrm{R}^i\,\Phi_{\sigma,*}\,\mathcal{V}_\sigma \cong \mathrm{R}^i\,\pi_{1,*}\,\mathcal{V}^{\mathrm{A}}$. From 3.7, we see that $\mathrm{R}^i\,\Phi_*\,(i^*_\mathrm{F}\,\mathcal{V}_{\Gamma,\Sigma})_{\Gamma_l} \cong \mathrm{R}^i_{\Gamma_l}\,\pi_{1*}\,\mathcal{V}^{\mathrm{A}}$, *i. e.*, the first and last terms in (3.9.5) are isomorphic.

We obtain towers of spectral sequences associated to (3.9.2), for each of $\mathrm{Z}=\mathrm{Z}_1$, $\mathrm{Z}=\mathrm{Z}_2$, and $\mathrm{Z}=\mathrm{Z}_3$:

$$(3.9.6)\qquad \begin{array}{ccc} \mathrm{H}^p\,(\mathrm{M}_\mathrm{F}, \mathrm{R}^q_{\Gamma_l}\,\tilde{\Phi}_*\,\mathcal{V}|_\mathrm{Z}) & \Rightarrow & \mathrm{H}^{p+q}_{\Gamma_l}(\mathrm{Z}, \mathcal{V}|_\mathrm{Z}) \\ \uparrow\wr & & \uparrow\wr \\ \mathrm{H}^p\,(\mathrm{M}_\mathrm{F}, \mathrm{R}^q_{\Gamma_l}\,\tilde{\Psi}_*\,(p_*\,\mathcal{V}|_\mathrm{Z})) & \Rightarrow & \mathrm{H}^{p+q}_{\Gamma_l}(\mathrm{Z}/\mathrm{T}^c, p_*\,\mathcal{V}|_\mathrm{Z}) \\ \uparrow\downarrow\, {\scriptstyle inv} & & \uparrow\downarrow\, {\scriptstyle inv} \\ \mathrm{H}^p\,(\mathrm{M}_\mathrm{F}, \mathrm{R}^q_{\Gamma_l}\,\tilde{\Psi}_*\,(p_*\,\mathcal{V}|_\mathrm{Z})^{\mathrm{T}^c}) & \Rightarrow & \mathrm{H}^{p+q}_{\Gamma_l}(\mathrm{Z}/\mathrm{T}^c, (p_*\,\mathcal{V}|_\mathrm{Z})^{\mathrm{T}^c}) \\ \uparrow\wr & & \uparrow\wr \\ \mathrm{H}^p\,(\mathrm{M}_\mathrm{F}, \mathrm{R}^q_{\Gamma_l}\,\pi_{1*}\,\mathcal{V}^{\mathrm{A}}) & \Rightarrow & \mathrm{H}^{p+q}_{\Gamma_l}(\mathrm{A}_\mathrm{F}, \mathcal{V}^{\mathrm{A}}) \end{array}$$

which are compatible with the restrictions from Z_2 to Z_1 and Z_3. The mappings denoted "*inv*" are projections onto T^c-invariants, and, as has already been noted, are isomorphisms for $Z=Z_1$. As we remarked at the end of Section 2, the lower half of (3.9.6) is the same in all three cases. By analogues of 3.7.7, the spectral sequences in (3.9.6) all degenerate at E_2.

3.9.7. *Remark.*

(i) It is not hard to see that one can use "$H^p_{\Gamma_l}(M_F, R^q$" and "$H^p(M_F, R^q_{\Gamma_l}$" interchangeably; the latter displays the fact that the action of Γ_l on M_F is trivial. In particular, the bottom line in (3.9.6) is the same as (3.7.7.1).

(ii) For $Z=Z_3$ (or $Z=Z_2$), classes in $H^\bullet_{\Gamma_l}(Z, \mathcal{V}|_Z)$ can be represented by $\bar\partial$-closed Γ_l-invariant C^∞-forms on Z (with growth conditions), with values in $\mathcal{V}$. This follows from (2.9.2) and the appropriate version of the Dolbeault lemma.

3.10. Main theorem. – We can now state and prove a provisional form of our main result. Start with a $\bar\partial$-closed $(0, i)$-form η on M_Γ belonging to the complex $\mathcal{A}^\bullet_{\text{sia}} \otimes \mathcal{V}_{\lambda,\Gamma}$. The isomorphism (3.8.3.2) identifies η as an element of

$$(3.10.1)\qquad (C^\infty(\Gamma\backslash G(\mathbb{R})^+)_{\text{sia}} \otimes \Lambda^i(\mathfrak{p}^-)^* \otimes V_\lambda)^{K_p}.$$

By Prop. 3.8.3.1, η determines a cohomology class $[\eta] \in H^1(M_{\Gamma,\Sigma}, \mathcal{V}_{\Gamma,\Sigma})$. This admits a natural "boundary value", by restriction:

$$(3.10.2)\qquad \begin{aligned} r_F[\eta] &\in H^i({}^{<}Z_{F,\Sigma}, i_F^* \mathcal{V}_{\Gamma,\Sigma}) \cong H^i_{\Gamma_l}(Z_1, \mathcal{V}|_{Z_1}) \\ &\cong H^i_{\Gamma_l}(Z_1/T^c, p_* \mathcal{V}|_{Z_1}) \cong H^i_{\Gamma_l}(A_F, \mathcal{V}^A). \end{aligned}$$

On the other hand, we can take the constant term η_F of η with respect to P_F, producing a $\bar\partial$-closed form on $\Gamma_P^+\backslash D$, where $\Gamma_P^+ = \Gamma \cap P(\mathbb{Q})^+$, by averaging the coefficient functions in (3.10.1) over $\Gamma_W\backslash W(\mathbb{R})$, where $\Gamma_W = \Gamma \cap W(\mathbb{Q})$. This can be restricted to Z_3, which can be viewed as a subset of $\Gamma_P^+\backslash D$ by reduction theory. We have been leading up to the following:

3.10.3. Theorem. – *Under the identifications available from* (3.9.4), *we have* $[\eta_F|_{Z_3}] = r_F[\eta]$ *in* $H^i_{\Gamma_l}(A_F, \mathcal{V}^A)$.

Proof. – If one writes η as an element of (3.10.1), then η_F is by definition

$$(3.10.3.1)\qquad \eta_F = \int_{\Gamma_W\backslash W(\mathbb{R})} \eta \in [C^\infty(\Gamma_P^+\backslash G(\mathbb{R})^+/W(\mathbb{R})) \otimes \Lambda^\bullet(\mathfrak{p}^-)^* \otimes V_\lambda]^{K_p},$$

where $\Gamma_W\backslash W(\mathbb{R})$ has total measure one. This can be restricted to $P(\mathbb{R})^+$ (which acts transitively on D), yielding an element

$$(3.10.3.2)\qquad \begin{aligned} \operatorname{Res}(\eta_F) &\in [C^\infty(\Gamma_P^+\backslash P(\mathbb{R})^+/W(\mathbb{R})) \otimes \Lambda^\bullet(\mathfrak{p}^-)^* \otimes V_\lambda]^{K_p\cap P(\mathbb{R})}, \\ &\cong [C^\infty((\Gamma_P^+/\Gamma_W)\backslash L(\mathbb{R})^+) \otimes \Lambda^\bullet\mathfrak{p}^+ \otimes V_\lambda]^{K_h\cdot K_l}, \end{aligned}$$

where $L = G_h \cdot G_l$ is the Levi subgroup of P as in 1.2. Now, it follows from 1.8.3 and Lemma 1.8.6 that there is a $K_h \cdot K_l$-equivariant isomorphism

$$\text{(3.10.3.3)} \qquad \mathrm{Ad}(c_F): \quad \mathfrak{p}^+ \xrightarrow{\sim} \mathfrak{u}_{\mathbb{C}} \oplus \mathfrak{v}^+ \oplus \mathfrak{p}_h^+$$

preserving $\mathfrak{p}_h^+$. When inserted in (3.10.3.2), we get

$$\text{(3.10.3.4)} \quad \mathrm{Res}(\eta_F) \in [C^\infty((\Gamma_P/\Gamma_W)\backslash L(\mathbb{R})^+) \otimes \Lambda^\bullet(\mathfrak{u}_{\mathbb{C}} \oplus \mathfrak{p}_h^-)^* \otimes \Lambda^\bullet(\mathfrak{v}^-)^* \otimes V_\lambda]^{K_h \cdot K_l},$$

since $\mathfrak{u}_{\mathbb{C}}$ is self-dual as $K_h \cdot K_l$-module; this (double) complex is quasi-isomorphic to (compare the proof of 3.5.12)

$$\text{(3.10.3.5)} \qquad [C^\infty((\Gamma_P/\Gamma_W)\backslash L(\mathbb{R})^+) \otimes \Lambda^\bullet(\mathfrak{u}_{\mathbb{C}} \oplus \mathfrak{p}_h^-)^* \otimes H^\bullet(\mathfrak{v}^-, V_\lambda)]^{K_h \cdot K_l}.$$

3.10.3.6. *Remark.*

(i) Let $\mathfrak{g} = \mathrm{Lie}(K_p) \oplus \mathfrak{p}_p$ be the usual Cartan decomposition, where

$$\mathfrak{p}_p = \mathfrak{g}(\mathbb{R}) \cap (\mathfrak{p}^+ \oplus \mathfrak{p}^-),$$

and let $\tilde{\mathfrak{p}}_l = \mathfrak{p}_p \cap \mathrm{Lie}(G_l)$. Recall [AMRT], III, 4.2, Thm. 1 that the adjoint action of G_l on U presents the cone C_F as a model of the symmetric space of $G_l(\mathbb{R})^0$. Thus, let p_l be a fixed point in C_F of K_l; then $\tilde{\mathfrak{p}}_l \cong T_{C_F, p_l} \cong \mathfrak{u}$. Here we are implicitly identifying differentials on $i\,\mathfrak{u}(\mathbb{R})$ (or $2\pi i\,\mathfrak{u}(\mathbb{R})$) with differentials on $\tilde{\mathfrak{p}}_l$. We return to this point in 4.9.

(ii) Let $G_h^0 = G_h(0)(\mathbb{R})^0$, $G_l^0 = G_l(\mathbb{R})^0$. We can replace $L_F(\mathbb{R})^+$ in (3.10.3.5) first by $L_F(\mathbb{R})^0$, then its derived group $G_h^0 \cdot G_l^{0,\mathrm{der}}$, and finally by the latter's finite cover $G_h^0 \times G_l^{0,\mathrm{der}}$, by adjusting K_h and K_l correspondingly and eliminating the Lie algebra of the contractible central factor of G_l^0. Specifically, let $\bar{K}_h$ and $\bar{K}_l$ denote maximal compact subgroups of the respective factors. Assume first that $\Gamma_h \cdot \Gamma_l = \Gamma_P/\Gamma_W$, where $\Gamma_h = \Gamma \cap G_h^0$. Then (3.10.3.5) is quasi-isomorphic to

$$[C^\infty((\Gamma_h\backslash G_h^0) \times (\Gamma_l\backslash G_l^{0,\mathrm{der}}) \otimes \Lambda^\bullet(\mathfrak{p}_{l,\mathbb{C}} \oplus \mathfrak{p}_h^-)^* \otimes H^\bullet(\mathfrak{v}^-, V_\lambda)]^{\bar{K}_h \cdot \bar{K}_l},$$

where $\mathfrak{p}_l = \mathfrak{p}_p \cap \mathrm{Lie}(G_l^{\mathrm{der}})$, and then to

$$\{C^\infty(\Gamma_h\backslash G_h^0) \otimes \Lambda^\bullet(\mathfrak{p}_h^-)^* \otimes [C^\infty(\Gamma_l\backslash G_l^{0,\mathrm{der}}) \otimes \Lambda^\bullet(\mathfrak{p}_{l,\mathbb{C}}^*) \otimes H^\bullet(\mathfrak{v}^-, V_\lambda)]^{\bar{K}_l}\}^{\bar{K}_h},$$

and then to

$$\{C^\infty(\Gamma_h\backslash G_h^0) \otimes \Lambda^\bullet(\mathfrak{p}_h^-)^* \otimes H^\bullet(\Gamma_l, H^\bullet(\mathfrak{v}^-, V_\lambda))\}^{\bar{K}_h},$$

whose cohomology is

$$\text{(3.10.2)} \qquad H^\bullet(M_F, \mathbf{R}^\bullet_{\Gamma_l}\tilde{\pi}_{1*}\mathcal{V}^A) \cong H^\bullet_{\Gamma_l}(A_F, \mathcal{V}^A).$$

One can always reduce to the preceding case by finding Γ'_h, a normal subgroup of finite index in Γ_h, such that $\Gamma'_h \cdot \Gamma_l \subset \Gamma_P/\Gamma_W$. This generates a finite covering of A_F of the preceding type, without changing Γ_l.

We break up the process that defines η_{F} in (3.10.3.1) into two steps. First, take the average over $\mathrm{U_F}(\mathbb{R})$ only, yielding

$$\eta^c = \int_{\Gamma_{\mathrm{U}}\backslash \mathrm{U_F}(\mathbb{R})} \eta \in [\mathrm{C}^\infty(\Gamma_{\mathrm{P}}\backslash \mathrm{G}(\mathbb{R})/\mathrm{U}(\mathbb{R})) \otimes \Lambda^\bullet(\mathfrak{p}^-)^* \otimes \mathrm{V}_\lambda]^{\mathrm{K}_{\mathrm{P}}}, \tag{3.10.3.7}$$

which can be restricted, as in (3.10.3.2), to give an element of

$$\begin{aligned}(3.10.3.8)\quad &[\mathrm{C}^\infty(\Gamma_{\mathrm{P}}\backslash \mathrm{P}(\mathbb{R})/\mathrm{U}(\mathbb{R})) \otimes \Lambda^\bullet(\mathfrak{p}^-)^* \otimes \mathrm{V}_\lambda]^{\mathrm{K}_h\cdot \mathrm{K}_l}\\ &\cong [\mathrm{C}^\infty(\Gamma_{\mathrm{P}}\backslash \mathrm{P}(\mathbb{R})/\mathrm{U}(\mathbb{R})) \otimes \Lambda^\bullet(\mathfrak{u}_{\mathrm{C}} \oplus \mathfrak{p}_h^-)^* \otimes \Lambda^\bullet(\mathfrak{v}^-)^* \otimes \mathrm{V}_\lambda]^{\mathrm{K}_h\cdot \mathrm{K}_l},\end{aligned}$$

which is, in turn, quasi-isomorphic to (3.10.3.5). Then, averaging η^c with respect to $(\mathrm{W}/\mathrm{U})(\mathbb{R})$ results in η_{F}, for the Haar measure of $\mathrm{W}(\mathbb{R})$ decomposes.

From (3.10.2) and (3.9.4), we have that

$$r_{\mathrm{F}}[\eta] = r_{\mathrm{F}}[\eta^c] \quad \text{in } \mathrm{H}^i_{\Gamma_l}(\mathrm{Z}_1, \mathcal{V}|_{\mathrm{Z}_1}) \cong \mathrm{H}^i_{\Gamma_l}(\mathrm{A_F}, \mathcal{V}^{\mathrm{A}})$$

so we may replace η by η^c, also an "*sia*" form. Then $\eta^c|_{\mathrm{Z}_3}$ represents

$$[\eta^c|_{\mathrm{Z}_2}] \in \mathrm{H}^i_{\Gamma_l}(\mathrm{A_F}, \mathcal{V}^{\mathrm{A}}) \subset \mathrm{H}^i_{\Gamma_l}(\mathrm{Z}_2, \mathcal{V}|_{\mathrm{Z}_2}) \quad \text{(sic)}.$$

Thus both $r_{\mathrm{F}}[\eta^c]$ and $[\eta^c|_{\mathrm{Z}_3}]$ are restrictions of $[\eta^c|_{\mathrm{Z}_2}] \in \mathrm{H}^i_{\Gamma_l}(\mathrm{Z}_2, \mathcal{V}|_{\mathrm{Z}_2})$, so

$$r_{\mathrm{F}}[\eta^c] = [\eta^c|_{\mathrm{Z}_3}] \quad \text{in } \mathrm{H}^i_{\Gamma_l}(\mathrm{A_F}, \mathcal{V}^{\mathrm{A}}).$$

Finally $[\eta^c|_{\mathrm{Z}_3}] = [\eta_{\mathrm{F}}|_{\mathrm{Z}_3}]$, because averaging a form over a compact torus preserves its cohomology class in Lie algebra cohomology (*see* also 3.5.12). Thus, the desired assertion follows.

3.11. Incorporating growth conditions. – In 3.10, we considered $\bar{\partial}$-closed $(0, i)$-forms with logarithmic growth and with values in $\mathcal{V}_{\lambda,\Gamma}$, from the point of view of a fixed boundary component F. However, these forms have moderate growth in *all* directions. Therefore, we can expect to control the behavior of the forms and cohomology classes that entered in 3.10 as one approaches the boundary of F, *i. e.*, $\mathrm{F}' < \mathrm{F}$.

To get started, we can improve upon (3.10.2). Recall that the closure of ${}^{<}\mathrm{Z}_{\mathrm{F},\Sigma}$ in $\mathrm{M}_{\Gamma,\Sigma}$ is $\mathrm{Z}_{\mathrm{F},\Sigma}$; let i_{F} denote the inclusion of $\mathrm{Z}_{\mathrm{F},\Sigma}$ in $\mathrm{M}_{\Gamma,\Sigma}$, as in 3.7. Then we have the restriction

$$\bar{r}_{\mathrm{F}}[\eta] \in \mathrm{H}^i(\mathrm{Z}_{\mathrm{F},\Sigma}, i_{\mathrm{F}}^* \mathcal{V}_{\Gamma,\Sigma}), \tag{3.11.1}$$

whose restriction to ${}^{<}\mathrm{Z}_{\mathrm{F},\Sigma}$ is $r_{\mathrm{F}}[\eta]$.

After refining Σ, if necessary, we can construct the diagram (1.6.4). We need to specify small neighborhoods Z_2^* of $\tilde{\mathrm{Z}}_{\mathrm{F},\Sigma}$ in $\mathrm{M}'_{\mathrm{F},\Sigma}$. To this end, it is convenient to abandon the Siegel domain picture of D, and revert to the real one (the one that would make sense even for non-Hermitian groups), though the answer is equivalent to what is given in (2.5.2). Their intersection with M_Γ is conveniently described in terms of the face $e(\mathrm{P})$ on the manifold-with-corners $\bar{\mathrm{D}}$ of Borel-Serre [BS]. Consider the orbit of p under $({}^0\mathrm{P}(\mathbb{R}))^0$, the identity component of the kernel in $P(\mathbb{R}) \cap G^{der}(\mathbb{R})$ of the determinant of the canonical action of $P(\mathbb{R})$ on $W(\mathbb{R})$ (${}^0P$ is as in [BS], § 1.1; it has the same identity component as $P(\mathbb{R})/A(\mathbb{R})$). This is a cross-section to the so-called geodesic action [BS], § 3 (see also [Z2], (1.2)) of $\tilde{A}(\mathbb{R})^0 = A(\mathbb{R})^0 \cap G^{der}(\mathbb{R})$ (notation 1.2.2), so is diffeomorphic to the face $e(P) = ({}^0P(\mathbb{R}))^0/K_p \cap ({}^0P(\mathbb{R}))^0$. These two commuting actions determine a decomposition of D:

$$D \cong \tilde{A} \times e(P), \tag{3.11.2}$$

with respect to which all $({}^0P(\mathbb{R}))^0$-orbits are of the form $\{t\} \times e(P)$.

3.11.3. PROPOSITION (cf. [Z2]: (3.19)). – *There is a* $W(\mathbb{R})$- *and* Γ_p*-invariant function* g *in* $\mathcal{A}^0_{\text{sia}}(e(P))$ (*cf.* Remark 3.8.3.4), *such that the desired neighborhoods* Z_2^* *are the interiors of the closures of the images of*

$$\tilde{Y}_t = \{(a, x) \in \tilde{A}(\mathbb{R})^0 \times e(P) | a^\beta > tg(x)\} \quad (t \geqq 1)$$

in $M'_{F,\Sigma}$, *where* β *denotes the one simple* $\mathbb{Q}$*-root that is non-zero on* $\tilde{A}$.

3.11.4. *Remarks.*

(i) It is only for products of $\mathbb{Q}$-rank one groups that the ${}^0P(\mathbb{R})$-orbits define such neighborhoods.

(ii) One places $e(P)$ at the boundary of D in the manifold-with-corners by letting $a^\beta = \infty$.

(iii) One can view g as a function on $({}^0P(\mathbb{R}))^0$. Then in (2.5.3), the dependence on G_h appears as the dependence of the defining condition for D_F on the F-coordinate, and the dependence on G_l is reflected in the choice of Y (or core).

We can change variables and write

$$\tilde{Y}_t = \{r \in \mathbb{R} | t < r < \infty\} \times e(P) \quad (r = a^\beta/g(x)). \tag{3.11.5}$$

From 3.11.3, we easily deduce:

3.11.6. LEMMA. – *If a function on* $\tilde{Y}_t$ *is* sia *in the sense of* 3.8.1, *then it is* sia *as a function of r and* $({}^0P(\mathbb{R}))^0$. *The corresponding assertion holds for* rda *functions.*

Next, we observe that since taking the constant term involves only averaging over $\Gamma_W \backslash W(\mathbb{R})$, it follows from 3.11.6 that η_F is slowly increasing with respect to (r and) $({}^0P(\mathbb{R}))^0$. Likewise, the argument that reduces this to cohomology on M_F (3.5.12) involves only K_h-equivariant projections, which certainly preserve the growth condition. We conclude:

3.11.7. PROPOSITION. – *If* η *is slowly increasing, then the class*

$$[\eta_F] \in \bigoplus_r \bigoplus_{w \in W^{F,p}} H^{i-r-l(w)}(M_F, \mathcal{V}_{\lambda(h,w)}) \otimes H^r(X(\Gamma_l), \tilde{\mathbf{V}}_{\lambda(l,w)})$$

is represented by a collection of slowly increasing forms on M_F.

Combining this with 3.8.2, we deduce:

3.11.8. COROLLARY. – *If η is slowly increasing, then η_F defines a class*

$$[\eta_F]_\Xi \in \bigoplus_r \bigoplus_{w \in W^{F,p}} H^{i-r-l(w)}(M_{F,\Xi}, \mathcal{V}_{\lambda(h,w),\Xi}) \otimes H^r(X(\Gamma_l), \tilde{\mathbf{V}}_{\lambda(l,w)}),$$

for any toroidal compactification $M_{F,\Xi}$ *of* M_F.

3.12. MAIN THEOREM WITH CONDITIONS AT INFINITY. – It remains to compare the classes $[\eta_F]_\Xi$ and $\bar{r}_F[\eta]$. As in (1.6.3), the tower (3.9.2) extends:

(3.12.1)
$$\bar{\bar{\Phi}} \left[\begin{array}{c} Z^* \\ \downarrow \bar{p} \\ Z^*/T^c \\ \downarrow \bar{q} \\ A_{F,\Xi} \\ \downarrow \bar{\pi}_1 \\ M_{F,\Xi} \end{array} \right] \bar{\bar{\Psi}}$$

for all of $Z^* = Z_1^* = \tilde{Z}_{F,\Sigma}$, $Z^* = Z_2^*$, $Z^* = Z_3^* = Z_2^* - Z_1^*$, and $Z^* = Z_4^* = M'_{F,\Sigma(\Xi)}$, inside of which Z_2^* is a neighborhood of Z_1^*.

Let $\bar{\mathcal{V}}'_{F,\Sigma}$ denote the canonical extension of $\mathcal{V}'_{F,\Sigma}$ to Z_4^* (for simplicity of notation, we use the same symbols here for the restrictions to Z_i^*, $i=1, 2, 3$), and, for any $\sigma \in \Sigma_F$, write $\bar{\mathcal{V}}_\sigma$ for its restriction to $\bar{Z}_\sigma$; likewise for $\bar{\psi}_\sigma$, $\bar{p}_\sigma$, and $\bar{q}_\sigma$. For $i=1$, $\bar{\mathcal{V}}'_{F,\Sigma}$ is isomorphic to the pullback to $\tilde{Z}_{F,\Sigma}$ of $i_F^* \mathcal{V}_{\Gamma,\Sigma}$ (notation 3.11). Extending 3.2.1, 3.4.1, and 2.8.7, we have:

3.12.2. PROPOSITION.

(i) $\bar{\mathcal{V}}'_{F,\Sigma} \cong \bar{\pi}_2^*(\mathcal{V}^A)_\Xi$ *for some vector bundle* $(\mathcal{V}^A)_\Xi$ *on* $A_{F,\Xi}$ *whose restriction to* A_F *is* $\mathcal{V}^A$.

(ii) *For any component* $\bar{Z}_\sigma$ *of* Z_1^*, $\mathbf{R}\bar{\psi}_{\sigma,*}(\bar{\mathcal{V}}_\sigma) \approx (\mathcal{V}^A)_\Xi$.

(iii) *For any* σ, *the natural mappings*

$$\bar{q}_\sigma^{-1}(\mathcal{V}^A)_\Xi \to (\bar{p}_{\sigma,*}\bar{\mathcal{V}}_\sigma)^{T^c}; \qquad \bar{q}^{-1}(\mathcal{V}^A)_\Xi \to (\bar{p}_* \bar{\mathcal{V}}'_{F,\Sigma})^{T^c}$$

are isomorphisms.

Proof. – Assertion (i) is proved locally. As recalled above, every rational boundary component of (P', D_F) is of the form (P'_1, D_{F_1}), for some rational boundary component F_1 of F. In a neighborhood of the corresponding boundary stratum M'_{F_1} of $M_{F,\Xi}$, it follows from (3.3.5) that $\bar{\mathcal{V}}'_{F,\Sigma}$ is even the pullback of a vector bundle on A_{F_1}, which is the quotient of (the degenerating family of abelian varieties) $A_{F,\Xi}$ near M'_{F_1}.

Lemma 1.6.8 (iii) enables us to prove assertion (ii) by the same arguments used to prove Proposition 3.4.1. With the description of the boundary given in 1.6, assertion (iii) follows from Propositions 2.8.7 and 2.8.12.

To put us in the setting of Proposition 2.9.4 and (2.9.9), we also need:

3.12.3. LEMMA. – *In all four cases of* (3.12.1), *we have*

$$(\mathcal{V}^A)_\Xi \approx \mathbf{R}\bar{q}_*\{\bar{q}^{-1}(\mathcal{V}^A)_\Xi\}.$$

Proof. – The local structure of $\bar{\pi}_2$ is a morphism of torus embeddings (*see* the end of 1.6). Using 1.3.6 and 2.8.11, we have for the higher direct images

$$0 = \mathrm{R}^i\, \bar{\pi}_{2,*}\, (\bar{\mathcal{V}}'_{\mathrm{F},\,\Sigma}) = \mathrm{R}^i\, (\bar{q} \circ \bar{p})_*\, (\bar{\mathcal{V}}'_{\mathrm{F},\,\Sigma}) \Rightarrow \mathrm{R}^i\, \bar{q}_*\, \{\bar{p}_*\, (\bar{\mathcal{V}}'_{\mathrm{F},\,\Sigma})\} = 0$$
$$\Rightarrow \mathrm{R}^i\, \bar{q}_*\, \{\bar{p}_*\, (\bar{\mathcal{V}}'_{\mathrm{F},\,\Sigma})\}^{\mathrm{T}^c} = 0 \Rightarrow \mathrm{R}^i\, \bar{q}_*\, \{\bar{q}^{-1}\, (\mathcal{V}^{\mathrm{A}})_\Xi\} = 0,$$

and similarly $(\mathcal{V}^{\mathrm{A}})_\Xi \longrightarrow \bar{q}_*\, \{\bar{q}^{-1}\, (\mathcal{V}^{\mathrm{A}})_\Xi\}$ is an isomorphism.

