Annales scientifiques de l'É.N.S.

Anthony Joseph

Annihilators and associated varieties of unitary highest weight modules

Annales scientifiques de l'É.N.S. 4e série, tome 25, nº 1 (1992), p. 1-45

<http://www.numdam.org/item?id=ASENS_1992_4_25_1_1_0>

Ann. scient. Éc. Norm. Sup.,
4[e] série, t. 25, 1992, p. 1 à 45.

ANNIHILATORS AND ASSOCIATED VARIETIES OF UNITARY HIGHEST WEIGHT MODULES

BY ANTHONY JOSEPH

1. Introduction

1.1. This paper is a sequel to [10], hereafter referred to as EJ. We shall adopt the same notation, which will nevertheless be redefined unless it is completely standard. Let $\mathfrak{g}$ be a complex simple Lie algebra, with $\mathfrak{g}=\mathfrak{n}^+ \oplus \mathfrak{h} \oplus \mathfrak{n}$ a triangular decomposition in the sense of [7], 1.10.14.

Fix a non-compact real form $\mathfrak{g}_0$ of $\mathfrak{g}$. The classification of unitary highest weight modules has been studied by several authors (*see* EJ, Introduction) and was in particular completed in [9] and in [14]. In EJ we cast this into a new and quite intrinsic form simplying both the formulae and calculations involved. Let us recall briefly some details of the classification. First one may assume that the reductive subalgebra $\mathfrak{k}$ corresponding to a maximal compact subalgebra of $\mathfrak{g}_0$ is the Levi factor of a maximal parabolic subalgebra $\mathfrak{p}^+$ of $\mathfrak{g}$ whose nilradical $\mathfrak{m}^+$ is commutative.

Let α denote the simple (non-compact) root not occurring in $\mathfrak{k}$ and ω the corresponding fundamental weight. Let P_c^+ denote the set of $\mathfrak{k}$-dominant integral weights. For each $\tau \in P_c^+$ and each $u \in \mathbb{R}$ we let $V(\tau) \otimes \mathbb{C}_{u\omega}$ denote the simple finite dimensional $\mathfrak{p}^+$ module with highest weight $\tau+u\omega$ and $N(\tau+u\omega)$ the corresponding induced $\mathfrak{g}$ module. All unitary highest weight modules occur as the unique simple quotient $L(\tau+u\omega)$ of some $N(\tau+u\omega)$. Indeed let s denote the level of τ (EJ, 1.6). Then there exist real parameters $u_s^\tau < u_{s-1}^\tau < \ldots < u_1^\tau$ such that $N(\tau+u\omega)$ is simple and unitary if and only if $u < u_s^\tau$, the $L(\tau+u_i^\tau\omega), i=1,2,\ldots,s$ are unitary and this list exhausts all unitary highest weight modules. The parameters u_i^τ were given in [9] but can also be derived from [14]. Here we shall use the formula (EJ, 4.2) which is both simple and intrinsic. For $s \geqq 2$ one has that $u_i^\tau - u_{i+1}^\tau = \varepsilon_{\mathfrak{g},\alpha}\omega$, $\forall i=1,2,\ldots,s-1$. Remarkably $\varepsilon_{\mathfrak{g},\alpha}$ is independent of τ and i. We call this the *equal spacing rule*. We also use a result of M. G. Davidson, T. J. Enright and R. J. Stanke ([6], Thm. 3.1) which asserts in particular that the maximal submodule $\overline{N(\lambda)}$ of $N(\lambda)$ is generated by a highest weight vector.

1.2. Now take $\tau=0$ in 1.1 and denote u_i^0 simply by u_i. One has $u_i = -(i-1)\varepsilon_{\mathfrak{g},\alpha}$ in the notation of EJ, 3.6. Let $\mathscr{V}_i$ denote the associated variety of $L(u_i\omega)$. These are interesting subvarieties of $\mathfrak{m}^+$, singular for $i>1$. Set $J_i = \mathrm{Ann}_{U(\mathfrak{g})} L(u_i\omega)$. Continuing the work of T. Levasseur, S. P. Smith and J. T. Stafford [25] who studied the case $i=2$, T. Levasseur and J. T. Stafford [26] showed for $\mathfrak{g}$ classical that J_i is always a maximal ideal and remarkably that $U(\mathfrak{g})/J_i$ identifies with the ring $\mathscr{D}_i$ of differential operators on $\mathscr{V}_i$. This was important as it meant one could say rather a lot about $\mathscr{D}_i$ a situation which is remarkable considering that $\mathscr{V}_i$ is singular. A difficulty in the work of Levasseur-Stafford is that it involved rather long case by case analysis using in particular Howe theory. Here following mainly [18] and the analysis in EJ we shall give a short intrinsic proof of their results which furthermore applies to arbitrary $\mathfrak{g}$ simple (Theorems 4.2, 4.5).

We remark that the varieties $\mathscr{V}_i$ occurred earlier in the work of M. Harris and H. P. Jakobsen [12]. They describe a constant coefficient differential operator on $\mathfrak{m}^+$ and use it to construct the unitary highest weight modules in the case $\tau=0$ ([12], Sect. 3). This may be viewed as giving the space of regular functions on $\mathscr{V}_i$ a $U(\mathfrak{g})$ module structure (which is furthermore a unitary highest weight module). This was a first step in [26] whose authors were unaware of this connection with unitary. However the two main problems in [26] mentioned above were not considered in [12].

1.3. The second aim of our work concerns the associated variety $\mathscr{V}_i^\tau$ of an arbitrary unitary highest weight module $L(\tau+u_i^\tau\omega)$ which is not induced. Assume again for the moment that $\mathfrak{g}$ is classical. T. J. Enright pointed out to me the following remarkable result obtained in [6], Sect. 7. Let $\mathfrak{m}$ be the subalgebra of $\mathfrak{g}$ opposed to $\mathfrak{m}^+$. For any such unitary module L and any $0\neq f\in L$ the ideal $\text{Ann}_{U(\mathfrak{m})} f$ in the (commutative) ring $U(\mathfrak{m})$ is prime! Although this is also true for the induced module $N(\lambda)$ it is almost never true for any non-trivial simple quotient $L(\lambda)$ of $N(\lambda)$. Indeed setting $J_i^\tau=\text{Ann}_{U(\mathfrak{g})} L(\tau+u_i^\tau\omega)$ the above property implies [Lemma 6.5 (iii)] that the Goldie rank $\text{rk}(U(\mathfrak{g})/J_i^\tau)$ of the quotient ring is bounded by $\dim V(\tau)$. Recalling that for a finite dimensional simple module L one has $\text{rk}(U(\mathfrak{g})/\text{Ann}\, L)=\dim L$, one sees that this result never holds when $\tau\neq 0$ and u is chosen so that $L(\tau+u\omega)$ is finite dimensional. This is consistent with the classical fact that a non-compact real semi-simple Lie group with trivial centre admits no non-trivial finite dimensional unitary representations.

The above result of M. G. Davidson, T. J. Enright and R. J. Stanke is obtained by a quite complicated procedure involving Howe theory and the construction of harmonic polynomials. Here we give a simple intrinsic proof (Theorem 5.16). This not only extends the result to arbitrary $\mathfrak{g}$ but also gives a quite explicit method for determining $\mathscr{V}_i^\tau$. In more detail, let t denote the level of the zero weight (EJ, 1.4, 1.6). This is always an upper bound on the level of any other $\mathfrak{k}$ dominant weight τ. By convention we define $\mathscr{V}_j=\mathfrak{m}^+$ for $j>t$. Fix τ. Then there exists $j\in\mathbb{N}^+$ such that $\mathscr{V}_1^\tau=\mathscr{V}_j$. We first show (Theorem 2.5) that $\mathscr{V}_i^\tau=\mathscr{V}_{j+i-1}$ for all $i=1, 2, \ldots, s$. This is a rather easy consequence of the Jakobsen-Vergne tensor product construction in [15]. In type A_n a comparison result of a similar nature but concerning annihilation by constant coefficient differential operators, can be found in [13], Introduction and Corollary 3.6. Secondly in Section 7 we explicitly compute $\mathscr{V}_1^\tau$ for each τ. The latter depends in a quite complicated way on τ. For example we had first guessed that $\mathscr{V}_s^\tau=\mathscr{V}_t$ but this fails badly. One has $\mathscr{V}_1^\tau=\mathscr{V}_{l(\tau)}$ where $l(\tau)$ is given in the Table. In type A_l, we may view $l(\tau)$ as the length of the support of $\tau+\omega$.

1.4. Set $Q_i = \mathrm{Ann}_{U(\mathfrak{m})} L(u_i \omega)$ which is a prime ideal of $U(\mathfrak{m})$. The result described in 1.3 can be expressed (*see* 2.4) as saying that for each pair τ, j there exists $i \in \{1, 2, \ldots, t+1\}$ such that $L(\tau + u_j^\tau \omega)$ is a torsion-free $U(\mathfrak{m})/Q_i$ module. One can ask if the only non-trivial simple quotients of $N(\tau + u\omega)$ satisfying the above condition are the unitary ones. *A priori* this would seem rather optimistic. However the above inequality condition on Goldie rank shows that it is generically true. This is because by [21], 5.1, the degree of the Goldie rank polynomial defined by the coherent family attached to $L(\tau + u\omega)$ strictly exceeds the degree of Goldie rank polynomial defining $V(\tau)$ — the latter being the product of the compact positive roots. It is hence quite accidental that the higher degree polynomial takes a smaller value as in the case of the unitary quotients. Besides it would be rather exciting to have a Goldie rank criterion for unitary; but the naive inequality fails (8.3). It is also perhaps interesting to recall that despite his initial scepticism to the idea, D. A. Vogan ([29], Prop. 7.12) actually proved a result in this direction for complex groups.

1.5. Assume $\mathrm{Ann}_{U(\mathfrak{m})} L(\tau + u_j^\tau \omega) \neq 0$. In section 6 we give a necessary (Theorem 6.8) condition for J_j^τ to be maximal. In section 8 we give several examples of non-maximal annihilators including in type D_{2l}, $l \geqq 2$ an ideal $l-1$ steps from being maximal. Given the truth of a certain simplicity conjecture (6.11) we also derive a sufficient condition (Proposition 6.14) for J_j^τ to be maximal. Unfortunately this is not quite the converse to Theorem 6.8.

1.6. It could be a rather difficult matter to find an example for which $\mathrm{rk}\, U(\mathfrak{g})/J_j^\tau = 1$, yet $\tau \neq 0$. Fortunately we found quite accidentally examples in type D_{2l+1} (8.9).

Acknowledgements

This work was started whilst the authors was a guest of Professors T. J. Enright and L. W. Small at the University of California, San Diego during July-August 1989. In particular he would like to thank T. J. Enright for a preview of his work with M. G. Davidson and R. J. Stanke. The results described in 1.3 were formulated in a rough fashion at that time. The final results were presented at a seminar in Paris during February 1990.

2. Primeness and a tensor product reduction

2.1. Define Q_i, $i = 1, 2, \ldots, t$ as in 1.4, equivalently as in EJ, 8.1 and set $Q_r = \{0\}$, $r \geqq t+1$. Correspondingly (*see* 1.2) we set $u_r = -(r-1)\varepsilon_{\mathfrak{g}, \alpha}$, $r \in \mathbb{N}^+$. Our immediate aim is to prove that Q_i is a prime ideal of $S(\mathfrak{m})$. This is essentially well-known; but the usual proofs involve case by case analysis (*cf.* [12], Sect. 4; [25], Chap. II). Here we give an intrinsic proof based on the following easy and perhaps known lemma.

2.2. Define the sequence $\beta_1, \beta_2, \ldots, \beta_t$ of strongly orthogonal positive non-compact roots as in EJ, 1.4, and recalling (EJ, 2.1) set

$$\mu_i = \sum_{j=1}^{i} \beta_j.$$

LEMMA. — *Let* $-\nu$ *be a weight of* $S(\mathfrak{m})$. *The equation* $k\mu_j = \nu + \mu_i$, $k \in \mathbb{N}$ *has no solution for* $j < i$.

We must obviously have $k \geqq 1$. Now ν is a sum of positive non-compact roots and we can assume of these exactly l_s be in $\Gamma_n^s \setminus \{\beta_s\}$, $s = 1, 2, \ldots, t$ (notation 3.2). Assume $j < i$. Cancelling off the β_s, $s \leqq j$ occurring in both sides of the above equation we can write

$$(*) \qquad \sum_{s=1}^{j} k_s \beta_s = \nu + \beta_{j+1} + \ldots + \beta_i, \qquad k_s \in \mathbb{Z}.$$

Equating coefficients of the non-compact simple root α it follows from $(*)$ that

$$(**) \qquad \sum_{s=1}^{j} k_s \geqq (i-j) + \sum_{s=1}^{j} l_s > \sum_{s=1}^{j} l_s.$$

Take $\gamma \in \Gamma_n^s \setminus \{\beta_s\}$. Then $(\beta_r, \gamma) = 0$ for $r < s$ whilst $(\beta_s, \gamma) = (1/2)(\beta_s, \beta_s)$ by [18], 2.2 (iv). Finally suppose $r > s$. If $(\beta_r, \gamma) \geqq 0$, then $\gamma - \beta_r \in \Gamma_c^s$ and so $\gamma - \beta_r - \beta_{r'}$ cannot be a root for $r' > s$. Hence there is at most one $r > s$ such that $(\beta_r, \gamma) > 0$ and since $\gamma + \beta_r$ is not a root we further have $(\beta_r, \gamma) = (1/2)(\beta_r, \beta_r)$. Let $l_{r,s}$, $r > s$ denote the number of $\gamma \in \Gamma_n^s$ occurring in ν for which $(\beta_r, \gamma) > 0$. By the above $\sum_r l_{r,s} \leqq l_s$. Then by $(*)$ for all $r \leqq j$ we obtain

$$2k_r = (\beta_r^\vee, \nu) \leqq l_r + \sum_{s<r} l_{r,s}.$$

Summing over $r \leqq j$ gives

$$\sum_{r=1}^{j} k_r \leqq \sum_{r=1}^{j} l_r,$$

in contradiction to $(**)$. This proves the lemma.

2.3. Recall (EJ, 8.1) that $Q_j \subset Q_i$ for $j \geqq i$. For $j > i$ this inclusion is obviously strict. Furthermore this extends to the case $j = t+1$. Let $\mathrm{Spec}_{\mathfrak{k}} S(\mathfrak{m})$ denote the set of $\mathfrak{k}$ stable prime ideals of $S(\mathfrak{m})$.

PROPOSITION. — $\mathrm{Spec}_{\mathfrak{k}} S(\mathfrak{m}) = \{Q_i\}_{i=1}^{t+1}$.

Obviously each Q_i is a $\mathfrak{k}$ stable ideal of $S(\mathfrak{m})$. Conversely let Q be a non-zero $\mathfrak{k}$ stable prime ideal of $S(\mathfrak{m})$. By commutativity of $\mathfrak{m}$ and finiteness of $\mathfrak{k}$ action $Q^{\mathfrak{n}} = Q^{\mathfrak{n}_c} \neq 0$, where $\mathfrak{n}_c = \mathfrak{n} \cap \mathfrak{k}$. By semisimplicity of $\mathfrak{h}$ action there exists $\mu \in \mathfrak{h}^*$ such that $Q^{\mathfrak{n}}_{-\mu} \neq 0$. Let v_i denote (EJ, 2.1) the unique up to scalars vector of weight $-\mu_i$ in $S(\mathfrak{m})^{\mathfrak{n}}$. Then by EJ, 2.1 (iv), one has $\mu = \sum k_i \mu_i$, $k_i \in \mathbb{N}$ and up to a scalar a non-zero vector in $Q^{\mathfrak{n}}_{-\mu}$ is the product of the $v_i^{k_i}$. Certainly $\mu \neq 0$ for otherwise $Q = S(\mathfrak{m})$. Hence $v_i \in Q$ for some i. If i is the least integer with this property it follows by EJ, 8.1, that $Q = Q_i$.

It remains to show that all the Q_i are prime ideals. This is proved by induction on i. Since $V_1 = \mathfrak{m}$ it follows that Q_1 is the augmentation ideal of $S(\mathfrak{m})$ and so it holds for $i = 1$. Suppose we have shown that $Q_1, Q_2, \ldots, Q_{i-1}$ are prime and consider Q_i. The radical $\sqrt{Q_i}$ of Q_i contains Q_l for $l \geqq i$ and is an intersection of $\mathfrak{k}$ stable prime ideals of $S(\mathfrak{m})$. By EJ, 8.1, we conclude that $\sqrt{Q_i} = Q_j$ for some $j \leqq i$.

If $j = i$ we are done. Otherwise $j < i$ and there exists a positive integer k such that $v_j^k \in Q_i^{\mathfrak{n}}$.

Now consider a non-zero weight vector $a \in Q_i^{\mathfrak{n}} = (V_i S(\mathfrak{m}))^{\mathfrak{n}}$. We can write

$$a = \sum_{r=1}^{\dim V_i} b_r c_r$$

with $b_r \in V_i$, $c_r \in S(\mathfrak{m})$ being weight vectors. We can assume the indexing to be chosen so that b_1 is of lowest weight amongst the b_i (and that $b_1 c_1 \neq 0$). Then $[(\operatorname{ad} x) b_1]\, c_1 = 0$ for all $x \in \mathfrak{n}$ since a is $\mathfrak{n}$ invariant. Since $S(\mathfrak{m})$ is an integral domain, it follows that $b_1 \in V_i^{\mathfrak{n}} = \mathbb{C} v_i$. Taking $a = v_j^k$ and letting $-\nu$ denote the weight of c_1 we conclude that

$$k \mu_j = \mu_i + \nu.$$

By 8.2 this equation has no solution. This contradiction proves the proposition.

Remark. — Let K denote the connected algebraic subgroup of $GL(\mathfrak{m})$ with Lie algebra $\mathfrak{k}$. Let $\mathscr{V}$ denote the closure of a K orbit in $\mathfrak{m}$ and Q its ideal of definition. Obviously $Q \in \operatorname{Spec}_{\mathfrak{k}} S(\mathfrak{m})$ and so $\mathscr{V} = \mathscr{V}_i$ for some i. Thus there are finitely many K orbits in $\mathfrak{m}$ and by the irreducibility of the $\mathscr{V}_i$ each of the latter is the closure of a K orbit. Notice that we can also deduce Proposition 2.3 if we can show that the number of K orbits in $\mathfrak{m}$ is at most $t+1$. All this is well-known; but we point it out anyway.

2.4. Fix $\tau \in P_c^+$ of level s and $i = \{1, 2, \ldots, s\}$. Set $\lambda = \tau + u_i^{\tau} \omega$ and identify $V(\lambda) := V(\tau) \otimes \mathbb{C}_{u_i^{\tau} \omega}$ with its image in the quotient $L(\lambda)$ of the induced module $N(\lambda) := U(\mathfrak{g}) \otimes_{U(\mathfrak{p})} V(\lambda)$. It then makes sense to consider $\operatorname{Ann}_{U(\mathfrak{m})} V(\lambda)$ and this identifies with a $\mathfrak{k}$ stable ideal of $S(\mathfrak{m})$. We shall eventually prove the remarkable fact that this ideal is prime and hence by 2.3 one of the Q_j. For the moment observe the

LEMMA. — *Assume* $\operatorname{Ann}_{U(\mathfrak{m})} V(\lambda) = Q_j$ *for some* $j \in \{1, 2, \ldots, t+1\}$. *Then for each* $0 \neq f \in L(\lambda)$ *one has* $\operatorname{Ann}_{U(\mathfrak{m})} f = Q_j$. *Equivalently* $L(\lambda)$ *is a torsion-free* $U(\mathfrak{m})/Q_j$ *module.*

Since $\mathfrak{m}$ is commutative and $U(\mathfrak{m}) V(\lambda) = L(\lambda)$ we obtain the inclusion $Q := \operatorname{Ann}_{U(\mathfrak{m})} f \supset Q_j$. Suppose this inclusion is strict. Since $\mathfrak{p}^+$ acts finitely on f we have for the canonical filtration of $U(\mathfrak{g})$ that $\mathfrak{p}^+ \subset \sqrt{\operatorname{gr} \operatorname{Ann}_{U(\mathfrak{g})} f}$. It follows that the associated variety of $U(\mathfrak{g})f$ identifies with the subvariety $\mathscr{V}(Q)$ of $\mathfrak{m}^+$ of zeros of Q. On the other hand the associated variety of $L(\lambda)$ is just $\mathscr{V}(Q_j)$. Since Q_j is prime, we have a strict inclusion $\mathscr{V}(Q) \subsetneq \mathscr{V}(Q_j)$. Yet $L(\lambda)$ is simple, so $U(\mathfrak{g})f = L(\lambda)$ and the resulting contradiction proves the lemma.

