ANNALES SCIENTIFIQUES DE L'É.N.S.

Y. GUIVARC'H
Y. LE JAN

Asymptotic winding of the geodesic flow on modular surfaces and continuous fractions

Annales scientifiques de l'É.N.S. 4e série, tome 26, nº 1 (1993), p. 23-50
<http://www.numdam.org/item?id=ASENS_1993_4_26_1_23_0>

Ann. scient. Éc. Norm. Sup.,
4e série, t. 26, 1993, p. 23 à 50.

ASYMPTOTIC WINDING OF THE GEODESIC FLOW ON MODULAR SURFACES AND CONTINUOUS FRACTIONS

BY Y. GUIVARC'H AND Y. LE JAN

ABSTRACT. — We study the statistical behaviour of renormalized integrals of harmonic 1-forms along geodesics on modular surfaces, using their coding by continuous fractions. We are led to prove a similar result for the continuous fractions transformation, by perturbation of the associated transfer operator.

Introduction

The chaotic behavior of geodesics on surfaces of constant negative curvature and finite volume has been known since Hadamard (1898). Later were proved the ergodicity of the geodesic flow and central limit theorems ([Si], Rat]). The general theory of Anosov flows was developed in order to include various examples, *e.g.* geodesic flows on manifold with non constant negative curvature. Independently, the geometry of Brownian paths was studied by Lévy and his followers. Clearly the existence of a central limit theorem suggests an analogy between geodesics in negative curvature and Brownian motion already observed by Sullivan ([Su]). Here we present, in a very specific case, a result analogous to Spitzer's theorem which describes the asymptotic law of the windings of a two-dimensional Brownian motion around a point *cf.* ([Spi], [LM], [RY], [F]).

The spaces we consider are modular surfaces obtained as quotients of the hyperbolic plane by normal subgroups of finite index in the modular group $SL_2(Z)$. In fact such a modular surface has finite hyperbolic area and is naturally compactified in a compact Riemann surface of genus g by adding c points (cusps) [Sch]. Then, roughly speaking, our main result says that the normalized homological winding of the geodesic flow converges toward the product of two non degenerate probability laws. The first one is a $2g$-dimensional Gaussian law associated with the compactification; the second one is a $(c-1)$-dimensional Cauchy law which is itself the convolution of c elementary Cauchy laws corresponding to the cusps.

This result should be viewed as a first step since the problem of asymptotic laws of normalized integrals and winding we address is meaningful on manifolds of finite volume (possibly with different normalizations and with respect to suitable finite invariant measure).

In our proof, an important idea is to reduce the problem to the study of harmonic forms integrated along the geodesic and change the contour of integration along the lines of the Farey tesselation. We transform the problem into the study of an additive functional of a Markov chain involving continuous fractions, which occurs from the coding of the geodesic flow given by the Farey tesselation. We are lead to prove a result on continuous fractions, of independent interest, by perturbation of the transfer operator.

These results were announced in a note [GL], together with other results which we will detail in a forthcoming paper.

They were presented in the conference held in Paris, in Spring 1990, to honour Professor K. Itô.

The authors wish to thank F. Ledrappier for indicating them useful references and the referee for his advices.

1. Framework and notations

1.1. Modular spaces. – Let $G=SL^2(\mathbb{R})/\pm I$ be the group of projective transformations of the projective line $\mathbb{P}$.

Set

$$R_\theta=\pm\begin{pmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{pmatrix}\ \theta\in\mathbb{R}/_{\pi\mathbb{Z}}\qquad\text{and}\qquad U_t=\pm\begin{pmatrix}e^{t/2} & 0\\ 0 & e^{-t/2}\end{pmatrix},$$

$$K=\{R_\theta,\ \theta\in R/_{\pi\mathbb{Z}}\},\qquad K^+=\left\{R_\theta,\ \theta\in\left(0,\frac{\pi}{2}\right)\right\},\qquad D=\{U_t,\ t\in\mathbb{R}\}.$$

K and D are subgroups of G.

$\mathbb{H}=G/K$ can be identified with the hyperbolic space, the hyperbolic distance between gK and hK being $2\,\mathrm{Log}\|h^{-1}g\|$, $\|\ \ \|$ being the euclidean operator norm. The half plane representation is obtained by mapping gK onto

$$g(i)\equiv\frac{ai+b}{ci+a}\qquad\text{if}\quad g=\pm\begin{pmatrix}a & b\\ c & d\end{pmatrix}$$

$\forall g\in G$, $\{gU_t(i),\ t\in\mathbb{R}\}$ is a geodesic of $\mathbb{H}$. G/D can be identified with the set of (oriented) geodesics and G with the set of contact elements $T_1\,\mathbb{H}$ (unitary tangent bundle).

$$g\text{ is associated to }\left(g(i),\ \lim_{\varepsilon\downarrow 0}\frac{g(e^\varepsilon i)-g(i)}{\varepsilon}\right)=\left(\frac{ai+b}{ci+d},\ i(ci+d)^{-2}\right).$$

If

$$g=\begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix}U_t R_\theta,\qquad g(i)=x+ie^t\qquad\text{and}\qquad i(c+d)^{-2}=e^t e^{i(\pi/2-2\theta)}$$

The flow of mappings on $G: g \to g U_t$ is the geodesic flow. The end points of the geodesic defined by g are $g(0)$ and $g(\infty)$, and therefore the space G/D can be identified with $\mathbb{P} \times \mathbb{P} - \Delta$. The left action of g onto G induces an isometry of $\mathbb{H}$, so that G can be represented by a group of isometries of $\mathbb{H}$. They map the geodesic (α, β) onto $(g(\alpha), g(\beta))$. The Haar measure of G, denoted μ (which is also the Liouville measure on $T_1 \mathbb{H}$) can be defined as follows

$$\forall f \in C_K(G), \qquad \int f \, d\mu = \int \tilde{f}(\alpha, \beta) \frac{d\alpha \, d\beta}{(\alpha - \beta)^2}$$

with $\tilde{f}(\alpha, \beta) = \int_{-\infty}^{+\infty} f(g U_t) \, dt$, for any g such that $g(0) = \infty$ and $g(\infty) = \beta$. Denote by Γ_0 the modular group $SL_2(\mathbb{Z})/\pm I$.

Let Γ be a normal subgroup of Γ_0, such that $\Sigma = \Gamma \backslash \Gamma_0$ is finite. For $g \in G$, we set $\hat{g} = \Gamma g$. In particular, $\hat{g} \in \Sigma$ if $g \in \Gamma_0$.

We are interested in the geodesic flow on the modular space $\Gamma \backslash H$. The geodesic flow on G clearly induces the geodesic flow on the unitary tangent bundle to $\Gamma \backslash H$ which is isomorphic to $\Gamma \backslash G$.

If $\alpha \in Q \cup \{\infty\}$, set $\hat{\alpha} = \{\beta \in Q \cup \infty, \exists g \in \Gamma, g(\alpha) = \beta\}$.

Let C be the space of cusps $\{\hat{\alpha}, \alpha \in Q \cup \{\infty\}\}$.

Σ acts transitively on C (as Γ_0 on $Q \cup \{\infty\}$), hence C is finite. The conductor of Γ, denoted N, is the order of the stabilizer of any cusp. In particular $N = \inf \left\{ n, \pm \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} \in \Gamma \right\}$. We have $\Sigma^\# = N C^\#$.

1.2. Modular forms. — To any holomorphic function f on the Poincaré half plane, we can associate a mapping $\hat{f}$ from G into $\mathbb{C}$ given by $\hat{f}(g) = f(g(i)) g'(i) = f(ai + b/ci + d)(ci + d)^{-2}$, for $g = \pm \begin{pmatrix} a & b \\ c & d \end{pmatrix}$.

Note that $\hat{f}(g R_\theta) = \hat{f}(g) e^{-2i\theta}$ and $\int_0^t \hat{f}(g U_s) \, ds = \int_{g(i)}^{g U_t(i)} f(z) \, dz$.

If $\gamma = \pm \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G$, $\hat{f}(\gamma g) = \hat{f}(g)$ for all g iff $f(\gamma z) \, d(\gamma z) = f(z) \, dz$, *i.e.* $f(\gamma z) = f(z)(cz + d)^2$.

The space of entire modular forms of dimension -2, denoted $\mathcal{M}$ is defined as follows (*cf.* [S]).

$\mathcal{M} = \left\{ f \text{ holomorphic in H}, \hat{f}(\gamma g) = \hat{f}(g), \text{ for any } \gamma \in \Gamma \text{ and for any } \gamma = \pm \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0, \right.$

$\left. f(\gamma z) \text{ admits a development of the form } (cz + d)^2 \sum_0^\infty c_n e^{inz/N} \text{ for } \mathrm{Im}(z) \text{ large enough} \right\}$. We define f on the cusps by setting $f(\hat{\gamma} \infty) = c_0$ with the notations above.

$\Gamma\backslash H \cup C$ is a compact Riemann surface $\mathscr{R}$ whose genus g can be computed (*cf.* [Sch]).

Differentials on $\mathscr{R}$ having no poles outside the cusps and degree at most -1 at the cusps are represented by entire modular forms of dimension -2. A differential has a residue at a cusp $\hat{\alpha}$ if the corresponding form does not vanish at $\hat{\alpha}$. Entire forms vanishing at every cusp (*i.e.* cusp forms) are associated with holomorphic differentials. Their space has a finite $\mathbb{C}$-dimension equal to the genus g.

From the Riemann-Roch theorem, it is known that the dimension of the space of entire modular forms is $C^{\#}-1+g$. Moreover, it is known that the sum of the residues of a differential is zero ([Spr]). Hence, for any complex function g on C, such that $\sum_{\alpha \in C} g(\hat{\alpha})=0$, their exists an entire form φ, determined up to a cusp form, such that $\varphi=g$ on C. The $\mathbb{R}$-dimension of the space $\mathscr{M}_{\mathbb{R}}$ of entire forms taking real values at the cusps is $2g+C^{\#}-1$.

We now fix an entire form φ, taking real values at the cusps. The real part of $\varphi(z)\,dz$ is a harmonic 1-form on H denoted ω. It induces a harmonic 1-form on $\Gamma\backslash H$ also denoted by ω.

Given $g \in G$ and $t>0$, denote $\gamma_t(g)$ the arc of geodesic $\{g\,U_s(i),\ s \leqq t\}$ and by $\hat{\gamma}_t(g)$ the corresponding arc on $\Gamma\backslash H$.

We are interested in the asymptotic study of the integral of ω along geodesics, *i.e. of* $\int_{\gamma_t(g)} \omega$ as $t \uparrow +\infty$.

2. Asymptotic winding

THEOREM 2.1. — *Under the normalized Liouville measure* $\hat{\mu}$ *on* $\Gamma\backslash G$,

a. $1/t \int_{\hat{\gamma}_t} \omega$ *converges in law towards a Cauchy distribution with parameter* $3/\pi C^{\#} \sum_{\hat{\alpha} \in C} |\varphi(\hat{\alpha})|$ *as* $t \to \infty$.

b. If φ_c *is a cusp form,* $1/\sqrt{t} \int_{\hat{\gamma}_t} \omega_c$ *converges in law towards a non degenerate gaussian distribution with variance* $\sigma^2(\omega_c)=2\int_{\Gamma\backslash H} \|\omega_c\|^2\,dv$ *(dv being the normalized volume element).*

c. $1/t \int_{\hat{\gamma}_t} \omega$ *and* $1/\sqrt{t} \int_{\hat{\gamma}_t} \omega_c$ *are asymptotically independent.* ($\omega_c=\operatorname{Re}(\varphi_c(z)\,dz)$).