3.12.4. Proposition (cf. 3.9.5). – *In all four cases, we have mappings*

$$\mathrm{R}^i\, \bar{\Phi}_*\, (\bar{\mathcal{V}}'_{\mathrm{F},\,\Sigma})_{\Gamma_l} \cong \mathrm{R}^i_{\Gamma_l}\, \bar{\bar{\Psi}}_*(\bar{p}_*\, \bar{\mathcal{V}}'_{\mathrm{F},\,\Sigma}) \overset{inv}{\to} \mathrm{R}^i_{\Gamma_l}\, \bar{\bar{\Psi}}_*(\bar{p}_*\, \bar{\mathcal{V}}'_{\mathrm{F},\,\Sigma})^{\mathrm{T}^c}$$
$$\mathrm{R}^i_{\Gamma_l}\, (\bar{\pi}_1 \circ \bar{q})_*\, \{\bar{q}^{-1}\, (\mathcal{V}^{\mathrm{A}})_\Xi\} \cong \mathrm{R}^i_{\Gamma_l}\, \bar{\pi}_{1*}\, (\mathcal{V}^{\mathrm{A}})_\Xi.$$

We obtain, as in (3.9.6), towers of spectral sequences associated to (3.12.1), in each of the four cases:

$$\text{(3.12.5)}\qquad \begin{array}{ccc} \mathrm{H}^p\, (\mathrm{M}_{\mathrm{F},\,\Xi},\, \mathrm{R}^q\, \bar{\bar{\Phi}}_*\, \bar{\mathcal{V}}'_{\mathrm{F},\,\Sigma}) & \Rightarrow & \mathrm{H}^{p+q}_{\Gamma_l}\, (\mathrm{Z}^*,\, \mathcal{V}'_{\mathrm{F},\,\Sigma}) \\ \uparrow\wr & & \uparrow\wr \\ \mathrm{H}^p_{\Gamma_l}\, (\mathrm{M}_{\mathrm{F},\,\Xi},\, \mathrm{R}^q\, \bar{\bar{\Psi}}_*\, (\bar{p}_*\, \bar{\mathcal{V}}'_{\mathrm{F},\,\Sigma})) & \Rightarrow & \mathrm{H}^{p+q}_{\Gamma_l}\, (\mathrm{Z}^*/\mathrm{T}^c,\, \bar{p}_*\, \bar{\mathcal{V}}'_{\mathrm{F},\,\Sigma}) \\ \uparrow\downarrow\ inv & & \uparrow\downarrow\ inv \\ \mathrm{H}^p_{\Gamma_l}\, (\mathrm{M}_{\mathrm{F},\,\Xi},\, \mathrm{R}^q\, \bar{\bar{\Psi}}_*\, (\bar{p}_*\, \bar{\mathcal{V}}'_{\mathrm{F},\,\Sigma})^{\mathrm{T}^c}) & \Rightarrow & \mathrm{H}^{p+q}_{\Gamma_l}\, (\mathrm{Z}^*/\mathrm{T}^c,\, (\bar{p}_*\, \bar{\mathcal{V}}'_{\mathrm{F},\,\Sigma})^{\mathrm{T}^c}) \\ \uparrow\wr & & \uparrow\wr \\ \mathrm{H}^p_{\Gamma_l}\, (\mathrm{M}_{\mathrm{F},\,\Xi},\, \mathrm{R}^q\, \bar{\bar{\pi}}_{1*}\, (\mathcal{V}^{\mathrm{A}})_\Xi) & \Rightarrow & \mathrm{H}^{p+q}_{\Gamma_l}\, (\mathrm{A}_{\mathrm{F},\,\Xi},\, (\mathcal{V}^{\mathrm{A}})_\Xi) \end{array}$$

which are compatible with the restrictions from Z_2^* to Z_1^* and Z_3^*. Moreover, in the case of Z_1^*, we again have that on both sides the first and last terms are isomorphic, which implies that *inv* is an isomorphism. It follows immediately from (3.12.2) that [*cf.* (3.7.5)]

$$\text{(3.12.6)}\qquad \mathrm{H}^i\, (\mathrm{Z}_{\mathrm{F},\,\Sigma},\, (\bar{\mathcal{V}}'_{\mathrm{F},\,\Sigma})_{\Gamma_l}) \cong \mathrm{H}^i_{\Gamma_l}\, (\mathrm{A}_{\mathrm{F},\,\Xi},\, (\mathcal{V}^{\mathrm{A}})_\Xi).$$

3.12.7. Theorem. – *Under the identifications available from* (3.12.4), $[\eta_{\mathrm{F}}]_\Xi = \bar{r}_{\mathrm{F}}\, [\eta]$ in $\mathrm{H}^{\mathrm{i}}_{\Gamma_l}\, (\mathrm{A}_{\mathrm{F},\,\Xi},\, (\mathcal{V}^{\mathrm{A}})_\Xi)$.

Proof. – Now that we are down to $\mathrm{A}_{\mathrm{F},\,\Xi}$, we can use 3.8 to revert to complexes of slowly increasing forms to compute the cohomology groups. The idea is to repeat the argument used for 3.10.3, making sure that whenever we asserted before that an inclusion of complexes induces an isomorphism on cohomology, the isomorphism can be effected by means of a projection and a homotopy operator under which the growth condition is preserved.

It is convenient to write $C^\bullet(A_F, \mathcal{V}^A)_{si}$, etc., for the Dolbeault complex of $\mathcal{V}^A$-valued forms of moderate growth in the sense of Borel [B2], *cf.* (3.8.1), and $H^\bullet_{\Gamma_l}(A_F, \mathcal{V}^A)_{si}$ for its Γ_l-equivariant cohomology. More precisely, there is a complex of sheaves $\mathcal{C}^\bullet(A_{F,\Xi}, \mathcal{V}^A)_{si}$ on $A_{F,\Xi}$, and we take *its* Γ_l-equivariant hypercohomology. By a small generalization of 3.8.3, this complex is seen to coincide with $\mathcal{A}^\bullet_{si}(A_{F,\Xi}, (\mathcal{V}^A)_\Xi)$, –in view of 3.8.2, we sometimes write $(\mathcal{V}^A)_{si}$ for $(\mathcal{V}^A)_\Xi$– hence

(3.12.7.1) $$H^\bullet_{\Gamma_l}(A_{F,\Xi}, (\mathcal{V}^A)_\Xi) \cong H^\bullet_{\Gamma_l}(A_F, \mathcal{V}^A)_{si}.$$

We had defined $[\eta_F]_\Xi$ as an element of the right-hand side. Indeed, if η is slowly increasing, then so are η^c (the T^c-invariant projection of η) and η_F, as we have already observed. As before, $\bar{r}_F[\eta]$ and $[\eta^c]$ are equal in $H^\bullet_{\Gamma_l}(A_{F,\Xi}, (\mathcal{V}^A)_\Xi)$, for they are both restrictions of $[\eta|_{Z_2^*}]$.

On the other hand, η_F is obtained from η^c by averaging over a compact torus, which has an L^∞-bounded homotopy operator that thus respects our growth condition. It follows that η_F and η^c define the same class in $H^\bullet_{\Gamma_l}(A_F, \mathcal{V}^A)_{si}$, and we are done.

3.12.8. *Remark.* – The construction of $(\mathcal{V}^A)_\Xi$ (essentially Lemma 1.6.8) shows that it can be viewed as a *canonical extension* of $\mathcal{V}^A$, in a sense which can be made more precise using the methods of paragraph 4.6 below.

3.13. REDUCTION TO COHOMOLOGY ON M_F (WITH GROWTH CONDITION). – We begin by noting that (3.12.5) and (3.12.7.1) can be combined to give

(3.13.1) $$H^\bullet(Z_{F,\Sigma}, (\bar{\mathcal{V}}'_{F,\Sigma})_{\Gamma_l}) \cong H^\bullet_{\Gamma_l}(A_F, \mathcal{V}^A)_{si}.$$

We wish to express the right-hand side as cohomology on M_F, as we did in 3.7. Recall from (3.6.3) that

(3.13.2) $$R^q\pi_{1,*}\mathcal{V}^A \xrightarrow{\sim} \bigoplus_{w\in W^{F,p}(q)} \mathcal{V}_{\lambda(h,w)} \otimes V_{\lambda(l,w)},$$

with Γ_l acting on the right-hand factor. This gives at once on $(M_F)_\Xi$,

(3.13.3) $$(R^q\pi_{1,*}\mathcal{V}^A)_{si} \xrightarrow{\sim} \bigoplus_{w\in W^{F,p}(q)} (\mathcal{V}_{\lambda(h,w)})_{si} \otimes V_{\lambda(l,w)}.$$

We have the following version of the results in 3.5 and 3.7, which pays attention to the growth conditions:

3.13.4. PROPOSITION.

(i) $R^q\bar{\pi}_{1,*}((\mathcal{V}^A)_{si}) \cong (R^q\pi_{1,*}\mathcal{V}^A)_{si}$;

(ii) *The following spectral sequences degenerate at* E_2:

(3.13.4.1) $$E_2^{p,q} = H^p(M_F, R^q\pi_{1,*}\mathcal{V}^A)_{si} \Rightarrow H^{p+q}(A_F, \mathcal{V}^A)_{si},$$

$$\text{(3.13.4.2)} \qquad E_2^{p,q} = H^p_{\Gamma_l}(M_F, R^q\pi_{1,*}\mathcal{V}^A)_{si} \Rightarrow H^{p+q}_{\Gamma_l}(A_F, \mathcal{V}^A)_{si},$$

$$\text{(3.13.4.3)} \qquad E_2^{r,s} = H^r(\Gamma_l, H^s(A_F, \mathcal{V}^A)_{si}) \Rightarrow H^{r+s}_{\Gamma_l}(A_F, \mathcal{V}^A)_{si},$$

$$\text{(3.13.4.4)} \qquad E_2^{r,t} = H^r(\Gamma_l, H^t(M_F, R^q\pi_{1,*}\mathcal{V}^A)_{si}) \Rightarrow H^{r+t}_{\Gamma_l}(M_F, R^q\pi_{1,*}\mathcal{V}^A)_{si},$$

as do their restrictions over subsets of $(M_F)_\Xi$.

Proof. – Recall that the main point in the proof of Proposition 3.5.12 was the Γ_l-equivariant embedding of the complex (3.5.12.1) in (3.5.12.2), which induced an isomorphism on cohomology. One actually knows much more: we claim it is induced by the composite of Γ_l-equivariant, quasi-isomorphic embeddings of complexes of sheaves

$$\text{(3.13.4.5)} \quad \mathcal{C}^\bullet(M_{F,\Xi}, \mathcal{H}^\bullet(\mathfrak{s}_p, V_\lambda))_{si} \hookrightarrow \mathcal{C}^\bullet(M_{F,\Xi}, \Lambda^\bullet(\mathfrak{s}_p)^* \otimes \mathcal{V}_\lambda)_{si} \hookrightarrow \mathcal{C}^\bullet(A_{F,\Xi}, \mathcal{V}^A)_{si},$$

given by averaging over the fibers of π_1, and by K-equivariant projection respectively. These operations respect the growth conditions, and moreover, they come with cochain homotopy operators that also respect the growth conditions. Thus we have (3.13.4.5) (*cf.* proof of 3.12.7). Now, this and (3.13.3) give (i) and the degeneration of the first two spectral sequences in (ii). As in the proof of Proposition 3.7.7, the degeneration of the last two become equivalent; they do in fact degenerate, for the same reason as before.

Rewriting the above assertions in the alternate notation, we obtain:

3.13.5. Corollary.

(i) $R^q\bar{\pi}_{1*}((\mathcal{V}^A)_\Xi) \cong \bigoplus_{w\in W^{F,p}(q)} (\mathcal{V}_{\lambda(h,w)})_\Xi \otimes V_{\lambda(l,w)}$,

(ii) *The following spectral sequences degenerate at* E_2*:*

$$\text{(3.13.5.1)} \qquad E_2^{p,q} = H^p(M_{F,\Xi}, R^q\bar{\pi}_{1,*}(\mathcal{V}^A)_\Xi) \Rightarrow H^{p+q}(A_{F,\Xi}, (\mathcal{V}^A)_\Xi),$$

$$\text{(3.13.5.2)} \qquad E_2^{p,q} = H^p_{\Gamma_l}(M_{F,\Xi}, R^q\bar{\pi}_{1,*}(\mathcal{V}^A)_\Xi) \Rightarrow H^{p+q}_{\Gamma_l}(A_{F,\Xi}, (\mathcal{V}^A)_\Xi),$$

$$\text{(3.13.5.3)} \qquad E_2^{r,s} = H^r(\Gamma_l, H^s(A_{F,\Xi}, (\mathcal{V}^A)_\Xi)) \Rightarrow H^{r+s}_{\Gamma_l}(A_{F,\Xi}, (\mathcal{V}^A)_\Xi),$$

$$\text{(3.13.5.4)} \quad E_2^{r,t} = H^r(\Gamma_l, H^t(M_{F,\Xi}, R^q\bar{\pi}_{1,*}(\mathcal{V}^A)_\Xi) \Rightarrow H^{r+t}_{\Gamma_l}(M_{F,\Xi}, R^q\bar{\pi}_{1*}(\mathcal{V}^A)_\Xi).$$

Finally, (3.13.1) can now be expressed as:

3.13.6. Corollary. – $H^i(Z_{F,\Sigma}, (\bar{\mathcal{V}}'_{F,\Sigma})_{\Gamma_l})$ *is isomorphic to*

$$\bigoplus_r \bigoplus_{w\in W^{F,p}} H^{i-r-l(w)}(M_{F,\Xi}, \mathcal{V}_{\lambda(h,w),\Xi}) \otimes H^r(X(\Gamma_l), \tilde{\mathbf{V}}_{\lambda(l,w)}).$$

3.14. Cohomology of rapidly decreasing forms. – We continue the discussion of the last two sections, with $Z_1^* = \tilde{Z}_{F,\Sigma}$, which we denote simply $\tilde{Z}$; write $\partial = \partial\tilde{Z}_{F,\Sigma}$, ${}^>\partial = {}^>\partial\tilde{Z}_{F,\Sigma}$, ${}^<\partial = {}^<\partial\tilde{Z}_{F,\Sigma}$, in the notation of 1.5, 1.6. Let $\tilde{\mathcal{I}}_\partial$, $\tilde{\mathcal{I}}_{>\partial}$, $\tilde{\mathcal{I}}_{<\partial}$ be the corresponding (invertible) sheaves of ideals in $\mathcal{O}_{\tilde{Z}}$; they are the pullbacks from $Z_{F,\Sigma}$ of invertible sheaves $\mathcal{I}_\partial$, $\mathcal{I}_{>\partial}$, $\mathcal{I}_{<\partial}$, defined in the obvious way. As in 3.12, we let $\bar{\mathcal{V}}'_{F,\Sigma}$ denote the pullback to $\tilde{Z}$ of $i_F^* \mathcal{V}_{\Gamma,\Sigma}$, and let $\bar{\mathcal{V}}'_{F,\Sigma}(-\partial)$ (resp. $\bar{\mathcal{V}}'_{F,\Sigma}(-{}^{>}\partial)$, resp. $\bar{\mathcal{V}}'_{F,\Sigma}(-{}^{<}\partial)$) be $\bar{\mathcal{V}}'_{F,\Sigma} \otimes \mathcal{I}_\partial$ (resp. $\bar{\mathcal{V}}'_{F,\Sigma} \otimes \mathcal{I}_{>\partial}$, resp. $\bar{\mathcal{V}}'_{F,\Sigma} \otimes \mathcal{I}_{<\partial}$). Note that $\bar{\mathcal{V}}'_{F,\Sigma}(-\partial)$ is the pullback from $Z_{F,\Sigma}$ of the vector bundle $i_F^* \mathcal{V}_{\Gamma,\Sigma}(-\partial) = i_F^* \mathcal{V}_{\Gamma,\Sigma} \otimes \mathcal{I}_\partial$.

On the other hand, let $\partial A_{F,\Xi} = A_{F,\Xi} - A_F$, $\partial M_{F,\Xi} = M_{F,\Xi} - M_F$. These are again divisors with normal crossings on $A_{F,\Xi}$ and $M_{F,\Xi}$, defined respectively by invertible sheaves of ideals $\mathcal{I}_{\partial A_{F,\Xi}}$, $\mathcal{I}_{\partial M_{F,\Xi}}$. We let $(\mathcal{V}^A)_\Xi(-\partial) = (\mathcal{V}^A)_\Xi \otimes \mathcal{I}_{\partial A_{F,\Xi}}$; if $\mathcal{W}_\Xi$ is the canonical extension to $M_{F,\Xi}$ of an automorphic vector bundle $\mathcal{W}$ on M_F, we let $\mathcal{W}^{sub} = \mathcal{W}_\Xi \otimes \mathcal{I}_{\partial M_{F,\Xi}}$ (the *subcanonical extension*, as in [H5]). In view of Lemma 1.6.8 and Proposition 3.12.2, we have

$$\mathbf{R}\,\bar{\pi}_{2,*}\,\bar{\mathcal{V}}'_{F,\Sigma}(-\partial) \approx [(\mathcal{V}^A)_\Xi(-\partial)] \otimes \mathbf{R}\,\bar{\pi}_{2,*}(\mathcal{I}_{>\partial}) \tag{3.14.1}$$

In the last two sections we made no use of conditions of moderate growth on G_l, since they are not necessary in the computation of the cohomology of its associated locally symmetric spaces. When we consider the coherent cohomology of $\tilde{Z}$ with coefficients in $\bar{\mathcal{V}}'_{F,\Sigma}(-\partial)$ or $\bar{\mathcal{V}}'_{F,\Sigma}(-{}^{>}\partial)$, however, the condition of rapid decrease along G_l becomes relevant. Indeed, $\tilde{Z}$ is a toroidal compactification of ${}^0\tilde{Z}_{F,\Sigma}$ and $\bar{\mathcal{V}}'_{F,\Sigma}$ is a canonical extension, in the sense of 3.2, of its restriction to ${}^0\tilde{Z}_{F,\Sigma}$. Thus the considerations of 3.8 apply to this situation. In analogy with Corollary 3.7.8, we have

3.14.2. PROPOSITION. – *In the notation of Corollary* 3.7.8, *for each integer t, there is a natural isomorphism*

$$H^t(Z_{F,\Sigma}, i_F^* \mathcal{V}_{\Gamma,\Sigma}(-\partial)) \tilde{\to} \bigoplus_r \bigoplus_{w \in W^{F,p}} H^{t-r-l(w)}(M_{F,\Xi}, (\mathcal{V}^{sub}_{\lambda(h,w)})) \otimes H^r_c(X(\Gamma_l), \tilde{\mathbf{V}}_{\lambda(l,w)}),$$

where $H^\bullet_c$ *denotes cohomology with compact supports.*

Remark. – A combinatorial analysis of the left-hand side, along the lines of 3.7, will be carried out in Part II (in greater generality).

Proof. – We argue as in 3.9. We need to compute

$$H^t(Z_{F,\Sigma}, i_F^* \mathcal{V}_{\Gamma,\Sigma}(-\partial)) \cong H^t_{\Gamma_l}(\tilde{Z}, \bar{\mathcal{V}}'_{F,\Sigma}(-\partial))$$

(*cf.* 3.7). By (3.14.1), this can be identified with

$$H^t_{\Gamma_l}(A_{F,\Xi}, [(\mathcal{V}^A)_\Xi(-\partial)] \otimes \mathbf{R}\,\bar{\pi}_{2,*}\mathcal{I}_{>\partial}). \tag{3.14.1}$$

To compute $\mathbf{R}\,\bar{\pi}_{2,*}\mathcal{I}_{>\partial}$, we factor $\bar{\pi}_2 = \bar{q} \circ \bar{p}$, as in (3.12.1). Let $Y = \tilde{Z}/T^c$, ${}^{>}\partial Y = {}^{>}\partial/T^c$, ${}^{>}Y = Y - {}^{>}\partial Y$; let $j : {}^{>}Y \to Y$ be the open immersion. At this point we make use of the quasi-isomorphism

$$\mathcal{I}_{>\partial} \approx \mathrm{Cone}\,\{\mathcal{O}_{\tilde{Z}} \to \mathcal{O}_{>\partial}\}\,[-1].$$

It will be convenient to view $\mathcal{O}_{\tilde{Z}}$ and $\mathcal{O}_{>\partial}$ (resp. $\mathbb{C}_Y$ and $\mathbb{C}_{>\partial Y}$) as simplicial sheaves on $\tilde{Z}$ (resp. Y) as in 2.7, corresponding to the closed covers by smooth irreducible components (resp. quotients by T^c of smooth irreducible components). Then

$$\begin{aligned} \mathbf{R}\,\bar{\pi}_{2,*}\,\mathcal{I}_{>\partial} &\approx \mathrm{Cone}\,\{\mathbf{R}\,\bar{\pi}_{2,*}\,\mathcal{O}_{\tilde{Z}} \to \mathbf{R}\,\bar{\pi}_{2,*}\,\mathcal{O}_{>\partial}\}\,[-1] \\ &\approx \mathrm{Cone}\,\{\mathbf{R}\,\bar{q}_*\,(\bar{p}_*\,\mathcal{O}_{\tilde{Z}}) \to \mathbf{R}\,\bar{q}_*\,(\bar{p}_*\,\mathcal{O}_{>\partial})\}\,[-1] \end{aligned}$$

by (2.9.6). By Proposition 3.12.2 (ii), the assertion of Proposition 3.4.4 remains valid with p replaced by $\bar{p}$. Thus, by Proposition 3.12.2 (iii), and the five-lemma, this gives the same equivariant cohomology as

$$\begin{aligned} &\mathrm{Cone}\,\{\mathbf{R}\,\bar{q}_*\,(\bar{p}_*\,\mathcal{O}_{\tilde{Z}})^{T^c} \to \mathbf{R}\,\bar{q}_*\,(\bar{p}_*\,\mathcal{O}_{>\partial})^{T^c}\}\,[-1] \\ &\approx \mathrm{Cone}\,\{\mathbf{R}\,\bar{q}_*\,(\bar{q}^{-1}\,\mathcal{O}_{A_{F,\Xi}})_Y \to \mathbf{R}\,\bar{q}_*\,(\bar{q}^{-1}\,\mathcal{O}_{A_{F,\Xi}})_{>\partial Y}\}\,[-1] \\ &\approx \mathcal{O}_{A_{F,\Xi}} \otimes \mathrm{Cone}\,\{\mathbf{R}\,\bar{q}_*\,\mathbb{C}_Y \to \mathbf{R}\,\bar{q}_*\,\mathbb{C}_{>\partial Y})\}\,[-1] \cong \mathcal{O}_{A_{F,\Xi}} \otimes \mathbf{R}\,\bar{q}_*\,j_!\,\mathbb{C}_{>Y}. \end{aligned}$$

We obtain that

$$\text{(3.14.1)} \qquad H^t\,(Z_{F,\Sigma}, i_F^*\,\mathcal{V}_{\Gamma,\Sigma}\,(-\partial)) \cong H^t_{\Gamma_l}\,(A_{F,\Xi}, (\mathcal{V}^A)_\Xi\,(-\partial) \otimes \mathbf{R}\,\bar{q}_*\,j_!\,\mathbb{C}_{>Y}).$$

The latter is the abutment of the Leray spectral sequence for $\bar{\pi}_1$, whose E_2 term is

$$\text{(3.14.3)} \qquad \begin{aligned} E_2^{p,q} &= H^p\,(M_{F,\Xi}, R^q_{\Gamma_l}\,\bar{\pi}_{1,*}\,(\mathcal{V}^A)_\Xi\,(-\partial) \otimes \mathbf{R}\,\bar{q}_*\,j_!\,\mathbb{C}_{>Y}) \\ &\cong H^p\,(M_{F,\Xi}, R^q_{\Gamma_l}\,\bar{\pi}_{1,*}\,(\mathcal{V}^A)_{\mathrm{rda}} \otimes \mathbf{R}\,\bar{q}_*\,j_!\,\mathbb{C}_{>Y}) \end{aligned}$$

where as in (3.12.7.1) we have replaced $(\mathcal{V}^A)_\Xi\,(-\partial)$ by the canonically quasi-isomorphic Dolbeault complex on A_F of forms which are *rda* near $\partial A_{F,\Xi}$. As in 3.13, the last line is isomorphic to:

$$\cong \bigoplus_r \bigoplus_{w \in W^{F,p}} H^{t-r-l(w)}\,(M_{F,\Xi}, (\mathcal{V}^{\mathrm{sub}}_{\lambda(h,w)})) \otimes H^r_c\,(X\,(\Gamma_l), \tilde{\mathbf{V}}_{\lambda(l,w)}).$$

Finally, the spectral sequence (3.14.3) is seen to degenerate, and be canonically split, by the argument of 3.5.12 (applied to *rda* forms).

4. Adelization and canonical models

The constructions in the previous sections have been purely complex analytic, and have made no reference to the arithmetic structure of canonical models for the Shimura varieties and automorphic vector bundles in question. For applications to Eisenstein cohomology classes and the eventual construction of mixed motives, it is necessary to take this arithmetic structure into account. The most convenient language for this purpose is that of adelic Shimura varieties, which was briefly introduced in Sections 1.1, 1.7, and 3.1-2. The boundary cohomology groups computed in 3.13 will first be replaced by their adelic versions. We then show how to construct the canonical models of automorphic vector bundles and their canonical extensions, and show how each of the steps in the computation of cohomology in paragraph 3 respects the arithmetic structures.

4.1. ADELIC COMPUTATION OF BOUNDARY COHOMOLOGY.

Fix a rational boundary component F and the corresponding maximal parabolic subgroup $P_F \subset G$ as in 1.2.1. In order to compute, as painlessly as possible, the cohomology of the adelic F-stratum of $Sh(G, X)^{\sim}$ (notation 1.7), we introduce several mixed Shimura varieties treated by Pink [P], which arise at intermediate stages of the computation. We do so informally, without recalling Pink's formalism, much of which is devoted to problems arising from disconnectedness of the symmetric spaces; in particular, our symmetric spaces will in (4.1.1) and (4.1.2) be taken to be connected components of those used by Pink. For us it will only be necessary to recall that the mixed Shimura varieties have natural algebraic structures and canonical models over the reflex field E(G, X), compatible with all morphisms introduced and with the actions of the various adele groups.

We fix F for the remainder of paragraph 4, and abbreviate $U=U_F$, $W=W_F$, $V=V_F$, $G_h=G_{h,F}$, $G_l=G_{l,F}$, $P=P_F$, etc. Define

$$\mathcal{M}'_F = \overset{\lim}{\longleftarrow} P'(\mathbb{Q})^+\backslash D_F \times P(\mathbf{A}^f)/K_P; \tag{4.1.1}$$

$$\mathcal{A}_F = \overset{\lim}{\longleftarrow} P'(\mathbb{Q})^+\backslash [D_F/U(\mathbb{C}) \times P(\mathbf{A}^f))/U(\mathbf{A}^f)]/K_{\mathcal{A}}; \tag{4.1.2}$$

Here (4.1.1) and (4.1.2) are slightly modified versions of the mixed Shimura varieties $Sh(P', D_F)$ and $Sh(P'/U, D_F/U(\mathbb{C}))$ defined by Pink (this is a slight abuse of notation, since D_F and $D_F/U(\mathbb{C})$ are connected). The modifications are adapted to the P-stratum of $Sh(G, X)^*$, *see* below.

The inverse limits in (4.1.1-2) are taken with respect to the indicated families of compact open subgroups of the relevant adele group, which are intrinsically defined. However, it will be necessary to work with these objects at finite level. Thus, let $K \subset G(\mathbf{A}^f)$ be a neat compact open subgroup, and let

$$K_P = K \cap P(\mathbf{A}^f), \qquad K_{\mathcal{A}} = K \cap P(\mathbf{A}^f)/K \cap U(\mathbf{A}^f) \subset (P/U)(\mathbf{A}^f),$$
$$K_F = K \cap P(\mathbf{A}^f)/K \cap W(\mathbf{A}^f) \subset L(\mathbf{A}^f)$$

We make the simplifying assumption that the given Levi decomposition of P induces a product decomposition

$$K_P \cong K_F \times (K \cap W(\mathbf{A}^f)). \tag{4.1.3}$$

It is easy to see that the set of K's with this property are cofinal; we leave it to the reader to reduce the arguments in this Section to this case.

Now define

(4.1.4) ${}_K Sh(G, X)^P = {}_{(K_F)} Sh(G, X)^P = Sh(G, X)/K_F$ (for typographical reasons), ${}_K\mathcal{M}'_F = \mathcal{M}'_F/K_P$, ${}_K\mathcal{A}_F = \mathcal{A}_F/K_{\mathcal{A}}$.