2.5. We now reduce primeness to the case $i=1$, referred to generally as the last place of unitary. For this we use the tensor product construction introduced by H. P. Jakobsen and M. Vergne, [15]. Set $\lambda_i^\tau = \tau + u_i^\tau \omega$, $\xi_i = u_i \omega$. Recall that $\xi_i = (i-1)\,\varepsilon_{\mathfrak{g},\alpha}\,\omega$, $\forall i \leqq t$ (EJ, 4.3). By the equal spacing principle we have $\lambda_i^\tau = \lambda_1^\tau + \xi_i - \xi_1 = \lambda_1^\tau + \xi_i$.

THEOREM. — *Fix* $\tau \in P_c^+$ *of level s. Suppose* $\mathrm{Ann}_{U(\mathfrak{m})} V(\lambda_1^\tau) = Q_j$ *for some* $j \in \mathbb{N}^+$. *Then* $\mathrm{Ann}_{U(\mathfrak{m})} V(\lambda_i^\tau) = Q_{j+i-1}$, $\forall i \in \{1, 2, \ldots, s\}$.

The positive definite forms on $L(\lambda_1^\tau)$ and $L(\xi_i)$ give a positive definite product form on $L(\lambda_1^\tau) \otimes L(\xi_i)$. Let e_1^τ (resp. f_i) denote the canonical generator of $L(\lambda_1^\tau)$ [resp. $L(\xi_i)$]. The restriction of a positive definite form to a submodule is again positive definite and so we conclude that the $U(\mathfrak{g})$ submodule of $L(\lambda_1^\tau) \otimes L(\xi_i)$ generated by $e_1^\tau \otimes f_i$ is unitary. Since it is a highest weight module of highest weight λ_i^τ we conclude that it identifies with $L(\lambda_i^\tau)$. Taking account of the $\mathfrak{p}^+$ and $\mathfrak{k}$ actions we conclude [noting $V(\xi_i) = \mathbb{C} f_i$] that

$$U(\mathfrak{m})(V(\lambda_1^\tau) \otimes V(\xi_i)) = L(\lambda_i^\tau)$$

for this identification. Moreover $V(\lambda_1^\tau) \otimes V(\xi_i)$ identifies with $V(\lambda_i^\tau)$.

Now take $v' \in V(\lambda_i^\tau)$. We can write $v' = v \otimes f_i$ for some $v \in V(\lambda_1^\tau)$. By the hypothesis and 2.4 we have $\mathrm{Ann}_{U(\mathfrak{m})} v = Q_j$, whilst $\mathrm{Ann}_{U(\mathfrak{m})} f_i = Q_i$. We claim that this implies that $\mathrm{Ann}_{U(\mathfrak{m})} v' = Q_{i+j-1}$. Normally such a result would be very difficult to prove as it involves analysis of a diagonal action of $\mathfrak{m}$. However here we can obtain the result by applying the tensor product argument to the case $\tau = 0$. Indeed the latter implies that $L(\xi_{i+j-1})$ is just the submodule of $L(\xi_j) \otimes L(\xi_i)$ generated over $U(\mathfrak{m})$ by $f_j \otimes f_i$. It follows that $\mathrm{Ann}_{U(\mathfrak{m})}(f_j \otimes f_i)$ for the diagonal action of $U(\mathfrak{m})$, which is what we want to compute, is just $\mathrm{Ann}_{U(\mathfrak{m})} f_{i+j-1} = Q_{i+j-1}$ as required. Note that there it does not matter if $i+j-1$ exceeds t. This is because the module $L(\xi_r)$, $\xi_r = (r-1)\,\varepsilon_{\mathfrak{g},\alpha}\,\omega$ is still unitary for $r > t$ and moreover in that case is just the induced module $N(\xi_r)$ which is a free $U(\mathfrak{m})$ module. This proves the claim which in turn implies the assertion of the theorem.

3. Reduction to smaller rank

3.1. The proof of the main results described in the introduction obtains via a reduction technique introduced in [18], Sect. 4. The method is quite elementary, the key point being to realize the generators v_i of the $\mathfrak{k}$ stable prime ideals of $S(\mathfrak{m})$ as related to the lowest weight vectors of simple Lie subalgebras of a localization of $U(\mathfrak{g})$. Unfortunately this is somewhat obscurred by the complicated notation and induction technique that we have to introduce.

3.2. Let us recall the notation of EJ, 1.3, 1.4. Let $\Delta \subset \mathfrak{h}^*$ denote the set of non-zero roots, Δ^+ (resp. Δ^-) the set of positive (resp. negative) roots corresponding to the triangular decomposition of $\mathfrak{g}$ introduced in 1.1. We define subsets $\Gamma^i \subset \Delta^-$, $\Delta^i \subset \Delta$, $i=1, 2, \ldots, t$, inductively as follows. Set $\Delta^1=\Delta$. Assume Δ^i is defined and is a simple root system. Then $\{\gamma \in \Delta^i \mid (\gamma, \beta_i)=0\}$ is a root subsystem of Δ^i. By definition of t, if $i<t$ then it admits a unique simple root subsystem containing α, which we define to be Δ^{i+1}. Observe that $\beta_i \in \Delta^i$ and is the unique highest root. Finally set $\Gamma^i=\{\gamma \in \Delta^i \mid (\gamma, \beta_i)<0\}$.

3.3. Recall (EJ, 1.3) that the subscript c (resp. n) refers to compact (resp. non-compact) roots, etc. Let $\mathfrak{a}^i$ (resp. $\mathfrak{a}^i_n$) denote the subalgebras of $\mathfrak{n}$ spanned by the root vectors x_γ, $\gamma \in \Gamma^i$ (resp. $\gamma \in \Gamma^i_n$). As noted in [17], 4.8, the $\mathfrak{a}^i$ are Heisenberg Lie algebras with centre $\mathbb{C}x_{-\beta_i}$. Obviously $\mathfrak{a}^i_n=\mathfrak{a}^i \cap \mathfrak{m}$ and so is commutative. It is convenient to take $\mathfrak{a}^0=\{0\}$. Set

$$\mathfrak{b}^i=\sum_{j=0}^{i} \mathfrak{a}^j, \qquad \mathfrak{b}^i_n=\sum_{j=0}^{i} \mathfrak{a}^j_n.$$

Let $\mathfrak{g}^i$ denote the simple subalgebra of $\mathfrak{g}$ spanned by $\mathfrak{h}$ and the root vectors x_γ, $\gamma \in \Delta^i$. Set $\mathfrak{n}^i=\mathfrak{g}^i \cap \mathfrak{n}$. Set $\mathfrak{k}^i=\mathfrak{k} \cap \mathfrak{g}^i$ which is the Levi factor of a maximal parabolic subalgebra $\mathfrak{p}^i$ of $\mathfrak{g}^i$ whose nilradical we denote by $\mathfrak{m}^i$. One has

$$(*) \qquad \mathfrak{m}^i \oplus \mathfrak{a}^{i-1}_n=\mathfrak{m}^{i-1}, \qquad \forall\, i=2, 3, \ldots, t.$$

Again $\mathfrak{m}^i$ is a simple $\mathfrak{k}^i$ module with $x_{-\beta_i}$ as its lowest weight vector. Let $\mathfrak{l}^i$ denote the subalgebra of $\mathfrak{h}$ spanned by the coroots 0, $\beta_1^\vee, \beta_2^\vee, \ldots, \beta_i^\vee$ and set $\mathfrak{c}^i=\mathfrak{l}^i \oplus \mathfrak{b}^i$.

3.4. Set $y_1=1$ and for $1<i\leq t+1$ set $y_i=v_1 v_2 \ldots v_{i-1}$. Let Y_i denote the multiplicative subset of $S(\mathfrak{m})^{\mathfrak{n}}=U(\mathfrak{m})^{\mathfrak{n}}$ generated by y_i. Since the adjoint action of $\mathfrak{m}$ and hence of each y_i on $U(\mathfrak{g})$ is locally nilpotent it follows from [4], 6.1, that Y_i is Ore in any subalgebra of $U(\mathfrak{g})$ containing $U(\mathfrak{b}^{i-1}_n)$. [The induction argument in the lemma below gives $y_i \in U(\mathfrak{b}^{i-1}_n)$.]

We apply the construction of [18], 4.1, 4.9, to the semi-direct product $\mathfrak{g}^i \oplus \mathfrak{a}^{i-1}$. This gives a sequence of $\mathfrak{g}^i$ modules and Lie algebra embeddings $\Theta^1=\mathrm{Id}_{\mathfrak{g}}$, $\Theta^i: \mathfrak{g}^i \to Y_i^{-1} U(\mathfrak{g}^i \oplus \mathfrak{b}^{i-1})^{\mathfrak{b}^{i-1}}$ having the form $\Theta^i(x)=x-\theta^i(x)$, where $\theta^1=0$ and $\theta^i(\mathfrak{g}^i) \subset Y_i^{-1} U(\Theta^{i-1}(\mathfrak{a}^{i-1}))$ for $i>1$. The image of $\mathfrak{g}^i$ under Θ^i is a copy of $\mathfrak{g}^i$ in the localized algebra $Y_i^{-1} U(\mathfrak{g})$ commuting with the sum $\mathfrak{b}^{i-1}$ of Heisenberg algebras. The possibility for doing this follows from the quite general principles discussed in [7], 10.1.4. However in the present simpler situation we can give quite explicit formulae for the $\theta^i(x)$. It will be enough to analyse these in the case $i=2$, since the general situation is similar. Setting $\beta=\beta_1$, $\Gamma=\Gamma^1$ we recall ([18], 4.9) that $\theta(x_\delta)$, $\delta \in \Delta^2$ has denominator $x_{-\beta}$ and numerator a sum of terms of the form $x_{\gamma_1} x_{\gamma_2}$, $\gamma_1, \gamma_2 \in \Gamma$ with $\gamma_1+\gamma_2+\beta=\delta$. In particular $\gamma_1+\gamma_2$ is never a root and so x_{γ_1}, x_{γ_2} commute. A similar result holds for $\theta^2(h)$, $h \in \mathfrak{h}$ except that this time $\gamma_1+\gamma_2=-\beta$.

We now prove the result referred to in 3.1. Set $v_0=1$.

LEMMA. — *For each* $i \in \{1, 2, \ldots, t\}$ *one has* $\Theta^i(x_{-\beta_i}) = v_{i-1}^{-1} v_i$. *Moreover* $v_i \in U(\mathfrak{b}_n^i)$.

One has $\theta^i(\mathfrak{a}^i) \subset Y_i^{-1} U(\Theta^{i-1}(\mathfrak{a}^{i-1}))$. Hence $\Theta^i(\mathfrak{a}^i) \subset \mathfrak{a}^i + Y_i^{-1} U(\Theta^{i-1}(\mathfrak{a}^{i-1}))$. It follows by an easy induction argument that $U(\Theta^i(\mathfrak{a}^i)) \subset Y_i^{-1} U(\mathfrak{b}^i)$. In particular $\Theta^i(x_{-\beta_i}) \subset Y_i^{-1} U(\mathfrak{b}^i)^{\mathfrak{b}^i \oplus \mathfrak{n}^{i+1}}$ and has weight $-\beta_i$. Although one need not have $\mathfrak{b}^i \oplus \mathfrak{n}^{i+1} \supset \mathfrak{n}$ because $\{\delta \in \Delta^{i-1} \mid (\gamma, \beta_{i-1}) = 0\} \setminus \Delta^i$ can have simple compact factors, a similar analysis taking account of these terms shows that $\Theta(x_{-\beta_i})$ in $\mathfrak{n}$ invariant. Since $\mathfrak{b}^i \subset \mathfrak{n}$ and $y_i \in Z(\mathfrak{n})$ it follows that $\Theta^i(x_{-\beta_i}) \in \text{Fract } Z(\mathfrak{n})$ and has weight $-\beta_i$. By [18], 4.12, we conclude that $\Theta^i(x_{-\beta_i}) = v_{i-1}^{-1} v_i$ up to a non-zero scalar which can be absorbed in the definition of v_i. In particular $v_i \in (Y_i^{-1} U(\mathfrak{b}^i)) \cap U(\mathfrak{m}) = U(\mathfrak{b}_n^i)$. This implies that $y_{i+1} \in U(\mathfrak{b}_n^i)$. By an easy induction argument this can be used to justify the formation of the localizations at y_{i+1}.

3.5. Let $\mathscr{V}_i$ denote the subvariety of $\mathfrak{m}^+$ of zeros of Q_i. This is just the associated variety of $L(u_i \omega)$. Let $\mathscr{R}(\mathscr{V}_i)$ denote the ring of regular functions on $\mathscr{V}_i$ and $\mathscr{D}(\mathscr{V}_i)$ the ring of differential operators on $\mathscr{V}_i$. (For general definitions—*see* [27], Chap. 15, for example.) We have the

PROPOSITION. — *For all* $i \in \{1, 2, \ldots, t\}$ *one has*

(i) $Y_i^{-1} \mathscr{R}(\mathscr{V}_i) \cong Y_i^{-1} S(\mathfrak{b}_n^{i-1})$.

(ii) $Y_i^{-1} \mathscr{D}(\mathscr{V}_i) \cong Y_i^{-1} U(\mathfrak{c}^{i-1})$.

(iii) $Y_i^{-1} \mathscr{D}(\mathscr{V}_i)$ *is a simple ring.*

(iv) $\dim \mathscr{V}_i = \sum_{j<i} |\Gamma_n^j|$.

(i) Recall that Θ^i is a $\mathfrak{g}^i$ module homomorphism and that v_{i-1} is $\mathfrak{g}^i$ invariant. Then the simplicity of $\mathfrak{m}^i$ as a $\mathfrak{k}^i$ module implies by 3.4 that

$$\Theta^i(\mathfrak{m}^i) \subset v_{i-1}^{-1} V_i \subset Y_i^{-1} Q_i \subset Y_i^{-1} S(\mathfrak{m}).$$

Yet $\Theta^i(\mathfrak{m}^i) \subset Y_i^{-1} U(\Theta^{i-1}(\mathfrak{a}^i)) \subset Y_i^{-1} U(\mathfrak{b}^{i-1})$. We conclude that

$$\theta^i(\mathfrak{m}^i) \subset Y_i^{-1} S(\mathfrak{b}_n^{i-1}).$$

From 3.3 (*) we obtain

$$(*) \qquad \mathfrak{m}^i \oplus \mathfrak{b}_n^{i-1} = \mathfrak{m}.$$

From 2.3 it follows that the image of v_j, $j<i$, is a non-zero divisor in $S(\mathfrak{m})/Q_i$ and so this ring embeds in $Z_i := Y_i^{-1}(S(\mathfrak{m})/Q_i)$. Since $v_i \in Q_i$ it follows from 3.4 that $\Theta^i(x_{-\beta_i})$ is zero in Z_i. Using the $\mathfrak{k}^i$ action as above, it follows that $\Theta^i(\mathfrak{m}^i)$ is also zero in Z_i, equivalently that $x = \theta^i(x) \in Y_i^{-1} S(\mathfrak{b}_n^{i-1})$, $\forall x \in \mathfrak{m}^i$ in Z_i. Combined with (*) it follows that $Y_i^{-1}(S(\mathfrak{m})/Q_i) = Y_i^{-1} \mathscr{R}(\mathscr{V}_i)$ identifies with an image of $Y_i^{-1}(\mathfrak{b}_n^{i-1})$.

To complete the proof of (i) and indeed of the proposition we use the fact (EJ, 5.2) that $S(\mathfrak{m})/Q_i$ admits a $U(\mathfrak{g})$ module structure extending the left $\mathfrak{m}$ module action by multiplication. This gives additional information which would be otherwise rather hard to obtain. Setting $\xi_i = u_i \omega$ we identify $S(\mathfrak{m})/Q_i$ with $L(\xi_i)$. Note that this identification also preserves the $\mathfrak{k}$ module structure up to a shift defined by u_i.

The action of $U(\mathfrak{g})$ on $L(\xi_i)$ defines an embedding of $U(\mathfrak{g})/\mathrm{Ann}\, L(\xi_i)$ into $\mathrm{End}_{\mathbb{C}} L(\xi_i) = \mathrm{End}_{\mathbb{C}} \mathscr{R}(\mathscr{V}_i)$. Since ad $\mathfrak{m}$ has a locally nilpotent action on $U(\mathfrak{g})$, it follows that the image is contained in $\mathscr{D}(\mathscr{V}_i)$. Take $j<i$. As the image of v_j is a non-zero divisor in $S(\mathfrak{m})/Q_i$ it follows that $v_j m=0$, $m \in L(\xi_i)$ implies $m=0$. It follows that the image of v_j (and hence of y_i) in $U(\mathfrak{g})/\mathrm{Ann}\, L(\xi_i)$ is a non-zero divisor. This gives an embedding $U(\mathfrak{g})/\mathrm{Ann}\, L(\xi_i) \hookrightarrow Y_i^{-1}(U(\mathfrak{g})/\mathrm{Ann}\, L(\xi_i))$. Let $\pi: U(\mathfrak{g}) \to U(\mathfrak{g})/\mathrm{Ann}\, L(\xi_i)$ denote the canonical projection.

Since each $\mathfrak{a}^i$ is a Heisenberg Lie algebra with centre $\mathbb{C} x_{-\beta_i}$, the construction in 3.4 shows that $Y_i^{-1} U(\mathfrak{c}^{i-1})$ is a localized Weyl algebra and hence a simple ring. (This is discussed in further detail in [18], Sect. 6.) Now $\mathfrak{c}^{i-1} \subset \mathfrak{h} \oplus \mathfrak{n} \subset \mathfrak{k} \oplus \mathfrak{m}$, and so we conclude that $Y_i^{-1} U(\mathfrak{c}^{i-1})$ identifies with a subring of $Y_i^{-1} \pi(U(\mathfrak{k} \oplus \mathfrak{m}))$. Now the action of $\mathfrak{k} \oplus \mathfrak{m}$ on $L(\xi_i)$ results from the identification of the latter as a quotient of the induced module $N(\xi_i)$. From this it is easy to check that the action of the subring $Y_i^{-1} U(\mathfrak{c}^{i-1})$ on the image of the map $\varphi: Y_i^{-1} U(\mathfrak{b}_n^{i-1}) \to Y_i^{-1}(S(\mathfrak{m})/Q_i) = Y_i^{-1} L(\xi_i)$ identifies $Y_i^{-1} U(\mathfrak{b}_n^{i-1})$ with the standard (and hence simple) module over this localized Weyl algebra. (In other words $\mathfrak{b}_n^{i-1}$ acts by multiplication and the remaining *dim* $\mathfrak{c}^{i-1} - \dim \mathfrak{b}_n^{i-1} = \dim \mathfrak{b}_n^{i-1}$ generators by appropriate differentiation.) This proves that φ is injective and so completes the proof of (i). Notice that we have also proved (ii), (iii) and furthermore that the embedding $U(\mathfrak{g})/\mathrm{Ann}\, L(\xi_i) \hookrightarrow \mathscr{R}(\mathscr{V}_i)$ gives rise to an isomorphism

$$(**) \qquad Y_i^{-1}(U(\mathfrak{g})/\mathrm{Ann}\, L(\xi_i)) \xrightarrow{\sim} Y_i^{-1} \mathscr{D}(\mathscr{V}_i).$$

Finally

$$\dim \mathscr{V}_i = \dim \mathfrak{b}_n^{i-1} = \sum_{j=1}^{i-1} \dim \mathfrak{a}_n^j = \sum_{j=1}^{i-1} |\Gamma_n^j|$$

which is (iv).

