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THE ADMISSIBLE DUAL OF SL(N). I

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This is the first of a short series of papers devoted to describing the admissible dual of the special linear group $G' = SL(N, F)$ of a non-Archimedean local field F, using the method of restriction to compact open subgroups. Since our approach here will be quite close to that taken in [BK] for the case of the general linear group, a brief description of that situation is in order.

To this end, we recall that a simple type in $G = GL(N, F)$ is a pair (J, λ), where J is a compact open subgroup of G and λ is an irreducible smooth representation of J, both components being of a very special kind. Attached to the simple type (J, λ) is a field extension K/F, $K \subset A = \mathbb{M}(N, F)$. Write G_K for the G-centraliser of $K^\times$, so that $G_K \cong GL(M, K)$, $M[K:F] = N$. The G-intertwining of the representation λ is then $JG_K J$, and $J \cap G_K$ is an Iwahori subgroup of G_K.

There are two particular examples of simple types. The first of these is the case where J is an Iwahori subgroup of G, and λ is the trivial character of J. In this case, we have $K = F$. At the other extreme, we have the *maximal* simple types: – a simple type (J, λ) is maximal if the associated field K is a maximal subfield of A, so that $G_K = K^\times$. Any irreducible smooth representation π of G containing a maximal simple type (J, λ) is supercuspidal. Indeed, there is a unique extension Λ of λ to a representation of the group $K^\times J$ which induces to π. It is one of the main results of [BK] that every irreducible supercuspidal representation of G contains a maximal simple type.

The general simple type (J, λ) in G is, in a precise sense, built from these two extremes. First, there is a maximal simple type (J_0, λ_0) [in a group $G_0 = GL(N_0, F)$] invariantly associated to (J, λ). Here, $N_0 = N/e$, for some integer e. If we choose an irreducible (necessarily supercuspidal) representation π_0 of G_0 which contains λ_0, then an irreducible representation π of G contains the simple type λ if and only if the supercuspidal support of π (in the sense of [BZ]) consists of unramified twists of π_0. In the other direction, the Hecke algebra $\mathscr{H}(G, \lambda)$ of compactly supported λ-spherical functions on G is isomorphic to the Hecke algebra $\mathscr{H}(G_K, 1)$, where 1 denotes here the

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trivial character of the Iwahori subgroup $J \cap G_K$ of G_K. Thus the classification of the irreducible representations of G containing λ is reduced to that of the irreducible representations of G_K containing the "trivial" type, *i. e.* the trivial character of an Iwahori subgroup. This reduction process respects virtually all of the interesting structures associated with representations. In particular, it preserves square-integrability, and is compatible with induction from parabolic subgroups. Moreover, an irreducible representation of G can contain at most one simple type (up to conjugacy), so we do have an effective classification of those irreducible representations of G which contain some simple type. An irreducible representation of G which contains no simple type is irreducibly induced from a proper parabolic subgroup of G, and the inducing data may be chosen to have an explicit description in terms of simple types.

We shall show in these papers that the admissible dual of $G' = SL(N, F)$ exhibits similar structures. There is, however, one extra complication. In the case of GL (N), the dual is governed by Hecke algebras $\mathscr{H}(G_K, 1)$ (as above) which are all *affine*: – they have a uniform presentation in terms only of the dimension $M = N/[K:F]$ and the size of the residue class field of K. Similar affine Hecke algebras appear in association with the dual of G′ (they arise from "simple types" in G′ which come directly from simple types in G), but certain degenerate algebras also arise. For example, some of the tame intertwining algebras in [M] are of this form, as are the algebras associated with the unitary principal series representations discussed in [GK1,2]. These seem to be intimately connected with elliptic representations (*see*, e.g. [C]).

In this paper, we confine our attention to the two extreme cases of supercuspidal representations and representations with Iwahori fixed vector. As in the case of GL (N), the starting point is the notion of "fundamental stratum" in G′. In section 1, we show that there is a quite exact relation between the fundamental strata occurring in a given irreducible representation π of G and those occurring in its restriction $\pi \mid G'$. From there, it is comparatively easy to give a general (*i. e.* characteristic-independent) treatment of the basic properties of the restriction functor on smooth representations induced by the inclusion $G' \to G$. Tadić's recent paper [T] also considers this topic. Our method is completely different, and yields essential extra detail. Nonetheless, some of the principal results of our section 1 [notably (1.17), (1.20) and parts of (1.7)] are proved in [T]. ([T] also contains a convenient survey of the history of this topic, to which we refer the interested reader.)

In section 2, we show that every irreducible supercuspidal representation of G′ is induced from a compact open subgroup. This is a direct consequence of the fundamental property "intertwining implies conjugacy" of simple types in G. We do not consider here, in any detail, the structure of the inducing representation. This is better treated, in greater generality, in the next paper of the series.

In section 3, we consider representations with Iwahori fixed vector. The trivial character of an Iwahori subgroup is a simple type (in the context of either G or G′), and, using this viewpoint, we show directly that the category of smooth representations of G′ which are generated by Iwahori fixed vectors is equivalent to the category of all modules over the appropriate Hecke algebra. Moreover, this classification is compatible with the functor of restriction from G to G′. This classification (for admissible representations, and any semisimple group) is given in [Bo], and the admissibility condition can be removed using the "Bernstein Centre". Our approach, however, is totally different, and provides the model for more general subsequent arguments. The one major topic of [Bo] not covered here, namely the characterisation of representations with Iwahori fixed vector in terms of parabolic induction, will be treated more generally in a later paper.

In conclusion, we have to thank our colleagues David Keys, Lawrence Morris and Roger Plymen for stimulating conversations, and also the Center for Advanced Studies of the University of Iowa, in whose unique surroundings this project was conceived.

Notation. — Throughout, F denotes a non-Archimedean local field. We write $\mathfrak{o}_F$ for the discrete valuation ring in F, $\mathfrak{p}_F$ for the maximal ideal of $\mathfrak{o}_F$, $k_F=\mathfrak{o}_F/\mathfrak{p}_F$ for the residue class field of F. We use V to denote an F-vector space of finite dimension N, and set $A=\mathrm{End}_F(V)$, $G=\mathrm{Aut}_F(V)$. We write $\det_G$, or just det, for the determinant homomorphism $G \to F^\times$, and $G'=\mathrm{Ker}(\det)$. Thus $G\cong GL(N, F)$, $G'\cong SL(N, F)$. Further, if K is any subgroup of G, we write $K'=K\cap G'$.

0. Preliminaries

We start by collecting together a few basic techniques which we use throughout this paper and its successors. We will always work in the context of a locally profinite group G (*i.e.*, one which is locally compact and totally disconnected). If K_1, K_2 are compact subgroups of G, and ρ_i is a smooth representation of K_i, for $i=1,2$, we say that an element $g\in G$ *intertwines* ρ_1 with ρ_2 if there is a nonzero $K_1^g\cap K_2$-homomorphism between ρ_1^g and ρ_2. Here, $K_1^g=g^{-1}K_1 g$ and ρ_1^g denotes the representation $x\mapsto \rho_1(g x g^{-1})$ of K_1^g.

From now on, assume that the compact subgroups K_i are also *open*. In this context, if $(\pi, \mathscr{V})$ is a smooth representation of G and ρ_i is irreducible, we say that ρ_i *occurs in* π if $\mathscr{V}$ has a K_i-subspace equivalent to ρ_i. As usual, the sum of all such subspaces of $\mathscr{V}$ is called the ρ_i-*isotypic subspace of* $\mathscr{V}$, and it is denoted by $\mathscr{V}^{\rho_i}$.

(0.1) *Let* $(\pi, \mathscr{V})$ *be a smooth representation of the locally profinite group* G. *For* $i=1,2$, *let* ρ_i *be an irreducible smooth representation of a compact open subgroup* K_i *of* G. *Suppose that*

(*a*) $\mathscr{V}$ *is generated, as* G-*space, by* $\mathscr{V}^{\rho_1}$, *and*

(*b*) ρ_2 *occurs in* $\mathscr{V}$.

Then some element $g\in G$ *intertwines* ρ_1 *with* ρ_2.

In particular, hypothesis (*a*) here holds if $\mathscr{V}$ is irreducible and ρ_1 occurs in $\mathscr{V}$. Under this more restrictive condition, (0.1) is proved in [H] Lemma 2.2. However, the proof there remains valid in this extra generality.

Now we specialise to our standard situation $G = GL(N, F)$, where F is some non-Archimedean local field. We also write $G' = SL(N, F)$, and $\det : G \to F^\times$ for the standard determinant homomorphism. If K is a subgroup of G, we always write $K' = K \cap G'$. We shall repeatedly need the following property:

(0.2) *Let* K *be a compact open subgroup of* G, ρ *an irreducible smooth representation of* K, *and* ρ' *an irreducible component of the restriction* $\rho \mid K'$. *Let* $\bar{\rho}$ *be an irreducible smooth representation of* K *such that* $\bar{\rho} \mid K'$ *also contains* ρ'. *Then there exists a quasicharacter* χ *of* $F^\times$ *such that* $\bar{\rho} \cong \rho \otimes \chi \circ \det$.

Proof. – Let $U = \mathrm{Ker}(\rho)$, so that, in particular, U is an open normal subgroup of K of finite index. The restriction $\bar{\rho} \mid U'$ contains the trivial character, so $\bar{\rho} \mid U$ contains a character of the form $\chi \circ \det \mid U$, for some quasicharacter χ of $F^\times$. Since $\bar{\rho}$ is irreducible and $U \lhd K$, it follows that $\bar{\rho} \mid U$ is a multiple of this character. We may therefore adjust $\bar{\rho}$ to be trivial on U.

We are now effectively working on the finite discrete group K/U. By hypothesis, we have $\mathrm{Hom}_{K'U}(\rho, \bar{\rho}) \neq 0$, so Frobenius Reciprocity implies that $\bar{\rho}$ occurs in the representation $\mathrm{Ind}(\rho \mid K'U : K'U, K)$. This induced representation is just $\rho \otimes \mathrm{Ind}(1)$, where 1 denotes the trivial character of $K'U$. However, Ind(1) is a sum of representations of the form $\chi_i \circ \det \mid K$, for various quasicharacters χ_i of $F^\times$. ■

(0.3) *For* $i = 1, 2$, *let* K_i *be a compact open subgroup of* G *and* ρ_i *an irreducible component of* $\rho_i \mid K_i'$, *and suppose that* $x \in G$ *intertwines* ρ_1' *with* ρ_2'. *Then there exists a quasicharacter* χ *of* $F^\times$ *such that* x *intertwines* ρ_1 *with* $\rho_2 \otimes \chi \circ \det$.

Proof. – Replacing (K_1, ρ_1, ρ_1') by $(K_1^x, \rho_1^x, (\rho_1')^x)$ allows us to assume that $x = 1$. We then have to show that, for some χ, the representations ρ_1, $\rho \otimes \chi \circ \det$ have a component in common on $K_1 \cap K_2$, given that the ρ_i' have a component in common on $K_1' \cap K_2' = (K_1 \cap K_2)'$.