The proof of this theorem will be given in the next chapters. Coding and contour deformation will allow us to show that this result can be proved via a theorem on continuous fractions that will be proved in the final part. Before that we shall discuss the scope of the theorem and give an equivalent formulation in terms of probabilities on homology spaces.

Remark. — The integral of the form ω on an elementary loop $l_{\hat{\alpha}}$ around the cusp $\hat{\alpha}$ is $N\varphi(\hat{\alpha})$ and the volume of Γ_0/H is $|\Gamma_0/H|=\pi/3$, so that the parameter of the Cauchy law can be rewritten as $1/|H/\Gamma| \sum_{\hat{\alpha}\in C} |\langle \omega, l_{\hat{\alpha}}\rangle|$.

Let $\mathscr{H}$ be the space of harmonic forms which are the real part of an element of $\mathscr{M}_{\mathbb{R}}$.

To show that the asymptotic winding of the geodesics is completely described by the integrals $\int_{\gamma_t(g)} \omega$, it is important to remark the following:

PROPOSITION 2.2. — *Every* C^∞ *closed form on* $\Gamma\backslash H$ *is cohomologous to an element of* $\mathscr{H}$.

Proof. — Note first that $\mathscr{H}$ contains no exact forms. The existence of such forms would imply the existence of holomorphic modular functions on $\mathscr{R}$, which is known to be impossible (*cf.* [Leh]).

Except three exceptional cases in which $\Gamma\backslash H$ is simply connected, the signature of Γ is always (2, 3, N). Hence it is known that Γ has no elliptic elements (*cf.* [S], p. 81-83, 94-97). Therefore, Γ is a free group with $d=2g+C^{\#}-1$ generators (*cf.* [S], p. 202, [Leh], p. 362).

Fix a base point b in $\Gamma\backslash H$ and let l be a loop from b to b in $\Gamma\backslash H$. Its lift in H defines an element $\bar{l}$ in Γ and $\overline{ll'}=\bar{l}\bar{l'}$. If $\bar{l'}=\bar{l}$, $\int_l \omega = \int_{l'} \omega$ since H is simply connected and $\int_l \omega$ equals the integral of the pullback of ω on any lift of l.

Hence, $\hat{\omega}$ defined by $\hat{\omega}(\bar{l})=\int_l \omega$ is an additive character of Γ, and $\hat{\omega}=\hat{\omega}'$ clearly implies that ω and ω' are cohomologous, since $\Gamma\backslash H$ is arcwise connected. Now the result follows from the fact that the space of additive characters on a free group with d generators has dimension $d=\dim \mathscr{H}$.

Let $\mathscr{H}_c$ be the space of real harmonic forms on $\mathscr{R}=\Gamma\backslash H \cup C$ induced by cusp forms. It has dimension $2g$. There are no exact forms in $\mathscr{H}_c$ on $\Gamma\backslash H$, hence on $\mathscr{R}$. Besides it is well known that the dimension of the first cohomology space of $\mathscr{R}$ is $2g$. Hence we have:

PROPOSITION 2.2 *bis*. — *Each* C^∞*-closed form on* $\mathscr{R}$ *is cohomologous to an element of* $\mathscr{H}_c$.

Recall now that the first real singular homology space of a manifold M is the dual of the first cohomology space (defined in terms of classes of C^∞ differential forms. (*Cf.* [Die] 24-32-2.)

Proposition 2.2 shows the existence of a unique specific representative $\tilde{\omega}$ in $\mathscr{H}$ for each cohomology class $\bar{\omega}$. Hence to each contact element u in $\Gamma\backslash G$, we associate $T_t u$ in $H_1^{\mathbb{R}}(\Gamma\backslash H)$ defined by $\langle T_t u, \bar{\omega}\rangle = \int_{\gamma_t(u)} \tilde{\omega}$.

Let π be the natural projection from $H_1^{\mathbb{R}}(\Gamma\backslash H)$ onto $H_1^{\mathbb{R}}(\mathscr{R})$. The kernel $\mathscr{K}$ of π is the orthogonal of the space $\mathscr{H}_c$ identified with the first cohomology space of $\mathscr{R}$ in proposition 2.2 *bis*.

The vector space $H_1^{\mathbb{R}}(\mathscr{R})$ has a natural scalar product. In fact an euclidean norm $\|.\|$ is defined on $\mathscr{H}_c$ by $\|\omega_c\|^2 = \int \|\omega_c\|^2 \, dv$ where dv is the volume element of Γ/H and $\|\omega_c\|$ is the norm of the linear form ω_c, on the tangent space of Γ/H. This corresponds to the so-called Petersson scalar product. By duality a scalar product is defined on $H_1^{\mathbb{R}}(\mathscr{R})$. Again the corresponding norm is denoted by $\|.\|$.

On the other hand, the group $\mathscr{K}$ is $|C|-1$ dimensional and is naturally generated by $|C|$ elementary loops $l_{\hat{\alpha}}$ around each cusp $\hat{\alpha}$. For each cusp $\hat{\alpha}$ we consider the cauchy law $C_{\hat{\alpha}}^{A}$ with support equal to the subspace

$$\mathbb{R}\, l_{\hat{\alpha}} = \{ h \in \mathscr{K} = t l_{\hat{\alpha}};\ t \in \mathbb{R} \}$$

and defined by $C_{\hat{\alpha}}^{A}(dh) = (1/\pi)(A\, dt/(A^2 + t^2)$ we can then reformulate theorem 2.1. as follows.

THEOREM 2.3. — *The image distribution of* $\hat{\mu}$ *by* $(T_t/t, \pi T_t/\sqrt{t})$ *converge towards a probability distribution* ν *on* $\mathscr{K} \times H_1^{\mathbb{R}}(\mathscr{R})$.

This law is the product of a law of Cauchy type and a Gaussian law. The first law is the convolution of the elementary Cauchy laws $C_{\hat{\alpha}}^{A}$ $(\hat{\alpha} \in C)$ *with* $A = 1/|H/\Gamma|$. *The second law is the Gaussian law with density*

$$\frac{1}{(2\pi\sigma^2)^g} e^{-(\|h\|^2/2\sigma^2)} \qquad \textit{with} \quad \sigma^2 = \frac{2}{|H/\Gamma|},$$

and $\|h\|$ *is the norm associated with the canonical scalar product on* $H_1^{\mathbb{R}}(\mathscr{R})$.

3. The modular coding

Our presentation is very close (*cf.* [Se]), with minor variations. According to [Se] this method appears to be very old.

There is a one to one mapping between Γ_0 and the (oriented) Farey geodesics $\Gamma_0 D\left(\pm \begin{pmatrix} a & b \\ c & d \end{pmatrix} \to \left(\frac{a}{c}, \frac{b}{a}\right) \text{ and the fractions are irreducible since the determinant is } 1\right)$.

A geodesic belongs to $\Gamma_0 D$ if its two endpoints are adjacent in one of the Farey sequences $F_n = \{p/q, p, q \in \mathbb{Z}, |q| \leq n\}$, arranged in increasing order (*cf.* [HW]). Set $X = \{g \in G, g(0) \text{ and } g(\infty) \text{ are irrationals}\}$. For $g \in X$, consider the set of $(t, \gamma) \in \mathbb{R} \times \Gamma_0$

such that the geodesic $g\mathrm{D}$ cuts at $g\,\mathrm{U}_t(i)$ the geodesic $\gamma\mathrm{D}$, and the angle between $g\mathrm{D}$ and $\gamma\mathrm{D}$ is less than π, *i.e.* $\{(t, \gamma), g\,\mathrm{U}_t \in \gamma\,\mathrm{DK}^+\}$. Elementary considerations on the Farey sequences show that this set is countable and can be written as a sequence $\{(t_n(g)), \gamma_n(g)), n \in \mathbb{Z}\}$, with t_n increasing, $\lim_{n \to \pm\infty} t_n = \pm\infty$, $t_0 \leqq 0$ and $t_1 > 0$. Also, an easy geometrical argument shows that if $g \in \mathrm{DK}^+$, *i.e.* if $t_0(g) = 0$ and $\gamma_0(g) = \mathrm{I}$, $\gamma_1(g)\mathrm{D}$ is either the geodesic with endpoints $(1, \infty)$, either the geodesic $(0, 1)$. Hence $\gamma_1(g)$ is either $\tau_1 = \pm\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, either $\tau_{-1} = \pm\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$.

Note that $t_n(\gamma g) = t_n(g)$ and $\gamma_n(\gamma g) = \gamma\gamma_n(g)$ for any $\gamma \in \Gamma_0$. Also $\gamma_{n+1}(g) = \gamma_1(g\,\mathrm{U}_{t_n(g)})$.

Set $\sigma_n(g) = \gamma_n(g)^{-1}\gamma_{n+1}(g) = \gamma_n^{-1}(g)\gamma_1(g\,\mathrm{U}_{t_n(g)}) = \gamma_1(\gamma_n^{-1}(g)\,g\,\mathrm{U}_{t_n(g)})$.

Since $(\gamma_n^{-1}(g)\,g\,\mathrm{U}_{t_n(g)}) \in \mathrm{DK}^+$, σ_n equals τ_ε for $\varepsilon = +1$ or -1.

Let $m_k(g)$ be the sequence of integers n such that $\sigma_n(g) \neq \sigma_{n-1}(g)$, arranged in increasing order, with $m_0 \leqq 0$ and $m_1 > 0$. m_k is a doubly infinite sequence. For example, to show that $m_1(g)$ is finite note that $m_1(g) \geqq \mathrm{N}$ implies that $\gamma_0(g)^{-1}g(\infty)$ belongs to $(\mathrm{N}\,\infty)$ if $\sigma_0 = \tau_1$ and to $(0, 1/\mathrm{N})$ if $\sigma_0 = \tau_{-1}$.

The geodesic defined by g is obviously determined by the sequence γ_n and by the subsequence γ_{m_k} as well (since it is doubly infinite).

Denote $\mathrm{T}(g) = -t_{m_0}(g)$ and $\mathrm{S} = \{\xi \in \mathrm{X}, \mathrm{T}(\xi) = 0\}$. $g\,\mathrm{U}_{-\mathrm{T}(g)}$ belongs to S and will be denoted $p(g)$ so that p maps X onto S.

A shift $\bar{\theta}$ is naturally defined by the return map on S. Precisely, $\bar{\theta}\xi = \xi\,\mathrm{U}_{t_{m_1}(\xi)}$. Set $h(\xi) = d(\xi, \bar{\theta}\xi)$. Note that $\bar{\theta}$ preserve $p(\mu)$, and that $\mathrm{T}(g)$ is less than $h(p(g))$.

