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# CONVERGENCE OF RIEMANNIAN MANIFOLDS WITH INTEGRAL BOUNDS ON CURVATURE I

BY DEANE YANG

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## 1. Introduction

Fix  $p \geq n/2$  and a smooth compact  $n$ -dimensional manifold  $M$ . What happens if we try to minimize the  $L^p$  norm of curvature over all Riemannian metrics of fixed volume? This is an intriguing question, and to answer it, we must understand exactly when a minimizing sequence of metrics converges and when it does not. In this paper and its sequel [22] I obtain theorems that describe what is needed for the metrics to converge. In another paper [24] I prove a collapsing theorem—generalizing results of Cheeger-Gromov—that describes what is happening when convergence fails. The consequences of these results for an energy-minimizing sequence of Riemannian metrics are presented in an announcement [25].

Convergence theorems for sequences of Riemannian manifolds were first obtained by J. Cheeger [8] and M. Gromov [16] (see also [15], [19]). The key assumption in these results is a pointwise bound on sectional curvature, *i. e.* an  $L^\infty$  bound on curvature.

In [14], [23] the Ricci flow is used to smooth a Riemannian metric on a compact manifold, converting a metric with bounds on the Sobolev constant and the  $L^p$  norm of curvature,  $p > n/2$ , into one with pointwise bounds on sectional curvature. Using this, one obtains pinching and compactness theorems for compact manifolds with integral bounds on curvature and a bound on the global Sobolev constant.

On the other hand, L. Z. Gao ([14], [12]) has obtained convergence theorems assuming a local lower volume bound, a pointwise bound on Ricci curvature, and a local  $L^{n/2}$  bound on the Riemann curvature. M. T. Anderson [2] has also found simple proofs of Gao's theorems.

The results cited above, however, are inadequate for attacking the minimisation problem. First, we expect a minimizing sequence of metrics to converge only on an open subset of the manifold and to collapse on the complement. To deal with this, it is necessary to have a local convergence theorem in contrast to the global convergence result obtained in [23]. Gao and Anderson both obtain local convergence theorems, but require pointwise bounds on the Ricci curvature.

Both of these issues are addressed in this paper and its sequel [22]. The key contribution of this paper lies in Section 7. I prove that given an  $L^p$  bound on the negative part of Ricci curvature,  $p > n/2$ , the volume of a geodesic cone can be bounded from above. Combining this with an isoperimetric inequality of Chris Croke shows that a lower bound on the volume of a large geodesic ball and the  $L^p$  bound on Ricci curvature imply an isoperimetric inequality on a smaller geodesic ball. These bounds are used to obtain elliptic and parabolic estimates on a Riemannian manifold under weaker assumptions than those required by Gao and Anderson.

A second key idea is a new approach towards smoothing Riemannian manifolds, using what I call the “local Ricci flow”. The global Ricci flow has two major shortcomings. First, it is useless for proving local convergence theorems, since it tries to smooth the metric globally. Second, when studying the critical power  $p = n/2$ , what matters is not the global  $L^{n/2}$  bound on curvature but the local bound, *i. e.* the  $L^{n/2}$  norm of curvature on each geodesic ball of fixed radius. A global heat flow will not control such a local bound. The local Ricci flow smooths the metric only on a given subset of the manifold and leaves the metric fixed elsewhere. This overcomes both of these difficulties.

The third idea is addressed in [22]. This is the existence of harmonic co-ordinates on a geodesic ball of uniform size. The elliptic estimates used here follow closely those of Gao [12]. However, instead of the blow-up argument used by Gao and Anderson, I use the local Ricci flow to construct the harmonic co-ordinates.

The local Ricci flow is itself of much interest. It can be used to study the Ricci flow on a complete Riemannian manifold. In Section 9.3 I give a new proof of a recent result of W. X. Shi [20]. In [24] the local Ricci flow is used to obtain a new characterization of manifolds that collapse with bounded geometry. The local isoperimetric inequality proved here also plays a crucial role in this result.

Also, in Section 6 counterexamples are described, demonstrating the need to assume local lower volume bounds. The Riemannian metric can collapse locally, *i. e.* on only part of the manifold but not everywhere. In particular, for any  $p < \infty$ , exact  $L^p$  analogues of the Cheeger finiteness [8] and Gromov convergence theorems cannot hold, and that an additional assumption—such as a bound on the Sobolev (*i. e.* isoperimetric) constant or a lower bound on the volume of small geodesic balls—is needed. This contrasts with the situation when pointwise bounds on curvature are assumed, where collapsing must occur globally and therefore a global lower volume bound suffices for convergence and finiteness theorems.

### Acknowledgments

I would like to thank I. Z. Gao for his help. Although the approach taken here is quite different from [13], [14], [12], these papers indicated the right directions to aim at and inspired all of the work in this paper. I also appreciate his help in finding errors and gaps in earlier versions of this work.

I also would like to thank Jeff Cheeger, Chris Croke, Sylvestre Gallot, and Mikhael Gromov for helpful discussions.

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## 2. Statement of main theorems

Given a subset  $\Omega$  of a Riemannian manifold and  $\varepsilon > 0$ , let

$$\Omega_\varepsilon = \{ x \in \Omega \mid d(x, M \setminus \Omega) > \varepsilon \}.$$

Let  $\omega$  be the volume of the unit sphere in  $\mathbf{R}^n$ . For other notation and definitions, see Section 4.

The goal of this paper is to prove the following generalization of the local Gromov convergence theorem (see Theorem 5.1 in Section 4 and compare with results in [14], [12]) :

**THEOREM 2.1.** — *Let  $n \geq 3$ ,  $p > n/2$ , and  $0 < \eta < 1$ . There exist constants  $\varepsilon(n) > 0$  and  $\kappa(n, p, \eta) > 0$  such that the following holds:*

*Let  $M_1, \dots$  be a sequence of complete  $n$ -dimensional Riemannian manifolds,  $\Omega_i \subset M_i$  open subsets, and  $D, \rho > 0, K \geq 0$ , constants satisfying the following:*

$$(2.2) \quad \text{vol}(B(x, \rho)) \geq \eta^n n^{-1} \omega \rho^n, \quad \text{for all } x \in \Omega_i$$

$$(2.3) \quad \text{diam}(\Omega_i) < D$$

$$(2.4) \quad \|\text{Rm}\|_{n/2, B(x, \rho)} \leq \varepsilon(n) \eta^{2(n+1)}, \quad \text{for all } x \in \Omega_i$$

$$(2.5) \quad \rho^{2-(n/p)} \|\text{Rc}\|_{p, \Omega_i} \leq \kappa(n, p, \eta)^2$$

*Given  $\varepsilon > 0$ , assume that there is a  $v > 0$  such that  $\text{vol}(\Omega_{i, \varepsilon}) > v$ . Then there exists a subsequence  $\Omega_{i, \varepsilon}$  converging in Hausdorff distance to an open  $C^1$  manifold with a  $C^0$  Riemannian metric.*

*Remark.* — In [22] stronger conclusions, namely Lipschitz convergence and better regularity for the limiting metric, are obtained under the same assumptions. Also, based upon the general discussion on convergence theorems for Riemannian manifolds given in the introduction to [3], the proof presented here already implies Lipschitz convergence.

It is probably worth explaining the assumptions in the theorem a little. Given  $\rho > 0$  satisfying (2.2) and (2.5), then by Theorem 7.4 the isoperimetric constant on any ball of radius  $\eta\rho/2$  is uniformly bounded from below by a positive number, or equivalently, the Sobolev constant is uniformly bounded from above. Given a compact Riemannian manifold, any sufficiently small  $\rho > 0$  satisfies (2.2) and (2.5). For the theorem, we assume that there is a fixed  $\rho$  satisfying (2.2) and (2.5) for all of the manifolds in the

sequence. By Corollary 7.7 this bound plus the upper bound on diameter (2.3) imply the existence of a subsequence that converges to a metric space with respect to Hausdorff distance. The local structure of the metric space is then studied using the local Ricci flow. The uniform bound on the Sobolev constant and the bound (2.4), which translates via Theorem 7.4 into

$$\| \text{Rm} \|_{n/2, B(x, \rho)} \leq \varepsilon'(n) C_s(B(x, \rho))^{-1},$$

are used to obtain parabolic estimates for the curvature.

**COROLLARY 2.6.** — *Let  $M_1, \dots$  be a sequence of compact Riemannian manifolds and  $D, E, \rho, K > 0, p > n/2$  constants satisfying the following:*

$$(2.7) \quad \text{vol}(B(x, \rho)) \geq \eta^n n^{-1} \omega \rho^n, \quad \text{for all } x \in \Omega_i$$

$$(2.8) \quad \text{diam}(M_i) < D$$

$$(2.9) \quad \| \text{Rm} \|_{n/2} \leq E$$

$$(2.10) \quad \rho^{2-(n/p)} \| \text{Rc} \|_{p, \Omega_i} \leq \kappa(n, p, \eta)^2$$

*Then there exists a subsequence converging in Hausdorff distance to a metric space  $M$  such that  $M \setminus \{p_1, \dots, p_N\}$  is an open  $C^1$  manifold with a  $C^0$  Riemannian metric for some finite set of points  $p_1, \dots, p_N \in M$ .*

*Remark.* — M. Anderson and J. Cheeger [3] have proven a finiteness theorem, assuming upper bounds on diameter,  $L^\infty$  norm of Ricci curvature, and  $L^{n/2}$  norm of Riemann curvature, and a lower bound on volume. They also observe that the counterexamples described here in Section 6 show that the theorem does not hold if the  $L^\infty$  norm on Ricci curvature is replaced by a  $L^p$  norm. On the other hand, their proof and therefore their conclusion of finiteness seem to hold under the assumptions of Corollary 2.6. The only changes in the proof are that the upper and lower bounds on the volume of a geodesic ball obtained in Section 7 should be used instead of the Bishop-Gromov relative volume comparison theorem and that when they discuss  $C^{1,\alpha}$  bounds on the metric, only  $C^\alpha$  bounds would be obtained here.