Let σ' be an irreducible representation of $K_1' \cap K_2'$ occurring in both $\rho_i' \mid K_1' \cap K_2'$. Choose an irreducible representation σ of $K_1 \cap K_2$ whose restriction to $K_1' \cap K_2'$ contains σ'. By (0.2), there exist quasicharacters χ_i of $F^\times$ such that $\sigma \otimes \chi_1 \circ \det$ occurs in $\rho_i \mid K_1 \cap K_2$, $i = 1, 2$. Thus the restrictions $\rho_1 \mid K_1 \cap K_2$ $\rho_2 \otimes (\chi_1 \chi_2^{-1}) \circ \det \cap K_1 \cap K_2$ have the common component $\sigma \otimes \chi_1 \circ \det$, as required. ■

1. Fundamental strata and the restriction functor

We start this section by showing that G' has a theory of "fundamental strata" analogous to that [Bu] for G. (The relevant background is also reviewed in [BK], particularly sections 1.1, 2.3) This gives some extremely useful information about the way irreducible smooth representations of G' restrict to certain compact open subgroups. With this to hand, we derive the basic properties of the functor of restriction from representations of G to representations of G'.

We fix once for all a continuous character ψ_F of the additive group of F, with conductor $\mathfrak{p}_F$, and write $\psi_A = \psi_F \circ \mathrm{tr}$, where tr is the trace map $A \to F$.

Let $\mathfrak{A}$ be some hereditary $\mathfrak{o}_F$-vector in A, with Jacobson radical $\mathfrak{B}$. As in [BK] section 1.1, we write $\mathbf{U}(\mathfrak{A}) = \mathfrak{A}^{\times}$, $\mathbf{U}^n(\mathfrak{A}) = 1 + \mathfrak{P}^n$, $n \geqq 1$. We recall that there is an isomorphism between $\mathfrak{P}^{-n}/\mathfrak{P}^{1-n}$ and the Pontrjagin dual of the finite abelian group $\mathbf{U}^n(\mathfrak{A})/\mathbf{U}^{n+1}(\mathfrak{A})$, $n \geqq 1$, denoted $b + \mathfrak{P}^{1-n} \mapsto \psi_b$, $b \in \mathfrak{P}^{-n}$, where ψ_b is the character $x \mapsto \psi_A(b(x-1))$, $x \in \mathbf{U}^n(\mathfrak{A})$, of $\mathbf{U}^n(\mathfrak{A})/\mathbf{U}^{n+1}(\mathfrak{A})$. In the language of **[BK]**, the correspondence $b \mapsto \psi_b$ gives a bijection between equivalence classes of strata $[\mathfrak{A}, n, n-1, b]$ and characters of the finite abelian group $\mathbf{U}^n(\mathfrak{A})/\mathbf{U}^{n+1}(\mathfrak{A})$, when $n \geqq 1$.

We recall that this parametrisation gives a useful description of intertwining of certain characters in G. In particular, suppose we have two strata $[\mathfrak{A}, n, n-1, b]$, $[\mathfrak{B}, m, m-1, c]$. Write $\mathfrak{P} = \mathrm{rad}(\mathfrak{A})$, $\mathfrak{Q} = \mathrm{rad}(\mathfrak{B})$, and let $g \in G$. We say that these strata are *intertwined formally* by g if

$$(b + \mathfrak{P}^{1-n}) \cap g^{-1}(c + \mathfrak{Q}^{1-m})\, g \neq \varnothing.$$

Then, if we also have, $n, m \geqq 1$, the element g intertwines the strata formally if and only if it intertwines the characters ψ_b of $\mathbf{U}^n(\mathfrak{A})$, ψ_c of $\mathbf{U}^m(\mathfrak{B})$ (*cf.* [H] Lemma 2.3).

There is an analogous structure in G′, given as follows.

(1.1) PROPOSITION. — *Let $\mathfrak{A}$ be a hereditary $\mathfrak{o}_F$-order in* A, *with Jacobson radical $\mathfrak{P}$, and let $n \geqq 1$ be an integer. The map $b \mapsto \psi_b \mid \mathbf{U}^n(\mathfrak{A})'$ induces an isomorphism between $\mathfrak{P}^{-n}/((F \cap \mathfrak{P}^{-n}) + \mathfrak{P}^{1-n})$ and the Pontrjagin dual of* $\mathbf{U}^n(\mathfrak{A})'/\mathbf{U}^{n+1}(\mathfrak{A})'$.

Proof. — Write $e = e(\mathfrak{A} \mid \mathfrak{o}_F)$ for the period of $\mathfrak{A}$, so that $\mathfrak{p}_F \mathfrak{A} = \mathfrak{P}^e$. The canonical embedding $\mathbf{U}^n(\mathfrak{A})'/\mathbf{U}^{n+1}(\mathfrak{A})' \to \mathbf{U}^n(\mathfrak{A})/\mathbf{U}^{n+1}(\mathfrak{A})$ is an isomorphism when n is not divisible by e (*see* [BF] (2.8.3)) and, in this case, we have $F \cap \mathfrak{P}^{-n} \subset \mathfrak{P}^{1-n}$. So suppose that $n = em$, for some integer m. Composition with the determinant gives an injection

$$(\mathbf{U}^m(\mathfrak{o}_F)/\mathbf{U}^{m+1}(\mathfrak{o}_F))\hat{} \to (\mathbf{U}^{em}(\mathfrak{A})/\mathbf{U}^{em+1}(\mathfrak{A}))\hat{}$$

whose cokernel is $(\mathbf{U}^{em}(\mathfrak{A})'/\mathbf{U}^{em+1}(\mathfrak{A})')\hat{}$. On the other hand, the image of the composite

$$(\mathbf{U}^m(\mathfrak{o}_F)/\mathbf{U}^{m+1}(\mathfrak{o}_F))\hat{} \to (\mathbf{U}^{em}(\mathfrak{A})/\mathbf{U}^{em+1}(\mathfrak{A}))\hat{} \to \mathfrak{P}^{-em}/\mathfrak{P}^{1-em}$$

is $((F \cap \mathfrak{P}^{-em}) + \mathfrak{P}^{1-em})/\mathfrak{P}^{1-em}$. The Proposition follows. ■

When convenient, we shall often abbreviate $\psi_b \mid \mathbf{U}^n(\mathfrak{A})' = \psi'_b$. Now let $[\mathfrak{A}, n, n-1, b]$ be some stratum in A. We say that it is *G′-fundamental* if the coset $b + \mathfrak{P}^{1-n} + (F \cap \mathfrak{P}^{-n})$, where $\mathfrak{P} = \mathrm{rad}(\mathfrak{A})$, contains no nilpotent element of A. This is equivalent to saying that all of the strata $[\mathfrak{A}, n, n-1, b+c]$ are fundamental, as c ranges over $F \cap \mathfrak{P}^{-n}$.

(1.2) PROPOSITION. — *Let $[\mathfrak{A}, n, n-1, b]$ be a G′-fundamental stratum in* A, *with $n \geqq 1$. Suppose that the character ψ'_b intertwines in G′ with the trivial character of* $\mathbf{U}^m(\mathfrak{B})'$, *where $\mathfrak{B}$ is some hereditary order in* A, *and $m \geqq 1$. Then $n/e(\mathfrak{A}) \leqq (m-1)/e(\mathfrak{B})$.*

This follows from the (slightly) more general result

(1.3) PROPOSITION. — *Let* $[\mathfrak{A}, n, n-1, b]$, $n \geqq 1$, *be a* G'*-fundamental stratum in* A, *let* $\mathfrak{B}$ *be a hereditary order in* A, *and* $m \geqq 1$ *an integer. Suppose that the character* ψ_b *of* $\mathbf{U}^n(\mathfrak{A})$ *intertwines in* G *with a character* θ *of* $\mathbf{U}^m(\mathfrak{B})$ *which factors through* $\det_G$. *Then* θ *is trivial and* $n/e(\mathfrak{A}) \leqq (m-1)/e(\mathfrak{B})$.

Proof. — Suppose first that the character θ of $\mathbf{U}^m(\mathfrak{B})$ is trivial. Then the fundamental stratum $[\mathfrak{A}, n, n-1, b]$ intertwines formally with the stratum $[\mathfrak{B}, m, m-1, 0]$, and the assertion follows from [BK] (2.6.2). So, let $k \geqq m$ be the least integer such that the character θ of $\mathbf{U}^m(\mathfrak{B})$ is trivial on $\mathbf{U}^{k+1}(\mathfrak{B})$. Then $\theta \mid \mathbf{U}^k(\mathfrak{B})$ is of the form ψ_c, for some $c \in \mathrm{F}$. The stratum $[\mathfrak{B}, k, k-1, c]$ is fundamental, and intertwines formally with the fundamental stratum $[\mathfrak{A}, n, n-1, b]$. This implies $n/e(\mathfrak{A}) = k/e(\mathfrak{B})$, so $c \in \mathfrak{P}^{-n}$. The strata $[\mathfrak{A}, n, n-1, b-c]$, $[\mathfrak{B}, k, k-1, 0]$ therefore intertwine formally, and, since $[\mathfrak{A}, n, n-1, b-c]$ is fundamental, [BK] (2.6.2) shows that $n/e(\mathfrak{A}) \leqq (k-1)/e(\mathfrak{B})$. This contradiction completes the proof. ■

We now have the analogue in G' of the main result of [Bu].

(1.4) PROPOSITION. — *Let* π' *be an irreducible smooth representation of* G'. *Then either*

(i) *there is a hereditary* $\mathfrak{o}_F$*-order* $\mathfrak{A}$ *in* A *such that* $\pi' \mid$ G' *contains the trivial character of* $\mathbf{U}^1(\mathfrak{A})'$, *or*

(ii) *there is a* G'*-fundamental stratum* $[\mathfrak{A}, n, n-1, b]$ *in* A, *with* $n \geqq 1$, *such that* $\pi' \mid \mathbf{U}^n(\mathfrak{A})'$ *contains the character* ψ'_b.

These two possibilities are mutually exclusive. Moreover, if (ii) *holds, and if* $[\mathfrak{B}, m, m-1, c]$ *is a further* G'*-fundamental stratum in* A *such that* $\pi' \mid \mathbf{U}^{m+1}(\mathfrak{B})'$ *contains* ψ'_c, *then we have* $m/e(\mathfrak{B} \mid \mathfrak{o}_F) = n/e(\mathfrak{A} \mid \mathfrak{o}_F)$.

Proof. — Suppose that condition (i) does not hold. Choose a hereditary $\mathfrak{o}_F$-order $\mathfrak{A}$ in A, and an integer $n \geqq 1$, such that

(1.5) (*a*) π' *contains the trivial character of* $\mathbf{U}^{n+1}(\mathfrak{A})'$, *and*

(*b*) *the quantity* $n/e(\mathfrak{A} \mid \mathfrak{o}_F)$ *is minimal subject to* (*a*).

