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HELICES ON SOME FANO THREEFOLDS: CONSTRUCTIVITY OF SEMIORTHOGONAL BASES OF K_0

BY D. YU NOGIN

ABSTRACT. — The paper deals with the properties of helices on the level of the Grothendieck group $K_0(X)$. For Fano threefolds with Pic $X \cong \mathbb{Z}$ (the simplest from the point of view of helix theory) there are considered the *semiorthogonal bases* of $K_0(X)$, viewed as a $\mathbb{Z}$-module, which arise as a natural generalization of images in K_0 of foundations of helices.

For these threefolds equations are derived, which play the role analogous to that of Markov equation for helices on $\mathbb{P}^2$. With the help of these equations the semiorthogonal bases of K_0 are classified modulo action of mutations, *i.e.* the "K_0-constructivity" of helices is proved.

Preface

The notion of a helix was first introduced by A. L. Gorodentsev and A. N. Rudakov in [5] in connection with the problem of constructing of exceptional bundles on $\mathbb{P}^n$. It is shown in this paper and also in [3], [8], [15], that the exceptional bundles can be obtained one from another by canonical mutations in exceptional pairs. The principal difficulty therewith is to provide a large enough store of pairs for which the mutations are defined. The concept of a helix is used to avoid these difficulties.

According to [3], [5], a *helix on* $\mathbb{P}^n$ is an infinite periodical exceptional collection $\sigma = \{E_i\}$ of vector bundles (or coherent sheaves) of period $n+1$, such that for any element of it there exist multiple (left) mutations $L_\sigma^{(k)} E_i$ for $1 \leqq k \leqq n$, while $L_\sigma^{(n)} E_i = E_{i-n-1}$ and the collections $L_{E_i}^{(k)} \sigma$ are exceptional (we use here the notations of [3]). With such a definition, nontrivial is the fact proved in [3], that any mutation of a helix is also a helix, which means that some initial store of mutations provides the existence of further ones.

In the succeeding works ([2], [4], [14], [16]) the helix theory got its further progress. Thus, very promising was found the approach of A. L. Gorodentsev [4], who suggested to consider helices in the derived category of coherent sheaves over an arbitrary manifold. Such an approach on the one hand generalizes the notion of a helix on $\mathbb{P}^n$, and on the other hand allows to consider helices in any triangled category, where $\mathrm{Hom}^\bullet(E, F)$ for any two objects E, F has a structure of finite-dimensional graded vector space over $\mathbb{C}$. In particular, in [2] from this point of view there are studied the categories of representations of some classes of algebras.

Exceptional vector bundles were first considered in [9], where they were used for the description of moduli space of stable vector bundles on $\mathbb{P}^2$. There the exceptional bundles arose as stable bundles with discriminant $\Delta(E)<1/2$. Later J.-M. Drezet proved in [8] that this definition is equivalent to the following: a bundle E is called *exceptional*, if

$$\begin{cases} {}^0\langle E, E\rangle \cong \mathbb{C} \\ {}^i\langle E, E\rangle = 0 \quad \text{for} \quad i\neq 0, \end{cases}$$

where ${}^i\langle E, F\rangle$ denotes (as everywhere below) the space $\mathrm{Ext}^i(E, F)$.

Such a cohomological description of exceptional bundles allowed A. L. Gorodentsev and A. N. Rudakov to extend the notion of an exceptional bundle to $\mathbb{P}^n$ and other manifolds (*see*, for example, [3], [5], [16]). It is possible thereby to consider exceptional objects not only in the category of vector bundles, but in more general cases: in the category of coherent sheaves, and in the derived category of coherent sheaves. When studying vector bundles, such a transition to the greater amount of objects we consider has a certain purport because of the notion of a mutation of an exceptional pair and the concept of a helix (*see*, for example, [4]).

Remind that a pair (E, F) of exceptional objects is called *exceptional* if ${}^i\langle F, E\rangle = 0$ for all i. It is one of the most important properties of exceptional bundles on $\mathbb{P}^2$ that for an exceptional pair there can be defined a (left) mutation: for an exceptional pair (E, F) the kernel $L_E F$ of the canonical morphism

$${}^0\langle E, F\rangle \otimes E \xrightarrow{\text{lcan}} F$$

is exceptional, and the morphism itself is surjective. Moreover, the pair $(L_E F, E)$ is also exceptional; it is called the *left mutation* of the pair (E, F), and the object $L_E F$ itself is called the *left shift* of F in the pair. The mapping $(E, F) \mapsto (L_E F, E)$ is also called the left mutation, the left shift of F or the *transfer* of F over E.

The *right mutation* is defined dually with the help of the morphism $\text{rcan} = \text{lcan}^*(F^*, E^*)$. The mapping rcan is inverse to lcan, and vice versa.

Mutations of exceptional vector bundles on $\mathbb{P}^2$ first appeared in [8], although without the terminology.

For an exceptional pair of bundles on $\mathbb{P}^n$ the surjectivity of lcan (correspondingly, injectivity of rcan) is proved only provided that the pair is included in a constructive helix [3]. Here the helix is called *constructive*, if it can be obtained by successive mutations from the canonical helix $\{\mathcal{O}(i)\}$.

When studying exceptional pairs of sheaves on a two-dimensional quadric $\mathbb{P}^1 \times \mathbb{P}^1$ (*see* [16], [4]), one can note that the canonical morphism lcan can be not surjective. It can be injective; in this case one can easily check that the cokernel of the morphism is exceptional, and it can be regarded as a left shift in the pair. The mutation of such type is called a *recoil*, as distinguished from the case of the mutation of *division* type, described above. The case is also possible, when ${}^0\langle E, F\rangle = 0$, then the result of a mutation is an exceptional sheaf $\mathscr{E}$, obtained as the universal extension

$$0 \to F \to \mathscr{E} \to {}^1\langle E, F\rangle \otimes E \to 0$$

Such a mutation of *extension* type is inverse to a mutation of recoil type.

These three cases were combined by A. L. Gorodentsev [4] into the concept of a mutation in the derived category of coherent sheaves. An object A of the derived category is called *exceptional*, if

$$\mathrm{Hom}^{\bullet}(A, A) = \mathbb{C}^0,$$

and the result of a (left) mutation is the cone of the (left) canonical morphism, which, as Gorodentsev proves, is also an exceptional object.

An ordered pair (A, B) of exceptional objects is called *exceptional*, if $\mathrm{Hom}^{\bullet}(B, A) = 0$.

Undoubtedly important is the concept of a helix in the derived category, introduced by Gorodentsev. Remind that an ordered collection of objects is called *exceptional*, if any ordered pair, being a subcollection of it, is exceptional. The *helix* of period n in the derived category of coherent sheaves on a manifold X is an infinite ordered collection $\sigma = \{A_i\}$, which satisfy the following conditions:

1) Any n successive elements of the collection σ form an exceptional collection (called a *foundation* of a helix),

2) $A_{i-n} = L_{\sigma}^{(n-1)} A_i$, where $L_{\sigma}^{(n-1)} A_i$ is a $(n-1)$-th left shift of A_i in σ.

The result of a shift of A_i in a helix is a collection, obtained from it by corresponding shifts of all elements A_k, $k \equiv i \pmod n$. Therefore, condition 2) may be written as $L_{A_i}^{(n-1)} \sigma = \sigma$.

The connection between the concept of a helix and the structure of the derived category was revealed by A. I. Bondal [2]. He has proved the following theorem:

Let X be a Fano manifold, $(A_0, \ldots, A_{n-1})$ be an exceptional collection of objects of $D^b(\mathrm{Sh}(X))$ be the bounded derived category of coherent sheaves on X. Then the following conditions are equivalent:

1) The collection $(A_0, \ldots, A_{n-1})$ generates $D^b(\mathrm{Sh}(X))$,

2) An infinite collection $\{A_i\}$, defined periodically by the condition $A_{i-n} = A_i \otimes \omega_X [\dim X]$, is a helix of period n.

Here ω_X is a canonical class of X, and a number in square brackets denotes the multiplicity of the shift of an object to the left viewed as a graded complex. Such helices are called the *complete* helices on a manifold X.

If all the objects A_i have only one non-zero cohomology, *i.e.* if they are represented by sheaves, then the corresponding helix $\{E_i\}$ in the category of sheaves obey the property $E_{i-n} = E_i(\mathscr{K}_X)$, where $\mathscr{K}_X$ is the canonical divisor. An example of a complete sheaf helix may be given by the helix $\{\mathcal{O}(i)\}$ of period $n+1$ on $\mathbb{P}^n$: according to [1], the collection $\{\mathcal{O}, \ldots, \mathcal{O}(n-1)\}$ generates $D^b(\mathrm{Sh}(\mathbb{P}^n))$, and according to [3] all mutation of this helix are possible in the category of sheaves. Collections which generate the derived category are described also by M. M. Kapranov ([10], [11], [12]) for Grassmannians, quadrics and flag manifolds. A symmetric sheaf helix on the Grassmannian $G(2, 4)$ is constructed by B. V. Karpov [13].

For a foundation of a complete helix on a manifold X there is valid an analogue of the Beilinson theorem – there exists a spectral sequence associated with the foundation, which generalizes the Beilinson spectral sequence for the helix $\{\mathcal{O}(i)\}$ on $\mathbb{P}^n$ ([3], [5], compare with [1], [8]).

The possibility to chose suitable foundations of helices and to mutate the foundations opens wide prospects in problems of constructing of moduli spaces, since it allows to obtain the most convenient representations for a given sheaf, and also to mutate the moduli spaces themselves. Thus, on this way J.-M. Drezet achieved a considerable success in constructing moduli spaces of stable vector bundles on $\mathbb{P}^2$ and studying their geometry ([6], [7]).

The existence of Beilinson-type spectral sequences, associated with a foundation of a complete helix, provide the fact that the images in $K_0(X)$ of elements of a foundation of a complete helix form a basis of $K_0(X)$ viewed as a $\mathbb{Z}$-module. On $K_0(X)$ there is defined an integer bilinear form $\langle \bar{E}, \bar{F} \rangle = \chi(E, F)$ where $\chi(E, F)$ is the Euler characteristic, equal to

$$\sum_i (-1)^i \cdot \dim^i \langle E, F \rangle$$

for classes $\bar{E}$, $\bar{F}$ represented by sheaves E, F; or, in more general case, to

$$\sum_i (-1)^i \cdot \dim H^i(\mathrm{Hom}^\cdot(E, F))$$

for objects of the derived category.

Since the elements of a foundation of a helix form an exceptional collection, any basis $(e_0, \ldots, e_n)$ of the module $K_0(X)$, obtained as an image of some foundation of a helix, satisfy the conditions

$$\langle e_i, e_i \rangle = 1, \qquad \langle e_j, e_i \rangle = 0 \qquad \text{for} \quad j > i.$$

The bases, which satisfy these conditions, are called *semiorthogonal*.

On the set of all semiorthogonal bases, as well as on the set of helices (*see* [4, (4.4.1)]), there acts by left mutations the Artin braid group: the action of a generator of the group on a basis $(e_0, \ldots, e_n)$ is defined as an elementary mutation, which maps the pair (e_i, e_{i+1}) to

$$(\langle e_i, e_{i+1} \rangle e_i - e_{i+1}, e_i).$$

This definition of mutation in K_0 correlates with the corresponding definition in the category of sheaves up to changing sign:

$$L_{\bar{E}} \bar{F} = \pm \overline{L_E F}.$$

(The sign depends on the type of a sheaf mutation). The line denotes here the class of a sheaf in K_0; further on we as a rule omit it.

In connection with the problem of classifying the complete helices it is natural to consider that of to classify the semiorthogonal bases, *i. e.* to select the set of the simplest bases (further on we call them *canonical* bases) and to describe the set of *constructive* bases, which can be obtained from these canonical ones by successive mutations.

In this paper there is studied the problem of constructivity of the semiorthogonal bases of K_0 for Fano threefolds, which are the simplest from the point of view of helix theory, – for the threefolds with $\operatorname{Pic} X \cong \mathbb{Z}$. A complete helix on such a threefold must have period 4. There exist four kinds of such manifolds: the projective space $\mathbb{P}^3$, a smooth quadric Q, the manifold V_5 and the family V_{22}.

For $\mathbb{P}^3$, Q and manifolds V_5 there are known the examples of complete sheaf helices. For $\mathbb{P}^3$ it is the helix $\{\mathcal{O}(i)\}$ according to [3]. For a quadric Q M. M. Kapranov [10] has constructed exceptional collections, which generate the derived category of coherent sheaves and therefore, taking into account the theorem of A. I. Bondal [2] given above, are foundations of some complete helices. The example of such a collection gives

$$(\mathcal{O}, \mathcal{S}^*, \mathcal{O}(1), \mathcal{O}(2)),$$

where $\mathcal{S}$ is the spinor bundle. One can easily check using some formulas from [10] that this helix is a sheaf one.

The example of a complete sheaf helix on V_5 was recently constructed by D. O. Orlov – it is the helix with the foundation

$$(\mathcal{O}, \mathcal{Q}, \mathcal{S}^*, \mathcal{O}(1)),$$

where $\mathcal{S}$ and $\mathcal{Q}$ are correspondingly the restrictions to V_5 of the universal bundle and the factor bundle on the Grassmannian $G(2, 5)$ in case of V_5 realized as the intersection of the image of the Plücker embedding $G(2, 5) \subsetneq \mathbb{P}^9$ with a general subspace $\mathbb{P}^6 \subset \mathbb{P}^9$. The examples of complete helices on V_{22} are not yet known.

The problem of the classification of the semiorthogonal bases of $K_0(X)$ is, in fact, a Diophantine one. In the simplest cases it can be reduced to that of to solve one Diophantine equation. Thus, A. N. Rudakov [15] discovered that the ranks of the elements of a foundation of a helix on $\mathbb{P}^2$ satisfy the Markov equation

$$x^2 + y^2 + z^2 = 3xyz, \tag{1}$$

moreover, a mutation of a helix corresponds to a mutation of a numerical solution of the equation $((x, y, z) \mapsto (3yz - x, y, z))$. Thereby, in order to prove the constructivity of all helices it is sufficient to prove that all numerical solutions of the Markov equation form one orbit modulo action of mutations (up to multiplying two elements of a Markov triple by -1, which in geometrical situation corresponds to the shift of graduation in the derived category). In similar case, when describing symmetric helices on a quadric [16], there is used the equation

$$x^2 + y^2 + 2z^2 = 4xyz, \tag{2}$$

which holds for the ranks of elements of a symmetric helix. Certainly, the transition from solving a Diophantine problem to geometrical constructivity is possible only when the corresponding geometrical properties of exceptional objects are studied.

The main tool when proving constructivity of the semiorthogonal bases is the method of Markov-type equations, which generalizes the equations (1), (2) for helices of period 3. For helices of greater period a Markov-type equation for an exceptional pair (E_1, E_2) is a Diophantine one, which holds for the ranks of elements of the pair and some invariant of the pair. An invariant of a pair is an integer function of elements of it, which is invariant under mutations.

For equations (1), (2) such an invariant is

$$\dim \operatorname{Hom}(E_1, E_2),$$

which is equal correspondingly to $3z$, tripled rank of the third element of the foundation of the helix in equation (1), and to $4z$, taken four times rank of the non-diagonal bundle in the foundation of the symmetric helix in equation (2). And if by z we denote $\dim \operatorname{Hom}(E_1, E_2)$, then equations (1), (2) would be rewritten in the form

$$x^2+y^2+\frac{z^2}{\mathcal{K}^2}=xyz, \tag{3}$$

where $\mathcal{K}^2$ is the selfintersection number of the canonical divisor, equal to 9 and 8 correspondingly. Similar equation for rational ruled surfaces was obtained by the author in [14].