By (p, T), the geodesic flow on H can be identified to the special flow over S with height function h. An element ξ of S is characterized by the sequence $\gamma_{m_k}(\xi)$ or equivalently by a triple $(n_k)), \varepsilon, \gamma)$ where

$$n_k = m_k(\xi) - m_{k-1}(\xi),\ k \in \mathbb{Z}, \qquad \sigma_0(\xi) = \tau_\varepsilon \qquad \text{and} \qquad \gamma_0(\xi) = \gamma.$$

The shift is given by $\bar{\theta}((n_k), \varepsilon, \gamma) = ((n_{k+1}), -\varepsilon, \gamma\tau_\varepsilon^{n_1})$.

The endpoints of the geodesic defined by ξ are $\xi(0) = -\gamma(\chi_-^\varepsilon)$ with

$$\chi_- = 1/n_0 + 1/n_{-1} + 1/n_{-2} + \ldots \qquad \text{and} \qquad \xi(\infty) = \gamma(\chi_+^\varepsilon)$$

with $\chi_+ = 1/n_1 + 1/n_2 + 1/n_3 + \ldots$ $(n_k)_{k \in \mathbb{Z}}$ can be replaced by (χ_+, χ_-) and S identified to $[0, 1]^2 \times (\mathbb{Z}/_{2\mathbb{Z}} \times \Gamma_0)$.

The shift $\bar{\theta}$ appears to be an extension of the continuous fraction transformation

$$\theta(\chi_-, \chi_+) = (\chi_- + [\chi_+^{-1}]^{-1}, \chi_+^{-1} - [\chi_+^{-1}])$$

$h(\xi)$ equals $\Phi(\chi_+, \chi_-) \equiv -(1/2)\,\mathrm{Log}(\chi_+\chi_-(\chi_+\chi_- \circ \theta))$ and $p(\mu)$ equals, up to a multiplicative constant, $\nu \otimes$ (counting measure), where ν is the θ invariant probability $(1/\mathrm{Log}\,2)(1/(\chi_+\chi_- + 1))^2\,d\chi_+\,d\chi_-$.

Note that $T(\gamma g)=Tg$ for any γ in Γ_0 and that $p(\gamma g)=\gamma p(g)$. Hence $\hat{p}$ maps $\Gamma\backslash X$ onto $\Gamma\backslash S$ and T is defined on $\Gamma\backslash X$.

The geodesic flow on $\Gamma\backslash G$ is identified by $(\hat{p}, T)$ to a special flow over $\Gamma\backslash S \simeq [0, 1]^2 \times (\mathbb{Z}/_{2\mathbb{Z}} \times \Sigma)$. The height function is given by Φ, the shift by $\hat{\theta}((\chi^+, \chi^-), (\varepsilon, \rho)) = (\theta(\chi_+, \chi_-), (-\varepsilon, \rho\hat{\tau}_\varepsilon^{n(\chi_+)}))$ where $n(\chi)=[\chi^{-1}]$, and the invariant probability by $\hat{\nu}=\nu\otimes$ (equiprobability).

4. Contour deformation

Given any $\xi=((n_k), \varepsilon, \gamma)$ in S, note that $\xi(i)$ lies on the Farey geodesic γD.

Set $C^+(\xi)=\gamma(\infty)$, $C^-(\xi)=\gamma(0)$ if $\varepsilon=1$, and $C^+(\xi)=\gamma(0)$, $C^-(\xi)=\gamma(\infty)$ if $\varepsilon=-1$.

It is easy to check that $C^+(\xi)=C^-(\bar{\theta}\xi)$ (*).

Note that the integrals of ω along γD between $\xi(i)$ and $C^\pm(\xi)$ are well defined and equal to

$$\int_{\gamma^{-1}\xi(i)}^{\pm i\infty} \gamma^*\omega \qquad \text{with} \quad \gamma^*(\omega)=\operatorname{Re}(\varphi(\gamma z)\, d(\gamma z))$$

this improper integral along the imaginary axis converges exponentially since φ is real and holomorphic at the cusps.

A contour deformation (Fig. 1) shows that

$$\int_{\xi(i)}^{\bar{\theta}\xi(i)} \omega = \int_{\xi(i)}^{C^+(\xi)} \omega + \varepsilon n_1 \varphi(C^+(\xi)) + \int_{C^+(\xi)}^{\bar{\theta}\xi(i)} \omega$$

Then by (*), the integral of ω along the geodesic defined by ξ until its n-th change of winding orientation, $\tau_n = t_{m_n}$,

$$\int_{\gamma_{\tau_n}(\xi)} \omega = \int_{\xi(i)}^{\bar{\theta}^n\xi(i)} \omega \text{ equals } \sum_{k=0}^{n-1}\left[n_{k+1}\varphi(C^+(\bar{\theta}^k\xi))\,\varepsilon(-1)^k + \int_{C^-(\bar{\theta}^k\xi)}^{C^+(\bar{\theta}^k\xi)} \omega \right] - \int_{C^-(\xi)}^{\xi(i)} \omega + \int_{C^-(\bar{\theta}^n\xi(i)}^{\bar{\theta}^n\xi(i)} \omega.$$

Note that $\displaystyle\int_{C^-(\xi)}^{C^+(\xi)} \omega = \varepsilon \int_0^{i\infty} \gamma^*\omega$ and $\varepsilon n_1 \varphi(C^+(\xi))$ depends only of $\hat{\xi}$.

Define ψ and η on $\hat{S}$ by:

$$\psi(\hat{\xi}) = \varepsilon n_1 \varphi(C^+(\xi)) + \varepsilon\int_0^{i\infty} \gamma^*\omega$$

$$\eta(\hat{\xi}) = \varepsilon\int_0^{i\infty} \gamma^*\omega_c.$$

Except two boundary terms, $\displaystyle\int_{\hat{\gamma}_{\tau_n}(\hat{\xi})} \omega$ equals

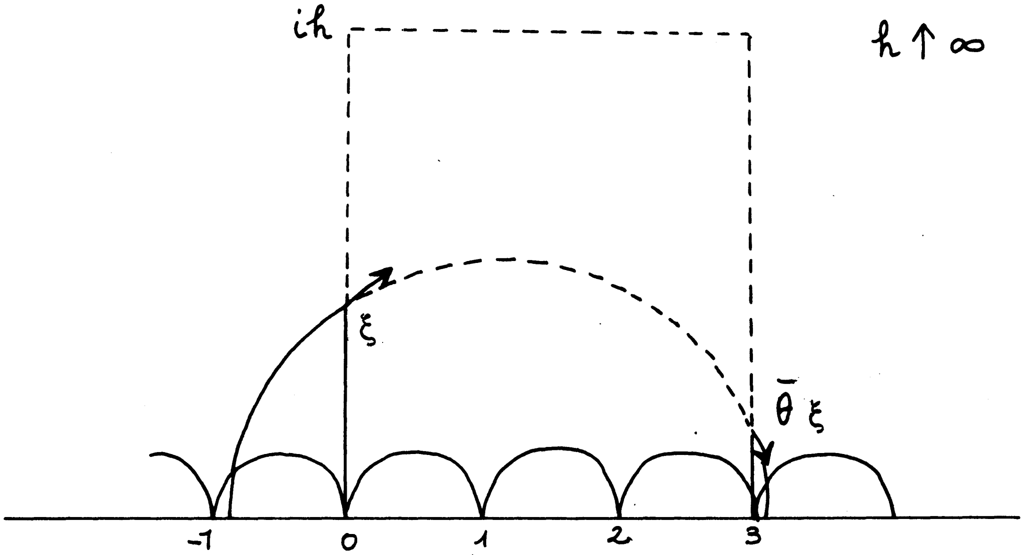

Fig. 1. — Contour deformation with $\gamma=I$, $n_1=3$, $\varepsilon=1$.
The case $\varepsilon=-1$ is reduced to this one by applying $\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}$
and by symmetry with respect to the imaginary axis.

$$\sum_{0}^{n-1}\psi\circ\hat\theta^k\xi \quad \text{and} \quad \int_{\hat\gamma_{\tau_n}(\hat\xi)}\omega_c \text{ equals } \sum_{0}^{n-1}\eta\circ\hat\theta^k\xi.$$

Hence it follows easily that to prove theorem 2.1., $(\Gamma\diagdown H, \hat\mu)$ can be replaced by

$$(\Gamma\diagdown S, \hat\nu) \quad \text{and} \quad \left(\frac{1}{t}\int_{\hat\gamma_t}\omega, \frac{1}{\sqrt{t}}\int_{\hat\gamma_t}\omega_c\right), \qquad t\uparrow\infty$$

by

$$\left(\frac{1}{\tau_n}\sum_{1}^{n}\psi\circ\hat\theta^m, \frac{1}{\sqrt{\tau_n}}\sum_{1}^{n}\eta\circ\hat\theta^m\right), \qquad n\uparrow\infty.$$

Moreover we have the two following easy lemmas.

LEMMA 4.1. — $\sum_{1}^{n}(\Phi\circ\hat\theta^k/n)\to(\pi^2/6\,\mathrm{Log}\,2)$ $\hat\nu$-*a. s.*

LEMMA 4.2. — *Let* Z_k *be a sequence of real r.v's such that for some* $\alpha>0$, *the variables* $k^{-\alpha}Z_k$ *converge in law towards a distribution* μ. *Let* S_k *be a sequence of random integers* *such that* $S_k/k \to 1$ *in probability as* $k \uparrow \infty$. *Then* $k^{-\alpha} Z_{S_k}$ *converge in law towards* μ *as* $k \uparrow \infty$.

Hence the factors $(\tau_n)^\alpha$ can be replaced by $(n\pi^2/6 \operatorname{Log} 2)^\alpha$. Therefore, except for the calculation of σ^2, which will be done in the last chapter, the proof of theorem 2.1. can be reduced to the following proposition proved in the next chapter.

PROPOSITION 4.3. – *Under* $\hat{\nu}$, $\left(1/n \sum_1^n \psi \circ \hat{\theta}^m, 1/\sqrt{n} \sum_1^n \eta \circ \hat{\theta}^m\right)$ *converge in law towards the product of a Cauchy distribution of parameter* $(1/2\pi \operatorname{Log}(2)) (\Sigma |\varphi(\hat{\alpha})|/C^\#)$ *and a non degenerate centered gaussian distribution.*

Remark. – The theorem given in the next chapter proves the non degeneracy given the fact that η is not identically 0, which can be proved as follows: We observe that η cannot vanish unless φ_c vanishes. Since the signature of Γ is alway (2, 3, N) (*cf.* the proof of Proposition 2.2), we can find a fundamental domain for $\Gamma \diagdown H$ limited by geodesics of the Farey tessellation, *i.e.* images of geodesics between the cusps.

If φ_c is non zero, there is a loop γ on $\Gamma \diagdown H$ along which its integral does not vanish.

Clearly γ has to cut some geodesic between the cusps. (Otherwise it would be homotopic to 0 by our remark on the fundamental domain).

In $\mathscr{R} = (= \Gamma \diagdown H \cup C)$, γ is clearly homotopic to a loop $\tilde{\gamma}$ cutting the Farey geodesics at the cusps (add a thin loop along the geodesic arc between the intersection point and the cusp). The vanishing of η then yields a contradiction since $\int_{\tilde{\gamma}} \omega^c$ is clearly a finite sum of $[\varphi_c]$ (ρ)'s.

But σ^2 will be computed in the last chapter via another method.