Thus $\pi' \mid \mathbf{U}^n(\mathfrak{A})'$ contains some character ψ'_b, with $b \in \mathfrak{P}^{-n}$, where $\mathfrak{P} = \mathrm{rad}(\mathfrak{A})$. We show that the stratum $[\mathfrak{A}, n, n-1, b]$ is G'-fundamental. Suppose otherwise. We may therefore choose b within its $(\mathrm{F} \cap \mathfrak{P}^{-n}) + \mathfrak{P}^{1-n}$-coset so that the stratum $[\mathfrak{A}, n, n-1, b]$ is not fundamental. By [Bu] Th. 2 (or [BK] (2.3.1)), there is a hereditary order $\mathfrak{A}_1$ and an integer $n_1 \geqq 1$ such that $\mathbf{U}^{n_1+1}(\mathfrak{A}_1) \subset \mathrm{Ker}(\psi_b) \subset \mathbf{U}^n(\mathfrak{A})$ and $n_1/e(\mathfrak{A}_1) < n/e(\mathfrak{A})$. This means that π' contains the trivial character of $\mathbf{U}^{n_1+1}(\mathfrak{A}_1)'$, contrary to (1.5).

We conclude that if (i) of the Proposition fails, then (ii) holds. Now we show that these possibilities are mutually exclusive. Suppose, for a contradiction, that π' contains the trivial character of $\mathbf{U}^1(\mathfrak{A}_1)'$ for some hereditary order $\mathfrak{A}_1$, and the character ψ'_b of $\mathbf{U}^n(\mathfrak{A})'$ for some G'-fundamental stratum $[\mathfrak{A}, n, n-1, b]$ in A. Thus (by (0.1)) ψ'_b and the trivial character of $\mathbf{U}^1(\mathfrak{A}_1)'$ intertwine in G'. Now (1.2) implies $n/e(\mathfrak{A}) \leqq 0$, which is impossible.

The final statement follows easily from (1.2) and symmetry. ■

The uniqueness part of (1.4) allows us to define a concept of *level* for irreducible representations of G', just as for G. To be precise, let π' (resp. π) be an irreducible smooth representation of G' (resp. G). Define $l_{G'}(\pi')$ [resp. $l_G(\pi)$] to be the minimum value of $n/e(\mathfrak{U})$, as n ranges over non-negative integers and $\mathfrak{U}$ over hereditary $\mathfrak{o}_F$-orders in A such that π' contains the trivial character of $U^{n+1}(\mathfrak{U})'$ (resp. π contains the trivial character of $U^{n+1}(\mathfrak{U})$). Further, if π is an irreducible smooth representation of G, we put

$$(1.6)\qquad l_G^0(\pi)=\min_{\chi}\{l_G(\pi\otimes\chi\circ\det)\},$$

where χ ranges over all quasicharacters of $F^\times$.

We next have to compare the fundamental strata which occur in a given irreducible smooth representation π of G, and the analogous structures over G' when π is considered as a smooth representation of G' via restriction.

First we need some more notation. If $(\pi, \mathscr{V})$ is a smooth representation of G, we write $(\check{\pi}, \check{\mathscr{V}})$ for the G-smooth dual or contragredient of $(\pi, \mathscr{V})$. Likewise, if $(\sigma, \mathscr{W})$ is a smooth representation of G', we write $(\sigma^*, \mathscr{W}^*)$ for the G'-smooth dual of $(\sigma, \mathscr{W})$. Thus, if $(\pi, \mathscr{V})$ is a smooth representation of G, we can view it as a smooth representation of G' via restriction, and form the G'-dual $(\pi^*, \mathscr{V}^*)$. We then have $\check{\mathscr{V}}\subset\mathscr{V}^*$ (as spaces of linear forms on $\mathscr{V}$).

Also, write $\mathscr{Z}(G)$ for the centre of G. Thus $\mathscr{Z}(G)\cong F^\times$, and is a closed subgroup of G. We can form the closed subgroup $\mathscr{Z}(G)G'$, which can also be described as the subgroup of $g\in G$ such that $\det(g)$ is an N-th power in $F^\times$, where $N=\dim_F(V)$. When the characteristic of F does not divide N, the subgroup $\mathscr{Z}(G)G'$ is open and of finite index in G, which makes most of what follows completely trivial in that case. For (i) of the following Proposition, *see* [T] and the references cited there.

(1.7) PROPOSITION. — *Let $(\pi, \mathscr{V})$ be a smooth irreducible representation of* G.

(i) *The restriction $\pi|G'$ is a finite direct sum of irreducible smooth representations of* G'. *Moreover, the* G*-stabiliser of any irreducible* G'*-subspace of $\mathscr{V}$ is open, normal and of finite index in* G.

(ii) *The canonical embedding $\check{\mathscr{V}}\to\mathscr{V}^*$ is an isomorphism.*

(iii) *We have $l_G^0(\pi)=l_{G'}(\pi')$, for any irreducible* G'*-component π' of $\pi|G'$.*

Proof. — Nothing is changed if we replace π by $\pi\otimes\chi\circ\det$, for some quasicharacter χ of $F^\times$. Therefore, we assume that π satisfies

$$(1.8)\qquad l_G(\pi)=l_G^0(\pi).$$

The main effect of this hypothesis is:

(1.9) LEMMA. — *Let $(\pi, \mathscr{V})$ be an irreducible smooth representation of* G *which satisfies* (1.8). *Let $\mathfrak{U}$ be a hereditary order in* A, *and $n\geq 0$ an integer. Suppose that $\pi|U^{n+1}(\mathfrak{U})$ contains a character ρ which factors through* $\det_G$. *Then ρ is trivial.*

Proof. — Let χ be a quasicharacter of $F^\times$ such that $\rho=\chi\circ\det|U^{n+1}(\mathfrak{U})$. Let $m\geq n$ be the least integer such that $\rho|U^{m+1}(\mathfrak{U})$ is trivial. Then $l_G(\pi)=m/e(\mathfrak{U})$, while $l_G(\pi\otimes\chi^{-1}\circ\det)\leq n/e(\mathfrak{U})$, and the assertion follows from (1.8). ■

To avoid horrifying superscripts, we now introduce another notation. If K is a subgroup of G (or G′), and $\mathscr{V}$ is some G-space (or G′-space), we write

$$\mathscr{F}(\mathscr{V}, \mathrm{K}) = \mathscr{V}^{\mathrm{K}} \tag{1.10}$$

for the space of K-fixed vectors in $\mathscr{V}$.

(1.11) LEMMA. – *Let* $(\pi, \mathscr{V})$ *be an irreducible smooth representation of* G *which satisfies* (1.8). *Let* $\mathfrak{A}$ *be a hereditary order in* A *and* $n \geqq 0$ *an integer such that* $l_{\mathrm{G}}(\pi) = n/e(\mathfrak{A})$. *Then*

$$\mathscr{F}(\mathscr{V}, \mathbf{U}^{n+1}(\mathfrak{A})') = \mathscr{F}(\mathscr{V}, \mathbf{U}^{n+1}(\mathfrak{A})).$$

Proof. – We certainly have $\mathscr{F}(\mathscr{V}, \mathbf{U}^{n+1}(\mathfrak{A})') \supset \mathscr{F}(\mathscr{V}, \mathbf{U}^{n+1}(\mathfrak{A}))$. For the opposite containment, take $v \in \mathscr{F}(\mathscr{V}, \mathbf{U}^{n+1}(\mathfrak{A})')$, and let $\mathscr{U}$ denote the $\mathbf{U}^{n+1}(\mathfrak{A})$-space generated by v. Then $\mathscr{U}$ is a finite-dimensional representation of $\mathbf{U}^{n+1}(\mathfrak{A})$. By (1.9), these are all trivial, and the lemma follows. ■

(1.12) LEMMA. – *Let* $(\pi, \mathscr{V})$ *be an irreducible smooth representation of* G *which satisfies* (1.8). *Let* $\mathscr{V}_1 \supset \mathscr{V}_2$ *be* G′*-subspaces of* $\mathscr{V}$ *such that* $\mathscr{W} = \mathscr{V}_1/\mathscr{V}_2$ *is irreducible over* G′. *Write* σ *for the implied representation of* G′ *on* $\mathscr{W}$. *Then* $l_{\mathrm{G}'}(\sigma) = l_{\mathrm{G}}(\pi)$.

Proof. – (1.9) implies straightaway that $l_{\mathrm{G}}(\pi) \leqq l_{\mathrm{G}'}(\sigma)$. Thus, if $l_{\mathrm{G}'}(\sigma) = 0$, there is nothing to prove. We therefore assume that $l_{\mathrm{G}'}(\sigma) > 0$ and $l_{\mathrm{G}'}(\sigma) > l_{\mathrm{G}}(\pi)$. There is a G′-fundamental stratum $[\mathfrak{A}, n, n-1, b]$ in A such that $n \geqq 1$ and σ contains ψ'_b. Choose $v \in \mathscr{V}_1$, $v \notin \mathscr{V}_2$, such that v transforms according to ψ'_b under $\mathbf{U}^n(\mathfrak{A})'$, and let $\mathscr{U}$ be the $\mathbf{U}^n(\mathfrak{A})$-space generated by v. As a representation of $\mathbf{U}^n(\mathfrak{A})$, this is a finite direct sum of characters ρ such that $\rho \mid \mathbf{U}^{n+1}(\mathfrak{A})$ factors through det. (1.9) implies that $\rho \mid \mathbf{U}^{n+1}(\mathfrak{A})$ is trivial, so $\rho = \psi_c$, where $c \in \mathfrak{P}^{-n}$ and $c \equiv b \pmod{(\mathrm{F} \cap \mathfrak{P}^{-1}) + \mathfrak{P}^{1-n}}$ (and $\mathfrak{P} = \mathrm{rad}(\mathfrak{A})$). Since the stratum $[\mathfrak{A}, n, n-1, b]$ is G′-fundamental, it follows that $[\mathfrak{A}, n, n-1, c]$ is fundamental, and $l_{\mathrm{G}}(\pi) = n/e(\mathfrak{A})$, which gives the desired contradiction. ■

Taking these two results together, we get the following:

(1.13) *Let* $(\pi, \mathscr{V})$ *be an irreducible smooth representation of* G *which satisfies* (1.8). *Let* $\mathscr{V}_1 \supset \mathscr{V}_2$ *be* G′*-subspaces of* $\mathscr{V}$ *such that* $\mathscr{V}_1/\mathscr{V}_2$ *is irreducible over* G′. *There exists a hereditary order* $\mathfrak{A}$ *and an integer* $n \geqq 0$ *such that* $n/e(\mathfrak{A}) = l_{\mathrm{G}}(\pi)$ *and the space* $\mathscr{F}(\mathscr{V}_1, \mathbf{U}^{n+1}(\mathfrak{A})') = \mathscr{F}(\mathscr{V}, \mathbf{U}^{n+1}(\mathfrak{A})) \cap \mathscr{V}_1$ *has nonzero image in* $\mathscr{V}_1/\mathscr{V}_2$.