3.6. The irreducible varieties $\mathscr{V}_i$, $i \in \{1, 2, \ldots, t+1\}$ arise as associated varieties of the highest weight modules $L(\xi_i)$ and so are the closures of orbital varieties (*see* [23], Sect. 7). To show that every orbital variety arises in such a fashion (*see* [21], Sect. 8.1) is a difficult and as yet unsolved problem. The present simple case is already quite subtle and has a significant history (EJ, 5.2) and [12].

4. The maximal ideal and surjectivity theorems

4.1. As discussed in 1.2 we now recover the results of Levasseur-Stafford in [26] and further extend them to the exceptional cases (actually only E_7 remained open). Our analysis is furthermore case by case free.

4.2. Retain the notation of Section 3.

THEOREM. — *Fix* $i \in \{1, 2, \ldots, t\}$ *and set* $\xi_i = u_i \omega$. *Then* $J_i := \operatorname{Ann} L(\xi_i)$ *is a maximal ideal of* $U(\mathfrak{g})$.

We can assume $i > 1$ for J_1 is just the augmentation ideal of $U(\mathfrak{g})$. If J_i were not maximal then by 3.5(*) and 3.5(iii) it would be contained in some maximal ideal J satisfying $J \cap Y_i \neq \varnothing$. Since trivially $J \cap Y_1 = \varnothing$ there exists a largest integer j, $1 \leqq j \leqq i-1$ such that $J \cap Y_j = \varnothing$. We recall an argument in [21], 4.4, to show that $v_j^l \in J$ for l sufficiently large.

Since a maximal ideal is primitive, Duflo's theorem ([16], 7.4) gives a simple highest weight module L such that $J = \operatorname{Ann} L$. Let e be a choice of highest weight vector for L. Take k, $1 \leqq k \leqq j$. If $v_k^l e \neq 0$, $\forall l \in \mathbb{N}$, then the torsion submodule of the simple $U(\mathfrak{g})$ module L with respect to the Ore subset $\{v_k^l\}_{l \in \mathbb{N}}$ of $U(\mathfrak{g})$ is zero. In this case $v_k m = 0$, $m \in L$ implies $m = 0$, so r_k is a non-zero divisor in $U(\mathfrak{g})/J$. If this were to hold for all k, $1 \leqq k \leqq j$ it would contradict the hypothesis $J \cap Y_{j+1} \neq \varnothing$. Hence $v_k^l e = 0$ for some k, $1 \leqq k \leqq j$ and some $l \in \mathbb{N}^+$. Since $L = U(\mathfrak{n}) e$ and $v_k \in U(\mathfrak{g})^{\mathfrak{n}}$ it follows that $v_k^l \in J$. Then $J \cap Y_{k+1} \neq \varnothing$ and so by choice of j we obtain $k = j$. This proves the required assertion.

Take j as above. Since $L(\xi_i)$ is already Y_i torsion-free (*see* proof of 3.5 for example) it is necessarily Y_j torsion-free. Hence $L(\xi_i)$ embeds in $Y_j^{-1} L(\xi_i)$ which we may consider as a $Y_j^{-1} U(\mathfrak{g})$ module and hence (*cf.* 3.4) as a $U(\Theta^j(\mathfrak{g}^j))$ module. Let f denote the image in $Y_j^{-1} L(\xi_i)$ of a non-zero vector of weight ξ_i of $L(\xi_i)$. Set $\tilde{\mathfrak{g}} = \Theta^j(\mathfrak{g}^j)$, $\tilde{\mathfrak{k}} = \Theta^j(\mathfrak{k}^j)$. We claim that $\tilde{\mathfrak{k}}$ acts on f by scalars.

The case $j = 1$ is trivial. Take $j = 2$ and recall the description of θ^2 given in 3.4. Take $\delta \in \Delta_c^2$. Then the numerator of $\theta^2(x_\delta)$ takes the form $x_{\gamma_1} x_{\gamma_2}$ with $\gamma_1, \gamma_2 \in \Gamma$ and $\gamma_1 + \gamma_2 + \beta = \delta$. It follows that either γ_1 or γ_2 is compact. Since $x_\eta f = 0$, $\forall \eta \in \Delta_c$ we conclude that $\Theta^2(x_\delta) f = 0$, $\forall \delta \in \Delta_c^2$. Take $h \in \mathfrak{h}$. Then the numerator of $\theta^2(h)$ takes the form $x_{\gamma_1} x_{\gamma_2}$ with $\gamma_1, \gamma_2 \in \Gamma$ and $\gamma_1 + \gamma_2 + \beta = 0$. Hence again either γ_1 or γ_2 is compact. However this time $[x_{\gamma_1}, x_{\gamma_2}]$ is a multiple of $x_{-\beta}$ and so $\theta^2(h) f$ can be a non-zero multiple of f. Consequently f viewed as a weight vector for $\tilde{k}$ will have a weight which may differ from ξ_i. We could in principle calculate the resulting shift of weight directly; but this would be a messy error-prone calculation. We shall find a more devious method to calculate this shift. Taking account of the stepwise nature of the construction in 3.2-3.4, repetition of the above analysis establishes the claim for arbitrary j.

Given $\gamma \in \Delta^j$, we set $\tilde{x}_\gamma = \Theta^j(x_\gamma)$. Now assume $\gamma \in \Delta^+$ and let us show that $\tilde{x}_\gamma f = 0$. As above we are reduced to the case $j = 2$ and furthermore we can assume that γ is non-compact. Then the numerator of $\theta^2(x_\gamma)$ has terms of the form $x_{\gamma_1} x_{\gamma_2}$ with $\gamma_1, \gamma_2 \in \Gamma$ and $\gamma_1 + \gamma_2 + \beta = \gamma$. Hence both γ_1, γ_2 are compact and so $\theta^2(x_\gamma) f = 0$. Since $x_\gamma f = 0$, we obtain $\tilde{x}_\gamma f = 0$ as required.

We conclude from the above that $L_2 := U(\tilde{\mathfrak{g}}) f$ is an image of a module N_2 induced from a 1 dimensional representation of the parabolic subalgebra $\tilde{\mathfrak{p}}$ of $\tilde{\mathfrak{g}}$ with Levi factor $\tilde{\mathfrak{k}}$ and nilradical $\tilde{\mathfrak{m}}^+ := \mathbb{C}\{\Theta^j(x_\gamma), \gamma \in \Gamma_n^j \cap \Delta^+\}$.

We now compute the highest weight of L_2 (which we recall differs slightly from ξ_i). Extend Θ^j to an algebra homomorphism of $U(\mathfrak{g}^j)$ into $Y_j^{-1} U(\mathfrak{g}^j \oplus \mathfrak{b}^{j-1})^{\mathfrak{b}^{j-1}}$. Let $\mathfrak{n}^j$ denote the subalgebra of $\mathfrak{g}^j$ spanned by the $x_{-\gamma}$, $\gamma \in \Delta^j \cap \Delta^+$. The result in EJ, 2.1, applies to the pair $\mathfrak{g}^j$, $\mathfrak{k}^j$ and so we obtain a unique up to scalars element $v_{i-j+1}^{(j)} \in S(\mathfrak{m}^j)^{\mathfrak{n}^j}$ of weight $-(\beta_j+\beta_{j+1}+\ldots+\beta_i)$. Exactly as in the proof of 3.4 one checks that $\tilde{v}_{i-j+1}:=\Theta^j(v_{i-j+1}^{(j)}) \in Y_j^{-1} S(\mathfrak{m})^{\mathfrak{n}}$ and has weight $-(\mu_i-\mu_{j-1})$ which equals the above sum. By EJ, 2.1, such an element is necessarily proportional to $v_{j-1}^{-1} v_i$. Since $v_i f=0$, we conclude that $\tilde{v}_{i-j+1} f=0$. By EJ, 5.3, we conclude that L_2 has highest weight $\tilde{\xi}_i:=u_{i-j+1}\,\omega$ viewed as a weight of the simple Lie algebra $[\tilde{\mathfrak{g}}, \tilde{\mathfrak{g}}]$. This does not fully calculate the highest weight of L_2 but is sufficient for our present purposes.

Now recall the maximal ideal $J \supset \operatorname{Ann} L(\xi_i)$ of $U(\mathfrak{g})$. Recall further that for all $k<j$, v_k is a non-zero divisor in $U(\mathfrak{g})/J$ and that $v_j^l \in J$ for some $l \in \mathbb{N}^+$. Then $\tilde{J}:=Y_j^{-1} J \cap U(\tilde{\mathfrak{g}}) \supset \operatorname{Ann} L_2$. Moreover $\tilde{v}_1=\Theta^j(x_{-\beta_j})=v_{j-1}^{-1} v_j$ and so $\tilde{v}_1^l \in \tilde{J}$. Then by Borho's lemma ([18], 6.11) applied to the simple Lie algebra $[\tilde{\mathfrak{g}}, \tilde{\mathfrak{g}}]$ with lowest weight vector $\tilde{x}_{-\beta_j}=\tilde{v}_1$ it follows that $\tilde{J}$ has finite codimension in $U(\tilde{\mathfrak{g}})$. If we let $\tilde{\rho}$ denote the half sum of roots in $\Delta^j \cap \Delta^+$, this in turn implies that $\tilde{\rho}+\tilde{\xi}_i$ is integral and regular for Δ^j. The final step in the proof of the theorem consists showing that the above condition is never satisfied. Recalling that $j \leqq i-1$ it suffices to proves the lemma below.

4.3. Define $u_i \in \mathbb{R}$ as in 1.2 and EJ, 3.4, 4.3.

LEMMA. — *Take* $i \in \{1,2,\ldots,t\}$. *If* $\rho+u_i\omega$ *is both integral and regular, then* $i=1$.

We adopt the normalization of EJ, Table, that is $(\alpha,\alpha)=2$ or equivalently $(\alpha,\omega)=1$. One has $u_1=0$ and so u_i can be computed from EJ, Table. We can assume without loss of generality that u_i is an integer. It is then enough to show that for i, $2 \leqq i \leqq t$ there exists a non-compact positive root γ such that $0=(\gamma,\rho+u_i\omega)=(\gamma,\rho)+u_i$ equivalently that $\{(\gamma,\rho)\mid\gamma\in\Delta_n^+\} \supset \{1,2,\ldots,[u_i-u_t]\}$.

Suppose all the roots in Δ have the same length. Since $\mathfrak{m}$ is a simple $\mathfrak{k}$ module, the left hand side takes all positive integer values up to (β,ρ). Yet by EJ, 4.2, one has $u_1-u_t=-u_t<(1/t)\,(\mu_t,\rho) \leqq (\beta,\rho)$, which concludes the proof in this case.

It remains to consider $\mathfrak{g}$ of type C_l. Then $u_1-u_t=(l-1)/2$. Using the Bourbaki convention ([5], *Pl.* III) we have $\alpha=2\varepsilon_l$, $\omega=\varepsilon_1+\varepsilon_2+\ldots+\varepsilon_l$ and so $(\alpha,\omega)=2$. Also $2\varepsilon_i$, $i=1, 2, \ldots, l$ is a non-compact positive root and $(2\varepsilon_i,\rho)=2(l-i+1)$. Taking account of our present normalization, this shows that the left hand side above contains the set $\{1,2,\ldots,l\}$ which is all we require.

4.4. As in 4.2 we set $\xi_i=u_i\omega$, $i\in\{1,2,\ldots,t\}$. Let $F(\xi_i)$ [resp. $A(\xi_i)$] denote the $\mathbb{C}$-endomorphisms of $L(\xi_i)$ which are locally finite under the diagonal action of $\mathfrak{g}$ (resp. $\mathfrak{n}$). One has embeddings $U(\mathfrak{g})/J_i \subsetneq F(\xi_i) \subsetneq A(\xi_i)$. Recall that $\mathfrak{n}=\mathfrak{m}\oplus\mathfrak{n}_c$. Since $\mathfrak{n}_c$ has a locally nilpotent action on $L(\xi_i)$ it follows that $A(\xi_i)$ contains all $\mathbb{C}$-endomorphisms of $L(\xi_i)$ which are only required to be $\mathfrak{m}$ locally finite. Since $\mathfrak{m}$ is commutative, identifying $L(\xi_i)$ with $S(\mathfrak{m})/Q_i$ gives the

LEMMA. — *One has* $A(\xi_i)=\mathscr{D}(\mathscr{V}_i)$, $\forall\, i\in\{1,2,\ldots,t+1\}$. *In particular* $A(\xi_i)$ *is an integral domain and* J_i *is completely prime.*

4.5. We now extend the main surjectivity result of Levasseur-Stafford ([26], 0.3) to arbitrary $\mathfrak{g}$.

THEOREM. — *Take $i \in \{1, 2, \ldots, t\}$. Then the embeddings $U(\mathfrak{g})/J_i \subsetneq F(\xi_i) \subsetneq A(\xi_i)$ are all isomorphisms.*

By (∗∗) of 3.5 these are isomorphisms up to localization with respect to the Ore set Y_i. By 4.2 it then follows from [20], 9.1, that the first embedding is an isomorphism.

To show that the second embedding is an isomorphism we apply [24], 5.8. Since $L(\xi_i)$ is simple (EJ, 8.2) it remains to show that $L(\xi_i)$ is rigid in the sense of [24], 1.2. As discussed in [24], 1.5, this last property is an immediate consequence of the fact that the associated variety of $L(\xi_i)$ is $\mathscr{V}_i$ and so is a proper closed subvariety of the nilradical $\mathfrak{m}^+$ of a maximal parabolic subalgebra $\mathfrak{p}^+$ of $\mathfrak{g}$ (and hence cannot be induced). This proves the theorem.

Remark. – For $i = t+1$, the second embedding cannot be an isomorphism since $A(\xi_{t+1})$ is a Weyl algebra – this is of course the induced case (*cf.* [24], 7.6 and [25], 3.9).

4.6. COROLLARY. — *Take $i \in \{1, 2, \ldots, t+1\}$. The ring $\mathscr{D}(\mathscr{V}_i)$ of differential operators on $\mathscr{V}_i$ is simple and noetherian.*

For $i \leqq t$, this follows from 4.2 and 4.5. For $i = t+1$, it follows from the remark in 4.5.

Remark. – The above result for $i=2$ is due to Levasseur-Smith-Stafford ([25], 5.3) and for $\mathfrak{g}$ classical to Levasseur-Stafford ([26], 0.3). The only new case is E_7 for $i=3$. Yet our present proof is much simpler and essentially case by case free. For $2 \leqq i \leqq t$, the $\mathscr{V}_i$ are all singular so the result is not an immediate consequence of general considerations (as in say [27], 15.3.8).

5. Transference of unitarity

5.1. We now establish the main results claimed in 1.3. For this we use the construction of 3.2-3.4 and adopt the notation there. We use $\mathscr{O}$ to denote the well-known Bernstein-Gelfand-Gelfand category (*see* [7], 7.8.15 for example).

5.2. Fix $\tau \in P_c^+$, $u \in \mathbb{R}$ and set $\lambda = \tau + u\omega$. Set $N(\lambda) = U(\mathfrak{g}) \otimes_{U(\mathfrak{p}^+)} (V(\tau) \otimes \mathbb{C}_{u\omega})$ and let $L(\lambda)$ be the unique simple quotient of $N(\lambda)$. Denote again by $V(\tau)$ the image of $1 \otimes V(\tau) \otimes 1$ in $L(\lambda)$. Assume that $L(\lambda)$ is not finite dimensional. By Borho's lemma ([18], 6.11) this implies that $L(\lambda)$ in Y_2 torsion-free. Set $\tilde{\mathfrak{g}} = \Theta(\mathfrak{g}^2) \subset Y_2^{-1} U(\mathfrak{g}^2 \oplus \mathfrak{a}^1)^{\mathfrak{a}^1}$, $\tilde{\mathfrak{k}} = \Theta^2(\mathfrak{k}^2)$, $\tilde{x} = \Theta^2(x)$, $\bar{x} = \theta^2(x)$, $\forall x \in \mathfrak{g}^2$. Consider $Y_2^{-1} L(\lambda)$ as a $\tilde{\mathfrak{g}}$ module.

Since $V(\lambda)$ is a simple $\mathfrak{k}$ module, its lowest $\mathfrak{k}$ weight space is one dimensional and generates a simple $\mathfrak{k}^2$ module V_1. From now on fix a highest $\mathfrak{k}^2$ weight vector $f \in V_1$.

LEMMA.

(i) $\mathfrak{a}_c^1 V_1 = 0$.

(ii) $\bar{x}_\delta V_1 = 0$, $\forall \delta \in \Delta_c^2 \cup (\Delta_n^2 \cap \Delta^+)$.

(iii) $\bar{h} f \in \mathbb{C} f$, $\forall h \in \mathfrak{h}$.

(iv) *f is a highest* $\tilde{\mathfrak{g}}$ *weight vector.*

The restriction to $\mathfrak{k}^2$ of the adjoint action of $\mathfrak{g}^2$ on $\mathfrak{a}^1$ leaves $\mathfrak{a}_c^1$ (and $\mathfrak{a}_n^1$) invariant. By choice of f this proves (i).

Recall that each term in the numerator of $\bar{x}_\delta$ takes the form $x_{\gamma_1} x_{\gamma_2}$ with $\gamma_1, \gamma_2 \in \Gamma$ and $\gamma_1+\gamma_2+\beta=\delta$. If both γ_1, γ_2 are non-compact, then $\delta \in \Delta_n^-$ which is excluded by the hypothesis of (ii). Hence (ii) follows from (i). The proof of (iii) is similar to (ii) except that we have to remember the scale factor coming from $[x_{\gamma_1}, x_{\gamma_2}]$ which is proportional to the denominator of $\theta^2(h)$.

Finally (iv) obtains from (ii) and (iii) since for say x_δ, $\delta \in \Delta^2 \cap \Delta^+$ we have $\tilde{x}_\delta = x_\delta - \bar{x}_\delta$ and $x_\delta f=0$ by choice of f.

5.3. Retain the notation and hypotheses of 5.2. Let W_c^i denote the compact Weyl group for $\Theta^i(\mathfrak{g}^i)$. This is just the Weyl group for $\mathfrak{k}^i$ defined by the root system Δ_c^i. Let w_c^i be the unique longest element in W_c^i. Let $\tilde{\mathfrak{g}}'$ denote the derived algebra of $\tilde{\mathfrak{g}}$. This is a simple Lie algebra with root system Δ^2. Set $\tilde{\mathfrak{h}}'=\tilde{\mathfrak{h}} \cap \tilde{\mathfrak{g}}'$. Recall (EJ, 3.6) the definition of $\varepsilon_{\mathfrak{g}, \alpha}$.

LEMMA. — *As an* $\tilde{\mathfrak{h}}'$ *vector f has weight* $\lambda_2 := w_c^2 w_c^1 \lambda + \varepsilon_{\mathfrak{g}, \alpha} \omega$.

It is immediate that f is an $\mathfrak{h}$ weight vector of weight $w_c^2 w_c^1 \lambda$. This is not quite the weight of f as an $\tilde{\mathfrak{h}}$ weight vector because of the scale factor occurring in each $\theta^2(h)$, $h \in \mathfrak{h}$. To compute the contribution of these scale factors it is enough to compute λ_1 for a special choice of λ. In the notation of 4.2, we take $\lambda=\xi_2=u_2\omega=(u_2-u_1)\omega = -\varepsilon_{\mathfrak{g}, \alpha}\omega$. In this case $\tilde{\mathfrak{g}}' f=0$. This is because by 5.2 we have $v_2 L(\xi_2)=0$, so in particular $v_1^{-1} v_2 = \Theta^2(x_{-\beta_1}) = \tilde{x}_{-\beta_1} \in \operatorname{Ann} U(\tilde{\mathfrak{g}}) f$. This implies that $U(\tilde{\mathfrak{g}}) f$ is the trivial $\tilde{\mathfrak{g}}'$ module. Consequently f has zero $\tilde{\mathfrak{h}}'$ weight. We conclude that the scale factors add on the term $\varepsilon_{\mathfrak{g}, \alpha}\omega$ when f is viewed as a weight vector for $\tilde{\mathfrak{h}}'$.