Proof of Lemma 4.1. – Since $\hat{\theta}$ is ergodic, it follows from the calculation of

$$\int \Phi \, d\hat{\nu} = \frac{1}{2 \operatorname{Log} 2} \int_0^1 d\alpha \int_1^\infty d\beta \frac{1}{(\alpha-\beta)^2} \operatorname{Log}\left(\frac{\beta([\beta]+\alpha)}{\alpha(\beta-[\beta])}\right)$$

$$= \frac{1}{2 \operatorname{Log} 2} \sum_{n=1}^\infty \int_0^1 d\alpha \int_0^1 d\beta \frac{1}{(\alpha+\beta+n)^2} \operatorname{Log}\left(\frac{(\beta+n)(\alpha+n)}{\alpha\beta}\right)$$

$$= \sum_1^\infty \frac{1}{\operatorname{Log} 2} \iint \frac{1}{(\alpha+\beta+n)^2} (\operatorname{Log}(\alpha+n) - \operatorname{Log}(\alpha))$$

$$= \sum_1^\infty \frac{1}{\operatorname{Log} 2} \int_0^1 \left(\frac{1}{\alpha+n} - \frac{1}{\alpha+n+1}\right) (\operatorname{Log}(\alpha+n) - \operatorname{Log}(\alpha))$$

$$= \frac{1}{\operatorname{Log} 2} \left(\int_1^\infty \frac{1}{x(1+x)} \operatorname{Log} x \, dx - \int_0^1 \frac{\operatorname{Log} x}{1+x} dx\right)$$

$$= \frac{-2}{\operatorname{Log} 2} \int_0^1 \frac{\operatorname{Log} x}{1+x} dx = \frac{\pi^2}{6 \operatorname{Log} 2}.$$

Proof of lemma 4.2. — Since the laws of the $r.v$'s $n^{-\alpha}Z_n$ are tight, for any $\varepsilon>0$, $\exists M_\varepsilon>0$ such that $P(n^{-\alpha}Z_n>M_\varepsilon)<\varepsilon$ for every n.
Set $\Delta_k=|E(e^{ik^{-\alpha}Z_{S_k}}-e^{ik^{-\alpha}Z_k})|$.

Then $\Delta_k\leqq\sum_{-N}^{N}E(|e^{ik^{-\alpha}Z_n}-1|\,1_{S_k=n+k})+P(|S_k-k|>N)$.

Each term of the first sum is dominated by

$$\|e^{ik^{-\alpha}Z_n}-1\|_2\,P(S_k=n+k)<\sqrt{\varepsilon+\frac{1}{2}\left(M_\varepsilon\left(\frac{n}{k}\right)^\alpha\right)^2}\,P(S_k=n+k)$$

$$<\sqrt{\varepsilon+\varepsilon^2/_2}\,P(S_k=n)\qquad\text{if}\quad N=[(\varepsilon/M)^{1/\alpha}]\,k$$

Hence $\Delta_k\leqq|\sqrt{\varepsilon+\varepsilon^2/2}+P(|(S_k/k)-1|>(\varepsilon/M)^{1/\alpha})$

$$\leqq\varepsilon+\sqrt{\varepsilon+\varepsilon^2/2}\qquad\text{for}\quad k\text{ large enough.}$$

5. The transfer operator and limit theorems for continuous fractions

Here we prove the proposition 4.3 and the more general theorems 5.1, 5.10 which have independent interest in the context of continuous fractions.

We consider the one-sided shift θ on N^N; this shift will be identified with the continuous fraction transformation and the correspondance is given by $x=1/(n_1+(1/n_2+\ldots)$ where $x\in[0,1]$ and $(n_k)_{k\geqq1}\in N^N$. We shall write $\theta x=\{1/x\}=1/n_2+(1/n_3+\ldots)$, $n_1=n(x)$. The Gauss measure m on I is the projection of $\hat{\nu}$ and is given by $dm(x)=(1/\text{Log}\,2)\,(1/1+x)$. We shall consider also a finite set F and a family of permutations $t_k=s_k^{-1}$ of $F\,(k\geqq1)$. Then we can define a skew product transformation $\tilde{\theta}$ on $I\times F$ by

$$\tilde{\theta}(x,\sigma)=[\theta x,\,t_n(x)\,\sigma].$$

We consider a function f from $I\times F$ to $\mathbb{R}^2$ which is given by

$$f(x,\sigma)=[n^\beta(x)\,a(\sigma),\,b(\sigma)]=[\psi(x,\sigma),\,b(\sigma)]$$

with $\beta>(1/2)$ and a, b are two functions from F to $\mathbb{R}$.

The following theorem gives in case $\beta=1$ the asymptotic behaviour in law of the Birkhoff sum $S_n=\sum_0^{n-1}f\circ\tilde{\theta}^k=(S_n^1,S_n^2)$ under the product measure of m and the counting measure on F.

THEOREM 5.1. — *With the above notations suppose that a is non zero but $\sum_{\sigma\in F}a(\sigma)=\sum_{\sigma\in F}b(\sigma)=0$ and take $\beta=1$.*

Suppose also that the group generated by the permutations s_k is transitive on F. *Denote by* γ *the number*

$$\frac{1}{F^{\#} \operatorname{Log} 2} \sum_{\sigma \in F} a(\sigma) \operatorname{Log} |a(\sigma)|.$$

Then the sequence $[(S_n^1 - n\gamma)/n, (S_n^2/\sqrt{n})]$ *converges in law toward the product of a Cauchy law and a Gaussian law. The Fourier transform* $r(\lambda, \mu)$ *of this law is given by*

$$r(\lambda, \mu) = e^{-D|\lambda| - (\sigma^2/2)\mu^2}$$

with $D = (\pi/2 F^{\#} \operatorname{Log} 2) \sum_{\sigma \in F} |a(\sigma)|$ *and* σ^2 *is described below* (*lemmas* 6, 7). *If* $b \neq 0$, *then* σ^2 *is positive*.

Remarks. — *a*. In the case when the law of a is symmetric we have $\gamma = 0$. The proposition 4.3. is obtained with $F = Z_{/2Z} \times \Sigma$, $\sigma = (\varepsilon, \rho)$ $\varepsilon = \pm 1$ $s_k(\varepsilon, \rho) = (-\varepsilon, \rho\tau_\varepsilon^k)$. It is clear that τ_1 and τ_{-1} generate the group Γ_0, the quotient group $\Gamma \diagdown \Gamma_0$ is Σ and consequently the group generated by the permutations s_k is transitive on F. In this case, $a(1, \rho) = \varphi(\rho(\infty))$, $a(-1, \rho) = \varphi(\sigma(\infty))$ and $b(\varepsilon, \rho) = \varepsilon \int_0^{i\infty} \rho^* \omega_c$. a and b have clearly a symmetric law.

If we consider a more general function f' of the form $f'(\chi, \sigma) = [n(x) a(\sigma) + c(\sigma), b(\sigma)]$ with $c(\sigma) \in \mathbb{R}$ and $\sum_{\sigma \in F} c(\sigma) = 0$ we get the same result as a corollary of the theorem because the normalisation by $1/n$ destroys the Birkhoff sum associated with $c(\sigma)$. This is valid too in proposition 4.3.

b. This type of limit theorem for continuous fractions and positive functions of $n(x)$ have been considered by P. Levy in [L], but in a more qualitative form. See [JK] for general theorems on the topic of convergence toward stable laws for sums of stationary random variables. Here in theorem 5.1 when $b = 0$, we get only the Cauchy law and not the other stable laws of index 1, because of the condition $\sum_{\sigma \in F} a(\sigma) = 0$. Otherwise the Fourier transform of the limit law would have a logarithmic term. An example of application in this area is given by the convergence in law of $1/n \sum_1^n (-1)^{k+1} n_k(x)$ towards the Cauchy law with Fourier transform $e^{-(\pi/2 \operatorname{Log} 2)|\lambda|}$.

c. A more general result for $\beta > (1/2)$ is given below; then the constant γ does not occur. Also much more precise results for limit laws can be obtained by the same method.

d. The method of proof relies on the use of a transfer operator like in [GH]. This transfer operator Q is the adjoint of $\tilde{\theta}$ relative to the measure $\tilde{m}$; it plays the role of $\tilde{\theta}^{-1}$ and defines a Markov chain on $I \times F$. In fact Q is given by the conditionnal expectation relative to the future $\mathscr{F}^+$ on $N^Z \times F$ with respect to the natural $\hat{\theta}$-invariant measure: $Qu = E[u \circ \hat{\theta}^{-1} / \mathscr{F}^+]$. This appearance of N^Z may seems to be redondant at first sight but it is the natural way to use stationarity. Because of stationarity, the sums $S_n=\sum_0^{n-1} f\circ\tilde{\theta}_k$ and $S'_n=\sum_{-n}^{-1} f\circ\hat{\theta}^k$ have the same law. If we consider the trajectories $\omega=(x_k)_{k\in\mathbb{N}}=(x_k, \sigma_k)_{k\in\mathbb{N}}$ of the Markov chain Q under the Markov measure with initial distribution $\tilde{m}$ on $I\times F$, the law of S'_n is the law of $\sum_0^{n-1} f(x_k)=S''_n$. The Fourier transform $\rho_n(\lambda, \mu)$ of this law is $\rho_n(\lambda, \mu)=\int e^{i\langle(\lambda, \mu), S''_n(\omega)\rangle} dQ_x(\omega)\, d\tilde{m}(x)$ where Q_x is the canonical Markov measure on the trajectories starting from x. If we denote by $Q_{\lambda, \mu}$ the operator defined by $Q_{\lambda, \mu}u=Q[e^{i\langle(\lambda, \mu), f\rangle}u]$ the Fourier transform above is expressed via the iteration of $Q_{\lambda, \mu}$:

$$\rho_n(\lambda, \mu)=\int Q^n_{\lambda, \mu}1(x)\, d\tilde{m}(x)=\langle Q^n_{\lambda, \mu}1, 1\rangle$$

using the natural scalar product on $\mathbb{L}^2(I\times F)$. In order to prove limit theorems for S_n we are thus reduced to a spectral study of $Q_{\lambda, \mu}$ and in this case, it is sufficient to take (λ, μ) small and then to apply perturbation results in convenient functional spaces. This is in fact an extension of the classical method of characteristic functions.

We shall consider the space L of Lipschitz fonctions on $I\times F$; the uniform norm of u we be denoted by

$$|u|_\infty=\sup_{x, \sigma}|u(x, \sigma)|$$

and the Lipschitz coefficient of u will be denoted by

$$[u]=\sup_{x, x', \sigma}\frac{|u(x, \sigma)-u(x', \sigma)|}{|x-x'|}.$$

The space L is normed by

$$\|u\|=|u|_\infty+[u]$$

and it is then a Banach space included in the space B of continuous fonctions; the natural injection of L into B is compact.