Continuing in the same situation, it follows that any nontrivial G′-subquotient $\mathscr{W}$ of $\mathscr{V}$ has a fixed vector under $\mathbf{U}^{n+1}(\mathfrak{A})'$, for some pair $(\mathfrak{A}, n)$ such that $n/e(\mathfrak{A}) = l_{\mathrm{G}}(\pi)$. Moreover, if $\mathscr{F}(\mathscr{W}, \mathbf{U}^{n+1}(\mathfrak{A})') \neq \{0\}$, then

$$\mathscr{F}(\mathscr{W}, \mathbf{U}^{n+1}(x^{-1}\mathfrak{A}x)') \neq \{0\}$$

for any $x \in \mathrm{G}'$. By the following lemma (1.14), the hereditary orders in A fall into a finite number of G′-conjugacy classes. Since π is admissible, the spaces $\mathscr{F}(\mathscr{V}, \mathbf{U}^{n+1}(\mathfrak{A}))$ are all finite-dimensional, whence $\pi \mid \mathrm{G}'$ has finite length bounded by

$$\sum_{\mathfrak{A}} \dim \mathscr{F}(\mathscr{V}, \mathbf{U}^{n+1}(\mathfrak{A})),$$

where $\mathfrak{U}$ ranges over a set of representatives of the G′-conjugacy classes of hereditary orders in A, and $n/e(\mathfrak{U})=l_G(\pi)$.

(1.14) LEMMA. — *There are only finitely many* G′*-conjugacy classes of hereditary* $\mathfrak{o}_F$*-orders in* A.

Proof. — The hereditary orders in A form finitely many G-conjugacy classes. Indeed, these conjugacy classes are in bijection with the set of ordered partitions of $N=\dim_F(V)$ modulo cyclic permutation: — *see* [Re] (39.14). The G′-classes of orders within the G-conjugacy class of $\mathfrak{U}$ are in bijection with the double cosets $\mathfrak{K}(\mathfrak{U})\backslash G/G'=G/\mathfrak{K}(\mathfrak{U})\,G'$, where, in standard notation, $\mathfrak{K}(\mathfrak{U})=\{x\in G : x^{-1}\mathfrak{U}x=\mathfrak{U}\}$. The determinant gives a bijection $G/\mathfrak{K}(\mathfrak{U})\,G'=F^\times/\det(\mathfrak{K}(\mathfrak{U}))$. However, $\det(\mathfrak{K}(\mathfrak{U}))$ contains $\det(U(\mathfrak{U}))=U(\mathfrak{o}_F)$ and π_F^N for any prime element π_F of F. Therefore the group $G/\mathfrak{K}(\mathfrak{U})\,G'$ is finite, of order dividing N. ■

Now we return to the proof of the Proposition, knowing that $\mathscr{V}$ has finite length as G′-space. It therefore has an irreducible G′-subspace.

(1.15) LEMMA. — *In the same situation, let* $\mathscr{V}_1$ *be an irreducible* G′*-subspace of* $\mathscr{V}$. *Let* $H=\{x\in G : \pi(x)\mathscr{V}_1=\mathscr{V}_1\}$. *Then* $H\supset \mathscr{Z}(G)\,G'$, *and* H *is open, normal and of finite index in* G.

Proof. — The space $\mathscr{V}_1$ is certainly stabilized by $\mathscr{Z}(G)$ and G′, so $H\supset\mathscr{Z}(G)\,G'$ and H is normal in G. Choose some nonzero $v\in\mathscr{V}_1$, and an open subgroup K of G which fixes v. Take $y\in K$, and consider the subspace $\mathscr{V}_2=\pi(y)\mathscr{V}_1$. Since G′ is normal in G, this is a G′-subspace of $\mathscr{V}$, and it is irreducible. Moreover, it meets $\mathscr{V}_1$ nontrivially, so we deduce that $\mathscr{V}_2=\mathscr{V}_1$ and that $K\subset H$. Therefore $\det(H)$ contains $\det(\mathscr{Z}(G)\,K)=F^{\times N}\det(K)$, which is open and of finite index in $F^\times$. The lemma follows. ■

Thus, if $\mathscr{V}_1$ is any irreducible G′-subspace of $\mathscr{V}$, with stabiliser H, then $\mathscr{V}$ is the sum of subspaces $\pi(x)\mathscr{V}_1$, as x ranges over the finite abelian group G/H. This proves (i) of the Proposition.

Continuing with the same $\mathscr{V}_1$, H, consider the group $\bar{H}$ of elements $g\in G$ such that $\pi(g)\mathscr{V}_1\cong\mathscr{V}_1$ as a representation of H. Thus $\bar{H}\supset H$ and $\bar{H}/H$ is finite abelian. Supposing that $\bar{H}\neq H$, we can choose a subgroup H′, $H\subsetneq H'\subset\bar{H}$ such that H′/H is cyclic. Then $\mathscr{V}_1$ admits extension to a representation of H′. The H′-subspace of $\mathscr{V}$ generated by $\mathscr{V}_1$ is thus a direct sum of irreducible H′-subspaces, each of which is irreducible (and equivalent to $\mathscr{V}_1$) on restriction to H. As a result of this, if we choose $\mathscr{V}_1$ so that its stabiliser H is maximal, the translates $\pi(x)\mathscr{V}_1$ are mutually inequivalent as representations of H. Writing π_1 for the implied representation of H on $\mathscr{V}_1$, we have

$$(1.16)\qquad \pi\cong\operatorname{Ind}(\pi_1 : H, G)=\prod_{x\in G/H}\pi(x)\mathscr{V}_1.$$

Taking contragredients, we have $\check{\pi}=\operatorname{Ind}(\check{\pi}_1 : H, G)$, using $\check{}$ to denote H-duals as well. We have $(\mathscr{V}_1)\check{}\subset(\mathscr{V}_1)^*$, and the latter is irreducible over G′, so $\check{\pi}_1\,|\,G'$ is irreducible. The spaces $\mathscr{V}$, $\check{\mathscr{V}}$ therefore have the same finite composition length $(G:H)$ over G'. However, $\mathscr{V}^*$ has the same G'-length as $\mathscr{V}$, so the inclusion $\check{\mathscr{V}} \subset \mathscr{V}^*$ is an equality. This proves (ii), and (iii) has already been done in (1.12). This completes the proof of the Proposition. ■

The next step is to show that all smooth irreducible representations of G' arise from G, by restriction, in this manner.

(1.17) Proposition. – *Let $(\pi', \mathscr{V}')$ be an irreducible smooth representation of* G'.

(i) *There exists an irreducible smooth representation $(\pi, \mathscr{V})$ of G such that $\mathscr{V}'$ is equivalent to a G'-subspace of $\mathscr{V}$.*

(ii) *Let $(\pi, \mathscr{V})$ be as in* (i), *and let $(\pi_1, \mathscr{V}_1)$ be some irreducible smooth representation of* G. *Then $\mathscr{V}_1$ has a G'-subspace equivalent to $\mathscr{V}'$ if and only if* $\pi_1 \cong \pi \otimes \chi \circ \det$, *for some quasicharacter χ of* $F^\times$.

Proof. – In (i), we shall in fact show that the contragredient $(\pi'^*, \mathscr{V}'^*)$ of $(\pi', \mathscr{V}')$ is equivalent to a subspace of some irreducible G-space. This will clearly suffice.

Choose a hereditary $\mathfrak{o}_F$-order $\mathfrak{A}$ in A and an integer $n \geqq 0$ such that $l_{G'}(\pi') = n/e(\mathfrak{A})$ and π' contains the trivial character of $\mathbf{U}^{n+1}(\mathfrak{A})'$. Abbreviate $\mathbf{U}^{n+1}(\mathfrak{A}) = \mathbf{U}$, and let ω' be the central quasicharacter of π'. In particular, ω' is trivial on $\mathscr{Z}(G') \cap \mathbf{U}'$. There exists a quasicharacter ω of $F^\times = \mathscr{Z}(G)$ which agrees with ω' on $\mathscr{Z}(G')$ and is trivial on $\mathscr{Z}(G) \cap \mathbf{U}$. We put $G'' = \mathscr{Z}(G)\, G'$, and extend π' to a representation of G'' on $\mathscr{V}'$ by setting $\pi''(zg) = \omega(z)\, \pi'(g)$, $z \in \mathscr{Z}(G)$, $g \in G'$. By construction, π'' contains the trivial character of $\mathbf{U}'' = \mathbf{U} \cap G''$. We form the smooth induced representation

$$(\pi, \mathscr{V}) = \operatorname{Ind}(\pi'' : G'', G).$$

Thus $\mathscr{V}$ is the space of all right G-smooth functions $\phi : G \to \mathscr{V}'$ such that $\phi(xg) = \pi''(x)\, \phi(g)$, $x \in G''$, $g \in G$. There is a canonical G''-surjection

$$\varepsilon : \quad \mathscr{V} \to \mathscr{V}'$$

given by $\varepsilon(\phi) = \phi(1)$.

We next choose a nonzero vector $u \in \mathscr{F}(\mathscr{V}', \mathbf{U}'')$, and define a function $\Phi \in \mathscr{F}(\mathscr{V}, \mathbf{U})$ by

(1.18) (i) Φ *has support* $G''\mathbf{U} = \mathbf{U}\, G''$;

(ii) $\Phi(gu) = \pi''(g)\, u$, $g \in G''$, $u \in \mathbf{U}$.

Then $\varepsilon(\Phi) = \Phi(1) = u \neq 0$. Now let $\mathscr{U}$ be the G-subspace of $\mathscr{V}$ generated by Φ. Then ε restricts to a G''-surjection $\mathscr{U} \to \mathscr{V}'$. Write $\mathscr{U}_1 = \mathscr{F}(\mathscr{U}, \mathbf{U})$. There is a uniquely determined $\mathbf{U}$-subspace $\mathscr{U}_2$ of $\mathscr{U}$ such that $\mathscr{U} = \mathscr{U}_1 \oplus \mathscr{U}_2$. Likewise, there is a uniquely $\mathbf{U}''$-decomposition $\mathscr{V}' = \mathscr{V}'_1 \oplus \mathscr{V}'_2$ of $\mathscr{V}'$, with $\mathscr{V}'_1 = \mathscr{F}(\mathscr{V}'_1, \mathbf{U}'')$.

(1.19) Lemma. – *We have $\varepsilon(\mathscr{U}_i) = \mathscr{V}'_i$, for $i = 1, 2$.*

Proof. – Since ε is surjective, it is enough to show that $\varepsilon(\mathscr{U}_i) \subset \mathscr{V}'_i$, for each i. We surely have $\varepsilon(\mathscr{U}_1) \subset \mathscr{V}'_1$, so it is enough to show that $\varepsilon(\mathscr{M}) \subset \mathscr{V}'_2$ for any irreducible $\mathbf{U}$-subspace $\mathscr{M}$ of $\mathscr{U}_2$. If this is false, $\mathscr{M}$ contains a nonzero $\mathbf{U}''$-fixed vector v. The

U-space spanned by v is then a direct sum of characters of U of the form $\chi \circ \det$ and trivial on U″. However, by the construction of $\mathscr{U}$ and (0.1), any such character must intertwine in G with the trivial character of U, which means it is trivial. This is impossible, so we have the lemma. ■

Now let f be a nontrivial G′-smooth linear from on $\mathscr{V}'$ which is null on $\mathscr{V}'_2$. Then $f \circ \varepsilon$ is a nonzero linear form on $\mathscr{U}$ which is null on $\mathscr{U}_2$. It follows that $f \circ \varepsilon$ is G-smooth. Of course, $f \circ \varepsilon = \varepsilon^*(f)$, where ε^* is the G′-injection $(\mathscr{V}')^* \to \mathscr{U}^*$ defined by ε and G′-duality. We have just shown that the G′-injection $(\mathscr{V}')^* \to \mathscr{U}^*$ defined by ε and G′-duality. We have just shown that the G′-irreducible space $\varepsilon^*((\mathscr{V}')^*)$ meets, and is therefore contained in, the G′-subspace $\check{\mathscr{U}}$ of $\mathscr{U}^*$.