Then there are natural morphisms (at "level K"):

(4.1.5) $$\pi_{2,\mathrm{K}}: \quad {}_{\mathrm{K}}\mathcal{M}'_{\mathrm{F}} \to {}_{\mathrm{K}}\mathcal{A}_{\mathrm{F}}; \qquad \pi_{1,\mathrm{K}}: \quad {}_{\mathrm{K}}\mathcal{A}_{\mathrm{F}} \to {}_{\mathrm{K}}\mathrm{Sh}(\mathrm{G}, \mathrm{X})^{\mathrm{P}};$$

Under the hypothesis (4.1.3), the map $\pi_{1,\mathrm{K}}$ makes ${}_{\mathrm{K}}\mathcal{A}_{\mathrm{F}}$ an abelian scheme over ${}_{\mathrm{K}}\mathrm{Sh}(\mathrm{G}, \mathrm{X})^{\mathrm{P}}$; let

$$\zeta_{\mathrm{K}}: \quad {}_{\mathrm{K}}\mathrm{Sh}(\mathrm{G}, \mathrm{X})^{\mathrm{P}} \to {}_{\mathrm{K}}\mathcal{A}_{\mathrm{F}}$$

be the zero section.

If $\Sigma = \bigcup \Sigma_{\mathrm{F}}$ is an adelic family of fans as in 1.7, then the (partial) toroidal compactification ${}_{\mathrm{K}}\mathcal{M}'_{\mathrm{F},\Sigma}$ is defined just as in the connected case (*see* [H3] for details). The results of Pink cited in 1.6 are actually adelic, and imply that, possibly after refining Σ, we may extend (4.1.5) to

(4.1.6) $$\pi_{2,\Sigma}: \quad {}_{\mathrm{K}}\mathcal{M}'_{\mathrm{F},\Sigma} \to {}_{\mathrm{K}}\mathcal{A}_{\mathrm{F},\Xi^{\mathcal{A}}}; \qquad \pi_{1,\Sigma}: \quad {}_{\mathrm{K}}\mathcal{A}_{\mathrm{F},\Xi^{\mathcal{A}}} \to {}_{\mathrm{K}}\mathrm{Sh}(\mathrm{G}, \mathrm{X})^{\mathrm{P}}_{\Xi},$$

where $\Xi^{\mathcal{A}}$ and Ξ are families of fans defined for the corresponding mixed Shimura varieties. Furthermore, these fans may be chosen in such a way that the spaces ${}_{\mathrm{K}}\mathcal{M}'_{\mathrm{F},\Sigma}$, ${}_{\mathrm{K}}\mathcal{A}_{\mathrm{F},\Xi^{\mathcal{A}}}$, and ${}_{\mathrm{K}}\mathrm{Sh}(\mathrm{G}, \mathrm{X})^{\mathrm{P}}_{\Xi}$ are smooth varieties defined over $\mathrm{E}(\mathrm{G}, \mathrm{X})$, the latter two projective; the boundaries are divisors with normal crossings, each of whose irreducible components is smooth; and the morphisms $\pi_{2,\Sigma}$ and $\pi_{1,\Sigma}$ are $\mathrm{E}(\mathrm{G}, \mathrm{X})$-rational. We also assume hypotheses (2.2.7) and (2.2.9) (in their adelic versions).

In most cases we will be content to work with the partial toroidal compactifications relative to $\bigcup \Sigma_{\mathrm{F}'}$ for $\mathrm{F}' \geqq \mathrm{F}$, as in 1.5 (actually, all information is already contained in Σ_{F}); in other words, boundary divisors corresponding to the boundary components of F are removed.

Until further notice, we fix a neat level subgroup K, and let $\mathrm{Sh} = {}_{\mathrm{K}}\mathrm{Sh}(\mathrm{G}, \mathrm{X})$. Let Sh_{Σ} be an admissible toroidal compactification of Sh which admits the morphisms (4.1.6), and let $\mathrm{Sh}_{\Sigma}(\mathrm{F})$ be the partial toroidal compactification relative to $\bigcup \Sigma_{\mathrm{F}'}$ for $\mathrm{F}' \geqq \mathrm{F}$. Let $\partial \mathrm{Sh}_{\Sigma} = \mathrm{Sh}_{\Sigma} - \mathrm{Sh}$, and let $\mathrm{Sh}^{\mathrm{P}}_{\Sigma}$ denote its P-stratum (1.7.11), which is the same as the P-stratum of $\mathrm{Sh}_{\Sigma}(\mathrm{F})$. Let $\widetilde{\mathrm{Sh}}^{\mathrm{P}}_{\Sigma}$ be the closure in ${}_{\mathrm{K}}\mathcal{M}'_{\mathrm{F},\Sigma} - {}_{\mathrm{K}}\mathcal{M}'_{\mathrm{F}}$ of

$$(\pi_{1,\Sigma} \circ \pi_{2,\Sigma})^{-1}({}_{\mathrm{K}}\mathrm{Sh}(\mathrm{G}, \mathrm{X})^{\mathrm{P}}),$$

via (4.1.6). Then $\widetilde{\mathrm{Sh}}^{\mathrm{P}}_{\Sigma}$ is, at least analytically, an etale covering of $\mathrm{Sh}^{\mathrm{P}}_{\Sigma}$:

(4.1.7) $$\mathrm{Sh}^{\mathrm{P}}_{\Sigma} \cong (\mathrm{P}(\mathbb{Q})^{+}/\mathrm{P}'(\mathbb{Q})^{+}) \backslash \widetilde{\mathrm{Sh}}^{\mathrm{P}}_{\Sigma}.$$

Indeed, over the connected component M_{Γ} this is essentially Lemma 1.5.3. It is proved by Pink ([P], Theorem 12.4) that the natural map $\widetilde{\mathrm{Sh}}^{\mathrm{P}}_{\Sigma} \to \mathrm{Sh}^{\mathrm{P}}_{\Sigma}$ is a *local isomorphism in the Zariski topology*, compatible with the $\mathrm{E}(\mathrm{G}, \mathrm{X})$-rational structures on both sides.

The computation of cohomology in paragraph 3 immediately extends to the adelic setting. It is most easily written down for the inverse limit (with respect to K) over the P-strata. We let

(4.1.8) $$\mathrm{X}(\mathrm{G}_{l}) = \overset{\lim}{\longleftarrow} \mathrm{G}_{l}(\mathbb{Q}) \backslash \mathrm{G}_{l}(\mathbf{A}) / \mathrm{K}_{l} \cdot \mathrm{A}(\mathbb{R}) \cdot \mathrm{K}_{l,f}$$

be the adelic locally symmetric space for G_l; the limit is taken with respect to open compact subgroups $K_{l,f} \subset G_l(\mathbf{A}^f)$. The local systems $\tilde{\mathbf{V}}_{\lambda(l,w)}$ on $X(G_l)$ are defined as in Corollary 3.7.8, and the cohomology groups $H^r(X(G_l), \tilde{\mathbf{V}}_{\lambda(l,w)})$ are defined as direct limits over $K_{l,f}$:

$$(4.1.9) \qquad H^r(X(G_l), \tilde{\mathbf{V}}_{\lambda(l,w)}) = \varinjlim H^r(X(G_l)/K_{l,f}, \tilde{\mathbf{V}}_{\lambda(l,w)}).$$

These have "topological" rational structures over the fields of definition of the algebraic representations $(\lambda(l, w), V_{\lambda(l,w)})$ of G_l.

Next, for any Shimura variety $Sh(G, X)$ and any automorphic vector bundle $\mathcal{V}$, we define as in [H5], § 2 the admissible $G(\mathbf{A}^f)$-modules

$$\tilde{H}^\bullet(\mathcal{V}^{can}) = \varinjlim_{K,\Sigma} H^\bullet({}_K Sh(G, X)_\Sigma, \mathcal{V}^{can}),$$

$$\tilde{H}^\bullet(\mathcal{V}^{sub}) = \varinjlim_{K,\Sigma} H^\bullet({}_K Sh(G, X)_\Sigma, \mathcal{V}^{sub});$$

$$\bar{H}^\bullet(Sh(G, X), \mathcal{V}) = \mathrm{Im}(\tilde{H}^\bullet(\mathcal{V}^{sub}) \to \tilde{H}^\bullet(\mathcal{V}^{can})).$$

Write $\mathcal{V}_\Sigma^P$ (resp. $\mathcal{V}_\Sigma^F$) for the pullback of $\mathcal{V}^{can}$ to the P- (resp. F-) stratum [from (1.7.10) and (1.7.11)]. Now we compute $\varinjlim_{K,\Sigma} H^i(\overline{Sh}_\Sigma^P, \mathcal{V}_\Sigma^P)$ by the spectral sequence of the closed cover $\mathcal{Z}_\Sigma^P$ by its irreducible components, as in 3.7. For any fixed K, Σ_F (resp. Σ_F^c) is a fan in $G(\mathbb{Q})^+ \times^{P(\mathbb{Q})^+} (\bar{C}_F \times G(\mathbf{A}^f)/K)$ (resp. $G(\mathbb{Q})^+ \times^{P(\mathbb{Q})^+} (C_F \times G(\mathbf{A}^f)/K)$), by (1.7.1). The P-stratum corresponds to the subcomplex of the fan given by $\Sigma_P = \Sigma_F \cap (\bar{C}_F \times P(\mathbf{A}^f) \cdot K/K)$, and two simplices in this subcomplex define the same divisor in $\overline{Sh}_\Sigma^P$ if and only if they are in the same orbit under $P(\mathbb{Q})^+$. If we let $\hat{\Sigma}_P = \{\hat{\sigma} | \sigma \in \Sigma_P\}$ (*cf.* 2.2), it follows that the nerve of $\mathcal{Z}_\Sigma^P$ is just $P(\mathbb{Q})^+ \backslash \hat{\Sigma}_P$ (*cf.* 3.7.2). Then the reasoning of paragraph 3 applies. Let Δ_1 be the kernel of the natural map $G_h(\mathbf{A}^f) \times G_l(\mathbf{A}^f) \to G_h(\mathbf{A}^f) \cdot G_l(\mathbf{A}^f)$, $\Delta_0 = L(\mathbb{Q})^+/(G_h(\mathbb{Q})^+ \cdot G_l(\mathbb{Q})^+)$, $\Delta = \Delta_0 \times \Delta_1$. Then Δ_1 acts naturally on

$$(4.1.10) \qquad \mathcal{H}^\bullet(w) := \tilde{H}^{\bullet - l(w)}((\mathcal{V}_{\lambda(h,w)})^{can}) \otimes H^\bullet(X(G_l), \tilde{\mathbf{V}}_{\lambda(l,w)})$$

for each w, through the natural $G_h(\mathbf{A}^f) \times G_l(\mathbf{A}^f)$ action on the tensor product. Moreover $\delta \in \Delta_0$ acts on the left on the space $Sh(G_h, X(F)) \times X(G_l)$ by

$$(4.1.11) \qquad \begin{cases} \delta \cdot (x, \gamma) = (\delta(x), \delta\gamma\delta^{-1}), \\ x \in F \times [G_l(\mathbb{R})/K_l \cdot Z_{G_l}(\mathbb{R})], \qquad \gamma \in G_h(\mathbf{A}^f) \cdot G_l(\mathbf{A}^f), \end{cases}$$

and the sheaves $\mathcal{V}_{\lambda(h,w)} \times \tilde{\mathbf{V}}_{\lambda(l,w)}$ are equivariant for this action. Thus Δ_0 also acts on $\mathcal{H}^\bullet(w)$, and we have

4.1.12. COROLLARY. – $\varinjlim_{K,\Sigma} H^i(\overline{Sh}_\Sigma^P, \mathcal{V}_\Sigma^P)$ *is isomorphic to*

$$\bigoplus_r \bigoplus_{w \in W^{F,p}} I^P\{\tilde{H}^{i-r-l(w)}((\mathcal{V}_{\lambda(h,w)})^{can}) \otimes H^r(X(G_l), \tilde{\mathbf{V}}_{\lambda(l,w)})\}.$$

Here $(\mathcal{V}_{\lambda(h,w)})^{\mathrm{can}}$ is viewed as a (family of) automorphic vector bundle(s) on toroidal compactification(s) of $\mathrm{Sh}(\mathrm{G}_h, \mathrm{X}(\mathrm{F}))$, with $\mathrm{X}(\mathrm{F})$ as in (1.7.4). Here

$$\text{(4.1.13)} \qquad \mathrm{I}^{\mathrm{P}}\{\mathcal{H}^\bullet(w)\} := \mathrm{Ind}_{[\mathrm{G}_h(\mathbf{A}^f)\cdot \mathrm{G}_l(\mathbf{A}^f)]\rtimes\Delta_0}^{\mathrm{L_F}(\mathbf{A}^f)}\{\mathcal{H}^\bullet(w)^{\Delta_1}\}.$$

This is easy to see on the level of Lie algebra cohomology [*cf.* (4.2.9), (4.2.10), below]. A direct topological proof is sketched in 4.8, below. Similarly, we can compute the cohomology of the F-stratum. Let $\overline{\mathrm{Sh}}_\Sigma^{\mathrm{F}}$ be the closure of the F-stratum of Sh_Σ (*cf.* 1.7.9). Then

4.1.14. COROLLARY. – $\varinjlim_{\mathrm{K},\Sigma} \mathrm{H}^i(\overline{\mathrm{Sh}}_\Sigma^{\mathrm{F}}, \mathcal{V}_\Sigma^{\mathrm{F}})$ *is* $\mathrm{G}(\mathbf{A}^f)$*-equivariantly isomorphic to*

$$\mathrm{Ind}_{\mathrm{P}(\mathbf{A}^f)}^{\mathrm{G}(\mathbf{A}^f)}[\bigoplus_r \bigoplus_{w\in \mathrm{W}^{\mathrm{F},p}} \mathrm{I}^{\mathrm{P}}\{\tilde{\mathrm{H}}^{i-r-l(w)}((\mathcal{V}_{\lambda(h,w)})^{\mathrm{can}})\otimes \mathrm{H}^r(\mathrm{X}(\mathrm{G}_l), \tilde{\mathbf{V}}_{\lambda(l,w)})\}].$$

The $\mathrm{G}(\mathbf{A}^f)$-equivariance of the isomorphism follows by comparing the descriptions of the F- and P-strata, and using the description ([H5], 2.5.6) of the $\mathrm{G}(\mathbf{A}^f)$-action on the cohomology.

Finally, we have the analogous fact for the cohomology of $\overline{\mathrm{Sh}}_\Sigma^{\mathrm{F}}$ relative to its boundary. Let $\partial = \bigcup_{\mathrm{F}'\neq \mathrm{F}} (\overline{\mathrm{Sh}}_\Sigma^{\mathrm{F}} \cap \overline{\mathrm{Sh}}_\Sigma^{\mathrm{F}'}) \subset \overline{\mathrm{Sh}}_\Sigma^{\mathrm{F}}$, and define $\mathcal{V}_\Sigma^{\mathrm{F}}(-\partial)$ as in 3.14. Then the adelic version of Proposition 3.14.2 is

4.1.15. COROLLARY. – $\varinjlim_{\mathrm{K},\Sigma} \mathrm{H}^i(\overline{\mathrm{Sh}}_\Sigma^{\mathrm{F}}, \mathcal{V}_\Sigma^{\mathrm{F}}(-\partial))$ *is* $\mathrm{G}(\mathbf{A}^f)$*-equivariantly isomorphic to*

$$\mathrm{Ind}_{\mathrm{P}(\mathbf{A}^f)}^{\mathrm{G}(\mathbf{A}^f)}[\bigoplus_r \bigoplus_{w\in \mathrm{W}^{\mathrm{F},p}} \mathrm{I}^{\mathrm{P}}\{\tilde{\mathrm{H}}^{i-r-l(w)}((\mathcal{V}_{\lambda(h,w)})^{\mathrm{sub}})\otimes \mathrm{H}_c^r(\mathrm{X}(\mathrm{G}_l), \tilde{\mathbf{V}}_{\lambda(l,w)})\}].$$

4.2. RELATION WITH ADELIC AUTOMORPHISM FORMS.

We also need an adelic version of our main theorem 3.10.3/3.12.7. Fix a point $p \in \mathrm{D}$, and define K_p and $\mathfrak{P}_p = \mathrm{Lie}(\mathcal{P}_p)$ as in 1.8. Let $\hat{\mathrm{K}}_h = \mathrm{K}_h \cdot \mathrm{A}(\mathbb{R})$, $\hat{\mathrm{K}}_l = \mathrm{K}_l \cdot \mathrm{A}(\mathbb{R})$. As in 3.10, there is an isomorphism ([H5], Cor. 3.4)

$$\text{(4.2.1)} \qquad \eta \to [\eta]: \quad \mathrm{H}^\bullet(\mathfrak{P}_p, \mathrm{K}_p; \mathrm{C}^\infty(\mathrm{G}(\mathbb{Q})\backslash \mathrm{G}(\mathbf{A}))_{\mathrm{sia}} \otimes \mathrm{V}_\lambda)) \xrightarrow{\sim} \tilde{\mathrm{H}}^\bullet(\mathcal{V}^{\mathrm{can}}),$$

with $\mathcal{V} = \mathcal{V}_\lambda$, where C^∞ means locally constant with respect to $\mathrm{G}(\mathbf{A}^f)$ and smooth with respect to $\mathrm{G}(\mathbb{R})$. On the other hand, for each $w \in \mathrm{W}^{\mathrm{F},p}$, there are isomorphisms

(4.2.2) $\beta \to [\beta]$:

$$\mathrm{H}^\bullet(\mathfrak{P}_h, \hat{\mathrm{K}}_h, \mathrm{C}^\infty(\mathrm{G}_h(\mathbb{Q})\backslash \mathrm{G}_h(\mathbf{A}))_{\mathrm{sia}} \otimes \mathrm{V}_{\lambda(h,w)})) \xrightarrow{\sim} \tilde{\mathrm{H}}^\bullet((\mathcal{V}_{\lambda(h,w)})^{\mathrm{can}})$$

(4.2.3) $\gamma \to [\gamma]$:

$$\mathrm{H}^\bullet(\mathfrak{g}_l, \hat{\mathrm{K}}_l, \mathrm{C}^\infty(\mathrm{G}_l(\mathbb{Q})\backslash \mathrm{G}_l(\mathbf{A}))_{\mathrm{sia}} \otimes \mathrm{V}_{\lambda(h,w)})) \xrightarrow{\sim} \mathrm{H}^\bullet(\mathrm{X}(\mathrm{G}_l), \tilde{\mathbf{V}}_{\lambda(l,w)});$$

the isomorphism (4.2.3) is due to Borel [B2], whereas (4.2.2) is a special case of (4.2.1).

Denote the left-hand side of (4.2.1) by $\mathcal{H}^*_{\bar{\partial}}(\mathrm{G}, \mathrm{K}_p; \mathrm{V}_\lambda)_{\mathrm{sia}}$, and the left-hand side of (4.2.3) by $\mathcal{H}^*_d(\mathrm{G}_l, \hat{\mathrm{K}}_l; \mathrm{V}_{\lambda(\lambda, w)})_{\mathrm{sia}}$; the left-hand side of (4.2.2) is denoted $\mathcal{H}^*_{\bar{\partial}}(\mathrm{G}_h, \hat{\mathrm{K}}_h; \mathrm{V}_{\lambda(h, w)})_{\mathrm{sia}}$, in conformity with the notation for (4.2.1). We write $\mathrm{I}^{\mathrm{G}}_{\mathrm{P}}[\cdot]$ for $\mathrm{Ind}^{\mathrm{G}(\mathbf{A}^f)}_{\mathrm{P}(\mathbf{A}^f)}[\cdot]$, and recall the definition of $\mathrm{I}^{\mathrm{P}}\{\cdot\}$ (4.1.13). Then the proof of Corollary 3.13.6 provides the existence of a commutative diagram

(4.2.4)

$$\begin{array}{ccc}
\mathcal{H}^\bullet_{\bar{\partial}}(\mathrm{G}, \mathrm{K}_p; \mathrm{V}_\lambda)_{\mathrm{sia}} & \to & \mathrm{I}^{\mathrm{G}}_{\mathrm{P}}[\bigoplus_w \mathrm{I}^{\mathrm{P}}\{\mathcal{H}^{\bullet - l(w)}_{\bar{\partial}}(\mathrm{G}_h, \hat{\mathrm{K}}_h; \mathrm{V}_{\lambda(h, w)})_{\mathrm{sia}} \otimes \mathcal{H}^\bullet_d(\mathrm{G}_l, \hat{\mathrm{K}}_l; \mathrm{V}_{\lambda(\lambda, w)})_{sia}\}] \\
\downarrow \eta \to [\eta] & & \downarrow \oplus\beta\otimes\gamma \to \oplus[\beta]\otimes[\gamma] \\
\tilde{\mathrm{H}}^\bullet(\mathcal{V}^{\mathrm{can}}) & \xrightarrow{r_{\mathrm{F}}} & \mathrm{I}^{\mathrm{G}}_{\mathrm{P}}[\bigoplus_w \mathrm{I}^{\mathrm{P}}\{\tilde{\mathrm{H}}^{\bullet - l(w)}((\mathcal{V}_{\lambda(h, w)})^{\mathrm{can}}) \otimes \mathrm{H}^\bullet(\mathrm{X}(\mathrm{G}_l), \tilde{\mathbf{V}}_{\lambda(l, w)})\}]
\end{array}$$

The first line needs to be clarified. Starting with an element

$$\begin{aligned}\eta &\in \mathrm{C}^\bullet(\mathfrak{P}_p, \mathrm{K}_p; \mathrm{C}^\infty(\mathrm{G}(\mathbb{Q})\backslash\mathrm{G}(\mathbf{A}))_{\mathrm{sia}} \otimes \mathrm{V}_\lambda)) \\ &= [\mathrm{C}^\infty(\mathrm{G}(\mathbb{Q})\backslash\mathrm{G}(\mathbf{A}))_{\mathrm{sia}} \otimes \Lambda^\bullet(\mathfrak{p}^-) \otimes \mathrm{V}_\lambda]^{\mathrm{K}_p},\end{aligned}$$

one first defines its constant term

$$\eta_{\mathrm{F}} = \int_{\mathrm{W}(\mathbb{Q})\backslash\mathrm{W}(\mathbf{A})} \eta \in [\mathrm{C}^\infty(\mathrm{P}(\mathbb{Q}) \cdot \mathrm{W}(\mathbf{A})\backslash\mathrm{G}(\mathbf{A}))_{\mathrm{sia}} \otimes \Lambda^\bullet(\mathfrak{p}^-)^* \otimes \mathrm{V}_\lambda]^{\mathrm{K}_p} \quad (4.2.5)$$

where the measure, which is applied to the coefficient functions of η, is normalized to have total volume 1. One then restricts η_{F} as in (3.10.3.2) to an element of

$$\mathrm{Res}(\eta_{\mathrm{F}}) \in [\mathrm{C}^\infty(\mathrm{P}(\mathbb{Q}) \cdot \mathrm{W}(\mathbf{A})\backslash\mathrm{P}(\mathbb{R}) \times \mathrm{G}(\mathbf{A}^f))_{\mathrm{sia}} \otimes \Lambda^\bullet(\mathfrak{p}^-)^* \otimes \mathrm{V}_\lambda]^{\mathrm{K}_p \cap \mathrm{P}(\mathbb{R})}. \quad (4.2.6)$$

Applying the Cayley transform, as in (3.10.3.3), we rewrite $\mathrm{Res}(\eta_{\mathrm{F}})$ as an element of the complex

$$[\mathrm{C}^\infty(\mathrm{L}_{\mathrm{F}}(\mathbb{Q}) \cdot \mathrm{W}(\mathbf{A})\backslash\mathrm{L}_{\mathrm{F}}(\mathbb{R}) \times \mathrm{G}(\mathbf{A}^f))_{\mathrm{sia}} \otimes \Lambda^\bullet(\mathfrak{p}^-_h \oplus \mathfrak{v}^- \oplus \mathfrak{u}_{\mathbb{C}})^* \otimes \mathrm{V}_\lambda]^{\mathrm{K}_p \cap \mathrm{L}_{\mathrm{F}}(\mathbb{R})}.$$

which is quasi-isomorphic to the double complex

$$\begin{aligned}&[\mathrm{C}^\infty(\mathrm{L}_{\mathrm{F}}(\mathbb{Q}) \cdot \mathrm{W}(\mathbf{A})\backslash\mathrm{L}_{\mathrm{F}}(\mathbb{R}) \\ &\times \mathrm{G}(\mathbf{A}^f))_{\mathrm{sia}} \otimes \Lambda^\bullet(\mathfrak{p}^-_h \oplus \mathfrak{u}_{\mathbb{C}})^* \otimes (\Lambda^\bullet(\mathfrak{v}^-)^* \otimes \mathrm{V}_\lambda)]^{\mathrm{K}_h \cdot \mathrm{K}_l}.\end{aligned} \quad (4.2.7)$$

As in [3.10.3.6 (i)], we let p_l be a point in C_{F} fixed by K_l, $\tilde{\mathfrak{p}}_l = \mathfrak{p}_p \cap \mathfrak{g}_l$, [notation (1.2.2)] and identity $\mathfrak{u}_{\mathbb{C}} \cong \mathrm{T}_{\mathrm{C}_{\mathrm{F}}, p_l} \cong \tilde{\mathfrak{p}}_{l, \mathbb{C}}$. Recall that A is not contained in K_l.

Since the $\mathrm{G}(\mathbf{A}^f)$-action commutes with the differentials in the complex, this last complex is canonically quasi-isomorphic to the complex

$$\mathrm{I}^{\mathrm{G}}_{\mathrm{P}}[[\mathrm{C}^\infty(\mathrm{L}_{\mathrm{F}}(\mathbb{Q})\backslash\mathrm{L}_{\mathrm{F}}(\mathbf{A}))_{\mathrm{sia}} \otimes \Lambda^\bullet(\mathfrak{p}^-_h \oplus \mathfrak{p}_l)^* \otimes \mathrm{H}^\bullet(\mathfrak{v}^-, \mathrm{V}_\lambda)]^{\mathrm{K}_h \cdot \mathrm{K}_l}]. \quad (4.2.8)$$

We write $\tilde{\mathfrak{p}}_l = \mathfrak{a} \oplus \mathfrak{p}_l$, with $\tilde{\mathfrak{a}} = \mathrm{Lie}(A) \cap \tilde{\mathfrak{p}}_l = \mathrm{Lie}(\tilde{A})$ [notation (1.2.2)] and $\mathfrak{p}_l = \tilde{\mathfrak{p}}_l \cap \mathfrak{g}_l^{\mathrm{der}}$. Then the complex in (4.2.8) is canonically quasi-isomorphic to

$$\text{(4.2.9)}\quad I_P^G [[\Lambda^\bullet (\mathfrak{p}_h^- \oplus \mathfrak{p}_l)^* \otimes H^\bullet (\tilde{\mathfrak{a}}, C^\infty (L_F(\mathbb{Q})\backslash L_F(\mathbf{A}))_{\mathrm{sia}} \otimes H^\bullet (\mathfrak{v}^-, V_\lambda))]^{K_h \cdot K_l}]$$
$$\cong I_P^G [[\Lambda^\bullet (\mathfrak{p}_h^- \oplus \mathfrak{p}_l)^* \otimes C^\infty (L_F(\mathbb{Q})\backslash L_F(\mathbf{A}))_{\mathrm{sia}} \otimes H^\bullet (\mathfrak{v}^-, V_\lambda)]^{K_h \cdot K_l \cdot A(\mathbb{R})}]$$

$$\text{(4.2.10)}\qquad \cong I_P^G [\bigoplus_w I^P \{C^\bullet_{\bar\partial} (G_h, \hat{K}_h; V_{\lambda(h,w)})_{\mathrm{sia}} \otimes C^\bullet_d (G_l, \hat{K}_l; V_{\lambda(\lambda,w)})_{sia}\}],$$

by Kostant's theorem, since $H^i(\tilde{\mathfrak{a}}, C^\infty (L_F(\mathbb{Q})\backslash L_F(\mathbf{A}))_{\mathrm{sia}} \otimes \cdot)$ is trivial for $i>0$. Here we have written

$$C^\bullet_{\bar\partial} (G_h, \hat{K}_h; V_{\lambda(h,w)})_{\mathrm{sia}} = C^\bullet (\mathfrak{P}_h, \hat{K}_h, C^\infty (G_h(\mathbb{Q})\backslash G_h(\mathbf{A}))_{\mathrm{sia}} \otimes V_{\lambda(h,w)})),$$
$$C^\bullet_d (G_l, \hat{K}_l; V_{\lambda(\lambda,w)})_{\mathrm{sia}} = C^\bullet (\mathfrak{g}_l, \hat{K}_l, C^\infty (G_l(\mathbb{Q})\backslash G_l(\mathbf{A}))_{\mathrm{sia}} \otimes V_{\lambda(h,w)})).$$

4.3. Automorphic vector bundles and periods of CM motives. – It follows from Theorem 3.2.4 that the cohomology restriction map r_F is defined over the field of definition of $[\mathcal{V}_\lambda]$, and the computation of the target cohomology groups in terms of automorphic vector bundles on $\mathrm{Sh}(G_h, X(F))$ is also purely algebraic. On the other hand, the latter cohomology groups likewise have canonical rational structures, and it is necessary to verify that the restriction maps on the bottom line of (4.2.4) are compatible with the canonical models on both sides. (Here we need to observe that the action of Δ_0 on the cohomology, defined by (4.1.11), preserves the rational structure (*cf.* Remark 4.8.4, below). Indeed, our automorphic vector bundles are given with canonical families of trivializations over CM points (in terms of the canonical local systems of [H2], or the period torsor, as in [Mi]), and it has to be shown that the commutative diagram (4.2.4) respects these trivializations. To this end, we need to recall how Theorem 3.1.3 is proved. We use a language halfway between those of [H2] and [Mi].