### 3. The key ideas

The proof consists of two parts. In Section 7, I show that an  $L^p$  bound on the negative part of Ricci,  $p > n/2$ , implies an upper bound on the volume of a geodesic cone. The argument is based on one by S. Gallot [11] to estimate the volume of a geodesic tubular neighborhood of a hypersurface.

Using an isoperimetric inequality of C. Croke [9], the volume bound yields a local isoperimetric inequality. This implies that given a sequence of Riemannian manifolds satisfying the assumptions of Theorem 2.1, a subsequence converges in Hausdorff distance to a metric space. It is also equivalent to a local Sobolev inequality that is used later in elliptic and parabolic estimates.

The second part consists involves using the local Ricci flow to smooth a Riemannian metric on a small geodesic ball. The estimates obtained are used to prove a local Lipschitz convergence theorem. This implies that the metric space to which the Riemannian manifolds are converging in Hausdorff distance is in fact a manifold with a continuous Riemannian metric.

The idea of using the Ricci flow,

$$\frac{\partial g}{\partial t} = -2 \operatorname{Rc}(g(t)),$$

to smooth a Riemannian metric was first proposed by Bemelmans-Min-Oo-Ruh [4], who studied the effect of the flow on metrics with pointwise bounded curvature. Using the flow to smooth metrics with  $L^p$  bounded curvature,  $p > n/2$ , was studied in [14], [23]. However, it does not work for  $p = n/2$ . In the critical case what matters is not the global  $L^{n/2}$  bound but the local concentration of the  $L^{n/2}$ -norm of curvature. The global nonlinear heat equation does not appear to control this.

What is needed is a localized version of the Ricci flow that allows us to smooth the metric only on a small open set. In this paper I introduce the following “local Ricci flow”:

$$\frac{\partial g}{\partial t} = -2 \chi^2 \operatorname{Rc}(g(t)),$$

where  $\chi$  is a smooth nonnegative function supported on a small geodesic ball. The standard facts about the Ricci flow, as described in [17], extend easily to the local Ricci flow. The advantage of the local Ricci flow is that the  $L^{n/2}$  norm of curvature satisfies an energy inequality. This is available for the global Ricci flow only if the global  $L^{n/2}$  norm of curvature is sufficiently small.

It is crucial to show that the local Ricci flow has a solution for a small but uniform time interval. The estimates needed to prove this are obtained by Moser iteration. Usually Moser iteration involves obtaining an iterative sequence of estimates on a shrinking sequence of domains. Here, it is important that the estimates be obtained on a fixed domain and with a fixed cutoff function. This, however, can be done simply by using higher and higher powers of the cutoff function. Such a version of Moser iteration was also used by Leon Simon in his work on minimal surfaces. The parabolic estimates needed here are stated and proved in the Appendix.

In Section 12 a local version of the convergence theorem obtained in [23] is stated and proved using the local Ricci flow.

The local Ricci flow can also be used to study the global Ricci flow on noncompact Riemannian manifolds. In Section 9.3 I give a new proof of a recent theorem of W.-X. Shi.

#### 4. Notation and definitions

Let  $M$  be a smooth  $n$ -dimensional Riemannian manifold. We shall denote the diameter of  $M$  by  $\text{diam}(M)$  and the volume by  $\text{vol}(M)$ . The Riemannian curvature tensor is  $Rm$ , and the Ricci tensor  $Rc$ .

Given  $\varepsilon > 0$  and  $x \in M$ , let  $B(x, \varepsilon)$  denote the geodesic ball of radius  $\varepsilon$  centered at  $x$  and  $S(x, \varepsilon) = \partial B(x, \varepsilon)$  the corresponding geodesic sphere.

A Riemannian manifold with the induced distance function is a metric space. Given two metric spaces there are two standard ways of defining the distance between the two spaces, Hausdorff distance and Lipschitz distance. Lipschitz distance is a much stronger metric; two metric spaces are a finite distance apart only if they are homeomorphic. For the definitions and basic facts about the two types of distance, see [16].

All norms in this paper are defined with respect to the given Riemannian metric (which may vary with time  $t$ ).

Suppose that the Riemannian metric of  $M$  depends on  $0 \leq t \leq T$ . Given  $1 \leq p < \infty$ , and open set  $U \subset M$ , and  $f \in C^\infty(U)$ , denote

$$\|f\|_{p,U} = \left( \int_U |f|^p dV_g \right)^{1/p}.$$

If  $p = \infty$ ,  $\|f\|_{\infty,U}$  denotes the essential supremum of  $f$  restricted to  $U$ . We shall denote  $\|f\|_p = \|f\|_{p,M}$ .

Given an open set  $U \subset M$ , we define the *local Sobolev constant*  $C_s(U)$  to be the smallest number  $A > 0$  such that

$$\|f\|_{2n/(n-2)}^2 \leq A \|\nabla f\|_2^2, \quad f \in C_0^\infty(U).$$

On the other hand, let the *local isoperimetric constant*  $C_1(U)$  be the largest number  $\alpha$  such that

$$\text{vol}_{n-1}(\partial\Omega) \geq \alpha \text{vol}(\Omega)^{(n-1)/n}$$

for any compact domain  $\Omega \subset U$  with smooth boundary  $\partial\Omega$ .

Recall from [6] that

$$(4.1) \quad C_s(U) = 4 \left( \frac{n-1}{n-2} \right)^2 C_1(U)^{-2}$$

Applying the isoperimetric inequality to geodesic balls  $B(x, r) \subset U$  and integrating the resulting differential inequality, we obtain the following lower bound on the volume of a geodesic ball:

LEMMA 4.2. — *Given a geodesic ball  $B(x, r) \subset U$ ,*

$$\text{vol}(B(x, r)) \geq (n^{-1} C_1(U) r)^n.$$

### 5. Local version of the Gromov convergence theorem

Given  $\Omega \subset M$  and  $\varepsilon > 0$ , let  $\Omega_\varepsilon = \{x \in \Omega \mid d(x, M \setminus \Omega) > \varepsilon\}$ . Given  $x \in M$ , let  $\text{inj}(x)$  denote the injectivity radius of  $x$  and  $\text{inj}(\Omega) = \inf_{x \in \Omega} \text{inj}(x)$ . The following version of the

Cheeger-Gromov convergence theorem will be used (see [1], [15], [16], [19]):

**THEOREM 5.1.** — *Let  $M_i$  be a sequence of smooth, complete Riemannian  $n$ -dimensional manifold and  $\Omega_i \subset M_i$  open subsets satisfying the following: there exist constants  $K, \delta, D > 0$  such that for all  $i$*

$$\begin{aligned} \|\text{Rm}\|_{\infty, \Omega_i} &\leq K; \\ \text{inj}(\Omega_i) &\geq \delta; \\ \text{diam}(\Omega_i) &\leq D. \end{aligned}$$

Given  $\varepsilon > 0$  assume that there exists  $v > 0$  such that for all  $i$ ,  $\text{vol}(\Omega_{i, \varepsilon}) \geq v$ . Then there exists an open manifold  $\Omega_{\infty, \varepsilon}$ , a subsequence of  $\{\Omega_{i, \varepsilon}\}$ , and diffeomorphisms  $\varphi_i : \Omega_{\infty, \varepsilon} \rightarrow \Omega_{i, \varepsilon}$  such that the metrics  $\varphi_i^* g_i$  converge to a  $C^{1, \alpha}$  Riemannian metric,  $0 < \alpha < 1$ .

By the following lemma the lower bound on the injectivity radius can be replaced by a local lower bound on volume:

**LEMMA 5.2.** — *Let  $M$  be a complete,  $n$ -dimensional Riemannian manifold,  $x \in M$ , and  $\kappa, \varepsilon, v_0 > 0$  such that the magnitude of sectional curvature on  $B(x, 2\varepsilon)$  is bounded by  $\kappa^2$  and such that*

$$\text{Vol}(B(x, 2\varepsilon)) \geq v_0$$

Then there exists  $i_0(n, \varepsilon, v_0, \kappa) > 0$  such that  $\text{inj}(y) > i_0(n, \varepsilon, v_0, \kappa)$  for any  $y \in B(x, \varepsilon)$ .

*Proof.* — This follows directly from Theorem 4.7 of [7].  $\square$

### 6. Counterexamples

Let  $N$  be a compact  $(n-1)$ -dimensional flat manifold with volume 1 and  $M = (-1, 1) \times N$ . Given  $\varepsilon > 0$  and a positive integer  $k$ , consider the following metric on  $M$ :

$$g = dr^2 + (\varepsilon + r)^{2k} g_N.$$

A straightforward calculation shows that the Riemannian curvature always satisfies  $|\text{Rm}| < k^2 r^{-2}$ . Therefore, given any  $p > 0$  and  $k > (2p-1)/(n-1)$ ,

$$\|\text{Rm}\|_p \leq \frac{1}{k(n-1) - 2p + 1} < \infty.$$

In particular, the  $L^p$  norm of  $Rm$  stays bounded as  $\varepsilon$  approaches zero, and a singularity forms at  $\varepsilon=0$ . By pasting this example into a given compact manifold, we obtain a contradiction to the statement obtained by replacing the  $L^\infty$  bound on curvature in the Gromov convergence theorem with an  $L^p$  bound.

It is even possible to have a point singularity with  $L^p$ -bounded curvature and infinite topology. Therefore, there is no exact  $L^p$  analogue of the Cheeger finiteness theorem. The following manifold was first described by J. Cheeger and Mikhael Gromov, who showed that it has a metric with bounded curvature and finite volume. By shrinking the end to a point, it also has a metric with  $L^p$ -bounded curvature.

Given a hyperbolic cusp with a toroidal end, when I say "torus", I will always mean a torus whose universal cover is a horosphere in hyperbolic space.