With respect to the measure $\tilde{m}$ the adjoint of $\tilde{\theta}$ in $\mathbb{L}^2(I\times F)$ is given by

$$Qu(x, \sigma)=\sum_1^\infty p(x, k)\, u\left(\frac{1}{k+x}, s_k\sigma\right)$$

with $\rho(x, k)=1+x/(k+x)(k+1+x)$. Clearly $\sum_{1}^{\infty} p(x, k)=1$ and we shall denote $1/(k+x)$ by $k.x$. In fact Q operates on L, as well as $Q_{\lambda, \mu}$ which is explicitly given by

$$Q_{\lambda, \mu} u(x, \sigma)=\sum_{1}^{\infty} p(x, k) e^{i\langle (\lambda, \mu), f(k.x, s_k \sigma)\rangle} u(k.x, s_k \sigma).$$

The spectral theory of these operators on L follows from the

PROPOSITION 5.2. – *For $u \in L$, we have*

$$|Q_{\lambda, \mu} u|_\infty \leqq |u|_\infty$$

$$[Q_{\lambda, \mu} u] \leqq \frac{5}{8}[u]+2|u|_\infty.$$

In particular $\|Q^n_{\lambda, \mu}\|$ is bounded.

Proof. – From the condition $\sum_{1}^{\infty} p(x, k)=1$ we get $|Q u|_\infty \leqq |u|_\infty$

$$|Q_{\lambda, \mu} u|_\infty \leqq |e^{i\langle (\lambda, \mu), f\rangle} u|_\infty=|u|_\infty.$$

From routine calculations we get the following inequalities

$$|p(x, k)-p(x', k)| \leqq |x-x'| \left[p(x, k)+\frac{1}{k^2}-\frac{1}{(k+1)^2}\right]$$

$$\sum_{1}^{\infty} |p(x, k)-p(x', k)| \leqq 2|x-x'|$$

$$\left(\sum_{2}^{\infty} \frac{1}{k^2}|p(x, k)|\right)+p(x, 1) \leqq \frac{1}{4}[1-p(x, 1)]+p(x, 1)=\frac{3}{4} p(x, 1)+\frac{1}{4} \leqq \frac{5}{8}.$$

Then we estimate $[Q u]$ as follows

$$|Q u(x, \sigma)-Q u(x', \sigma)| \leqq \sum_{1}^{\infty} |p(x, k)-p(x', k)|\, |u|_\infty+\sum_{1}^{\infty} p(x, k) \frac{1}{k^2}|x-x'|[u]$$

$$\leqq 2|u|_\infty |x-x'|+\frac{5}{8}[u]|x-x'|.$$

The same estimation works for $[Q_{\lambda, \mu} u]$ because of the expression

$$Q_{\lambda, \mu} u(x, \sigma)=\sum_{1}^{\infty} e^{i\langle (\lambda, \mu), f(k.x, s_k \sigma)\rangle} p(x, k) u(k.x, s_k \sigma)$$

and the special form of f which implies: $f(k.x, s_k \sigma)-f(k.x', s_k \sigma)=0$.

The boundedness of $\| Q^n_{\lambda,\mu} \|$ follows formally from the two properties already proved.

PROPOSITION 5.3. — *The equation* $Qu=u\,(u \in B)$ *is possible only if* $u=\mathrm{Cte}$. *The restriction of* $I-Q$ *to the subspace* H *of* L *given by* $H=\{u \in L; \tilde{m}(u)=0\}$ *is invertible. Furthermore* $\tilde{\theta}$ *is ergodic with respect to the measure* $\tilde{m}$.

Proof. — As u is continuous, the supremum M of u is attained at (x_0, σ_0). Then the equation $\sum_{k=1}^{\infty} p(x, k)\ u(k.x_0, s_k \sigma_0)=M$ implies, because $p(x, k)>0$, $\sum_1^{\infty} p(x, k)=1 : u(k.x_0, s_k\sigma_0)=M$ for every k. If $\hat{M}=\{x, \sigma), u(x, \sigma)=M\}$, then $\hat{M}$ is invariant under the transformations $(x, \sigma) \to (k.x, s_k \sigma)$. The projection of $\hat{M}$ on I, it also invariant under $x \to k.x$ and is consequently by equal to I. Denote by $F_x (x \in I)$ the set $F_x=\{\sigma \in F; u(x, \sigma)=M\}$. From the continuity of u we get

$$\lim_{x' \to x} F_{x'} \subset F_x$$

or the other hand by invariance

$$F_{k.x}=s_k F_x.$$

These two properties implies that F_x is constant and then $s_k F_x=F_x$ for every k. The transitivity of the group generated by the $s_k\,(k \in N)$ implies $F_x=F$. Finally $u=M$.

For the second property we use and argument from [N].

As $I-Q$ injective on H, we have only to show that $(I-Q)\,v=u$ has a solution for every $u \in H$, Because $\| Q^n \|$ is bounded we can solve $v_n(1+(1/n))-Q v_n=u$ in H.

From the inequality in the lemma above: $\| Q w \| \leqq \rho \| w \| + K | w |_\infty$ with $p<1$, it follows that

$$\| v_n \| \leqq \left\| v_n \left(1+\frac{1}{n}\right) \right\| \leqq \| u \| + \rho \| v_n \| + K | v_n |_\infty$$

$$\| v_n \| (1-\rho) \leqq \| u \| + K | v_n |_\infty.$$

Suppose for a moment $\overline{\lim_n} | v_n |_\infty = \infty$.

Then $v'_n = v_n / | v_n |_\infty$ is bounded in the norm $\| . \|$ and consequently relatively compact in the uniform norm. If $v' = \lim_n v'_n$, we get $v' - Q v' = 0$, $v' \in H$ because a uniform limit of Lipschitz functions with bounded Lipschitz norm is also Lipschitz. This is impossible because

$$\tilde{m}(v')=0, \qquad v'=\mathrm{Cte}, \qquad | v' |_\infty = 1.$$

Now we get from the relation

$$\| v_n \| (1-\rho) \leqq \| u \| + K | v_n |_\infty$$

the fact that $\|v_n\|$ is bounded.

Taking a convergent subsequence in the uniform norm we get

$$v - Qv = u$$

with $v \in H \subset L$ for the reason above and finally $I - Q$ is invertible on H.

Now L is the direct sum of the constants and H, and furthermore, 1 is not a spectral value of the restriction of Q to H. This implies the convergence of $1/n \sum_0^{n-1} Q^k \omega$ to $\tilde{m}(w)$ in L. The density of L in $\mathbb{L}^2(I \times F)$ implies the convergence in $\mathbb{L}^2(I \times F)$ of the same expression, and the ergodicity of the adjoint $\tilde{\theta}$ follows.

The following proposition is an abstract "operator" version of the convergence of normalised characteristic functions towards characteristic functions of stable laws.

PROPOSITION 5.4. — *Denote by* E *a Banach space,* Q *a bounded operator of* E. *Suppose that* 1 *is an isolated spectral value of* Q *corresponding to a simple eigenvalue with eigenvector* e.

Suppose that Q_t *is a continuous family of operators such that* $\|Q_t^n\| \leqq \text{Cte}\,(t \in \mathbb{R}^d,\ n \in \mathbb{N})$ *with* $Q_0 = Q$. *Denote by* $k(t)$ *the eigenvalue of* Q_t *defined by perturbation from* $k(0) = 1$. *Suppose that* a_n *is a family of endomorphisms of* $\mathbb{R}^d$ *with* $\lim_n a_n = 0$ *and* $\lim_n k[a_n(t)]^n = r(t)$ *for* t *small. Then the sequence of vectors* $Q_{a_n(t)}^n e$ *converges (for* t *small) towards* $r(t)e$.

Proof. — Denote by π the projection on the eigenspace generated by e, π_t the perturbed projection which exists by perturbation theory [*cf.* Ka]. Then $I = \pi_t + (I - \pi_t)$ and $I - \pi_t$ is also a projection. We can write $e = \pi_t e + (I - \pi_t) e$,

$$t_n = a_n(t),\ Q_{t_n}^n e = k(t_n)^n \pi_{t_n} e + Q_{t_n}^n (I - \pi_{t_n}) e$$

$$\| Q_{t_n}^n e - k(t_n)^n \pi_{t_n} e \| \leqq \| Q_{t_n}^n \| \, \| (I - \pi_{t_n}) e \|.$$

But: $\|Q_{t_n}^n\| \leqq \text{Cte}$, $\lim \pi_{t_n} e = \pi\, e = e$ and the right hand side tends to zero. On the other hand by hypothesis $\lim_n k(t_n)^n = r(t)$ and $\lim_n \|Q_{t_n}^n e - r(t) e\| = 0$, proving the claim.

We estimate in the following lemmas the operators $Q_{\lambda, \mu}$ for λ, μ small and the pertubed eigenvalue $k(\lambda, \mu)$ with $k(0, 0) = 1$.

In fact, for the theorem, the case $\beta = 1$ is sufficient but similar estimations are valid is the more general case $\beta > (1/2)$ and will be useful later. We define $\alpha = (1/\beta)$ $(0 < \alpha < 2)$ and write Q_λ, Q_μ instead of $Q_{\lambda, 0}$, $Q_{0, \mu}$.

LEMMA 5.5. — *There exist a constant* $C > 0$ *such that for* $\alpha \neq 1$:

$$\|Q_\lambda - Q\| \leqq C|\lambda|, \qquad \|Q_\mu - Q\| \leqq C|\mu|$$

$$\|Q_{\lambda\mu} - Q\| \leqq C[|\lambda|^\alpha + |\mu|]$$

$$\|Q_{\lambda\mu} - Q_\mu - (Q_\lambda - Q)\| \leqq C|\lambda|^\alpha |\mu|.$$

For $\alpha=1$, *the term* $|\lambda|^{\alpha}$ *in the above relations has to be replaced by* $|\lambda|\operatorname{Log}|\lambda|$.

Proof. — By definition

$$(Q_{\lambda} u-Q u)(x, \sigma)=\sum_{k} p(x, k)\left[e^{i \lambda \psi(k . x, s_{k} \sigma)}-1\right] u(k . x, s_{k} \sigma)$$

$$|Q_{\lambda} u-Q u|_{\infty} \leqq|u|_{\infty} \sum_{k} \frac{2}{k^{2}}|e^{i \lambda k^{\beta} a_{k}}-1|$$

with $\psi(k . x, s_{k} \sigma)=a(s_{k} \sigma) k^{\beta}=k^{\beta} a_{k}$.

By routine estimations we obtain

$$\sum_{k} \frac{1}{k^{2}}\left|\sin \frac{\lambda k^{\beta} a_{k}}{2}\right| \leqq \operatorname{Cte}|\lambda|^{\alpha} \qquad (\alpha \neq 1)$$

or

$$\sum_{k} \frac{1}{k^{2}}\left|\sin \frac{\lambda k a_{k}}{2}\right| \leqq \operatorname{Cte}|\lambda| \operatorname{Log}|\lambda| \qquad (\alpha=1).$$

For the Lipschitz coefficient, write:

$$|Q_{\lambda} u-Q u)(x, \sigma)-(Q_{\lambda} u-Q u)(x', \sigma)| \leqq \sum_{k}|p(x, k)-p(x', k||u|_{\infty}|e^{i \lambda k^{\beta} a_{k}}-1|$$

$$+\sum_{k} p(x, k)|e^{i \lambda k^{\beta} a_{k}}-1||u(k . x, s_{k} \sigma)-u(k . x', s_{k} \sigma)|$$

because $e^{i \lambda \psi(k . x, s_{k} \sigma)}$ is in fact independant of x. Also

$$|u(k . x, s_{k} \sigma)-u(k . x', s_{k} \sigma)| \leqq \frac{[u]}{k^{2}}|x-x'|$$

$$|p(x, k)-p(x', k)| \leqq \frac{2}{k^{2}}|x-x'|$$

from the expression of $p(x, k)$:

$$p(x, k)=\frac{1+x}{(x+k)(x+k+1)}.$$

Finally, estimating as above:

$$[Q_{\lambda} u-Q u] \leqq \operatorname{Cte}|\lambda|^{\alpha}\|u\| \qquad \textit{or} \qquad \operatorname{Cte}\|u\|\,|\lambda| \operatorname{Log}|\lambda|$$

according as $\alpha \neq 1$ or $\alpha=1$.