In this paper there is derived and applied when proving constructivity of the semiorthogonal bases the following Markov-type equation for threefolds listed above:

$$x^2+y^2+2\frac{(z+p)^2}{(-\mathcal{K})^3}=xyz. \tag{4}$$

Here also $z=\langle E_1, E_2 \rangle$, $x=\operatorname{rk} E_1$, $y=\operatorname{rk} E_2$; p is a parametre. This equation was first obtained by A. I. Bondal. However, for the proof of constructivity of the semiorthogonal bases it is more convenient to take as z some other invariant of a pair for which the corresponding Markov-type equations are:
for $\mathbb{P}^3$:

$$x^2+y^2+2z^2=(8z-p)xy \tag{5}$$

for Q:

$$x^2+y^2+3z^2=(9z-p)xy \tag{6}$$

for V_5:

$$x^2+y^2+5z^2=(10z-p)xy \tag{7}$$

for V_{22}:

$$x^2+y^2+11z^2=(11z-p)xy \tag{8}$$

In the paper there are derived some extra correlations between parametres of semi-orthogonal bases (foundations of helices), which are used when proving the constructivity. For each of these threefolds there exists a set of canonical bases, obtained one from another by elementwise tensoring by a power of the ideal sheaf of a point. Any semiorthogonal basis up to changing signs of some elements can be obtained by mutations from one of these canonical bases.

Let $\mathcal{J}_p$ denote the ideal sheaf of a point.

For $\mathbb{P}^3$ the canonical bases are

$$(\mathcal{O}\otimes\mathcal{J}_p^n,\ \mathcal{O}(1)\otimes\mathcal{J}_p^n,\ \mathcal{O}(2)\otimes\mathcal{J}_p^n,\ \mathcal{O}(3)\otimes\mathcal{J}_p^n), \qquad n\in\mathbb{Z}.$$

For a quadric Q the canonical bases are

$$(\mathcal{O}\otimes\mathcal{J}_p^n,\ \mathcal{S}^*\otimes\mathcal{J}_p^n,\ \mathcal{O}(1)\otimes\mathcal{J}_p^n,\ \mathcal{O}(2)\otimes\mathcal{J}_p^n), \qquad n\in\mathbb{Z},$$

where $\mathcal{S}$ is a spinor bundle.

For V_5 the canonical bases are

$$(\mathcal{O}\otimes\mathcal{J}_p^n,\ \mathcal{Q}\otimes\mathcal{J}_p^n,\ \mathcal{S}^*\otimes\mathcal{J}_p^n,\ \mathcal{O}(1)\otimes\mathcal{J}_p^n), \qquad n\in\mathbb{Z},$$

with $\mathcal{S}$ and $\mathcal{Q}$ described two pages above.

For V_{22} the canonical bases $(e_0, \ldots, e_3)$ are the bases with

$$\operatorname{rk} e_0=1, \qquad \operatorname{rk} e_1=4, \qquad \operatorname{rk} e_2=3, \qquad \operatorname{rk} e_3=2,$$
$$c_1 e_0=0, \qquad c_1 e_1=c_1 e_2=c_1 e_3=H,$$

where H is a positive generator of Pic X.

The study of geometrical properties of helices on these manifolds seems to be a field, worthy of the most intent consideration.

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1. Markov-type equations for exceptional pairs

Here and below we call a collection $(e_0, \ldots, e_n)$ of elements of $K_0(X)$ *exceptional* if it satisfies the conditions of semiorthogonality:

$$\langle e_i, e_i\rangle=1, \qquad \langle e_j, e_i\rangle=0 \qquad \text{for} \quad j>i.$$

In particular, we call $x\in K_0(X)$ an exceptional object if $\langle x, x\rangle=1$.

1.1. INVARIANTS OF EXCEPTIONAL COLLECTIONS. — An invariant of an exceptional collection is an integer function of elements of the collection, which is invariant under all possible mutations of it. The simple example is the invariant h of an exceptional pair (x, y), equal to the value of the form χ on the pair :

$$h(x, y)=\langle x, y\rangle.$$

Indeed, h is preserved under a mutation of the pair

$$h(\mathrm{L}_x y, x)=\langle \mathrm{L}_x y, x\rangle=\langle \langle x, y\rangle\cdot x-y, x\rangle$$
$$=\langle x, y\rangle\cdot\langle x, x\rangle-\langle y, x\rangle=\langle x, y\rangle=h(x, y).$$

The example of an invariant of an exceptional collection $(e_0, \ldots, e_k)$ of arbitrary length is given by

$$\det(s_i(e_j))_{i, j=0, \ldots, k},$$

where $(s_0, \ldots, s_k)$ is a collection of additive functions on $\mathrm{K}_0(\mathrm{X})$. Indeed, the transformation of the matrix under the left mutation of the pair (e_j, e_{j+1}) is, in fact, the column transformation, so the determinant is preserved.

The described above invariant h can also be represented as a sum of such invariants, since for an exceptional pair the Riemann-Roch theorem provides that

$$\langle x, y\rangle=\langle x, y\rangle-\langle y, x\rangle=\left(\mathrm{td}(\mathscr{T})\cdot\det\begin{pmatrix}\mathrm{ch}^* x & \mathrm{ch}^* y\\ \mathrm{ch}\, x & \mathrm{ch}\, y\end{pmatrix}\right)_n.$$

More complicated example of an invariant of a pair (x, y) is

$$\det\begin{pmatrix} s(x) & s(y)\\ s(\mathrm{L}_x y) & s(x)\end{pmatrix},$$

where s is an additive function. Indeed, the left mutation transforms the matrix to

$$\begin{pmatrix} s(\mathrm{L}_x y) & s(x)\\ hs(\mathrm{L}_x y)-s(x) & hs(x)-s(y)\end{pmatrix},$$

since

$$\mathrm{L}_{\mathrm{L}_x y} x=\langle \mathrm{L}_x y, x\rangle\cdot \mathrm{L}_x y-x=h\cdot \mathrm{L}_x y-x,$$

so the row transformation is made, which also preserves the determinant.

1.2. MARKOV-TYPE EQUATIONS. — Let there exist an equation, which establishes the relation between the ranks of elements of an exceptional pair (e_0, e_1), some invariant z of the pair and invariants $p_1, \ldots, p_n\, (n\geqq 0)$ of an exceptional collection $(e_2, \ldots, e_k)$, which complements the pair to a semiorthogonal basis. Consider this equation as a Diophantine one

$$\varphi(x, y, z; p_1, \ldots, p_n)=0$$

on tree variables

$$x = \operatorname{rk} e_0, \qquad y = \operatorname{rk} e_1, \qquad z = z(e_0, e_1)$$

with parametres $p_1, \ldots, p_n$. Let it be quadratic with respect to each variable. Then, with fixed values of parametres, on the set of all numerical solutions (x, y, z) of this Diophantine equation there act the mutations

$$x \mapsto x', \qquad y \mapsto y', \qquad z \mapsto z',$$

defined as following:

For any solution (x_0, y_0, z_0) consider

$$\varphi_x(t) = \varphi(t, y_0, z_0; p_1, \ldots, p_n)$$

– the quadratic polynomial on t; x_0 being one of its roots. The other root x' gives the mutated solution (x', y_0, z_0). The mutations $y \mapsto y'$, $z \mapsto z'$ are defined similarly.

We call such an equation $\varphi(x, y, z; p_1, \ldots, p_n) = 0$ *Markov-type*, if the following conditions hold:

1) the variables x and y are symmetrical in the equation,

2) numerical mutations $x \mapsto x'$, $y \mapsto y'$ correspond to right and left mutations of a pair (e_0, e_1) (up to the commutation of x and y, but they are symmetrical),

3) the value of the parametres $p_1, \ldots, p_n$ on the collections $(e_2, \ldots, e_k)$ and $(L_{e_1} e_2, \ldots, L_{e_1} e_k)$ coincide for any semiorthogonal basis $(e_0, \ldots, e_k)$,

4) the numerical mutation $z \mapsto z'$ corresponds to the transition from the basis $(e_0, \ldots, e_k)$ to the basis $(e_1 \otimes \omega_X, e_0, L_{e_1} e_2, \ldots, L_{e_1} e_k)$ (up to the commutation of x and y), where ω_X is the canonical sheaf on X.

Thus, the classical Markov equation (1)

$$x^2 + y^2 + z^2 = 3xyz,$$

which holds for the ranks of elements of a semiorthogonal basis of $K_0(\mathbb{P}^2)$ (a foundation of a helix on $\mathbb{P}^2$) is a Markov-type equation in the following sense: the equivalent equation (3)

$$x^2 + y^2 + \frac{h^2}{9} = hxy$$

is a Markov-type equation of variables x, y, h: conditions 1), 3) are obviously satisfied; $x' = hy - x$ and $y' = hx - y$ correspond to mutations in the pair; $h' = 9xy - h$ corresponds to the transition to the basis

$$(e_1 \otimes \omega, e_0, L_{e_1} e_2),$$

since for this basis

$$\langle e_1 \otimes \omega, e_0 \rangle = 3 \operatorname{rk} L_{e_1} e_2 = 3(\langle e_1, e_2 \rangle \cdot \operatorname{rk} e_1 - \operatorname{rk} e_2) = 3(3x \cdot y - z) = 9xy - h.$$

Note that when changing sign of one of the elements of a pair (it is the operation, which preserves the semiorthogonality of the basis and corresponds to the shift of grading in the derived category) the Markov equation (1) is no longer valid, while the equation (3) is: not only the sign of one of the ranks changes, but that of the invariant h also does.

The Markov-type equation (4) is derived in section 3.

1.3. REDUCTION METHOD. — Markov-type equations help to find the conditions, which provide that the corresponding mutations reduce the numerical solutions. Thus, let $\varphi(a, b, c)=0$ be a Markov-type equation with fixed values of parametres, (a_0, b_0, c_0) be one of its solutions. Let the top coefficient of the quadratic $\varphi_a(t)=\varphi(t, b_0, c_0)$ be positive. One of the roots of this polynomial is a_0. A mutation reduces a, if the other root is less than a_0. To prove it, it is sufficient to find $t_0 < a_0$ such that $\varphi_a(t_0)<0$.

Suppose that any numerical solution of a Markov-type equation can be reduced to one of the simplest, which correspond to some certain (canonical) semiorthogonal bases. It would mean that any semiorthogonal basis can be reduced by mutations to one of the canonical, *i.e.* the constructivity of the semiorthogonal bases is proved.

Below this method is applied to some concrete examples.

2. Properties of semiorthogonal bases

Let (e_0, e_1, e_2, e_3) be a semiorthogonal basis of $K_0(X)$, $\langle x, y\rangle=\chi(x, y)$ be the bilinear form on $K_0(X)$. Denote by κ the linear operator on $K_0(X)$ such that

$$\langle x, \kappa y\rangle=\langle y, x\rangle.$$

For an arbitrary manifold X this operator exists due to Serre duality, namely

$$\kappa E=(-1)^{\dim X} E\otimes\omega.$$

As above, ω denotes here the canonical sheaf on X. In particular, for threefolds $\kappa E=-E\otimes\omega$.

2.1. DUAL BASES OF $K_0(X)$. — A basis $\{e_j^\vee\}$ is called *(left) dual* to a basis $\{e_i\}$ if

$$\langle e_i, e_j^\vee\rangle=\delta_{ij}.$$

If $\{e_i\}$ is a semiorthogonal basis, which corresponds to some foundation of a complete helix $\{E_i\}$, then, as A. L. Gorodentsev [4, (4.4.2)] has shown, the dual basis exists and is given by the conditions

$$e_i^\vee=\overline{L_{E_0}\dots L_{E_{i-1}} E_i} \tag{9}$$

i.e. $e_i^\vee$ is the image in $K_0(X)$ of the i-th left sheaf of E_i in the helix in the derived category. Conditions (9) together with the definition of a complete helix imply that

$$e_0^\vee=e_0,\qquad e_1^\vee=\overline{L_{E_0} E_1}=-L_{e_0} e_1=e_1-\langle e_0, e_1\rangle\cdot e_0,$$
$$e_3^\vee=\overline{L^{(3)} E_3}=\overline{E_3\otimes\omega[\dim X]}=(-1)^{\dim X}\,\overline{E_3\otimes\omega}=\kappa e_3,$$

$$e_2^\vee=\overline{L^{(2)}E_2}=\overline{R(L^{(3)}E_2)}=\overline{R(E_2\otimes\omega)[\dim X]}=-R_{\kappa e_3}\kappa e_2$$
$$=-\kappa R_{e_3}e_2=\kappa e_2-\langle e_2, e_3\rangle\cdot\kappa e_3.$$

Further on we use the notation

$$h_{ij}=\langle e_i, e_j\rangle.$$

Thus, if a given basis corresponds to a foundation of a complete helix, then the elements of the dual basis are given by formulas

$$(10)\qquad e_0^\vee=e_0,\qquad e_1^\vee=e_1-h_{01}e_0,\qquad e_2^\vee=\kappa e_2-h_{23}\kappa e_3,\qquad e_3^\vee=\kappa e_3.$$

Now one can easily check that for an arbitrary semiorthogonal basis the elements $e_i^\vee$ defined by (10) satisfy the conditions $\langle e_i, e_j^\vee\rangle=\delta_{ij}$ as well, *i.e.* form a dual basis.

2.2. Products of additive functions. — Define for $x\in K_0(X)$ the additive function

$$\lambda x=\langle\cdot, x\rangle:K_0(X)\to\mathbb{Z}.$$

The existence of semiorthogonal bases implies, in particular, that the operator

$$\lambda: K_0(X)\to K_0^*(X)$$

is invertible, *i.e.* for any additive function $s\in K_0^*(X)$ there is defined the element $\lambda^{-1}s$.

Define for any additive functions s and t their product

$$\langle s, t\rangle=\langle\lambda^{-1}s, \lambda^{-1}t\rangle.$$

Note that it can be similarly defined by means of the operator

$$\rho: x\mapsto(\langle x, \cdot\rangle:K_0(X)\to\mathbb{Z}),$$

but $\langle x, \cdot\rangle=\langle\cdot, \kappa x\rangle$, so $\rho=\lambda\kappa$, and thereby

$$\langle\lambda^{-1}s, \lambda^{-1}t\rangle=\langle\kappa\rho^{-1}s, \kappa\rho^{-1}t\rangle=\langle\rho^{-1}t, \kappa\rho^{-1}s\rangle=\langle\rho^{-1}s, \rho^{-1}t\rangle,$$

i.e. the definitions are equivalent.

Fix the basis $\{\rho e_i\}$ in $K_0^*(X)$ and consider the decomposition of an additive function t:

$$t=\sum_i t^i\rho e_i=\sum_i t^i\langle e_i, \cdot\rangle.$$

Then

$$t(e_j^\vee)=\sum_i t^i\langle e_i, e_j^\vee\rangle,$$

i.e. $t^i=t(e_i^\vee)$. Thus,

$$t=\sum_i t(e_i^\vee)\rho e_i.$$

Similarly, using the basis $\{\lambda e_i^\vee\}$ one can easily obtain for an additive function s the identity

$$s=\sum_i s(e_i)\lambda e_i^\vee.$$

Then

$$\langle s, t\rangle=\sum_{i,j} s(e_i)t(e_j^\vee)\langle\lambda e_i^\vee, \rho e_j\rangle.$$

Here

$$\langle\lambda e_i^\vee, \rho e_j\rangle=\langle e_i^\vee, \lambda^{-1}\rho e_j\rangle=\langle e_i^\vee, \kappa e_j\rangle=\langle e_j, e_i^\vee\rangle=\delta_{ij},$$

i.e.

$$\langle s, t\rangle=\sum_i s(e_i)t(e_i^\vee). \tag{11}$$

2.3. Proposition (helix property of semiorthogonal bases). – *Let* $(e_0, \ldots, e_k)$ *be a semiorthogonal basis of* $K_0(X)$. *Consider the infinite collection* $\{e_i\}$ *defined by the condition*

$$e_{i-k-1}=e_i\otimes\omega.$$

Then

1) *Any* $k+1$ *successive elements of the collection* $\{e_i\}$ *form a semiorthogonal basis;*

2) $L^{(k)}e_i=(-1)^{k+\dim X}\cdot e_{i-k-1}$.

Proof. – To prove assertion 1) it is sufficient to verify that the collections $(e_k\otimes\omega, e_0, \ldots, e_{k-1})$ and $(e_1, \ldots, e_k, e_0\otimes\omega^{-1})$ are semiorthogonal bases. The verifying is obvious.