Let  $n \geq 3$ . Fix an  $(n-1)$ -manifold  $\bar{N}$  such that  $\partial N = T^{n-2} \cup T^{n-2}$ . Put a smooth, complete Riemannian metric  $g_N$  on the interior  $N \subset \bar{N}$  such that the two ends are hyperbolic cusps with torii conformal to

$$g_0 = d\theta_1^2 + 2^2 d\theta_2^2 + \dots + 2^{2(n-3)} d\theta_{n-2}^2.$$

Let  $K$  be an upper bound for the magnitude of sectional curvature on  $N$ . Also, observe that the volume  $V$  of  $N$  is finite.

Fix a torus on each cusp such that one torus is  $2^{n-1}$  times as large as the other. Denote the length of the smallest closed geodesic on the smaller torus by  $2\pi l$ . Let  $r \geq 0$ ,  $0 \leq \theta_1, \dots, \theta_{n-2} \leq 2\pi$  be co-ordinates on each cusp so that the given torii correspond to  $r=0$  and the metrics on the cusps are  $dr^2 + l^2 e^{-2r} g_0$  and  $dr^2 + 2^{2(n-1)} l^2 e^{-2r} g_0$ . Given  $\rho \geq 0$ , let  $N(\rho)$  be the compact manifold obtained by chopping the cusps off  $N$  at the torii corresponding to  $r=\rho$ . By choosing the original torii,  $r=0$ , sufficiently far apart, we can assume that the diameter of  $N(r)$  is the distance between the two cusps. Let  $\delta$  denote the diameter of  $N(0)$ , so that  $\text{diam}(N(\rho)) = \delta + 2\rho$ .

Given  $\tau, \rho > 0$ , let  $M(\tau, \rho)$  be the manifold  $N(r) \times S^1$  with the metric

$$\tau^2 (g_N + 2^{2(n-2)} l^2 e^{-2\rho} d\varphi^2),$$

where  $0 \leq \varphi \leq 2\pi$  is the co-ordinate on  $S^1$ . Observe that  $\partial M(\tau, \rho)$  consists of two disjoint torii, one  $2^{n-2}$  times as large as the other.

Given  $M(\tau, \rho)$  and  $M(\tau', \rho')$ , the smaller torus of  $\partial M(\tau, \rho)$  is isometric to the larger torus of  $\partial M(\tau', \rho')$  if

$$\tau e^{-\rho} = 2^{n-2} \tau' e^{-\rho'}.$$

Fix  $0 < \alpha < 1$ . Now let  $M_k = M(\tau_k, \rho_k)$ , where

$$\tau_k = 2^{-k(n-2)(1-\alpha)} \quad \text{and} \quad \rho_k = k(n-2)\alpha \log 2.$$

Then  $\tau_k e^{-\rho_k} = 2^{n-2} \tau_{k+1} e^{-\rho_{k+1}}$ , so that the sequence  $M_1, \dots$  can be glued together to obtain a manifold  $\tilde{M}$  with a continuous metric and piecewise constant curvature. Since the second fundamental form of the boundary of each piece is exponentially small

with respect to  $k$ , the metric can be made smooth without perturbing the curvature significantly.

To show that  $\tilde{M}$  has a point singularity, it suffices to show that  $\tilde{M}$  has finite diameter. The diameter is

$$\begin{aligned} \text{diam}(\tilde{M}) &= \sum_{k=0}^{\infty} \text{diam}(M_k) \\ &= \sum_{k=0}^{\infty} 2^{-k(n-2)(1-\alpha)} (\delta + k\alpha \log 2) < \infty \end{aligned}$$

Next, we compute the integral norm of curvature:

$$\begin{aligned} \int_{\tilde{M}} |\text{Rm}|^p dV &\leq \sum_{k=0}^{\infty} (2^{2k(n-2)(1-\alpha)} K)^p (2^{-k(n-2)(1-\alpha)})^n V(2^{n-2} 2\pi l e^{-k(n-2)\alpha \log 2}) \\ &= 2^{n-1} \pi l K^p V \sum_{k=0}^{\infty} 2^{k(n-2)[2p-n-\alpha(2p-2)]} \end{aligned}$$

Therefore, given  $p \geq n/2$  and  $\alpha > (2p-n)/(2p-2)$ , the curvature of  $\tilde{M}$  is bounded in  $L^p$ .

An easy calculation shows that the volume of a geodesic ball centered at the singularity is

$$V(B(r)) \sim r^{n+\alpha/(1-\alpha)} = o(r^n).$$

In particular, the local Sobolev constant  $C_s(B(r))$  is bounded.

The only singularities with  $L^p$  bounded curvature that I know are orbifold singularities and those like the one above constructed with F-structures studied by Cheeger-Gromov. An intriguing question is whether this is the only possibility.

### 7. An upper bound for the volume of a geodesic cone

Let  $S_x \subset T_x M$  denote the space of unit tangent vectors at  $x$ . Given a subset  $\hat{S} \subset S_x$  and  $\rho > 0$ , let

$$\Gamma(\hat{S}, \rho) = \{ y = \exp_x r\theta \mid 0 \leq r < \rho, \theta \in \hat{S}, d(x, y) = r \}$$

Given  $x \in M$ , let  $\lambda(x)$  denote the lowest eigenvalue of the Ricci tensor at  $x$  and let  $\lambda_-(x) = \max(0, -\lambda(x))$ .

**THEOREM 7.1.** — *Let  $M$  be a Riemannian manifold,  $x \in M$ ,  $\hat{S} \subset S_x$ ,  $p > n/2$ ,  $\varepsilon > 0$ . Set*

$$\begin{aligned} \delta &= \frac{2p-n}{2p-1} \\ \hat{\omega} &= \text{vol}(\hat{S}) \end{aligned}$$

$$\kappa^{2p} = \int_{\Gamma(\hat{S}, \rho)} \lambda_-^p dV_g$$

$$r_0 = \frac{\tau^{1/\delta}}{1+\tau} (C(n, p))^{-1} \hat{\omega} \kappa^{-2p})^{1/(2p-n)}$$

where  $C(n, p)$  is as given in Lemma 7.3. Then for any  $0 \leq r \leq \rho$ ,

$$\text{Vol}(\Gamma(\hat{S}, r)) \leq \begin{cases} (1+\tau)^{n-1} n^{-1} \hat{\omega} r^n, & 0 \leq r \leq r_0 \\ c(n, p, \tau) \kappa^{2p} [(1-\delta)r + \delta r_0]^{2p}, & r \geq r_0 \end{cases}$$

where

$$c(n, p, \tau) = (1+\tau^{-1})^{2p-1} \frac{2p-1}{2p(n-1)} C(n, p)$$

*Proof.* — We shall always assume that  $\kappa > 0$ . If  $\kappa = 0$ , then the volume bound follows directly from the Bishop-Gromov inequality [6].

The exponential map at  $x$  defines a map

$$\begin{aligned} E : (0, \rho) \times S^{n-1} &\rightarrow M \\ (r, \theta) &\mapsto \exp_x r \theta \end{aligned}$$

We define the function  $J : (0, \rho) \times S^{n-1} \rightarrow \mathbb{R}$ ,  $J \geq 0$ , by pulling back the volume form of  $M$ :

$$dV_M = J^{n-1} dr d\theta,$$

where  $d\theta$  is the standard volume form on  $S^{n-1}$ .

It suffices to consider only length-minimizing geodesics from  $x$ . Define  $\tilde{U} \subset (0, \rho) \times \hat{S}$  to be the set of all  $(r, \theta)$  such that the open geodesic segment  $E((0, r) \times \{\theta\})$  is length-minimizing for any two points on it. Observe that  $E(\tilde{U}) = \Gamma(\hat{S}, \rho)$ .

For convenience we define  $\lambda_-(r, \theta) = \lambda_-(E(r, \theta))$  and let  $'$  denote differentiation with respect to  $r$ . The following are well-known facts about the function  $J$  (see [5]):

LEMMA 7.2. — Given  $(r, \theta) \in \tilde{B}$ ,  $J(r, \theta) > 0$ . Moreover,  $J$  satisfies the following:

$$J'' - \lambda_- J \leq 0, \quad J(0, \theta) = 0, \quad J'(0, \theta) = 1.$$

The following estimate was inspired by a similar one used by Sylvestre Gallot [11] to obtain a volume bound for a tubular neighborhood of a hypersurface.

LEMMA 7.3. — Given  $p > n/2$ ,  $(r, \theta) \in \tilde{B}$ ,

$$\frac{J' - 1}{J^\delta}(r, \theta) \leq \left( C(n, p) \int_0^r \lambda_-^p J^{n-1} ds \right)^{1/(2p-1)},$$

where

$$C(n, p) = \left(2 - \frac{1}{p}\right)^p \left(\frac{p-1}{2p-n}\right)^{p-1}.$$

*Proof.* — A straightforward computation shows that, given  $\delta > 0$ ,

$$\left(\frac{J'-1}{J^\delta}\right)' + \delta \frac{(J'-1)J'}{J^{\delta+1}} \leq \lambda_- J^{1-\delta}.$$

We fix  $\theta$  and view  $J$  as a function of  $r$  only. Assume  $J'(r, \theta) > 1$ ; otherwise, the estimate holds trivially. Let  $(r_0, r)$  be the largest interval on which  $J' > 1$ . Then along this interval,  $(J'-1)J' = (J'-1)^2 + J'-1 > (J'-1)^2 > 0$ , and therefore,

$$\left(\frac{J'-1}{J^\delta}\right)' + \delta J^{-1+\delta} \left(\frac{J'-1}{J^\delta}\right)^2 \leq \lambda_- J^{1-\delta}.$$

Now applying the inequality  $\alpha x \leq (1+x)_+^p$ , where  $\alpha = p^p/(p-1)^{p-1}$ , we obtain

$$\alpha \delta^{p-1} J^{-(p-1)(1-\delta)} \left(\frac{J'-1}{J^\delta}\right)^{2(p-1)} \left(\frac{J'-1}{J^\delta}\right)' \leq \lambda^p J^{p(1-\delta)}.$$