This gives the announced result for $\|Q_{\lambda}-Q\|$.

For $Q_\mu - Q$ we can write:

$$(Q_\mu u - Q u)(x, \sigma) = \sum_k p(x, k)[e^{i\mu b(s_k \sigma)} - 1] u(k.x, s_k \sigma)$$

comparing with the calculation above we see that the sum $\sum_k p(x, k) | e^{i\lambda k^\beta a_k} - 1 |$ has to be replaced here by $\sum_k p(x, k) | e^{i\mu b_k} - 1 |$ with $b_k = b(s_k \sigma)$ bounded. Then the second result follows from

$$\sum_k \frac{1}{k^2} \left| \sin \frac{\mu b_k}{2} \right| \leqq \mathrm{Cte} | \mu |.$$

The third inequality is obtained from the following relations

$$\| Q_{\lambda\mu} - Q \| \leqq \| Q_{\lambda\mu} - Q_\lambda \| + \| Q_\lambda - Q \|$$
$$Q_{\lambda\mu} u = Q_\lambda [e^{i\mu b} u]$$
$$\| Q_{\lambda\mu} u - Q_\lambda u \| \leqq \| Q_\lambda (e^{i\mu b} - 1) u \| \leqq \mathrm{Cte} \| (e^{i\mu b} - 1) u \|$$
$$\| (e^{i\mu b} - 1) u \| \leqq | e^{i\mu b} - 1 |_\infty \| u \| \leqq \mathrm{Cte} | \mu | \, \| u \|$$
$$\| Q_{\lambda\mu} - Q_\lambda \| \leqq \mathrm{Cte} | \mu |.$$

In order to deduce the fourth inequality observe that:

$$(Q_{\lambda\mu} - Q_\mu) u = (Q_\lambda - Q)(e^{i\mu b} u)$$
$$(Q_{\lambda\mu} - Q_\mu) - (Q_\lambda - Q) = (Q_\lambda - Q)[(e^{i\mu b} - 1) u]$$
$$\| Q_{\lambda\mu} - Q_\mu - (Q_\lambda - Q) \| \leqq \| Q_\lambda - Q \| \, | e^{i\mu b} - 1 |_\infty \| u \| \leqq \mathrm{Cte} | \lambda |^\alpha | \mu |.$$

LEMMA 5.6. — *Set*

$$r_2(\lambda, \mu) = (|\lambda|^\alpha + |\mu|) |\lambda|^\alpha \qquad \textit{if} \quad \alpha \neq 1$$

or

$$r_2(\lambda, \mu) = (|\lambda| \operatorname{Log} |\lambda| + |\mu|) |\lambda| \operatorname{Log} |\lambda| \qquad \textit{if} \quad \alpha = 1.$$

Then there exist a constant such that for (λ, μ) *small*

$$| k(\lambda, \mu) - k(\lambda, 0) - k(0, \mu) + 1 | \leqq \mathrm{Cte}\, r_2(\lambda, \mu).$$

Proof. — By perturbation theory, $k(\lambda, \mu)$ exists and is continuous for (λ, μ) small [Ka]. Let us fix the eigenvector $e_{\lambda\mu}$ such that $\tilde{m}(e_{\lambda\mu}) = \tilde{m}(1) = 1$. Also we use the notations $\langle u, v \rangle = \tilde{m}(u\bar{v})$, $e_\lambda = e_{\lambda 0}$, $e = 1$, $e_\mu = e_{0\mu}$.

From the equation

$$Q_{\lambda\mu} e_{\lambda\mu} = k(\lambda, \mu) e_{\lambda\mu}$$

we get $k(\lambda, \mu) = \langle Q_{\lambda\mu} e_{\lambda\mu}, e \rangle$ $k(\lambda, 0) = \langle Q_\lambda e_\lambda, e \rangle$, $k(0, \mu) = \langle Q_\mu e_\mu, e \rangle$ and then

$$\begin{aligned}(k(\lambda, \mu) - k(\lambda, 0) - k(0, \mu) + 1 &= \langle (Q_{\lambda\mu})(e_{\lambda\mu} - e), e \rangle \\ &+ \langle (Q_{\lambda\mu} - Q_\mu) e_\mu, e \rangle - \langle (Q_\lambda - Q) e_\lambda, e \rangle \\ &= \langle (Q_{\lambda\mu} - Q)(e_{\lambda\mu} - e_\mu), e \rangle + \langle (Q_{\lambda\mu} - Q_\mu)(e_\mu - e), e \rangle \\ &+ \langle (Q_{\lambda\mu} - Q_\mu) - (Q_\lambda - Q) e, e \rangle + \langle (Q_\lambda - Q)(e_\lambda - e), e \rangle \\ |k(\lambda, \mu) - k(\lambda, 0) - k(0, \mu) + 1| &\leqq \| Q_{\lambda\mu} - Q \| \, \| e_{\lambda\mu} - e_\mu \| \\ &+ \| Q_{\lambda\mu} - Q_\mu \| \, \| e_\mu - e \| + \| Q_\lambda - Q \| \, \| e_\lambda - e \| + \| Q_{\lambda\mu} - Q_\lambda - Q_\mu + Q \|.\end{aligned}$$

Furthermore:

$$\| Q_{\lambda\mu} - Q_\mu \| \leqq \| Q_{\lambda\mu} - Q_\mu - Q_\lambda + Q \| + \| Q_\lambda - Q \|$$

and by perturbation theory:

$$\begin{aligned}\| e_{\lambda\mu} - e_\mu \| &\leqq \text{Cte} \, \| Q_{\lambda\mu} - Q_\mu \| \\ \| e_\mu - e \| &\leqq \text{Cte} \, \| Q_\mu - Q \| \\ \| e_\lambda - e \| &\leqq \text{Cte} \, \| Q_\lambda - Q \|.\end{aligned}$$

Now it is easy, by the above lemmas to see that each term in the estimation of $|k(\lambda, \mu) - k(\lambda) - k(\mu) + 1|$ is bounded by Cte $r_2(\lambda, \mu)$.

LEMMA 5.7. — *With the above notations, the following equations are valid:*

$$\begin{aligned}k(\lambda, 0) &= \langle Q_\lambda e, e \rangle + \langle (Q_\lambda - Q)(e_\lambda - e), \mathrm{e} \rangle \\ k(0, \mu) &= \langle Q_\mu e, e \rangle + \langle (Q_\mu - Q)(e_\mu - e), e \rangle.\end{aligned}$$

Proof. — Trivial from the definitions.

LEMMA 5.8. — *With the above notations, there exist* $C > 0$ *such that*

$$|\langle (Q_\lambda - Q)(e_\lambda - e), e \rangle| \leqq C \, r_2(\lambda, 0).$$

Furthermore if b' is defined by

$$b' = \sum_0^\infty Q^k b \qquad [i.e. \, (I - Q) b' = b]$$

then $\langle (Q_\mu - Q)\,(e_\mu - e), e \rangle = -\mu^2 \langle b, Q b' \rangle + 0(\mu^2)$.

Proof. — The first relation follows from the lemmas above.

Observe that the derivative D of Q_μ at 0 exist and is defined by $\lim_{\mu \to 0} (Q_\mu - Q) u/\mu = i Q [bu]$.

Furthermore

$$\| Q_\mu - Q - \mu D \| \leqq \text{Cte} \, |\mu|^2.$$

From the expression

$$k(0,\mu)-1=\langle (Q_\mu - Q)\, e_\mu, e\rangle$$

it follows that the derivative of $k(0,\mu)$ at $(0, 0)$ equals $i\langle Q(be), e\rangle = i\langle b, e\rangle = 0$.

Then, by perturbation theory we can write the expansions

$$k(0,\mu)=1+0(\mu)$$
$$e_\mu = e + i\mu c + 0(\mu)$$
$$Q_\mu = Q + \mu D + 0(\mu)$$

and by identification in the relation $Q_\mu e_\mu = k(0,\mu)\, e_\mu$ we get:

$$De + iQc = ic$$
$$(I-Q)\, c = Qb.$$

Because $\langle e, c\rangle = \langle e, b\rangle = 0$ we obtain

$$c = \sum_0^\infty Q^k b = Qb'.$$

Finally

$$\lim_{\mu\to 0} \frac{\langle (Q_\mu - Q)(e_\mu - e), e\rangle}{\mu^2} = c^2 \langle Q\, bc, e\rangle = -\langle b, Q b'\rangle.$$

LEMMA 5.9. — *With the above notations, set*

$$\sigma^2 = \langle b'^2 - (Q b')^2, 1\rangle$$

and

$$C = \frac{1}{F^\# \operatorname{Log} 2} \sum_{\sigma\in F} |a(\sigma)|, \qquad \gamma = \frac{1}{F^\# \operatorname{Log} 2} \sum_{\sigma\in F} a(\sigma)\, |\operatorname{Log} a(\sigma)|.$$

Then we have the following expansions, if $\beta = 1$

$$k(0,\mu) = 1 - \frac{\sigma^2}{2}\mu^2 + 0(\mu^2)$$

$$k(\lambda, 0) = 1 - C\frac{\pi}{2}|\lambda| + i\gamma\lambda + 0(\lambda).$$

Proof. — We use the two lemmas above.

First

$$\langle Q_\mu e, e\rangle = \frac{1}{F^\#}\sum_\sigma e^{i\mu b(\sigma)} = 1 + i\mu\frac{1}{F^\#}\sum_\sigma b(\sigma)$$
$$-\frac{\mu^2}{2}\frac{1}{F^\#}\sum_\sigma b^2(\sigma) + 0(\mu^2) = 1 - \frac{1}{F^\#}\frac{\mu^2}{2}\sum_\sigma b^2(\sigma) + 0(\mu^2)$$

so we are left to show

$$\sigma^2 = \langle b^2, e\rangle + 2\langle b, Qb'\rangle.$$

But $b = b' - Qb'$ gives

$$\langle b^2, e\rangle + 2\langle b, Qb'\rangle = \langle b'^2 - (Qb')^2, e\rangle.$$

For the second relation, we have only to expand $\langle Q_\lambda e, e\rangle$ up to order one. But this quantity is nothing else than the Fourier transform of the law of $\psi(\chi, \sigma)$ under the arithmetic mean of $F^\#$ laws of random variables which takes the value $na(\sigma)$ $(\sigma \in F)$ with probability

$$p_n = \frac{1}{\operatorname{Log} 2}\int_0^1 p(n, x)\frac{dx}{1+x} = \frac{1}{\operatorname{Log} 2}\operatorname{Log}\left[1 + \frac{1}{n(n+2)}\right].$$

The Fourier transform of $\sum_1^\infty p_n \delta_n$ is of the form

$$1 + iK\lambda - \frac{1}{\operatorname{Log} 2}|\lambda|\left[\frac{\pi}{2} + i \operatorname{sgn}\lambda \operatorname{Log}|\lambda|\right] + 0(\lambda)$$

with a numerical constant K because this law is one sided and satisfies the tail expansion $\sum_N^\infty p_n \sim 1/N \operatorname{Log} 2$.