5.4. Drop the superscripts on Γ^1, $\mathfrak{a}^1$, $\mathfrak{c}^1$. Set $\Delta_n^0=\Gamma_n \setminus \{-\beta\}$, and $q_\gamma = x_\gamma$, $\forall \gamma \in \Gamma_n$, $p_{-\gamma} = N_{\gamma, -\beta-\gamma}^{-1} x_{-\beta}^{-1} x_{-\beta-\gamma}$, $\forall \gamma \in \Gamma_n^0$, $p_\beta=(1/2) x_{-\beta}^{-1} h_\beta$. Then $[q_\gamma, p_{-\gamma}]=1$, $\forall \gamma \in \Gamma_n$ and all other commutators vanish. Set $A_0=\mathbb{C}[q_\gamma, p_{-\gamma}, \gamma \in \Gamma_n^0]$, $A'=\mathbb{C}[q_{-\beta}^{-1}, q_{-\beta}, p_\beta]$ $A=A_0 \otimes A'$ which are (localized) Weyl algebras. We define an antiautomorphism $\tilde{\sigma}$ of A by $\tilde{\sigma}(q_\gamma)=p_\gamma$, $\tilde{\sigma}(p_\gamma)=q_\gamma$, $\forall \gamma \in \Gamma_n^0$, $\tilde{\sigma}(q_{-\beta})=q_{-\beta}^{-1}$, $\tilde{\sigma}(p_\beta)=q_{-\beta} p_\beta q_{-\beta}$. Observe that

$$(*) \qquad \tilde{\sigma}(h_\beta)=h_\beta \quad \text{and} \quad \tilde{\sigma}(q_\gamma p_\gamma)=q_\gamma p_\gamma, \qquad \forall \gamma \in \Gamma_n^0.$$

Recall that the map $\theta^2 : \mathfrak{g}^2 \to Y_2^{-1} U(\mathfrak{a})$ is a Lie algebra homomorphism.

LEMMA.

(i) $\tilde{\sigma}(\bar{h})=\bar{h}$, $\forall h \in \mathfrak{h}$.

(ii) $\tilde{\sigma}(\bar{x}_\delta)=\bar{x}_{-\delta}$, $\forall \delta \in \Delta_c^2$.

(iii) $\tilde{\sigma}(\bar{x}_\delta)=-\bar{x}_{-\delta}$, $\forall \delta \in \Delta_n^2$.

(i) is an immediate consequence of (*) above. The proof of (ii) and (iii) which are in principle quite delicate are made enormously easier by the fact that for each $\gamma \in \Gamma_n^0$ and each $\delta \in \Delta^2$ the δ-string containing γ has at most two elements. This is obvious if all roots have the same length. It holds in type B_n because $\delta = \pm\alpha_1$ and is necessarily a long root. In type C_n the assertion is checked by explicit computation.

Take $\delta \in \Delta_c^2$. Recall that $[x-\bar{x}, A_0]=0, \forall x \in \mathfrak{g}^2$. On easily checks that

$$\bar{x}_\delta = \theta^2(x_\delta) = \sum_{\gamma \in \Gamma_n^0} c^\delta_{\gamma+\delta,\, -\gamma}\, q_{\gamma+\delta}\, p_{-\gamma}$$

where

$$c^\delta_{\gamma+\delta,\, -\gamma}\, q_{\gamma+\delta} = [x_\delta, q_\gamma],$$

and since we have a Chevalley basis, these coefficients are integers.

Now fix $\gamma \in \Gamma_n^0$ such that $\gamma+\delta$ is a root. By the first remark $\gamma-\delta$ is not a root and $(\delta^\vee, \gamma) = -1$. Thus

$$-q_\gamma = [h_\delta, q_\gamma] = -[x_{-\delta}, [x_\delta, q_\gamma]]$$

and so

$$c^{-\delta}_{\gamma,\, -(\gamma+\delta)}\, c^\delta_{\gamma+\delta,\, -\gamma} = 1.$$

We conclude that these coefficients are pairwise equal. A similar result holds if $\gamma-\delta$ is a root. Combined, this gives just what we require to prove (i).

Take $\delta \in \Delta_n^2 \cap \Delta^+$. One has

$$\bar{x}_\delta = \sum_{\gamma \in \Gamma_n^0} d^\delta_{-\gamma,\, \beta+\gamma+\delta}\, q_{-\beta}\, p_{-\gamma}\, p_{\beta+\gamma+\delta}$$

where

$$d^\delta_{-\gamma,\, \beta+\gamma+\delta}\, q_{-\beta}\, p_{\beta+\gamma+\delta} = [x_\delta, q_\gamma],$$

and as before these coefficients are integers. Again

$$\bar{x}_{-\delta} = \sum_{\gamma \in \Gamma_n^0} e^{-\delta}_{\gamma,\, -(\beta+\gamma+\delta)}\, q_{-\beta}^{-1}\, q_\gamma\, q_{-(\beta+\gamma+\delta)}$$

where

$$e^{-\delta}_{\gamma,\, -(\beta+\gamma+\delta)}\, q_{-\beta}^{-1}\, q_{\beta-\gamma-\delta} = -[x_{-\delta}, p_{\beta+\gamma+\delta}]$$

and as before these coefficients are integers.

Now fix $\gamma \in \Gamma_n^0$ such that $\gamma+\delta$ is a root. As before

$$-q_\gamma = [h_\delta, q_\gamma] = -[x_{-\delta}, [x_\delta, q_\gamma]]$$

and so

$$-e^{-\delta}_{\gamma,\, -(\beta+\gamma+\delta)} \cdot d^\delta_{-\gamma,\, \beta+\gamma+\delta} = 1.$$

We conclude that these coefficients differ pairwise exactly by a sign. This is just what is required to prove (iii).

5.5. We have $\mathfrak{a}_c f=0$ by 5.2 (i). Hence $p_{-\gamma} f=0$, $\forall \gamma \in \Gamma_n^0$. Hence

$$M_0 := A_0 f = \mathbb{C}[q_\gamma, \gamma \in \Gamma_n^0]$$

is the standard A_0 module in which q_γ, $\gamma \in \Gamma_n^0$ acts by multiplication and $p_{-\gamma}$ by differentiation with respect to q_γ.

Let $\varepsilon : A_0 \to \mathbb{C}$ be the projection defined by the canonical basis of A_0 in which the q_γ (resp. p_γ) appear to the left (resp. right). Let j denote complex conjugation. One checks that the map $(a, b) \mapsto \varepsilon((j\tilde{\sigma}(a))b)$ of $A_0 \times A_0 \to \mathbb{C}$ factors to a sesquilinear $\tilde{\sigma}$-contravariant form $\langle \ , \ \rangle$ on M_0. Moreover up to scalars the monomials in the q_γ form an orthonormal basis and hence this form is positive definite. Of course this construction is quite classical and known to physicists as the Fock space construction of the "unitary" representation of the Weyl algebra given by the Stone-von Neumann theorem. (One nevertheless needs to check that signs do work out correctly – for example

$$\langle q, q \rangle = \langle 1, \tilde{\sigma}(q)q1 \rangle = \langle 1, pq \rangle = \langle 1, [p, q]1 \rangle = \langle 1, 1 \rangle = 1,$$

as required.)

5.6. Since $h_\beta f \in \mathbb{C} f$ we may identify $M' := A' f$ with $\mathbb{C}[q_{-\beta}^{-1}, q_{-\beta}]$. On M' we let $\langle \ , \ \rangle$ denote the unique sesquilinear form which extends $\langle q_{-\beta}^k, q_{-\beta}^l \rangle = \delta_{kl}$ (where δ is the Kronecker delta). It is obviously positive definite and noting that $\tilde{\sigma}(h_\beta) = h_\beta$ and $A' = \mathbb{C}[q_{-\beta}^{-1}, q_{-\beta}, h_\beta]$, we easily conclude that it is $\tilde{\sigma}$-contravariant.

It is immediate that $M := Af = M_0 \otimes M'$. Hence the

LEMMA. — *The product form $\langle \ , \ \rangle$ on M is sesquilinear, $\tilde{\sigma}$-contravariant and positive definite.*

5.7. Let $\mathfrak{g}_0^2$ denote the centralizer of $x_{-\beta}$ in $\mathfrak{g}^2$. One has $\mathfrak{g}_0^2 \oplus \mathbb{C} h_\beta = \mathfrak{g}^2$. Set $\tilde{\mathfrak{g}}_0^2 = \Theta^2(\mathfrak{g}_0^2)$. One has $\tilde{\mathfrak{g}}' \subset \tilde{\mathfrak{g}}_0^2$ and is of at most codimension 1. Set $B = U(\tilde{\mathfrak{g}}_0^2)$. Set $L_2 = Bf = U(\tilde{\mathfrak{g}})f$. View A, B as subrings of $Y_2^{-1} U(\mathfrak{g})$.

(i) *The map $a \otimes b \mapsto ab$ of $A \otimes B$ is an isomorphism of rings.*

(ii) *The map of $\otimes bf \mapsto abf$ of $M \otimes L_2$ onto $L := ABf$ is an isomorphism of AB modules.*

(iii) $AB = Y_2^{-1} U(\mathfrak{a}) U(\mathfrak{g}^2) = U(\mathfrak{g}^2) Y_2^{-1} U(\mathfrak{c})$.

(i) follows from the fact that A is central, simple and B is the commutant of A in AB.

(ii) By say Quillen's lemma ([7], 2.6.4) $\mathrm{End}_A M$ reduces to scalars. Then a standard application of the Jacobson density theorem shows that any $A \otimes B$ submodule of $M \otimes L_2$ takes the form $M \otimes N$ where N is a B submodule of L_2. This proves (ii).

(iii) is an obvious consequence of the relation $[\mathfrak{g}^2, \mathfrak{a}] \subset \mathfrak{a}$ and the definition of Θ^2.

Remark. – (i), (ii) can also be proved by an elementary direct computation.

5.8. Let σ denote the "non-compact" Chevalley antimorphism of $\mathfrak{g}$ defined in EJ, 5.1. We define $\tilde{\sigma}$ on $\mathfrak{g}^2$ by taking it to be the restriction of σ. Set $\tilde{x}=\Theta(x)=x-\theta^2(x)=x-\bar{x}$, $\forall x\in\mathfrak{g}^2$. By 5.4 and the definition of σ we obtain the remarkable fact that 5.4 holds with $\bar{x}$ replaced by $\tilde{x}$. By 5.2, $L_2=U(\tilde{\mathfrak{g}})f$ is a highest weight module. From the standard construction (*cf.* EJ, 5.1) we obtain a sesquilinear $\tilde{\sigma}$-contravariant form $\langle\ ,\ \rangle$ on L_2.

5.9. Let $\bar{\mathfrak{g}}$ denote the image of $\mathfrak{g}^2$ under the Lie algebra map $\theta^2:\mathfrak{g}^2\rightarrow A$. Set $N=U(\bar{\mathfrak{g}})f$ which is a $\bar{\mathfrak{g}}$ submodule of M. By construction $x=\theta^2(x)+\Theta^2(x)=\bar{x}+\tilde{x}$, $\forall x\in\mathfrak{g}^2$ and moreover $[\bar{x},\tilde{y}]=0$, $\forall x,y\in\mathfrak{g}^2$. Recalling 5.7 we conclude that $N\otimes L_2$ identifies with a $\bar{\mathfrak{g}}\times\tilde{\mathfrak{g}}$ submodule of L, itself a submodule Y_2^{-1} L (λ). Moreover the action of $\mathfrak{g}^2$ on $Y_2^{-1}L(\lambda)$ restricted to $N\otimes L_2$ is just the diagonal action, for N, L_2 viewed as $\mathfrak{g}^2$ modules. Let n, e be a choice of highest weight vector for N, L_2. By 5.7 (ii), we may identify $n\otimes e$ with f and hence $U(\mathfrak{g}_2)f$ with the submodule of $N\otimes L_2$ generated by $n\otimes e$. It follows from 5.3 (or of course directly from the proof of 5.3) that n has highest weight $-\varepsilon_{\mathfrak{g},x}\omega=:\xi_1$. Moreover 5.4 just says that $\tilde{\sigma}$ restricted to $\theta^2(\mathfrak{g}^2)$ coincides with the "non-compact" Chevalley antiautomorphism σ defined on $\mathfrak{g}^2$ by EJ, 5.1. Thus the restriction of the contravariant form M to N coincides up to a scalar with that defined in EJ, 5.1. Since the former is positive definite on M, it is positive definite on N and so this construction reproves that N is a unitary highest weight module, hence simple and isomorphic to $L(\xi_1)$. Possibly a more elegant proof of 5.4 would obtain by using our prior knowledge of the unitarity of $L(\xi_1)$. Take $\lambda=-(i-1)\varepsilon_{\mathfrak{g},\alpha}\omega$ and recall 3.4. Then this construction also recovers (by an essentially elementary argument) the apparently deep fact, namely that $Q_{i+1}:=\mathrm{Ann}_{U(\mathfrak{m})}(f_2\otimes f_i)$, noted during the proof of 2.5.

5.10. Let $\langle\ ,\ \rangle$ denote the form on $N\otimes L_2$ which is the product of the $\tilde{\sigma}$-contravariant forms on N, L_2 given by 5.6 and 5.8. We should like to compare this with the σ-contravariant form on $L(\lambda)$. In particular to show that if $L(\lambda)$ is unitary then $\langle\ ,\ \rangle$ is positive definite and hence that L_2 is unitary. Unfortunately, we have been unable to do this, so in fact 5.4-5.6 will not be used in the sequel. We have the

LEMMA. — *Suppose* $L(\lambda)$ *is unitary* (*as a* $\mathfrak{g}$ *module*). *Then* $Y_2^{-1}L(\lambda)$ *is a direct sum of unitary highest weight* $\mathfrak{g}^2$ *modules and hence so is* $N\otimes L_2$.

It is sufficient and convenient to prove the corresponding assertion for $\mathfrak{g}_0^2$. Obviously $L(\lambda)$ is unitary as a $\mathfrak{g}_0^2$ module. Since the Borel subalgebra $\mathfrak{b}\cap\mathfrak{g}_0^2$ of $\mathfrak{g}_0^2$ acts locally finitely on $L(\lambda)$, then $Z(\mathfrak{g}_0^2)$ primary decomposition splits $L(\lambda)$ into a direct sum of modules in the $\mathcal{O}$ category and the latter all have finite length ([7], 7.8.15). Thus $L(\lambda)$ is a direct sum of unitary highest weight $\mathfrak{g}_0^2$ module. Since $x_{-\beta}$ commutes with $\mathfrak{g}_0^2$ the same holds for $x_{-\beta}^{-k}L(\lambda)$, $\forall k\in\mathbb{N}^+$. Then a standard argument on semisimplicity proves the assertion for $Y_2^{-1}L(\lambda)$ and then for the submodule $N\otimes L_2$.

5.11. We are now faced with the following question. Let N_1, N_2 be highest weight (not necessarily simple) $\mathfrak{g}$ modules such that $N_2\otimes N_1$ is unitary with respect to the diagonal action of $\mathfrak{g}$. Then are N_1, N_2 unitary? The first unpleasant fact is that this can fail even if we impose that N_2 be unitary. For example, take $\mathfrak{g}$ of type $\mathfrak{sl}(2)$ with α the non-compact simple root (in the conventions of EJ). Take $N_1 = L(u\omega)$, $u \in]0, 1[$, $N_2 = L(-\omega)$. Use of $Z(\mathfrak{g})$ shows that $N_1 \otimes N_2$ is a direct sum of the modules $L((u-(2k+1))\omega)$, $k \in \mathbb{N}$ and is hence unitary. Yet N_1 is not unitary. The second unpleasant fact is that this can fail even if all simple factors of N_1 and N_2 are unitary. For example, take $N_1 = N(0)$ which has length 2 having the trivial module $L(0)$ as quotient and $L(-\alpha)$ as a submodule. Both factors are unitary. Take N_2 as above. Then every simple factor of $N_2 \otimes N_1$ is unitary. Yet this module is semisimple because its submodule $L(-\alpha) \otimes L(-\omega)$ is unitary and the quotient $L(0)$ can only be non-trivially extended (from below) by a submodule isomorphic to $L(-\alpha)$ and obviously no such factor occurs in $L(-\alpha) \otimes L(-\omega)$.

Two further unpleasant facts are uncovered by this second example. This first is non-trivial extensions between unitary modules can exist and the second that such extensions can be annihilated by tensor product. Further examples are given by the following. Retain the notation of EJ, 1.6.

PROPOSITION. — *Let λ be a first reduction point. Then $\overline{N(\lambda)}$ is unitary.*

Let μ be a highest $\mathfrak{k}$ weight occurring in $N(\lambda)$ and suppose that $\|\mu+\rho\| \leqq \|\lambda+\rho\|$. Then from the calculation in EJ, 5.2, using EJ, 3.9, 4.3, 4.4, we see that either $\mu=\lambda$, or μ is the highest $\mathfrak{k}$ weight of $\overline{N(\lambda)}$. By [9], 3.9, this proves the required assertion.

The reader may now easily check that taking $\mathfrak{g} = \mathfrak{sl}(3)$, for which 0 is again a first reduction point, one also obtains that $N(0) \otimes L(-\omega)$ is semisimple via $Z(\mathfrak{g})$ primary decomposition and is hence unitary.

Remark. — In general the unique simple quotient of $\overline{N(\lambda)}$ need not be unitary — see section 8.6. This destroys a possible approach to establishing the main result of [8].

5.12. As before we set $\xi_1 = -\varepsilon_{\mathfrak{g},\alpha}\omega$. We call $L(\lambda)$ quasi-unitary if $\lambda = \tau + u\omega$ with $\tau \in P_c^+$ of level s and $u < u_1^\tau + \varepsilon_{\mathfrak{g},\alpha}\omega$ or $u = u_i^\tau$, $1 \leqq i \leqq s$.

LEMMA. — *Suppose* $L(\lambda) \otimes L(\xi_1)$ *is unitary. Then* $L(\lambda)$ *is quasi-unitary.*

Let e (resp. f) be a choice of highest weight vector for $L(\lambda)$ [resp. $L(\xi_1)$]. Then $L = U(\mathfrak{g})(e \otimes f)$ is a submodule of $L(\lambda) \otimes L(\xi_1)$ and so by the hypothesis is a direct sum of simple highest weight modules (recall argument in proof of 5.10). Yet L is indecomposable, because it is a highest weight module. We conclude that L is a simple unitary module of highest weight $\lambda + \xi_1$. From the equal spacing rule we conclude that either $L(\lambda)$ is quasi-unitary, or $\lambda + \xi_1$ is a last place of unitary. Suppose the latter holds and set $g = e \otimes f$. We claim that there exists $a \in \mathfrak{m} \otimes U(\mathfrak{k})$ such that $ag = 0$. This again follows from the classification theory and can be expressed by saying that there is a component, namely the PRV component P — *see* EJ, 1.5, such that P is a relation in $\mathfrak{m} \otimes V(\lambda)$.

We can write

$$a = \sum_{i=1}^{r} x_i \otimes y_i$$

$x_i \in \mathfrak{m}$, $y_i \in U(\mathfrak{k})/\mathrm{Ann}_{U(\mathfrak{k})} g$ satisfying the usual linear independence. Since f generates a one dimension $\mathfrak{k}$ module we can find $\tilde{y}_i \in U(\mathfrak{k})/\mathrm{Ann}_{U(\mathfrak{k})} e$ such that $y_i(e \otimes f) = \tilde{y}_i e \otimes f$. Since $x_i \in \mathfrak{m}$ we obtain

$$0 = ag = \sum (x_i \tilde{y}_i) e \otimes f + \sum \tilde{y}_i e \otimes x_i f.$$

Now as a $U(\mathfrak{m})$ module, $L(\xi_1)$ identifies with $S(\mathfrak{m})/Q_2$. We recall that Q_2 is a (homogeneous) ideal generated by quadratic elements. Then f, $\{x_i f\}$ are linearly independent in $L(\xi_1)$. Consequently, $\tilde{y}_i e = 0$, and so $y_i g = 0$, $\forall i$. This is clearly absurd and the contradiction proves the lemma.