This implies that

$$\frac{\partial}{\partial r} \left(\frac{J'-1}{J^\delta}\right)^{2p-1} \leq (2p-1) \alpha^{-1} \delta^{-p+1} \lambda_- J^{(2p-1)(1-\delta)}.$$

Now set  $\delta = (2p-n)/(2p-1)$  and integrate both sides from  $r_0$  to  $r$ . The lemma now follows easily.  $\square$

Let

$$J_+(r, \theta) = \begin{cases} J(r, \theta), & (r, \theta) \in \tilde{\mathbf{B}} \\ 0, & (r, \theta) \notin \tilde{\mathbf{B}} \end{cases},$$

and define

$$v(r) = \left( \int_{\tilde{\mathbf{S}}} J_+(r, \theta)^{n-1} d\theta \right)^{1/(n-1)}.$$

A straightforward calculation using Hölder's inequality and Lemma 7.3 yields

$$v'(r) \leq \alpha + \beta v^\delta,$$

where  $\alpha = \hat{\omega}^{1/(n-1)}$  and  $\beta = (C(n, p) \kappa^{2p})^{1/(2p-1)}$ . On the other hand, if we define

$$w(r) = \begin{cases} (1+\tau)\alpha r, & 0 \leq r \leq r_0, \\ [(1+\tau^{-1})(1-\delta)\beta r + \delta(\tau\alpha\beta^{-1})^{(1-\delta)/\delta}]^{1/(1-\delta)}, & r \geq r_0, \end{cases}$$

where

$$r_0 = \frac{1}{(1+\tau)\alpha} \left( \frac{\tau\alpha}{\beta} \right)^{1/\delta},$$

then it is easily checked that  $w' \geq \alpha + \beta w^\delta$ . It then follows that  $v(r) \leq w(r)$  and therefore for  $r \geq r_0$ ,

$$\begin{aligned} \text{vol}(\Gamma(\hat{S}, r)) &= \int_0^r v(r')^{n-1} dr' \leq \int_0^r w(r')^{n-1} dr' \\ &= \frac{\tau^{n/\delta}}{1+\tau} \left[ \frac{1}{n} - \frac{2p-1}{2p(n-1)} \right] \alpha^{(n/\delta)-1} \beta^{-n/\delta} + c(n, p, \tau) \kappa^{2p} [(1-\delta)r + \delta r_0]^{2p}. \quad \square \end{aligned}$$

Applying the isoperimetric inequality of Chris Croke [9], we obtain the following local isoperimetric inequality:

**THEOREM 7.4.** — *Given  $p > n/2$ ,  $\tau, R > 0$ , and  $x_0 \in M$ , let*

$$\eta = \left[ \frac{\text{vol}(\mathbf{B}(x_0, R))}{n^{-1} \omega R^n} \right]^{-1/n}$$

and

$$r = \frac{\eta}{1+\tau} R$$

Then if

$$C(n, p) R^{2p-n} \int_{\mathbf{B}(x_0, R+2r)} \lambda_-^p dV_g < \omega \min \left( \tau^{2p-1} \eta^{n-2p}, \frac{2p(n-1)}{n(2p-1)} \frac{\tau \eta^n}{(1+\tau+\eta)^{2p}} \right)$$

then

$$C_1(\mathbf{B}(x_0, r)) \geq C(n) \left( \frac{\tau \eta}{1+\tau+\eta} \right)^{n+1} C_1(\mathbf{R}^n)$$

*Proof.* — Let  $\Omega \subset \mathbf{B}(x_0, r)$  have smooth boundary  $\partial\Omega$ . Given  $x \in \Omega$ , let  $\hat{S}_x \subset S_x$  denote the set of unit tangent vectors  $v$  such that the corresponding geodesic  $\exp_x sv$ ,  $s > 0$ , is a minimal geodesic joining  $x$  to some point in  $\mathbf{B}(x_0, R) \setminus \mathbf{B}(x_0, r)$ . Choose  $x \in \Omega$  so that  $\hat{S}_x$  has minimal volume.

By Theorem 11 of [9],

$$\frac{\text{vol}(\partial\Omega)}{\text{vol}(\Omega)^{(n-1)/n}} \geq C(n) C_1(\mathbf{R}^n) \left( \frac{\hat{\omega}}{\omega} \right)^{1+(1/n)}.$$

It therefore suffices to obtain a lower bound for  $\hat{\omega}$ .

By Theorem 7.1,

$$\text{vol}(\mathbf{B}(x_0, r)) \leq (1+\tau)^{n-1} n^{-1} \omega r^n$$

It follows that

$$(7.5) \quad \begin{aligned} \text{vol}(\Gamma(\hat{S}_x, R+r)) &\geq \text{vol}(B(x_0, R) \setminus B(x_0, r)) \\ &\geq n^{-1} \omega(\eta R)^n \frac{\tau}{1+\tau} \end{aligned}$$

Setting  $\hat{\tau} = \tau^{-1}$ , we find that

$$\text{vol}(\Gamma(\hat{S}_x, R+r)) \geq c(n, p, \hat{\tau}) \kappa(R+r)^{2p}$$

and therefore by Theorem 7.1,

$$(7.6) \quad \text{vol}(\Gamma(\hat{S}_x, R+r)) \leq (1+\tau^{-1})^{n-1} n^{-1} \hat{\omega}(R+r)^n$$

Combining (7.6) and (7.5) and solving for  $\hat{\omega}$ , we obtain the inequality

$$\frac{\hat{\omega}}{\omega} \geq \left( \frac{\tau\eta}{1+\tau+\eta} \right)^n$$

Substituting this into Croke's inequality yields the theorem.  $\square$

Applying Proposition 5.2 in [16], we obtain the following precompactness theorem:

**COROLLARY 7.7.** — *Fix  $p > n/2$ ,  $r > 0$  and  $\eta > 0$ . There exists a constant  $\kappa(n, p, \eta) > 0$  such that given any sequence  $\{M_i\}$  of compact Riemannian manifolds satisfying the following bounds:*

$$\begin{aligned} \text{diam}(M_i) &\leq D \\ \text{vol}(B(x, r)) &> \eta^n n^{-1} \omega r^n, \quad x \in M_i \\ r^{2p-n} \int_{M_i} \lambda_-^p &\leq \kappa(n, p, \eta)^{2p} \end{aligned}$$

*there exists a subsequence that converges with respect to Hausdorff distance to a compact metric space.*

*Proof.* — Using the results given in [6], it suffices to obtain for each  $\varepsilon < r$ , an upper bound on

$$N_i(\varepsilon) = \frac{\text{vol}(M_i)}{\inf_{x \in M_i} \text{vol}(B(x, \varepsilon))}$$

that is independent of  $i$ . However, Theorem 7.1 bounds the numerator and Theorem 7.4 gives a lower bound for the denominator.  $\square$

### 8. Local Ricci flow

Let  $M$  be a smooth  $n$ -manifold without boundary. Given a smooth Riemannian metric  $g_0$  and a smooth compactly supported function  $\chi$ , we wish to study the following evolution equation:

$$(8.1) \quad \frac{\partial g}{\partial t} = -2\chi^2 \text{Rc}(g); \quad g(0) = g_0.$$

Observe that if  $M$  is compact and set  $\chi \equiv 1$ , this reduces to Hamilton's Ricci flow.

**THEOREM 8.2.** — *There exists  $T > 0$  such that (8.1) has a smooth solution for  $0 \leq t \leq T$ .*

*Proof.* — First, we can modify the manifold  $M$  any way we want outside the support of  $\chi$ . In particular, we may as well assume that  $M$  is compact. We describe two different approaches to proving this result.

**APPROACH 1.** — Although (8.1) is at heart a parabolic equation, its equivariance under the group of diffeomorphisms makes it highly degenerate. To reveal its parabolicity, it is necessary to “break the symmetry”. We use a trick of Dennis DeTurck [10] to do this.

Fix an invertible symmetric tensor  $S = S_{ij} dx^i dx^j$  (e. g.,  $g_0$ ). Given a metric  $\hat{g}$ , let  $\hat{\nabla}$  denote covariant differentiation with respect to  $\hat{g}$  and define

$$\begin{aligned} \hat{G}(S)_{ij} &= S_{ij} - \frac{1}{2} \hat{g}^{pq} S_{pq} \hat{g}_{ij} \\ \hat{\delta}(S)_i &= -\hat{g}^{pq} \hat{\nabla}_q S_{ip}. \end{aligned}$$

Let  $\hat{\delta}^*$  denote the formal adjoint of  $\hat{\delta}$ , so that given a 1-form  $\omega_i dx^i$ ,

$$\hat{\delta}^*(\omega)_{ij} = \frac{1}{2} (\hat{\nabla}_j \omega_i + \hat{\nabla}_i \omega_j).$$

Instead of solving for  $g(t)$  directly, we solve for a metric  $\hat{g}(t)$  and a 1-parameter family of diffeomorphisms  $\varphi_t: M \rightarrow M$ ,  $0 \leq t \leq T$  satisfying the following:

$$\begin{aligned} \frac{\partial \hat{g}}{\partial t} &= -2[(\chi \circ \varphi_t^{-1})^2 \text{Rc}(\hat{g}) - \hat{\delta}^*(\chi \circ \varphi_t^{-1})^2 S^{-1} \delta G(S)], \quad \hat{g}(0) = g_0 \\ \frac{\partial \varphi}{\partial t} &= -\chi^2 \varphi_{t*} [S^{-1} \delta G(S)]^*, \quad \varphi_0 = I, \end{aligned}$$

where  $(\omega^*)^i = \hat{g}^{ij} \omega_j$  and  $I: M \rightarrow M$  is the identity map. The existence of a smooth solution can be proved either using a fixed point argument for an appropriate Banach space ( $C^0([0, T], H^k(M))$ ,  $k$  sufficiently large) or, if one's taste runs to fancy machinery, the Nash-Moser implicit function theorem. In either case, the proof reduces to proving that the initial value problem for the linearized equation has a unique smooth solution and that the solution satisfies smooth tame estimates. The linearized equation takes the

following general form:

$$\begin{aligned} \frac{\partial h}{\partial t} &= 2\chi^2 \Delta h + \chi (B \nabla h + C \nabla \psi) + D h + E \psi, & h(0) &= 0; \\ \frac{\partial \psi}{\partial t} &= -\chi^2 (P \nabla \psi + Q \nabla h + R h), & \psi(0) &= 0, \end{aligned}$$

where  $h$  is the infinitesimal deformation of the metric  $\hat{g}$ ,  $\psi$  is the infinitesimal deformation of the diffeomorphism  $\varphi_t$ , and for convenience we've set  $\psi_t = I$ . The smooth tame estimates are obtained by standard arguments using energy integrals and interpolation inequalities. These estimates also directly imply uniqueness of the solution. The appendix in [18] contains all the essential ideas, although for a quasilinear hyperbolic system rather than a parabolic equation.