Replacing in this expansion λ by $a(\sigma)\lambda$ we obtain finally

$$\langle Q_\lambda e, e\rangle = 1 + |\lambda|\frac{\pi}{2F^\# \operatorname{Log} 2}\sum_\sigma |a(\sigma)| + \frac{i\lambda}{F^\# \operatorname{Log} 2}\sum_\sigma a(\sigma)\operatorname{Log}|a(\sigma)| + 0(\lambda).$$

LEMMA 5.10. — *$\sigma^2 = 0$ implies $b = 0$.*

Proof. — By definition

$$\sigma^2 = \langle b'^2 - (Qb')^2, e\rangle = \langle (b' \circ \hat\theta^{-1} - Qb')^2, e\rangle = \langle (b' \circ \hat\theta^{-1} + b - b')^2, e\rangle$$

using the natural scalar product in $N^Z \times F$.

Then the condition $\sigma^2=0$ implies $b=b'-b'\circ\hat{\theta}^{-1}$, $b'\circ\tilde{\theta}=b'\circ\tilde{\theta}-b'$. But then the sum $\sum_0^M b\circ\tilde{\theta}^k$ is uniformly bounded in (χ, σ, M).

In explicit form, for every sequence t_k and $\sigma\in F$, the sum $\sum_0^M b(t_k\ldots t_2 t_1 \sigma)$ is bounded.

This is clearly impossible if $b\neq 0$, because we can choose the t_k independantly so that $t_k\ldots t_2 t_1\sigma$ visits the whole of F and then the central limit theorem for a Markov chain on F implies $\sum_{\sigma\in F}|b(\sigma)|^2=0$, $b=0$.

PROOF OF THE THEOREM 5.1. — According to the beginning of this paragraph we have to show the convergence of

$$\rho'_n(\lambda, \mu)=\int e^{i\langle(\lambda/n,\, \mu/\sqrt{n}),\, S_n(\chi,\sigma)-n(\gamma,0)\rangle}\, d\tilde{m}(\chi,\sigma)=\langle Q_{\lambda/n,\,\mu/\sqrt{n}}1, 1\rangle e^{-i\lambda\gamma}.$$

We use proposition 5.3. with $E=L$, $Q_t=Q_{\lambda,\mu}$, $a_n(\lambda,\mu)=(\lambda/n, \mu/\sqrt{n})$, $e=1$. The spectral properties of Q are clear from proposition 5.3 and the continuity of the family $Q_{\lambda,\mu}$ follows from lemma 5.8. The boundedness of $Q^n_{\lambda,\mu}$ is stated in proposition 5.2.

From lemmas 5.6 and 5.9, we have

$$k(\lambda,\mu)=1-C\frac{\pi}{2}|\lambda|+i\gamma\mu-\frac{\sigma^2}{2}\mu^2+0(\lambda)+0(\mu^2)+0[\lambda\,\mathrm{Log}|\lambda|(|\mu|+|\lambda|\,\mathrm{Log}|\lambda|)]$$

and consequently

$$\lim_n k\left(\frac{\lambda}{n}, \frac{\mu}{\sqrt{n}}\right)^n=e^{-C(\pi/2)\mu|\lambda|+i\gamma\lambda-(\sigma^2/2)\mu^2}.$$

Proposition 5.4. then gives us

$$\lim_n Q_{\lambda/n,\,\mu/\sqrt{n}}1=e^{-C(\pi/2)|\lambda|+i\gamma\lambda-(\sigma^2/2)\mu^2},$$

$$\lim_n \rho'_n(\lambda,\mu)=e^{-C(\pi/2)|\lambda|-(\sigma^2/2)\mu^2}=r(\lambda,\mu).$$

The values of C, λ have been calculated before and lemma 5.10 implies the non degeneracy of the Gaussian part ($\sigma^2>0$) if $b\neq 0$.

We have obtained finally

$$\gamma=\frac{1}{F^{\#}\,\mathrm{Log}\,2}\sum_\sigma a(\sigma)\,\mathrm{Log}|a(\sigma)|$$

$$C=\frac{1}{F^{\#}\,\mathrm{Log}\,2}\sum_\sigma|a(\sigma)|.$$

In order to extend theorem 5.1. to functions f of the form

$$f(x, \sigma)=[n^{\beta}(x)\, a(\sigma), b(\sigma)] \qquad \beta>\frac{1}{2}$$

let us recall some notations for stable laws.

The Cauchy law with density $1/\pi\ A/(A^2+x^2)$ has Fourier transform $e^{-A|\lambda|}$ and tails of the form

$$\frac{1}{\pi}\int_t^{\infty}\frac{A\,dx}{A^2+x^2}\sim\frac{A}{t}.$$

It is the simplest example of stable laws (except the Gaussian law). Such a stable law, for a fixed index $\alpha\,(0<\alpha<2)$ is defined, modulo translation, by two parameters $C>0$ and $p\,(0\leqq p\leqq 1)$ and will be denoted $\gamma_{\alpha, C, p}$. For $\alpha\neq 1$, the Fourier transform of $\gamma_{\alpha, C, p}$ is given by

$$\hat{\gamma}_{\alpha, C, p}(\lambda)=e^{|\lambda|\, C\, \Gamma(3-\alpha)/\alpha(\alpha-1)\,[\cos\pi(\alpha/2)-i(p-q)\sin\pi(\alpha/2)]}$$

with $\lambda\geqq 0$, $p+q=1$.

Of course the change of λ in $-\lambda$ replaces $-i$ by i.

The constants C, p have the following sense. The tails of $\gamma_{\alpha, C, p}$ are of the form

$$\gamma_{\alpha, C, p}[t, \infty]\sim\frac{B_+}{t^{\alpha}} \qquad (t\to+\infty)$$

$$\gamma_{\alpha, C, p}[-\infty, t]\sim\frac{B_-}{|t|^{\alpha}} \qquad (t\to-\infty)$$

and we have $C(2-\alpha)/\alpha=B_-+B_+$, $p=B_-/(B_-+B_+)$.

In the case $\alpha=1$, the Fourier transform has in general a logarithmic term. For symmetric laws $(p=(1/2))$ that is to say the Cauchy laws, we have

$$\hat{\gamma}_{1, C, 1/2}(\lambda)=e^{-C(\pi/2)|\lambda|}$$

with $\gamma_{1, C, 1/2}[t, \infty]\sim(C/2t)$ $(t\to+\infty)$.

For the Cauchy law of density $(1/\pi)$ $A/(A^2+x^2)$ the parameters are consequently $C=(\pi/2)A$ and $p=(1/2)$.

We shall say that the stable laws considered above are the centered stable laws. The effect of a translation of amplitude γ, is to introduce a term $i\gamma\lambda$ in the Fourier transform.

THEOREM 5.11. – *Denote by $f(\chi, \sigma)$ a function from* $I\times F$ *to* $\mathbb{R}^2$ *of the form*

$$f(\chi, \sigma)=[n^{\beta}(x)\, a(\sigma), b(\sigma)]$$

with $\beta>(1/2)$ *and* $\beta\neq 1$, $\sum_{\sigma\in F} a(\sigma)=\sum_{\sigma\in F} b(\sigma)=0$ *and* $a(\sigma)\not\equiv 0$. *Write* $\alpha=(1/\beta)$ *and define*

$$p=\Big(\sum_{a(\sigma)>0} |a(\sigma)|^{\alpha}\Big)\Big(\sum_{\sigma\in F} |a(\sigma)|^{\alpha}\Big)^{-1} \qquad C=\frac{1}{F^{\neq}\operatorname{Log} 2}\sum_{\sigma\in F} |a(\sigma)|^{\alpha}.$$

Moreover let σ^2 *defined from* b *as in lemma* 7. *Suppose that the group generated by the permutations* s_k *is transitive on* F *and consider the Birkhoff sum* $S_n(\chi, \sigma)$ *associated with* f *and* $\tilde{\theta}: S_n=\sum_0^{n-1} f\circ\tilde{\theta}^k=(S_n^1, S_n^2)$. *Then the sequence* $(S_n^1/n^{\beta}, S_n^2/\sqrt{n})$ *converges in law towards the product of a centered stable law of index* α, *parameters* C, p *given above and a Gaussian law of variance* σ^2. *If* $b\neq 0$, *then* $\sigma^2>0$.

Remarks. — *a.* As an example take the special case $b=0$, $F=\{\pm 1\}$, $s_k(\varepsilon)=-\varepsilon$ $a(\varepsilon)=\varepsilon$. Then the theorem 5.11. applies to $S_{2p}(\chi, \varepsilon)=\sum_1^{2p}(-1)^{k-1} n_k^{\beta}(x)$ and consequently to $S''_{2p}(\chi)=\sum_1^{2p}(-1)^{k-1} n_k^{\beta}(x)$.

Here $p=(1/2)$, $C=(1/\operatorname{Log} 2)$, and the law $\gamma_{\alpha, C, p}$ is symmetric. The theorem implies the convergence in law of $1/(2p)^{\beta}\sum_1^{2p}(-1)^{k-1} n_k^{\beta}(x)$ towards $\gamma_{\alpha, C, p}$. In particular if we take $\beta=2$, then $S''_{2p}=1/(2p)^2\sum_1^{2p}(-1)^{k-1} n_k^2(x)$ converges towards the stable law of with Fourier transform $e^{-3\sqrt{\pi/2}\,|\lambda|}$. This law is the convolution the law on $\mathbb{R}^+$ with density $K^{1/2}/\sqrt{2\pi x^3}\, e^{-K/2x}$ $(K=3/2\sqrt{\pi/2})$ with the symmetric law on $\mathbb{R}_-$. It is easy to deduce that $\lim_p |S''_{2p}|=+\infty$ and S''_{2p} changes sign infinitely often a.e. On the contrary, in the situation of theorem 5.1, the sum $S'_{2p}(\chi)=\sum_1^{2p}(-1)^{k-1} n_k(\chi)$ takes the value zero infinitely often a.e. and furthermore:

$$\overline{\lim}\, S'_{2p}(\chi)=+\infty, \qquad \underline{\lim}\, S'_{2p}(\chi)=-\infty.$$

b. In theorems 5.1, 5.10, the form of the permutations s_k does not change the values of the constants in the limiting law.

This is because the random variables $s_{k(x)}$ define a stationnary process on F with invariant measure equal to the counting measure.

Proof of the theorem. — It is the same proof as theorem 5.1 except that lemma 5.9 has to be replaced by the following lemma 5.12. The expansion of $k(\lambda, \mu)$ is now changed into

$$k(\lambda, \mu)=\hat{\gamma}_{\alpha, C, p}(\lambda)\, e^{-(\sigma^2/2)\mu^2}[1+0(|\lambda^{\alpha}\mu|+|\lambda|^{2\alpha})]$$

by lemmas 5.6, 5.9, 5.12.