Let (ρ, V) be a faithful $\mathbb{Q}$-rational representation of G. Then $\tilde{V} = \check{M} \times V$ is naturally a homogeneous vector bundle over $\check{M}$, hence defines (by Theorem 3.1.3) an $E(G, X)$-rational automorphic vector bundle $[\tilde{V}]$ over $\mathrm{Sh}(G, X)$. The bundle $[\tilde{V}]$ is endowed with the following additional structure:

(4.3.1) (i) A $G(\mathbf{A}^f)$-invariant flat connection ∇.

(ii) A $G(\mathbf{A}^f)$-invariant filtration (Hodge filtration) by G-homogeneous vector subbundles, corresponding to the filtration on $\tilde{V}$ which, at the point $p \in \check{M}$, is the natural filtration defined by the restriction of ρ to the maximal parabolic $\mathcal{P}_p$ (notation 3.1).

(iii) For every G-invariant tensor $\gamma \in V^{\otimes m} \otimes (V^*)^{\otimes n}$ and corresponding line sub-bundle $\mathcal{L}(\gamma) \subset [\tilde{V}]^{\otimes m} \otimes [\tilde{V}^*]^{\otimes n}$, a $G(\mathbf{A}^f)$-equivariant isomorphism $i_\gamma : \mathcal{O}_{\mathrm{Sh}(G,X)} \xrightarrow{\sim} \mathcal{L}(\gamma)$.

The verification of this fact, which is a simple consequence of Theorem 3.1.3, is actually the main step in the latter's proof. Using this structure, we construct a principal G-bundle $I(G, X)$ over $\mathrm{Sh}(G, X)$, rational over $E(G, X)$: over any open subset $\mathcal{U} \subset \mathrm{Sh}(G, X)$,

$$\text{(4.3.2)}\qquad I(G, X)(\mathcal{U}) = \{f \in \mathrm{Isom}_{\mathcal{O}_{\mathcal{U}}} (V \otimes \mathcal{O}_{\mathcal{U}}, [\tilde{V}]_{\mathcal{U}}) \mid f(\gamma) = i_\gamma(1)\}.$$

This can be defined without reference to a particular V, by using tensor categories, and one then sees that I(G, X) is independent of the choice of V; however, this concrete realization will be useful.

The action of $G(\mathbf{A}^f)$ on $[\tilde{V}]$ defines an action on I(G, X), and the canonical morphism d: I(G, X) → Sh(G, X) is $G(\mathbf{A}^f)$-equivariant. On the other hand, at any geometric point f of I(G, X), with $d(f)=x$, the Hodge filtration on the fiber $[\tilde{V}]_x$ induces a filtration on V via f^{-1}, and one verifies that the stabilizer of this filtration is conjugate to $\mathcal{P}_p$ for $p \in X$, hence defines a point $\pi(f) \in \check{M} = \check{M}(G, X)$ (notation 3.1). It is easy to show that $\pi : I(G, X) \to \check{M}$ is also E(G, X) rational, and equivariant with respect to $G \times G(\mathbf{A}^f)$.

Now the functor of Theorem 3.1.3 is easily made explicit. Given $\mathcal{V}$ as in the statement of the theorem, $\pi^*(\mathcal{V})$ is a $G \times G(\mathbf{A}^f)$-homogeneous vector bundle on I(G, X), hence descends to a bundle

(4.3.3) $$[\mathcal{V}] = \pi^*(\mathcal{V})/G$$

on Sh(G, X). On the other hand, the set of complex points of I(G, X) may be identified ([H2], (3.4.2.4)) with

(4.3.4) $$\overset{\lim}{\leftarrow} G(\mathbb{Q})\backslash G(\mathbb{C}) \times X \times G(\mathbf{A}^f)/K,$$

such that the map π takes $(g, p, g') \in G(\mathbb{C}) \times X \times G(\mathbf{A}^f)$ to $g^{-1} \cdot \beta(p)$. Denote the class of the image in $\mathrm{Sh}(G, X)(\mathbb{C})$ (resp. $I(G, X)(\mathbb{C})$) of $(p, g') \in X \times G(\mathbf{A}^f)$ (resp. $(g, p, g') \in G(\mathbb{C}) \times X \times G(\mathbf{A}^f)$) by $[p, g']$ (resp. $[g, p, g']$. Now over the point $[p, g'] \in \mathrm{Sh}(G, X)(\mathbb{C})$, for any fixed g', there is a canonical lifting

$$f(p, g') = [1, p, g'] \in I(G, X)(\mathbb{C}),$$

whose image under π is $\beta(p)$, corresponding to the choice of local flat framings of $\mathcal{V}$. This is well-defined in the inverse limit, because hypotheses (1.1.3-4) imply that $\overset{\lim}{\leftarrow}(G(\mathbb{Q}) \cap K) = \{1\}$ ([Mi], p. 324); otherwise the point would only be well-defined in the quotient of I(G, X) by the Zariski-closure of this inverse limit in $G(\mathbb{C})$. There is a canonical composite "periods" isomorphism

(4.3.5) $$\mathrm{Per}_{(p, g')} : \quad \mathcal{V}_{\beta(p)} \xrightarrow{\sim} \pi^* \mathcal{V}_{f(p, g')} \xrightarrow{\sim} [\mathcal{V}]_{[p, g']}.$$

If $\gamma \in G(\mathbb{Q})$ then $[\gamma p, \gamma g'] = [p, g']$, but $f(p, g') \neq f(\gamma p, \gamma g')$, so $\mathrm{Per}_{(\gamma p, \gamma g')}$ and $\mathrm{Per}_{(p, g')}$ are not identical. For example, if $\mathcal{V} = \tilde{V}$, with (ρ, V) as above, then $\mathrm{Per}_{(\gamma p, \gamma g')}$ and $\mathrm{Per}_{(p, g')}$ differ by translation by $\rho(\gamma)$.

Suppose $(H, h) \subset (G, X)$ is a CM pair, with H a torus. Then $\check{M}(H, h)$ is the $E(H, h)$-rational point $\beta(h) \in \check{M}$. The fiber $\mathcal{V}_{\beta(h)}$ is the space of a representation $\chi = \chi_{\mathcal{V}, h}$ of H. Let $E(\mathcal{V}, h)$ denote the field of definition of χ; and let $\mathcal{V}_{\beta(h)}(E(\mathcal{V}, h))$ denote the space of $E(\mathcal{V}, h)$-rational points. Then (4.3.5) defines an $H(\mathbf{A}^f)$-equivariant local system of $E(\mathcal{V}, h)$-vector spaces:

$$H_B(\mathfrak{M}(\chi)/\mathrm{Sh}(H, h))_{[h, h']} := \mathrm{Per}_{(h, h')}(\mathcal{V}_{\beta(h)}(E(\mathcal{V}, h))) \subset [\mathcal{V}]_{[h, h']}(\mathbb{C}).$$

(Here $\mathfrak{M}$ stands for "motive", for reasons explained in [H2], § 3.15.) Then there is an element $p(h, \chi) \in \mathrm{Aut}([\mathcal{V}]_{|\mathrm{Sh}(\mathrm{H}, h)}(\mathbb{C}))$, which depends only on (H, h) and χ, and is canonical $\bmod \chi(\mathrm{H}(\mathbb{Q}))$, such that

$$\text{(4.3.6)} \qquad \mathrm{H}_{\mathrm{DR}}(\mathfrak{M}(\chi)/\mathrm{Sh}(\mathrm{H}, h))_{[h, h']} := p(h, \chi) \cdot \mathrm{H}_{\mathrm{B}}(\mathfrak{M}(\chi)/\mathrm{Sh}(\mathrm{H}, h))_{[h, h']}$$

is the canonical $\mathrm{E}(\mathcal{V}, h)$-rational structure on $[\mathcal{V}]_{|\mathrm{Sh}(\mathrm{H}, h)}$ [H2], 3.15; [Mi], p. 314. As (H, h) varies, the intersection of the $\mathrm{E}(\mathcal{V}, h)$ is just the field of definition $\mathrm{E}(\mathcal{V})$ of $\mathcal{V}$, and the union of the $\mathrm{G}(\mathbf{A}^f)$-translates of Sh(H, h) in Sh(G, X) is dense [De3]. In particular, the space of rational (meromorphic) sections of $[\mathcal{V}]$ over any field containing $\mathrm{E}(\mathcal{V})$ is completely determined by (4.3.3) and (4.3.4). We return to this point when we discuss canonical trivializations below.

4.4. Limit Hodge theory for flat automorphic vector bundles. – Define (ρ, V) and $[\tilde{\mathrm{V}}]$ as in 4.3. Then $[\tilde{\mathrm{V}}]$ extends canonically over Sh_Σ to a vector bundle $[\tilde{\mathrm{V}}]_\Sigma$ in such a way that the flat connection ∇ extends to a connection with regular singularities [De1]:

$$\text{(4.4.1)} \qquad \nabla_\Sigma : \quad [\tilde{\mathrm{V}}]_\Sigma \to [\tilde{\mathrm{V}}]_\Sigma \otimes_{\mathcal{O}} \Omega^1(\log(\partial \mathrm{Sh}_\Sigma)) \quad (\mathcal{O} = \mathcal{O}_{\mathrm{Sh}_\Sigma}, \text{ etc.})$$

This extension can be shown to coincide with the one entering in 3.8.2 [H3], (4.2.2); together with ∇_Σ, it is defined over E(G, X) (since $[\tilde{\mathrm{V}}]$ is). We may define $\mathrm{I}_\Sigma = {}_{\mathrm{K}}\mathrm{I}(\mathrm{G}, \mathrm{X})_\Sigma$ analogously to (4.3.2), relative to $[\tilde{\mathrm{V}}]_\Sigma$. For that, one needs to check that for all γ, $\mathcal{L}(\gamma)$ and i_γ: $\mathcal{O}_{\mathrm{Sh}} \tilde{\to} \mathcal{L}(\gamma)$ extend over Sh_Σ; but this follows from the functoriality of the canonical extension. Thus, we have in I_Σ an E(G, X)-rational principal G-bundle over the entire space.

4.4.2. Lemma. – *Let $\mathcal{V}$ be any* G*-homogeneous vector bundle over* $\check{\mathrm{M}}$. *Then:*

(i) *the mapping* $\pi : \mathrm{I}(\mathrm{G}, \mathrm{X}) \to \check{\mathrm{M}}$ *extends to a* G*-equivariant mapping*

$$\pi_\Sigma : \ \mathrm{I}_\Sigma \to \check{\mathrm{M}}.$$

(ii) *there is a canonical isomorphism*

$$[\mathcal{V}]_\Sigma \tilde{\to} [\mathcal{V}]_\Sigma^{\natural} := \pi_\Sigma^*(\mathcal{V})/\mathrm{G}$$

over Sh_Σ, *rational over the field of definition of* $\mathcal{V}$, *whose restriction to* Sh *is the isomorphism defined above.*

Proof.

(i) The Hodge filtration of $[\tilde{\mathrm{V}}]$ extends to $[\tilde{\mathrm{V}}]_\Sigma$, as the former is given by G-homogeneous bundles [*see* (4.3.1), (ii)], and one just takes their canonical extensions. One then defines π_Σ by the same prescription used to define π.

(ii) When $\mathcal{V}$ is of the form $\tilde{\mathrm{V}}$, this is a tautology. Since V is a faithful representation and the functor $\mathcal{V} \mapsto [\mathcal{V}]_\Sigma^{\natural}$ commutes with tensor products, it follows that the theorem is true for flat vector bundles. From here the proof is a paraphrase of the argument used to prove [H3], Theorem 4.2, (iii): one represents the general $\mathcal{V}$ as a subquotient of some flat $\tilde{\mathrm{W}}$ and then the assertion for $\mathcal{V}$ follows from that for $\tilde{\mathrm{W}}$.

It is useful to put (4.4.2), (i) and other particulars in the context of general asymptotic Hodge theory. First, we de-adelize (4.3.4) to give

$$(4.4.3)\qquad d:\ \mathrm{I}(\mathrm{G},\mathrm{X})_\Gamma = \Gamma\backslash(\mathrm{G}(\mathbb{C})\times \mathrm{X}) \to \Gamma\backslash \mathrm{X}, \qquad \pi:\ \mathrm{I}(\mathrm{G},\mathrm{X})_\Gamma \to \check{\mathrm{M}}.$$

Recall that the variation of Hodge structure of $\mathcal{V}_\Gamma$ is induced from the "tautological" one on $\mathrm{X}\times\mathrm{V}$, where the Hodge filtrations (4.3.1), (ii) satisfy

$$\mathrm{F}^\bullet_{p'} = \rho(g)\,\mathrm{F}^\bullet_p,$$

whenever $g\cdot p = p'$, corresponding to $(1, p')$ in (4.4.3).

We now use the Siegel domain coordinates associated to the boundary component F, coming from the group $\mathrm{H} = \mathrm{P}'(\mathbb{R})\cdot \mathrm{U}(\mathbb{C})$. The projection

$$\mathrm{H} \to \mathrm{P}'(\mathbb{R})/\mathrm{U}(\mathbb{R})$$

gives rise to the torus fibration (1.2.5), and by using local cross-sections, one sees that the essential issue is to treat a single fiber.

As in 2.3, let σ be a top-dimensional cone in the fan Σ^c_F. Let $\{e_1,\ldots,e_n\}$ be the generators of $\sigma\cap\Gamma_\mathrm{U}$, and let $\{q_1,\ldots,q_n\}$ be the dual basis. At first, we identify $\mathfrak{u}_\mathbb{C}$ and $\mathrm{U}(\mathbb{C})$, as is customary, and use only additive notation. The functions

$$\zeta_j(u) = \langle q_j, u\rangle \quad (u\in\mathfrak{u}_\mathbb{C})$$

define coordinates on $\mathfrak{u}_\mathbb{C} \cong \mathrm{U}(\mathbb{C})$, and

$$(4.4.4)\qquad t_j(u) = e^{2\pi i \zeta_j(u)} \quad 1\leqq j\leqq n$$

induces an isomorphism $\mathrm{T}_\mathrm{F} \cong (\mathbb{C}^*)^n$ [*cf.* (2.3.5)]. One notes that

$$u = u(\zeta) = \sum_{j=1}^n \zeta_j\, e_j;$$

If we now take care to make explicit the identification $\exp:\mathfrak{u}_\mathbb{C}\to\mathrm{U}(\mathbb{C})$ and let $\mathrm{N}_j = \log e_j \in \mathfrak{u}$, then we have

$$(4.4.5)\qquad u(\zeta) = \exp\left(\sum_{j=1}^n \zeta_j\,\mathrm{N}_j\right).$$

Of course, $\sigma_j = \langle e_j\rangle$ is an edge of σ. The part of the corresponding boundary divisor, which we denote here Z_j, that comes from σ is the locus of points where $t_j=0$ in (4.4.4); it is achieved in the limit as $\mathrm{Im}\,\zeta_j\to\infty$. It follows that $d\rho(\mathrm{N}_j)$ is the nilpotent logarithm of the local monodromy transformation around Z_j; for notational convenience, we use N_j to also denote the monodromy logarithm, here or in any variation of Hodge structure.

In general, frames for the Deligne canonical extension on a product of punctured discs, when the monodromy is unipotent, are given by the translates of flat frames by (4.4.5), and the extension of the Hodge filtration follows from the deep Nilpotent Orbit Theorem of Schmid [Sc], (4.12). In the case of a flat G-homogeneous bundle on a fiber of π_2 over A_F, one is taking the equivalence class of the point $(u(\zeta), u(\zeta)x_0)$ in (4.4.3), for some x_0, and thus *the image under* π *is independent of* ζ (for $\mathrm{Im}(\zeta_j)$ sufficiently large), thus giving the limit. (This was explained in another way for (4.4.2), (i).) Note that the limit depends on the choice of x_0 in its $\mathrm{U}(\mathbb{C})$-orbit (this recovers the usual ambiguity in the definition of the "limit Hodge filtration").

Next, a nilpotent endomorphism N of a finite-dimensional vector space V admits a *weight filtration* $\mathrm{W}(\mathrm{N})$, characterized by:

(4.4.6) (i) $\mathrm{N}\mathrm{W}(\mathrm{N})_k \subset \mathrm{W}(\mathrm{N})_{k-2}$,
(ii) N^k induces an isomorphism $\mathrm{Gr}_k^{\mathrm{W}(\mathrm{N})}\mathrm{V} \cong \mathrm{Gr}_{-k}^{\mathrm{W}(\mathrm{N})}\mathrm{V}$.

There is also the more elementary *kernel filtration* $\mathrm{K}(\mathrm{N})$, with

$$\mathrm{K}(\mathrm{N})_l\,\mathrm{V} = \ker \mathrm{N}^{l+1}; \tag{4.4.7}$$

whenever V is d-dimensional and indecomposable under N,

$$\mathrm{K}(\mathrm{N})_l\,\mathrm{V} = \mathrm{W}(\mathrm{N})_{2l-d}\,\mathrm{V}.$$

One can define, likewise, the *image filtration*, by

$$\mathrm{I}_{-m}\,\mathrm{V} = \mathrm{N}^m\,\mathrm{V}, \tag{4.4.8}$$

and $\mathrm{W}(\mathrm{N})$ admits the simple algebraic description as the convolution of K and I (*see* [SZ], (2.3)). If a "weight" $w \in \mathbb{Z}$ is attached to V, one recenters $\mathrm{W}(\mathrm{N})$ by putting $\mathrm{W}(\mathrm{N}, w) = \mathrm{W}(\mathrm{N})[-w]$ (the shift of $\mathrm{W}(\mathrm{N})$ by w).

Recall that the Cayley morphism w_{F} determines m_{F} [*see* (1.2.2.1)], which (*see* [D2]) is the restriction of a group homomorphism h_s (defined over $\mathbb{Q}$), given by the composite

$$\mathrm{SL}(2) \xrightarrow{\delta_s} \mathrm{SL}(2)^r \rightarrow \mathrm{G} \tag{4.4.9}$$

for some $s \leqq r$, where δ_s is the partial diagonal whose components are the identity mapping for the first s factors and are trivial for the rest. Let

$$\mathrm{N}_0 = dh_s\left(\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\right) \in \mathrm{C}_{\mathrm{F}}. \tag{4.4.10}$$

Then the weight filtration of N_0 is split by the weight spaces for w_{F}. Since G_l centralizes w_{F}, and acts transitively on C_{F}, the same holds for any $\mathrm{N} \in \mathrm{C}_{\mathrm{F}}$. Thus,

4.4.11. PROPOSITION. – *Let* V *be of weight* w. *For all* $\mathrm{N} \in \mathrm{C}_{\mathrm{F}}$, *the weight filtration* $\mathrm{W}(\mathrm{N}, w)$ *is the same,* viz. W^{F}.

According to [Sc], § 6, for a local monodromy logarithm the filtrations $\mathrm{W}(\mathrm{N})$ and the limit (in the above sense) of the Hodge filtration define a mixed Hodge structure on V; moreover, if the type of Hodge structure involved has a locally-symmetric classifying space, one can replace the limit Hodge filtration by that of any sufficiently nearby point. In the homogeneous case, the latter assertion is contained in the axioms that define an admissible Cayley morphism (1.2.2), and the assertion about the limit follows from that, as (4.4.5) acts trivially on $\mathrm{Gr}_\bullet^{\mathrm{W}(\mathrm{N})}\mathrm{V}$.

For general degenerations of Hodge structure, the weight filtrations of the monodromy logarithms N_j can differ, and the relations among them are quite complicated. A theorem of Cattani and Kaplan (*see* [SZ], (3.12)) asserts:

4.4.13. THEOREM. – *For each non-empty subset* J *of* $\{1, \ldots, n\}$, *let*

$$\tau_J = \{N = \sum_{j \in J} \lambda_j N_j : \lambda_j > 0 \textit{ for all } j\}.$$

Then (i) *for any* J, W(N) *is a single filtration for all* $N \in \tau_J$*; denote it by* W^J*;*

(ii) if $N \in \tau_J$, *then for any* J′, $W(N \operatorname{rel} W^{J'}) = W^{J \cup J'}$.

To explain the last assertion, we recall:

4.4.14. DEFINITION [D4], (1.6). – *Let* W′ *be an increasing filtration of the finite-dimensional vector space* V, *and* N *a nilpotent endomorphism of* V *that respects* W′. *The increasing filtration* M *of* V *is the* weight filtration *of* N relative to W′ *if:*

(i) $NM_k \subset M_{k-2}$,

(ii) N^k *induces an isomorphism* $\mathrm{Gr}^M_{l+k}\,\mathrm{Gr}^{W'}_l V \cong \mathrm{Gr}^M_{l-k}\,\mathrm{Gr}^{W'}_l V$ *for all l, i. e.,* M *induces on* $\mathrm{Gr}^{W'}_l V$ *the filtration* $W(\mathrm{Gr}^{W'}_l N, l)$.

4.4.15. *Remark.* – There is at most one filtration M of V satisfying these conditions [D4], (1.6.13). The criteria for its existence are given in [SZ], § 2. One writes $M = W(N \operatorname{rel} W')$.

We consider the conclusions of 4.4.13 in the homogeneous case, returning to the consideration of a top-dimensional cone in Σ_F. In light of 4.4.11, there is nothing serious in 4.4.14 when $\sigma \in \Sigma_F^c$: one simply takes $M = W^F \star W^F$ (convolution of filtrations, as in [SZ], (1.4)). Thus, one should have in mind that σ has at least one edge in $\partial\Sigma_F$. We may assume that the boundary component F is F_s, having Cayley morphism coming from (4.4.9), and that we are considering, for $j \in J$, $\sigma_j \subset C_{F_{s_j}}$. It is clear that $\tau_J \subset C_{F_\mu}$, where $\mu = \max\{s_j : j \in J\}$, and then $W^J = W^{F_\mu}$ in 4.4.13. Note that if $W(i)$ denotes the weight filtration of the i-th factor of SL(2) in (4.4.9), we have

$$W^{F_\mu} = W(1) \star \ldots \star W(\mu) \qquad (4.4.16)$$

(distributive in the sense of [Ka], (1.6.3)), *i. e.* the total weight under (4.4.9) is the sum of those of the individual non-trivial factors.

To make (4.4.13, (ii)) explicit, we may take $J = \{1, 2\}$ and assume that $s_1 < s_2$, $F(1) = F_{s_1}$, $F(2) = F_{s_2}$. Then the non-vacuous statement is that for $W' = W^{F(1)}$, $M = W^{F(2)}$ satisfies [4.4.14, (ii)]. If we use (4.4.16) to write

$$W^{F(2)} = W^{F(1)} \star R,$$

where R is the weight filtration for the complementary factors, it follows formally (*see* [SZ], (1.5)) that

$$\mathrm{Gr}^M_{l+k}\,\mathrm{Gr}^{W'}_l V \cong \mathrm{Gr}^R_k\,\mathrm{Gr}^{W'}_l V,$$

and the rest is easy.

4.5. REDUCTION OF THE STRUCTURE GROUP. – We now want to compare the canonically extended automorphic vector bundles on Sh_Σ to automorphic vector bundles on $\mathrm{Sh}(\mathrm{G}_h, \mathrm{X}(\mathrm{F}))$. We restrict our attention, until further notice, to the open subvariety $\mathrm{Sh}_\Sigma(\mathrm{F})$.

4.5.1. PROPOSITION. – *Over the* F*-stratum* $\mathrm{Sh}_\Sigma^\mathrm{F}$, *the structure group of* I_Σ *can be reduced to* $\mathrm{P}' = \mathrm{P}'_\mathrm{F}$*: there is an* E(G, X)*-rational principal* P′*-bundle* $\mathrm{I}_\Sigma^\mathrm{F}$ *over* $\mathrm{Sh}_\Sigma^\mathrm{F}$ *and an isomorphism* $\mathrm{I}_\Sigma|_{\mathrm{Sh}_\Sigma^\mathrm{F}} \xrightarrow{\sim} \mathrm{I}_\Sigma^\mathrm{F} \times^{\mathrm{P}'} \mathrm{G}$.

Proof. – In this section we reduce the proof of the Lemma to Lemma 4.6.9 of the following section. Parts of the following argument are probably familiar to many people, but it does not seem to be in print. Recall that Σ_F is a fan in $\mathrm{G}(\mathbb{Q})^+ \times^{\mathrm{P}(\mathbb{Q})^+} (\bar{\mathrm{C}}_\mathrm{F} \times \mathrm{G}(\mathbf{A}^f)/\mathrm{K})$. Let σ be a 1-simplex in Σ_F^c. Let $\bar{\mathrm{Z}}_\sigma \subset \mathrm{Sh}_\Sigma^\mathrm{F}$ be the corresponding complete divisor, and let $[\tilde{\mathrm{V}}]_\sigma = [\tilde{\mathrm{V}}]_\Sigma \otimes (\mathcal{O}/\mathcal{I}_\sigma)$ be the restriction of $[\tilde{\mathrm{V}}]_\Sigma$ to $\bar{\mathrm{Z}}_\sigma$ ($\mathcal{O}$ denotes the structure sheaf of $\mathrm{Sh}_\Sigma^\mathrm{F}$). Let $\mathrm{T}(-\log(\bar{\mathrm{Z}}_\sigma))$ be the logarithmic tangent bundle relative to $\bar{\mathrm{Z}}_\sigma$, *i. e.*, the vector fields that preserves the ideal sheaf of $\bar{\mathrm{Z}}_\sigma$; it is the dual of $\Omega^1(\log(\bar{\mathrm{Z}}_\sigma))$. The logarithmic normal bundle $\mathfrak{N}_\sigma$ is the quotient of $\mathrm{T}(-\log(\bar{\mathrm{Z}}_\sigma))|_{\bar{\mathrm{Z}}_\sigma}$ by the tangent bundle to $\bar{\mathrm{Z}}_\sigma$. The regular connection ∇_Σ of (4.3.5) determines a residue map

$$\mathrm{Res}: \quad \mathfrak{N}_\sigma \otimes_{\mathcal{O}_{\bar{\mathrm{Z}}_\sigma}} [\tilde{\mathrm{V}}]_\sigma \to [\tilde{\mathrm{V}}]_\sigma \tag{4.5.2}$$

by the contraction:

$$\mathrm{X} \otimes v \mapsto \langle \mathrm{X}', \nabla_\Sigma(v') \rangle \pmod{\mathcal{I}_\sigma} \tag{4.5.3}$$

where v' (resp. X′) is any extension of v (resp. X) to a section of $[\tilde{\mathrm{V}}]_\Sigma$ (resp. $\mathrm{T}(-\log(\bar{\mathrm{Z}}_\sigma))$) near $\bar{\mathrm{Z}}_\sigma$. Since X′ has a zero along $\bar{\mathrm{Z}}_\sigma$, the map (4.5.2) is $\mathcal{O}_{\bar{\mathrm{Z}}_\sigma}$-linear.