Thus we have a G′-embedding ε^* of $(\mathscr{V}')^*$ in the smooth G-space $\check{\mathscr{U}}$. Write $\mathscr{W}$ for the G-space generated by $\varepsilon^*((\mathscr{V}')^*)$, and choose a nonzero $v \in (\mathscr{V}')^*$. Let $\mathscr{W}_0$ be a G-subspace of $\mathscr{W}$ maximal for the property of not containing $\varepsilon(v)$. Then $\mathscr{W}/\mathscr{W}_0$ is an irreducible smooth G-space, and the implied G′-map $(\mathscr{V}')^* \to \mathscr{W}/\mathscr{W}_0$ is nonzero, hence injective. This completes the proof of (i).

To prove the uniqueness statement, again let $\mathfrak{U}$, n be such that π' contains the trivial character of $\mathrm{U}^{n+1}(\mathfrak{U})'$ and $n/e(\mathfrak{U}) = l_{G'}(\pi')$. Write $\mathrm{H} = \mathrm{U}^{n+1}(\mathfrak{U})\mathscr{Z}(\mathrm{G})\mathrm{G}'$. As in the proof of (1.15), π' admits extension to a smooth representation $\tilde{\pi}'$ of H, and any two such extensions must differ by a tensor factor of the form $\chi \circ \det | \mathrm{H}$, for some quasicharacter χ of $\mathrm{F}^\times$. Any irreducible representation of G which contains π' is thus a factor of an induced representation $\mathrm{Ind}(\tilde{\pi}' \otimes \chi \circ \det : \mathrm{H}, \mathrm{G}) = \mathrm{Ind}(\tilde{\pi}' : \mathrm{H}, \mathrm{G}) \otimes \chi \circ \det$. If σ is an irreducible component of $\mathrm{Ind}(\tilde{\pi}' : \mathrm{H}, \mathrm{G})$, one verifies directly that $\mathrm{Ind}(\sigma | \mathrm{H} : \mathrm{H}, \mathrm{G})$ is equivalent to $\sigma \otimes \mathrm{Ind}(1)$, where 1 denotes the trivial character of H. However, $\mathrm{Ind}(1 : \mathrm{H}, \mathrm{G})$ is a finite sum of characters $\xi \circ \det$. The result follows. ■

In all, restriction induces a functor from the category of smooth representations of G to the category of smooth representations of G′. It takes representations of finite length to representations of finite length. It will be useful to have some more precise statements.

(1.20) PROPOSITION (*see* also [T]). — *Let* $(\pi, \mathscr{V})$ *be an irreducible smooth representation of* G, *and* $(\pi', \mathscr{V}')$ *an irreducible* G′-*subspace of* $\mathscr{V}$. *Then*:

(i) π *is supercuspidal if and only if* π' *is supercuspidal*;

(ii) π *is discrete series if and only if* π' *is discrete series*.

Proof. — Any two irreducible G′-subspaces of $\mathscr{V}$ are G-conjugate, up to equivalence, so their coefficient functions have the same analytic properties. Thus π has an irreducible supercuspidal (resp. discrete series) G′-subspace if and only if every irreducible G′-subspace of π is supercuspidal (resp. discrete series).

To prove (i), let P be some proper parabolic subgroup of G with unipotent radical U. As usual, write $\mathscr{V}_\mathrm{U}$ for the maximal quotient of $\mathscr{V}$ on which U acts trivially. Then π is supercuspidal if and only if $\mathscr{V}_\mathrm{U} = 0$ for all P, and the same applies to irreducible representations of G′. Now write $\mathscr{V} = \mathscr{V}_1 \oplus \ldots \oplus \mathscr{V}_r$, where the $\mathscr{V}_i$ are irreducible G′-subspaces. We have $\mathrm{U} \subset \mathrm{G}'$, and $\mathrm{V}_\mathrm{U} = \sum (\mathscr{V}_i)_\mathrm{U}$. Thus, for given P, the space $\mathscr{V}_\mathrm{U}$ is zero if and only if all $(\mathscr{V}_i)_\mathrm{U}$ are zero. Allowing P to range over all proper parabolic subgroups of G (hence G′), we see that $\mathscr{V}$ is supercuspidal over G if and only if all $\mathscr{V}_i$ are supercuspidal over G″.

In part (ii), assume to start with that π is discrete series. Tensoring with a one-dimensional representation $\chi \circ \det$, we can assume that π is actually square-integrable. By (1.7) (ii), any coefficient function of π' is the restriction of a coefficient ϕ of π. If ϕ is square integrable on G, it is surely square-integrable on G′, so π' is discrete series.

Conversely, assume that π' is discrete series, hence square integrable. We can adjust π so that its central quasicharacter is unitary. Choose a nonzero $v \in \mathscr{V}'$, and a nonzero $\check{v} \in \check{\mathscr{V}}$ which is null on some G′-complement of $\mathscr{V}'$ in $\mathscr{V}$. It is enough to show that the function

$$g \mapsto \langle \pi(g) v, \check{v} \rangle, \qquad g \in G,$$

is absolutely square-integrable on $G/\mathscr{Z}(G)$. For suitable choices of measure, we have

$$\begin{aligned}\int_{G/\mathscr{Z}(G)} |\langle \pi(g) v, \check{v} \rangle|^2 \, d\dot{g} &= \int_{G/\mathscr{Z}(G)G'} \int_{\mathscr{Z}(G)G'/\mathscr{Z}(G)} |\langle \pi(x)\pi'(y) v, \check{v} \rangle|^2 \, dy dx \\ &= \int_{G/\mathscr{Z}(G)G'} \int_{\mathscr{Z}(G)G'/\mathscr{Z}(G)} |\langle \pi'(y) v, \check{\pi}(x^{-1})\check{v} \rangle|^2 \, dy dx.\end{aligned}$$

We can identify $\mathscr{Z}(G)G'/\mathscr{Z}(G) = G'/G' \cap \mathscr{Z}(G)$, so the inner integral converges (to a locally constant function) by hypothesis. The group $G/\mathscr{Z}(G)G' \cong F^\times/F^{\times N}$ is compact, so the outer integral always converges. ■

2. Supercuspidal representations of G′

We now prove a fundamental result on the structure of irreducible supercuspidal representations of G′. Special cases of this result are proved in [KS] and [MS].

(2.1) THEOREM. – *Let π' be an irreducible supercuspidal representation of* G′. *There exists an principal order* $\mathfrak{A}$ *in* A, *and an irreducible representation* τ' *of* $U(\mathfrak{A})'$ *such that* $\pi' \cong \operatorname{Ind}(\tau' : U(\mathfrak{A})', G')$.

Before proceeding, we need to recall a few things from [BK] about the supercuspidal representations of G. If π is an irreducible supercuspidal representation of G, then we know ([BK] (8.4.1)) that π contains a *maximal simple type* (J, λ). Thus, in particular, J is a compact open subgroup of G, and λ is an irreducible smooth representation of J which occurs in the restriction $\pi | J$. Attached to this maximal simple type is a principal $\mathfrak{o}_F$-order in A [with $J \subset U(\mathfrak{A})$] and also a field extension E/F such that $E^\times$ normalises the order $\mathfrak{A}$ and the pair (J, λ). Further, there is an extension Λ of λ to a representation of $E^\times J$ such that $\pi \cong \operatorname{Ind}(\Lambda : E^\times J, G)$. The pairs (J, λ), ($E^\times J$, Λ) are uniquely determined up to G-conjugation. (*See* [BK], § 6, for more detail.)

Now we prove (2.1). By (1.7) and (1.20), there is an irreducible supercuspidal representation π of G such that π' is equivalent to an irreducible G′-subspace of π. Let (J, λ) be a maximal simple type in π, and let $\mathfrak{A}$ be the principal $\mathfrak{o}_F$-order in A which underlies (J, λ). Write $\tau = \mathrm{Ind}(\lambda : J, \mathbf{U}(\mathfrak{A}))$. The $\mathbf{U}(\mathfrak{A})$-intertwining of the representation λ is precisely J, by [BK] (3.3.2), so τ is irreducible. Let τ' be some irreducible component of $\tau \mid \mathbf{U}(\mathfrak{A})'$. Some irreducible G′-subspace of π contains τ'. Since all such subspaces are G-conjugate up to equivalence, we may as well assume that π' contains τ'. It is enough, therefore, to show that the induced representation $\mathrm{Ind}(\tau' : \mathbf{U}(\mathfrak{A})', G')$ is irreducible.

To do this, we must show that any $x \in G'$ which intertwines τ' lies in $\mathbf{U}(\mathfrak{A})'$ (*cf.* [Ca] (1.5)). So, let $x \in G'$ intertwine τ'. By (0.3), the element x intertwines τ with $\tau \otimes \chi \circ \det$ for some quasicharacter χ of $F^\times$. Therefore, some element y in the double coset $\mathbf{U}(\mathfrak{A})\, x\, \mathbf{U}(\mathfrak{A})$ intertwines λ with $\lambda \otimes \chi \circ \det$. (*See* [BK] (4.1) for a discussion of this aspect of intertwining.) However, the pair $(J, \lambda \otimes \chi \circ \det)$ is again a maximal simple type in G, attached to the same field E and the same order $\mathfrak{A}$ as (J, λ). Let Λ be the extension of λ to $E^\times J$ which induces π. The element y then intertwines Λ with some extension of $\lambda \otimes \chi \circ \det$ to $E^\times J$. We may take this extension to be $\Lambda \otimes \chi \circ \det$. These two representations Λ and $\Lambda \otimes \chi \circ \det$ induce irreducibly to G, hence to equivalent representations of G. In other words, π contains the maximal simple type $(J, \lambda \otimes \chi \circ \det)$. Thus, by [BK] (6.2.4), there exists $z \in \mathbf{U}(\mathfrak{A})$ such that $J^z = J$ and $\lambda^z \cong \lambda \otimes \chi \circ \det$. The element yz^{-1} intertwines λ with itself, and hence lies in $E^\times J$. Therefore $x \in \mathbf{U}(\mathfrak{A})\, E^\times J\, \mathbf{U}(\mathfrak{A}) \cap G' = \mathbf{U}(\mathfrak{A})'$, as required. ■

3. Representations with Iwahori-fixed vector

We now consider another special class of representations, those with fixed vectors for an Iwahori subgroup. For this section, let $\mathfrak{A}$ denote a *minimal* hereditary $\mathfrak{o}_F$-order in A, so that $\mathbf{U} = \mathbf{U}(\mathfrak{A})$ is an Iwahori subgroup of G. Thus $\mathbf{U}' = \mathbf{U} \cap G'$ is an Iwahori subgroup of G′. Further, we have $\det(\mathfrak{K}(\mathfrak{A})) = F^\times$, so any two choices of $\mathfrak{A}$ are conjugate by an element of G′.