LEMMA 5.11. — *With the above notations, consider the centered stable law* $\gamma_{\alpha, C, p}$ *of index* $\alpha=(1/\beta)\neq 1$, *and parameters* C, p *given by* $C=(1/F^{\#}\,\mathrm{Log}\,2)\sum_{\sigma\in F}|a(\sigma)|^{\alpha}$

$$p=\Big(\sum_{a(\sigma)>0}|a(\sigma)|^{\alpha}\Big)\Big(\sum_{\sigma\in F}|a(\sigma)|^{\alpha}\Big)^{-1}.$$

Then we have the following expansion

$$k(\lambda, 0)=\hat{\gamma}_{\alpha, C, p}(\lambda)+0(|\lambda|^{\alpha}).$$

Proof. — We use the two lemmas as in lemma 5.8.:

$$k(\lambda, 0)=\langle Q_{\lambda} 1, 1\rangle+0(|\lambda|^{2\alpha}).$$

So we have only to show that, up terms of order $|\lambda|^{\alpha}$, the expansion of $\langle Q_{\lambda} 1, 1\rangle$ is the same as $\hat{\gamma}_{\alpha, C, p}$ with α, C, p given as above.

As in lemma 5.9. we are reduced to the arithmetic mean of $F^{\#}$ probability measures corresponding to random variables taking the value $n^{\beta} a(\sigma)$ with probability p_n.

Consequently we have

$$\tilde{m}\{(\chi, \sigma);\ \psi(\chi, \sigma)>t\}\sim\frac{C_+}{t^{\alpha}} \qquad (t\to+\infty)$$

$$\tilde{m}\{(\chi, \sigma);\ \psi(\chi, \sigma)<-t\}\sim\frac{C_-}{|t|^{\alpha}} \qquad (t\to-\infty)$$

with

$$C_+=\frac{1}{F^{\#}\,\mathrm{Log}\,2}\sum_{a(\sigma)>0}|a(\sigma)|^{\alpha}$$

$$C_-=\frac{1}{F^{\#}\,\mathrm{Log}\,2}\sum_{a(\sigma)<0}|a(\sigma)|^{\alpha}$$

so that we get the relevant expression for $C=C_-+C_+$ and $p=C_+/C$. If $\alpha<1$, we are done because the law of ψ has the same tails as $\gamma_{\alpha, C, p}$ and consequently $\langle Q_{\lambda} 1, 1\rangle$ has the same expansion as $\hat{\gamma}_{\alpha, C, p}$ up to order α. If $\alpha>1$, the same reason is valid but we have also to show that the derivative of $\langle Q_{\lambda} 1, 1\rangle$ for $\lambda=0$ is zero. From $\langle Q_{\lambda} 1, 1\rangle=(1/F^{\#})\sum_{\sigma\in F}\sum_{n}p_n e^{i\lambda n^{\beta} a(\sigma)}$ it follows:

$$(d/d\lambda\langle Q_{\lambda} 1, 1\rangle)_{\lambda=0}=0=(1/F^{\#})\sum_{\sigma\in F}\sum_{n} i n^{\beta} a(\sigma) p_n=(i/F^{\#})\sum_{\sigma\in F} a(\sigma)\sum_{1}^{\infty} n^{\beta} p_n=0 \qquad (\beta<1).$$

6. Calculation of $\sigma(\omega_c)$

For $z \in H$ and $t>0$, let ν_z^t be the uniform distribution on the (non euclidean) circle of radius t centered in z.

Let $\mathcal{F}$ be a fundamental domain for the action of Γ.

Let us denote by dv the normalized volume element on $\mathcal{F}$ and $\Gamma \backslash H$ and by $F_z(\xi)$ the function $\int_z^\zeta \operatorname{Re}(\varphi(\xi)\,d\xi$.

Then

$$\int_{\Gamma \backslash G} \exp\left(\frac{i}{\sqrt{t}} \int_{\gamma_t} \omega_c\right) d\hat{\mu} = \int_{\mathcal{F}} dv \int \exp\left(\frac{i}{\sqrt{t}} F_z\right) d\nu_z^t.$$

Similarly, let K_t denote the heat kernel on H, $K_t(x, dz) = \nu_x^t(dz)\,\rho_t(r)\,dr$, $\rho_t(r)\,dr$ being the law of the distance d_t of the Brownian motion on H at time t to its starting point.

Set $P_t^{(\alpha)}(x, dz) = K_t(x, dz)\,e^{i\alpha F_x(z)}$, then

$$\int_{\mathcal{F}} P_t^{1/\sqrt{t}}\,1\,dv = \int dr\,\rho_t(r) \int_{\mathcal{F}} dv \exp\left(\frac{i}{\sqrt{t}} F_z\right) d\nu_z^t.$$

Besides, we know that d_t is a diffusion with generator $1/2\,(d^2/dx^2 + \coth x\,(d/dx))$, that d_t/t converges a. s. towards $1/2$ as $t \uparrow \infty$, and hence that $t\,\rho_t(\alpha t)\,d\alpha$ converges weakly towards $\delta_{1/2}$, which allows to conclude that $\int_{\mathcal{F}} P_t^{1/\sqrt{t}}\,1\,dv$ converges to $\exp(-\sigma^2(\omega_c)/4)$.

Recall that the Laplacian on Γ/H has a non positive spectrum, 0 being a simple isolated eigenvalue (the continuous spectrum starts at $-1/4$).

For each α, $P_t^{(\alpha)}$ induces a semigroup of self adjoint hermitian contractions $\hat{P}_t^{(\alpha)}$, on $L^2(\Gamma \backslash H, dv)$ which verifies, for f bounded with two bounded derivatives,

$$\frac{d}{dt} \int_{\Gamma \backslash H} P_t^{(\alpha)} f\,dv \,\Big|_{t=0} = \frac{1}{2} \int (\Delta f + 2\alpha i \langle \omega_c, df \rangle - \alpha^2/2\,\|\omega_c\|^2 f)\,dv$$

$$= -\alpha^2/2 \int \|\omega_c\|^2 f\,dv \qquad (\text{since } \delta\omega_c = 0).$$

More generally, if $L^{(\alpha)}$ is the infinitesimal generator of $P_t^{(\alpha)}$, for any f in $\mathcal{D}(L^{(\alpha)})$ (since C_b^2 is dense in L^2 and mapped into itself by $(L^{(\alpha)} - 1)^{-1}$),

$$\int L^{(\alpha)} f\,dv = \frac{-\alpha^2}{2} \int \|\omega_c\|^2 f\,dv. \quad (*)$$

By perturbation theory, for α small enough, $L^{(\alpha)}$ admits a simple isolated eigenvalue $\lambda_\alpha \leq 0$, associated to a unique eigenvector e_α such that $\langle e_\alpha, 1 \rangle = 1$, with λ_α and e_α continuous in α, $\lambda_0 = 0$ and $e_0 = 1$.

From (*), $\lambda_\alpha \sim -\alpha^2/2 \int \| \omega_c \|^2 dv$ as $\alpha \downarrow 0$. Equivalently, $t \lambda_{t^{-1/2}} \sim -(1/2) \int \| \omega_c \|^2 dv$, *i.e.*

$$\langle P_t^{(t^{-1/2})} e_{t^{-1/2}}, e_{t^{-1/2}} \rangle \sim e^{-1/2} \int \| \omega_c \|^2 dv.$$

Since

$$\langle P_t^{t^{-1/2}} 1, 1 \rangle = \langle P_t^{(t^{-1/2})} e_{t^{-1/2}}, e_{t^{-1/2}} \rangle + 0(t^{-1/2}).$$

We conclude that $\sigma^2(\omega_c) = 2 \int_{\Gamma / \mathrm{H}} \| \omega_c \|^2 dv$.

This quantity is known as the Petersson scalar product (*cf.* [Leh] and [K-S] for a related result).

REFERENCES

[Die] J. Dieudonné, *Éléments d'analyse* 9, Gauthier-Villars, Paris, 1982.

[F] J. Franchi, *Théorèmes des résidus asymptotiques pour le mouvement Brownien sur une surface riemannienne compacte* (*Ann. I.H.P.*, Vol. 27, 1991, pp. 445-462).

[H] S. Hu, *Homotopy Theory*, Academy Press, 1959.

[G] Y. Guivarc'h and G. Hardy, *Théorèmes limites pour une classe de chaînes de Markov et applications aux difféomorphismes d'Anosov* (*Ann. I.H.P.*, Vol. 24, 1988, pp. 73-98).

[GL] Y. Guivarc'h and Y. Le Jan, *Sur l'enroulement du flot géodésique* (*C.R. Ac. Sc. Paris*, T. 311, Série I, 1990, pp. 645-648).

[HW] G. Hardy and E. M. Wright, *An Introduction to the Theory of Numbers*, University Press, Oxford, 1975.

[JK] A. Jakubowski and M. Kobus, α-*Stable Limit Theorems for Sums of Dependent Random Vectors* (*J. Mult. Analysis*, Vol. 29, 1989, pp. 219-251).

[Ka] T. Kato, *Perturbation Theory for Linear Operators*, Springer, Berlin, Heidelberg, New York, 1976.

[KS] A. Katsuda and T. Sunada, *Closed Orbits in Homology Classes* (*Pub. Math. I.H.E.S.*, Vol.71, 1990).

[LeH] J. Lehner, *Discontinuous Groups and Automorphic Functions* (*A.M.S.*, Providence, 1964).

[L] P. Lévy, *Fractions continues aléatoires* (Rend. Circ. Math. Palermo, 1952, pp. 1-39).

[LM] T. Lyons and H. P. Mc Kean, *Windings of the Plane Brownian Motion*, (*Advances in Maths*, Vol. 51, 1984, pp. 212-225).

[N] F. Norman, *Markov Processes and Learning Model* (Academic press, 1972).

[Rat] M. Ratner, *The Central Limit Theorem for Geodesic Flows on n Dimensional Manifolds of Negative Curvature* (*Israël J. of Math.*, Vol. 16, 1973, pp. 180-197).

[RY] D. Revuz and M. Yor, *Continuous Martingale Calculus and Brownian Motion* (to appear).

[Sch] B. Schoenberg, *Elliptic Modular Functions*, Springer, Berlin, Heidelberg, New York, 1974.

[Se] C. Series, *The Modular Surface and Continued Fractions* (*J. London Math. Soc.*, Vol. 31, 1985, pp. 69-80).

[Si] Y. G. Sinaï, *The Central Limit Theorem for Geodesic Flows on Manifolds of Constant Negative Curvature* (*Dokl. Akad. Nauk SSSR*, Vol. 133, 1960, pp. 1303-1306).

[Spi] F. Spitzer, *Some Theorems Concerning Two Dimensional Brownian Motion* (*Trans. Ann. Math. Soc.*, Vol. 87, 1958, pp. 187-197).

[Spr] G. Springer, *Introduction to Rieman Surfaces*, *Addison-Wesley* (Reading, 1957).

[Su] SULLIVAN, *Disjoint Spheres, Approximation by Imaginary Quadratic Numbers and the Logarithm Law for Geodesics*, (*Acta Math.*, Vol. 149, 1983, pp. 123-237).

(Manuscript received July 4, 1991;
accepted June 19, 1992).

Y. GUIVARC'H
Université de Rennes
Département de Mathématiques
Campus de Beaulieu
35042 Rennes Cedex,
France;

Y. LE JAN
Université Paris-VI
Laboratoire de Probabilités (CNRS)
Tour 56, 4, place Jussieu
75252 Paris Cedex 05,
France.