To obtain existence of a solution to the linearized equation, we regularize the linearized equation by fixing  $\varepsilon > 0$  and adding a new term,  $\varepsilon^2 \Delta h$  to the righthand side of the equation for  $h$ . The new equation is now a strictly parabolic equation, and the regularized initial value problem for  $h$  and  $\psi$  has a unique global smooth solution. That the solution to the regularized problem converges to a smooth solution of the original initial value problem as  $\varepsilon \rightarrow 0$  follows from the energy estimates obtained above.

Existence can also be obtained from scratch by observing that with the *a priori* energy inequalities that can be obtained from the equations, the proof of existence for a parabolic equation as given in [21] works here, too.

Finally, one sets  $g = \varphi_t^* \hat{g}$ .

APPROACH 2. — Given  $\varepsilon > 0$ , consider

$$\frac{\partial g}{\partial t} = -2(\varepsilon^2 + \chi^2) \text{Rc}(g); \quad g(0) = g_0.$$

Again, using DeTurck's trick this system can be reduced to a nonlinear, strictly parabolic system which has a smooth solution for some time interval  $[0, T)$ ,  $T > 0$ . The curvature and its covariant derivatives satisfy a local heat equation and can be shown to satisfy  $L^2$  energy bounds that are independent of  $\varepsilon > 0$ . Using this observation, one shows that  $T$  can be chosen independent of  $\varepsilon$  and that as  $\varepsilon \rightarrow 0$ , the solution to the regularized flow converges to a smooth solution of (8.1).  $\square$

### 9. Smoothing a Riemannian metric

Let  $M$  be a smooth  $n$ -manifold with Riemannian metric  $g_0$  and  $\Omega$  an open subset of  $M$ . Let  $\chi$  be a nonnegative smooth compactly supported function on  $\Omega$ . Consider the following evolution equation:

$$(9.1) \quad \frac{\partial g}{\partial t} = -2\chi^2 \text{Rc}(g), \quad g(0) = g_0.$$

We want to show that given appropriate integral bounds on the Riemann and Ricci curvatures, this equation has a solution for a uniform amount of time and the positive time metrics  $g(t)$  are regularizations of the initial metric  $g_0$ .

Three different situations will be considered:  $L^p$  bound on curvature with  $p = n/2$ ,  $p > n/2$ , and  $p = \infty$ .

9.1.  $p = n/2$ .

THEOREM 9.2. — *There exists a constants  $c(n)$  and  $C(n, p)$ ,  $p > n/2$ , such that if*

$$\left( \int_{\Omega} |\text{Rm}(g_0)|^{n/2} dV_{g_0} \right)^{2/n} < (c(n) C_S(\Omega))^{-1}$$

and

$$\left( \int_{\Omega} |\text{Rc}(g_0)|^p dV_{g_0} \right)^{1/p} < K,$$

then the equation (9.1) has a smooth solution for  $t \in [0, T)$ , where

$$T \geq \min(\|\nabla \chi\|_{\infty}^{-2}, C(n, p) K^{-2p/(2p-n)} C_S(\Omega)^{-n/(2p-n)}).$$

Moreover, for  $t \in (0, T)$ , the Riemannian curvature satisfies the following bound:

$$(9.3) \quad \|\chi^2 \text{Rm}\|_{\infty} \leq C(n) C_S(\Omega) (t \|\nabla \chi\|_{\infty}^2 + 1) t^{-1}.$$

*Proof.* — By Theorem 8.2, the equation (9.1) has a smooth solution on a sufficiently small time interval starting at  $t=0$ .

Let  $[0, T_{\max})$  be a maximal time interval on which (9.1) has a smooth solution and such that the following hold for each metric  $g(t)$ :

$$(9.4) \quad \|f\|_{2n/(n-2)}^{(n-2)/n} \leq 4 A_0 \|\nabla f\|_2^2, \quad f \in C_0^{\infty}(\Omega);$$

$$(9.5) \quad \frac{1}{2} g_0 \leq g(t) \leq 2 g_0;$$

$$(9.6) \quad \|\text{Rm}(g(t))\|_{n/2} \leq 2 (c(n) A_0)^{-1},$$

Suppose that  $T_{\max} < T_0 = \min(\|\nabla \chi\|_{\infty}^{-2}, C(n, p) K^{-2p/(2p-n)} A^{-n/(2p-n)})$ . We show that this leads to a contradiction.

The curvature tensor  $\text{Rm}$  satisfies the following equation (see [17]) :

$$(9.7) \quad \frac{\partial \text{Rm}}{\partial t} = \chi^2 (\Delta_g \text{Rm} + Q(\text{Rm}, \text{Rm})) \\ + 2 \chi a(\nabla \chi, \nabla \text{Rm}) + b(\nabla \chi, \nabla \chi, \text{Rm}) + \chi c(\nabla^2 \chi, \text{Rm}),$$

where  $Q$ ,  $a$ ,  $b$ , and  $c$  are multilinear functions of their arguments. Their definitions depend only on the dimension  $n$  of the manifold. Then using the energy inequality used in the proof of Theorem A.7, the assumption that  $t < T_0$  implies that strict inequality holds for (9.6).

Next, since the Ricci curvature satisfies an equation of the form

$$\frac{\partial \text{Rc}}{\partial t} = \chi^2 (\Delta_g \text{Rm} + \text{Q}(\text{Rm}, \text{Rc})) + 2\chi a(\nabla \chi, \nabla \text{Rc}) + b(\nabla \chi, \nabla \chi, \text{Rc}) + \chi c(\nabla^2 \chi, \text{Rc}),$$

Corollary A.10 implies that

$$|\chi^2 \text{Rc}(g(t))| \leq C(n, p) A_0^{n/2p} (1 + t \|\nabla \chi\|_\infty^2)^2 t^{-n/2p} K.$$

Applying the bound on Rc to the following

$$\left| \frac{d}{dt} \int_{\mathbf{M}} f^p dV_g \right| = \left| \int \chi^2 \text{S} f^p dV_g \right| \leq \|\chi^2 \text{Rc}\|_\infty \int f^p dV_g,$$

we find that

$$\left| \log \frac{\|f\|_p(t)}{\|f\|_p(0)} \right| \leq \log 2, \quad 0 \leq t \leq T_2.$$

The differential inequality

$$\left| \frac{d}{dt} \int_{\mathbf{M}} |\nabla f|^2 dV_g \right| = 2 \left| \int \text{Rc}(\nabla f, \nabla f) - \frac{1}{2} \text{S} |\nabla f|^2 dV_g \right| \leq 2 \|\text{Rc}\|_\infty \int_{\mathbf{M}} |\nabla f|^2 dV_g,$$

leads to an analogous estimate. It therefore follows that for any  $t \leq T_0$ , (9.4) holds with strict inequality.

We use Hamilton's trick to verify that (9.5) also holds with strict inequality for  $t \leq T_0$ . Simply fix a tangent vector  $v$ , compute derivative of the norm squared of  $v$  with respect to the metric  $g(t)$ , and integrate the resulting differential inequality.

Finally, by differentiating (9.7), we see that the covariant derivatives of Rm satisfy evolution equations for which  $L^2$  energy bounds can be obtained. Therefore, we can use Hamilton's argument in Section 14 of [17] to show that  $g(t)$  has a smooth limit as  $t \rightarrow T_{\max}$ . If  $T_{\max} < T_0$ , we would be able to extend the solution to (9.1) smoothly beyond  $T_{\max}$  with (9.4), (9.5) and (9.6) still holding. This contradicts the assumed maximality of  $T_{\max}$ . We conclude that  $T_{\max} \geq T_0$ .

The estimate (9.3) follows by applying Theorem A.7 to (9.7).  $\square$

9.2.  $n/2 < p < \infty$ . Here the argument follows Section 5 of [23] and yields the following:

**THEOREM 9.8.** — *Let  $A_0$  and  $B_0$  be constants such that*

$$\left( \int_{\Omega} |f|^{2n/(n-2)} dV_{g_0} \right)^{(n-2)/n} \leq A_0 \int_{\Omega} |\nabla f|^2 dV_{g_0} + B_0 \int_{\Omega} |f|^2 dV_{g_0},$$

for any  $f \in C_0^\infty(\Omega)$ .

Given  $q > n$ , assume that the curvature satisfies:

$$\left( \int_{\Omega} |\text{Rm}(g_0)|^{q/2} dV_{g_0} \right)^{2/q} \leq \mu_0.$$

Then there exists a constant  $c(n, q)$  such that the evolution equation (9.1) has a smooth solution for  $0 \leq t \leq c(n, q) \alpha^{-1}$ , where

$$\alpha = |\nabla \chi|^2 + A_0^{n/(q-n)} \mu_0^{q/(q-n)} + A_0^{-1} B_0.$$

9.3.  $p = \infty$ . Using the same ideas as before and Theorem C.1, we also obtain:

**THEOREM 9.9.** — *Let  $M$  be a noncompact  $n$ -manifold with a complete Riemannian metric  $g_0$ . Let  $\Omega \subset M$  be open and such that*

$$\|\text{Rm}\|_{\infty, \Omega} < K.$$

*Then there exists a constant  $c(n)$  such that the local Ricci flow (9.1) has a solution for  $0 \leq t \leq c(n) (\|\nabla \chi\|^2 + K)^{-1}$ .*

From this, we obtain the following result of Shi [20]:

**COROLLARY 9.10.** — *Let  $M$ ,  $g_0$ ,  $K$ , and  $c(n)$  be as above. Then there exists a smooth solution to*

$$\frac{\partial g}{\partial t} = -2 \text{Rc}(g), \quad g(0) = g_0,$$

for  $0 \leq t \leq c(n) K^{-1}$ .