We can see that for any (local) section X of $\mathfrak{N}_\sigma$, the operator $\mathrm{Res}(\mathrm{X} \otimes \bullet)$ is essentially a multiple of N_σ [recall the paragraphs following (4.4.5)]. In fact, we can make the identification natural by describing the Deligne extension and the residue map concretely in a complex analytic neighborhood of $\bar{\mathrm{Z}}_\sigma$ (*cf.* [H3], p. 19). We return temporarily to the notation of paragraph 1: Γ is an arithmetic subgroup of $\mathrm{G}(\mathbb{Q})$, etc. Then $\mathrm{D}_{\mathrm{F},\mathrm{Star}(\sigma)}$ is locally isomorphic to both an open neighborhood of $\bar{\mathrm{Z}}_\sigma$ in $(\mathrm{M}'_\mathrm{F})_{\Sigma_\mathrm{F}}$ and an open neighborhood of $\bar{\mathrm{Z}}_\sigma$ in Sh_Σ. Now the pullback of $[\tilde{\mathrm{V}}]_\Sigma$ to $\mathrm{D}_{\mathrm{F},\mathrm{Star}(\sigma)}$ extends as follows to $(\mathrm{M}'_\mathrm{F})_{\Sigma_\mathrm{F}}$. First, let $[\tilde{\mathrm{V}}]_\mathrm{F}$ be the local system $\Gamma'_\mathrm{F}\backslash(\mathrm{V} \times \mathrm{D}_\mathrm{F})$ over $\mathrm{M}'_\mathrm{F} = \Gamma'_\mathrm{F}\backslash\mathrm{D}_\mathrm{F}$, where the action of Γ'_F on V is by ρ. It follows from reduction theory (cf. 1.4 and 3.3) that $[\tilde{\mathrm{V}}]_\mathrm{F} \cong [\tilde{\mathrm{V}}]$ in a deleted neighborhood of $\bar{\mathrm{Z}}_\sigma$ in $(\mathrm{M}'_\mathrm{F})_{\Sigma_\mathrm{F}}$. If $\mathcal{U} \subset \mathrm{A}_\mathrm{F}$ is an open ball and

$$\mathcal{U}' = \pi_2^{-1}(\mathrm{U}) \cong \mathrm{T}_\mathrm{F}(\mathbb{C}) \times \mathcal{U} \subset \mathrm{M}'_\mathrm{F},$$

the restriction of $[\tilde{\mathrm{V}}]_\mathrm{F}$ to $\mathcal{U}'$ is just isomorphic to $\Lambda(\mathrm{V}) = (\Gamma_\mathrm{U}\backslash\mathrm{V} \times \mathrm{U}(\mathbb{C})) \times \mathcal{U}$, a local system over $\mathrm{T}_\mathrm{F}(\mathbb{C}) \times \mathcal{U}$ (*cf.* Prop. 3.3.6). Then the restriction of $[\tilde{\mathrm{V}}]_\Sigma$ to the closure $\bar{\mathcal{U}}'$ of $\mathcal{U}'$ in $\mathrm{D}_{\mathrm{F},\mathrm{Star}(\sigma)}$ is isomorphic to the Deligne canonical extension of $\Lambda(\mathrm{V})$ to $\bar{\mathcal{U}}'$.

The reduction of stucture group over $\mathbb{C}$ is easy in this model. Let $\mathrm{I}(\mathrm{G}, \mathrm{D}_\mathrm{F})$ be the G-torsor over M_F defined by (4.3.2) relative to $[\tilde{\mathrm{V}}]_\mathrm{F}$. Then locally on $\mathrm{M}'_\mathrm{F}(\mathbb{C})$ (in the analytic topology), $\mathrm{I}(\mathrm{G}, \mathrm{D}_\mathrm{F})$ defines isomorphisms between $[\tilde{\mathrm{V}}']_\mathrm{F}$ and the constant sheaf $V' \times M'_F(\mathbb{C})$, for any representation (ρ', V') of G. We take ρ' to be the adjoint representation on $\mathfrak{g}$, and let

(4.5.4) $I'_F = \{f \in I(G, D_F) | f$ respects W^P and is the standard flat isomorphism on $W^P_{-2} = \mathfrak{u}\} \cong \Gamma'_F \backslash P'(\mathbb{C}) \times D_F$.

This construction is $\Gamma_P = \Gamma \cap P$-equivariant when Γ_P acts by conjugation on $P'(\mathbb{C})$ and through its natural action on D_F. It then follows from (4.3.4) that I'_F does give rise to $I(G, X)(\mathbb{C})$ in a deleted neighborhood of F on $M_{\Gamma, \Sigma}$ by extending the structure group from $P'(\mathbb{C})$ to $G(\mathbb{C})$. The extension of I'_F to a $P'(\mathbb{C})$-torsor on $(M'_F)_{\Sigma_F}$ is carried out as in the discussion preceding Lemma 4.4.2; taking quotients by Γ_P then provides the reduction of structure group analytically on ${}^{<}Z_{F, \Sigma}$ (notation 1.5). Since $P'_F \supset P'_{F'}$ for $F > F'$, this reduction extends to the closure $Z_{F, \Sigma}$ of ${}^0Z_{F, \Sigma}$ in $M_{\Gamma, \Sigma}$, hence is algebraic by GAGA. We let $I^F_{\Sigma, \mathbb{C}}$ be the corresponding P′-torsor over Sh^F_Σ.

An integral generator $\tilde{\sigma}$ of the simplex σ corresponds to a 1-parameter subgroup $\mu_\sigma : \mathbb{G}_m \rightarrow T_F(\mathbb{C})$, given by the formula

$$\mu_\sigma(z) = (2\pi i)^{-1} \log(z) \cdot \tilde{\sigma} \,(\mathrm{mod}\, \Gamma_U) \quad (\text{any branch of log}).$$

Let X_σ denote the invariant vector field on $T_F(\mathbb{C})$ whose value at the identity is $d\mu_\sigma(x \cdot (d/dx))$. Then X_σ defines a local section of $\mathfrak{N}_\sigma$ on $\mathcal{U}'$. It follows from [H3], (4.2.3) that

(4.5.5) $$N_\sigma := \mathrm{Res}(X_\sigma \otimes \cdot)|_{\mathcal{U}'} = (2\pi i)^{-1} d\rho(\tilde{\sigma}).$$

Although the formula (4.5.5) depends on our choice of trivialization of $\mathcal{U}'$, because $\mathfrak{N}_\sigma$ is a line bundle, the 1-dimensional space of operators $\langle N_\sigma \rangle$ is well-defined on $\bar{Z}_\sigma$. This induces a weight filtration $W_\bullet[\tilde{V}]_\sigma$ of $[\tilde{V}]_\sigma$.

We now define a subfunctor I^F_Σ of the functor represented by I_Σ. First, let

(4.5.6) $${}^0I^F_\sigma(\mathcal{U}) = \{f \in I_\Sigma(\mathcal{U}) : f(W_i V \otimes \mathcal{O}_\mathcal{U}) = W_i[\tilde{V}]_\sigma(\mathcal{U})\};$$

this is clearly a principal P-bundle over $\bar{Z}_\sigma$. We next take

(4.5.7) $$I^F_\sigma(\mathcal{U}) = \{f \in {}^0I^F_\sigma(\mathcal{U}) : f \circ d\rho(\gamma\tilde{\sigma}) \in \langle N_{\gamma\sigma} \rangle \text{ for all } \gamma \in \Gamma_l\}.$$

Since Γ_l is Zariski-dense in G_l, which acts transitively on C_F, it follows that I^F_σ is a principal P″-bundle over $\bar{Z}_\sigma$, where P″ is the largest subgroup whose adjoint action on U_F is given by homotheties; P′ is the identity component of P″. Furthermore, for any two simplices σ and σ', the functors (4.5.7) coincide on the intersection $\bar{Z}_\sigma \cap \bar{Z}_{\sigma'}$, so the I^F_σ patch together to define a principal P″-bundle J^F_Σ over Sh^F_Σ. It is rational over E(G, X) because the functor (4.5.7) is defined over E(G, X). Since J^F_Σ is defined as a subfunctor of I_Σ, the isomorphism $I_\Sigma|_{Sh^F_\Sigma} \cong J^F_\Sigma \times^{P''} G$ is automatic. Also, when we replace (ρ, V) by the adjoint representation as above, the condition in (4.5.7) specifies the isomorphism on $\mathfrak{u}$, and this is as required by (4.5.4), giving that over $\mathbb{C}$, J^F_Σ is given by $I^F_{\Sigma, \mathbb{C}}$.

It remains to reduce the structure group from P″ to P′. This has already been accomplished over $\mathbb{C}$. Now $R_u(P') = R_u(P'') = W$. Thus it suffices to show that the quotient

$I^F_{\Sigma,1}$ of I^F_Σ by W is rational over E(G, X), as a subfunctor of the functor represented by J^F_Σ/W. But this is immediate from Lemma 4.6.9, below.

4.5.8. LEMMA.

(i) *There is an* E(G, X)*-rational,* P′*-equivariant morphism* $\pi^F_\Sigma : I^F_\Sigma \to \check{M}$ *such that* $\pi^F_\Sigma \times^{P'} G : I_\Sigma|_{Sh^F_\Sigma} \to \check{M}$ *coincides with the restriction of* π_Σ *to* Sh^F_Σ.

(ii) *The image of* π^F_Σ *equals* $\check{M}(F) = P' \cdot p \subset \check{M}$, *and is thus independent of* Σ.

(iii) *There is a* P′*-equivariant isomorphism*

$$W\backslash\check{M}(F) \cong \check{M}(G_h, X(F)),$$

induced by passing from V *to* $Gr^W_\bullet V$, *where the last space is the compact dual of* F.

Proof. – The first statement is trivial: since I^F_Σ is defined by a subfunctor of I_Σ, $\pi^F_\Sigma : I^F_\Sigma \to \check{M}$ is just the restriction of π_Σ. Let (ρ, V) be as in 4.3, and view $\check{M}$ as a family of filtrations of the category of representations of G. Then the image of π^F_Σ equals the $P'(\mathbb{C})$-invariant set of filtrations of V induced by the Hodge filtration on $[\tilde{V}]_\Sigma$ at the points of Sh^F_Σ. Thus, to prove (ii), it is enough to see that $\beta(p)$ is in the image of π^F_Σ. For this, we simply take $x_0 = p$ in 4.4. Assertion (iii) is clear.

4.6. COMPARISON OF RATIONAL STRUCTURES. – The normal subgroup W acts freely on I^F_Σ; let $I^F_{\Sigma,1}$ (resp. $I^F_{\Sigma,2}$) denote the quotient of I^F_Σ by W (resp. U). Then $I^F_{\Sigma,1}$ is a principal G_h-bundle on Sh^F_Σ, and $I^F_{\Sigma,2}$ is a principal P′/U-bundle, and there are natural morphisms

$$(4.6.1) \qquad \pi^F_{\Sigma,2} : I^F_{\Sigma,2} \to U\backslash\check{M}(F), \qquad \pi^F_{\Sigma,1} : I^F_{\Sigma,1} \to W\backslash\check{M}(F) \cong \check{M}(G_h, X(F))$$

defined by Lemma 4.5.8, which are P′-equivariant. We let $\tilde{I}^F_\Sigma$ denote the pullback of $I^F_\Sigma|_{Sh^{P'}_\Sigma}$ to $\widetilde{Sh}^{P'}_\Sigma$ by the map (4.1.7), and define $\tilde{I}^F_{\Sigma,1}$ and $\tilde{I}^F_{\Sigma,2}$ analogously.

It is clear that, on any $\bar{Z}_\sigma$, the formula (4.5.7) identifies the restriction of I^F_Σ with the principal P′-bundle attached [as in (4.3.1)] to $[\tilde{V}]_F$; in other words, to the (Deligne extension of the) flat bundle defined by the faithful representation ρ of the fundamental group Γ'_F of M'_F. Similarly, $I^F_{\Sigma,2}$ (resp. $I^F_{\Sigma,1}$) is the principal P′/U-bundle (resp. G_h-bundle) attached by the same sequence of constructions to a representation of Γ'_F which factors through a faithful rational representation (ρ_2, V_2) of P′/U (resp. a faithful rational representation (ρ_1, V_1) of G_h). Let $\mathcal{F}(V_2)$, $\mathcal{F}(V_1)$ be the corresponding flat vector bundles over ${}_K\mathcal{M}'_F$, let $\mathcal{F}(V_2)_\Sigma$, $\mathcal{F}(V_1)_\Sigma$ denote their Deligne canonical extensions, and let $\mathcal{F}(V_2)^P_\Sigma$, $\mathcal{F}(V_1)^P_\Sigma$ denote the pullbacks to $\widetilde{Sh}^P_\Sigma$.

But now ρ_2 and ρ_1 already define the flat vector bundles

$$[\tilde{V}_2] := \varprojlim P'(\mathbb{Q})^+\backslash V_2(\mathbb{C}) \times (D_F/U(\mathbb{C})) \times (P(\mathbf{A}^f)/U(\mathbf{A}^f))/K_{\mathcal{A}}$$

(notation as in 4.1) and $[\tilde{V}]_1$, on the mixed Shimura variety $\mathcal{A}_F$ and on $Sh(G_h, X(F))$, respectively. It is clear that, at least analytically, in the notation of 4.1

$$(4.6.2) \qquad \mathcal{F}(V_2)^P_\Sigma \cong \pi^*_{2,K}({}_{K_{\mathcal{A}}}[\tilde{V}_2]), \qquad \mathcal{F}(V_1)^P_\Sigma \cong \pi^*_{2,K} \circ \pi^*_{1,K}({}_{K_F}[\tilde{V}_1]).$$

The isomorphisms in (4.6.2) respect the algebraic structures on both sides determined by Deligne's existence theorem [D1], hence extend to isomorphisms of canonical extensions. It follows that, over $\widetilde{\mathrm{Sh}}^{\mathrm{P}}_{\Sigma}$,

$$\tilde{\mathrm{I}}^{\mathrm{F}}_{\Sigma,2} \cong \pi^*_{2,\mathrm{K}}(\mathrm{I}'_2), \qquad \tilde{\mathrm{I}}^{\mathrm{F}}_{\Sigma,1} \cong \pi^*_{2,\mathrm{K}} \circ \pi^*_{1,\mathrm{K}}(\mathrm{I}'_1), \tag{4.6.3}$$

for some principal G_h-bundle I'_1 (resp. $\mathrm{P}'/\mathrm{U}_\mathrm{F}$-bundle I'_2) on $\mathrm{Sh}(\mathrm{G}_h, \mathrm{X}(\mathrm{F}))$ (resp. $\mathcal{A}_\mathrm{F}$), with $\mathrm{W}\backslash\mathrm{I}'_2 \cong \pi^*_{1,\mathrm{K}}(\mathrm{I}'_1)$, Furthermore, the computations of 3.4 and 3.5 show that

$$\pi_{2,\mathrm{K},*}(\mathcal{F}(\mathrm{V}_2)^{\mathrm{P}}_{\Sigma}) \cong {}_{\mathrm{K}_\mathcal{A}}[\tilde{\mathrm{V}}_2], \qquad (\pi_{1,\mathrm{K}} \circ \pi_{2,\mathrm{K}})_*(\mathcal{F}(\mathrm{V}_1)^{\mathrm{P}}_{\Sigma}) \cong {}_{\mathrm{K}_\mathrm{F}}[\tilde{\mathrm{V}}_1] \tag{4.6.4}$$

(zeroth direct image). We thus obtain canonical $\mathrm{E}(\mathrm{G}, \mathrm{X})$-rational structures on ${}_{\mathrm{K}_\mathcal{A}}[\tilde{\mathrm{V}}_2]$ and ${}_{\mathrm{K}_\mathrm{F}}[\tilde{\mathrm{V}}_1]$, hence on I'_2, I'_1, which are compatible with the isomorphisms (4.6.3) and (4.6.4), and the morphisms of (4.6.1) factor through

$$\pi'_2 : \mathrm{I}'_2 \to \mathrm{U}\backslash\check{\mathrm{M}}(\mathrm{F}), \qquad \pi'_1 : \mathrm{I}'_1 \to \mathrm{W}\backslash\check{\mathrm{M}}(\mathrm{F}) \cong \check{\mathrm{M}}(\mathrm{G}_h, \mathrm{X}(\mathrm{F})) \tag{4.6.5}$$

via (4.6.3). We trivially have the $\mathrm{E}(\mathrm{G}, \mathrm{X})$-rational isomorphism

$$\mathrm{W}\backslash\mathrm{I}'_2 \cong \pi^*_{1,\mathrm{K}}(\mathrm{I}'_1). \tag{4.6.6}$$

With the zero section $\zeta_\mathrm{K} : {}_\mathrm{K}\mathrm{Sh}(\mathrm{G}, \mathrm{X})^\mathrm{P} \to {}_\mathrm{K}\mathcal{A}_\mathrm{F}$ defined as in 4.1, there is an isomorphism

$$\zeta^*_\mathrm{K}\, \pi_{2,\mathrm{K},*}(\mathcal{F}(\mathrm{V}_1)^{\mathrm{P}}_{\Sigma}) \cong {}_{\mathrm{K}_\mathrm{F}}[\tilde{\mathrm{V}}_1]$$

which is compatible with the $\mathrm{E}(\mathrm{G}, \mathrm{X})$-structures on both sides; this follows from the trivial isomorphism $\zeta^*_\mathrm{K} \circ \pi^*_{1,\mathrm{K}}(\mathcal{O}_1) \cong \pi_{1,\mathrm{K},*}\, \mathcal{O}_2$, where $\mathcal{O}_1$ and $\mathcal{O}_2$ are the structure sheaves of ${}_\mathrm{K}\mathrm{Sh}(\mathrm{G}_{h,\mathrm{F}}, \mathrm{X}(\mathrm{F}))$ and ${}_\mathrm{K}\mathcal{A}_\mathrm{F}$, respectively. Thus

$$\zeta^*_\mathrm{K}(\mathrm{W}\backslash\mathrm{I}'_2) \cong \mathrm{I}'_1 \quad (\text{rationally over } \mathrm{E}(\mathrm{G}, \mathrm{X})). \tag{4.6.7}$$

On the other hand, there is an obvious isomorphism

$$\mathrm{I}'_1(\mathbb{C}) \cong {}_{\mathrm{K}_\mathrm{F}}\mathrm{I}(\mathrm{G}_h, \mathrm{X}(\mathrm{F}))(\mathbb{C}) \tag{4.6.8}$$

compatible with (4.3.2) and (4.6.5) for the pair $(\mathrm{G}_h, \mathrm{X}(\mathrm{F}))$.

4.6.9. Lemma. – *The isomorphism* (4.6.8) *is rational over* $\mathrm{E}(\mathrm{G}, \mathrm{X})$.

Proof. – We drop the subscripts K in this proof. We use the strategy of [H2], § 6. First, define $(\mathrm{G}^{(2)}, \Delta(\mathrm{P}_\mathrm{F}))$ as in 1.8, and let $\eta : \mathrm{Sh}(\mathrm{G}^{(2)}, \Delta(\mathrm{P}_\mathrm{F})) \to \mathrm{Sh}(\mathrm{G}, \mathrm{X})$ be the natural map (*cf.* [H2], § 5). This extends to maps

$$\eta^{(*)} : \quad \mathrm{Sh}(\mathrm{G}^{(2)}, \Delta(\mathrm{P}_\mathrm{F}))^* \to \mathrm{Sh}^*, \qquad \eta_\Sigma : \quad \mathrm{Sh}(\mathrm{G}^{(2)}, \Delta(\mathrm{P}_\mathrm{F}))_{\Sigma^{(2)}} \to \mathrm{Sh}_\Sigma$$

of minimal and toroidal compactification, respectively, for some admissible $\Sigma^{(2)}$ [H3], Prop. 3.4. Since F is still a rational boundary component of $\Delta(\mathrm{P}_\mathrm{F})$ and C_F is again the corresponding cone, we may even assume $\Sigma_{\rm F}^{(2)} = \Sigma_{\rm F}$. It follows easily from the construction (4.3.1) that, in the obvious notation,

$$\eta^* \mathrm{I}_\Sigma^{\rm F} \cong \mathrm{I}\,(\mathrm{G}^{(2)}, \Delta\,(\mathrm{P_F}))_{\Sigma^{(2)}}^{\rm F} \times^{\mathrm{G}^{(2)}} \mathrm{G}. \tag{4.6.9.1}$$

Let π_2, $\mathrm{Sh}_\Sigma^{\mathrm{P}\,(2)}$, $\mathcal{A}_{\rm F}^{(2)}$, $\mathrm{Sh}\,(\mathrm{G}, \mathrm{X})^{\mathrm{P}\,(2)}$, $\mathrm{I}_1'^{(2)}$, and $\mathrm{I}_2'^{(2)}$ be the objects corresponding to π_2, $\mathrm{Sh}_\Sigma^{\rm P}$, $\mathcal{A}_{\rm F}$, $\mathrm{Sh}\,(\mathrm{G}, \mathrm{X})^{\rm P}$, I_1' and I_2' for $\mathrm{Sh}\,(\mathrm{G}^{(2)}, \Delta\,(\mathrm{P_F}))_{\Sigma^{(2)}}$. Then (*cf.* [H2], § 6)

$$\mathcal{A}_{\mathrm{F}^{(2)}} = \mathrm{Sh}\,(\mathrm{G}, \mathrm{X})^{\mathrm{P}\,(2)} \cong \zeta\,(\mathrm{Sh}\,(\mathrm{G}, \mathrm{X})^{\rm P}), \tag{4.6.9.2}$$

and

$$\begin{array}{ccc} \mathrm{Sh}_\Sigma^{\mathrm{P}\,(2)} & \longrightarrow & \mathrm{Sh}_\Sigma^{\rm P} \\ {\scriptstyle \pi_2}\downarrow & & {\scriptstyle \pi_2}\downarrow \\ \mathrm{Sh}\,(\mathrm{G}_h, \mathrm{X}\,(\mathrm{F}))^{(2)} & \stackrel{\zeta}{\hookrightarrow} & \mathcal{A}_{\rm F} \end{array} \tag{4.6.9.3}$$

is Cartesian, since $\mathrm{W}_{\rm F}^{(2)} = \mathrm{U_F}$. Hence

$$\mathrm{I}_1'^{(2)} = \mathrm{I}_2'^{(2)} \cong \zeta^*\,(\mathrm{W}\backslash \mathrm{I}_2') \cong \mathrm{I}_1', \tag{4.6.9.4}$$

where the last isomorphism is (4.6.7). Thus we may assume $(\mathrm{G}, \mathrm{X}) = (\mathrm{G}^{(2)}, \Delta\,(\mathrm{P_F}))$. Now the argument used to prove [H2], Corollary 6.4.3 shows that we may "factor out" the pair $(\mathrm{G}_h, \mathrm{X}\,(\mathrm{F}))$ and assume F to be a point and D to be a rational tube domain; the reduction in [*loc. cit.*, § 6.5] shows further that we may assume $\mathrm{G}^{\rm ab} = \mathbb{G}_{\rm m}$, $\mathrm{G}_h = \mathbb{G}_m \times \mathbb{G}_m$.

Finally, [H2], Lemma 6.5.1 shows that we may "approach" the boundary component $\mathrm{Sh}\,(\mathbb{G}_m \times \mathbb{G}_m, \{pt\})$ along an embedded $\mathrm{Sh}\,(\mathrm{GL}\,(2), \mathfrak{H}^\pm)$ in $\mathrm{Sh}(\mathrm{G}, \mathrm{X})$, where $\mathfrak{H}^\pm$ is the union of the upper and lower half-planes in $\mathbb{C}$. Thus we may replace (G, X) by the pair $(\mathrm{GL}\,(2), \mathfrak{H}^\pm)$. If for V we take the dual of the standard two-dimensional representation of GL(2), then $[\tilde{\mathrm{V}}]$ is canonically isomorphic to the relative de Rham H^1 of the universal family of elliptic curves with level structure over $\mathrm{Sh}\,(\mathrm{GL}\,(2), \mathfrak{H}^\pm)$. The lemma in this case is then a simple consequence of the theory of the Tate elliptic curve (*cf.* [H2], § 6.6).

4.7. COHOMOLOGY OF THE ABELIAN SCHEME. – In what follows, we let $\mathcal{W}$ be a homogeneous vector bundle on $\check{\mathrm{M}}$ as in 3.1 and let $[\mathcal{W}]$ be the corresponding automorphic vector bundle, $[\mathcal{W}]_\Sigma$ its canonical extension to Sh_Σ, $[\mathcal{W}]_\Sigma^{\rm P}$ its restriction to the P-stratum. Let $\gamma : \widetilde{\mathrm{Sh}}_\Sigma^{\rm P} \longrightarrow \mathrm{Sh}_\Sigma^{\rm P}$ be the map (4.1.7). We first observe:

4.7.1. LEMMA. – *Let* $\mathcal{W}\,(\mathrm{F})$ *denote the pullback of* $\mathcal{W}$ *to the* P′-*orbit* $\check{\mathrm{M}}\,(\mathrm{F})$. *Then* $\mathcal{W}\,(\mathrm{F})$ *descends to a* $\mathrm{P'/U_F}$-*homogeneous vector bundle* $\mathcal{W}\,(\mathrm{F})^{\rm A}$ *on* $\mathrm{U}\backslash\check{\mathrm{M}}\,(\mathrm{F})$, *and there are canonical isomorphisms*

$$[\mathcal{W}]_\Sigma^{\mathrm{P}'} \xrightarrow{\sim} \pi_\Sigma^{\mathrm{F},*}\,(\mathcal{W}\,(\mathrm{F}))/\mathrm{P}', \; \gamma^*\,[\mathcal{W}]_\Sigma^{\mathrm{P}'} \xrightarrow{\sim} (\pi_{2,\mathrm{K}})^*\,(\pi_2'^{,*}\,(\mathcal{W}\,(\mathrm{F})^{\rm A})\,/(\mathrm{P}'/\mathrm{U})).$$

Proof. – It follows from 4.5.8 (ii) that the vector group U acts freely on $\check{\mathrm{M}}\,(\mathrm{F})$, so the first assertion is clear. The isomorphisms are then tautological.

For clarity, we let $\mathrm{Q}' = \mathrm{P}'/\mathrm{U}$, $\check{\mathrm{M}}\,(\mathrm{F})^{\rm A} = \mathrm{U}\backslash\check{\mathrm{M}}\,(\mathrm{F})$, so that $\mathcal{W}\,(\mathrm{F})^{\rm A}$ is a Q′-homogeneous vector bundle over $\check{\mathrm{M}}\,(\mathrm{F})^{\rm A}$ and $\mathrm{I}_2' = \mathrm{I}_2'\,({}_{\rm K}\mathcal{A}_{\rm F})$ is a principal Q′-bundle over ${}_{\rm K}\mathcal{A}_{\rm F}$. Let

$\mathcal{A}_{\rm F} := \overset{\lim}{\leftarrow}\ {}_{\rm K}\mathcal{A}_{\rm F}$, the limit taken over K [or rather over $\mathrm{K}_{\mathcal{A}}$, *cf.* (4.1.4)]. If $\mathrm{K}' \subset \mathrm{K}$ and $\beta : {}_{\mathrm{K}'}\mathcal{A}_{\rm F} \to {}_{\rm K}\mathcal{A}_{\rm F}$ is the natural projection, then one verifies easily (using [H3], 4.3.2) that $\mathrm{I}'_2\,({}_{\mathrm{K}'}\mathcal{A}_{\rm F}) \cong \beta^*\,\mathrm{I}'_2\,({}_{\rm K}\mathcal{A}_{\rm F})$. In the limit, we obtain the diagram

$$\begin{array}{ccc} & \mathrm{I}'_2 & \\ {}^{\pi'_2}\swarrow & & \searrow \\ \check{\mathrm{M}}\,(\mathrm{F})^{\mathrm{A}} & & \mathcal{A}_{\rm F} \end{array} \tag{4.7.2}$$

from which we can construct a theory of automorphic vector bundles on $\mathcal{A}_{\rm F}$, starting from Q'-homogeneous vector bundles on $\check{\mathrm{M}}\,(\mathrm{F})^{\mathrm{A}}$. Namely, let $\tilde{q} = c_{\rm F}^{-1}\,(p) \in \check{\mathrm{M}}\,(\mathrm{F})$, q its image in $\check{\mathrm{M}}\,(\mathrm{F})^{\mathrm{A}}$, let x be the projection of p (or q) on F, let $\mathcal{P}_x^{\rm F} \subset \mathrm{G}_h$ denote its stabilizer, and let V_x^- be the algebraic subgroup of $\mathrm{V}_{\rm F}$ with Lie algebra $\mathfrak{v}_x^-$. Then the category of Q'-homogeneous vector bundles on the Q'-homogeneous space $\check{\mathrm{M}}\,(\mathrm{F})^{\mathrm{A}}$ is naturally equivalent to the category of representations of

$$\mathrm{Stab}_{\mathrm{Q}'}\,(q) = [\mathrm{Ad}\,(c_{\rm F})^{-1}\,(\mathcal{P}_p) \cap \mathrm{P}' \cdot \mathrm{U}]/\mathrm{U} = \mathcal{P}_x^{\rm F} \cdot \mathrm{V}_x^-,$$

where the last equality follows as in 1.8.6; note that $c_{\rm F}$ centralizes G_h, hence commutes with $\mathcal{P}_x^{\rm F}$. If $\mathcal{E}$ is a Q'-homogeneous vector bundle on $\check{\mathrm{M}}\,(\mathrm{F})^{\mathrm{A}}$ we let $\sigma_{\mathcal{E}}$ denote the corresponding representation of $\mathcal{P}_x^{\rm F} \cdot \mathrm{V}_x^-$. Let

$$[\mathcal{E}] = \pi_2'^{\,*}\,(\mathcal{E})/\mathrm{Q}' \tag{4.7.3}$$

be the corresponding automorphic vector bundle over the mixed Shimura variety $\mathcal{A}_{\rm F}$. One verifies as in the pure case that the morphism $\mathrm{I}'_2 \to \mathcal{A}_{\rm F}$ is $\mathrm{Q}'\,(\mathbf{A}^f)$-equivariant – this follows already from the corresponding fact for Sh and the equivariance properties of the canonical extension – so $[\mathcal{E}]$ is naturally $\mathrm{Q}'\,(\mathbf{A}^f)$-homogeneous. For any pair $(q, g) \in \mathrm{D}_{\rm F}/\mathrm{U}\,(\mathbb{C}) \times \mathrm{Q}'\,(\mathbf{A}^f)$, let $[q, g]$ be the corresponding point of $\mathcal{A}_{\rm F}\,(\mathbb{C})$. Then there is an isomorphism

$$\mathrm{Per}_{(q,\,g)} :\ \mathcal{E}_q \xrightarrow{\sim} [\mathcal{E}]_{[q,\,g]} \tag{4.7.4}$$

defined as in (4.3.3), where we identify q with its image in $\check{\mathrm{M}}\,(\mathrm{F})^{\mathrm{A}}$ under the obvious map.