To prove assertion 2) consider the collection

$$(L^{(k)}e_k, e_0, \ldots, e_{k-1}),$$

which is a semiorthogonal basis, since mutations preserve semiorthogonality. Therefore

$$\langle e_i, L^{(k)}e_k\rangle=0 \qquad \text{for} \quad 0\leqq i\leqq k-1. \tag{12}$$

Furthermore, for an exceptional pair (x, y) we have

$$\langle y, L_x y\rangle=\langle y, \langle x, y\rangle x-y>=-1.$$

Hence, by induction on j we obtain that $\langle e_k, L^{(j)}e_k\rangle=(-1)^j$. Then (12) together with the equality $\langle e_k, L^{(k)}e_k\rangle=(-1)^k$ implies that

$$\lambda L^{(k)}e_k=(-1)^k\lambda e_k^\vee,$$

hence,

$$L^{(k)}e_k=(-1)^k e_k^\vee=(-1)^k\kappa e_k=(-1)^{k+\dim X}e_{-1}.$$

The proposition implies, in particular, that the transition from a basis $(e_0, \ldots, e_k)$ to $(e_1\otimes\omega, e_0, L_{e_1}e_2, \ldots, L_{e_1}e_k)$, which appears in the definition of a Markov-type equation (1.2), can be obtained by mutations (up to changing signs).

3. Markov-type equations for X

The manifold X we consider is understood to be one of the threefolds $\mathbb{P}^3$, Q, V_5, V_{22}. The canonical class of X is denoted by $\mathcal{K}$.

3.1. Riemann-Roch formula. – Consider for a manifold X the Riemann-Roch theorem in Hirzebruch form:

$$\chi(E)=(\mathrm{ch}(E)\cdot \mathrm{td}(\mathcal{T}))_3,$$

where $\mathcal{T}$ is the tangent sheaf on X; $(\ )_3$ denotes the component of codimension 3 in $A(X)\otimes\mathbb{Q}$, where A (X) is the Chow ring; ch (E) is the Chern character;

$$\mathrm{td}(\mathcal{T})=1+\frac{c_1}{2}+\frac{(c_1^2+c_2)}{12}+\frac{c_1 c_2}{24}$$

is the Todd class of $\mathcal{T}$, $c_i=c_i(\mathcal{T})$. Here $c_1(\mathcal{T})=-\mathcal{K}$, and $c_1 c_2/24=\chi(\mathcal{O})=1$, so

$$c_2(\mathcal{T})=\frac{24 c_1^2}{c_1^3}=-\frac{24\mathcal{K}^2}{\mathcal{K}^3}.$$

Thus,

$$\mathrm{td}(\mathcal{T})=1-\frac{\mathcal{K}}{2}+\mathcal{K}^2\left(\frac{1}{12}-\frac{2}{\mathcal{K}^3}\right)+1.$$

Introduce the notations for "specific components" of Chern character:

$$\nu(E)=\frac{c_1(E)}{r(E)},\qquad q(E)=\frac{\mathrm{ch}_2(E)}{r(E)}=\frac{1}{2r(E)}(c_1^2(E)-2c_2(E)),$$

$$\eta(E)=\frac{\mathrm{ch}_3(E)}{r(E)}=\frac{1}{6r(E)}(c_1^3(E)-3c_1(E)c_2(E)+3c_3(E)),$$

then

$$\mathrm{ch}(E)=r(E)\cdot(1+\nu(E)+q(E)+\eta(E)),$$

and the Riemann-Roch formula may be written as

$$\begin{aligned}\chi(E)&=(\mathrm{ch}(E)\cdot\mathrm{td}(\mathcal{T}))_3\\ &=r(E)\cdot\left((1+\nu(E)+q(E)+\eta(E))\cdot\left(1-\frac{\mathcal{K}}{2}+\mathcal{K}^2\left(\frac{1}{12}-\frac{2}{\mathcal{K}^3}\right)+1\right)\right)_3\\ &=r(E)\cdot\left(1+\mathcal{K}^2\left(\frac{1}{12}-\frac{2}{\mathcal{K}^3}\right)\nu(E)-\frac{\mathcal{K}}{2}q(E)+\eta(E)\right).\end{aligned}$$

For locally free sheaves E, F we have the identity $\chi(E, F)=\chi(E^*\otimes F)$, and the multiplicativity of Chern class implies that

$$\nu(E^*\otimes F)=\nu(F)-\nu(E)=\Delta\nu,\qquad q(E^*\otimes F)=q(E)+q(F)-\nu(E).\nu(F),$$

$$\eta(E^*\otimes F)=\Delta\eta+q(E)\nu(F)-q(F)\nu(E),$$

where $\Delta\eta=\eta(\mathrm{F})-\eta(\mathrm{E})$. Then

$$(13)\quad \langle \mathrm{E}, \mathrm{F}\rangle=\chi(\mathrm{E}, \mathrm{F})=r(\mathrm{E})\cdot r(\mathrm{F}) \times\left(1+\mathscr{K}^2\left(\frac{1}{12}-\frac{2}{\mathscr{K}^3}\right)\Delta\nu-\frac{\mathscr{K}}{2}(q(\mathrm{E})+q(\mathrm{F})-\nu(\mathrm{E})\cdot\nu(\mathrm{F})) +\Delta\eta+q(\mathrm{E})\nu(\mathrm{F})-q(\mathrm{F})\nu(\mathrm{E})\right).$$

Since the Euler characteristic is additive, formula (13) is valid for an arbitrary sheaf E of non-zero rank.

In particular, from (13) one can derive the expression for $\langle \mathrm{E},\mathrm{F}\rangle_+=\langle \mathrm{E},\mathrm{F}\rangle+\langle \mathrm{F},\mathrm{E}\rangle$:

$$(14)\qquad \langle \mathrm{E}, \mathrm{F}\rangle_+=2r(\mathrm{E})\cdot r(\mathrm{F})\cdot\left(1-\frac{\mathscr{K}}{2}(q(\mathrm{E})+q(\mathrm{F})-\nu(\mathrm{E})\cdot\nu(\mathrm{F}))\right).$$

3.2. Riemann-Roch formula for an exceptional pair. — For an exceptional sheaf E formula (14) implies

$$2=\langle \mathrm{E}, \mathrm{E}\rangle_+=2r^2(\mathrm{E})\cdot\left(1-\frac{\mathscr{K}}{2}(2q(\mathrm{E})-\nu^2(\mathrm{E}))\right),$$

hence, $-(\mathscr{K}/2)(2q(\mathrm{E})-\nu^2(\mathrm{E}))=1/r^2(\mathrm{E})-1$, so

$$(15)\qquad -\frac{\mathscr{K}}{2}q(\mathrm{E})=\delta(\mathrm{E})-\frac{1}{2}-\frac{\nu^2(\mathrm{E})}{2}\cdot\frac{\mathscr{K}}{2},$$

where $\delta(\mathrm{E})$ denotes $1/(2r^2(\mathrm{E}))$.

Substituting (15) and the similar expression for $-(\mathscr{K}/2)q(\mathrm{F})$ into (14), we obtain the Riemann-Roch formula for an exceptional pair (E, F):

$$\langle \mathrm{E}, \mathrm{F}\rangle=\langle \mathrm{E}, \mathrm{F}\rangle_+=2r(\mathrm{E})\cdot r(\mathrm{F}) \times\left(1+\delta(\mathrm{E})-\frac{1}{2}-\frac{\nu^2(\mathrm{E})}{2}\cdot\frac{\mathscr{K}}{2}+\delta(\mathrm{F})-\frac{1}{2}-\frac{\nu^2(\mathrm{F})}{2}\cdot\frac{\mathscr{K}}{2}+\frac{\mathscr{K}}{2}\nu(\mathrm{E})\cdot\nu(\mathrm{F})\right)$$
$$=2r(\mathrm{E})\cdot r(\mathrm{F})\cdot\left(\delta(\mathrm{E})+\delta(\mathrm{F})-\frac{\mathscr{K}}{4}(\nu^2(\mathrm{E})+\nu^2(\mathrm{F})-2\nu(\mathrm{E})\cdot\nu(\mathrm{F}))\right),$$

i.e.

$$(16)\qquad \langle \mathrm{E}, \mathrm{F}\rangle=2r(\mathrm{E})\cdot r(\mathrm{F})\cdot\left(\delta(\mathrm{E})+\delta(\mathrm{F})-\frac{\mathscr{K}}{4}(\Delta\nu)^2\right).$$

Let H be the positive generator of the group $\operatorname{Pic}\mathrm{X}\cong\mathbb{Z}\cdot\mathrm{H}$. Denote by k the multiplicity of the anticanonical class (index of Fano threefold), *i.e.* $-\mathscr{K}=k\mathrm{H}$. Denote by d the integer additive function on $\mathrm{K}_0(\mathrm{X})$, equal to the multiplicity of the first Chern class, *i.e.* $c_1(\mathrm{E})=d(\mathrm{E})\cdot\mathrm{H}$. [In particular, $k=d(\omega^{-1})$.]

Consider (16) for an exceptional pair (e_i, e_j) using the above notations:

$$\langle e_i, e_j\rangle=2r_i r_j\left(\delta(e_i)+\delta(e_j)+\frac{k}{4}\mathrm{H}\left(\frac{d_j\mathrm{H}}{r_j}-\frac{d_i\mathrm{H}}{r_i}\right)^2\right).$$

Remind that we denote $\langle e_i, e_j \rangle$ by h_{ij}. Then

$$h_{ij} r_i r_j = 2 r_i^2 r_j^2 \left(\delta(e_i) + \delta(e_j) + \frac{k}{4} \left(\frac{d_j}{r_j} - \frac{d_i}{r_i} \right)^2 \mathrm{H}^3 \right) = r_j^2 + r_i^2 + \frac{k}{2} \mathrm{H}^3 (r_i d_j - r_j d_i)^2.$$

Here

$$r_i d_j - r_j d_i = \det \begin{pmatrix} r_i & r_j \\ d_i & d_j \end{pmatrix} = \mathrm{C}_{ij}$$

– an invariant of the pair (e_i, e_j).

Thus, we obtain the formula

$$h_{ij} r_i r_j = r_j^2 + r_i^2 + \frac{k}{2} \mathrm{H}^3 \cdot \mathrm{C}_{ij}^2. \tag{17}$$

Note that it is not a Markov-type equation, since it includes two invariants of a pair: h_{ij} and C_{ij}.

3.3. Proposition. – *An exceptional object on* X *is of non-zero rank.*

Proof. – Formula (14) for an exceptional object E of zero rank implies:

$$2 = \langle \mathrm{E}, \mathrm{E} \rangle_+ = 2 \cdot \frac{\mathscr{K}}{2} c_1^2 (\mathrm{E}) = -k \mathrm{H} (d(\mathrm{E}) \cdot \mathrm{H})^2 = -k \mathrm{H}^3 d^2 (\mathrm{E}) < 0,$$

which provides a contradiction.

3.4. Proposition. – *For an exceptional pair of objects of positive rank the inequalities* $h_{ij} \geqq 3$, $\mathrm{C}_{ij} \neq 0$ *hold.*

Proof. – Formula (17) written as

$$(h_{ij} - 2) r_i r_j = (r_i - r_j)^2 + \frac{k}{2} \mathrm{H}^3 \cdot \mathrm{C}_{ij}^2,$$

with $r_i, r_j > 0$ immediately implies that $h_{ij} \geqq 2$. If $h_{ij} = 2$, then $r_i = r_j$, $\mathrm{C}_{ij} = 0$. Conversely, under $\mathrm{C}_{ij} = 0$ we obtain

$$h_{ij} r_i r_j = r_j^2 + r_i^2,$$

hence $r_i \mid r_j$, $r_j \mid r_i$, *i.e.* $r_i = r_j$. Thus, in both cases $r_i = r_j$, $c_1(e_i) = c_1(e_j)$. Then (15) implies that $q(e_i) = q(e_j)$ as well. Then the condition $\langle e_j, e_i \rangle = 0$ together with (13) provide that $\eta(e_i) = \eta(e_j)$, therefore $\mathrm{ch}(e_i) = \mathrm{ch}(e_j)$, *i.e.* $e_1 = e_j$ as elements of $\mathrm{K}_0(\mathrm{X})$. But it would imply

$$1 = \langle e_i, e_i \rangle = \langle e_j, e_i \rangle = 0.$$

3.5. Corollary. – *Mutations of an exceptional pair of objects of positive rank preserve the ranks to be positive.*

Indeed, h is an invariant of a pair; thereby if an object obtained by a mutation has negative rank, then in equality (17) for the mutated pair we would have $r_i>0$, $r_j<0$, $h_{ij}\geqq 3$, which provides a contradiction.

When multiplying one of the elements of a semiorthogonal basis by -1, the basis remains semiorthogonal. Therefore, taking corollary 3.5 into account, we consider further on only the bases consisting of objects of positive rank.

All the following assertions of this section are formulated for such bases.

3.6. PROPOSITION. — *An exceptional pair with fixed values of invariants* C *and* h *can be reduced by mutations to a pair for which the lesser rank* (*among those of elements of the pair*) *is not greater than* $\mathrm{C}\cdot\sqrt{k\,\mathrm{H}^3/2\,(h-2)}$.

Proof. — We prove the proposition by induction on the sum of ranks of elements of a pair (assuming them to be positive). Denote the ranks by x and y; then (17) is written as

$$x^2+y^2+\frac{k}{2}\,\mathrm{H}^3\cdot\mathrm{C}^2=hxy.$$

Applying the reduction method (1.3), we obtain that when $x\geqq y$, the mutation does not reduce x only if

$$0\leqq\varphi_x(y)=2\,y^2+\frac{k}{2}\,\mathrm{H}^3\cdot\mathrm{C}^2-hy^2,$$

i.e. $(h-2)\,y^2\leqq(k/2)\,\mathrm{H}^3\cdot\mathrm{C}^2$.

Q.E.D.

3.7. THE DUAL BASES. — According to 2.1, the dual basis of $\mathrm{K}_0(\mathrm{X})$ is defined by formulas (10):

$$e_0^\vee=e_0,\qquad e_1^\vee=e_1-h_{01}\,e_0,\qquad e_2^\vee=\kappa\,e_2-h_{23}\,\kappa\,e_3,\qquad e_3^\vee=\kappa\,e_3,$$

where $\kappa\,\mathrm{E}=(-1)^{\dim \mathrm{X}}\,\mathrm{E}\otimes\omega=-\,\mathrm{E}\otimes\omega$.

For additive functions r (rank) and d introduced above, we obtain

$$r(\kappa\,e_i)=-r_i,\qquad d(\kappa\,e_i)=-d(e_i\otimes\omega)=-(d_i-kr_i)=kr_i-d_i,$$

so $r(e_i^\vee)$ and $d(e_i^\vee)$ can easily be computed:

$$re_0^\vee=r_0,\qquad re_1^\vee=r_1-h_{01}\,r_0,\qquad re_2^\vee=-r_2+h_{23}\,r_3,\qquad re_3=-r_3;$$
$$de_0^\vee=d_0,\qquad de_1^\vee=d_1-h_{01}\,d_0,\qquad de_2^\vee=kr_2-d_2-h_{23}\,(kr_3-d_3),$$
$$de_3^\vee=kr_3-d_3.$$

Furthermore, $\lambda^{-1}\,r=\mathcal{O}_p$, the class of a structure sheaf of a point (*see* 2.2), and $\lambda^{-1}\,d$ is represented by a one-dimensional cycle, so

$$\langle r, r\rangle=\langle r, d\rangle=\langle d, r\rangle=0.$$

The same products can be computed using (11):

$$(18)\qquad 0=\langle r, r\rangle=\sum_i r_i re_i^\vee = r_0^2-h_{01}r_0r_1+r_1^2-(r_2^2-h_{23}r_2r_3+r_3^2);$$

$$(19)\quad 0=\langle r, d\rangle-\langle d, r\rangle=\sum_i(r_i de_i^\vee - d_i re_i^\vee)$$
$$=h_{01}(r_0d_1-r_1d_0)+kr_2^2-h_{23}kr_2r_3+h_{23}(r_3d_2-r_2d_3)kr_3^2$$
$$=h_{01}C_{01}+h_{23}C_{23}+k(r_2^2-h_{23}r_2r_3+r_3^2).$$

The obtained equalities (18), (19) together with (17) provide

$$(20)\qquad h_{01}C_{01}+h_{23}C_{23}=\frac{k^2}{2}H^3\cdot C_{01}^2=\frac{k^2}{2}H^2\cdot C_{23}^2.$$

3.8. Proposition. — *For semiorthogonal bases consisting of objects of positive rank the equality* $|C_{01}|=|C_{23}|$ *holds, and at least one of the numbers* C_{01}, C_{23} *is positive.*

Proof. — The first part of the assertion follows immediately from (20). Furthermore, $h_{01}C_{01}+h_{23}C_{23}>0$ because of (20) and proposition 3.4, therefore when $h_{01}\geqq h_{23}$, we have

$$(21)\qquad h_{01}+(\text{sign } C_{23})h_{23}=\frac{k^2}{2}H^3\cdot C_{01}>0.$$

(Correspondingly, when $h_{01}\leqq h_{23}$, then $C_{23}<0$).