*Proof.* — Let  $\chi_i$  be a sequence of compact supported functions on  $M$  that converge to the constant function 1 on any compact subset and such that  $\|\nabla \chi_i\| \rightarrow 0$ . Let  $g_i(t)$  be the solution to the local Ricci flow. For any fixed  $t$  and bounded open subset  $\Omega \subset M$ , the sequence  $g_i(t)$  satisfies the assumptions of Theorem 5.1. Therefore, a subsequence converges. Now do this for an exhaustion of  $M$  by bounded open subsets and take a diagonal subsequence. The limit will be a smooth solution to the Ricci flow.  $\square$

## 10. A local convergence theorem

Given a Riemannian manifold  $M$ ,  $\Omega \in M$ , we define

$$\Omega_\varepsilon = \{x \in \Omega \mid d(x, M \setminus \Omega) > \varepsilon\}.$$

**THEOREM 10.1.** — *Let  $M_1, M_2, \dots$  be a sequence of complete Riemannian manifolds,  $\Omega_i \subset M_i$  open subsets, and  $A_0, D, v, K > 0, p > n/2$  constants satisfying the following:*

$$(10.2) \quad C_S(\Omega_i) \leq A_0$$

$$(10.3) \quad \text{diam}(\Omega_i) < D$$

$$(10.4) \quad \|\text{Rc}\|_{p, \Omega_i} < K$$

$$(10.5) \quad \|\text{Rm}\|_{n/2, \Omega_i} < (c(n) A_0)^{-1},$$

where  $c(n)$  is the constant appearing in Theorem 9.2.

Given  $\varepsilon > 0$  assume that there is a  $v > 0$  such that  $\text{vol}(\Omega_{i, \varepsilon}) > v$ . Then there exists an open manifold  $\Omega_{\infty, \varepsilon}$ , a subsequence  $\Omega_{i, \varepsilon}$ , and diffeomorphisms  $\varphi_i: \Omega_{\infty, \varepsilon} \rightarrow \Omega_{i, \varepsilon}$  such that the metrics  $\varphi_i^* g_i$  converge uniformly to a continuous metric on  $\Omega_{\infty, \varepsilon}$ .

*Proof.* — Let  $\chi_i$  be a smooth compact supported function on  $M_i$  such that  $\chi_i = 1$  on  $\Omega_{i, \varepsilon/2}$ ,  $\chi_i = 0$  on  $M_i \setminus \Omega_i$ , and  $|\nabla \chi_i| \leq 4\varepsilon^{-1}$ . Let  $g_i(t)$  be the smooth family of Riemannian metrics on  $\Omega_i$  obtained from the local Ricci flow. By Theorem 9.2, these are well-defined for  $t \in [0, T)$ , where  $T$  is fixed, independent of  $i$ .

Now fix  $t \in [0, T)$  and consider the sequence  $(\Omega_{i, \varepsilon/2}, g_i(t))$ . By Lemma 4.2 there is a local lower volume bound. It follows that the sequence satisfies the assumptions of Theorem 5.1.

Let  $t_k \rightarrow 0$ . By Theorem 5.1 there exists a subsequence  $\Omega_{i, \varepsilon}$  and diffeomorphisms  $\varphi_i: \Omega_{\infty, \varepsilon} \rightarrow \Omega_{i, \varepsilon}$  such that the metrics  $\varphi_i^* g_i(t_0)$  converge smoothly to a metric  $g(t_0)$  on  $\Omega_{\infty, \varepsilon}$ . Now for each  $t_k$  we restrict to a subsequence such that the metrics  $\varphi_i^* g_i(t_k)$  converge smoothly. We therefore obtain a sequence of smooth metrics  $g(t_k)$  on  $\Omega_{\infty, \varepsilon}$ . The estimates for the Ricci flow imply that

$$\|g(t_k) - g(t_l)\|_{\infty} \leq C |t_k - t_l|^{\alpha},$$

for some  $0 < \alpha < 1$ . Therefore, the metrics converge uniformly to a  $C^0$  Riemannian metric.  $\square$

## 11. Proof of main theorem and corollary

*Proof of Theorem 2.1.* — By Corollary 7.7 there exists a subsequence of the  $\Omega_{i, \varepsilon}$  converging in Hausdorff distance to a metric space  $\Omega_{\varepsilon}$ . The idea is to show that locally the manifolds are converging in Lipschitz distance to a manifold with a continuous Riemannian metric.

In what follows, when we say “ $B(x, r)$ ”, we really mean “ $B(x, r) \cap \Omega_{i, \varepsilon}$ ”.

First, let's recall some terminology and facts from [16]. Given  $\delta > 0$ , a  $\delta$ -net consists of a maximal set of points  $\{x_i\}$  such that  $d(x_i, x_j) > \varepsilon$  for all  $i \neq j$ . By Proposition 3.5 of [16], given  $\delta' < \delta$  and a  $\delta$ -net  $\{x_{\alpha}\} \subset \Omega_{\varepsilon}$ , there exists a sequence of  $\delta'$ -nets  $\{x_{i, \alpha}\} \subset \Omega_{i, \varepsilon}$  that converge to  $\{x_{\alpha}\}$  in Lipschitz distance. In particular, take  $\delta = \rho/4$  and  $\delta' = \rho/8$  and fix a corresponding sequence of nets.

Now fix one  $x_{\alpha} \in \Omega_{\varepsilon}$ . Then  $B(x_{i, \alpha}, \rho)$  converges to  $B(x_{\alpha}, \rho) \subset \Omega_{\varepsilon}$  in Hausdorff distance. On the other hand, Theorem 7.4, (2.2), and (2.5) imply that there is a constant  $c'(n) > 0$  such that

$$C_S(B(x_{i, \alpha}, \rho)) \leq c'(n) \eta^{-2(n+1)}$$

Choosing  $\varepsilon(n)$  sufficiently small in (2.4), the assumptions of Theorem 10.1 hold. Therefore, there exists a subsequence  $B(x_{i_\alpha}, \rho/2)$  that converges in Lipschitz distance to a  $C^1$  manifold  $B_\alpha$  with a  $C^0$  Riemannian metric. Since  $B(x_\alpha, \rho/2)$  must be isometric to  $B_\alpha$  as metric spaces, this puts a manifold and Riemannian structure on  $B(x_\alpha, \rho/2)$ .  $\square$

*Proof of Corollary 2.6.* — This involves the well-known phenomenon of concentrated compactness that occurs for scale-invariant functionals. This was first observed by Sacks-Uhlenbeck for harmonic maps of surfaces into Riemannian manifolds and for the Yamabe functional by T. Aubin. It has also been seen in the study of Yang-Mills gauge fields and of Einstein manifolds. The basic idea is that compactness can be lost if a sufficient amount of energy concentrates at a point. On the other hand, since the total amount of energy is, by assumption, uniformly bounded, the energy can only concentrate at a finite set of points. Outside of these points, one obtains convergence.

First, by Corollary 7.7 we can restrict to a subsequence that converges in Hausdorff distance to a metric space  $M$ . Fix  $r_0 > 0$  sufficiently small and let  $r_k = r_0/k$ . For each  $k$ , cover each manifold  $M_i$  by geodesic balls of radius  $r_k$ . A ball  $B$  is called *good* if

$$\left( \int_B |\text{Rm}|^{n/2} \right)^{2/n} \leq c(n) C_S(B)^{-1}$$

and *bad* otherwise. Let  $\Omega_i \subseteq M_i$  be the union of good balls. Observe that the number of bad balls is bounded, independent of both  $i$  and  $k$ . Apply Theorem 2.1 to this sequence, yielding a limiting manifold that we denote  $\Omega(k)$ . Do this for each integer  $k$ . We then observe that  $\Omega(1) \subset \Omega(2) \subset \Omega(3) \dots \subset M$ . Therefore,

$$\Omega = \bigcup_{k=1}^{\infty} \Omega(k)$$

is an open manifold with a continuous Riemannian metric.

To show that  $M \setminus \Omega$  consists of a finite set of points, it is necessary to show that punctured geodesic balls centered at a singularity are connected and that the distance between two points near a given singularity must be small. A clear discussion of how to prove this is given in Lemma 1.2 and Step 2 of the proof to Neck Theorem 1.3 in [3]. Their proof works here exactly as stated, except that the upper and lower bounds on the volume of a geodesic ball obtained in section 7 are used instead of the (Bishop-Gromov) relative volume comparison theorem.  $\square$

*Remark.* — In fact, as mentioned in Section 2, the arguments contained in [3] carry over to here without much change. In particular, this implies that the singularities of  $M$  are orbifold singularities.