We need to recall some matters concerning affine Weyl groups and Hecke algebras. The notation we use is chosen to be consistent with the summary in [BK], section 5.4. Let $\{L_k : k \in \mathbb{Z}\}$ be the $\mathfrak{o}_F$-lattice chain in V which defines $\mathfrak{A}$, and identify V with F^N via an $\mathfrak{o}_F$-basis of $\{L_k\}$, as in [BK] section 1.1. This identifies G with $GL(N, F)$, G′ with $SL(N, F)$, and $\mathfrak{A}$ with the order of all matrices with entries in $\mathfrak{o}_F$ and which are upper triangular mod $\mathfrak{p}_F$. We also fix a prime element π_F of F. These choices give rise to an affine Weyl group $\tilde{\mathbf{W}}$. This is the group of all monomial matrices in G whose entries are powers of π_F. It contains the subgroup $\mathbf{W}_0$ consisting of all permutations of the chosen basis, and the element

$$\Pi = \begin{pmatrix} 0 & 1 & 0 & 0 & \dots & 0 \\ 0 & 0 & 1 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \ddots & & \vdots \\ \vdots & \vdots & & \ddots & \ddots & 0 \\ 0 & 0 & \dots & \dots & 0 & 1 \\ \pi_F & 0 & 0 & \dots & \dots & 0 \end{pmatrix}$$

Then $\tilde{\mathbf{W}}$ is generated by $\mathbf{W}_0$ and the element Π. We have a bijection $\mathbf{w} \mapsto \mathbf{U}\,\mathbf{w}\mathbf{U}$ between $\tilde{\mathbf{W}}$ and $\mathbf{U}\backslash G/\mathbf{U}$.

Now we come to the first result of the section. Recall the notation (1.10): – if $\mathscr{V}$ is a smooth, say, G-space, and K is a subgroup of G, then $\mathscr{F}(\mathscr{V}, \mathrm{K})$ denotes the space $\mathscr{V}^{\mathrm{K}}$ of K-fixed vectors in $\mathscr{V}$.

(3.1) PROPOSITION. – (i) *Let $(\pi, \mathscr{V})$ be a smooth representation of* G *such that $\mathscr{V}$ is generated, as* G*-space, by the subspace $\mathscr{F}(\mathscr{V}, \mathbf{U})$. Then any irreducible* G*-subquotient $\mathscr{W}$ of $\mathscr{V}$ has $\mathscr{F}(\mathscr{W}, \mathbf{U}) \neq 0$.*

(ii) *Let $(\pi', \mathscr{V})$ be a smooth representation of* G' *such that $\mathscr{V}$ is generated, as* G'*-space, by $\mathscr{F}(\mathscr{V}, \mathbf{U}')$. Then any irreducible* G'*-subquotient $\mathscr{W}$ of $\mathscr{V}$ has $\mathscr{F}(\mathscr{W}, \mathbf{U}') \neq 0$.*

(iii) *Let $(\pi, \mathscr{V})$ be a smooth representation of* G *as in* (i). *Then $\mathscr{F}(\mathscr{V}, \mathbf{U}') = \mathscr{F}(\mathscr{V}, \mathbf{U})$ and $\mathscr{V}$ is generated, as* G'*-space, by $\mathscr{F}(\mathscr{V}, \mathbf{U}')$.*

Proof. – In (i), let $(\sigma, \mathscr{W})$ be some irreducible G-subquotient of $\mathscr{V}$. The hypothesis on $\mathscr{V}$ and (0.1) imply that, if K is any compact open subgroup of G, and ρ is an irreducible smooth representation of K which occurs in $\sigma \mid \mathrm{K}$, then ρ intertwines with the trivial character $\mathbf{1}_{\mathbf{U}}$ of U. If $[\mathfrak{U}_1, n_1, n_1-1, b]$ is a fundamental stratum in A with $n_1 \geqq 1$, the character ψ_b of $\mathbf{U}^{n_1}(\mathfrak{U}_1)$ does not intertwine with $\mathbf{1}_{\mathbf{U}}$ (by (1.3)), so it follows that σ contains the trivial character of $\mathbf{U}^1(\mathfrak{U}_1)$, for some hereditary order $\mathfrak{U}_1$ in A. If we choose $\mathfrak{U}_1$ minimal for this property, then σ contains a representation ρ of $\mathbf{U}(\mathfrak{U}_1)$ which is the inflation of an irreducible cuspidal representation $\bar{\rho}$ of the reductive k_F-group $\mathscr{G} = \mathbf{U}(\mathfrak{U}_1)/\mathbf{U}^1(\mathfrak{U}_1)$. In our present situation, this has to intertwine with $\mathbf{1}_{\mathbf{U}}$. There is no harm in replacing $(\mathfrak{U}_1, \rho)$ by a G-conjugate and assuming $\mathfrak{U}_1 \supset \mathfrak{U}$. Therefore ρ intertwines with some irreducible representation μ of $\mathbf{U}(\mathfrak{U}_1)$ which occurs in the representation of $\mathbf{U}(\mathfrak{U}_1)$ induced by $\mathbf{1}_{\mathbf{U}}$. Any such representation μ is trivial on $\mathbf{U}^1(\mathfrak{U}_1)$. Now we appeal to [BK] (5.5.5). The proof of this result shows the following:

Let ν be an irreducible representation of $\mathbf{U}(\mathfrak{U}_1)$ which is inflated from $\mathscr{G} = \mathbf{U}(\mathfrak{U}_1)/\mathbf{U}^1(\mathfrak{U}_1)$, and let $g \in \mathrm{G}$ intertwine ν with ρ. Then $g^{-1}\mathbf{U}(\mathfrak{U}_1)g \cap \mathbf{U}(\mathfrak{U}_1)$ maps onto $\mathscr{G}$ (under the natural quotient map).

In particular, the representations ρ, μ^g remain irreducible when restricted to $\mathbf{U}(\mathfrak{U}_1) \cap \mathbf{U}(\mathfrak{U}_1)^g$. However, the group $\mathbf{U}(\mathfrak{U}_1)$ has a unique maximal pro-p subgroup (up to conjugacy), namely $\mathbf{U}^1(\mathfrak{U})$, where p is the characteristic of k_F. It follows that the image of $\mathbf{U}(\mathfrak{U}_1) \cap \mathbf{U}^1(\mathfrak{U})^g$ in $\mathscr{G}$ is a p-Sylow subgroup, that is, the unipotent radical of a Borel subgroup, of the reductive group $\mathscr{G}$. However, by construction, the representation μ contains the trivial character of $\mathbf{U}^1(\mathfrak{U})$. so the cuspidal representation $\bar{\rho}$ contains the trivial character of the unipotent radical of a Borel subgroup of $\mathcal{G}$. This is impossible unless this radical is trivial, which amounts to saying that $\mathfrak{U}_1$ is a minimal hereditary order, *i.e.*, $\mathfrak{U}_1=\mathfrak{U}$ in this context.

Thus ρ is a character of $\mathbf{U}(\mathfrak{U})$ which is trivial on $\mathbf{U}^1(\mathfrak{U})$ and which intertwines with the trivial character $\mathbf{1}_\mathrm{U}$ of $\mathbf{U}=\mathbf{U}(\mathfrak{U})$. We identify G with GL (N, F) via a basis of the lattice chain of $\mathfrak{U}$ as above. Then we can assume that ρ, $\mathbf{1}_\mathrm{U}$ are intertwined by some $\mathbf{w}\in\tilde{\mathbf{W}}$. Immediately, we see that $\mathbf{U}\cap\mathbf{U}^w$ contains the group (call it D) of diagonal matrices in $\mathbf{U}$. Thus ρ is trivial on D. However, $\mathbf{U}=\mathrm{D}\,\mathbf{U}^1(\mathfrak{U})$, with the result that ρ is trivial, as required.

In (ii), let $(\boldsymbol{\sigma}, \mathscr{W})$ be an irreducible G'-subquotient of $\mathscr{V}$. As in part (i), (1.2) shows that $\boldsymbol{\sigma}$ cannot contain ψ'_b, for any G'-fundamental stratum $[\mathfrak{U}_1, n_1, n_1-1, b]$ with $n_1\geqq 1$. (1.4) implies that $\boldsymbol{\sigma}$ contains the trivial representation of $\mathbf{U}^1(\mathfrak{U}_1)'$, for some hereditary $\mathfrak{o}_\mathrm{F}$-order $\mathfrak{U}_1$ in A. We choose $\mathfrak{U}_1$ minimal for this property, and let ρ' be an irreducible representation of $\mathbf{U}(\mathfrak{U}_1)'$ which is trivial on $\mathbf{U}^1(\mathfrak{U}_1)'$ and occurs in $\boldsymbol{\sigma}$. There is an irreducible representation ρ of $\mathbf{U}(\mathfrak{U})$ which is trivial on $\mathbf{U}^1(\mathfrak{U}_1)$ and contains ρ'. By (0.3), $\mathbf{1}_\mathrm{U}$ intertwines with $\rho\otimes\chi\circ\det$, for some quasicharacter χ of $\mathrm{F}^\times$. By (1.3), the character $\chi\circ\det$ is trivial on $\mathbf{U}^1(\mathfrak{U}_1)$, so we may as well assume that $\mathbf{1}_\mathrm{U}$ intertwines with ρ. The minimality of $\mathfrak{U}_1$ implies that ρ and ρ' are both cuspidal [when taken mod $\mathbf{U}^1(\mathfrak{U}_1)$], so the argument of (i) shows that $\mathfrak{U}_1\cong\mathfrak{U}$ and that ρ, hence also ρ', is trivial.

In part (iii), we surely have $\mathscr{F}(\mathscr{V}, \mathbf{U})\subset\mathscr{F}(\mathscr{V}, \mathbf{U}')$ so take $v\in\mathscr{F}(\mathscr{V}, \mathbf{U}')$. Let $\mathscr{U}$ be the $\mathbf{U}$-space generated by v. This is equivalent to a finite direct sum of characters of $\mathbf{U}$ of the form $\chi\circ\det$. All of these characters must intertwine with $\mathbf{1}_\mathrm{U}$, hence they are all trivial. Thus $v\in\mathscr{F}(\mathscr{V}, \mathbf{U})$, as required.