Now $\mathcal{E}$ is of the form $\check{\pi}_1^*\,(\mathcal{Y})$, for some G_h-homogeneous vector bundle $\mathcal{Y}$ on $\check{\mathrm{M}}\,(\mathrm{G}_h,\,\mathrm{X}\,(\mathrm{F}))$, if and only if $\sigma_{\mathcal{E}}$ is trivial on V_x^-. In that case, let $[\mathcal{Y}]$ be the corresponding automorphic vector bundle on $\mathrm{Sh}\,(\mathrm{G}_h,\,\mathrm{X}\,(\mathrm{F}))$. It follows from 4.6.9 that the canonical isomorphism (4.6.6)

$$[\mathcal{E}] \xrightarrow{\sim} \pi_1^*\,[\mathcal{Y}], \tag{4.7.5}$$

where $\pi_1 : \mathcal{A}_{\rm F} \to \mathrm{Sh}\,(\mathrm{G}_h,\,\mathrm{X}\,(\mathrm{F}))$ is the limit of the morphisms $\pi_{1,\mathrm{K}}$ at finite level, respects the rational structures on both sides.

Since V_x^- is unipotent, every $\mathcal{E}$ has a filtration $0 = \varphi_0\,\mathcal{E} \subset \varphi_1\,\mathcal{E} \subset \ldots\ \varphi_r\,\mathcal{E} = \mathcal{E}$ by homogeneous subbundles such that $\mathrm{Gr}_i^{\varphi}\,\mathcal{E} \cong \check{\pi}_1^*\,(\mathcal{Y}_i)$, which induces a corresponding filtration $\varphi_i\,[\mathcal{E}]$. On the other hand, the inverse limit over $\mathrm{K}_{\mathcal{A}}$ of the zero section

$\zeta_K : {}_K\mathrm{Sh}(\mathrm{G}_h, \mathrm{X}(\mathrm{F})) \to {}_K\mathcal{A}_\mathrm{F}$ (*cf.* 4.1) is a morphism $\zeta : \mathrm{Sh}(\mathrm{G}_h, \mathrm{X}(\mathrm{F})) \to \mathcal{A}_\mathrm{F}$. There is also a section

$$\check{\zeta} : \quad \check{\mathrm{M}}(\mathrm{G}_h, \mathrm{X}(\mathrm{F})) \to \check{\mathrm{M}}(\mathrm{F})^\mathrm{A}$$

of $\check{\pi}_1$, which, like ζ, corresponds by functoriality to the inclusion of the pair $(\mathrm{G}^{(2)}, \Delta(\mathrm{P}_\mathrm{F}))$ in (G, X). Evidently, $\mathrm{Stab}_{\mathrm{Q}'}(q) \cap \mathrm{G}_h = \mathrm{Stab}_{\mathrm{G}_h}(q) = \mathcal{P}_x^\mathrm{F}$ if q is in the image of $\check{\zeta}$; in particular,

$$\check{\zeta}^*(\mathcal{E}) \cong \bigoplus_i \check{\zeta}^*(\mathrm{Gr}_i^\varphi \mathcal{E}) \cong \bigoplus_i \mathcal{Y}_i. \tag{4.7.6}$$

4.7.7. Lemma. – *Let* $(\mathrm{H}, x) \subset (\mathrm{G}_{h,\mathrm{F}}, \mathrm{X}(\mathrm{F}))$ *be a* CM *pair, with* H *a torus. Let* $\mathcal{E}$ *be a* Q′*-homogeneous vector bundle over* $\check{\mathrm{M}}(\mathrm{F})^\mathrm{A}$, *and let* $\sigma_{\mathcal{E},x}$ *denote the restriction of* $\sigma_\mathcal{E}$ *to* $\mathrm{H} \subset \mathrm{Stab}_{\mathrm{Q}'}(\check{\zeta}(x))$; *denote by* $\mathrm{E}(\mathcal{E}, x)$ *the field of definition of* $\sigma_{\mathcal{E},x}$. *Define the period element* $p(x, \sigma_{\mathcal{E},x}) \in \mathrm{Aut}([\mathcal{E}]|_{\zeta(\mathrm{Sh}(\mathrm{H},x))})$ *as in* 4.3. *For any* $g \in \mathrm{Q}'(\mathbf{A}^f)$, *we identify* Sh(H, *x*) *with its g-translate* $\zeta(\mathrm{Sh}(\mathrm{H}, x)) \cdot g \subset \mathcal{A}_\mathrm{F}$, *and let* $p(x, \sigma_{\mathcal{E},x}; g)$ *be the element of* $\mathrm{Aut}([\mathcal{E}]|_{\zeta(\mathrm{Sh}(\mathrm{H},x))\cdot g})$ *defined by transport of structure. Then*

$$p(x, \sigma_{\mathcal{E},x}; g) \cdot \mathrm{Per}_{(x,g)}(\mathcal{E}_x(\mathrm{E}(\mathcal{E}, x))) = [\mathcal{E}]_{\zeta(\mathrm{Sh}(\mathrm{H},x))\cdot g}(\mathrm{E}(\mathcal{E}, x)),$$

where the right-hand side is the rational structure defined by (4.7.3).

Proof. – Since both sides are homogeneous with respect to $\mathrm{Q}'(\mathbf{A}^f)$, it suffices to verify this for $g=1$, where it is a consequence of Lemma 4.6.9.

Now the representation $\sigma_\mathcal{E} : \mathcal{P}_x^\mathrm{F} \cdot \mathrm{V}_x^- \to \mathrm{GL}(\mathcal{E}_x)$ defines a $\mathcal{P}_x^\mathrm{F}$-equivariant homomorphism

$$\mathfrak{v}_x^- \otimes \mathrm{Gr}_\bullet^\varphi(\mathcal{E}_x) \to \mathrm{Gr}_{\bullet-1}^\varphi(\mathcal{E}_x) \tag{4.7.8}$$

Let $\mathfrak{V}^-$ be the G_h-homogeneous vector bundle on $\check{\mathrm{M}}_{h,\mathrm{F}}$ associated to the adjoint representation of $\mathcal{P}_x^\mathrm{F}$ on $\mathfrak{v}_x^-$; then (4.7.8) defines a homomorphism

$$\mathfrak{V}^- \otimes \mathrm{Gr}_\bullet^\varphi(\mathcal{E}) \to \mathrm{Gr}_{\bullet-1}^\varphi(\mathcal{E}),$$

thus a homomorphism of automorphic vector bundles

$$\mathrm{Gr}_\bullet^\varphi[\mathcal{E}] \to [\mathfrak{V}^{-,*}] \otimes \mathrm{Gr}_{\bullet-1}^\varphi[\mathcal{E}]. \tag{4.7.9}$$

Now for any $x \in \mathrm{X}(\mathrm{F})$, $g \in \mathrm{G}_h(\mathbf{A}^f)$, define $[x, g] \in \mathrm{Sh}_{h,\mathrm{F}}$ as in 4.3. Then $\mathfrak{v}_x^{-,*}$ is canonically isomorphic as $\mathcal{P}_x^\mathrm{F}$-module, in the notation 3.5, to

$$\mathrm{Gr}_1^\mathrm{F}\,\mathrm{H}_{\mathrm{DR}}^1(\mathrm{A}_{[x,g]}) = \mathrm{H}_{\mathrm{DR}}^1(\mathrm{A}_{[x,g]})/\mathrm{H}^0(\mathrm{A}_{[x,g]}, \Omega^1_{\mathrm{A}_{[x,g]}}) \cong \mathrm{H}^1(\mathrm{A}_{[x,g]}, \mathcal{O}_{\mathrm{A}_{[x,g]}})$$

(*cf.* [P, 3.22]); thus there is a canonical isomorphism over $\mathbb{C}$

$$[\mathfrak{V}^{-,*}] \xrightarrow{\sim} \mathrm{R}^1\pi_{1,*}\mathcal{O}_{\mathcal{A}_\mathrm{F}}. \tag{4.7.10}$$

4.7.11. Lemma. – *The isomorphism* (4.7.10) *respects the* E(G, X)*-rational structures on both sides. More precisely, if* $(\mathrm{H}, x) \subset (\mathrm{G}_h, \mathrm{X}(\mathrm{F}))$ *is a* CM *point,* $g \in \mathrm{G}_h(\mathbf{A}^f)$, *and*

$\sigma_x^- : \mathrm{H} \to \mathrm{GL}(\mathfrak{v}_x^{-,*})$ *is induced by the restriction of the adjoint representation then* [*in the notation of* (4.3.5)]

$$p(x, \sigma_x^-; g) \cdot \mathrm{Per}_{(x,g)}(\mathfrak{v}_x^{-,*}(\mathrm{E}(\mathrm{H}, x))) = \mathrm{H}^1(\mathrm{A}_{[x,g]}, \mathcal{O}_{\mathrm{A}_{[x,g]}})(\mathrm{E}(\mathrm{H}, x)))_{[x,g]}$$

Proof. – We may replace the pair $(\mathrm{G}_h, \mathrm{X}(\mathrm{F}))$ by (H, x), and write $\mathcal{A}_x$ instead of $\mathcal{A}_\mathrm{F}$. We identify the vector groups V and U with rational vector spaces. Now the adjoint representation $\mathrm{H} \to \mathrm{GL}(\mathrm{V})$ factors through a representation

$$\mathrm{H} \to \mathrm{GSp}(\mathrm{V}, \langle \cdot, \cdot \rangle_\lambda), \tag{4.7.11.1}$$

where $\langle \cdot, \cdot \rangle_\lambda$ is a symplectic form defined by composing the Lie bracket $\mathrm{V} \otimes \mathrm{V} \to \mathrm{U}$ with some rational linear form $\lambda \in \mathrm{U}^*$. If λ is positive on the homogeneous cone C_F, then the map (4.7.11.1) defines a morphism of basic pairs $(\mathrm{H}, x) \to (\mathrm{GSp}(\mathrm{V}, \langle \cdot, \cdot \rangle_\lambda), \mathfrak{H}^\pm)$, where $\mathfrak{H}^\pm$ is the Siegel double space, and $\mathcal{A}_x$ is the pullback to $\mathrm{Sh}(\mathrm{H}, x)$ of the universal abelian scheme (with level N structure for all N) over $\mathrm{Sh}(\mathrm{GSp}(\mathrm{V}, \langle \cdot, \cdot \rangle_\lambda), \mathfrak{H}^\pm)$ (For all this, *cf.* [Br1], or [P], 3.20, 10.7).

Now the period invariant $p(x, \sigma_x^-; g)$ depends only on the representation σ_x^- of H. We have just shown that this is the representation associated to the (0, 1)-part of the Hodge structure associated to the isogeny class of abelian varieties obtained by restricting the universal abelian scheme to $\mathrm{Sh}(\mathrm{H}, x)$, which is none other than $\mathcal{A}_x$. But by construction [H2], 3.15 $p(x, \sigma_x^-; g)$ is the period invariant associated to the (0, 1)-part of the Hodge structure associated to the isogeny class of abelian varieties. Now the lemma is a tautology.

Let $q \in \check{\mathrm{M}}(\mathrm{F})^\mathrm{A}$, and let $x = x(q)$ denote its image in $\check{\mathrm{M}}_{h,\mathrm{F}}$. For $i = 0, 1, \ldots$, we let $\mathcal{E}^{(i)}$ be the G_h-homogeneous vector bundle on $\check{\mathrm{M}}_{h,\mathrm{F}}$ associated to the representation of $\mathcal{P}_x^\mathrm{F}$ on $\mathrm{H}^i(\mathfrak{v}_x^-, \mathcal{E}_q)$, with q and x as above. We now restrict our attention to $\mathcal{E}$ of the form $\mathcal{W}(\mathrm{F})^\mathrm{A}$, as at the beginning of this section, where $\mathcal{W}$ is moreover attached to a representation λ of K_p. In 3.5 we have constructed canonical isomorphisms over $\mathbb{C}$:

$$\mathrm{R}^i \pi_{1,*}[\mathcal{W}(\mathrm{F})^\mathrm{A}] \cong [\mathcal{W}(\mathrm{F})^{\mathrm{A},(i)}]. \tag{4.7.12}$$

The same argument works word for word for the subquotients $\mathrm{Gr}_j^\varphi[\mathcal{W}(\mathrm{F})^\mathrm{A}]$:

$$\mathrm{R}^i \pi_{1,*} \mathrm{Gr}_j^\varphi [\mathcal{W}(\mathrm{F})^\mathrm{A}] \cong [(\mathrm{Gr}_j^\varphi \mathcal{W}(\mathrm{F})^\mathrm{A})^{(i)}], \quad i, j \geqq 0. \tag{4.7.13}$$

If $\mathcal{E} = \mathcal{W}(\mathrm{F})^\mathrm{A}$ or $\mathrm{Gr}_j^\varphi[\mathcal{W}(\mathrm{F})^\mathrm{A}]$, $(\mathrm{H}, x) \subset (\mathrm{G}_h, \mathrm{X}(\mathrm{F}))$ is a CM pair, and $g \in \mathrm{G}_h(\mathbf{A}^f)$, the isomorphism (4.7.12/13) is defined as in Proposition 3.5.8: we let

$$\mathrm{C}^\bullet(\mathfrak{v}_x^-, \mathcal{E}_q) = \Lambda^\bullet(\mathfrak{v}_x^-)^* \otimes \mathcal{E}_q \subset \Lambda^\bullet(\mathfrak{v}_x^-)^* \otimes \mathrm{C}^\infty(\mathrm{V}(\mathbb{Q})\backslash\mathrm{V}(\mathbf{A})) \otimes \mathcal{E}_q,$$

where the differential on the last term is defined by the complex structure on $\mathrm{V}_\mathrm{F}(\mathbb{R})$. Then the composite map

$$\mathrm{H}^i(\mathfrak{v}_x^-, \mathcal{E}_q) = \mathrm{H}^i(\mathrm{C}^\bullet(\mathfrak{v}_x^-, \mathcal{E}_q)) \to \mathrm{H}^i(\Lambda^\bullet(\mathfrak{v}_x^-)^*$$
$$\otimes \mathrm{C}^\infty(\mathrm{V}(\mathbb{Q})\backslash\mathrm{V}(\mathbf{A}) \cdot g) \otimes \mathcal{E}_q) \cong \mathrm{R}^i \pi_{1,*}[\mathcal{E}]_{[x,g]}$$

is an isomorphism. Denote this composite $\psi^{(i)}$, and let $\sigma^i_{x,\mathcal{E}}$ be the representation of $\mathcal{P}^{\mathrm{F}}_x$ on $\mathrm{H}^i(\mathfrak{v}^-_x, \mathcal{E}_q)$. Let $\mathrm{E}(\mathcal{W})$ be the field of definition of the G-homogeneous bundle $\mathcal{W}$, and let $\mathrm{E}(\mathcal{W}, x)$ be the field of definition of $\mathcal{W}(\mathrm{F})_q$, which depends only on $x(q)$.

4.7.14. LEMMA. – *The isomorphisms* (4.7.12) *and* (4.7.13) *respect the* $\mathrm{E}(\mathcal{W})$*-rational structures on both sides. More precisely, for any* CM *pair* (H, x) *as above, and with* $\mathcal{E} = \mathcal{W}(\mathrm{F})^{\mathrm{A}}$ *or* $\mathrm{Gr}^{\varphi}_j\,\mathcal{W}(\mathrm{F})^{\mathrm{A}}$, *we have*

$$p(x, \sigma^i_{x,\mathcal{E}}) \cdot \psi^{(i)}(\mathrm{H}^i(\mathfrak{v}^-_x, \mathcal{E}_q(\mathrm{E}(\mathcal{W}, x)))) = \mathrm{R}^i\,\pi_{1,*}[\mathcal{E}]_{[x,g]}(\mathrm{E}(\mathcal{W}, x)). \tag{4.7.14.1}$$

Proof. – The fact that (4.7.14.1) is a more precise version of the first statement is a consequence of (1.8.2). We first assume $\mathcal{E} = \mathrm{Gr}^{\varphi}_j\,\mathcal{W}(\mathrm{F})^{\mathrm{A}}$. Then $[\mathcal{E}] = \pi_1^*[\mathcal{Y}_j]$, so

$$\mathrm{R}^i\,\pi_{1,*}[\mathcal{E}] \cong \mathrm{R}^i\,\pi_{1,*}[\mathcal{O}] \otimes [\mathcal{Y}_j] \cong \Lambda^i\,\mathrm{R}^1\,\pi_{1,*}[\mathcal{O}] \otimes [\mathcal{Y}_j], \tag{4.7.14.2}$$

with $\mathcal{O} = \mathcal{O}_{\mathcal{A}_{\mathrm{F}}}$ [Mu1], § 1. In this case, the Lemma is an immediate consequence of 4.7.11.

Now to treat the general case, we consider the spectral sequence for the filtration φ:

$$\begin{aligned}\mathrm{E}_1^{r,s} = \mathrm{R}^s\,\pi_{1,*}[\mathrm{Gr}^{\varphi}_r\,\mathcal{W}(\mathrm{F})^{\mathrm{A}}] \Rightarrow \mathrm{E}_{\infty}^{r,s} &= \mathrm{Gr}^{\varphi}_r\,\mathrm{R}^{r+s}\,\pi_{1,*}[\mathcal{W}(\mathrm{F})^{\mathrm{A}}] \\ &\cong \mathrm{Gr}^{\varphi}_r\,[\mathcal{W}(\mathrm{F})^{\mathrm{A},(r+s)}].\end{aligned} \tag{4.7.14.3}$$

By our hypothesis on $\mathcal{W}$, $[\mathcal{W}(\mathrm{F})^{\mathrm{A},(i)}]$ is associated to a representation of $\mathcal{P}^{\mathrm{F}}_x$ which factors through K_x, for each i (Corollary 3.6.3). There is thus a canonical isomorphism $\mathrm{Gr}^{\varphi}_{\bullet}[\mathcal{W}(\mathrm{F})^{\mathrm{A},(i)}] \cong [\mathcal{W}(\mathrm{F})^{\mathrm{A},(i)}]$ splitting the filtration. Furthermore, the differentials in the spectral sequence respect this rational structure; indeed, they are all derived by tensor operations from the morphism (4.7.9), and the isomorphisms (4.7.10) and (4.7.14.2), all of which respect the rational structure. Since (as we have already seen) the rational structure on the E_1-term is given by (4.7.14.1), we are done.

4.8. RATIONALITY OF THE CONSTANT TERM.

We are now ready to prove the main result of this section. Let Q′ = P′/U, as before, and let Q = P/U. Let $\mathcal{V}$ be an automorphic vector bundle on Sh(G, X), attached to a representation of K_p, and let $\mathrm{E}(\mathcal{V})$ be its field of definition. The two sides of the bottom line of diagram (4.2.4) have canonical $\mathrm{E}(\mathcal{V})$-rational structures, which are determined as explained before. In the first place, $\mathcal{V}$ and $\mathcal{V}_{\lambda(h,w)}$, for all w, have canonical models relative to their restrictions to CM Shimura subvarieties Sh(H, x), determined by the condition that the right-hand side of (4.3.6) define the $\mathrm{E}(\mathcal{V}, x)$-rational structure on the restriction to $\mathrm{G}(\mathbf{A}^f)$-translates (resp. $\mathrm{G}_h(\mathbf{A}^f)$-translates) of Sh(H, x). On the other hand, the local system $\tilde{\mathbf{V}}_{\lambda(l,w)}$ is associated to the vector space $\mathrm{V}_{\lambda(l,w)}$, which has a unique $\mathrm{E}(\mathcal{V})$-rational structure compatible with its realization, for every pair (H, x) as above, as a direct summand of $\mathrm{H}^{\bullet}(\mathfrak{v}^-_x, \mathcal{V}_q)$. Here $\mathcal{V}_q$, defined (relative to $\mathcal{V}$) as in 4.7, above, has a natural $\mathrm{E}(\mathcal{V}, x)$-rational structure, as does $\mathfrak{v}^-_x$; this defines an $\mathrm{E}(\mathcal{V}, x)$-rational structure on $V_{\lambda(l,w)}$ for each (H, x), and these are obviously compatible and descend to define an $E(\mathcal{V})$-rational structure on $V_{\lambda(l,w)}$.

4.8.1. THEOREM. – *The homomorphism* r_F *of diagram* (4.2.4) *is rational with respect to the* $E(\mathcal{V})$*-rational structures just defined on the two sides.*

Proof. – It suffices to consider the restriction to the P-stratum, since the action of $G(\mathbf{A}^f)$ preserves the rational structures. The cohomology of the P-stratum is given by Corollary 4.1.12.

Now the discrete group $L_F(\mathbb{Q})^+$ acts naturally on $D_F/U(\mathbb{C})$ and by conjugation on $P(\mathbb{Q})^+ \cdot Q'(\mathbf{A}^f)$; thus there is a holomorphic action of $L_F(\mathbb{Q})^+$ on $\mathcal{A}_F$ [notation (4.1.2)]. By [P], Prop. 11.10, this action respects the canonical model of $\mathcal{A}_F$. Furthermore, L_F acts algebraically on $\check{M}(F)$ (*e.g.*, by the concrete description given in 4.5.8). Say $\mathcal{V} = [\mathcal{W}]$, for some homogeneous vector bundle $\mathcal{W}$ on $\check{M}$. Since $\mathcal{W}$ is homogeneous with respect to G, it is *a fortiori* homogeneous with respect to L_F. We have the following Lemma:

4.8.3. LEMMA. – *The action of* $L_F(\mathbb{Q})^+$ *on* $\mathcal{A}_F$ *lifts to an* $E(G, X)$*-rational action on* I'_2*, with respect to which the morphism* π'_2 *of* (4.7.2) *is equivariant.*

Proof. – It is clear how to define the action over $\mathbb{C}$: writing

$$I'_2(\mathbb{C}) = \overset{\lim}{\longleftarrow} P'(\mathbb{Q})^+ \backslash Q'(\mathbb{C}) \times (D_F/U(\mathbb{C})) \times Q(\mathbf{A}^f)/K_{\mathcal{A}} \quad (cf.\ 4.6),$$

the action of $L_F(\mathbb{Q})^+$ is defined by conjugation, and it is obvious that this action commutes with π'_2. It has to be verified that the action respects the rational structure on I'_2. Now the natural right action of $G(\mathbb{Q}) \times G(\mathbf{A}^f)$ on $I(G, X) = \overset{\lim}{\longleftarrow} G(\mathbb{Q}) \backslash G(\mathbb{C}) \times X \times G(\mathbf{A}^f)/K$ restricts to an $E(G, X)$-rational action of $L_F(\mathbb{Q})^+$, which is equivalent to the (right)-action:

$$(4.8.4) \qquad \begin{cases} [g, x, g_f] \cdot \gamma = [\gamma^{-1} g \gamma, \gamma^{-1} \cdot x, \gamma^{-1} g_f \gamma], \\ g \in G(\mathbb{C}), \quad x \in X, \quad g_f \in G(\mathbf{A}^f), \quad \gamma \in L_F(\mathbb{Q})^+. \end{cases}$$

In the limit (over K), (4.8.4) evidently preserves the P′-stratum, hence descends to an $E(G, X)$-rational action of $L_F(\mathbb{Q})^+$ on I'_2. One verifies directly that this action is the one defined analytically above.

4.8.5. *Remark.* – It follows easily from 4.8.3 that the action of Δ_0, defined by (4.1.11), preserves the rational structure on the cohomology groups $\mathcal{H}^\bullet(w)$.

The morphism $\pi_{2,K}$ of (4.1.5) is a torus fibration with fiber $\mathcal{T}$ which may vary from one connected component to another (*cf.* [H3], 2.5). Let $\mathcal{V}^A$ denote the bundle $\pi_{2,K,*}(\mathcal{V}^P_\Sigma)^{\mathcal{T}}$ over $\mathcal{A}_F$ (the adelic version of the definition in 3.2). In what follows, we let $\Gamma = P(\mathbb{Q})^+/P'(\mathbb{Q})^+$. For any G-equivariant sheaf $\mathcal{W}$ on $\check{M}$, let $\mathcal{W}(F)^A$ be as in 4.7.1. The automorphic vector bundle $[\mathcal{W}(F)^A]$ on $\mathcal{A}_F$, defined by 4.7.3, is endowed with a Γ-action which preserves the rational structure. Let $\mathcal{V} = [\mathcal{W}]$ as before. By comparing (3.2.1) with 4.7.1 one sees that $[\mathcal{W}(F)^A] \cong \mathcal{V}^A$. We may thus define Γ-equivariant cohomology $H^\bullet_\Gamma(\mathcal{A}_F, \mathcal{V}^A)$. Now we can factor the isomorphism (4.1.12):

$$(4.8.6) \qquad \underset{K,\Sigma}{\overset{\lim}{\longrightarrow}} H^\bullet(Sh^P_\Sigma, \mathcal{V}^P_\Sigma) \tilde{\to} H^\bullet_\Gamma(\mathcal{A}_F, \mathcal{V}^A) \tilde{\to} \bigoplus_w I^P\{\mathcal{H}^\bullet(w)^{\Delta_1}\}.$$

Here the first isomorphism is defined as in the proof of Lemma 3.7.5, using (4.1.7). The relation between $H^\bullet_\Gamma(\widetilde{Sh}^P_\Sigma, \mathcal{V}^A)$ and the $H^\bullet_{\Gamma_l}({}^{<}\tilde{Z}_{F,\Sigma}, \tilde{i}^*_F \mathcal{V}'_{F,\Sigma})$ of paragraph 3 can be derived from Corollary 2.10.3, but the derivation of the first isomorphism can be carried out without reference to connected components. The second isomorphism is as in Corollary 3.7.8. More precisely, the argument used to prove Proposition 3.7.7 shows that

$$\text{(4.8.7)} \qquad H^\bullet_\Gamma(\mathcal{A}_F, \mathcal{V}^A) \cong H^\bullet_\Gamma(\mathrm{Sh}(G, X)^P, (\mathcal{V}_{\lambda(h,w)})^{\mathrm{can}}) \otimes V_{\lambda(l,w)})$$

where $\mathrm{Sh}(G, X)^P$ is the P-stratum of $\mathrm{Sh}(G, X)^*$, as in 1.7:

$$\mathrm{Sh}(G, X)^P \cong [\overset{\lim}{\leftarrow} \mathrm{Sh}(G, X)^{P'} \times L_F(\mathbf{A}^f) \cdot K/K]/G_h(\mathbf{A}^f),$$

with Γ acting on the left on the second factor and $G_h(\mathbf{A}^f)$ acting on the right on both factors. If in Proposition 2.9.4 we take $B = \mathrm{Sh}(G, X)^P$, $Q = L_F(\mathbb{R})^+/G_h(\mathbb{R})^+ \cdot K_l \times \mathrm{Sh}(G, X)^P$, and π the projection on the second factor, we can identify (4.8.7) with

$$[\overset{\lim}{\rightarrow} H^\bullet(\Gamma\backslash L_F(\mathbb{R})^+ \times L_F(\mathbf{A}^f) \cdot K/G_h(\mathbb{R})^+ K_l K,$$
$$H^\bullet(\mathrm{Sh}(G, X)^{P'}, (\mathcal{V}_{\lambda(h,w)})^{\mathrm{can}}) \otimes V_{\lambda(h,w)})]^{G_h(\mathbf{A}^f)},$$

which is easily identified with the rightmost term of (4.8.6).