**12. Local  $L^p$  convergence theorem,  $p > n/2$**

The estimates in Section B can be used to obtain the following local version of the convergence theorem in [23]:

**THEOREM 12.1.** — *Let  $M_1, \dots$  be a sequence of complete Riemannian manifolds,  $\Omega_i \subseteq M_i$  open subsets, and  $A, B, D, K > 0, p > n/2$  constants such that the following hold:*

$$(12.2) \quad \|f\|_{2n/(n-2)}^2 \leq A \|\nabla f\|_2^2 + B \|f\|_2^2, \quad f \in C^\infty(\Omega_i)$$

$$(12.3) \quad \text{diam}(\Omega_i) < D;$$

$$(12.4) \quad \|\text{Rm}\|_{p, \Omega_i} \leq K,$$

*Given  $\varepsilon > 0$ , assume that there is a  $v > 0$  such that  $\text{vol}(\Omega_{i, \varepsilon}) > v$ . Then there exists a subsequence  $\Omega_{i, \varepsilon}$  converging in Lipschitz distance to an open  $C^1$  manifold with a  $C^\infty$  Riemannian metric.*

APPENDIX A

**Moser iteration for a local heat flow**

Here, a simple form of Moser iteration is applied to a local nonlinear heat equation. Since the estimates are slightly different from the standard iteration, we provide the details. Although the estimates will be applied to the systems satisfied by the curvature and its covariant derivatives, it is more convenient here to work with a single equation and scalar functions. Everything extends easily to vector- (or tensor-) valued functions satisfying a local nonlinear heat flow of the appropriate type.

Fix an open set  $B_0 \subseteq M$  and a smooth compactly supported function  $\chi \in C_0^\infty(B_0)$ .

Let  $g(t), 0 \leq t \leq T$ , be a 1-parameter family of smooth Riemannian metrics. Let  $\nabla$  denote covariant differentiation with respect to the metric  $g(t)$  and  $-\Delta$  be the corresponding Laplace-Beltrami operator.

Let  $A > 0$  be a constant that satisfies the standard Sobolev inequality

$$\left( \int_{B_0} f^{2n/(n-2)} dV_g \right)^{(n-2)/n} \leq A \int_{B_0} |\nabla f|^2 dV_g, \quad f \in C_0^\infty(B_0),$$

with respect to each metric  $g(t), 0 \leq t \leq T$ .

Assume that for each  $t \in [0, T]$ ,

$$\frac{1}{2} g_{ij}(0) \leq g_{ij}(t) \leq 2 g_{ij}(0) \quad \text{on } B_0.$$

All geodesic balls in this section are defined with respect to the metric  $g(0)$ , and therefore, are fixed open subsets of  $M$ , independent of  $t$ .

First, we study the linear heat equation:

THEOREM A. 1. — *Let  $q > n$ ,  $p_0 > n/2$ , and  $f$  and  $u$  be nonnegative function on  $B_0 \times [0, T]$  such that*

$$\frac{\partial}{\partial t} dV \leq c \chi^2 u dV_g,$$

for some constant  $c$ , and

$$(A.2) \quad \frac{\partial f}{\partial t} \leq \chi^2 (\Delta f + uf) + 2a \chi |\nabla \chi| |\nabla f| + b (|\nabla \chi|^2 - \chi \Delta \chi) f, \quad 0 \leq t \leq T.$$

Assume that

$$\left( \int_{B_0} \chi^{q-n} u^{q/2} \right)^{2/q} \leq \mu t^{-(q-n)/q}.$$

Then given  $(x, t) \in B_0 \times [0, T]$ .

$$|\chi(x)^2 f(x, t)| \leq C A^{n/2 p_0} [\|\nabla \chi\|_\infty^2 + t^{-1} (1 + A^{n/(q-n)} \mu^{q/(q-n)})]^{(1+(n/2))/p_0} \left( \int_0^t \int_{B_0} \chi^{2p_0-n} f^{p_0} \right)^{1/p_0},$$

where  $C$  depends on  $n, q, p_0, a$ , and  $b$ .

*Proof.* — Throughout this section  $C$  is a constant depending only on  $n, q, p_0, a, b$ , and  $c$ . The following is easily proved using integration by parts and the Cauchy-Schwarz inequality:

LEMMA A. 3. — *Given  $p > 1$ ,  $\varphi \in C_0^\infty(B_0)$ ,  $f \in C^\infty(M)$ ,  $f \geq 0$ ,*

$$\int_{B_0} |\nabla(\varphi f^{p/2})|^2 \leq \frac{p^2}{2(p-1)} \int_{B_0} \varphi^2 f^{p-1} (-\Delta f) dV_g + \left(1 + \frac{1}{(p-1)^2}\right) \int_{B_0} |\nabla \varphi|^2 f^p dV_g.$$

Given  $p \geq p' \geq p_0 > 1$ , we combine the lemma with (A.2) and use the Cauchy-Schwarz, Hölder, and Sobolev inequalities to obtain

$$\begin{aligned} \frac{\partial}{\partial t} \int \chi^{2p'} f^{p+2} \left(1 - \frac{1}{p}\right)^2 \int |\nabla(\chi^{(p'+1)} f^{p/2})|^2 \\ \leq (p'+1)^2 C \int |\nabla \chi|^2 \chi^{2p'} f^p + p \int u \chi^{2(p'+1)} f^p \\ \leq (p'+1)^2 C \int |\nabla \chi|^2 \chi^{2p'} f^p + p \mu t^{-(q-n)/q} \left( \int \chi^{2p'} f^p \right)^{1-(n/q)} \end{aligned}$$

$$\begin{aligned}
 & \times \left( \int (\chi^{2p'+2} f^p)^{n-(n-2)} \right)^{(n-2)/n(n/q)} \\
 & \leq (p'+1)^2 C \int |\nabla \chi|^2 \chi^{2p'} f^p + \varepsilon^{1-n/q} A \int |\nabla (\chi^{p'+1} f^{p/2})|^2 \\
 & \quad + \varepsilon^{-n/q} p \mu t^{-(q-n)/q} \int \chi^{2p'} f^p.
 \end{aligned}$$

Setting  $\varepsilon = A^{-q/(q-n)}$ , we obtain the following estimate:

$$\begin{aligned}
 \text{(A.4)} \quad \frac{\partial}{\partial t} \int \chi^{2p'} f^p + \int |\nabla (\chi^{p'+1} f^{p/2})|^2 \\
 \leq [(p'+1)^2 C \|\nabla \chi\|_\infty^2 + (p \mu A^{n/q})^{q/(q-n)} t^{-1}] \int \chi^{2p'} f^p.
 \end{aligned}$$

Now given  $0 < \tau < \tau' < T$ , let

$$\psi(t) = \begin{cases} 0, & 0 \leq t \leq \tau \\ (t-\tau)/(\tau'-\tau), & \tau \leq t \leq \tau' \\ 1, & \tau' \leq t \leq T \end{cases}$$

Multiplying (A.4) by  $\psi$ , we obtain

$$\frac{\partial}{\partial t} \left( \psi \int \chi^{2p'} f^p \right) + \psi \int |\nabla (\chi^{p'+1} f^{p/2})|^2 \leq p^{2q/(q-n)} (\hat{C}(t) \psi + \psi') \int_{B'} \chi^{2p'} f^p,$$

where  $\hat{C}(t) = C \|\nabla \chi\|_\infty^2 + (\mu A^{n/q})^{q/(q-n)} t^{-1}$ . Integrating this with respect to  $t$ , we get

LEMMA A.5:

$$\int_t \chi^{2p'} f^p + \int_\tau^t \int |\nabla (\chi^{p'+1} f^{p/2})|^2 \leq \left( p^{q/(q-n)} \hat{C}(\tau') + \frac{1}{\tau' - \tau} \right) \int_\tau^T \int_{B'} \chi^{2p'} f^p, \quad \tau' \leq t \leq T.$$

Given  $p \geq p' \geq p_0$ ,  $0 \leq \tau < T$ , denote

$$H(p, p', \tau) = \int_\tau^T \int_{B_0} \chi^{2p'} f^p.$$

LEMMA A.6. — Given  $p \geq p_0$ ,  $0 \leq \tau < \tau' < T$ ,

$$H\left(p \left(1 + \frac{2}{n}\right), p' \left(1 + \frac{2}{n}\right) + 1, \tau'\right) \leq AC [(\tau' - \tau)^{-1} + p^{q/(q-n)} \hat{C}(\tau')]^{1+(2/n)} H(p, p', \tau)^{1+(2/n)}.$$

*Proof:*

$$\begin{aligned} \int_{\tau'}^T \int \chi^2 (\chi^{2p'} f^p)^{1+(2/n)} &\leq \int_{\tau'}^T \int (\chi^{2p'} f^p)^{2/n} \left( \int (\chi^{2p'} + 2f^p)^{n/(n-2)} \right)^{(n-2)/n} dt \\ &\leq \left( \sup_{\tau' \leq t \leq T} \int_B \chi^{2p'} f^p \right)^{2/n} A \int_{\tau'}^T \int |\nabla (\chi^{p'+1} f^{p/2})|^2 dt \end{aligned}$$

Applying Lemma A.5, we obtain the desired estimate.  $\square$

Now, denote

$$v = 1 + \frac{2}{n}, \quad \eta = v^{2n/(q-n)}.$$

Fix  $0 < t < T$ , and set

$$\begin{aligned} p'_k &= \left( p_0 - \frac{n}{2} \right) v^k + \sum_{j=0}^{k-1} v^j \\ p_k &= p_0 v^k; \\ \tau_k &= t(1 - \eta^{-k}); \\ \Phi_k &= H(p_k, p'_k, \tau_k)^{1/p_k}. \end{aligned}$$

Applying Lemma A.6, we obtain

$$H(p_{k+1}, p'_{k+1}, \tau_{k+1}) \leq AC \left( \|\nabla \chi\|_\infty^2 + [(\mu A^{n/q})^{q/(q-n)} + 1] \frac{\eta}{\eta-1} t^{-1} \right)^v \eta^{kv} H(p_k, p'_k, \tau_k)^v,$$

Therefore

$$\begin{aligned} \Phi_{k+1} &\leq (AC)^{(\sigma_{k+1}-1)/p_0} \left( \|\nabla \chi\|_\infty^2 + [(\mu A^{n/q})^{q/(q-n)} + 1] \frac{\eta}{\eta-1} t^{-1} \right)^{\sigma_k/p_0} \\ &\quad \times \eta^{\sigma'_k/p_0} H\left(p_0, p_0 - \frac{n}{2}, 0\right)^{1/p_0}, \end{aligned}$$

where

$$\sigma_k = \sum_{i=0}^k v^{-i}, \quad \sigma'_k = \sum_{i=0}^k i v^{-i}.$$

We let  $k \rightarrow \infty$  to obtain

$$\begin{aligned} |\chi^2 f(x, t)| CA^{n/2 p_0} [\|\nabla \chi\|_\infty^2 + t^{-1} (1 + (\mu A^{n/q})^{q/(q-n)})]^{1+(n/2)/p_0} \left( \int_0^T \int_{B_0} \chi^{2p_0-n} f^{p_0} \right)^{1/p_0}, \\ 0 < t < T. \end{aligned}$$