Let C_{34} denote $C(e_3, e_0\otimes\omega^{-1})$. Then similarly $|C_{12}|=|C_{34}|$.

3.9. Proposition. — *For any semiorthogonal basis of* $K_0(X)$ *one of the following cases holds: either*

$$(+)\quad C_{01}=C_{23}\quad \textit{and}\quad C_{12}=C_{34}$$

or

$$(-)\quad C_{01}=-C_{23}\quad \textit{and}\quad C_{12}=-C_{34}.$$

Proof. — According to 3.8, in all the other cases excepts (+) and (−) exactly one of the numbers $C_{i,i+1}$ $(0\leqq i\leqq 3)$ is negative. Assume without loss of generality that $C_{23}<0$, since all the other cases can be brought to this by taking into consideration one of the bases

$$(e_3\otimes\omega, e_0, e_1, e_2),\quad (e_2\otimes\omega, e_3\otimes\omega, e_0, e_1)\quad \text{or}\quad (e_1, e_2, e_3, e_0\otimes\omega^{-1}).$$

So, let

$$C_{01}=-C_{23}=C>0,\qquad C_{12}=C_{34}=C'>0.$$

Consider the basis

$$f=(f_0, f_1, f_2, f_3)=(e_0, e_2, R_{e_2}e_1, e_3).$$

For this basis $C_{12}^f=C_{12}=C'$, since C' is preserved under the mutation of the pair (e_1, e_2). Denote $C_{01}^f=\tilde{C}$. Since

$$C_{ij}=r_ir_j\left(\frac{d_j}{r_j}-\frac{d_i}{r_i}\right),$$

the sign of C_{ij} equals to that of $(\mu_j-\mu_i)$, where $\mu=d/r$ is the slope. Then for the starting basis $\mu_0<\mu_1<\mu_2$ and $\mu_2>\mu_3$, thus, $\mu_2<\mu(f_2)$, $\mu(f_2)>\mu_3$. It implies that $\tilde{C}>0$, $C_{23}^f=-\tilde{C}$.

Furthermore,

$$\tilde{C}=r_0 r_2(\mu_2-\mu_0)=r_0 r_2((\mu_2-\mu_1)+(\mu_1-\mu_0))=r_0\frac{C'}{r_1}+r_2\frac{C}{r_1} \tag{22}$$

Moreover,

$$-C=C_{23}=r_2 r_3(\mu_3-\mu_2)=r_2 r_3((\mu_3-\mu(f_2))+(\mu(f_2)-\mu_2))=r_2\cdot\frac{-\tilde{C}}{r_2^f}+r_3\cdot\frac{C'}{r_2^f},$$

where $r_2^f=h_{12}r_2-r_1$. Together with (22) it gives

$$-C(h_{12}r_2 r_1-r_1)r_1=r_2(-C'r_0-Cr_2)+r_1 r_3 C',$$

i.e.

$$-C(-r_2^2+h_{12}r_2 r_1-r_1^2)=-C'r_0 r_2+C'r_1 r_3,$$

where

$$-r_2^2+h_{12}r_2 r_1-r_1^2=\frac{k}{2}H^3\cdot C^2=\frac{k}{2}H^3\cdot(C')^2$$

according to (17), (18); therefore, recalling $C'\neq 0$, we obtain

$$\frac{k}{2}H^3\cdot C\cdot C'=r_0 r_2-r_1 r_3. \tag{23}$$

Now consider the basis

$$g=(g_0, g_1, g_2, g_3)=(e_1^*\otimes\omega, e_0^*\otimes\omega, e_3^*, e_2^*).$$

For it

$$C_{01}^g=C_{01}=C,\qquad C_{12}^g=C_{30}=C',\qquad C_{23}^g=C_{23}=-C,$$

then (23) implies that

$$\frac{k}{2}H^3\cdot C\cdot C'=r_0^g\cdot r_2^g-r_1^g\cdot r_3^g=r_1 r_3-r_0 r_2.$$

Together with (23) it gives $C\cdot C'=0$, which contradicts to 3.4.

3.10. Corollary. – *The classes of* (+)*-bases and* (−)*-bases are closed under the action of mutations.*

Indeed, any mutation preserves one of the pairs of numbers: (C_{01}, C_{23}) or (C_{12}, C_{34}).

3.11. Corollary. – *Formula* (23) *is valid for* (−)*-bases with* $h_{01}\geqq h_{23}$, $h_{12}\geqq h_{34}$.

Indeed, when deriving this formula we used the conditions $C_{01}=-C_{23}>0$, $C_{12}>0$ only, which hold for such (−)-bases, since 3.8 for $h_{01}\geqq h_{23}$ implies $C_{01}>0$.

3.12. PROPOSITION. — *For (+)-bases there is valid the formula*

$$\frac{k}{2}\mathrm{H}^3\cdot \mathrm{C}\cdot \mathrm{C}' = r_0 r_2 + r_1 r_3. \tag{24}$$

It can be deduced similarly as the formula (23) in 3.9 bearing in mind that a mutation transforms a (+)-basis to a (+)-basis (3.10).

3.13. MARKOV-TYPE EQUATIONS FOR (+)-BASES. — (Below we prove for each of the manifolds we consider that (−)-bases consisting of objects of positive rank do not exist).

For a (+)-basis (20) holds in the form

$$\frac{h_{01}+h_{23}}{(k^2/2)\,\mathrm{H}^3} = \mathrm{C}_{01} = \mathrm{C}_{23}. \tag{25}$$

Consider (17) with $i=0, j=1$. Let $h_{01}=z$, $h_{23}=p$. Then (25) gives $\mathrm{C}_{01}=z+p/(k^2/2)\,\mathrm{H}^3$, thus we obtain equation (4) :

$$x^2+y^2+\frac{(z+p)^2}{(k^3/2)\,\mathrm{H}^3} = xyz,$$

i.e.

$$x^2+y^2+2\,\frac{(z+p)^2}{(-\mathscr{K})^3} = xyz.$$

Conversely, taking the invariant C as z, *i.e.* considering $\mathrm{C}_{01}=z$, $h_{23}=p$, obtain

$$x^2+y^2+\frac{k}{2}\mathrm{H}^3 z^2 = xy\left(\frac{k^2}{2}\mathrm{H}^3 z - p\right) \tag{26}$$

(compare with equations (5-8) of the preface).

For equation (26) the numerical mutation of z is $z \mapsto z' = kxy - z$. This mutation corresponds to the transition to the basis $(e_1\otimes\omega, e_0, \mathrm{L}_{e_1}e_2, \mathrm{L}_{e_1}e_3)$, since

$$\mathrm{C}(e_1\otimes\omega, e_0) = \det\begin{pmatrix} r_1 & r_0 \\ d(e_1\otimes\omega) & d_0 \end{pmatrix} = \det\begin{pmatrix} r_1 & r_0 \\ d_1 - kr_1 & d_0 \end{pmatrix} = -\mathrm{C}_{01} + kr_0 r_1 = kxy - z.$$

Hence, equation (26) is indeed a Markov-type equation (verifying other properties is obvious).

For equation (4) the numerical mutation of z is

$$z \mapsto z' = \frac{(-\mathscr{K})^3}{2}\,xy - 2p - z.$$

On the other hand,

$$h(e_1\otimes\omega, e_0) = \frac{k^2}{2}\mathrm{H}^3\,\mathrm{C}(e_1\otimes\omega, e_0) - p$$

according to (25), so

$$h(e_1 \otimes \omega, e_0) = \frac{k^2}{2} \mathrm{H}^3 (kxy - \mathrm{C}_{01}) - p = \frac{k^3}{2} \mathrm{H}^3 xy - \frac{k^2}{2} \mathrm{H}^3 \mathrm{C}_{01} - p = \frac{(-\mathscr{K})^3}{2} xy - (z+p) - p,$$

Q.E.D.

Thus, (4) is also a Markov-type equation.

4. Two lemmas

4.1. LEMMA. – *Let (e_0, e_1, e_2, e_3) and (f_0, f_1, f_2, f_3) be two semiorthogonal bases of $\mathrm{K}_0(\mathrm{X})$ with $r(e_i) = r(f_i)$, $d(e_i) = d(f_i)$. Then there exists $\alpha \in \mathbb{Z}$ such that $f_i = e_i \otimes \mathscr{J}_p^{\alpha}$, where $\mathscr{J}_p$ is an ideal sheaf of a point.*

Proof. – For an exceptional object formula (15) implies that q is determined by r and c_1, so $q(e_i) = q(f_i)$. Hence, $f_i = e_i + n_i \mathcal{O}_p$, where $n_i \in \mathbb{Z}$. Conversely, any object of such a kind as $e_i + n\mathcal{O}_p$ is exceptional, since

$$\langle e_i + n\mathcal{O}_p, e_i + n\mathcal{O}_p \rangle = 1 + n \langle e_i, \mathcal{O}_p \rangle + n \langle \mathcal{O}_p, e_i \rangle = 1 + n \cdot r_i - n \cdot r_i = 1.$$

On the other hand, semiorthogonality implies that for $j > i$ we have

$$0 = \langle f_j, f_i \rangle = \langle e_j + n_j \mathcal{O}_p, e_i + n_i \mathcal{O}_p \rangle = n_i r_j - n_j r_i,$$

so $n_i/r_i = n_j/r_j = -\alpha \in \mathbb{Q}$. Here $\alpha \cdot r_i \in \mathbb{Z}$ for any i, *i.e.*

$$\alpha \cdot \mathrm{g.c.d.}(r_0, r_1, r_2, r_3) \in \mathbb{Z}.$$

But r_0, r_1, r_2, r_3 are coprime as the values of the function r on the elements of a basis. Thus, $\alpha \in \mathbb{Z}$ and

$$f_i = e_i - r_i \cdot \alpha \mathcal{O}_p = e_i \otimes (\mathcal{O} - \alpha \mathcal{O}_p) = e_i \otimes \mathscr{J}_p^{\alpha},$$

Q.E.D.

Further we need the following technical result:

4.2. LEMMA. – *Let a $(+)$-basis (e_0, e_1, e_2, e_3) satisfy the inequalities $\mathrm{C} \geqq p \cdot r_0 r_1$, $\mathrm{C}' \geqq q \cdot r_1 r_2$, where $\mathrm{C} = \mathrm{C}_{01} = \mathrm{C}_{23}$, $\mathrm{C}' = \mathrm{C}_{12} = \mathrm{C}_{34}$, $p, q \in \mathbb{R}$, with $\mathrm{A}pq > 1$, where $\mathrm{A} = k/2\,\mathrm{H}^3$. Besides, let both possible mutations do not reduce the rank of e_3,* i.e. *$r_3 \leqq h_{23} r_2 - r_3$ and $r_3 \leqq h_{34} r_0 - r_3$. Then*

$$r_2 \leqq \frac{\sqrt{\mathrm{A}p^2+1}}{\mathrm{A}pq-1} \quad \text{or} \quad r_0 r_1 < \frac{\sqrt{\mathrm{A}p^2+1}}{\mathrm{A}pq-1},$$

and also

$$r_0 \leqq \frac{\sqrt{\mathrm{A}q^2+1}}{\mathrm{A}pq-1} \quad \text{or} \quad r_1 r_2 < \frac{\sqrt{\mathrm{A}q^2+1}}{\mathrm{A}pq-1}.$$

Proof. – Formula (23) provides

$$r_0 r_2 + r_1 r_3 = \mathrm{A} \cdot \mathrm{C} \cdot \mathrm{C}' \geqq \mathrm{A}pq \cdot r_0 r_1^2 r_2 \geqq \mathrm{A}pq \cdot r_0 r_2,$$

so $r_1 r_3 \geqq (A pq-1) r_0 r_2$. Then $r_0 r_2 \leqq r_1 r_3/(A pq-1)$, *i.e.*

$$r_1 r_3 \left(\frac{1}{A pq-1}+1\right) \geqq r_0 r_2 + r_1 r_3 = A \cdot C \cdot C',$$

hence,

$$r_1 r_3 \frac{A pq}{A pq-1} \geqq A \cdot C \cdot C' \geqq A pq \cdot r_0 r_1^2 r_2,$$

i.e.

(27)
$$r_3 \geqq (A pq-1) \cdot r_0 r_1 r_2.$$

Furthermore,

$$r_1 r_3 \frac{A pq}{A pq-1} \geqq A \cdot C \cdot C' \geqq A \cdot C' \cdot p r_0 r_1,$$

hence,

(28)
$$C' \leqq \frac{q}{A pq-1} \cdot \frac{r_3}{r_0}.$$

Similarly,

(29)
$$C \leqq \frac{p}{A pq-1} \cdot \frac{r_3}{r_2}.$$

Since the mutation does not reduce the rank of e_3, we have

(30)
$$h_{23} r_2 \geqq 2 r_3,$$

(31)
$$h_{34} r_0 \geqq 2 r_3.$$

Consider equation (17) for the pair (e_2, e_3):

$$r_2^2 + r_3^2 + A \cdot C^2 = h_{23} r_2 r_3 \geqq 2 r_3^2$$

according to (30); substituting (29) into it we obtain

$$r_2^2 + \frac{A p^2}{(A pq-1)^2} \cdot \frac{r_3^2}{r_2^2} \geqq r_3^2,$$

so

$$r_3^2 \left(r_2^2 - \frac{A p^2}{(A pq-1)^2}\right) \leqq r_2^4,$$

then either the expression in brackets is non-positive, *i.e.*

$$r_2 \leqq \frac{\sqrt{A p^2}}{A pq-1} < \frac{\sqrt{A p^2+1}}{A pq-1},$$

or

(32)
$$r_3^2 \leqq \frac{r_2^2}{(r_2^2 - A p^2/(A pq-1)^2)} r_2^2.$$

On the other hand, (27) implies

$$r_0^2 r_1^2 r_2^2 (\mathrm{A}pq-1)^2 \leqq r_3^2,$$

which together with (32) provides

$$r_0^2 r_1^2 \leqq \frac{r_2^2}{(\mathrm{A}pq-1)^2 r_2^2 - \mathrm{A}p^2},$$

hence,

$$r_0 r_1 \leqq \frac{r_2}{\sqrt{(\mathrm{A}pq-1)^2 r_2^2 - \mathrm{A}p^2}}. \tag{33}$$

Consider the function

$$f(t) = \frac{t}{\sqrt{(\mathrm{A}pq-1)^2 t^2 - \mathrm{A}p^2}}.$$

On the set $t > \sqrt{\mathrm{A}p^2}/(\mathrm{A}pq-1)$ this function monotonically decreases, and $f(t)=t$ when $t=t_0=\sqrt{(\mathrm{A}p^2+1)}/(\mathrm{A}pq-1)$. It means that either $t \leqq t_0$ or $f(t)<t_0$. Recalling inequality (32), we obtain that either $r_2 \leqq t_0$ or $r_1 r_2 < t_0$, *i.e.* the first part of the assertion is proved.

Similarly, considering equation (17) for the pair $(e_3, e_0 \otimes \omega^{-1})$ and using inequalities (28), (31), we obtain the second part of it.

5. Constructivity problem for $\mathbb{P}^3$

For a semiorthogonal basis of $\mathrm{K}_0(\mathbb{P}^3)$ formula (17) takes the form

$$x^2 + y^2 + 2z^2 = hxy, \tag{34}$$

where $x=r_i$, $y=r_j$, $z=\mathrm{C}_{ij}$, $h=h_{ij}$. The constant $(k^2/2)\mathrm{H}^3$ in formulas (21), (25), (26) equals 8.