It remains to be shown that the two isomorphisms of (4.8.6) are rational. Now Lemma 4.7.1, taken in the limit over K, implies that the first isomorphism

$$\underset{K,\Sigma}{\overset{\lim}{\rightarrow}} H^\bullet(\mathrm{Sh}^P_\Sigma, \mathcal{V}^P_\Sigma) \xrightarrow{\sim} H^\bullet_\Gamma(\mathcal{A}_F, \mathcal{V}^A),$$

of (4.8.6) respects the rational structures of the two sides. On the other hand, it follows from Lemma 4.7.14 that the second isomorphism of (4.8.6) also respects the rational structures.

4.9. Canonical trivializations. – The upper row of (4.2.4) is usually written in terms of complex vector-valued functions. The results of paragraph 4 show how to calculate the restriction map r_F in terms of specific systems of coordinates, which derive from the coordinates of the compact dual and homogeneous vector bundles thereupon. These systems of coordinates are used implicitly in paragraph 5 in proving that the Eisenstein map, which goes in the reverse direction, preserves rationality. In order to interpret the results of paragraph 5 for explicit automorphic forms, the definitions of these coordinate systems have to be recalled. In the first author's work on holomorphic and non-holomorphic cuspidal cohomology classes on GL(2), these coordinate systems are called *canonical trivializations*.

The proper way to look at the rational structure on the boundary cohomology is to view the factor $H^\bullet(X(G_l), \cdot)$ as being purely topological. A number of people seem to arrived at the conclusion, from different starting points, that the cohomology of $H^\bullet(X(G_l), \cdot)$ only contributes Tate twists. In the present formulation, even the Tate twists have been incorporated into the $H^\bullet(\mathrm{Sh}_{h,F}, \cdot)$ factor. Indeed, the different Weyl group elements w determine different automorphic vector bundles, to which are attached different motivic weights (in terms of the weight map $\mathbb{G}_m \rightarrow G_{h,F}$, whose relation to the Cayley morphism is described in [H2], § 5). Thus when F is a point and D is a rational tube domain over F,

G_h is a torus, and the cohomology of automorphic vector bundles on $Sh_{h,F}$ is essentially given by Tate motives, twisted by finite Dirichlet characters. We return to this point in Part II, where our approach is based on Hodge theory.

5. Cohomology classes defined by Eisenstein series

We use the method of Eisenstein series to associate coherent cohomology classes on Sh(G, X) to classes on the boundary stratum attached to a maximal parabolic subgroup P, and prove their non-triviality and arithmeticity under certain additional hypotheses. Emphasis is placed on the absolutely convergent case, since this is where rationality theorems can most easily be proved. It is probably safe to say that the non-convergent case remains at the experimental stage. More examples need to be worked out for low rank groups before the general pattern can be discerned. We hope to return to this topic in a future paper.

In section 5.4, we also assume that P is a *cuspidal* parabolic subgroup; in other words, it is assumed that the real Levi component $L(\mathbb{R})$ possesses a discrete series. This is a technical hypothesis which permits us to quote a theorem of Schmid (unpublished), reproved by Blank [B]), on the embedding of discrete series representations in representations induced from the discrete series. The effect, as explained in [H4], is to kill the intertwining operator – there is only one, since we are starting with cusp forms on maximal parabolic subgroups – in the theory of the constant term of the Eisenstein series. Thus, just as in the holomorphic case [H1], [H2], the map defining the Eisenstein series exactly inverts the constant term map. Together with the results of the preceding sections, this permits us to apply the strategy of [H1] to prove arithmeticity of (most) absolutely convergent Eisenstein classes.

Apart from the identification of the constant term with restriction to the boundary, most of the ideas of this section were already explained in [H4], § 6.

5.1. EISENSTEIN SERIES ATTACHED TO CUSP FORMS. – The results described in this section are standard, and we summarize them quickly. Fix a point $p \in X$ as before, and let $K_p \subset G(\mathbb{R})$ denote its stabilizer. We write L for the Levi subgroup of $P = P_F$ containing A, $W = W_F$ the unipotent radical, and let $L = MAW$ be the corresponding Langlands decomposition, with $L = MA$. Let $\delta_P : P(\mathbf{A}) \to \mathbb{R}^\times$ be the square root of the modulus character. Using the Iwasawa decomposition, one extends the function δ_P to a positive-valued function $\tilde{\delta}_P$ on $G(\mathbf{A})$ in the usual way (*cf.* [A], p. 254).

Let (Π, H_Π) be a cuspidal automorphic representation of $L(\mathbf{A})$, $\Pi \cong \otimes \Pi_v$, where v runs through the places of $\mathbb{Q}$. For $s \in \mathbb{C}$, define the (normalized) induced representation

$$\begin{aligned} (5.1.1) \qquad I_P(\Pi, s) = \{\varphi \in C^\infty(G(\mathbf{A}), H_\Pi) | \varphi(pg) \\ = \Pi(p)\, \delta_P(p)^{s+1} \cdot \varphi(g),\ p \in P(\mathbf{A}),\ g \in G(\mathbf{A})\}, \end{aligned}$$

where $\Pi(p)$ acts on H_Π through the projection of p modulo $W_P(\mathbf{A})$.

Let $I_P(\Pi, s)_0$ be the space of K_p-finite vectors in $I_P(\Pi, s)$; then for any $\varphi \in I_P(\Pi, s)_0$, $\varphi(g)(\cdot)$ is a cusp form on $L(\mathbf{A})$. Let $f_\varphi(g, s) = \varphi(g)(1)$; then f_φ is a function on

$W(\mathbf{A}) \cdot P(\mathbb{Q})\backslash G(\mathbf{A})$. For any $s \in \mathbb{C}$, multiplication by $(\tilde{\delta}_P)^{-s}$ defines a vector space isomorphism $\delta_s : I_P(\Pi, s)_0 \xrightarrow{\sim} I_P(\Pi, 0)_0 \cong \mathcal{I}$, say. We call a C^∞ function $f(g, s)$ on $G(\mathbf{A}) \times \mathbb{C}$ an *admissible section* if (i) for all s, $f(g, s) = \varphi_s(g)(1)$, for some $\varphi_s \in I_P(\Pi, s)_0$ (as a function of g) and (ii) $s \mapsto \delta_s(\varphi_s)$ is a holomorphic map from $\mathbb{C}$ to $\mathcal{I}$. Let $f(g, s)$ be an admissible section. If $\mathrm{Re}(s)$ is sufficiently large, the Eisenstein series

$$E(f, s, g) = \sum_{\gamma \in P(\mathbb{Q})\backslash G(\mathbb{Q})} f(\gamma g, s) \tag{5.1.2}$$

converges absolutely to an automorphic form on $G(\mathbb{Q})\backslash G(\mathbf{A})$, holomorphic in s. Thus there is a homomorphism

$$E : \quad I_P(\Pi, s)_0 \to \mathcal{A}(G) \tag{5.1.3}$$

of $(\mathfrak{g}, K_p) \times G(\mathbf{A}^f)$-modules, where $\mathcal{A}(G)$ denotes the space of automorphic forms on $G(\mathbb{Q})\backslash G(\mathbf{A})$.

The constant term $E(f, s, g)_F$ along F, defined by the integral (4.2.5), is computed as follows:

$$E(f, s, \cdot)_F(h) = f(h, s) + [M(w_0, s) \cdot f(\cdot, s)](h), \tag{5.1.4}$$

where $w_0 \in W^{F,p}$ is the unique element that normalizes L and such that $w_0(P) \cap P = L$, and $M(w_0, s) : \Pi \otimes \delta_P^s \to \Pi' \otimes \delta_P^{-s}$ is the corresponding intertwining operator [A], with $\Pi' = \Pi^{w_0}$. Indeed, f is a cusp form, so all remaining intertwining operators vanish on f.

Write $I_P(\Pi, s)_0 \cong I_P(\Pi_\infty, s)_0 \otimes \bigotimes I_P(\Pi_v, s)$, where v runs through the finite places and for v finite (resp. $v = \infty$) $I_P(\Pi_v, s) = \mathrm{Ind}_{P(\mathbb{Q}_v)}^{G(\mathbb{Q}_v)}[\Pi_v \otimes \delta_P^s]$ (resp. the K_p-finite vectors in $\mathrm{Ind}_{P(\mathbb{R})}^{G(\mathbb{R})}[\Pi_\infty \otimes \delta_P^s]$. Then for $\mathrm{Re}(s)$ sufficiently large –in particular, if $E(\star, s, \cdot)$ is absolutely convergent (*cf.* § 5.3), below– $I_P(\Pi_v, s)$ has a unique irreducible quotient $J_P(\Pi_v, s)$, the *Langlands quotient*, for all v. Define $J'(\Pi_v, s)$ by the exact sequence

$$0 \to J'(\Pi_v, s) \to I_P(\Pi_v, s) \to J_P(\Pi_v, s) \to 0, \tag{5.1.5}$$

and let $J'(\Pi, s)_v = J'(\Pi_v, s) \otimes \bigotimes_{w \neq v} I_P(\Pi_w, s)$. The theory of local intertwining operators then implies that, for $\mathrm{Re}(s)$ sufficiently large as above,

$$\sum_v J'(\Pi, s)_v \subset \mathrm{Ker}(M(w_0, s)). \tag{5.1.6}$$

For all this (*cf.* [A]; Sch, § 6).

5.2. Lifting cohomology classes.

We start with an element $w \in W^{F,p}$ and a non-zero cohomology class

$$[\omega] \in I_P^G[I^P\{\tilde{H}^{i-l(w)}((\mathcal{V}_{\lambda(h,w)})^{\mathrm{can}}) \otimes H^j(X(G_l), \tilde{\mathbf{V}}_{\lambda(l,w)})\}]$$

represented as in paragraph 4.2 by a differential form

$$\omega \in \mathrm{I}_{\mathrm{P}}^{\mathrm{G}}\,[\mathrm{I}^{\mathrm{P}}\,\{\mathrm{C}^{\bullet}_{\bar\partial}(\mathrm{G}_h,\,\mathrm{K}_h;\,\mathrm{V}_{\lambda\,(h,\,w)})_{\mathrm{sia}}\otimes \mathrm{C}^{\bullet}_{d}(\mathrm{G}_l,\,\mathrm{K}_l\cdot \mathrm{A},\,\mathrm{V}_{\lambda\,(l,\,w)})_{\mathrm{sia}}\}].$$

We assume ω to be a *cuspidal automorphic form*, associated to an automorphic representation $\Pi(\omega)$ of $\mathrm{L}(\mathbf{A})$, which is not necessarily irreducible. However, we assume for simplicity that $\Pi(\omega)\cong \Pi_\infty\otimes\Pi_f$, where Π_∞ is an irreducible $(\mathfrak{g}_h\times\mathfrak{g}_l,\,\mathrm{K}_h\cdot\mathrm{K}_l\cdot\mathrm{A})$-module,

Tracing back the identifications in paragraph 4.2, we can realize ω as an element of the space (4.2.6):

$$\omega\in[\mathrm{C}^\infty(\mathrm{P}(\mathbb{Q})\cdot\mathrm{W}(\mathbf{A})\backslash\mathrm{P}(\mathbb{R})\times\mathrm{G}(\mathbf{A}^f))_{\mathrm{sia}}\otimes\Lambda^{i+j}(\mathfrak{p}^-)^*\otimes\mathrm{V}_\lambda]^{\mathrm{K}_p\cap\mathrm{P}(\mathbb{R})},$$

and therefore (since $\mathrm{G}(\mathbb{R})=\mathrm{P}(\mathbb{R})\cdot\mathrm{K}_p$) as an element

$$\omega(\cdot)\in[\mathrm{C}^\infty(\mathrm{P}(\mathbb{Q})\cdot\mathrm{W}(\mathbf{A})\backslash\mathrm{G}(\mathbf{A}))_{\mathrm{sia}}\otimes\Lambda^{i+j}(\mathfrak{p}^-)^*\otimes\mathrm{V}_\lambda]^{\mathrm{K}_p}. \tag{5.2.1}$$

Let $\omega(g,\,s)=\omega(g)\cdot\tilde{\delta}_{\mathrm{P}}(g)^s$. For $\mathrm{Re}(s)$ sufficiently large we can perform Eisenstein summation, and define the Eisenstein series $\mathrm{E}(\omega,\,s,\,g)\in\mathcal{A}(\mathrm{G})$ by (5.1.2). If $\mathrm{E}(\omega,\,s,\,g)$ converges absolutely at $s=0$ then

$$\begin{aligned}\mathrm{E}(\omega,\,0,\,g)&\in[\mathrm{E}(\mathrm{I}_{\mathrm{P}}(\Pi(\omega),\,0)_0)\otimes\Lambda^{i+j}(\mathfrak{p}^-)^*\otimes\mathrm{V}_\lambda]^{\mathrm{K}_p}\\&=\mathrm{C}^{i+j}(\mathfrak{P}_p,\,\mathrm{K}_p;\,\mathrm{E}(\mathrm{I}_{\mathrm{P}}(\Pi(\omega),\,0)_0)\otimes\mathrm{V}_\lambda))\\&\subset\mathrm{C}^{i+j}(\mathfrak{P}_p,\,\mathrm{K}_p;\,\mathrm{C}^\infty(\mathrm{G}(\mathbb{Q})\backslash\mathrm{G}(\mathbf{A}))_{\mathrm{sia}}\otimes\mathrm{V}_\lambda)).\end{aligned}\tag{5.2.2}$$

Consider the following hypothesis on the intertwining operator $\mathrm{M}(w_0)$:

5.2.3. HYPOTHESIS. – The sum defining $\mathrm{E}(\omega,\,s)$ converges absolutely at $s=0$, and the constant term $\mathrm{E}(\omega,\,0)_{\mathrm{F}}$ along F equals ω.

It follows from diagram (4.2.4) that

5.2.4. PROPOSITION. – *Under hypothesis* 5.2.3, *suppose* $\mathrm{E}(\omega,\,0,\,g)$ *is a closed form in* $\mathrm{C}^{i+j}(\mathfrak{P}_p,\,\mathrm{K}_p;\,\mathrm{C}^\infty(\mathrm{G}(\mathbb{Q})\backslash\mathrm{G}(\mathbf{A}))_{\mathrm{sia}}\otimes\mathrm{V}_\lambda))$. *Then the cohomology class* $[\mathrm{E}(\omega,\,0)]\in\tilde{\mathrm{H}}^{i+j}(\mathcal{V}^{\mathrm{can}})$ *is not equal to zero, and in fact its restriction* $r_{\mathrm{F}}[\mathrm{E}(\omega,\,0)]$ *to the* F*-stratum of the boundary coincides with* $[\omega]$.

5.3. RATIONALITY IN THE ABSOLUTELY CONVERGENT RANGE.

The contents of the present section were developed in part in discussions with J. Franke. Let π be an automorphic representation of G. Although π need not be irreducible, we assume $\pi=\bigotimes\pi_v$ to be factorizable over the places of $\mathbb{Q}$, with π_v irreducible and unramified for almost all v. Furthermore, we assume that there exists a finite-dimensional irreducible representation $(\mu,\,\mathrm{W}_\mu)$ of G such that center $\mathrm{Z}(\mathfrak{g}_{\mathbb{C}})$ of the enveloping algebra of $\mathfrak{g}_{\mathbb{C}}$ acts on π_∞ and W_μ through the same character χ_∞. We write $\chi_\infty=\chi_\alpha$, for some $\alpha\in\mathfrak{h}^*_{\mathbb{C}}$, in terms of the Harish-Chandra homomorphism; thus $\chi_\alpha=\chi_{w(\alpha)}$ for all w in the Weyl group $\mathrm{W}(\mathfrak{g}_{\mathbb{C}},\,\mathfrak{h}_{\mathbb{C}})$. The set $\{w(\alpha)\}$ can be identified with the set of extreme weights of μ.

Let Q be any standard rational parabolic subgroup of G, not necessarily maximal; let L_Q be its Levi component, $A_Q = Z_{L_Q}$. We say Q *belongs to* (the rational boundary component) F if $G_{h,F} \subset Q \subset P_F$. Then $Q_l := Q \cap G_{l,F}$ is a parabolic subgroup of G_l, and the set of standard rational parabolics that belong to F is in one-one correspondence with the set of standard rational parabolics of $G_{l,F}$. If Q belongs to F, then the sum $\mathfrak{h}_{h,\mathbb{C}} + c_F(\mathfrak{h}_{l,\mathbb{C}})$ is a Cartan subalgebra of L_Q (*cf.* 3.6). It follows from our conventions that $\mathrm{Lie}(Z_G)_{\mathbb{C}} = \mathfrak{h}_{h,\mathbb{C}} \cap c_F(\mathfrak{h}_{l,\mathbb{C}})$ but that $\mathfrak{a}_{Q,\mathbb{C}} := \mathrm{Lie}(A_Q)_{\mathbb{C}} = \mathfrak{a}^0_{Q,\mathbb{C}} \oplus \mathrm{Lie}(Z_G)_{\mathbb{C}}$, with $\mathfrak{a}^0_{Q,\mathbb{C}} = \mathrm{Lie}(A_Q)_{\mathbb{C}} \cap \mathrm{Lie}(G^{\mathrm{der}})_{\mathbb{C}}$. For $* = h, l$, we write $\mathfrak{h}_{*,\mathbb{C}} = (\mathfrak{h}_{*,\mathbb{C}} \cap \mathfrak{a}^0_{Q,\mathbb{C}}) \oplus \mathfrak{h}_{*,\mathbb{C}}$, where $\mathfrak{h}_{*,\mathbb{C}} = \mathfrak{h}_{*,\mathbb{C}} \cap \mathrm{Lie}(L_Q^{\mathrm{der}})_{\mathbb{C}}$. Given $\chi_\infty = \chi_\alpha$ as above, write

$$(5.3.1)\qquad \begin{cases} \alpha = \alpha(h, Q) + c_F(\alpha(l, Q) + \nu(\alpha, Q)), \\ \text{with} \\ \alpha(h, Q) \in \mathfrak{h}^*_{h,\mathbb{C}},\ \alpha(l, Q) \in \mathfrak{h}^*_{l,\mathbb{C}},\ \nu(\alpha, Q) \in \mathfrak{a}^*_{Q,\mathbb{C}}. \end{cases}$$

5.3.2. DEFINITION. – *We say α is* very convergent *if, for every $\mathbb{Q}$-parabolic* $Q \subset G$, $\nu(\alpha, Q) > \rho_Q$, *relative to the ordering on $\mathfrak{a}^*_{Q,\mathbb{C}}$ given by the roots of* Q. *Here ρ_Q is the half-sum of roots of* (*the unipotent radical of*) Q. *The automorphic representation π is* very convergent *if the corresponding α is. Finally, the automorphic vector bundle $\mathcal{V} = \mathcal{V}_\lambda$, where λ is the representation of K_p with highest weight Λ, is called* very convergent *if its associated infinitesimal parameter $-\Lambda - \rho \in \mathfrak{h}^*_{\mathbb{C}}/W(\mathfrak{g}_{\mathbb{C}}, \mathfrak{h}_{\mathbb{C}})$ is very convergent, with ρ as in* 3.6.

5.3.3. *Remark*. – Here we recall that, if an automorphic representation π contributes to the coherent cohomology of $\mathcal{V}_\lambda$, then the infinitesimal character of π_∞ equals $\chi_{(-\Lambda-\rho)}$ ([H5], Prop. 4.3.2).

The complex space $\mathfrak{h}^*_{\mathbb{C}}$ has a $\mathbb{Q}$-structure defined by the lattice of algebraic characters of H. The parameter α belongs to this lattice. The following Lemma was pointed out to us by J. Franke.

5.3.4. LEMMA. – *There is a finite set of hyperplanes $\{H_i\}$ in $\mathfrak{h}^*_{\mathbb{C}}$, rational relative to the $\mathbb{Q}$-structure defined above, such that α is very convergent if and only if $\alpha \notin H_i$ for all i.*

Proof. – We may take α to be in the positive Weyl chamber. By hypothesis, α belongs to the lattice of characters of H. Definition 5.3.2 thus excludes a finite set of hyperplanes for each Q.

As Franke pointed out, the excluded hyperplanes are not necessarily root hyperplanes.

The analogous definition can be made for spherical functions on the groups of p-adic points of G. Let $P_{0,p} \subset G_p$ be a minimal $\mathbb{Q}_p$-parabolic subgroup, with respect to which the standard $\mathbb{Q}$-parabolic subgroups are still standard. Let $A^0_{0,p}$ be a split component which is compatible with the A_Q's above, $A_{0,p} = A^0_{0,p} \cdot Z_{G_p}$. The Satake transform associates to any spherical representation π_p of G_p (relative to a special maximal compact subgroup) a vector $\alpha(\pi_p)$ in $X^*(A_{0,p}) \otimes \mathbb{C}$. If $Q \subset G$ is a standard $\mathbb{Q}$-parabolic subgroup as above, we can write

$$(5.3.5)\qquad X^*(A_{0,p}) \otimes \mathbb{C} \cong X^*(A_{0,p} \cap L_Q^{\mathrm{der}}(\mathbb{Q}_p)) \otimes \mathbb{C} \bigoplus X^*(A_Q) \otimes \mathbb{C}.$$

5.3.6. DEFINITION ([H1], 1.1). – *Let π_p be as above. We say π_p is* convergent *for* Q *if, in the decomposition* (5.3.5), $\alpha(\pi_p) = \alpha(\pi_p, Q) + \nu(\alpha(\pi_p), Q)$, *where*

(i) $\alpha(\pi_p, Q)$ *corresponds to a bounded spherical function on* $L_Q^{der}(\mathbb{Q}_p)$, *and*

(ii) $\mathrm{Re}(\nu(\alpha(\pi_p), Q)) > \rho_Q$.

5.3.7. *Remark.* – We could also require $\alpha(\pi_p, Q)$ to correspond to a *unitarizable* spherical representation. When $G(\mathbb{Q}_p)$ has Kazhdan's property (T), this imposes a significantly stronger constraint on π_p than the one used in [H1], and this in turn implies a stronger rationality theorem than Theorem 5.3.11, below. We leave this for another occasion.

It follows from [H1], Proposition 1.2 that

5.3.8. LEMMA. – *Any given π_p is convergent for at most one* (*not necessarily proper*) *parabolic* Q.

Langlands' theory of Eisenstein series defines a natural decomposition of the space $\mathcal{A}(G)$ of automorphic forms on $G(\mathbb{Q})\backslash G(\mathbf{A})$, the *Q-decomposition*:

$$\mathcal{A}(G) = \bigoplus_{Q} \mathcal{A}(G)_Q. \tag{5.3.9}$$

Here $\mathcal{A}(G)_Q$ is the space of all automorphic forms arising as constituents of the space of Eisenstein series attached to cusp forms on $L_Q(\mathbf{A})$, and the sum is taken over associate classes of standard parabolic subgroups Q. In particular, $\mathcal{A}(G)_G = \mathcal{A}_0(G)$ is the space of cusp forms on G.

5.3.10. LEMMA. – *Let $\pi \subset \mathcal{A}(G)_Q$ and $\pi' \subset \mathcal{A}(G)_{Q'}$ be automorphic representations with $\pi_p \cong \pi'_p$ for some unramified place p. Suppose π_∞ and π'_∞ are very convergent. Then* $Q = Q'$.

Proof. – Suppose not. Write the Langlands decomposition $Q = M_Q \cdot A_Q \cdot N_Q$, with $M_Q = L_Q^{der}$; $Q' = M_{Q'} \cdot A_{Q'} \cdot N_{Q'}$. It follows from the hypothesis that π (resp. π') is isomorphic to a constituent of the normalized induced representation $\mathrm{Ind}_Q^G(\tau \otimes \nu \otimes 1_{N_Q})$ (resp. $\mathrm{Ind}_{Q'}^G(\tau' \otimes \nu' \otimes 1_{N_{Q'}})$), where τ (resp. τ') is a cuspidal automorphic representation of M_Q (resp. $M_{Q'}$), $\nu \in X^*(A_Q) \otimes \mathbb{C}$ satisfies $\nu > \rho_Q$ (resp. $\nu' > \rho_{Q'}$), and 1_{N_Q} and $1_{N_{Q'}}$ are the trivial representations. In particular, for every unramified prime *p*, the local factor π_p (resp. π'_p) is convergent for Q (resp. for Q′). Now apply Lemma 5.3.8.

Lemma 5.3.10 provides an effective form of what Harder calls the *Manin-Drinfeld principle*, for the cohomology of Sh(G, X) with coefficients in the local system attached to W_μ, provided the infinitesimal character of μ does not lie on one of the hyperplanes in 5.3.4. In particular, the lemma shows that the action of $G(\mathbf{A}^f)$ separates the pieces of automorphic cohomology coming from different pieces of the Q-decomposition. Since the action of $G(\mathbf{A}^f)$ respects the rational structure of cohomology, this implies that the corresponding pieces of cohomology are rational. In particular, this provides a way of showing that Eisenstein cohomology classes are rational.

The same argument applies to coherent cohomology; the case of holomorphic automorphic forms was the subject of [H1]. Here is the result, which is certainly not the strongest possible. Our proof makes use of the recent difficult theorem of Franke [F], which shows that all cohomology of Sh(G, X) (coherent or with twisted coefficients) is represented by automorphic forms. This was not necessary in the holomorphic case, and it may not be necessary in general. It also makes use of results to be proved in Part III of this paper, and in this sense should be considered provisional, except when G^{der} is of $\mathbb{Q}$-rank 1, where the arguments of Part I already suffice.

5.3.11. THEOREM. – *Suppose the automorphic vector bundle $\mathcal{V} = \mathcal{V}_\lambda$ is very convergent, in the sense of Definition* 5.3.2. *Let*

$$[\omega] \in I_P^G [I^P \{\tilde{H}^{i-l(w)} ((\mathcal{V}_{\lambda(h,w)})^{can}) \otimes H^j (X(G_l), \tilde{\mathbf{V}}_{\lambda(l,w)})\}],$$

be defined by a cuspidal automorphic form ω, as in paragraph 5.2. *Suppose*

(*a*) $\omega(g, 0) \in J'(\Pi(\omega), 0)_\infty$;

(*b*) *the sum defining* $E(\omega, s)$ *converges absolutely at* $s=0$;

(*c*) $E(\omega, 0, \cdot)$ *is a closed form in* $C^{i+j}(\mathfrak{P}_p, K_p; C^\infty(G(\mathbb{Q})\backslash G(\mathbf{A}))_{sia} \otimes V_\lambda))$;

(*d*) *The class* $[\omega]$ *is rational over a finite extension* $\mathcal{L}$ *of* $E(\mathcal{V}_\lambda)$.

Then for all $\sigma \in \mathrm{Aut}(\mathbb{C}/\mathcal{L})$, $[E(\omega, 0)]^\sigma = [E(\omega, 0)] \in \tilde{H}^{i+j}((\mathcal{V}_\lambda)^{can})$.