5.13. Corollary. — *Take* $M \in \mathrm{Ob}\, \mathcal{O}$ *(for example, a highest weight module). Suppose* $M \otimes L(\xi_1)$ *is unitary. Then every simple subquotient of* M *is quasi-unitary.*

5.14. Let $H(\lambda)$ denote a *not* necessarily simple, highest weight module of highest weight λ. Recall that if $\lambda \in P_c^+$ then $N(\lambda)$ is defined EJ, 1.5.

Proposition. — *Suppose* $H(\lambda) \otimes L(\xi_1)$ *is unitary. Then*

(i) $\lambda \in P_c^+$ *and* $H(\lambda)$ *is a quotient of* $N(\lambda)$.

Write $\lambda = \lambda_0 + u\omega$ *with* λ_0 *the first reduction point. Assume* $u \notin]0, \varepsilon_{\mathfrak{g},\alpha}[$.

(ii) *If* $\lambda \neq \lambda_0$ *then* $H(\lambda)$ *is unitary.*

(iii) *If* $H(\lambda)$ *is not unitary, then* $H(\lambda) = N(\lambda)$.

By 5.13 every simple factor of $H(\lambda)$ is quasi-unitary. By identification of $\mathfrak{k}$ with the complexification of a maximal compact subalgebra of the real form $\mathfrak{g}_0$ of $\mathfrak{g}$ it follows that every such factor is $\mathfrak{k}$ locally finite and hence so is $H(\lambda)$. This proves (i).

By 5.13 again, $L(\lambda)$ is unitary. Let e (resp. f) denote the canonical generator of $H(\lambda)$ [resp. $L(\xi_1)$]. Then $L := U(\mathfrak{g})(e \otimes f)$ is a submodule of a unitary module, hence unitary. It is also a highest weight module. Consequently $L \cong L(\lambda + \xi_1)$. Set $g = e \otimes f$.

Suppose the hypothesis of (ii) holds. Then $N(\lambda + \xi_1)$ is not simple and hence for some $i <$ level of j the PRV component of $V(\lambda + \xi_1) \otimes V_{i+1}$ (notation EJ, 2.5) is a relation in $L(\lambda + \xi_1)$. As in 5.12 we can choose an a of the form

$$a = \sum x_j \otimes y_j$$

$x_j \in S(\mathfrak{m})$ homogeneous of degree $i+1$ and $y_j \in U(\mathfrak{k})/\mathrm{Ann}_{U(\mathfrak{k})} V(\lambda + \xi_1)$ such that $ag = 0$. Then as in 5.12 we obtain

$$0 = a(e \otimes f) = \sum_{j,k} (x'_{jk} \tilde{y}_j e) \otimes x''_{jk} f$$

where

$$\sum_{k=0}^{i+1} x'_{jk} \otimes x''_{jk}$$

is the image of x under the diagonal map using the usual Hopf algebra convention on sums and taking the x'_{jk} (resp. x''_{jk}) to be homogeneous of degree $i+1-k$ (resp. k).

Now as in 5.12 using also that Q_2 is homogeneous we conclude that

$$\sum_j x'_{j1} \tilde{y}_j e = 0.$$

This means that $H(\lambda)$ has level of reduction $\leqq i$ (*cf.* EJ, 6.4). Yet again by the classification theory (EJ, 6.4) the induced module $N(\lambda + \xi_1)$ has level of reduction $i+1$ and so $N(\lambda)$ and hence $H(\lambda)$ has level of reduction i. Now we use the even harder fact (*cf.* EJ, 6.6, 8.3) that $\overline{N(\lambda)}$ is *generated* by the PRV component (which is *simple* as a $\mathfrak{k}$ module) of $V(\lambda) \otimes V_i$. This forces $H(\lambda)$ to be the simple quotient of $N(\lambda)$ proving (ii).

(iii) follows from (ii) and 5.11.

Remark. — One may also give an elementary proof of (i) using only $\mathfrak{k}$ structure.

5.15. Now return to the situation of 5.1-5.8. In particular define f as in 5.2 and L_2 as in 5.7.

THEOREM. — *Suppose that* $L(\lambda)$ *is unitary and that* $\operatorname{Ann}_{U(\mathfrak{m})} L(\lambda) \neq 0$. *Then* L_2 *is unitary.*

Let $L_2(\lambda_2)$ denote the simple quotient of L_2. Unfortunately 5.12 is not quite strong enough to say that $L_2(\lambda_2)$ is unitary. Yet λ_2 is given by 5.3 so this can be checked from the classification of unitary highest weight modules. This is a case by case analysis which we relegate to Section 7, so now we assume $L_2(\lambda_2)$ to be unitary.

The assertion now follows from 5.9 and 5.14 unless L_2 is the induced module $N_2(\lambda_2)$ defined relative to $\tilde{\mathfrak{g}}$. The latter means that L_2 is a free $U(\tilde{\mathfrak{m}})$ module. Since the Weyl algebra module M is free over $U(\mathfrak{a}_n)$ we conclude that $L \cong M \otimes L_2$ is free over $U(\tilde{\mathfrak{m}} \times \mathfrak{a}_n)$. In particular $\operatorname{Ann}_{U(\tilde{\mathfrak{m}} \times \mathfrak{a}_n)} f = 0$. Now $\mathfrak{a}_c f = 0$ by 5.2 (i) so this just means that $\operatorname{Ann}_{U(\mathfrak{m}^2 + \mathfrak{a}_n)} f = 0$. Recalling that $\mathfrak{m}^2 + \mathfrak{a}_n = \mathfrak{m}$, we obtain $\operatorname{Ann}_{U(\mathfrak{m})} f = 0$ contradicting the hypothesis.

Remark. — We could obviously do better; but not quite that $L(\lambda)$ itself is induced. This is because we have no control over the compact roots not in Δ^2.

5.6. We now prove the remarkable result promised in 1.3.

THEOREM. — *Suppose* $L(\lambda)$ *is unitary. Then* $Q := \operatorname{Ann}_{U(\mathfrak{m})} L(\lambda)$ *is a prime ideal.*

The proof is by induction on rank $\mathfrak{g}$. It is trivial if rank $\mathfrak{g} = 0$. If $\tau = 0$, then the assertion is just a consequence of the classification (EJ, 8.2) of the unitary modules in this case and 2.3. If $\tau \neq 0$, then $L(\lambda)$ is infinite dimensional and hence Y_2 torsion-free. Define L_2 as in 5.7. We can obviously assume $Q \neq 0$. Then by 5.15 L_2 is a unitary module for the strictly lower rank simple Lie algebra $\tilde{\mathfrak{g}}'$, so we can assume that the assertion holds for L_2.

Define $\tilde{v}_k := \Theta^2(v_k^{(2)}) = v_1^{-1} v_{k+1}$ as in the proof of 4.2. Then by 2.3, 2.4 the assertion for L_2 means that there exists j, $1 \leqq j \leqq t$ such that L_2 is torsion-free with respect to the $\tilde{v}_k$, $k < j$ and if $j \leqq t-1$ that $\tilde{v}_j L_2 = 0$. Define A as in 5.4 and L as in 5.7 (ii). Then $[\tilde{v}_k, A] = 0$, whereas $L = AL_2$ by 5.7. We conclude that L is v_{k+1} torsion-free for $k < j$ and if $j \leqq t-1$ that $v_{j+1} L = 0$.

We claim that the above assertions hold with L replaced by $L(\lambda)$. Suppose first that there exists $0 \neq m \in L(\lambda)$ such that $v_{k+1}^l m = 0$ for some $l \in \mathbb{N}^+$. Since $\{v_{k+1}^l\}_{l \in \mathbb{N}}$ is Ore in $U(\mathfrak{g})$ and $L(\lambda)$ is a simple $U(\mathfrak{g})$ module if follows that $v_{k+1}^{l'} f = 0$ for some $l' \in \mathbb{N}^+$. We conclude that $k \geqq j$. It remains to show that $v_{j+1} L(\lambda) = 0$ when $j \leqq t-1$.

Let e be a choice of highest weight for $L(\lambda)$. It is clearly enough to show that $v_{j+1} e = 0$. Unfortunately $e \notin L$ in general, so this is far from obvious. However since $U(\mathfrak{k}) f = V(\lambda) \ni e$ there is one case when this assertion does hold, namely when $\mathbb{C} v_{j+1}$ is $\mathfrak{k}$ stable. This arises (except possibly in types A_n, E_6) when $j = t-1$. Remarkably we can reduce to this case.

Set $v = v_{j+1}$ and let u be a highest weight vector in the simple (EJ, 2.1) $\mathfrak{k}$ submodule of $S(\mathfrak{m})$ generated by v. By our first argument it follows that there exists $l \in \mathbb{N}^+$ such that v^l annihilates every vector in the finite dimensional subspace $V(\lambda)$ of $L(\lambda)$. Hence $v^l L(\lambda) = 0$ as v is $\mathfrak{m}$ invariant. Consequently $u^l L(\lambda) = 0$.

Recall that v has weight $-\mu_{j+1}$ and so u has weight $-w_c \mu_{j+1}$. Suppose first that $\beta_t = \alpha$. We claim that $-w_c \mu_{j+1} = -(\beta_{t-j} + \beta_{t-j+1} + \ldots + \beta_t)$. First observe that (EJ, 2.3) this weight does in fact belong to $W_c \mu_{j+1}$, so it is enough to prove it to be $\mathfrak{k}$ dominant. Let α_i, α_i' be the possibly two simple roots non-orthogonal to β_i (*cf.* EJ, 2.1 or [18], 2.2 (vi)). Since $\beta_r \in \Delta^s$ for $r \geqq s$, we have $(\alpha_i, \beta_r) \leqq 0$, $(\alpha_i', \beta_r) \leqq 0$, $\forall r > i$. It follows we can assume $i \geqq t-j$ without loss of generality. However in this case α_i, $\alpha_i' \in \Delta^{t-j}$. Yet we know (*cf.* EJ, 2.1 or [16], 2.8) that $\beta_{t-j} + \ldots + \beta_t$ is orthogonal to every simple root of Δ^{t-j} except those non-orthogonal to β_t (in this case just α). Thus $-w_c \mu_{j+1}$ is only non-dominant with respect to the non-compact simple root α and hence is $\mathfrak{k}$ dominant.

Now because the weight of u lies in Δ^{t-j} and $u \in \mathfrak{m}$, it follows that $u \in S(\mathfrak{m} \cap \mathfrak{g}^{t-j})$. Applying (EJ, 2.1) to $\mathfrak{g}^{t-j}$ we may find $\mathfrak{k}^{t-j}$ lowest weight vectors $v_1', v_2', \ldots, v_{j+1}' \in S(\mathfrak{m} \cap \mathfrak{g}^{t-j})$ of weights

$$-\beta_{t-j},\ -(\beta_{t-j} + \beta_{t-j+1}),\ \ldots,\ -(\beta_{t-j} + \beta_{t-j+1} + \ldots + \beta_t).$$

By EJ, 2.3, the simple $\mathfrak{k}$ modules generated by $v_1', v_2', \ldots, v_{j+1}'$ are respectively V_1, $V_2, \ldots, V_{j+1}$. We conclude that $L(\lambda)$ is v_{k+1}' torsion-free for all $k < j$. Observe that $\mathbb{C} u = \mathbb{C} v_{j+1}'$.

Now consider $L(\lambda)$ as a $\mathfrak{g}^{t-j}$ module. It is unitary and by the argument in 5.9, a direct sum of unitary highest weight modules $L(\lambda_i)$ each of which satisfy the hypothesis of 5.15. Fix i and let $f_i \in L(\lambda_i)$ be defined as in 5.2. By the previous paragraph f_i is v_{k+1}' torsion free for all $k < j$. Hence $v_{j+1}' f_i = 0$ by 5.15 (as in the first step). Yet $v_{j+1}' = u$ up to scalar and so $u f_i = 0$. It follows that u also annihilates a highest weight vector e_i of $V(\lambda_i)$ which may also be identified with the highest weight vector $L(\lambda_i)$. Hence $u L(\lambda_i) = 0$. Since i was arbitrary we conclude that $u L(\lambda) = 0$. Recalling that u generates V_{j+1} as a $\mathfrak{k}$ module we obtain $Q_{j+1} L(\lambda) = U(\mathfrak{m}) V_{j+1} L(\lambda) = 0$. Recalling 2.3, this proves that $Q = Q_{j+1}$ which is prime.

The cases $\alpha \neq \beta_t$ can only occur in types A_n, E_6. The argument is essentially the same for these cases.

First assume $\mathfrak{g}$ of type A_n and $\alpha=\alpha_t$, in the Bourbaki notation ([5], *Pl. I*). We can assume $t \leqq (n+1)/2$ without loss of generality and then this definition of t coincides with that used above. If $t=(n+1)/2$, then $\beta_t=\alpha_t$ and so the above argument applies. Otherwise let $\mathfrak{g}_0$ denote the Levi factor of $\mathfrak{g}$ defined by the simple roots $\alpha_1, \alpha_2, \ldots, \alpha_{2t-1}$. Observing that $u \in S(\mathfrak{g}_0 \cap \mathfrak{m})$ we see the above analysis with $\mathfrak{g}_0$ replacing $\mathfrak{g}$ applies and proves the theorem in this case.

Finally suppose $\mathfrak{g}$ of type E_6. We can assume $\alpha=\alpha_1$ without loss of generality. Then $t=2$ and we can assume $j=1$ without loss of generality. One checks that $w_c(\beta_1+\beta_2)=-(2\alpha_1+\alpha_2+2\alpha_3+2\alpha_4+\alpha_5)$ in the Bourbaki notation ([5], *Pl. V*). Let $\mathfrak{g}_0$ denote the Levi factor of $\mathfrak{g}$ defined by $\alpha_1, \alpha_2, \ldots, \alpha_5$. Then $u \in S(\mathfrak{g}_0 \cap \mathfrak{m})$, so we can replace $\mathfrak{g}$ by $\mathfrak{g}_0$ in the above. Finally $w_c(\beta_1+\beta_2)$ is orthogonal to the compact roots $\alpha_2, \alpha_3, \ldots, \alpha_5$ and so $\mathbb{C}u$ is $\mathfrak{k}_0 := \mathfrak{k} \cap \mathfrak{g}_0$ stable. Thus our previous analysis applies and proves the theorem in type E_6.

5.17. The argument in [29], 7.13 suggests an easy proof of 5.16 and it is perhaps worth mentioning why this cannot work. Define $Q=\mathrm{Ann}_{U(\mathfrak{m})} L(\lambda)$. Obviously Q is $\mathfrak{k}$ stable. Hence by EJ, 8.1, and 2.3 we conclude that the radical $\sqrt{Q}$ of Q is prime. It therefore suffices to show that $a^2 \in Q$ implies $a \in Q$. Suppose $a^2 \in Q$ and that we have $m \in L(\lambda)$ such that $am \neq 0$. Assume for simplicity that a is real, that is $a=j(a)$. Then $\langle am, am \rangle \neq 0$ by unitary and so $\sigma(a)am \neq 0$. Repeating this argument we conclude $\sigma(a)a\sigma(a)m \neq 0$. Had we been able to push a past $\sigma(a)$ then we would have got the desired contradiction. We can see why such an analysis is hopeless by taking unitary highest weight modules relative to the compact real form of $\mathfrak{g}$. All such modules are finite dimensional. If we let σ_0 denote the "compact" Chevalley antiautomorphism (EJ, 2.4) they are just those modules which admit a positive definite σ_0-contravariant form. Yet σ_0 hardly differs from σ and in any case such an argument is hardly likely to show up the different. Of course for a finite dimensional module, $\sqrt{Q}$ is the augmentation ideal of $S(\mathfrak{m})$ and so will coincide with Q only if $\dim L(\lambda)=\dim V(\lambda)$. Since $L(\lambda)$ is a simple $\mathfrak{g}$ module and $V(\lambda)$ is a simple $\mathfrak{k}$ module the latter only holds for the trivial module.

6. Maximality and Goldie rank

6.1. Let $L(\lambda)$ be a unitary highest weight module and set $J(\lambda)=\mathrm{Ann}\, L(\lambda)$. If λ is a multiple of ω then $J(\lambda)$ is maximal (Theorem 4.2) and completely prime (Lemma 4.4). Here we study how these conclusions are modified in the general case.

6.2. Define $V(\lambda)$ as in 5.1. Recall that $U(\mathfrak{m})$ is commutative and so identifies with $S(\mathfrak{m})$. Set $Q=\mathrm{Ann}_{U(\mathfrak{m})} L(\lambda)$. By 5.16, Q is a prime ideal and furthermore by 2.3, $L(\lambda)$ is torsion-free over the integral domain $U(\mathfrak{m})/Q$. Since $L(\lambda)=U(\mathfrak{m})V(\lambda)$ we conclude that $L(\lambda)$ has finite rank $r(\lambda) \leqq \dim V(\lambda)$ as a $U(\mathfrak{m})/Q$ module. Furthermore we can choose $0 \neq \bar{x} \in U(\mathfrak{m})/Q$ such that $\bar{X}^{-1} L(\lambda)$ is a free rank $r(\lambda)$ module over $\bar{X}^{-1}(U(\mathfrak{m})/Q)$, where $\bar{X}$ denotes the multiplicative set generated by $\bar{x}$.

6.3. As in 4.4 we let $F(\lambda)$ [resp. $A(\lambda)$] denote the subring of $\mathrm{End}_{\mathbb{C}} L(\lambda)$ on which the diagonal action of $\mathfrak{g}$ (resp. $\mathfrak{n}$) is locally finite. It is immediate that we have embeddings $U(\mathfrak{g})/J(\lambda) \hookrightarrow F(\lambda) \hookrightarrow A(\lambda)$. Since the action of $\mathfrak{k}$ on $L(\lambda)$ is locally finite, the diagonal action of $\mathfrak{k}$ on $\mathrm{End}_{\mathbb{C}} L(\lambda)$ is also locally finite. Consequently $A(\lambda)$ identifies with the subring of $\mathrm{End}_{\mathbb{C}} A(\lambda)$ on which the diagonal action of just $\mathfrak{m}$ is locally finite.

6.4. Choose a preimage $x \in U(\mathfrak{m})$ of $\bar{x}$ and set $X = \{x^k\}_{k \in \mathbb{N}}$. Obviously $\bar{X}^{-1} L(\lambda)$ is isomorphic to $X^{-1} L(\lambda)$ as a $U(\mathfrak{m})$ module. Since $\mathfrak{m}$ is commutative and its diagonal action on $A(\lambda)$ is locally nilpotent, then the diagonal action of x on $A(\lambda)$ is again locally nilpotent and so X is Ore in $A(\lambda)$. This gives $X^{-1} L(\lambda)$ the structure of an $X^{-1} A(\lambda)$ module. Moreover it is clear that we have a commutative square

$$\begin{array}{ccc} U(\mathfrak{g})/J(\lambda) & \hookrightarrow & A(\lambda) \\ \downarrow & & \downarrow \\ X^{-1}(U(\mathfrak{g})/J(\lambda)) & \hookrightarrow & X^{-1} A(\lambda) \end{array}$$

of ring embeddings.

6.5. Identify $\mathfrak{g}$ with its dual $\mathfrak{g}^*$ through the Killing form. Then $\mathfrak{m}^*$ identifies with $\mathfrak{m}^+$. Let $\mathscr{V}$ denote the subvariety of $\mathfrak{m}^+$ of zeros of Q and set $\mathscr{V}_0 = \mathscr{V} \setminus \{x=0\}$. Let $\mathscr{R}$ denote the ring of regular functions on $\mathscr{V}$. Then $X^{-1}\mathscr{R}$ identifies with the ring $\mathscr{R}_0$ of regular functions on $\mathscr{V}_0$. Again let $\mathscr{D}$ (resp. $\mathscr{D}_0$) denote the ring of differential operators on $\mathscr{V}$ (resp. $\mathscr{V}_0$). Then $\mathscr{D}_0$ identifies with $X^{-1}\mathscr{D}$ ([27], 15.1.25). Set $r = r(\lambda)$ and let $M_r(\mathscr{D}_0)$ denote the ring of $r \times r$ matrices over $\mathscr{D}_0$. Let rk denote Goldie rank.