5.1. PROPOSITION. — *(−)-bases of* $\mathrm{K}_0(\mathbb{P}^3)$ *do not exist.*

Proof. — It is sufficient to prove that for any (−)-basis consisting of objects of positive rank there exists a mutation, which reduces the sum of ranks of elements of the basis. Indeed, according to 3.5 and 3.10, ranks remain positive under mutations, and a basis remains to be a (−)-basis, *i.e.* the proposition will thereby be proved by induction on the sum of ranks of elements of a basis.

Assume without loss of generality that in the given basis $h_{01} \geqq h_{23}$, $h_{12} \geqq h_{34}$, for otherwise we may consider instead of it one of the bases

$$(e_3 \otimes \omega, e_0, e_1, e_2), \quad (e_2 \otimes \omega, e_3 \otimes \omega, e_0, e_1), \quad (e_1 \otimes \omega, e_2 \otimes \omega, e_3 \otimes \omega, e_0),$$

which, according to 2.3, can be obtained from the given one by mutations.

Then (21) holds in the form $h_{01}-h_{23}=8\,\mathrm{C}_{01}>0$; similarly, $h_{12}-h_{34}=8\,\mathrm{C}_{12}>0$. It implies, in particular, that in (34) for the pairs (e_0, e_1) and (e_1, e_2) there holds the condition $h>8\,z$.

Consider (34) with $h>8\,z$. Use the reduction method 1.3 for variables x and y.

1) Let $x\geqq y\geqq z$. Then $x'<y$, *i.e.* $\varphi_x(y)<0$, for otherwise

$$0\leqq\varphi_x(\mathrm{y})=2\,y^2+2\,z^2-hy^2<4\,y^2-8\,zy^2,$$

which provides a contradiction.

2) Let $x\geqq z>y$. Then $x'<z$, for otherwise

$$0\leqq\varphi_x(z)=y^2+3\,z^2-hyz<4\,z^2-8\,z^2\,y,$$

which also provides a contradiction.

3) Hence, both x and y can not be reduced only if $z>x$, $z>y$. Then

$$4\,z^2>x^2+y^2+2\,z^2=hxy>8\,xyz,$$

i.e. $z>2\,xy$.

So, the mutations of the pair (e_0, e_1) do not reduce the sum of the ranks only if $\mathrm{C}=\mathrm{C}_{01}>2\,xy=2\,r_0\,r_1$. Similarly, in the pair (e_1, e_2) the ranks can not be reduced only if $\mathrm{C}'=\mathrm{C}_{12}>2\,r_1\,r_2$. Then formula (23) (*see* corollary 3.11) gives

$$r_0\,r_2-r_1\,r_3=2\,\mathrm{C}\cdot\mathrm{C}'>8\,r_0\,r_1\cdot r_1\,r_2\geqq 8\,r_0\,r_2,$$

which provides a contradiction.

5.2. Corollary. — *The elements of a helix on* $\mathbb{P}^3$ *are ordered by slopes.*

Proof. — Shifting when necessary the grading of some elements of a helix, we obtain the helix consisting of objects of positive rank; the slopes are not changed. Then the image in $\mathrm{K}_0(\mathbb{P}^3)$ of any foundation of the obtained helix is a semiorthogonal basis of $\mathrm{K}_0(\mathbb{P}^3)$, which according to 3.9 and 5.1 can be only a (+)-basis. Then

$$\mu_{i+1}-\mu_i=\frac{\mathrm{C}_{i,\,i+1}}{r_i\,r_{i+1}}>0.$$

Now we start proving constructivity of semiorthogonal bases.

5.3. Theorem. — *Any semiorthogonal basis of* $\mathrm{K}_0(\mathbb{P}^3)$, *up to changing signs of some elements of it, can be reduced by mutations to one of the canonical bases*

$$(\mathcal{O}\otimes\mathcal{I}_p^n,\ \mathcal{O}(1)\otimes\mathcal{I}_p^n,\ \mathcal{O}(2)\otimes\mathcal{I}_p^n,\ \mathcal{O}(3)\otimes\mathcal{I}_p^n),\qquad n\in\mathbb{Z}.$$

Proof. — Changing the signs when necessary, we obtain a basis consisting of objects of positive rank. According to 5.1, this basis is a (+)-basis.

We prove the theorem by induction on the sum of ranks of elements of a basis: show that if the basis is not a canonical one (up to tensoring by an invertible sheaf), then always a mutation exists, which reduces the sum of ranks.

For a (+)-basis (25) gives

$$h_{01}+h_{23}=8\,C_{01}, \qquad h_{12}+h_{34}=8\,C_{12}.$$

As in the proof of 5.1, assume without loss of generality that $h_{01}\geqq h_{23}$, $h_{12}\geqq h_{34}$. Then in equation (34) for the pairs (e_0, e_1) and (e_1, e_2) we have $h\geqq 4z$.

Use the reduction method for x and y in (34) with $h\geqq 4z$.

1) Let $x\geqq y\geqq z$. Either $x'<y$ or

$$0\leqq \varphi_x(y)=2y^2+2z^2-hy^2\leqq 4y^2-4zy^2,$$

hence, $z=1$, then

$$0\leqq \varphi_x(y)=2y^2+2-hy^2\leqq 2-2y^2,$$

hence, $y=1$, then

$$0\leqq \varphi_x(y)=4-h,$$

so $h\leqq 4$. By proposition 3.4, we have $h\geqq 3$, but (34) immediately implies that for an exceptional pair on $\mathbb{P}^3$ the invariant h must be even. Thus, $h=4$. Then equation (34) takes the form

$$x^2+3=4x,$$

the lesser root is $x=1$.

2) Let $x\geqq z>y$. Then $x'<z$, for otherwise

$$0\leqq \varphi_x(z)=y^2+3z^2-hzy<4z^2-4z^2y,$$

which provides a contradiction.

Hence, either x (or y) can be reduced or $x=y=1$ or

3) $z>x$, $z>y$. In this case

$$4z^2>x^2+y^2+2z^2=hxy\geqq 4zxy,$$

i.e. $z>xy$.

Thus, the mutations of the pair (e_0, e_1) do not reduce the sum of ranks only if $r_0=r_1=1$ or $C=C_{01}>r_0r_1$. Similarly, the mutations of the pair (e_1, e_2) do not reduce the sum of ranks only if $r_1=r_2=1$ or $C=C_{12}>r_1r_2$.

Now, it remains to consider two cases:

1. $C>r_0r_1$, $C'>r_1r_2$.
2. The basis includes at least two objects of rank 1.

Case 1. – Apply lemma 4.2. Under the conditions of the lemma we have $p=q=1$, $A=2$, *i.e.*

$$\frac{\sqrt{Ap^2+1}}{Apq-1}=\frac{\sqrt{Aq^2+1}}{Apq-1}=\frac{\sqrt{3}}{2}<2.$$

Then both possible mutations do not reduce r_3 only if

$$(r_2 \leqq 1 \quad \text{or} \quad r_0 r_1 \leqq 1) \quad \text{and} \quad (r_0 \leqq 1 \text{ or } r_1 r_2 \leqq 1).$$

Therefore, in this case there also exist two elements of the basis of rank 1, *i.e.* it is sufficient to consider case 2.

Case 2. — Assume without loss of generality that $r_0 = r_1 = 1$. Then the Markov-type equation (26) for the pair (e_0, e_1) is

$$2 + 2z^2 = 8z - p,$$

hence, $z = 2 \pm \sqrt{3 - (1/2)p}$, where $p = h_{23}$, so either $h_{23} = 4$ or $h_{23} = 6$.

Let $h_{23} = 6$, then $C_{23} = z = 2$. Then lemma 3.6 for the pair (e_2, e_3) provides that under $r_i \geqq r_j$ (here $\{i, j\} = \{2, 3\}$) the mutation does not reduce r_i only if $r_j \leqq 2\sqrt{1/2}$, *i.e.* $r_j = 1$. Then equation (17) for the pair (e_2, e_3) is

$$r_i^2 + 9 = 6r_i,$$

hence, $r_i = 3$. Then condition (24) gives $2C' \cdot 2 = 1 + 3$, *i.e.* $C' = C_{12} = C_{34} = 1$. If $i = 2$, $j = 3$, then (17) for the pair (e_1, e_2) implies $h_{12} = 4$, so the mutation $L_{e_1} e_2$ reduces the sum of ranks. And if $i = 3$, $j = 2$, then, similarly, the right mutation of the pair $(e_3, e_0 \otimes \omega^{-1})$ reduces the sum of ranks.

Let $h_{23} = 4$, then $z = 1$ or $z = 3$. Consider the basis corresponding to the lesser root $z = 1$. Then lemma 3.6 for the pair (e_2, e_3) implies that under $r_i \geqq r_j$ the mutation does not reduce r_i only if $r_j \leqq 1$, *i.e.* $r_j = 1$. Then (17) for (e_2, e_3) is

$$r_i^2 + 3 = 4r_i,$$

the lesser root is $r_i = 1$.

So we come to the case $r_0 = r_1 = r_2 = r_3 = 1$, $C_{01} = C_{23} = 1$. Then (24) gives $2C' = 2$, so $C' = C_{12} = C_{34} = 1$. For the obtained basis consisting of objects of rank 1 we have

$$1 = C_{i,\,i+1} = d_{i+1} - d_i,$$

then $d_i = d_0 + i$. Then the « helix » $\{e_i'\}$ in the sense of 2.3 determined by this basis satisfies the condition $d(e_{i+1}') = d(e_i') + 1$ for all i. Hence, it includes a foundation

$$(e_{i_0}', e_{i_0+1}', e_{i_0+2}', e_{i_0+3}')$$

such that $d(e_{i_0}') = 0$.

Thus, we obtain a basis (e_0, e_1, e_2, e_3) with $r_i = 1$, $d_i = i$. The assertion of the theorem just follows from lemma 4.1, applied to this basis and the basis corresponding to a foundation of the helix $\{\mathcal{O}(i)\}$.

5.4. COROLLARY. — *The Gram matrix of the form χ for a canonical basis is*

$$\begin{pmatrix} 1 & 4 & 10 & 20 \\ 0 & 1 & 4 & 10 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Proof. — It is easy to compute h_{ij} for a semiorthogonal basis using formula (17). Indeed, for a canonical basis we obtain

$$h_{i,\,i+1}=2+2\,C_{i,\,i+1}^2=4;\qquad h_{i,\,i+2}=2+2\,C_{i,\,i+2}^2=10;\qquad h_{i,i+3}=2+2\,C_{i,\,i+3}^2=20.$$

(Here $C_{ij}=d_j-d_i$).

6. Constructivity problem for a quadric

Consider a smooth three-dimensional quadric Q. For a semiorthogonal basis of $K_0(Q)$ formula (17) takes the form

$$x^2+y^2+3\,z^2=hxy, \tag{35}$$

where $x=r_i$, $y=r_j$, $z=C_{ij}$, $h=h_{ij}$. The constant $(k^2/2)\,H^3$ equals 9.

6.1. Proposition. — *$(-)$-bases of $K_0(Q)$ do not exist.*

Proof. — As in 5.1, it is sufficient to show that for any $(-)$-basis consisting of objects of negative rank there exists a mutation, which reduces the sum of ranks. As above, assume without loss of generality that $h_{01}\geqq h_{23}$, $h_{12}\geqq h_{34}$. Then (21) gives $h_{01}-h_{23}=9\,C_{01}>0$; similarly, $h_{12}-h_{34}=9\,C_{12}>0$. Hence, for the pairs (e_0, e_1) and (e_1, e_2) in (35) there holds the condition $h>9\,z$.

Use the reduction method for x and y in (35) with $h>9\,z$.

1) Let $x\geqq y\geqq z$. Then $x'<y$, for otherwise

$$0\leqq\varphi_x(y)=2\,y^2+3\,z^2-hy^2<5\,y^2-9\,zy^2,$$

which provides a contradiction.

2) Let $x\geqq z>y$. Then $x'<z$, for otherwise

$$0\leqq\varphi_x(z)=y^2+4\,z^2-hyz<5\,z^2-9\,z^2\,y,$$

which also provides a contradiction.

3) Finally, if $z>x$, $z>y$, then

$$5\,z^2>x^2+y^2+3\,z^2=hxy>9\,xyz,$$

i.e. $z>(9/5)\;xy>xy$.

Thus, the mutations of the pairs (e_0, e_1) and (e_1, e_2) do not reduce the sum of ranks only if $C=C_{01}>r_0\,r_1$, $C'=C_{12}>r_1\,r_2$. Then (23) (corollary 3.11) implies

$$r_0\,r_2-r_1\,r_3=3\,C\cdot C'>3\,r_0\,r_1\cdot r_1\,r_2\geqq 3\,r_0\,r_2,$$

which provides a contradiction.

6.2. Corollary. — *The elements of a helix on a smooth three-dimensional quadric are ordered by slopes.*

6.3. THEOREM. — *Any semiorthogonal basis of* $K_0(Q)$, *up to changing signs of some elements of it, can be reduced by mutation to one of the canonical bases*

$$(\mathcal{O}\otimes\mathcal{J}_p^n,\ \mathcal{S}^*\otimes\mathcal{J}_p^n,\ \mathcal{O}(1)\otimes\mathcal{J}_p^n,\ \mathcal{O}(2)\otimes\mathcal{J}_p^n),\qquad n\in\mathbb{Z},$$

where $\mathcal{S}$ is the spinor bundle.

Proof. — As in the proof of the theorem 5.3, use induction on the sum of ranks of elements of a semiorthogonal (+)-basis consisting of objects of positive rank.

For a (+)-basis condition (25) gives

$$h_{01}+h_{23}=9\,C_{01},\qquad h_{12}+h_{34}=9\,C_{12}.$$

As above, assume without loss of generality that $h_{01}\geqq h_{23}$, $h_{12}\geqq h_{34}$. Then in equation (35) for the pairs (e_0, e_1) and (e_1, e_2) we have $h\geqq(9/2)\,z$.

Use the reduction method for x and y.

1) Let $x\geqq y\geqq z$. Either $x'<y$ or

$$0\leqq\varphi_x(y)=2y^2+3z^2-hy^2\leqq 5y^2-\frac{9}{2}zy^2,$$

hence, $z\leqq 10/9$, *i.e.* $z=1$. Then

$$0\leqq\varphi_x(y)=2y^2+3-hy^2\leqq 3-\frac{5}{2}y^2,$$

hence, $y^2\leqq 6/5$, *i.e.* $y=1$. Then

$$0\leqq\varphi_x(y)=5-h,$$

i.e. $h\leqq 5$. On the other hand, $h\geqq 9/2\ z=9/2$, hence, $h=5$. Then equation (35) takes the form

$$x^2+4=5x,$$

the lesser root is $x=1$.

Thus, in this case x is not reduced only if $x=y=1$.

2) Let $x\geqq z>y$. Either $x'<z$ or

$$0\leqq\varphi_x(z)=y^2+4z^2-hzy<5z^2-\frac{9}{2}z^2y,$$

hence, $y=1$. Then

$$0\leqq\varphi_x(z)=1+4z^2-hz\leqq 1-\frac{1}{2}z^2,$$

hence, $z^2\leqq 2$, *i.e.* $z=1$, which contradicts to the considered case $z>y$. Hence, in this case x can always be reduced.

3) Let $z>x$, $z>y$. Then

$$5z^2>x^2+y^2+3z^2=hxy\geqq\frac{9}{2}zxy,$$

i.e. $z>9/10\,xy$.

Thus, the mutations of the pairs (e_0, e_1) and (e_1, e_2) do not reduce the sum of ranks only in the following cases:

1. $C=C_{01}>(9/10)\,r_0 r_1$, $C'=C_{12}>9/10\,r_1 r_2$.
2. The basis includes at least two objects of rank 1.

Case 1. — Apply lemma 4.2. Under the conditions of the lemma we have $p=q=9/10$, $A=3$. Then

$$\frac{\sqrt{Ap^2+1}}{Apq-1}=\frac{\sqrt{Aq^2+1}}{Apq-1}=\frac{\sqrt{3\cdot 81/100+1}}{3\cdot 81/100-1}<2.$$

Hence, both possible mutations do not reduce r_3 only if

$$(r_2\leqq 1 \text{ or } r_0 r_1\leqq 1) \quad\text{and}\quad (r_0\leqq 1 \text{ or } r_1 r_2\leqq 1).$$

Therefore, in this case there also exist two elements of the basis of rank 1.