Proof. – We may assume ω belongs to a finite sum of irreducible cuspidal representations $\bigoplus \tau_i \otimes \nu_i$ of $L = L_P$, where τ_i and ν_i are as in the proof of Lemma 5.3.10. Since ω is realized in coherent cohomology, it follows easily that ν_i is an algebraic Hecke character of $A = A_P$ for all *i*, and since $\mathcal{V}_{\lambda(h,w)}$ is associated to an irreducible representation, we may even assume that the archimedean components of the ν_i all coincide. For any prime *p*, let $\tau_{i,p}$ and $\nu_{i,p}$ be the corresponding representations of $L(\mathbb{Q}_p)$. It follows from our hypotheses that, for each *p*, the natural action of $\mathrm{Aut}(\mathbb{C}/\mathcal{L})$ on representations of $L(\mathbb{Q}_p)$ permutes the sets of pairs $\{(\tau_{i,p}, \nu_{i,p})\}$. Thus,

(5.3.11.1) For all *p* at which $E(\omega, 0)$ is unramified, $\mathrm{Aut}(\mathbb{C}/\mathcal{L})$ permutes the local components $\{I_{p,i}\}$ of the induced representations $\mathrm{Ind}_P^G(\tau_i \otimes \nu_i \otimes 1_{N_P})$.

Let $\sigma \in \mathrm{Aut}(\mathbb{C}/\mathcal{L})$, and let $E' = [E(\omega, 0)]^\sigma - [E(\omega, 0)]$. We claim that E' is represented by cohomology classes in $\mathcal{A}(G)$. Indeed, Franke has proved that $\tilde{H}^\bullet((\mathcal{V}_\lambda)^{can})$ is entirely represented by automorphic forms, and that

$$\tilde{H}^\bullet((\mathcal{V}_\lambda)^{can}) \cong \bigoplus_Q H^\bullet(\mathfrak{P}_p, K_p; \mathcal{A}(G)_Q \otimes V_\lambda)). \tag{5.3.11.2}$$

This implies the claim. Moreover, since the action of $G(\mathbf{A}^f)$ on $\tilde{H}^\bullet((\mathcal{V}_\lambda)^{can})$ is rational over $E(\mathcal{V}_\lambda)$, it follows from (5.3.11.1) that

(5.3.11.3) For all $\sigma \in \mathrm{Aut}(\mathbb{C}/\mathcal{L})$ and almost all *p*, E' belongs to a sum of automorphic representations π_j whose local components at *p* are convergent for P.

We can ignore the π_j that do not contribute to cohomology. Our hypothesis that $\mathcal{V}_\lambda$ is very convergent then implies that, for all *j*, the archimedean component $\pi_{j,\infty}$ of π_j is very convergent. It now follows from (5.3.11.3) and Lemma 5.3.10 that the π_j all belong to $\mathcal{A}(G)_P$.

Let $P = P_F$. By Theorem 4.7.1, $r_F([E(\omega, 0)]^\sigma) = r_F([E(\omega, 0)])^\sigma$, and by Proposition 5.2.4 the latter equals $[\omega]^\sigma = [\omega]$. Thus $r_F(E') = 0$. The Theorem is now a consequence of the following lemma:

5.3.12. Lemma. – *Let* $P = P_F$, *and let* $c \in H^\bullet(\mathfrak{P}_p, K_p; \mathcal{A}(G)_P \otimes V_\lambda)) \subset \tilde{H}^\bullet((\mathcal{V}_\lambda)^{can})$ *have the property that* $r_F(c) = 0$. *Suppose* $\mathcal{V}_\lambda$ *is very convergent. Then* $c=0$ *as a cohomology class.*

The proof relies on an argument familiar from the theory of Eisenstein cohomology with twisted coefficients, but which in the context of coherent cohomology depends on results to be proved in Part III. Since $\mathcal{V}_\lambda$ is very convergent, the interior cohomology $\bar{H}^\bullet(Sh(G, X), \mathcal{V}_\lambda)$ is represented entirely by cusp forms. Indeed, it was proved in [H5], § 5 that this is true provided the highest weight λ is sufficiently regular. But in fact, as Franke pointed out to us, it suffices to assume that, for every root β of G we have

$$|\langle \lambda, \beta \rangle| > 1 \quad (cf. \text{[F]}). \tag{5.3.12.1}$$

This is clearly the case when $\mathcal{V}_\lambda$ is very convergent. (A similar suggestion was made by F. L. Williams.)

Since c is orthogonal to cusp forms, it follows that, if $c \neq 0$, then the image of c in $\overset{\lim}{\rightarrow} H^\bullet(\partial Sh_\Sigma, \mathcal{V}_\lambda \otimes \mathcal{O}_{\partial Sh_\Sigma})$ is non-trivial; here the limit is taken over compact open $K \subset G(\mathbf{A}^f)$ and admissible toroidal compactifications ${}_K Sh(G, X)_\Sigma$. As in [H4], we write $H^\bullet(\mathcal{V}_\lambda)(\infty) = \overset{\lim}{\rightarrow} H^\bullet(\partial Sh_\Sigma, \mathcal{V}_\lambda \otimes \mathcal{O}_{\partial Sh_\Sigma})$.

Now at finite level $Sh_\Sigma = {}_K Sh(G, X)_\Sigma$, we can compute the coherent cohomology of ∂Sh_Σ by the closed cover of its Q-strata $\overline{Sh}_\Sigma^Q$, where Q runs through the set of standard maximal parabolics. Thus, if Q is any standard parabolic, let $\overline{Sh}_\Sigma^Q = \bigcap_{\substack{Q' \supset Q \\ Q' \text{ maximal}}} \overline{Sh}_\Sigma^{Q'}$. Let $i_Q : \overline{Sh}_\Sigma^Q \hookrightarrow Sh_\Sigma$ be the corresponding closed embedding. Let $r(Q)$ denote the parabolic rank of Q. Then the nerve spectral sequence for this closed cover of ∂Sh_Σ takes the form

$$E_1^{r,s} = \bigoplus_{r(Q)=r} H^s(\overline{Sh}_\Sigma^Q, i_Q^*(\mathcal{V}_{\lambda,\Sigma})) \Rightarrow H^{r+s}(\partial Sh_\Sigma, \mathcal{V}_\lambda \otimes \mathcal{O}_{\partial Sh_\Sigma}). \tag{5.3.12.2}$$

Taking the limit over K and Σ, we obtain

$$E_1^{r,s} = \bigoplus_{r(Q)=r} \overset{\lim}{\rightarrow} H^s(\overline{Sh}_\Sigma^Q, i_Q^*(\mathcal{V}_{\lambda,\Sigma})) \Rightarrow H^{r+s}(\mathcal{V}_\lambda)(\infty). \tag{5.3.12.3}$$

The terms $E_1^{1,s}$ have been computed in the previous sections. Write

$$H^{s,Q}(\mathcal{V}_\lambda) = \overset{\lim}{\rightarrow} H^s(\overline{Sh}_\Sigma^Q, i_Q^*(\mathcal{V}_{\lambda,\Sigma})).$$

In Part II we will show that, for all Q,

$$\begin{aligned} &H^{\bullet,Q}(\mathcal{V}_\lambda) \\ &\quad \cong \bigoplus_{w \in W^Q} I_Q^G [I^Q \{\tilde{H}^{\bullet - l(w)}((\mathcal{V}_{\lambda(h_Q,w)})^{can}) \otimes H^\bullet(X(G_{l,Q}), \tilde{\mathbf{V}}_{\lambda(l,w)})\}]. \end{aligned} \tag{5.3.12.4}$$

Here the terms are defined as follows. Order the standard maximal parabolics P_F by setting $P_F < P_{F'}$ if $F < F'$. For simplicity we assume G to be $\mathbb{Q}$-simple; then this defines a total ordering. Let $P_{F(0)}(Q)$ be the standard maximal parabolic containing Q which is minimal for this ordering. Then $G_{l, F(0)} \cap Q$ is a maximal parabolic in $G_{l, F(0)}$, $G_{l, Q}$ is a standard Levi component, and $G_{h_Q} = G_{h, F(0)}$. Then $G_{l, F(0)} \cap Q$ determines a maximal parabolic subalgebra of $\mathfrak{k}_{p, \mathbb{C}}$ by Cayley transform, as in 3.6, and W^Q is the corresponding subset of $W(\mathfrak{k}_{p, \mathbb{C}}, \mathfrak{h}_{\mathbb{C}})$. For each $w \in W^Q$, $\mathcal{V}_{\lambda(h_Q, w)}$ is an automorphic vector bundle on $\mathrm{Sh}(G_{h, F(0)}, X(F(0)))$, and $\tilde{V}_{\lambda(l, w)}$ is a local system on the (adelic) locally symmetric space $X(G_{l, Q})$ associated to $G_{l, Q}$. The intermediate induction I^Q and the induction I_Q^G are defined by analogy with 4.1-2.

The differentials in the spectral sequence (5.3.12.3) have a simple expression in terms of the description (5.3.12.4). However, for our purposes, it suffices to remark that, if $\mathcal{V}_\lambda$ is very convergent, and if π_f is the representation of $G(\mathbf{A}^f)$ corresponding to $\pi \subset \mathcal{A}(G)_P$, then π_f does not intertwine with the $G(\mathbf{A}^f)$-action on $H^{s, Q}(\mathcal{V}_\lambda)$ for any $Q \neq P$. Indeed, by Franke's theorems [F] applied to G_{h_Q} (for coherent cohomology) and $G_{l, Q}$ (for cohomology with twisted coefficients), the right-hand side of (5.3.12.4) is a sum of representations $\tilde{\pi}_f$ of $G(\mathbf{A}^f)$ such that, for almost all p, the local component $\tilde{\pi}_p$ is convergent for some standard parabolic $\tilde{Q} \subset Q$. At the same time, each $\tilde{\pi}_f$ is the finite part of an automorphic representation $\tilde{\pi}$ whose archimedean part is very convergent, by our hypothesis on $\mathcal{V}_\lambda$. It thus follows from Lemma 5.3.8 that, for almost all p, $\tilde{\pi}_p$ is not convergent for P. But our hypothesis on π implies that π_p is convergent for P, so the claim is clear.

We can project the spectral sequence (5.3.12.3) on its π_f-isotypic component. More precisely, we fix a compact open subgroup $K \subset G(\mathbf{A}^f)$ such that π_f has a K-fixed vector, project all terms in (5.3.12.3) on their subspaces of K-fixed vectors, which are finite dimensional. Then π_f corresponds to a finite-dimensional representation $V(\pi_f)$ of the Hecke algebra $\mathcal{H}(K)$ of $G(\mathbf{A}^f)$ relative to K; we then project on the maximal $\mathcal{H}(K)$-quotient all of whose Jordan-Holder components are isomorphic to $V(\pi_f)$. It follows from the previous remarks that this procedure annihilates all $H^{s, Q}(\mathcal{V}_\lambda)$ for $Q \neq P$. But the map from $\tilde{H}^\bullet((\mathcal{V}_\lambda)^{\mathrm{can}})$ to the image of $H^{s, P}(\mathcal{V}_\lambda)$ in $H^\bullet(\mathcal{V}_\lambda)(\infty)$ is given by r_F. Thus c maps trivially to $H^\bullet(\mathcal{V}_\lambda)(\infty)$, so $c=0$.

5.3.13. *Remark.* – The same proof shows that the analogue of Theorem 5.3.11 holds when, in hypothesis (*a*), $J'(\sigma, 0)_\infty$ is replaced by $J'(\sigma, 0)_v$ for any finite place v. The present formulation was chosen on the basis of a (perhaps misplaced) analogy with the case of holomorphic Eisenstein series, and on an absence of examples satisfying both (*a*) and (*c*).

5.4. A CONJECTURE ABOUT EMBEDDINGS OF DISCRETE SERIES.

It remains to show that Theorem 5.3.11 is not vacuous; in other words, that rational cohomology classes on the boundary can, in some instances, be lifted to elements of $J'(\sigma, 0)_\infty$ which define closed forms. (We note that this difficulty does not arise if $J'(\sigma, 0)_\infty$ is replaced by $J'(\sigma, 0)_v$ for some finite place v, as in Remark 5.3.13, since the latter space is defined as a rational subspace of cohomology). The basis of our construction was already explained in [H4], § 6. We assume henceforward that $P(\mathbb{R})$ is a *cuspidal* maximal parabolic subgroup of $G(\mathbb{R})$; this means that its Levi component $L(\mathbb{R})$

has a Cartan subgroup which is compact modulo the center $Z_L(\mathbb{R})$ of L. The hypothesis is restrictive but not empty. The set of $\mathbb{R}$-irreducible examples can be enumerated easily. If $G(\mathbb{R})^{\mathrm{der},0}$ is $\mathbb{R}$-simple and $\mathrm{rank}_{\mathbb{Q}} G = \mathrm{rank}_{\mathbb{R}} G$, then there is always a P_F, corresponding to a maximal boundary component F, for which G_l is a torus; such a P_F is cuspidal. Each hermitian symmetric domain of the form $\mathrm{Sp}(n, \mathbb{R})/\mathrm{U}(n)$, $\mathrm{SO}^*(2n)/\mathrm{U}(n)$, or $\mathrm{SO}(2p+1, 2)^0/\mathrm{SO}(2p+1) \times \mathrm{SO}(2)$ ([He], p. 518) has an additional boundary component stabilized by a cuspidal parabolic subgroup, for which $G_l(\mathbb{R})^{\mathrm{der}}$ is isogenous, respectively, to SL(2), SL(2), and $\mathrm{SO}(2p, 1)^0$. This exhausts the list of irreducible examples.

Since P is cuspidal, $L(\mathbb{R})$ has a discrete series. Let $\mathfrak{h}$ be the Cartan subalgebra of K_p chosen in 3.6, $K_{p,L} = K_p \cap L(\mathbb{R})$; we assume $\mathfrak{h}_L = \mathfrak{h} \cap \mathrm{Lie}(K_{p,L})$ is a Cartan subalgebra of $\mathrm{Lie}(K_{p,L})$. We use the same notation to designate discrete series representations of $G(\mathbb{R})$ and their associated $(\mathfrak{g}, K_p)$-modules; likewise for $L(\mathbb{R})$. For every discrete series representation π_∞ of $G(\mathbb{R})$, Blank constructs in [Bl] a discrete series representation Π_∞ of $L(\mathbb{R})$ and an explicit embedding $S: \pi_\infty \hookrightarrow I_P^G(\Pi_\infty \otimes 1_{N_P})$ as $(\mathfrak{g}, K_p)$-modules.

The existence of the embedding was previously proved by Schmid (unpublished). The identification of Π_∞ depends on three factors. Write L=MA, as in 5.3, and let M^0 and A^0 be the identity components of $M(\mathbb{R})$ and $A(\mathbb{R})$, respectively. Write $\Pi_\infty = \tau(\pi_\infty) \otimes \nu(\pi_\infty)$, where $\tau(\pi_\infty)$ is a discrete series representation of $M(\mathbb{R})$ and $\nu(\pi_\infty)$ is a character of A^0. Now the infinitesimal character of π_∞ determines that of $\tau(\pi_\infty)$, and determines $\nu(\pi_\infty)$ up to sign. The set of discrete series representations τ with given infinitesimal character is determined by two additional data: the Harish-Chandra parameter $\Lambda(\tau^0)$ of an irreducible constituent τ^0 of $\tau|_{M^0}$, which is an element of $\mathfrak{h}^*_{L,\mathbb{C}}$, and a certain sign character of $M(\mathbb{R})/M^0$, which determines an extension of τ^0 to an irreducible representation of $M(\mathbb{R})$.

Starting with the Harish-Chandra parameter $\Lambda = \Lambda(\pi_\infty) \in \mathfrak{h}^*_{\mathbb{C}}$ of π_∞, Blank lets $\Lambda(\tau^0) = \Lambda|_{\mathfrak{h}^*_{L,\mathbb{C}}}$. The sign character is defined explicitly, and is determined by the condition that the central characters of π_∞ and $I_P^G(\Pi_\infty \otimes 1_{N_P})$ coincide. Finally, of the two possible choices of $\nu(\pi_\infty)$, Blank chooses the one with the property that $I_P^G(\Pi_\infty \otimes 1_{N_P})$ has a (non-tempered) Langlands quotient. The explicit formulas can be found in [Bl], p. 128; however, Blank actually constructs a map from the dual of $I_P^G(\Pi_\infty \otimes 1_{N_P})$ onto the dual of π_∞, so there is a change of sign.

The discrete series representation π_∞ has $\bar{\partial}$-cohomology [H5] in a single degree $q(\Lambda)$, and only with coefficients in the representation V_λ with highest weight $\Lambda^* - \rho$, where Λ^* is the Harish-Chandra parameter of $(\pi_\infty)^*$ and ρ is the half-sum of positive roots. (N.B.: In [H4], [H5], Λ is used to designate highest weights of K_p-modules and λ denotes Harish-Chandra parameters.) Let $B(\Lambda)$ be the (unique) lowest K_p-type of π_∞; we use the same notation to denote its highest weight in $\mathfrak{h}^*_{\mathbb{C}}$ (*Blattner parameter*, cf. [BW], II, § 5). Let $b(\Lambda)$ (resp. v_λ) be a highest weight vector in $B(\Lambda)$ (resp. V^*_λ). Recall that $q(\Lambda)$ is the cardinality of

(5.4.1) $$Q(\Lambda) = \{\alpha \in R_n^+ | \langle \alpha, \Lambda^* \rangle > 0\}.$$

Let $\beta(\Lambda) = \bigwedge_{\alpha \in Q(\Lambda)} v^{-\alpha} \in \Lambda^{q(\Lambda)} \mathfrak{p}^-$, where $v^{-\alpha} \in \mathfrak{p}^-$ is a root vector for $-\alpha$. Then the cocycle $\tilde{\omega}_\infty \in C^{q(\Lambda)}(\mathfrak{P}_p, K_p; \pi_\infty \otimes V_\lambda) \cong \mathrm{Hom}_{K_p}(\Lambda^{q(\Lambda)} \mathfrak{p}^- \otimes V^*_\lambda, \pi_\infty)$, which generates $H^{q(\Lambda)}(\mathfrak{P}_p, K_p; \pi_\infty \otimes V_\lambda)$, is the K_p-equivariant extension of the unique $\mathfrak{h}_{\mathbb{C}}$-homomorphism which takes $\beta(\Lambda) \otimes v_\lambda$ to $b(\Lambda) \in B(\Lambda) \subset \pi_\infty$ (*cf.* [W] for these computations, but note that the conventions are not identical).

To guarantee hypothesis (*c*) of Theorem 5.3.11, we have to assume $i + j = q(\Lambda)$. At the same time, to guarantee the existence of ω with non-trivial cohomology, we have to assume $\tau(\pi_\infty) \cong \tau_h \otimes \tau_l$, corresponding to the decomposition $M = M_h \cdot M_l$, where $M_h = G_h \cap M$, $M_l = G_l \cap M$. Then $q(\Lambda(\tau_h)) = i - l(w)$, $\Lambda(\tau_h^*) = \lambda(h, w) + \rho_h$, and τ_l has $(\mathfrak{m}_l, K_l \cap M)$-cohomology in degree j with values in $V_{\lambda(l, w)}$, $\mathfrak{m}_l = \mathrm{Lie}(M_l)$. Since τ_l is a discrete series module, we must have $j = q_l := 1/2 \dim X(G_l)$ [BW]: II, 5.3 (dimension as a real manifold).

Blank's example corresponds to the case $w = 1$; we write $\lambda(h) = \lambda(h, 1)$, $\lambda(l) = \lambda(l, 1)$. Define $\Lambda(h)$, $\beta(\Lambda(h))$, $v_{\lambda(h)}$, and $b(\Lambda(h))$ for τ_h in analogy with the definitions for G, and let $\omega_h \in \mathrm{Hom}_{K_p}(\Lambda^{q(\Lambda(h))}\mathfrak{p}_h^- \otimes V_{\lambda(h)}^*, \tau_h)$ in analogy with the definition of $\tilde{\omega}_\infty$ above. The cocycle corresponding to τ_l lies in a different Lie algebra complex, namely $C^\bullet(\mathfrak{m}_l, K_l \cap M; \tau_l \otimes V_{\lambda(l)})$. Let $G_l^1 = G_l(\mathbb{R})^0 \cdot Z_{G_l(\mathbb{R})}$, $K_l^1 = K_l \cap M_l^1$, and let τ_l^1 be the irreducible $(\mathfrak{g}_l, K_l^1)$-submodule of Π_l corresponding to τ^0 (introduced above). Then (in the notation of 4.2)

(5.4.2)
$$\begin{aligned} &C^\bullet(\mathfrak{m}_l, K_l \cap M; \tau_l \otimes V_{\lambda(l)}) \\ &\cong C^\bullet(\mathfrak{m}_l, K_l^1; \tau_l^1 \otimes V_{\lambda(l)}) \cong \mathrm{Hom}_{K_l^1}(\Lambda^\bullet \mathfrak{p}_l \otimes V_{\lambda(l)}^*, \tau_h^1). \end{aligned}$$

Let $\Lambda(l) \in \mathfrak{h}_{l,\mathbb{C}}^*$ be the Harish-Chandra parameter of the discrete series representation τ_l^1, let $B(\Lambda(l)) \subset \tau_l^1$ be the Blattner K_l^1-type, $b(\Lambda(l)) \in B(\Lambda(l))$ (resp. $v_{\lambda(l)} \in V_{\lambda(l)}^*$) a highest weight vector. Finally, let

(5.4.2)
$$\beta(\Lambda(l)) = \bigwedge_{\alpha \in P_n^+(\Lambda(l))} w^{-\alpha} \in \Lambda^{q(l)} \mathfrak{p}_l,$$

where $P_n^+(\Lambda(l))$ is the set of *all* non-compact roots α for $\mathfrak{h}_{l,\mathbb{C}}$ such that $\langle \alpha, \Lambda(l) \rangle > 0$, and $w^{-\alpha}$ is a basis of the root space corresponding to $-\alpha$. In terms of (5.4.2), $H^{q(l)}(\mathfrak{m}_l, K_l \cap M; \tau_l \otimes V_{\lambda(l)})$ is generated by the K_l^1-equivariant extension ω_l of the unique $\mathfrak{h}_{l,\mathbb{C}} \cap \mathfrak{m}$-homomorphism which takes $\beta(\Lambda(l)) \otimes v_{\lambda(l)}$ to $b(\Lambda(l)) \in B(\Lambda(l)) \subset \tau_l^1$ (*cf.* [BW], proof of Theorem 5.3).

Let $\pi_\infty^1 = \tau_h \otimes \tau_l^1$. The local computation underlying the restriction maps (4.2.5-10) is given by the composite of three maps:

$$S: \quad \mathrm{Hom}_{K_p}(\Lambda^{q(\Lambda)}\mathfrak{p}^- \otimes V_\lambda^*, \pi_\infty) \mapsto C_1 := \mathrm{Hom}_{K_p}(\Lambda^{q(\Lambda)}\mathfrak{p}^- \otimes V_\lambda^*, I_P^G(\Pi_\infty \otimes 1_{N_P}))$$

(derived functorially from Blank's embedding);

$$T: \quad C_1 \to C_2 := \mathrm{Hom}_{K_h \cdot K_l^1}(\Lambda^{q(\Lambda)}\mathfrak{p}^- \otimes V_\lambda^*, \tau_\infty^1)$$

(restriction of functions from $G(\mathbb{R})$ to $M(\mathbb{R})$); and

$$\mathrm{U}: \quad \mathrm{C}_2 \to \mathrm{C}_3 := \mathrm{Hom}_{\mathrm{K}_h \cdot \mathrm{K}_l^1}([\Lambda^{q(\Lambda(h))}\, \mathfrak{p}_h^- \otimes \mathrm{V}^*_{\lambda(h)}] \otimes [\Lambda^\bullet \mathfrak{p}_l \otimes \mathrm{V}^*_{\lambda(l)}], \tau_h \otimes \tau_l^1),$$

obtained by composing Cayley transform with restriction; since $l(w) = 1$, $\mathrm{V}^*_{\lambda(h)} \otimes \mathrm{V}^*_{\lambda(l)}$ is naturally (and uniquely) a $\mathrm{K}_h \cdot \mathrm{K}_l^1$-submodule of V^*_λ.

Since $\tilde{\omega}_\infty$ is (up to scalars) the unique cohomology class in $\mathrm{H}^{q(\Lambda)}(\mathfrak{P}_p, \mathrm{K}_p; \pi_\infty \otimes \mathrm{V}_\lambda)$, Hypothesis (*c*) of 5.3.11 is equivalent to the conclusion of the following proposition:

5.4.3. PROPOSITION. – *Suppose* $q(\Lambda(h)) + q(l) = q(\Lambda)$. *Then* $\mathrm{U} \circ \mathrm{T} \circ \mathrm{S}(\tilde{\omega}_\infty)$ *is a non-zero multiple of* $\omega_h \otimes \omega_l$.

Proof. – Blank has verified [Bl], Prop. 4.1 that $\mathrm{B}(\Lambda(h)) \otimes \mathrm{B}(\Lambda(l))$ is representable uniquely as a $\mathrm{K}_h \cdot \mathrm{K}_l^1$-invariant subspace (or quotient) of $\mathrm{B}(\Lambda)$, in such a way that $b(\Lambda) = b(\Lambda(h)) \otimes b(\Lambda(l))$. We may similarly write $v_\lambda = v_{\lambda(h)} \otimes v_{\lambda(l)}$. It therefore suffices to verify that Cayley transform identifies $\beta(\Lambda)$ with $\beta(\Lambda(h)) \otimes \beta(\Lambda(l))$; equivalently, that

$$\mathrm{Q}(\Lambda) = \mathrm{Q}(\Lambda(h)) \amalg \mathrm{Ad}(c_\mathrm{F})^{-1}(\mathrm{P}_n^+(\Lambda(l))). \tag{5.4.4}$$

Our hypothesis is that both sides of (5.4.4) have the same cardinality, so it suffices to verify that the left-hand side contains the right-hand side. But this follows immediately from the definitions.

We have shown that Theorem 5.3.11 is non-vacuous, but the examples proposed have two obvious drawbacks. In the first place, we had to assume P_F cuspidal, which is quite rare; the theory of holomorphic Eisenstein series gives examples of embeddings for arbitrary F, where the representation of G_l is a character (*cf.* [H1]). Furthermore, the examples only work for $w = 1$. Little is known in general about the embedding of discrete series modules in induced modules, or rather the most general results are expressed in terms of the Kazhdan-Lusztig polynomials, and are therefore not directly accessible. Our arguments in paragraph 3 lead us to suspect that Blank's embeddings have a direct interpretation in terms of the geometry of the flag variety for G, and that they admit the following generalization. For brevity, write G for $\mathrm{G}(\mathbb{R})$, etc., and assume G semi-simple and connected. Let $\mathcal{F} \subset \mathfrak{h}_\mathbb{C}^*$ be the set of differentials of algebraic characters of H. Assume $\lambda \in \mathcal{F}$ is R_c^+-dominant, and $\Lambda := \lambda + \rho$ is $(\mathfrak{g}_\mathbb{C}, \mathfrak{h}_\mathbb{C})$-regular. Let $\Lambda(h, w) = \lambda(h, w) + \rho_h$. Let $\pi(\Lambda)$ (resp. $\pi(\Lambda(h, w))$) be the discrete series representation of G (resp. of G_h) in the Harish-Chandra parametrization. Define $q(\Lambda)$ and $q(\Lambda(h, w))$ as above. For $w \in \mathrm{W}^{\mathrm{F}, p}$, let

$$d(\Lambda, w) = q(\Lambda) - q(\Lambda(h, w)) - l(w).$$

Finally, let $\lambda(0, w) = \lambda(l, w)|_\mathfrak{a}$, and write $\mathrm{P} = \mathrm{P}_\mathrm{F}$.

5.4.5. CONJECTURE. – Suppose $\lambda(0, w) - \rho_\mathrm{P}$, as a character of $\mathfrak{a}$, is in the negative chamber relative to the parabolic subgroup P. Let σ be an irreducible $(\mathfrak{g}_l, \mathrm{K}_l)$-module such that $\mathrm{H}^{d(\Lambda, w)}(\mathfrak{g}_l, \mathrm{K}_l, \sigma \otimes \mathrm{V}_{\lambda(l, w)}) \neq \{0\}$. Then the discrete series module $\pi(\Lambda)^*$ embeds as a subrepresentation of $\mathrm{Ind}_\mathrm{P}^\mathrm{G}(\pi(\Lambda(h, w))^* \otimes (\rho_\mathrm{P} - \Lambda_0(w)) \otimes \sigma)$ (normalized induction).

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