LEMMA.

(i) $X^{-1} A(\lambda) \xrightarrow{\sim} M_r(\mathscr{D}_0)$.

(ii) $X^{-1} A(\lambda)$ *is a simple ring*.

(iii) $\mathrm{rk}(U(\mathfrak{g})/J(\lambda))$ *divides* r. *In particular it is bounded by* $\dim V(\lambda)$.

(i) It is clear that $X^{-1} A(\lambda)$ identifies with the subring of $\mathrm{End}_{\mathbb{C}} X^{-1} L(\lambda)$ on which the diagonal action of $\mathfrak{m}$ is locally finite. Yet $X^{-1} L(\lambda)$ is just $\mathscr{R}_0^r$ as an $\mathscr{R}_0$ module and so this proves (i).

(ii) By 4.6, $\mathscr{D}$ is a simple ring. Hence so are $\mathscr{D}_0$ and $M_r(\mathscr{D}_0)$.

(iii) By [22], 7.11, the embedding $F(\lambda) \hookrightarrow A(\lambda)$ localizes to an isomorphism of rings of fractions. Hence $\mathrm{rk}\, F(\lambda) = \mathrm{rk}\, A(\lambda) = r$. Then (iii) results from [19], I. 5.12(iii).

Remark. — In 8.8 we give an example of $\mathrm{rk}(U(\mathfrak{g})/J(\lambda)) = 1$, when $\dim V(\lambda) > 1$.

6.6. We need the following fact which holds for any simple highest weight module $L(\mu)$. Recall ([16], 6.31) that $F(\mu)$ has finite length as a $U(\mathfrak{g})$ bimodule.

LEMMA. — *The socle* $\mathrm{Soc}\, F(\mu)$ *of* $F(\mu)$ *as a* $U(\mathfrak{g})$ *bimodule is an ideal of* $F(\mu)$ *considered as a ring*.

Set $F = F(\mu)$, $S = \mathrm{Soc}\, F(\mu)$. Let J denote the annihilator of F/S considered as a left $U(\mathfrak{g})$ module. By definition $JF \subset S$. Yet JF is a $U(\mathfrak{g})$ bisubmodule of S and hence a direct summand of S. Suppose $JF \subsetneq S$. Then J annihilates a non-zero direct summand of S. This is excluded by [16], 10.9, 10.12, concerning Gelfand-Kirillov dimension $d(\ )$ and the remarkable fact that by [21], II, 4.13, and the truth of the Kazhdan-Lusztig conjectures one has $d(F/S) < d(F)$. Hence $JF = S$. Then $SF = JFF = JF = S$. Similarly $FS = S$.

6.7. Recall that we are assuming $L(\lambda)$ to be unitary.

PROPOSITION. — *Suppose* $Q = \operatorname{Ann}_{U(\mathfrak{m})} L(\lambda) \neq 0$. *Then* $X^{-1} J(\lambda)$ *is a maximal ideal of* $X^{-1} U(\mathfrak{g})$.

The hypothesis $Q \neq 0$ implies by 2.3 and 5.15 that $Q = Q_i$ for some $i \leqq t$. Then $\mathscr{V} = \mathscr{V}_i$ and as discussed in the proof of 4.5 we may apply ([22], 5.8) to conclude that $F(\lambda) = A(\lambda)$. Then by 6.5, $X^{-1} F(\lambda)$ is a simple ring and so by 6.6 we have $X^{-1} F(\lambda)/X^{-1} \operatorname{Soc} F(\lambda) = X^{-1} (F(\lambda)/\operatorname{Soc} F(\lambda)) = 0$.

We conclude that $X^{-1} F(\lambda)$ is semisimple as an $X^{-1}(U(\mathfrak{g})/J(\lambda))$ bimodule. It contains the latter as an indecomposable direct summand and hence as a simple bisubmodule. However the latter conclusion is just what is required for the assertion of the proposition.

6.8. We may combine 6.7 with the analysis of 4.2 to give a simple combinatorial condition for $J(\lambda)$ to be maximal. Let ρ_i denote the half sum of the roots of $\Delta^i \cap \Delta^+$.

THEOREM. — *Assume* $L(\lambda)$ *unitary and* $Q := \operatorname{Ann}_{U(\mathfrak{m})} L(\lambda) \neq 0$ (*so then* $Q = Q_i$ *for some* $i \in \{1, 2, \ldots, t\}$). *Suppose* $J(\lambda)$ *is not maximal. Then there exists* j, $1 \leqq j \leqq i-1$ *such that* $w_c^j w_c^1 \lambda + (j-1)\varepsilon_{\mathfrak{g}, \alpha}\omega + \rho_j$ *is regular, integral for* Δ^j.

Let J be a maximal ideal of $U(\mathfrak{g})$ properly containing $J(\lambda)$. Then $J \cap X \neq \varnothing$ by 6.5. Suppose $x^l \in J$ and set $Z = U(\mathfrak{k}).x^l \subset U(\mathfrak{m})$. Obviously $Z \subset J$ and in particular $Z^{n_c} \subset J$. By the commutativity of $\mathfrak{m}$ one has $Z^n = Z^{n_c}$. By EJ, 2.1, the weight vectors of Z^n are products of the v_k, $k \in \{1, 2, \ldots, t\}$. If every such product has a factor of v_k with $k \leqq i$ we conclude by EJ, 8.1, that $Z^n \subset Q_i$ and so $x^l \in Z \subset Q_i$ which contradicts that x^l has a non-zero image in $U(\mathfrak{m})/Q_i$. This proves that there exists j, $1 \leqq j \leqq i-1$ such that $J \cap Y_j = \varnothing$ and $J \cap Y_{j+1} \neq \varnothing$.

Since $j \leqq i$ we have an embedding $L(\lambda) \hookrightarrow Y_j^{-1} L(\lambda)$. Set $\tilde{\mathfrak{g}} = \Theta^j(\mathfrak{g}^j)$. By the repeated application of the construction of Section 5 we obtain a highest weight $U(\tilde{\mathfrak{g}})$ submodule L_j of $Y_j^{-1} L(\lambda)$ of highest weight λ_j. Noting the combinatorial fact that $\varepsilon_{\mathfrak{g}^k, \alpha} = \varepsilon_{\mathfrak{g}, \alpha}$, $\forall k = 1, 2, \ldots, t$ which can for example be checked using EJ, Table, and that $w_c^k w_c^{k-1} (w_c^{k-1} w_c^1 \lambda + (k-2)\varepsilon_{\mathfrak{g}, \alpha}\omega) + \varepsilon_{\mathfrak{g}, \alpha}\omega = w_c^k w_c^1 \lambda + (k-1)\varepsilon_{\mathfrak{g}, \alpha}\omega$, we conclude from 5.3 that $\lambda_j = w_c^j w_c^1 \lambda + (j-1)\varepsilon_{\mathfrak{g}, \alpha}\omega$ on Δ^j. Set $\tilde{J} = Y_j^{-1} J \cap U(\tilde{\mathfrak{g}})$. Since $J \supset J(\lambda) = \operatorname{Ann} L(\lambda)$ and because Y_j is Ore in $U(\mathfrak{g})$ we obtain $\tilde{J} L_j = 0$. Yet by 3.4 $\tilde{x}_{-\beta_j} := \Theta^j(x_{-\beta_j}) = v_{j-1}^{-1} v_j$ and so $\tilde{x}^l_{-\beta_j} \in \tilde{J}$ for some $l \in \mathbb{N}$ by the hypothesis on j. By Borho's lemma ([18], 6.11) $\tilde{J}$ has finite codimension in $U(\tilde{\mathfrak{g}})$. Since $\tilde{J} L_j = 0$, this forces $\lambda_j + \rho_j$ to be integral, regular for Δ^j. Hence the theorem.

6.9. One can ask if the converse to 6.8 holds. For $j = 1$ the criterion is just that $\lambda + \rho$ be regular, integral. Since $i \geqq 2$, we cannot have $\lambda = 0$. One has $(\lambda + \rho, \alpha) < 1$ for the non-compact simple root α at the last place of unitarity and hence for all $L(\lambda)$ for otherwise by EJ, 7.9, one should have dim $L(\lambda)<\infty$ which implies $\lambda=0$. Since λ is assumed integral this forces $(\lambda+\rho, \alpha)\leqq 0$. Consequently $\lambda+\rho$ is not dominant. Then $(\lambda+\rho)$ being regular implies that $J(\lambda)$ is not maximal. We shall eventually obtain non maximal $J(\lambda)$ satisfying the hypotheses of the theorem in this fashion (Sect. 8). In such cases $L(\lambda)$ is *not* free over $U(\mathfrak{g})/Q$ because this would then contradict 6.7.

6.10. There are two difficulties in extending 6.9 to the case $j>1$. The first is a combinatorial result which is rather strange.

LEMMA. — *Take* $j\in\{1,2,\ldots,t\}$. *Then*

$$w_c^j w_c^1 \rho_1 - \rho_j = 2(j-1)\,\varepsilon_{\mathfrak{g},\alpha}\,\omega$$

on Δ^j.

It is obvious that $w_c^1 w_c^j$ sends a simple root of Δ_c^j to a simple root of Δ_c^1 and so the left hand side restricts to zero on Δ_c^j. It remains to show that both sides agree on α. Observe that $w_c^1 w_c^j \beta_j = w_c^1 \alpha = \beta_1$. Recall that α, β_j are both long roots. Then by the first result $(\alpha^\vee, w_c^j w_c^1 \rho_1 - \rho_j) = (\beta_j^\vee, w_c^j w_c^1 \rho_1 - \rho_j) = (\beta_1^\vee - \beta_j^\vee, \rho)$. By EJ, 3.6, one has $\varepsilon_{\mathfrak{g}^k,\alpha} = (1/2)(\beta_k^\vee - \beta_{k+1}^\vee, \rho)$ and we already remarked in 6.8 that they are all equal. We conclude that $(\beta_1^\vee - \beta_j^\vee, \rho) = 2(j-1)\,\varepsilon_{\mathfrak{g},\alpha}$ as required.

Remark. — The consequence of this unfortunate fact is the following. Let λ_j denote the highest weight of the $\tilde{\mathfrak{g}} := \Theta^j(\mathfrak{g}^j)$ module L_j considered in the proof of 6.8. Let L_j' denote a second such module obtained from some $L(\lambda')$. Then $\lambda_j + \rho_j = w_c^j w_c^1 (\lambda+\rho) - (j-1)\,\varepsilon_{\mathfrak{g},\alpha}\,\omega$, whereas had it not been for the presence of the factor of 2 above the second term would not have appeared. This in turn would have meant that if $\text{Ann}_{Z(\mathfrak{g})} L(\lambda) = \text{Ann}_{Z(\mathfrak{g})} L(\lambda')$, then $\text{Ann}_{Z(\tilde{\mathfrak{g}})} L_j = \text{Ann}_{Z(\tilde{\mathfrak{g}})} L_j'$. In fact this pleasant conclusion does not necessarily hold. Perhaps this is because Section 5 ignored the contribution of the opposite copy $\sigma(\mathfrak{a})$ of $\mathfrak{a}$. In any case the latter leads to the second of our difficulties noted below.

6.11. As in Section 5 we let $N(\mu)$ denote the module induced from a finite dimensional simple $\mathfrak{p}$ module $V(\mu)$. Let $L(\mu)$ be the unique simple quotient of $N(\mu)$. Set $L_1 = L(\mu)$ and let j be the largest integer $\leqq t+1$ such that L_1 is Y_j torsion-free. Let L_j be the $\tilde{\mathfrak{g}} := \Theta^j(\mathfrak{g}^j)$ submodule of $Y_j^{-1} L_1$ obtained by a repeated application of the construction in Section 5. Our second difficulty in proving the converse to 6.8 comes from not knowing if the following holds.

$(\mathscr{C}_0)$ L_j *is a simple* $\tilde{\mathfrak{g}}$ *module.*

We let $\mathscr{C}$ denote the corresponding question when we further impose that $\text{Ann}_{U(\mathfrak{m})} L(\mu) \neq 0$. By 5.15, $\mathscr{C}$ holds for unitary modules, since a unitary highest weight module is necessarily simple.

6.12. We first need the following fact which holds for any simple highest weight module $L(\mu)$. Let $d_A(M)$ denote the Gelfand-Kirillov dimension of a module M over a $\mathbb{C}$-algebra A. Let L be a non-zero $U(\mathfrak{n})$ submodule of $L(\mu)$ and set $d = d_{U(\mathfrak{n})}$.

LEMMA. — *One has* $d(L)=d(L(\mu))$.

We can assume without loss of generality that L is cyclic, say $L=U(\mathfrak{n})f$. The Borel subalgebra $\mathfrak{b}=\mathfrak{n}^+\oplus\mathfrak{h}$ acts locally finitely on $L(\mu)$. Thus $U(\mathfrak{b})f$ is finite dimensional and so by Lie's theorem admits a one-dimensional submodule $\mathbb{C}e$. Obviously $\mathbb{C}e$ is the unique highest weight space of the simple module $L(\mu)$ and so $U(\mathfrak{n})e=L(\mu)$. Again we can choose a finite dimensional subspace V_0 of $U(\mathfrak{b})$ such that $V_0 f=U(\mathfrak{b})f$. Set $V=U(\operatorname{ad}\mathfrak{n})V_0\subset U(\mathfrak{g})$ which is again finite dimensional. Since V is $\operatorname{ad}\mathfrak{n}$ stable we obtain $L(\mu)=U(\mathfrak{n})e=U(\mathfrak{n})Vf=VU(\mathfrak{n})f=VL$. Thus $L(\mu)$ is an image of $V\otimes L$ viewed as a $U(\mathfrak{n})$ module for diagonal action. Then $d(L(\mu))\leqq d(V\otimes L)=d(L)\leqq d(L(\mu))$, as required.

6.13. Now consider the situation described in 6.10 and set $J_j=Y_j^{-1}J(\mu)\cap U(\tilde{\mathfrak{g}})$. It is clear that $J_j L_j=0$; but it is not obvious if $J_j=\operatorname{Ann}L_j$. For any $\mathbb{C}$-algebra A, set $d(A)=d_A(A)$ and recall $d(U(\mathfrak{g})/\operatorname{Ann}L(\mu))=2\,d_{U(\mathfrak{g})}(L(\mu))=2\,d_{U(\mathfrak{n})}(L(\mu))$ by [16], 10.9.

LEMMA.

(i) $d(U(\tilde{\mathfrak{g}})/J_j)=d(U(\tilde{\mathfrak{g}})/\operatorname{Ann}L_j)$.

(ii) $d(U(\mathfrak{g})/J(\mu))=2\left(\sum_{i=1}^{j-1}|\Gamma_n^i|\right)+d(U(\tilde{\mathfrak{g}})/\operatorname{Ann}L_j)$.

Let A be the localized Weyl algebra $Y_j^{-1}U(\mathfrak{c}^j)$. Set $K=Y_j^{-1}U(\mathfrak{a}_n^j)$ which can be considered both as a ring and a standard A module. The construction of Section 5 gives a ring embedding $B:=A\otimes U(\tilde{\mathfrak{g}})/J_i\subsetneq Y_j^{-1}(U(\mathfrak{g})/J(\mu))$ and a B module embedding $K\otimes L_j\subsetneq Y_j^{-1}L(\mu)$. The first inclusion gives us

$$\begin{aligned}d(A)+d(U(\tilde{\mathfrak{g}})/J_j)&\leqq d(Y_j^{-1}(U(\mathfrak{g})/J(\mu))\\&=d(U(\mathfrak{g})/J(\mu)),\quad\text{by [4], 6.1}\\&=2\,d_{U(\mathfrak{g})}(L(\mu)).\end{aligned}$$

Now

$$d(A)=2\,d_A(K)=2\dim\mathfrak{a}_n^j=2\sum_{i=1}^{j-1}|\Gamma_n^i|$$

whereas by [16], 10.9

$$d(U(\tilde{\mathfrak{g}})/J_j)\geqq d(U(\tilde{\mathfrak{g}})/\operatorname{Ann}L_j)=2\,d(L_j).$$

Yet $L':=L(\mu)\cap(K\otimes L_j)$ is a non-zero $U(\mathfrak{n})$ submodule of $L(\mu)$ and so by 6.12 we obtain

$$d_{U(\mathfrak{g})}(L(\mu))=d_{U(\mathfrak{n})}(L(\mu))=d_{U(\mathfrak{n})}(L')\leqq d_{U(\mathfrak{n})}(K\otimes L_j)\leqq d_A(K)+d(L_j).$$

Hence

$$\begin{aligned}d(A)+d(U(\tilde{\mathfrak{g}})/J_j)&\leqq 2(d_A(K)+d(L_j))\\&\leqq d(A)+d(U(\tilde{\mathfrak{g}})/J_j).\end{aligned}$$

This forces all the above inequalities to be equalities and then inspection verifies (i), (ii).

6.14. Let $L(\mu)$ denote a simple but not necessarily unitary quotient of the induced module $N(\mu)$. Set $\varepsilon=\varepsilon_{\mathfrak{g},\alpha}$. Suppose for some j, $1\leqq j\leqq t$ that $yw_c^j w_c^1(\mu+\rho)-(j-1)\varepsilon\omega$ is dominant, regular and integral on Δ^j, for some $y\in W$ such that $w_c^1 w_c^j y w_c^j w_c^1(\mu+\rho)-\rho$ is $\mathfrak{k}$ dominant.

PROPOSITION. — *Let j be the least positive integer with the above property. Assume that $\mu_i+\rho_i:=w_c^i w_c^1(\mu+\rho)-(i-1)\varepsilon\omega$ is not dominant, regular and integral on Δ^i for all i, $1\leqq i\leqq j$. Then if $\mathscr{C}_0$ holds, $J(\mu)$ is not maximal.*

Let i be the largest positive integer $\leqq j$ such that $L(\mu)$ is Y_i torsion-free. Then we can define $L_i\subset Y_i^{-1}L(\mu)$ as in 6.10. By our calculation in the remark following 6.9, we find that L_i has highest weight μ_i. Thus the second hypothesis just means that L_i is not $\Theta^i(x_{-\beta_i})$ torsion and so $L(\mu)$ is not Y_{i+1} torsion. We conclude that L_j is defined and has no $\Theta^j(x_{-\beta_j})$ torsion. In particular L_j is not finite dimensional.

Set $x=w_c^j w_c^1$ and $\mu'=x^{-1}yx(\mu+\rho)-\rho$. Then by the hypothesis on y the construction of Section 5 applies to $L(\mu')$. Suppose $L(\mu_i)$ has Y_i torsion for some positive integer $i\leqq j$ and let i be the least integer with this property. Then $L'_i\subset Y_{i-1}^{-1}L(\mu')$ constructed as above [but with respect to $L(\mu')$] has $\Theta^i(x_{-\beta_i})$ torsion and so is finite dimensional. Applying 6.13 to this and our previous assertion we obtain $d(U(\mathfrak{g})/J(\mu))>d(U(\mathfrak{g})/J(\mu'))$. Since $J(\mu)\cap Z(\mathfrak{g})=J(\mu')\cap Z(\mathfrak{g})\in \operatorname{Max} Z(\mathfrak{g})$ we conclude from say [16], 5.21, that $J(\mu)$ is not maximal [but not necessarily contained in $J(\mu')$].

Now assume that $L(\mu')$ has no Y_j torsion. Then L'_j is defined and has highest weight $x\mu'+(j-1)\varepsilon\omega=yw_c^j w_c^1(\mu+\rho)-(j-1)\varepsilon\omega-\rho_j$. Thus the first hypothesis exactly means that L'_j has a finite dimensional quotient. Now given the $\mathscr{C}_0$ holds we obtain that L'_j itself is finite dimensional. Finally we apply 6.13 as above to obtain the required conclusion.

6.15. Even admitting $\mathscr{C}_0$, the above result is not a precise converse to 6.8. We remark that if $\operatorname{Ann}_{U(\mathfrak{m})}L(\mu)\neq 0$, then we need only assume $\mathscr{C}$ holds. This is not quite trivial, but follows by the reasoning in 5.16. Finally suppose $\operatorname{Ann}_{U(\mathfrak{m})}L(\lambda)=0$. Then even for $L(\lambda)$ unitary one would expect that $J(\lambda)$ could fail to be maximal without the conclusion of 6.8 being satisfied.