Case 2. — Let $r_0=r_1=1$. Then the Markov-type equation (26) for (e_0, e_1) is

$$2+3z^2=9z-p,$$

so $z=3/2\pm\sqrt{(19-4p)/12}$, where $p=h_{23}$. Hence, $h_{23}=4$, then $z=1$ or $z=2$. Consider the basis corresponding to the lesser root $z=1$. Then lemma 3.6 for (e_2, e_3) provides that under $r_i\geqq r_j$ the mutation does not reduce r_i only if $r_j\leqq\sqrt{3/2}$, *i.e.* $r_j=1$. Then (17) for the pair (e_2, e_3) is

$$r_i^2+4=4r_i,$$

i.e. $r_i=2$.

Thus, $r_0=r_1=r_j=1$, $r_i=2$, $C_{01}=C_{23}=1$. Then (24) takes the form $3C'=3$, *i.e.* $1=C'=C_{12}=C_{34}$. Since $C_{34}=C(e_3, e_0\otimes\omega^{-1})$, the "helix" $\{e_i'\}$ in the sense of 2.3 determined by this basis satisfies the condition $C_{i,i+1}=1$ for all i. Then this "helix" includes a foundation (e_0, e_1, e_2, e_3) such that $r_1=2$, $r_0=r_2=r_3=1$. Here $\mu_1-\mu_0=C_{01}/r_0 r_1=1/2$, similarly, $\mu_2-\mu_1=1/2$, $\mu_3-\mu_2=1$.

Moreover, since the foundation $(e_{-4}, e_{-3}, e_{-2}, e_{-1})$ can be obtained from the considered one when tensoring by the canonical class $\mathcal{K}=-3H$, we may assume without loss of generality that $-1\leqq d_0\leqq 1$. Besides, $h_{01}=h_{12}=4$, which follows from (35), so $r(L_{e_0}e_1)=r(R_{e_2}e_1)=2$.

Then for $d_0=1$ consider the basis

$$f=(e_3\otimes\omega,\ L_{e_0}e_1,\ e_0,\ e_2),$$

and for $d_0=-1$, the basis

$$f=(e_2,\ R_{e_2}e_1,\ e_3,\ e_0\otimes\omega^{-1}).$$

In each case we have

$$r(f_1)=2, \qquad r(f_0)=r(f_2)=r(f_3)=1, \qquad d_0=0, \qquad C'=C=1.$$

Then

$$d_1=r_1\cdot\mu_1=1, \qquad d_2=r_2\cdot\mu_2=r_2\left(\mu_1+\frac{1}{2}\right)=1, \qquad d_3=r_3\cdot\mu_3=r_3(\mu_2+1)=2.$$

Thus, we obtain a basis for which

$$r_0=r_2=r_3=1, \quad r_1=2, \qquad d_0=0, \quad d_1=d_2=1, \quad d_3=2.$$

The assertion of the theorem follows from lemma 4.1, applied to this basis and the basis corresponding to the foundation of a helix

$$(\mathcal{O}, \mathcal{S}^*, \mathcal{O}(1), \mathcal{O}(2)).$$

6.4. Corollary. — The Gram matrix of the form χ for a canonical basis is

$$\begin{pmatrix} 1 & 4 & 5 & 14 \\ 0 & 1 & 4 & 16 \\ 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Proof. — Indeed, formula (17) implies :

$$h_{01}=h_{12}=\frac{1+4+3}{2}=4, \qquad h_{02}=h_{23}=\frac{1+1+3}{1}=5,$$

$$h_{03}=\frac{1+1+3\cdot 4}{1}=14, \qquad h_{13}=\frac{1+4+3\cdot 9}{2}=16.$$

7. Constructivity problem for V_5

For a semiorthogonal basis of $K_0(V_5)$ formula (17) with $x=r_i$, $y=r_j$, $z=C_{ij}$, $h=h_{ij}$ takes the form

$$x^2+y^2+5z^2=hxy. \tag{36}$$

The constant $(k^2/2)H^3$ equals 10.

7.1. Proposition. — *(−)-bases of* $K_0(V_5)$ *do not exist.*

Proof is similar to those of propositions 5.1, 6.1. Assume without loss of generality that $h_{01}\geqq h_{23}$, $h_{12}\geqq h_{34}$, then for the pairs (e_0, e_1) and (e_1, e_2) in equation (36) we have $h>10z$.

Use the reduction method for x and y.

1) Let $x \geqq y \geqq z$. Then $x' < y$, for otherwise

$$0 \leqq \varphi_x(y) = 2y^2 + 5z^2 - hy^2 < 7y^2 - 10zy^2,$$

which provides a contradiction.

2) Let $x \geqq z > y$. Then $x' < z$, for otherwise

$$0 \leqq \varphi_x(z) = y^2 + 6z^2 - hyz < 7z^2 - 10z^2 y,$$

which also provides a contradiction.

3) Finally, if $z > x$, $z > y$, then

$$7z^2 > x^2 + y^2 + 5z^2 = hxy > 10xyz,$$

i. e. $z > (10/7)\, xy > xy$.

Thus, the mutations of (e_0, e_1) and (e_1, e_2) do not reduce the sum of ranks only if $C = C_{01} > r_0 r_1$, $C' = C_{12} > r_1 r_2$. Then corollary 3.11 implies:

$$r_0 r_2 - r_1 r_3 = 5\, C \cdot C' > 5 r_0 r_1 \cdot r_1 r_2 \geqq 5 r_0 r_2,$$

which provides a contradition.

7.2. Corollary. — *The elements of a helix on* V_5 *are ordered by slopes.*

7.3. Theorem. — *Any semiorthogonal basis of* $K_0(V_5)$, *up to changing signs of some elements of it, can be reduced by mutation to one of the canonical bases*

$$(\mathcal{O} \otimes \mathcal{J}_p^n,\ \mathcal{Q} \otimes \mathcal{J}_p^n,\ \mathcal{S}^* \otimes \mathcal{J}_p^n,\ \mathcal{O}(1) \otimes \mathcal{J}_p^n), \qquad n \in \mathbb{Z},$$

where $\mathcal{S}$ *and* $\mathcal{Q}$ *are correspondingly the restrictions to* V_5 *of the universal bundle and the factor bundle on the Grassmannian* $G(2, 5)$ *in case of* V_5 *realized as the intersection of the image of the Plücker embedding* $G(2, 5) \subsetneq \mathbb{P}^9$ *with a general subspace* $\mathbb{P}^6 \subset \mathbb{P}^9$.

Proof. — When proving the theorem, instead of the basis (e_0, e_1, e_2, e_3) we consider sometimes, for convenience, the adjoint basis $(e_3^*, e_2^*, e_1^*, e_0^*)$, proving the constructivity of it. Because of the identities

$$(L_e f)^* = R_{e^*} f^*, \qquad (R_e f)^* = L_{e^*} f^*$$

the constructivity of the adjoint basis is equivalent to the fact that the basis (e_0, e_1, e_2, e_3) itself can be reduced by mutations to a basis, which is adjoint to some canonical. Therefore, to prove the constructivity of (e_0, e_1, e_2, e_3), it will be sufficient to verify that a basis adjoint to a canonical one is constructive.

As before, use induction on the sum of ranks of elements of a semiorthogonal $(+)$-basis consisting of objects of positive rank. Condition (25) gives

$$h_{01} + h_{23} = 10\, C_{01}, \qquad h_{12} + h_{34} = 10\, C_{12}.$$

Assume without loss of generality that $h_{01} \geqq h_{23}$, $h_{12} \geqq h_{34}$; then in equation (36) for (e_0, e_1) and (e_1, e_2) we have $h \geqq 5z$.

Use the reduction method for x and y.

1) Let $x \geqq y \geqq z$. Either $x' < y$ or

$$0 \leqq \varphi_x(y) = 2y^2 + 5z^2 - hy^2 \leqq 7y^2 - 5zy^2,$$

hence, $z = 1$. Then

$$0 \leqq \varphi_x(y) = 2y^2 + 5 - hy^2 \leqq 5 - 3y^2,$$

hence, $y = 1$. Then

$$0 \leqq \varphi_x(y) = 7 - h,$$

i.e. $h \leqq 7$. On the other hand, $h \geqq 5z = 5$. Equation (36) gives

$$x^2 + 6 = hx,$$

hence, either $h = 5$, the lesser root is $x = 2$, or $h = 7$, the lesser root is $x = 1$.

Thus, in this case x is not reduced only if

$$x = 1, \quad y = 1, \quad z = 1, \quad h = 7 \qquad \text{or} \qquad x = 2, \quad y = 1, \quad z = 1, \quad h = 5.$$

2) Let $x \geqq z > y$. Either $x' < z$ or

$$0 \leqq \varphi_x(z) = y^2 + 6z^2 - hzy < 7z^2 - 5z^2 y,$$

hence, $y = 1$. Assume now that $x \geqq 2z$. Then $x' < 2z$, for otherwise

$$0 \leqq \varphi_x(2z) = 1 + 9z^2 - 2hz \leqq 1 - z^2,$$

i.e. $z = 1$, which contradicts to the considered case $z > y$. Hence, it remains to consider $x < 2z$. Then $z > 1/2\, x = 1/2\, xy$.

Thus, in this case x is not reduced only if $y = 1$, $z > 1/2\, xy$.

3) Let $z > x$, $z > y$. Then

$$7z^2 > x^2 + y^2 + 5z^2 = hxy \geqq 5zxy,$$

i. e. $z > (5/7)\, xy$.

Thus, the mutations of (e_0, e_1) and (e_1, e_2) do not reduce the sum of rank only in the following cases (as above, denote $C = C_{01}$, $C' = C_{12}$):

1. The basis includes a pair with $x = 2$, $y = 1$, $z = 1$, $h = 5$.
2. $C > (5/7)\, r_0 r_1$, $C' > (5/7)\, r_1 r_2$.
3. $C > (5/7)\, r_0 r_1$, $C' > (1/2)\, r_1 r_2$, $\{r_1, r_2\} \ni 1$.
4. $C > (1/2)\, r_0 r_1$, $C' > (1/2)\, r_1 r_2$, $r_1 = 1$.
5. The basis includes at least two objects of rank 1.

(The possible case $C>(1/2)\, r_0 r_1$, $C'>(5/7)\, r_1 r_2$, $\{r_0, r_1\} \ni 1$ is reduced to case 3 by considering the basis $(e_2^*, e_1^*, e_0^*, e_3^* \otimes \omega^{-1})$, taking into account the remark at the beginning of the proof.)

Case 1. — Assume without loss of generality that $r_0=1$, $r_1=2$, $C_{01}=1$, $h_{01}=5$, then $C_{23}=1$, $h_{23}=5$. Then lemma 3.6 for (e_2, e_3) implies that under $r_i \geqq r_j$ the mutation does not reduce r_i only if $r_j \leqq \sqrt{5/3}$, *i. e.* $r_j=1$. But then e_0 and e_i are of rank 1 and the case is reduced to case 5.

Case 2. — Under the conditions of lemma 4.2 we have $p=q=5/7$, $A=5$. Then

$$\frac{\sqrt{A p^2+1}}{A pq-1}=\frac{\sqrt{A q^2+1}}{A pq-1}=\frac{\sqrt{5\cdot 25/49+1}}{5\cdot 25/49\text{-}1}<2.$$

Hence, both possible mutations do not reduce r_3 only if

$$(r_2 \leqq 1 \quad \text{or} \quad r_0 r_1 \leqq 1) \qquad \text{and} \qquad (r_0 \leqq 1 \quad \text{or} \quad r_1 r_2 \leqq 1).$$

Therefore, in this case there also exist two elements of rank 1, which corresponds to case 5.

Case 3. — Under the conditions of lemma 4.2 we have $p=5/7$, $q=1/2$, $A=5$. Then

$$\frac{\sqrt{A q^2+1}}{A pq-1}=\frac{\sqrt{5/4+1}}{25/14-1}=\frac{21}{11}<2.$$

Hence, he mutations do not reduce r_3 only if $r_0=1$ or $r_1 r_2=1$; besides, $\{r_1, r_2\} \ni 1$, so in the basis there exist two elements of rank 1 (case 5).

Case 4. — Under the conditions of lemma 4.2 we have $p=q=1/2$, $A=5$. Then

$$\frac{\sqrt{A p^2+1}}{A pq-1}=\frac{\sqrt{A q^2+1}}{A pq-1}=\frac{\sqrt{5/4+1}}{5/4-1}=6.$$

Here the assumptions of the lemma hold as strict inequalities $C>p\cdot r_0 r_1$, $C'>q\cdot r_1 r_2$, which implies that the inequalities in the conclusion of the lemma are also strict (*see* the proof of 4.2). Thus, the mutations do not reduce r_3 only if $r_0<6$ or $r_2<6$; thereto, $r_1=1$.

Assume without loss of generality [considering when necessary the basis $(e_2^*, e_1^*, e_0^*, e_3^* \otimes \omega^{-1})$] that $r_0<6$. If $r_0=1$, then we are under conditions of case 5; therefore consider only $2 \leqq r_0 \leqq 5$.

Recall that the considered case $C>(1/2)\, r_0 r_1$ corresponds to the situation $y=1$, $x \geqq z>(1/2)\, x$, so the condition

$$r_0 \geqq C > \frac{1}{2} r_0$$

must hold. Moreover, (36) under $y=1$ provides that $x \mid (5 z^2+1)$, hence, $r_0 \neq 5$, $r_0 \neq C$.

Thus, we consider a basis satisfying the following conditions:

$$r_1=1, \qquad 2\leqq r_0\leqq 4, \qquad \frac{1}{2}r_0<C<r_0, \qquad r_0 \mid (5\,C^2+1).$$

If $r_0=2$, then $1/2<C<1$, which provides a contradiction. If $r_0=4$, then $2<C<4$, *i.e.* $C=3$, but then $5\,C^2+1=46$, which is not divisible by $r_0=4$.

Finally, if $r_0=3$, then $3/2<C<3$, *i.e.* $C=C_{01}=C_{23}=2$. Then (17) for (e_0, e_1) implies $h_{01}=10$, and (25) gives $h_{23}=10$. Then lemma 3.6 for (e_2, e_3) provides that under $r_i\geqq r_j$ the mutation does not reduce r_i only if $r_j\leqq 2\sqrt{5/8}$, *i.e.* $r_j=1$. Thus, e_0 and e_j are of rank 1, which corresponds to case 5.

Case 5. – Let $r_0=r_1=1$. Then the Markov-type equation (26) for the pair (e_0, e_1) takes the form

$$2+5\,z^2=10\,z-p,$$

hence, $z=1\pm\sqrt{(3-p)/5}$, *i.e.* $p=h_{23}=3$, so $z=C_{01}=C_{23}=1$. Then lemma 3.6 for (e_2, e_3) implies that under $r_i\geqq r_j$ the mutation does not reduce r_i only if $r_j\leqq\sqrt{5}$, *i.e.* $r_j=1$ or $r_j=2$. Under $r_j=1$ equation (17) for (e_2, e_3) is

$$r_i^2+6=3\,r_i,$$

which has no integer roots. If $r_j=2$, then (17) for (e_2, e_3) is

$$r_i^2+9=6\,r_i,$$

hence, $r_i=3$. Then (24) gives $5\,C'=2+3$, *i.e.* $C'=1$. Considering if necessary the basis $(e_1^*\otimes\omega, e_0^*\otimes\omega, e_3^*, e_2^*)$, without loss of generality assume $r_2=3$, $r_3=2$.