Now let  $T \rightarrow t$ .  $\square$

Applying this estimate to the nonlinear equation, we obtain:

THEOREM A.7. — *Let  $f \geq 0$  solve*

$$(A.8) \quad \frac{\partial f}{\partial t} \leq \chi^2 (\Delta f + c_0 f^2) + 2a\chi |\nabla \chi| |\nabla f| + b(|\nabla \chi|^2 - \chi \Delta \chi) f, \quad 0 \leq t \leq T.$$

on  $B_0 \times [0, T]$ . Assume that

$$\frac{\partial}{\partial t} dV_g \leq c \chi^2 f dV_g,$$

and that

$$\left( \int_{B_0} f_0^{n/2} \right)^{2/n} \leq \left[ \left( \frac{n}{2} + 1 \right) e c_0 A \right]^{-1},$$

where  $f_0(x) = f(x, 0)$ . Then

$$|\chi(x)^2 f(x, t)| \leq CA [t \|\nabla \chi\|_\infty^2 + 1]^2 t^{-1}, \quad 0 < t < \min(T, \|\nabla \chi\|_\infty^{-2}),$$

where  $C$  depends only on  $n$  and  $c_0, a$ , and  $b$ .

*Proof.* — Let  $[0, T'] \subset [0, T]$  be the maximal interval such that

$$e_0 = \sup_{0 \leq t \leq T'} \left( \int_{B_0} f^{n/2} \right)^{2/n} \leq \left[ \left( \frac{n}{2} + 1 \right) c_0 A \right]^{-1}.$$

Applying Lemma A.3 to (A.8), we obtain

$$\frac{\partial}{\partial t} \int f^p + 2 \left( 1 - \frac{1}{p} \right)^2 \int |\nabla \chi f^{p/2}|^2 \leq p \int |\nabla \chi|^2 f^p + p c_0 A \left( \int f^{n/2} \right)^{2/n} \int |\nabla (\chi f^{p/2})|^2.$$

Therefore, for  $p \leq (n/2) + 1$ , the bound on the  $L^{n/2}$  norm of  $f$  implies that for  $0 \leq t \leq T'$ ,

$$(A.9) \quad \frac{\partial}{\partial t} \int f^p + \int |\nabla (\chi f^{p/2})|^2 \leq p \|\nabla \chi\|_\infty \int f^p.$$

Set  $p = n/2$ , throw away the second term on the left, and integrate the resulting differential inequality. This implies that

$$\int f^{n/2} \leq e^{(n/2) \|\nabla \chi\|_\infty^2 t} \int f_0^{n/2}.$$

In particular, if  $T' < \|\nabla \chi\|_\infty^{-2}$ , then  $e_0 < [(n/2) + 1] c_0 A$ . Since  $e_0$  depends continuously on  $T'$ , this contradicts the assumed maximality of  $[0, T']$ . We can therefore assume that  $T' \geq \min((\log 2) \|\nabla \chi\|_\infty^{-2}, T)$ .

Arguing as before, (A.9) leads to an estimate of the form:

$$\int_{\mathbf{B}} f^p, \quad \int_0^t \int |\nabla(\chi f^{p/2})|^2 \leq C(t^{-1} + \|\nabla\chi\|_\infty) \int_0^t \int f^p.$$

Therefore,

$$\begin{aligned} \int \chi^2 f^{1+(n/2)} &\leq C(t^{-1} + \|\nabla\chi\|_\infty) \int_0^t \int \chi^2 f^{1+(n/2)} \\ &\leq C(t^{-1} + \|\nabla\chi\|_\infty) \int_0^t \left( \int_{\mathbf{B}_0} f^{n/2} \right)^{2/n} \left( \int (\chi f^{n/4})^{2n/(n-2)} \right)^{(n-2)/n} dt \\ &\leq C e_0 A(t^{-1} + \|\nabla\chi\|_\infty) \int_0^t \int |\nabla(\chi f^{n/4})|^2 \\ &\leq C e_0 A(t^{-1} + \|\nabla\chi\|_\infty)^2 \int_0^t \int_{\mathbf{B}_0} f^{n/2} \\ &\leq CA t(t^{-1} + \|\nabla\chi\|_\infty)^2 e_0^{1+(n/2)}. \end{aligned}$$

We can then apply Theorem A.1 with  $p_0 = n/2$ ,  $q = n + 2$ , and

$$\mu^{q/2} = CA(1 + t\|\nabla\chi\|_\infty^2)^2 e_0^{1+(n/2)},$$

obtaining the desired estimate.  $\square$

The argument also implies the following:

**COROLLARY A.10.** — *Let  $f$  satisfy the assumptions of Theorem A.7. Then given  $u \geq 0$  such that*

$$\frac{\partial u}{\partial t} \leq \chi^2(\Delta u + c_0 f u) + a \cdot \nabla u + bu,$$

*the following estimate holds for  $0 \leq t < \min(T, (\log 2)\|\nabla\chi\|_\infty^{-2})$ :*

$$|\chi(x)^2 x(x, t)| \leq CA^{n/2 p_0} [1 + t\|\nabla\chi\|_\infty^2]^2 t^{-(n/2)/p_0} \left( \int_{\mathbf{B}_0} u^{p_0} \right)^{1/p_0},$$

*where  $u_0(x, t) = u(x, 0)$  and  $C$  depends on  $n, p_0, a, b$ .*

## APPENDIX B

### Moser iteration for $p > n/2$

The estimate obtained by global Moser iteration in [23] is recalled here in a version adapted for the local heat equation.

Fix an open set  $\mathbf{B}_0 \subset \mathbf{M}$  and a smooth compact supported function  $\chi \in C_0^\infty(\mathbf{B}_0)$ .

Let  $g(t)$ ,  $0 \leq t \leq T$ , be a 1-parameter family of smooth Riemannian metrics. Let  $\nabla$  denote covariant differentiation with respect to the metric  $g(t)$  and  $-\Delta$  be the corresponding Laplace-Beltrami operator.

Assume that with respect to the metric  $g = g(t)$ ,  $0 \leq t \leq T$ ,  $f \in C_0^\infty(B_0)$ ,

$$\left( \int_M |f|^{2n/(n-2)} dV_g \right)^{(n-2)/n} \leq A \int_M |\nabla f|^2 dV_g + B \int_M |f|^2 dV_g.$$

Assume that for each  $t \in [0, T]$ ,

$$\frac{1}{2} g_{ij}(0) \leq g_{ij}(t) \leq 2 g_{ij}(0) \quad \text{on } B_0.$$

**THEOREM B. 1.** — *Let  $q > n$ ,  $p_0 > n/2$ , and  $f$  and  $u$  be nonnegative functions on  $B_0 \times [0, T]$*

$$\frac{\partial}{\partial t} dV_g \leq c \chi^2 u dV_g,$$

for some constant  $c$ , and

$$\frac{\partial f}{\partial t} \leq \chi^2 (\Delta f + uf) + 2a\chi |\nabla \chi| |\nabla f| + b(|\nabla \chi|^2 - \chi \Delta \chi) f, \quad 0 \leq t \leq T.$$

Assume that

$$\left( \int_{B_0} \chi^{q-n} u^{q/2} \right)^{2/q} \leq \mu.$$

Then given  $(x, t) \in B_0 \times [0, T]$ ,

$$|\chi(x)^2 f(x, t)| \leq C A^{n/2p_0} [t^{-1} + \alpha]^{(1+(n/2)/p_0)} e^{C\alpha t} \left( \int_0^t \int_{B_0} \chi^{2p_0-n} f^{p_0} \right)^{1/p_0},$$

where  $\alpha = |\nabla \chi|^2 + A^{n/(q-n)} \mu^{q/(q-n)} + A^{-1} B$ ;  $C$  depends on  $n$ ,  $q$ ,  $p_0$ ,  $a$ , and  $b$ ; and  $f_0(x) = f(x, 0)$ .

The proof is a combination of the proofs given in Section 4 in [23] and Section A in this paper.

## APPENDIX C

### Moser iteration for $p = \infty$

**THEOREM C. 1.** — *Let the assumptions of Theorem B.1 hold with  $q = p_0 = \infty$ . Then given  $(x, t) \in B_0 \times [0, T]$ ,*

$$|\chi(x)^2 f(x, t)| \leq C e^{\nu t} \|f_0\|_\infty,$$

where  $C$  depends on  $n$ ,  $a$ , and  $b$  and  $v = \|\nabla \chi\|_\infty^2 + \mu$ .

*Proof.* – The proof of Theorem B.1 yields the following estimate:

$$(C.2) \quad |\chi(x)^2 f(x, t)| \leq CA^{n/2 p_0} [t^{-1} + \alpha]^{(1 + (n-2)/p_0)} \left( \int_0^t \int_{B_0} f^{p_0} \right)^{1/p_0}.$$

Now the crucial point is that the integral on the righthand side can be bounded by the initial data without using the Sobolev inequality. Given an  $L^\infty$  bound on  $u$ , we can simply throw away the gradient term and obtain the following differential inequality:

$$\frac{\partial}{\partial t} \int f^{p_0} \leq C(\|\nabla \chi\|^2 + \mu) \int f^{p_0}.$$

Integrating this twice, we get

$$\int_0^t \int f^{p_0} \leq t e^{v t} \int f_0^{p_0}.$$

Substituting this into (C.2) and letting  $p_0 \rightarrow \infty$  proves the theorem.  $\square$

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