6.16. One may ask if one can have a strict inclusion $J(\lambda)\supsetneq J(\mu)$ with both $L(\lambda)$, $L(\mu)$ unitary. The above methods essentially reduce such questions to the case when $\lambda=0$ and so $\mu=y^{-1}\rho-\rho$ for some $y\in W$. Notice however that $y^{-1}\rho-\rho$ *can* be a unitary parameter. Indeed write $\mu:=y^{-1}\rho-\rho=\tau+u\omega$ in the conventions of EJ, Sect. 3. Then $u=(\beta^\vee, y^{-1}\rho)-(\beta^\vee, \tau+\rho)$, whereas by EJ, 4.1, we require $u=1+2(\beta^\vee, \rho_\tau)-(\beta^\vee, \tau+\rho)$ for μ to be a (last) place of unitarity. Now take $\mathfrak{g}$ simple of type A_n and $y=s_{\alpha_1}s_{\alpha_2}\ldots s_{\alpha_t}$ where α_t is the non-compact simple root. Then $y\alpha_i=\alpha_{i+1}$, $i<t$, $y\alpha_{t+1}=\alpha_1+\alpha_2+\ldots+\alpha_{t+1}$, $y\alpha_j=\alpha_j$, $j>t+1$. Thus τ is a multiple of ω_{t+1} and so $2(\beta^\vee, \rho_\tau)=n-2$. Again $(\beta^\vee, \rho-y^{-1}\rho)=(\beta^\vee, \alpha_1+\alpha_2+\ldots+\alpha_t)=1$, so $(\beta^\vee, y^{-1}\rho)=n-1$. Thus the required identity is satisfied.

7. Computation of varieties

7.1. Let $L(\lambda)$ be a unitary highest weight module. In this section we compute the associated variety $\mathscr{V}(L(\lambda))$ of $L(\lambda)$. Here it is perhaps helpful to recall some definitions. If L is a $U(\mathfrak{g})$ module generated by a finite dimensional subspace L^0, then we let $\mathscr{V}(L^0)$ denote the subvariety of $\mathfrak{g}^*$ of zeros of the graded ideal $\operatorname{gr}\operatorname{Ann}_{U(\mathfrak{g})} L^0$. By an old result of I. N. Bernstein ([16], 17.2) this is independent of the choice of L^0 and so we may define the associated variety $\mathscr{V}(L)$ of L to be $\mathscr{V}(L^0)$. Identify $\mathfrak{g}$ with $\mathfrak{g}^*$ through the Killing form. If L is the image of a module induced from a finite dimensional module of a parabolic subalgebra $\mathfrak{p}=\mathfrak{k}\oplus\mathfrak{m}^+$, then it is easy to see that $\mathscr{V}(L)$ is a closed $\mathfrak{k}$ stable subvariety of $\mathfrak{m}^+$. In general it is false that $\mathscr{V}(L)$ is irreducible even for L simple ([23], 10.1 and [24] note added in proof). However in our present situation this holds by EJ, 8.1, and 2.3 which in the notation of 3.5 implies that $\mathscr{V}(L(\lambda))=\mathscr{V}_i$ for some $i=\{1,2,\ldots,t+1\}$. Our aim is to calculate i as a function of λ. By 2.4 and 5.16 it is sufficient to do this when λ is at a last place of unitarity, that is when $\lambda=\tau+u_1^\tau\omega$ in the notation of EJ, 1.6.

7.2. Set $L_1=L(\lambda_1)$ with $\lambda_1=\tau_1+u_1^\tau\omega$, $\tau_1\in P_c^+$ and $u_1^{\tau_1}$ as given by EJ, 7.1. Assume $\lambda\neq 0$, so that L_1 has no Y_2 torsion, equivalently that $\tau_1\neq 0$. Set $\tilde{\mathfrak{g}}=\Theta^2(\mathfrak{g}^2)$ and let L_2 be the highest weight $\tilde{\mathfrak{g}}$ module constructed in Section 5. Then L_2 has highest weight λ_2 which equals $w_c^2 w_c^1\lambda+\varepsilon_{\mathfrak{g},\alpha}\omega$ on Δ^2. Clearly we can write

$$\lambda_2=\tau_2+u\omega,\ u\in\mathbb{R}$$

where τ_2 is a dominant $\mathfrak{k}^2$ weight which we can choose so that $(\tau_2,\alpha)=0$. We show that if s_2 denotes the level of τ_2 then either $u<u_{s_2}^{\tau_2}$ or $u=u_i^{\tau_2}$ for some positive integer $i\leqq s_2$. (Both situations can arise.) To compute u it is enough to compare it with $u_1^{\tau_2}$. Here we set $\varepsilon=\varepsilon_{\mathfrak{g},\alpha}$ which we recall also equals $\varepsilon_{\tilde{\mathfrak{g}},\alpha}$. One has the

LEMMA (Notation, EJ, 3.4, 4.2.)

(i) $\tau_2=w_c^2 w_c^1\tau_1-r\omega$, $r=(\alpha^\vee, w_c^2 w_c^1\tau_1)$.

(ii) $u-u_1^{\tau_2}=(1/2)(|S_{1,\tau_1}|-|S_{1,\tau_2}|)+2(\rho_{\tau_1},\beta_1^\vee)-2(\rho_{\tau_2},\beta_2^\vee)-\varepsilon$.

(i) is immediate [noting that r is chosen so that $(\tau_2,\alpha)=0$]. For (ii) we recall from (EJ, 4.2) that

$$u_1^{\tau_1}=1+\frac{1}{2}|S_{1,\tau_1}|+2(\rho_{\tau_1},\beta_1^\vee)-(\rho_1+\tau_1,\beta_1^\vee)$$

$$u_1^{\tau_2}=1+\frac{1}{2}|S_{1,\tau_2}|+2(\rho_{\tau_2},\beta_2^\vee)-(\rho_2+\tau_2,\beta_2^\vee).$$

By definition $u=\varepsilon+u_1^{\tau_1}+r$ and so $u-u_1^{\tau_2}=\varepsilon+u_1^{\tau_1}-(u_1^{\tau_2}-r)$. Hence (ii) results from the above if we note that $(\tau_2,\beta_2^\vee)+r=(w_c^2 w_c^1\tau_1,\beta_2^\vee)=(w_c^1\tau_1,\alpha^\vee)=(\tau_1,\beta_1^\vee)$ and $(\rho_1,\beta_1^\vee)-(\rho_2,\beta_2^\vee)=(\rho_1,\beta_1^\vee-\beta_2^\vee)=2\varepsilon$ by EJ, 3.6.

7.3. Define α_i, α_i', $i=1,2,\ldots,t$ as in EJ, 2.1. By [18], 2.2 (iv), one has $2(\beta_1^\vee,\alpha_1)=2(\beta_1^\vee,\alpha_1')=1$ whilst β_1 vanishes on the remaining simple roots. This leads to the following simple rule. Let c_i, c'_i, $i=1, 2$ denote the coefficients of α_i, α'_i in $2\rho_{\tau_i}$ expanded in terms of the simple roots of Δ^i. Then

LEMMA.

$$2(\rho_{\tau_1}, \beta_1^\vee)-2(\rho_{\tau_2}, \beta_2^\vee)=(c_1+c'_1)-(c_2+c'_2).$$

7.4. We compute the right hand side of 7.2 (ii) for each simple Lie algebra $\mathfrak{g}$ and each choice of non-compact simple root α. First assume $\mathfrak{g}$ of type A_l. Adopt the Bourbaki notation ([5], *Pl. I*) and take $\alpha=\alpha_t$ where we can assume $2t\leqq n+1$ and $t>1$. (This assures that t is as defined in EJ, 1.4.) One has $S_{1,\tau}=\varnothing$ and $\varepsilon=1$ in this case. Set

$$\pi^l=\{\alpha_1, \alpha_2, \ldots, \alpha_{t-1}\}, \qquad \pi^r=\{\alpha_{t+1}, \ldots, \alpha_l\}.$$

Obviously $\pi_c=\pi^l\cup\pi^r$. Given $\tau_1\in P_c^+$ we set $\operatorname{Supp}\tau_1=\{\gamma\in\pi_c\,|\,(\tau, \gamma)\neq\varnothing\}$ and define $\tau_2\in P_{2,c}^+$ by 7.2 (i).

LEMMA. — *Assume* $\mathfrak{g}$ *of type* A_n *and* $\alpha=\alpha_t$ *(as above). Then for all* $0\neq\tau_1\in P_c^+$ *one has*

$$u-u_1^{\tau_2}=\begin{cases} -\varepsilon_{\mathfrak{g},\alpha}, & (\operatorname{Supp}\tau_1)\cap\pi^l\neq\varnothing,\quad (\operatorname{Supp}\tau_1)\cap\pi^r\neq\varnothing\\ 0, & \textit{otherwise}.\end{cases}$$

One easily checks for all $j\in\{1, 2, \ldots, l\}\setminus\{t\}$ that

$$-w_c^1\alpha_j=\begin{cases}\alpha_{t-j}, & j\leqq t-1,\\ \alpha_{l+t+1-j}, & j\geqq t+1.\end{cases}$$

Combined with a similar result for $w_c^2\alpha_j$ in the A_{l-2} system $\{\alpha_2, \ldots, \alpha_{l-1}\}$ this gives

$$(*)\qquad w_c^1 w_c^2\alpha_j=\begin{cases}\alpha_{j-1}, & 1<j<t,\\ \alpha_{j+1}, & t<j<l.\end{cases}$$

Now let ω_j^1, $j=1, 2, \ldots, l$ denote the fundamental weights corresponding to π^1 and ω_j^2, $j=2, 3, \ldots, l-1$ the fundamental weights corresponding to π^2. Then from $(*)$ we obtain

$$w_c^2 w_c^1\omega_j^1=\begin{cases}\omega_{j+1}^2, & 1\leqq j<t-1.\\ \omega_{j-1}^2, & t+1<j\leqq n.\\ 0, & j=t-1 \quad\text{or}\quad t+1.\end{cases}$$

computed on Δ_c^2. This allows us to compute τ_2 from τ_1. If we view τ_1 as given by a Dynkin diagram weighted by the coefficients of ω_j^1, then τ_2 is obtained from τ_1 by deleting the extreme vertices and letting the weights move by one step towards α_t. It easily follows from this in the notation of 7.3 that

$$c_1-c_2=\begin{cases}1, & \pi^l\cap\operatorname{Supp}\tau_1=\varnothing.\\ 0, & \textit{otherwise}.\end{cases}$$

with a similar expression for the primed quantities but replacing π^l by π^r. Then by 7.3 we obtain the assertion of the lemma.

7.5. Assume $\mathfrak{g}$ simple of type B_l. In the Bourbaki notation ([5], *Pl. II*) we can only have $\alpha=\alpha_1$. Again α_2 is the unique simple root not orthogonal to β_1. Furthermore $\Delta^2=\{\pm\alpha\}$ and so we always have $\tau_2=0$. From EJ, Table, we obtain $\varepsilon=l-3/2$. Let $k\in\{2,\ldots,l\}$ be the smallest integer such that $\alpha_k\in\operatorname{Supp}\tau_1$ (recall $\tau_1\neq 0$). Then the connected component of Δ_{τ_1} containing α_2 is the A_{k-2} system $\{\alpha_2,\ldots,\alpha_{k-1}\}$ and so by 7.3 we obtain $2(\rho_{\tau_1},\beta_1^\vee)-2(\rho_{\tau_2},\beta_2^\vee)=k-2$.

Finally $S_{1,\tau_1}\neq\varnothing$ exactly when $\tau_1=\omega_l$ and then $|S_{1,\tau_1}|=1$ and $k=l$ above. Putting all this together we obtain the

LEMMA. — *Assume* $\mathfrak{g}$ *of type* B_l *and* $\alpha=\alpha_1$. *For all* $0\neq\tau_1\in P_c^+$ *one has*

(i) $\tau_2=0$.

(ii) $u=\begin{cases} l-k-(1/2)<0, & \tau_1\neq\omega_l \\ 0, & \tau_1=\omega_l. \end{cases}$

7.6. Assume $\mathfrak{g}$ simple of type C_l. In the Bourbaki notation ([5], *Pl. III*) we can only have $\alpha=\alpha_n$. Again α_1 is the unique simple root not orthogonal to β_1. Let $k=\{1,2,\ldots,l-1\}$ be the smallest integer such that $\alpha_k\in\operatorname{Supp}\tau_1$. Then the connected component of Δ_{τ_1} containing α_1 is the A_{k-1} system $\{\alpha_1,\alpha_2,\ldots,\alpha_{k-1}\}$. Now Δ_c^1 (resp. Δ_c^2) is the A_{l-1} (resp. A_{l-2}) system $\{\alpha_1,\alpha_2,\ldots,\alpha_{l-1}\}$ (resp. $\{\alpha_2,\alpha_3,\ldots,\alpha_{l-1}\}$). Exactly as in 7.4 this gives

$$(*)\qquad w_c^1\,w_c^2\,\alpha_j=\alpha_{j-1},\qquad \forall j,\quad 1<j<l.$$

Define ω_j^i, $i=1,2;\ j\in\{i,i+1,\ldots,l-1\}$ as in 7.4. Then from $(*)$ we obtain

$$w_c^2\,w_c^1\,\omega_j^1=\begin{cases}\omega_{j+1}^2, & i\leqq j<l-1.\\ 0, & j=l-1.\end{cases}$$

on Δ^2. View τ_1, τ_2 as weighted Dynkin diagrams. Then τ_2 is obtained from τ_1 by removing the left hand vertex and letting the weights move one step towards α_l. It easily follows that the connected component of Δ_{τ_2} containing α_2 is the A_{k-1} system $\{\alpha_2,\alpha_3,\ldots,\alpha_k\}$. From 7.3, we conclude that

$$2(\rho_{\tau_1},\beta_1^\vee)-2(\rho_{\tau_2},\beta_2^\vee)=0.$$

Now assume that τ_1 is not generic in the sense of EJ, 3.4. Then as noted in EJ, 4.3, either $\tau_1=\omega_m$ for some positive integer $m<l$ (in this case set $n=l$) or there exists n, $m\leqq n<l$ such that

$$\tau_1=\omega_m+\omega_n+\sum_{i=n}^{l-1}r_i\,\omega_i,\ r_i\in\mathbb{N}.$$

Moreover

$$|S_{1,\tau_1}|=n-m.$$

We conclude that

$$|S_{1,\tau_1}|-|S_{1,\tau_2}|=\begin{cases} 1, & \tau_1=\omega_m,\quad 1\leqq m<l. \\ 0, & otherwise. \end{cases}$$

Finally from EJ, Table, we obtain $\varepsilon=1/2$. Putting all this together we obtain the

LEMMA. — *Assume* $\mathfrak{g}$ *of type* C_l *and* $\alpha=\alpha_l$. *Then for all* $0\neq\tau_1\in P_c^+$ *one has*

$$u-u_1^{\tau_2}=\begin{cases} 0, & \tau_1=\omega_m,\quad 1\leqq m<l. \\ -\varepsilon_{\mathfrak{g},\alpha}, & otherwise. \end{cases}$$

7.7. Assume $\mathfrak{g}$ of type D_l with $\alpha=\alpha_1$. This case is very similar to type B_l. We get $\tau_2=0$ and $\varepsilon=l-2$. We can choose $k\in\{2,\ldots,l\}$ so that the connected component of Δ_{τ_1} containing α_2 is a system of type A_{k-2}. Then $2(\rho_{\tau_1},\beta_1^\vee)-2(\rho_{\tau_2},\beta_2^\vee)=k-2$. Consequently

LEMMA. — *Assume* $\mathfrak{g}$ *of type* D_l *with* $\alpha=\alpha_1$. *For all* $0\neq\tau_1\in P_c^+$ *one has*

(i) $\tau_2=0$.

(ii) $u=k-l$.

In particular $u<0$ *unless* Supp $\tau_1=\{\alpha_{l-1}\}$ *or* $\{\alpha_l\}$.

7.8. Assume $\mathfrak{g}$ of type D_l with $\alpha=\alpha_{l-1}$ or α_l. These cases are equivalent so we shall assume $\alpha=\alpha_l$. Again α_2 is the unique simple root orthogonal to β_1. As in 7.4 one checks that

$$(*)\qquad w_c^1 w_c^2\alpha_j=\alpha_{j-2},\qquad \forall j,\quad 2<j<l-1.$$

Define ω_j^i, $i=1,2$; $j\in\{2i-1, 2i,\ldots,l-1\}$ as in 7.4. Then (*) gives

$$w_c^2 w_c^1\omega_j^1=\begin{cases} \omega_{j+2}^2, & 1\leqq j\leqq l-3. \\ 0, & j=l-2,\ l-1. \end{cases}$$

on Δ^2. Thus τ_2 is obtained from τ_1 as weighted Dynkin diagrams by removing the vertices at α_1, α_2 and moving weights by *two* steps towards α_{l-1}.

If Supp $\tau_1=\{\alpha_1\}$ set $k=l$. Otherwise let $k\in\{2,\ldots,l-1\}$ be the smallest integer such that $\alpha_k\in$ Supp τ_1. We must distinguish four cases

1) $\alpha_1\in$ Supp τ_1, $k<l-1$.

In this case Δ_{τ_1} (resp. Δ_{τ_2}) is the A_{k-2} system $\{\alpha_2,\ldots,\alpha_{k-2}\}$ (resp. $\{\alpha_4,\ldots,\alpha_{k+1}\}$). Then by 7.3 we obtain

$$(*)\qquad 2(\rho_{\tau_1},\beta_1^\vee)-2(\rho_{\tau_2},\beta_2^\vee)=0.$$

2) $\alpha_1\notin$ Supp τ_1, $k<l-1$.

Here the only difference is we adjoin α_1 (resp. α_3) to the above description of Δ_{τ_1} (resp. Δ_{τ_2}). Thus (*) also holds in this case.

3) $\alpha_1 \in \operatorname{Supp} \tau_1$, $k=l-1$ or l.

In this case Δ_{τ_1} (resp. Δ_{τ_2}) is the A_{k-2} (resp. A_{l-4}) system $\{\alpha_2, \ldots, \alpha_{k-1}\}$ (resp. $\{\alpha_4, \ldots, \alpha_{l-1}\}$). Then by 7.3 recalling that $(\alpha_2, \beta_1) \neq 0$ we obtain

$$(**) \qquad 2(\rho_{\tau_1}, \beta_1^\vee) - 2(\rho_{\tau_2}, \beta_2^\vee) = (k-2)-(l-4) = 2-l+k.$$

4) $\alpha_1 \notin \operatorname{Supp} \tau_2$, $k=l-1$.