Thus, we obtain a basis for which

$$r_0=r_1=1, \qquad r_2=3, \qquad r_3=2, \qquad C=C'=1.$$

Then the "helix" $\{e_i'\}$ in the sense of 2.3 determined by this basis satisfies the condition $C_{i,\,i+1}=1$ for all i. In the "helix" there exists a foundation (e_0, e_1, e_2, e_3) with $r_0=1$, $r_1=3$, $r_2=2$, $r_3=1$, and $0\leqq d_0\leqq 1$ (tensoring of a foundation by $\mathscr{K}$ reduces d_0 by $k=2$). One can easily check using (17) that under $d_0=1$ the basis

$$f=(e_3\otimes\omega, L_{e_0}e_2, L_{e_0}R_{e_2}e_1, e_0)$$

satisfies

$$r(f_0)=1, \qquad r(f_1)=3, \qquad r(f_2)=2, \qquad r(f_3)=1,$$
$$d(f_0)=0, \qquad C'=C=1.$$

Thus, we obtain a basis with

$$r_0=r_3=1, \qquad r_1=3, \qquad r_2=2, \qquad d_0=0, \qquad C=C'=1.$$

Then for it

$$\mu_1-\mu_0=\frac{C}{r_0 r_1}=\frac{1}{3}, \qquad \mu_2-\mu_1=\frac{C'}{r_1 r_2}=\frac{1}{6}, \qquad \mu_3-\mu_2=\frac{C}{r_2 r_3}=\frac{1}{2},$$

so $\mu_1=1/3$, $\mu_2=1/2$, $\mu_3=1$, hence, $d_1=d_2=d_3=1$. Applying lemma 4.1, conclude that any basis of such a kind is

$$(\mathcal{O}\otimes\mathcal{J}_p^n, \mathcal{Q}\otimes\mathcal{J}_p^n, \mathcal{S}^*\otimes\mathcal{J}_p^n, \mathcal{O}(1)\otimes\mathcal{J}_p^n).$$

According to the remark at the beginning of the proof of the theorem now it remains to verify that a basis adjoint to a canonical one is constructive. Indeed, for a basis adjoint to a canonical we have

$$r_0=r_3=1, \qquad r_1=2, \qquad r_2=3, \qquad C=C'=1.$$

Formula (17) gives $h_{12}=3$, then the basis

$$f=(e_0, e_2, R_{e_2} e_1, e_3)$$

provides the already considered case

$$r(f_0)=r(f_3)=1, \qquad r(f_1)=3, \qquad r(f_2)=2, \qquad C'=C=1.$$

7.4. Corollary. — *The Gram matrix of the form* χ *for a canonical basis is*

$$\begin{pmatrix} 1 & 5 & 5 & 7 \\ 0 & 1 & 3 & 10 \\ 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Proof. — Indeed, formula (17) implies:

$$h_{01}=\frac{1+9+5}{3}=5, \qquad h_{02}=h_{23}=\frac{1+4+5}{2}=5, \qquad h_{03}=\frac{1+1+5}{1}=7,$$

$$h_{12}=\frac{4+9+5}{2\cdot 3}=3, \qquad h_{13}=\frac{9+1+5\cdot 4}{3}=10.$$

8. Constructivity problem for V_{22}

For a semiorthogonal basis of $K_0(V_{22})$ formula (17) with $x=r_i$, $y=r_j$, $z=C_{ij}$, $h=h_{ij}$ takes the form

$$x^2+y^2+11z^2=hxy. \tag{37}$$

The constant $(k^2/2)\, H^3$ equals 11.

8.1. PROPOSITION. — *(−)-bases of* $K_0(V_{22})$ *do not exist.*

Proof is similar to those of propositions 5.1, 6.1, 7.1. Assume without loss of generality that $h_{01} \geqq h_{23}$, $h_{12} \geqq h_{34}$, then for the pairs (e_0, e_1) and (e_1, e_2) in equation (37) we have $h > 11\,z$. Moreover, since $h_{01} = 11\,C + h_{23}$ and $h_{12} = 11\,C + h_{34}$, recalling 3.4, we obtain $h \geqq 11\,z + 3$. Use the reduction method:

1) Let $x \geqq y \geqq z$. Then $x' < y$, for otherwise

$$0 \leqq \varphi_x(y) = 2y^2 + 11z^2 - hy^2 \leqq 2y^2 + 11z^2 - (11z+3)\,y^2 \leqq 10y^2 - 11zy^2,$$

which provides a contradiction.

2) Let $x \geqq z > y$. Either $x' < z$ or

$$0 \leqq \varphi_x(z) = y^2 + 12z^2 - hzy < 13z^2 - 11z^2 y,$$

hence, $y = 1$. Assume that $x \geqq 2z$. Then $x' < 2\,z$, for otherwise

$$0 \leqq \varphi_x(2z) = 1 + 15z^2 - 2hz \leqq 1 - 7z^2,$$

which provides a contradiction.

If $x < 2\,z$, then $z > (1/2)\,x = (1/2)\,xy$.

3) Let $z > x$, $z > y$. Then

$$13z^2 > x^2 + y^2 + 11z^2 = hxy > 11\,xyz,$$

hence, $z > (11/13)\,xy > (1/2)\,xy$.

Thus, the mutations of (e_0, e_1) and (e_1, e_2) do not reduce the sum of ranks only if $C = C_{01} > 1/2\; r_0 r_1$, $C' = C_{12} > 1/2\; r_1 r_2$. Then corollary 3.11 implies:

$$r_0 r_2 - r_1 r_3 = 11\,C \cdot C' > \frac{11}{4} r_0 r_1 \cdot r_1 r_2 > r_0 r_2,$$

which also provides a contradiction.

8.2. COROLLARY. — *The elements of a helix on* V_{22} *are ordered by slopes.*

8.3. THEOREM. — *Any semiorthogonal basis of* $K_0(V_{22})$, *up to changing signs of some elements of it, can be reduced by mutations to a canonical basis* $(e_0, \ldots, e_3)$ *such that*

$$r_0 = 1, \quad r_1 = 4, \quad r_2 = 3, \quad r_3 = 2, \qquad d_0 = 0, \quad d_1 = d_2 = d_3 = 1.$$

Proof. — As above, use induction on the sum of ranks of elements of a semiorthogonal (+)-basis consisting of objects of positive rank. As when proving theorem 7.3, consider sometimes the adjoint basis $(e_3^*, e_2^*, e_1^*, e_0^*)$; further verify that a basis adjoint to a canonical one is constructive.

Assume without loss of generality that $h_{01} \geqq h_{23}$, $h_{12} \geqq h_{34}$; then in equation (37) for (e_0, e_1) and (e_1, e_2) we have $h \geqq (11/2) z$. Use the reduction method:

1) Let $x \geqq y \geqq z$. Either $x' < y$ or

$$0 \leqq \varphi_x(y) = 2y^2 + 11z^2 - hy^2 \leqq 13y^2 - \frac{11}{2} zy^2,$$

hence, $z \leqq 26/11$, *i.e.* $z = 1$ or $z = 2$.

If $z = 1$, then

$$0 \leqq \varphi_x(y) = 2y^2 + 11 - \frac{11}{2} y^2,$$

hence, $y^2 \leqq 22/7$, *i.e.* $y = 1$. Then equation (37) takes the form

$$x^2 + 12 = hx,$$

hence, $x \mid 12$, *i.e.* $x = 1$ or $x = 2$ or $x = 3$ (the other factors of 12 correspond to greater roots of the quadratic, which can be transformed by mutations to lesser ones).

If $z = 2$, then

$$0 \leqq \varphi_x(y) = 2y^2 + 44 - hy^2 \leqq 44 - 9y^2,$$

hence, $y \leqq 2$, but since we consider the case $y \geqq z$, then $y = 2$. Then

$$0 \leqq \varphi_x(y) = 52 - h \cdot 4,$$

i.e. $h \leqq 13$. On the other hand, $h \geqq (11/2) z = 11$. Equation (37) takes the form

$$x^2 + 48 = 2hx,$$

hence, $h = 13$, the lesser root is $x = 2$.

Thus, in this case x can not be reduced only if

$$(x; y) \in \{(1; 1), (1; 2), (1; 3), (2; 2)\}.$$

2) Let $x \geqq z > y$. Either $x' < z$ or

$$0 \leqq \varphi_x(z) = y^2 + 12z^2 - hzy < 13z^2 - \frac{11}{2} z^2 y,$$

hence, $y \leqq 26/11$, *i.e.* $y = 1$ or $y = 2$.

If $y = 1$, then $x' = hy - x = h - x$, *i.e.* $x' \geqq x$ implies $h \geqq 2x$. Then

$$x^2 + y^2 + 11z^2 = hxy \geqq 2x^2,$$

i.e. $11z^2 + 1 \geqq x^2$. Since $z > y = 1$, we have $z \geqq 2$, then $1 \leqq z^2/4$, *i.e.* $11z^2 + (z^2/4) \geqq x^2$, hence,

$$z \geqq \frac{2}{\sqrt{45}} x = \frac{2}{\sqrt{45}} xy.$$

Let $y=2$. Assume that $x \geqq (3/2)\, z$. Then $x' < (3/2)\, z$, for otherwise

$$0 \leqq \varphi_x\left(\frac{3}{2} z\right) = 4 + \frac{9}{4} z^2 + 11\, z^2 - h \cdot \frac{3}{2} z \cdot 2 < 4 + \frac{53}{4} z^2 - \frac{11}{2} \cdot 3\, z^2 = 4 - \frac{13}{4} z^2,$$

hence, $z=1$, which contradicts to the considered case $z>y=2$.

Thus, in this case x is not reduced only if

$$y=1, \quad x \geqq z \geqq \frac{2}{\sqrt{45}} xy \qquad \text{or} \qquad y=2, \quad z > \frac{2}{3} x = \frac{1}{3} xy.$$

3) Let $z>x$, $z>y$. Then

$$13\, z^2 > x^2 + y^2 + 11\, z^2 = hxy \geqq \frac{11}{2} zxy,$$

hence, $z > (11/26)\, xy$.

Thus, the mutations of (e_0, e_1) and (e_1, e_2) do not reduce the sum of ranks only in the following cases (with account of the transition to the adjoint basis, as in the proof of 7.3):

1. $C > (11/26)\, r_0 r_1$, $C' > (11/26)\, r_1 r_2$.
2. $C > (11/26)\, r_0 r_1$, $C' > (1/3)\, r_1 r_2$, $\{r_1, r_2\} \ni 2$.
3. $C > (11/26)\, r_0 r_1$, $C' \geqq (2/\sqrt{45})\, r_1 r_2$, $\{r_1, r_2\} \ni 1$.
4. $C > (2/3)\, r_0$, $C' > (2/3)\, r_2$, $r_1 = 2$.
5. $r_0 \geqq C \geqq (2/\sqrt{45})\, r_0$, $r_2 \geqq C' \geqq (2/\sqrt{45})\, r_2$, $r_1 = 1$.
6. The basis includes a pair with $x=y=2$.
7. The basis includes a pair with $x=y=1$.
8. The basis includes a pair with $x=1$, $y=2$.
9. The basis includes a pair with $x=1$, $y=3$.

Case 1. — Under the conditions of lemma 4.2 we have $A=11$, $p=q=(11/26)$. Then

$$\frac{\sqrt{A p^2 + 1}}{A pq - 1} = \frac{\sqrt{A q^2 + 1}}{A pq - 1} < 2,$$

i.e. both possible mutations do not reduce r_3 only if

$$(r_2 \leqq 1 \quad \text{or} \quad r_0 r_1 \leqq 1) \qquad \text{and} \qquad (r_0 \leqq 1 \quad \text{or} \quad r_1 r_2 \leqq 1);$$

hence, the basis includes a pair with $x=y=1$ (case 7).

Case 2. — Under the conditions of lemma 4.2 we have $A=11$, $p=11/26$, $q=1/3$. Then

$$\frac{\sqrt{A q^2 + 1}}{A pq - 1} = \frac{\sqrt{11/9 + 1}}{121/78 - 1} < 3,$$

i.e. the mutations do not reduce r_3 only if $r_0 \leqq 2$ or $r_1 r_2 \leqq 2$; moreover, $\{r_1, r_2\} \ni 2$. Hence, the conditions of cases 6 or 8 hold.

Case 3. — Under the conditions of lemma 4.2 we have $A=11$, $p=(11/26)$, $q=(2/\sqrt{45})$. Then $(\sqrt{A q^2+1})/(A pq-1)<4$, *i.e.* the mutations do not reduce r_3 only if $r_0 \leqq 3$ or $r_1 r_2 \leqq 3$; thereto, $\{r_1, r_2\} \ni 1$. Hence, the conditions of cases 7, 8 or 9 hold.

Case 4. — We can directly apply lemma 4.2, but better estimations can be obtained when repeating the proof of it using the condition $r_1=2$.

Formula (23) gives

$$r_0 r_2 + 2 r_3 = 11\ C \cdot C' > 11 \cdot \frac{4}{9} r_0 r_2,$$

hence

$$r_3 > \frac{35}{18} r_0 r_2. \tag{38}$$

Then

$$\frac{88}{35} r_3 > r_0 r_2 + r_3 = 11\ C \cdot C' > 11\ C \cdot \frac{2}{3} r_2,$$

i.e.

$$C < \frac{12}{35} \cdot \frac{r_3}{r_2}. \tag{39}$$

Equation (17) for the pair (e_2, e_3) implies

$$r_2^2 + r_3^2 + 11\ C^2 = h_{23} r_2 r_3 \geqq 2 r_3^2,$$

since $h_{23} r_2 \geqq 2\ r_3$, assuming that r_3 is not reduced by the corresponding mutation. Using (39) we obtain

$$r_2^2 + 11 \cdot \left(\frac{12}{35}\right)^2 \cdot \frac{r_3^2}{r_2^2} > r_3^2;$$

and using the estimation $11 \cdot (12/35)^2 < 4/3$ obtain

$$r_2^4 > r_3^2 \cdot \left(r_2^2 - \frac{4}{3}\right).$$

If $r_2 \geqq 2$, then

$$r_3^2 < r_2^2 \cdot \frac{r_2^2}{(r_2^2-(4/3))} \leqq r_2^2 \cdot \frac{4}{(4-(4/3))} = \frac{3}{2} r_2^2 < \frac{25}{16} r_2^2,$$

i.e. $r_3 < (5/4)\ r_2$. Together with (38) it gives

$$r_2 > \frac{4}{5} \cdot \frac{35}{18} r_0 r_2,$$

hence, $r_0<9/14$, which provides a contradiction. Hence, $r_2=1$, *i.e.* the conditions of case 8 hold.

Case 5. — In this case we can not apply the methods of lemma 4.2, since $\mathrm{A}pq=44/45<1$.

If $r_0 \leqq 3\,\mathrm{C}$ or $r_2 \leqq 3\,\mathrm{C}'$ (case 5 *a*), *i.e.* $\mathrm{C} \geqq (1/3)\, r_0 r_1$ or $\mathrm{C}' \geqq (1/3)\, r_1 r_2$, then

$$\mathrm{A}pq=11\cdot\frac{2}{\sqrt{45}}\cdot\frac{1}{3}>1,$$

and lemma 4.2 can be applied. Consider at first the case $r_0>3\,\mathrm{C}$, $r_2>3\,\mathrm{C}'$ (case 5 *b*).

Case 5 *b*. So, let

$$\frac{r_0}{3}>\mathrm{C}\geqq\frac{2}{\sqrt{45}}r_0,\qquad \frac{r_2}{3}>\mathrm{C}'\geqq\frac{2}{\sqrt{45}}r_2,\qquad r_1=1.$$

Use the fact that $\mu(e_0\otimes\omega^{-1})-\mu(e_0)=k=1$. On the other hand,

$$\mu(e_0\otimes\omega^{-1})-\mu(e_0)=\sum_{i=0}^{3}(\mu_{i+1}-\mu_i)=\sum_{i=0}^{3}\frac{\mathrm{C}_{i,\,i+1}}{r_i r_{i+1}},$$

where e_4 denotes $e_0\otimes\omega^{-1}$. Thus,

$$1=\frac{\mathrm{C}}{r_0 r_1}+\frac{\mathrm{C}'}{r_1 r_2}+\frac{\mathrm{C}}{r_2 r_3}+\frac{\mathrm{C}'}{r_3 r_0},$$

where $r_1=1$, *i.e.*

$$\mathrm{C}(r_0+r_2 r_3)+\mathrm{C}'(r_2+r_0 r_3)=r_0 r_2 r_3.$$

Using the inequalities $r_0/3>\mathrm{C}$, $r_2/3>\mathrm{C}'$ we obtain

$$\frac{1}{3}(r_0^2+r_0 r_2 r_3+r_2^2+r_0 r_2 r_3)>\mathrm{r}_0 r_2 r_3,$$

hence

$$r_0^2+r_2^2>r_0 r_2 r_3. \tag{40}$$

Furthermore, the Markov-type equation (26) for the pair (e_0, e_1) is

$$x^2+y^2+11\,z^2=x\,(11\,z-p),$$

hence

$$z=\frac{x}{2}\pm\sqrt{\frac{x^2}{4}-\frac{x^2+px+1}{11}}.$$

Then the inequality $r_0/3 > C$, *i.e.* $x/3 > z$, implies that

$$\frac{x}{2} - \sqrt{\frac{x^2}{4} - \frac{x^2+px+1}{11}} < \frac{x}{3},$$

hence

$$\frac{x^2}{4} > \frac{x^2+px+1}{11} + \frac{x^2}{36} > x\left(\frac{x+p}{11} + \frac{x}{36}\right),$$

therefore, $(2/9)\,x > (x+p)/11$, *i.e.* $p < (13/9)\,x$. Thus,

$$h_{23} < (13/9)\, r_0. \tag{41}$$

Finally, assuming that the transfer of e_2 over e_3 does not reduce the sum of ranks we obtain the inequality $r_2 \leqq h_{23} r_3 - r_2$, hence, $h_{23} \geqq 2(r_2/r_3)$. Together with (41) it gives $(13/9)\, r_0 > 2(r_2/r_3)$, *i.e.*

$$r_0 r_3 > \frac{18}{13} r_2.$$

Substituting it into (40) we obtain

$$r_0^2 + r_2^2 > \frac{18}{13} r_2^2,$$

i.e. $r_0^2 > (5/13)\, r_2^2 > (1/4)\, r_2^2$, hence, $r_0 > (1/2)\, r_2$.