For the final assertion, take any $v\in\mathscr{V}$. By hypothesis, there exist $g_i\in\mathrm{G}$, $u_i\in\mathscr{F}(\mathscr{V}, \mathbf{U})$, $1\leqq i\leqq r$, such that $v=\sum_{i=1}^{r}\pi(g_i)\,u_i$. Since $\det(\mathfrak{K}(\mathfrak{U}))=\mathrm{F}^\times$, we can write $g_i=h_i x_i$, with $h_i\in\mathrm{G}'$, $x_i\in\mathfrak{K}(\mathfrak{U})$. The vectors $\pi(x_i)\,u_i$ then all lie in $\mathscr{F}(\mathscr{V}, \mathbf{U})=\mathscr{F}(\mathscr{V}, \mathbf{U}')$, and the result follows. ■

Returning to the situation before (3.1), we also have the subgroup $\mathbf{W}$ of $\tilde{\mathbf{W}}$. This is generated by the conjugates $\Pi^i\mathbf{W}_0\Pi^{-i}$, $0\leqq i\leqq \mathrm{N}-1$. Equivalently, $\mathbf{W}$ is the set of $\mathbf{w}\in\tilde{\mathbf{W}}$ such that $\mathbf{U}\mathbf{w}\mathbf{U}\cap\mathrm{G}'\neq\varnothing$. For any $\mathbf{w}\in\tilde{\mathbf{W}}$, we have $\mathbf{U}\mathbf{w}\mathbf{U}=\mathbf{U}'\mathbf{w}\mathbf{U}=\mathbf{U}\mathbf{w}\mathbf{U}'$, so the intersection $\mathbf{U}\mathbf{w}\mathbf{U}\cap\mathrm{G}'$, if non-empty, consists of a single $(\mathbf{U}', \mathbf{U}')$-double coset, and we have a bijection $\mathbf{w}\mapsto\mathbf{U}\mathbf{w}\mathbf{U}\cap\mathrm{G}'$ between $\mathbf{W}$ and $\mathbf{U}'\backslash\mathrm{G}'/\mathbf{U}'$.

Now let us write $\mathbf{1}_\mathrm{U}$ for the trivial character of $\mathbf{U}$, $\mathbf{1}_{\mathrm{U}'}$ for that of $\mathbf{U}'$. (In subscripts, we abbreviate $\mathbf{1}_\mathrm{U}=\mathbf{1}$, $\mathbf{1}_{\mathrm{U}'}=\mathbf{1}'$.) Let $\mathscr{H}(\mathrm{G}, \mathbf{1}_\mathrm{U})$ denote the convolution algebra of compactly supported functions $\phi:\mathrm{G}\to\mathbb{C}$ such that $\phi(uxv)=\phi(x)$, $x\in\mathrm{G}$, u, $v\in\mathbf{U}$. We assume that convolution is taken relative to a Haar measure on G for which $\mathbf{U}$ has mass 1. As vector space, $\mathscr{H}(\mathrm{G}, \mathbf{1}_\mathrm{U})$ has basis $\phi_\mathbf{w}$, where $\phi_\mathbf{w}$ is the characteristic function of $\mathbf{U}\mathbf{w}\mathbf{U}$, $\mathbf{w}\in\tilde{\mathbf{W}}$. Likewise, we can form the algebra $\mathscr{H}(\mathrm{G}', \mathbf{1}_{\mathrm{U}'})$ (relative to a Haar measure on G' giving $\mathbf{U}'$ mass 1). This has basis $\phi'_\mathbf{w}$, where $\phi'_\mathbf{w}$ is the characteristic function of

$\mathbf{U}\,\mathbf{w}\,\mathbf{U}\cap\mathrm{G}'$, $\mathbf{w}\in\mathbf{W}$. Then $\phi'_{\mathbf{w}}\mapsto\phi_{\mathbf{w}}$, $\mathbf{w}\in\mathbf{W}$, induces an injective vector space map

$$\mathscr{H}(\mathrm{G}', \mathbf{1}_{\mathrm{U}'})\to\mathscr{H}(\mathrm{G}, \mathbf{1}_{\mathrm{U}}). \tag{3.2}$$

(3.3) Proposition. — *The map* (3.2) *is an algebra homomorphism.*

Proof. — The map (3.2) surely takes the identity element of $\mathscr{H}(\mathrm{G}', \mathbf{1}_{\mathrm{U}'})$ (*i.e.* the characteristic function of $\mathbf{U}'$) to that of $\mathscr{H}(\mathrm{G}, \mathbf{1}_{\mathrm{U}})$. We therefore have only to show that it preserves multiplication. For $i=1, 2$, let $\mathbf{w}_i\in\mathbf{W}$, and let ϕ_i (resp. Φ_i) denote the characteristic function of $\mathbf{U}\,\mathbf{w}_i\,\mathbf{U}\cap\mathrm{G}'$ (resp. $\mathbf{U}\,\mathbf{w}_i\,\mathbf{U}$). We have to show that $\Phi_1 * \Phi_2$ is the image of $\phi_1*\phi_2$. For each i, we choose $\mathbf{w}'_i\in\mathrm{G}'$ such that $\mathbf{w}_i^{-1}\mathbf{w}'_i\in\mathbf{U}'$, is diagonal, and has eigenvalues ± 1. Thus ϕ_i is the characteristic function of $\mathbf{U}'\,\mathbf{w}'_i\,\mathbf{U}'$.

For $g\in\mathrm{G}$, we have

$$\Phi_1*\Phi_2(g)=\int_{\mathrm{G}}\Phi_1(x)\,\Phi_2(x^{-1}g)\,dx.$$

The integral is effectively taken over $x\in\mathbf{U}\,\mathbf{w}_1\,\mathbf{U}$, and is identically zero unless $g\in\mathbf{U}\,\mathbf{w}_1\,\mathbf{U}\,\mathbf{w}_2\,\mathbf{U}$. Every $\mathbf{w}\in\mathbf{W}$ normalises the diagonal subgroup of $\mathbf{U}$, so $\mathbf{U}\,\mathbf{w}_1\,\mathbf{U}\,\mathbf{w}_2\,\mathbf{U}=\mathbf{U}\,\mathbf{w}_1\,\mathbf{U}'\,\mathbf{w}_2\,\mathbf{U}=\mathbf{U}\,\mathbf{w}'_1\,\mathbf{U}'\,\mathbf{w}'_2\,\mathbf{U}$. We may therefore take $g\in\mathbf{w}'_1\,\mathbf{U}'\,\mathbf{w}'_2$. This forces $g\in\mathrm{G}'$, and indeed $g\in\operatorname{supp}(\phi_1*\phi_2)$. We now write $\mathbf{U}\,\mathbf{w}_1\,\mathbf{U}=\mathbf{U}'\,\mathbf{w}_1\,\mathbf{U}=\bigcup y\,\mathbf{w}_1\,\mathbf{U}$, where y ranges over $\mathbf{U}'/(\mathbf{U}'\cap\mathbf{w}_1\,\mathbf{U}'\,\mathbf{w}_1^{-1})$. This gives us

$$\begin{aligned}\Phi_1*\Phi_2(g)&=\sum_y\Phi_1(y\,\mathbf{w}'_1)\,\Phi_2(\mathbf{w}_1'^{-1}\,y^{-1}\,g)\\&=\sum_y\phi_1(y\,\mathbf{w}'_1)\,\phi_2(\mathbf{w}_1'^{-1}\,y^{-1}\,g)\\&=\phi_1*\phi_2(g),\end{aligned}$$

as required. ■

Remark. — There is an explicit presentation [Iw], [IM], for the algebras $\mathscr{H}(\mathrm{G}, \mathbf{1}_{\mathrm{U}})$, $\mathscr{H}(\mathrm{G}', \mathbf{1}_{\mathrm{U}'})$ as "affine Hecke algebras". These are denoted respectively $\mathscr{H}(\mathrm{N}, q)$, $\mathscr{H}'(\mathrm{N}, q)$, where $q=|\mathrm{k}_{\mathrm{F}}|$ is the cardinality of the residue class field of F. The embedding of (3.3) then appears as a natural inclusion $\mathscr{H}'(\mathrm{N}, q)\subset\mathscr{H}(\mathrm{N}, q)$.

Now let $\mathfrak{S}\mathfrak{R}_1(\mathrm{G})$ denote the category whose object class $|\mathfrak{S}\mathfrak{R}_1(\mathrm{G})|$ consists of all smooth representations $(\pi, \mathscr{V})$ such that $\mathscr{V}$ is generated, as G-space, by $\mathscr{F}(\mathscr{V}, \mathbf{U})$, with morphism sets consisting of all G-homomorphisms. Likewise define $\mathfrak{S}\mathfrak{R}_{1'}(\mathrm{G}')$. According to (3.1), restriction gives us a functor

$$\mathrm{Res}_{\mathrm{G}/\mathrm{G}'}:\quad \mathfrak{S}\mathfrak{R}_1(\mathfrak{G})\to\mathfrak{S}\mathfrak{R}_{1'}(\mathrm{G}'). \tag{3.4}$$

If $\mathbb{A}$ is an associative $\mathbb{C}$-algebra with 1, we write $\mathbb{A}$-$\mathfrak{Mod}$ for the category of left $\mathbb{A}$-modules and all $\mathbb{A}$-homomorphisms. If $(\pi, \mathscr{V})\in|\mathfrak{S}\mathfrak{R}_1(\mathrm{G})|$, then $\mathscr{F}(\mathscr{V}, \mathbf{U})$ is, in a natural way, an $\mathscr{H}(\mathrm{G}, \mathbf{1}_{\mathrm{U}})$-module, and $(\pi, \mathscr{V})\mapsto\mathscr{F}(\mathscr{V}, \mathbf{U})$ gives us a functor

$$\mathbf{I}_{\mathrm{G}}:\quad \mathfrak{S}\mathfrak{R}_1(\mathrm{G})\to\mathscr{H}(\mathrm{G}, \mathbf{1}_{\mathrm{U}})\text{-}\mathfrak{Mod}. \tag{3.5}$$

Likewise we get a functor

$$(3.6)\qquad \mathbf{I}_{G'}:\quad \mathfrak{S}\mathfrak{R}_{1'}(G')\to\mathscr{H}(G',\mathbf{1}_{U'})\text{-}\mathfrak{Mod}.$$

On the other hand, restriction along the algebra homomorphism (3.2) gives us a functor

$$\mathscr{H}\text{-res}:\quad \mathscr{H}(G,\mathbf{1}_U)\text{-}\mathfrak{Mod}\to\mathscr{H}(G',\mathbf{1}_{U'})\text{-}\mathfrak{Mod}.$$

(3.7) THEOREM. — (i) *The functor* $\mathbf{I}_G$ *is an equivalence of categories. Under* $\mathbf{I}_G$, *irreducible representations in* $\mathfrak{S}\mathfrak{R}_1(G)$ *correspond to simple* $\mathscr{H}(G,\mathbf{1}_U)$*-modules, and admissible representations in* $\mathfrak{S}\mathfrak{R}_1(G)$ *correspond to* $\mathscr{H}(G,\mathbf{1}_U)$*-modules of finite* $\mathbb{C}$*-dimension.*

(ii) *The functor* $\mathbf{I}_{G'}$ *is an equivalence of categories. Under* $\mathbf{I}_{G'}$, *irreducible representations in* $\mathfrak{S}\mathfrak{R}_1(G')$ *correspond to simple* $\mathscr{H}(G',\mathbf{1}_{U'})$*-modules, and admissible representations in* $\mathfrak{S}\mathfrak{R}_{1'}(G')$ *correspond to* $\mathscr{H}(G',\mathbf{1}_{U'})$*-modules of finite* $\mathbb{C}$*-dimension.*

(iii) *The following square of functors commutes*:

$$\begin{array}{ccc}\mathfrak{S}\mathfrak{R}_1(G) & \xrightarrow{\mathbf{I}_G} & \mathscr{H}(G,\mathbf{1}_U)\text{-}\mathfrak{Mod}\\ {\scriptstyle\mathrm{Res}_{G/G'}}\downarrow & & \downarrow{\scriptstyle\mathscr{H}\text{-res}}\\ \mathfrak{S}\mathfrak{R}_{1'} & \xrightarrow[\mathbf{I}_{G'}]{} & \mathscr{H}(G',\mathbf{1}_{U'})\text{-}\mathfrak{Mod}\end{array}$$

Proof. — Parts (i) and (ii) are both special cases of the following general phenomenon. Let G be a locally profinite group, and K a compact open subgroup of G. Write simply 1 for the trivial character of K. Just as above, we can form the category $\mathfrak{S}\mathfrak{R}_1(G)$: — this is the full subcategory of the category of all smooth representations of G whose objects $(\pi,\mathscr{V})$ have the property of being generated by K-fixed vectors. We get a functor $\mathbf{I}_G$ from $\mathfrak{S}\mathfrak{R}_1(G)$ to the category of $\mathscr{H}(G,1)$-modules, just as before.