Similar computations for the pair (e_1, e_2) provide $r_2 > (1/2)\, r_0$. Hence, $2 > r_0/r_2 > 1/2$. Then (40) implies

$$r_3 < \frac{r_0}{r_2} + \frac{r_2}{r_0} < 2 + \frac{1}{2},$$

i.e. $r_3 \leqq 2$.

Thus, $r_1 = 1$, $r_3 \leqq 2$, so the conditions of cases 7 or 8 hold. It remains to consider.

Case 5 *a*. Assume without loss of generality that $p = 1/3$, $q = 2/\sqrt{45}$. Then $\sqrt{(A\,q^2+1)}/(A\,pq-1) < 16$. Hence, lemma 4.2 implies that the mutations do not reduce r_3 only if $r_0 \leqq 15$ or $r_2 \leqq 15$, since $r_1 = 1$.

To lessen the number of variants, make some extra estimations. Assuming that the transfers of e_0 over e_1 and of e_3 over e_2 do not reduce the sum of ranks we obtain the inequalities $h_{01} \geqq 2\, r_0$, $h_{23} \geqq 2(r_3/r_2)$. Then (25) implies

$$11\,C = h_{01} + h_{23} \geqq 2\left(r_0 + \frac{r_3}{r_2}\right) = \frac{2}{r_2}(r_0 r_2 + r_3) = \frac{2}{r_2} \cdot 11\,C \cdot C'$$

[last equality is equality (24)]. Hence, $C' \leqq (1/2)\, r_2$. Similarly, $C \leqq (1/2)\, r_0$.

Thus, we obtain that the considered basis includes the pair for which [using the notations of formula (37)]

$$x \leqq 15, \qquad y = 1, \qquad \frac{x}{2} \geqq z > \frac{2}{\sqrt{45}} x.$$

Furthermore, (37) implies that

$$h=\frac{11z^2+1}{x}+x,$$

hence, in particular, $x\,|\,(11\ z^2+1)$. Thereto, $z\leqq\frac{x}{2}$, *i.e.* $z\leqq 7$.

Now consider z lying in the interval $1\leqq z\leqq 7$ and for them x such that

$$2\ z\leqq x\leqq\frac{\sqrt{45}}{2}z<\frac{7}{2}z,$$

with $x\,|\,(11\,z^2+1)$.

Let $z=1$. Then $2\leqq x\leqq 3$, *i.e.* the conditions of the cases 8 or 9 hold.

Let $z=2$. Then $4\leqq x\leqq 6$; $11\,z^2+1=45$, hence, $x=5$. Then $h=14$.

Let $z=3$. Then $6\leqq x\leqq 10$; $11\,z^2+1=100$, hence, $x=10$. Then $h=20$.

Let $z=4$. Then $8\leqq x\leqq 13$; $11\,z^2+1=177=3\cdot 59$, so such x do not exist.

Let $z=5$. Then $10\leqq x\leqq 15$; $11\,z^2+1=276$, hence, $x=12$. Then $h=35$.

Let $z=6$. Then $12\leqq x\leqq 15$; $11\,z^2+1=397$, so such x do not exist.

Let $z=7$. Then $14\leqq x\leqq 15$; $11\,z^2+1=540$, hence, $x=15$. Then $h=51$.

Assume without loss of generality that $x=r_0$, $y=r_1$. Then for the pair (e_2, e_3) we have $C_{23}=z$, $h_{23}=11\,z-h$ [according to (25)]. Then lemma 3.6 for (e_2, e_3) implies that under $r_i\geqq r_j$ the mutation does not reduce r_i only if $r_j^2\leqq 11\,z^2/(11\ z-h-2)$.

For $z=2$, $h=14$, we obtain $r_j^2\leqq 44/6$, *i.e.* $r_j\leqq 2$.

For $z=3$, $h=20$, we obtain $r_j^2\leqq 99/11$, *i.e.* $r_j\leqq 3$.

For $z=5$, $h=35$, we obtain $r_j^2\leqq 275/18$, *i.e.* $r_j\leqq 3$.

Finally, for $z=7$, $h=51$, we obtain $r_j^2\leqq 539/24$, *i.e.* $r_j\leqq 4$.

Thus, either the pair (e_1, e_j) satisfies the conditions of the cases 7, 8 or 9, or (e_1, e_j) is a pair with $x=1$, $y=4$.

In last case consider for convenience a basis with $r_0=1$, $r_1=4$. Then the Markov-type equation (26) for (e_2, e_3) is

$$17+11\ z^2=4\,(11\ z-p),$$

hence, $z=2\pm\sqrt{(27-4p)/11}$, *i.e.* $p=4$, then $z=1$ or $z=3$. Consider the basis corresponding to the lesser root $z=1$. Then for it

$$h_{01}=11\,C-h_{23}=11\,z-p=7.$$

Thus, for the pair (e_0, e_1) we have $h_{01}=7$, and the ranks of elements of the pair are 1 and 4. Then the corresponding mutation of the pair leads to a pair with elements of ranks 1 and 3, *i.e.* to case 8.

Case 6. — Let $r_0=r_1=2$. Then (26) for (e_0, e_1) takes the form

$$8+11\ z^2=4\ (11\,z-p),$$

hence, $z=2\pm 2\sqrt{(9-p)/11}$, *i.e.* $h_{23}=p=9$, then $C=z=2$. Then lemma 3.6 for (e_2, e_3) implies that under $r_i \geqq r_j$ the mutation does not reduce r_i only if $r_j \leqq 2\sqrt{11/7}$, hence, $r_j=1$ or $r_j=2$. Under $r_j=1$ equation (17) for the pair (e_2, e_3) is

$$r_i^2+45=9\,r_i,$$

which provides a contradiction. Under $r_j=2$ equation (17) for the pair is

$$r_i^2+48=18\,r_i,$$

which also provides a contradiction.

Thus, the case is impossible.

Case 7. – Let $x=y=1$. The (37) and (25) imply

$$2+11\,z^2=h=h_{01}\leqq 11\,C_{01}=11\,z,$$

so, this case is also impossible.

Case 8. – Let $r_0=1$, $r_1=2$. Then (26) for (e_0, e_1) is

$$5+11\,z^2=2\,(11\,z-p),$$

hence, $z=1\pm\sqrt{(6-2\,p)/11}$, *i.e.* $h_{23}=p=3$, $C=z=1$. Then lemma 3.6 for (e_2, e_3) implies that under $r_i \geqq r_j$ the mutation does not reduce r_i only if $r_j \leqq \sqrt{11}$, hence, $r_j \leqq 3$.

The cases $r_j=1$ and $r_j=2$ because of $r_0=1$ and $r_1=2$ provide the cases 7 and 6 respectively, *i. e.* are impossible. Hence, $r_j=3$; then the pair (e_1, e_j) satisfies the conditions of case 9.

Case 9. – Let $r_0=1$, $r_1=3$. Then (26) for the pair (e_0, e_1) is

$$10+11\,z^2=3\,(11\,z-p),$$

hence, $z=(3/2)\pm(1/2)\sqrt{(59-12\,p)/11}$, *i.e.* $h_{23}=p=4$. Consider the basis corresponding to the lesser root $z=1$. Then lemma 3.6 for (e_2, e_3) implies that under $r_i \geqq r_j$ the mutation does not reduce r_i only if $r_j \leqq \sqrt{11/2}$, *i.e.* $r_j=1$ or $r_j=2$. Recalling case 7 we obtain $r_j=2$. Then (17) for (e_2, e_3) takes the form

$$r_i^2+15=8\,r_i,$$

the lesser root is $r_i=3$. Assume without loss of generality that $r_0=1$, $r_i=3$. Then the case $r_2=3$, $r_3=2$ is impossible, since formula (24) gives

$$11\,C\cdot C'=r_0\,r_2+r_1\,r_3=9.$$

Conversely, under $r_2=2$, $r_3=3$ formula (24) implies $11\,C\cdot C'=11$, hence, $C=C'=1$.

For the pair (e_3, e_4), where $e_4=e_0\otimes\omega^{-1}$, we have $C_{34}=C'=1$, and (17) implies $h_{34}=7$. Then $R_{e_4}e_3$ is of rank $h_{34}\,r_4-r_3=4$. Hence, for the basis

$$f=(e_0,\ R_{e_0}(e_3\otimes\omega),\ e_1,\ e_2)$$

we have

$$r(f_0)=1, \qquad r(f_1)=4, \qquad r(f_2)=3, \qquad r(f_3)=2,$$

and (24) implies $11\,\mathrm{C}\cdot\mathrm{C}'=3+4\cdot 2=11$, hence, $\mathrm{C}=\mathrm{C}'=1$. Then the "helix" $\{f_i\}$ in the sense of 2.3 determined by this basis satisfies the condition $\mathrm{C}_{i,\,i+1}=1$ for all i. This "helix" includes a foundation (e_0, e_1, e_2, e_3) for which $r_0=1$, $r_1=4$, $r_2=3$, $r_3=2$, and $d_0=0$, since the tensoring of a foundation by $\mathcal{K}$ reduces d_0 by $k=1$. Then

$$\mu_1-\mu_0=\frac{\mathrm{C}}{r_0 r_1}=\frac{1}{4}, \qquad \mu_2-\mu_1=\frac{\mathrm{C}'}{r_1 r_2}=\frac{1}{12}, \qquad \mu_3-\mu_2=\frac{\mathrm{C}}{r_2 r_3}=\frac{1}{6},$$

i.e. $\mu_1=1/4$, $\mu_2=1/3$, $\mu_3=1/3$, hence, $d_1=d_2=d_3=1$, as we need.

To finish the proof, it remains to show that a basis adjoint to a canonical one is constructive. Indeed, for a basis (e_0, e_1, e_2, e_3) adjoint to a canonical we have $r_0=2$, $r_1=3$, $r_2=4$, $r_3=1$. Then (17) implies that for this basis $h_{23}=(16+1+11)/4=7$, $h_{13}=(9+1+11)/3=7$. Then for the basis

$$f=(e_3, \mathrm{R}_{e_3} e_1, \mathrm{R}_{e_3} e_2, e_3\otimes\omega^{-1})$$

we have

$$r(f_0)=1, \qquad r(f_1)=7\cdot 1-3=4,$$
$$r(f_2)=7\cdot 1-4=3, \qquad r(f_3)=2, \qquad d(f_0)=0.$$

Then according to the computations in case 9 this basis is a canonical one.

8.4. COROLLARY. – *The Gram matrix of the form χ for a canonical basis is*

$$\begin{pmatrix} 1 & 7 & 7 & 8 \\ 0 & 1 & 3 & 8 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Proof. – Indeed, formula (17) implies:

$$h_{01}=\frac{16+1+11}{4}=7, \qquad h_{02}=\frac{9+1+11}{3}=7, \qquad h_{03}=\frac{4+1+11}{2}=8,$$

$$h_{12}=\frac{16+9+11}{3\cdot 4}=3, \qquad h_{13}=\frac{16+4+44}{2\cdot 4}=8, \qquad h_{23}=\frac{4+9+11}{2\cdot 3}=4.$$

REFERENCES

[1] A. A. BELINSON, Coherent Sheaves on $\mathbf{P}^n$ and Problems in Linear Algebra (in russian) (*Funkts. analiz. i ievo priloj.*, 1978, vol. 12, p. 68-69); english translation in *Functional Anal. Appl.*, vol. 12, 1978.

[2] A. I. BONDAL, Representations of Associative Algebras and Coherent Sheaves (in russian) (*Izv. AN SSSR. Ser. Matem*, 1989, vol. 53, p.23-42); english translation in *Math. USSR Izv.*, vol. 34, 1990.

[3] A. L. Gorodentsev, Transformations of Exceptional Vector Bundles on $\mathbf{P}^n$ (in russian) (*Izv. AN SSSR. Ser. Matem.*, 1988, vol. 52, p. 3-15); english translation in *Math. USSR Izv*, vol. 32, 1989.

[4] A. L. Gorodentsev, Exceptional Vector Bundles on Surfaces with a Moving Anticanonical Class (in russian) (*Izv. AN SSSR. Ser. Matem.*, 1988, vol. 52, p. 740-756); english translation in *Math. USSR Izv.*, vol. 33, 1989.

[5] A. L. Gorodentsev and A. N. Rudakov, Exceptional Vector Bundles on the Projective Spaces (*Duke Math. J.*, vol. 54, 1987, p. 115-130).

[6] J.-M. Drezet, Fibrés exceptionnels et variétés de modules de faisceaux semi-stables sur $\mathbb{P}_2(\mathbb{C})$ (*J. Reine und Angew. Math.*, vol. 380, p. 14-58).

[7] J.-M. Drezet, Cohomologie des variétés de modules de hauteur nulle (*Ann. of Math.*, vol. 281, 1988, p. 43-85).

[8] J.-M. Drezet, Fibrés exceptionnels et suite spectrale de Beilinson généralisés sur $\mathbb{P}_2(\mathbb{C})$ (*Ann. of Math.*, vol. 275, 1986, p. 25-48).

[9] J.-M. Drezet and J. Le Potier, Fibrés stables et fibrés exceptionnels sur $\mathbb{P}_2$ (*Ann. Scient. Éc. Norm. Sup.*, 1985, vol. 18, p. 193-244).

[10] M. M. Kapranov, *On the Derived Category of Coherent Sheaves on Grassmann Manifolds* (in russian) (*Izv. AN SSSR. Ser. Matem.*, vol. 48, 1984, p. 192-202); english translation in *Math. USSR Izv.*, vol. 24, 1985.

[11] M. M. Kapranov, *Derived Category of Coherent Sheaves on a Quadric* (in russian) (*Funkts. analiz. i ievo priloj.*, vol. 20, 1986, p. 67); english translation in *Functional Anal. Appl.*, vol. 20, 1986.

[12] M. M. Kapranov, On the Derived Category of Coherent Sheaves on some Homogeneous Spaces (*Invt. Math.*, vol. 92, 1988, p. 479-508).

[13] B. V. Karpov, A Symmetric Helix on the Plücker Quadric, in: *Helices and vector bundles*, A. N. Rudakov *et al.*, *Lond. Math. Soc. Lect. Notes Ser.*, vol. 148, 1990, p. 119-138.

[14] D. Yu Nogin, Spirals of Period four and Equations of Markov Type (in russian) (*Izv. AN SSSR. Ser. Matem.*, 1990, vol. 54, p. 862-878).

[15] A. N. Rudakov, The Markov Numbers and Exceptional Vector Bundles on $\mathbf{P}^2$ (in russian) (*Izv. AN SSSR. Ser. Matem.*, 1988, vol. 52, p. 100-112); english translation in *Math. USSR Izv.*, vol. 32, 1989.

[16] A. N. Rudakov, Exceptional Vector Bundles on a Quadric (in russian) (*Izv. AN SSSR. Ser. Mathem.*, 1988, vol. 52, p. 782-812); english translation in *Math. Izv.*, vol. 33, 1989.

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