(3.8) LEMMA. — *In the situation above, the following are equivalent*:

(i) *the functor* $\mathbf{I}_G$ *is an equivalence of categories*;

(ii) *if* $(\pi,\mathscr{V})\in|\mathfrak{S}\mathfrak{R}_1(G)|$, *then any irreducible* G*-subquotient of* $\mathscr{V}$ *has a nonzero* K*-fixed vector.*

Proof. — Assume first that (i) holds, and take $\mathscr{V}\in|\mathfrak{S}\mathfrak{R}_1(G)|$. Let $\mathscr{V}_2\subset\mathscr{V}_1\subset\mathscr{V}$ be G-subspaces of $\mathscr{V}$ such that $\mathscr{V}_1/\mathscr{V}_2$ is irreducible. We have to show that $\mathscr{V}_1/\mathscr{V}_2$ has a nontrivial K-fixed vector. The quotient $\mathscr{V}/\mathscr{V}_2$ is still generated by its K-fixed vectors, so we may factor out the subspace $\mathscr{V}_2$ and assume that $\mathscr{V}_1$ is an irreducible subspace of $\mathscr{V}$. Suppose, for a contradiction, that $\mathscr{V}_1$ does not contain the trivial character of K. Therefore the canonical quotient map $\mathscr{V}\to\mathscr{V}/\mathscr{V}_1$ induces an isomorphism f: $\mathbf{I}_G(\mathscr{V})\cong\mathbf{I}_G(\mathscr{V}/\mathscr{V}_1)$. Since $\mathbf{I}_G$ is an equivalence, there must be a G-homomorphism $\phi:\mathscr{V}/\mathscr{V}_1\to\mathscr{V}$ such that $\mathbf{I}_G(\phi)=f^{-1}$. In particular, ϕ splits the quotient map $\mathscr{V}\to\mathscr{V}/\mathscr{V}_1$, and we have a decomposition $\mathscr{V}=\mathscr{V}_1\oplus\phi(\mathscr{V}/\mathscr{V}_1)$ as G-space. Thus $\mathscr{V}_1$ appears as a quotient of $\mathscr{V}$, and must therefore be generated by its K-fixed vectors, of which it has none. This proves that (i) ⇒ (ii).

Now we assume that (ii) holds. We construct a functor $\mathbf{T}_G : \mathscr{H}(G, 1)\text{-}\mathfrak{Mod} \to \mathfrak{S}\mathfrak{R}(G)$, inverse to $\mathbf{I}_G$ as follows. Let $\mathscr{H}(G)$ denote the convolution algebra of locally constant, compactly supported functions $\phi : G \to \mathbb{C}$, relative to the same Haar measure used in the definition of $\mathscr{H}(G, 1)$. Let M be some $\mathscr{H}(G, 1)$-module. Form the $\mathscr{H}(G)$-module $\mathbf{T}_G(M) = \mathscr{V} = \mathscr{H}(G) \otimes_{\mathscr{H}(G)} M$. View this as a smooth representation of G, with G acting on the tensor factor $\mathscr{H}(G)$ by left translation. Then $\mathscr{F}(\mathscr{V}, K) = e \otimes M \cong M$, where $e \in \mathscr{H}(G, 1) \subset \mathscr{H}(G)$ is the identity element of $\mathscr{H}(G, 1)$. By hypothesis, $\mathscr{F}(\mathscr{V}, K)$ generates $\mathscr{V}$ as G-space. The action of $\mathbf{T}_G$ on morphisms is to be the obvious one, so $\mathbf{T}_G$ is a functor $\mathscr{H}(G, 1)\text{-}\mathfrak{Mod} \to \mathfrak{S}\mathfrak{R}_1(G)$. We have just observed that $\mathbf{I}_G(\mathbf{T}_G(M))$ is naturally isomorphic to M, for $M \in |\mathscr{H}(G, 1)\text{-}\mathfrak{Mod}|$. In the opposite direction, take $(\pi, \mathscr{V}) \in |\mathfrak{S}\mathfrak{R}_1(G)|$, and write $\mathscr{V}_0 = \mathscr{F}(\mathscr{V}, K)$. Extend π to a homomorphism $\mathscr{H}(G) \to \mathrm{End}_{\mathbb{C}}(\mathscr{V})$ in the usual way. We have a canonical G-homomorphism $\mathscr{H}(G) \otimes_{\mathscr{H}(G, 1)} \mathscr{V}_0 \to \mathscr{V}$ by $\phi \otimes v \mapsto \pi(\phi) v$. This is surjective, since $\mathscr{V} \in |\mathfrak{S}\mathfrak{R}_1(G)|$. It is also injective on the subspace $\mathscr{V}_0 = e \otimes \mathscr{V}_0$. This subspace generates $\mathscr{H}(G) \otimes \mathscr{V}_0$ over G, so any proper G-subspace of the tensor product meets $\mathscr{V}_0$ nontrivially. In particular, this applies to the kernel of the map $\mathscr{H}(G) \otimes \mathscr{V}_0 \to \mathscr{V}$. This map is therefore an isomorphism. It is easy to see it is natural in $\mathscr{V}$, and we have proved the lemma. ■

Since irreducible representations of G, G′ are admissible, the remaining assertions of (i) and (ii) are immediate. We now take $(\pi, \mathscr{V}) \in |\mathfrak{S}\mathfrak{R}_1(G)|$. Part (iii) simply asserts that the natural action of $\mathscr{H}(G', \mathbf{1}_{U'})$ on $\mathscr{F}(\mathscr{V}, \mathbf{U}')$, when $\mathscr{V}$ is viewed as a G′-space, coincides with the action implied by the homomorphism $\mathscr{H}(G', \mathbf{1}_{U'}) \to \mathscr{H}(G, \mathbf{1}_U)$ and the natural action of $\mathscr{H}(G, \mathbf{1}_U)$ on $\mathscr{F}(\mathscr{V}, \mathbf{U}) = \mathscr{F}(\mathscr{V}, \mathbf{U}')$. Write

$$\mathscr{V}_0 = \mathscr{F}(\mathscr{V}, \mathbf{U}) = \mathscr{F}(\mathscr{V}, \mathbf{U}').$$

We have $\mathscr{V} = \mathscr{V}_0 \oplus \mathscr{V}_1$, for a uniquely determined $\mathbf{U}$-subspace $\mathscr{V}_1$ of $\mathscr{V}$. Indeed, $\mathscr{V}_1$ is the unique $\mathbf{U}'$-complement of $\mathscr{V}_0$ in $\mathscr{V}$. Write $\mathrm{pr} : \mathscr{V} \to \mathscr{V}_0$ for the projection with kernel $\mathscr{V}_1$. Let $\mathbf{w} \in \tilde{\mathbf{W}}$, and let $\phi_{\mathbf{w}} \in \mathscr{H}(G, \mathbf{1}_U)$ be the characteristic function of $\mathbf{U}\mathbf{w}\mathbf{U}$. Then, for $v \in \mathscr{V}_0$, we have $\pi(\phi_{\mathbf{w}}) v = \mathrm{pr}(\pi(\mathbf{w}) v)$. On the other hand, let $\mathbf{w} \in \mathbf{W}$, and choose $\mathbf{w}' \in \mathbf{w}\mathbf{U} \cap G'$. Then $\mathbf{U}\mathbf{w}\mathbf{U} \cap G' = \mathbf{U}'\mathbf{w}'\mathbf{U}'$ and $\pi(\mathbf{w}) v = \pi(\mathbf{w}') v$ whenever $v \in \mathscr{V}_0$. Let $\phi' \in \mathscr{H}(G', \mathbf{1}'_{U'})$ be the characteristic function of $\mathbf{U}'\mathbf{w}'\mathbf{U}'$, so that the image of ϕ' under the map (3.2) is the function $\phi_{\mathbf{w}}$. Writing $\pi' = \pi | G'$, we have $\pi'(\phi') v = \mathrm{pr}(\pi'(\mathbf{w}') v) = \mathrm{pr}(\pi(\mathbf{w}) v) = \pi(\phi_{\mathbf{w}}) v$ for every $v \in \mathscr{V}_0$, and the result follows. ■

(3.9) REMARK. — (i) The argument of (3.8) holds in greater generality. Again let G be a locally profinite group, and K a compact open subgroup of G. Now take an irreducible smooth representation ρ of K. We can form the category $\mathfrak{S}\mathfrak{R}_\rho(G)$ of smooth representations $(\pi, \mathscr{V})$ of G with the property that $\mathscr{V}$ is generated over G by its space $\mathscr{V}^\rho$ of ρ-isotypic vectors. Mapping $(\pi, \mathscr{V})$ to $\mathscr{V}^\rho$ and composing with the Morita equivalence of [BK], section 4.2, we get a functor $\mathbf{I}_\rho : \mathfrak{S}\mathfrak{R}_\rho(G) \to \mathscr{H}(G, \rho)\text{-}\mathfrak{Mod}$. Using exactly the same argument, one can show that $\mathbf{I}_\rho$ is an equivalence if and only if any G-subquotient of any $(\pi, \mathscr{V}) \in |\mathfrak{S}\mathfrak{R}_\rho(G)|$ contains ρ.

(ii) The conditions on the pair (G, K) in (3.8) are very strong. For example, if we take $G = GL(N, F)$, for any $N \geqq 2$, and $K = GL(N, \mathfrak{o}_F)$, then the conditions fail.

The categorial equivalences in (3.7) strengthen those of [Bo], and hence imply most of the results of that paper for the group G′. However, one of the major preoccupations of [Bo] is the characterisation of representations with Iwahori fixed vector in terms of parabolic induction. There is no advantage for us in treating this here, since we later prove a result of this kind valid for all simple types in G′, of which the trivial character of an Iwahori subgroup is merely the first example.

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