In this case Δ_{τ_1} (resp. Δ_{τ_2}) is the A_{l-2} (resp. A_{l-3}) system $\{\alpha_1, \alpha_2, \ldots, \alpha_{l-2}\}$ (resp. $\{\alpha_3, \alpha_4, \ldots, \alpha_{l-1}\}$. Then by 7.3 recalling that $(\alpha_2, \beta_1) \neq 0$ we obtain

$$(***) \qquad 2(\rho_{\tau_1}, \beta_1^\vee) - 2(\rho_{\tau_2}, \beta_2^\vee) = 2(l-3)-2(l-4) = 2.$$

Finally $\varepsilon = 2$. Summarizing we obtain the

LEMMA. — *Suppose* $\mathfrak{g}$ *of type* D_l *with* $\alpha = \alpha_l$. *Then for all* $0 \neq \tau_1 \in P_c^+$ *one has*

$$u - u_1^{\tau_2} = \begin{cases} 0, & \operatorname{Supp} \tau_1 = \{\alpha_1\} \text{ or } \{\alpha_{l-1}\}. \\ -\dfrac{1}{2}\varepsilon_{\mathfrak{g},\alpha}, & \operatorname{Supp} \tau_1 = \{\alpha_1, \alpha_{l-1}\}. \\ -\varepsilon_{\mathfrak{g},\alpha}, & \textit{otherwise}. \end{cases}$$

7.8. Now assume $\mathfrak{g}$ of type E_6. We can assume $\alpha = \alpha_1$ without loss of generality. Taking

$$\tau_1 = \sum_{i=2}^{6} k_i \omega_i^1$$

we obtain

$$\tau_2 = k_6 \omega_3^2 + k_5 \omega_4^2 + k_4 \omega_5^2 + k_2 \omega_6^2.$$

Finally $\varepsilon = 3$ for type E_6. An easy computation gives the

LEMMA. — *Suppose* $\mathfrak{g}$ *of type* E_6 *with* $\alpha = \alpha_1$. *Then for all* $0 \neq \tau_1 \in P_c^+$ *one has*

$$u - u_1^{\tau_2} = \begin{cases} 0, & \operatorname{Supp} \tau_1 = \{\alpha_6\}. \\ -\dfrac{2}{3}\varepsilon_{\mathfrak{g},\alpha}, & \operatorname{Supp} \tau_1 = \{\alpha_5\} \text{ or } \{\alpha_5, \alpha_6\}. \\ -\varepsilon_{\mathfrak{g},\alpha}, & \textit{otherwise}. \end{cases}$$

7.9. Finally assume $\mathfrak{g}$ of type E_7. We must have $\alpha = \alpha_7$. Taking

$$\tau_1 = \sum_{i=1}^{6} k_i \omega_i^1$$

we obtain

$$\tau_2 = k_5\,\omega_2^2 + k_2\,\omega_3^2 + k_4\,\omega_4^2 + k_3\,\omega_5^2 + k_1\,\omega_6^2.$$

Finally $\varepsilon = 4$ for type E_7. An easy computation gives the

LEMMA. — *Suppose* $\mathfrak{g}$ *of type* E_7 *with* $\alpha = \alpha_7$. *Then for all* $0 \neq \tau_1 \in P_c^+$ *one has*

$$u - u_1^{\tau_2} = \begin{cases} -\dfrac{3}{4}\,\varepsilon_{\mathfrak{g},\alpha}, & \operatorname{Supp}\tau_1 = \{\alpha_2\}. \\ -\varepsilon_{\mathfrak{g},\alpha}, & \textit{otherwise}. \end{cases}$$

7.10. We can now complete the one verification needed to establish 5.15. Let $L(\lambda_1)$ be unitary and define λ_2 considered as weight of Δ^2 as in 5.3.

PROPOSITION. — *The* $\mathfrak{g}^2$ *module* $L(\lambda_2)$ *is unitary.*

By 5.3, $\lambda^2 = w_c^2\, w_c^1\, \lambda_1 + \varepsilon_{\mathfrak{g},\alpha}\,\omega$ which we must show is a unitary place for $\mathfrak{g}^2$. [By viewing $L(\lambda_1)$ as a unitary $\mathfrak{g}^2$ module it is trivial that $w_c^2\, w_c^1\, \lambda_1$ is a unitary place for $\mathfrak{g}^2$; but this is not quite enough!]

Recall that $\varepsilon_{\mathfrak{g},\alpha} = \varepsilon_{\mathfrak{g}^2,\alpha}$ and that W_c^1 stabilizes ω. By the equal-spacing principle, it follows that it is enough to establish the assertion when λ_1 is a last place of unitarity. Write $\lambda_2 = \tau_2 + u\,\omega$ as in 7.2. By the equal spacing principle it is then enough to check that $u - u_1^{\tau_2}$ is always a non-positive integer. From 7.4 to 7.8 we see that this is sometimes false! However in all the bad cases (*i.e.* when $u - u_1^{\tau_2} < 0$ and is not integer) we check below that $\tau_2 + u_1^{\tau_2}\,\omega$ is always the first reduction point, so in fact λ_2 is a unitary place. This verification is quite trivial in type B_l since $\mathfrak{g}^2$ is of type A_1 in that case. In the remaining bad cases it is enough to observe that τ_2 is always of level 1. Thus for D_l with $\alpha = \alpha_l$ (or α_{l-1}) we have $\alpha_1 \in \operatorname{Supp}\tau_1$. Then $\alpha_3 \in \operatorname{Supp}\tau_2$ so τ_2 is of level 1. In type E_6 with $\alpha = \alpha_1$ we have $\alpha_5 \in \operatorname{Supp}\tau_1$. Thus $\alpha_4 \in \operatorname{Supp}\tau_2$, so τ_2 is of level 1. In type E_7 with $\alpha = \alpha_7$ we have $\alpha_2 \in \operatorname{Supp}\tau_1$. Thus $\alpha_3 \in \operatorname{Supp}\tau_2$. Recalling that Δ^2 is a D_6 system with non-compact simple root α_7 in the E_7 labels (using D_6 labels the non-compact simple root becomes α_1 and $\operatorname{Supp}\tau_2 \ni \alpha_5$) we again see that τ_2 has level 1. This proves the proposition and completes the proof of 5.15.

7.11. Fix $\tau \in P_c^+$ and let $\mathscr{V}_i^\tau$ denote the associated variety of the unitary module $L(\tau + u_i^\tau\,\omega)$. By 2.5 and 5.16 it is enough to compute $\mathscr{V}_1^\tau$. We are now almost ready to do this but there is one more catch. In 5.15 we need to make the hypothesis that $Q := \operatorname{Ann}_{U(\mathfrak{m})} L(\lambda) \neq 0$. Assuming this holds we can then compute Q. Should our computed value of this ideal be zero, there is a contradiction; but there is no difficulty. Quite simply the correct hypothesis was that $Q = 0$ from the start. However should the assumption that L_2 and so on be unitary lead to a computed value of Q being different from zero, we cannot conclude that $Q \neq 0$; because we have to make this hypothesis to conclude that L_2 is unitary! Now by 7.9, λ_2 is a unitary place; but the trouble is that L_2 need not be simple, through it is a highest weight module of highest weight λ_2. Now by 5.10, $L_2 \otimes L(-\varepsilon_{\mathfrak{g}^2,\alpha}\,\omega)$ is unitary as a $\mathfrak{g}^2$ module and so by 5.14 (iii) the only difficulty occurs if λ_2 is a first reduction point. Of course we must repeat this procedure constructing L_3 with highest weight λ_3 and so on. Now let j be the smallest positive integer (which we can assume $\leqq t$) such that λ_j is a first reduction point. Consider the simple $\Theta^j(\mathfrak{g}^j)$ quotient $L(\lambda_j)$ of L_j. By 7.9, $L(\lambda_j)$ is unitary. By 2.5 and 5.16, either $Q' := \mathrm{Ann}_{U(\Theta^j(\mathfrak{m}^j))} L(\lambda_j) = 0$ or it is generated [through the adjoint action of $\mathfrak{k}$ and multiplication by $U(\mathfrak{m})$] by the highest degree fundamental invariant, namely $\Theta^j(v_{t-j+1}^{(j)})$ in the notation of 4.2. This in turn equals $v_{j-1}^{-1} v_t$ and so either $Q=0$ (if $Q'=0$ or L_j is not simple) or $Q=Q_t$. This means that we have proved the following result. Let $l(\tau)$ denote the value assigned to i satisfying $Q_i = \mathrm{Ann}_{U(\mathfrak{m})} L(\tau + u_1^{\tau} \omega)$ calculated under the assumption that the conclusion of 5.15 holds, but without the hypothesis that this annihilator be non-zero. (That is by using 7.4-7.8 as discussed below) and let $l'(\tau)$ denote the true value of i above.

LEMMA.

(i) *If* $l(\tau) \neq t$, *then* $l(\tau) = l'(\tau)$.

(ii) *If* $l(\tau) = t$, *either* $l'(\tau) = t$ *or* $l'(\tau) > t$ [i.e. $\mathrm{Ann}_{U(\mathfrak{m})} L(\tau + u_1^{\tau} \omega) = 0$].

7.12. We shall eventually prove that $l'(\tau) = l(\tau)$. However first let us clarify how $l(\tau)$ is computed. First take $\mathfrak{g}$ of type A_l and adopt the notation of 7.4. Fix $\tau_1 \in P_c^+$. If $\mathrm{Supp}\, \tau_1 \cap \pi^l = \varnothing$ set $s_1 = t$. Otherwise let s_1 be the smallest positive integer $< t$ such that $\alpha_{s_1} \in \mathrm{Supp}\, \tau_1$. If $\mathrm{Supp}\, \tau_1 \cap \pi^r = \varnothing$ set $s_1' = t$. Otherwise let s_1' be the largest positive integer $> t$ such that $\alpha_{s_1'} \in \mathrm{Supp}\, \tau_1 \cap \pi^r$. We claim that $l(\tau_1) = s_1' - s_1 + 1$ (one may also remark that τ_1 has level equal to min $\{s_1, l - s_1' + 1\}$). If the reader has absorbed all that has been said so far this will be completely obvious to him. Otherwise suppose for example that $s_1 < t < s_1'$. Recall 7.4 how τ_2 is obtained from τ_1 by shifting weights in the Dynkin diagram. Defining s_2, s_2' in a like fashion for τ_2 we obtain $s_2' - s_2 + 1 = s_1' - s_1 - 1$. It remains to show that $l(\tau_1) = l(\tau_2) + 2$. Now from 7.4 we have $u - u_1^{\tau_2} = -\varepsilon_{\mathfrak{g}, \alpha}$, so λ_2 is a second to last place of unitarity. By 2.5 this adds 1 to the degree of the fundamental invariant generating $\mathrm{Ann}_{U(\mathfrak{m})} L(\lambda_1)$. Since $\Theta^2(v_{i-1}^{(2)}) = v_1^{-1} v_i$ a further increase of degree by 1 is obtained on passing from $\Theta^2(\mathfrak{g}^2)$ back to $\mathfrak{g}^1$. This all means that $l(\tau_1) = l(\tau_2) + 2$ as required. (By our convention in 2.1 this still holds if $i > t$.) Hopefully this is now all clear and we can state the following lemma which the above analysis proves. Define t as in EJ, 1.3.

LEMMA. — *Fix* $\tau_1 \in P_c^+$ *and define* τ_2 *by* 7.2 (*). *If* $u - u_1^{\tau_2} \notin \mathbb{Z}\, \varepsilon_{\mathfrak{g}, \alpha}$, *then* $l(\tau_1) > t$. *Otherwise*

$$l(\tau_1) - l(\tau_2) = 1 + (u_1^{\tau_2} - u)/\varepsilon_{\mathfrak{g}, \alpha}.$$

7.13. One can easily compute $l(\tau)$ in all cases using 7.12 and the result is given in the Table. In type A_l, $l(\tau)$ coincides in an obvious sense with the length of the support of $\tau + \omega$. If Δ has two roots lengths, $l(\tau)$ does not only depend on $\mathrm{Supp}\, \tau$; but one can obtain a similar formula by splitting τ into two pieces, assigned respectively to Dynkin diagrams joined at the non-compact vertex. In types D_l, E_6, E_7 a good interpretation of $l(\tau)$ is less obvious.

TABLE

This Table describes $l(\tau)$ for all $\tau \in P_c^+$ except that we may omit some cases for which $\tau = 0$, or $l(\tau) > t$, or when in type D_l, the result can be deduced from the symmetry of the Dynkin diagram. For this reason E_6 disappears from our Table. By convention $s \leqq s' \leqq l$ and the notation $\omega_r + \ldots + \omega_{r'}$ means any sum $\sum_{i=r}^{r'} k_i \omega_i$ with k_r, $k_{r'}$ non-zero, $k_i \in \mathbb{N}$.

Type	α	t	$\tau+\omega$	$l(\tau)$
A_l	$\alpha_i,\ i \leqq \left[\frac{l+1}{2}\right]$	i	$\omega_s + \ldots + \omega_{s'}$	$s'-s+1$
B_l	α_1	2	$\omega_1 + \omega_l$	2
C_l	α_l	l	$\omega_s + \omega_{s'} + \ldots + \omega_l$	$2l-s-s'+1$
D_l	α_1	2	$\omega_1 + k\omega_l,\ k \in \mathbb{N}^+$	2
D_l	α_l	$\left[\frac{l}{2}\right]$	$\omega_s + \ldots + \omega_l$	$l-s+1$
D_{2l}	α_{2l}	l	$k\omega_1 + \omega_{2l},\ k \in \mathbb{N}^+$	l
E_7	α_7	3	$k\omega_6 + \omega_7,\ k \in \mathbb{N}^+$	3

7.14. We now show how to prove that $l(\tau) = l'(\tau)$ when $l(\tau) = t$. For this we embed $\mathfrak{g}$ in a larger simple Lie algebra $\mathfrak{g}^0$ (if it exists) such that $(\mathfrak{g}^0)^2$ (defined as in 3.2) coincides with $\mathfrak{g}$ and so that the non-compact simple root of its root system Δ^0 already lies in Δ^1. Thus in type A_l add one further vertex at each end. In type C_l add one further vertex on the left and in type D_l (with $\alpha = \alpha_{l-1}$ or α_l) add two further vertices on the left. We use a zero subscript of superscript (put in parentheses if it is necessary to avoid ambiguity) to denote the objects for $\mathfrak{g}^0$ defined as for $\mathfrak{g}^1$. It is trivial to verify that $t^{(0)} = t+1$. Furthermore as shown in the proof of 5.16 the $\mathfrak{k}^0$ highest weight vector in the $S(\mathfrak{m}^{(0)})$ module generated by $\mathscr{V}_t^{(0)}$ is up to a non-zero scalar the $\mathfrak{k}$ highest weight vector in the $S(\mathfrak{m})$ module generated by $\mathscr{V}_t$.

Now fix $\tau_1 \in P_c^+$ for which $l(\tau_1) = t$. Extend τ_1 to a $\mathfrak{k}^0$ highest weight τ_0 by taking τ_1 to vanish on the new compact simple roots. Set $\lambda_0 = \tau_0 + u_1^{\tau_0}\omega$. Then by definition $L(\lambda_0)$ is a unitary highest weight $\mathfrak{g}^0$ module and so its highest weight vector generates a unitary $\mathfrak{g}^1$ submodule of highest weight $\lambda_1 = \tau_0 + u_1^{\tau_0}\omega = \tau_1 + u_1^{\tau_0}\omega$. One checks from the Table that $l(\tau_0) = l(\tau_1)$ in all cases, except in type D_{2l} with $\operatorname{Supp}\tau = \{\alpha_1\}$. (We stress that this is not immediate and anyway not always true because the second term defined by 5.3 in the sequence obtained from τ_0 is not τ_1) Yet $l(\tau_0) = l(\tau_1) = t < t^{(0)}$ and so by 7.10 (i) we obtain that $\operatorname{Ann}_{U(\mathfrak{m}^0)} L(\lambda_0) = Q_t^{(0)}$. Yet $v_t \in Q_t^{(0)}$ and so $v_t L(\lambda_1) = 0$ as required.

Apart from a few special cases which we shall analyse in the next section the above result proves the following

THEOREM. — *Let* $L(\tau + u_i^\tau \omega)$ *be a unitary highest weight module with* τ *of level* s. *If* $u < u_i^\tau$, *then* $\operatorname{Ann}_{U(\mathfrak{m})} L(\tau + u_i^\tau \omega) = 0$. *Otherwise* $u = u_i^\tau$ *for some* $i \in \{1, 2, \ldots, s\}$ *and*

$$\operatorname{Ann}_{U(\mathfrak{m})} L(\tau + u_i^\tau \omega) = Q_{l(\tau)+i-1}$$

where $l(\tau)$ is as given in the Table.

7.15. It may seem strange that the construction of 7.13 does give the required additional information. However in this I had been encouraged by a remark of W. M. McGovern that he has also used in [28], *see* discussion following Proposition 6.1, a sly trick of this nature which he has drawn from work of D. Barbasch [1].

8. The exceptional cases. Examples

8.1. We now complete the proof of 7.14 by analyzing in detail the few remaining cases not covered there. Recall that we must show that at a last point of unitarity $\tau+u_1^\tau\omega$ with $l(\tau)=t$ one has $Q:=\operatorname{Ann}_{U(\mathfrak{m})}L(\tau+u_1^\tau\omega)\neq 0$. From 7.10 we already know that $Q=0$ or $Q=Q_t$. In all remaining cases (or in general by using the trick discussed in 5.16), Q_t is a principle ideal generated by v_t which happens to be $\mathfrak{k}$ invariant and is of degree t. All we need to show is that $v_t V(\tau)\subset S_{t-1}(\mathfrak{m})P_1$, where P_1 is the PRV component of $\mathfrak{m}\otimes V(\tau)$. This is a question in elementary linear algebra. One can easily check it in some simple examples. Thus in type A_3 we have $t=2$, and $v_t=ad-bc$ for a suitable basis of $\mathfrak{m}$. Taking $\tau=\omega_1$ we have $l(\tau)=2$. Again $V(\tau)$ is the 2 dimensional simple $A_1\times A_1$ module with basis (x, y) which we can choose so that $ay-bx$, $cy-dx$ is a basis for P_1. Then the required assertion follows from the identities $(ad-bc)x=c(ay-bx)-a(cy-dx)$ and $(ad-bc)y=d(ay-bx)-b(cy-dx)$. However in say E_7 it can be that v_t is a polynomial of degree 3 in 27 variables which itself is not too easy to write down [11].

8.2. We have to consider $\mathfrak{g}$ of type B_l, $l\geqq 3$, type D_l, $l\geqq 4$ with $\alpha=\alpha_1$, type E_7 with $\alpha=\alpha_7$ and type D_{2l} with $\alpha=\alpha_{2l-1}$, Supp $\tau=\{\alpha_1\}$. In the first two cases $t=2$. Let G denote the adjoint group of $\mathfrak{g}$. It is well-known that $G\mathscr{V}_2$ is just the closure of the so-called minimal non-zero nilpotent orbit **O**. This makes these two cases a little easier.

What we have to show is equivalent to the estimate $d_{U(\mathfrak{g})}(L(\lambda))<\dim\mathfrak{m}$. However this need not be too easy. Let $\mathscr{V}(J(\lambda))$ denote the associated variety of $U(\mathfrak{g})/J(\lambda)$. Since

$$\dim\mathscr{V}(L(\lambda))=d_{U(\mathfrak{g})}(L(\lambda))=\frac{1}{2}d(U(\mathfrak{g})/J(\lambda))=\frac{1}{2}\dim\mathscr{V}(J(\lambda))$$

we shall be able to achieve our aim by some rudimentary primitive ideal theory (at least in the first three cases). This will compare $J(\lambda)$ to $J(\xi_t)$ when $\xi_t:=-(t-1)\varepsilon_{\mathfrak{g},\alpha}\omega$. Below we let $J(\mu)$ denote the annihilator of the (*not* necessarily unitary) simple highest weight module $L(\mu)$. Given $w\in W$ we set $w.\mu=w(\mu+\rho)-\rho$.

8.3. First assume $\mathfrak{g}$ of type B_l, $l\geqq 3$. Then $t=2$. From the Table we see that $l(\tau)=2$ in just one case, namely when $\tau=\omega_l$. Set $\lambda=\tau+u_1^\tau\omega$. By EJ, 4.2, we have

$$(*)\qquad u_1^\tau=1+\frac{1}{2}|S_{1,\tau}|+2(\beta^\vee,\rho_\tau)-(\beta^\vee,\tau+\rho).$$

As noted in EJ, 7.1, the first three terms sum to $l-(1/2)$. Yet $(\beta^\vee, \rho)=2(l-1)$ and $(\tau, \beta^\vee)=1$ so in the Bourbaki convention ([5], *Pl. II*) we obtain

$$\lambda+\rho=-\left(l-\frac{3}{2}\right)\omega_1+\omega_2+\ldots+\omega_{l-1}+2\omega_l.$$

Let s_γ denote the reflection corresponding to $\gamma\in\Delta$. An old result of Duflo ([16], 5.14) asserts that if $\gamma\in\pi$ and $(\mu+\rho, \gamma^\vee)\notin\mathbb{Z}$, then $J(s_\gamma.\mu)=J(\mu)$. Taking $s_i=s_{\alpha_i}$ and setting

$$\begin{aligned}\mu+\rho&=s_{l-1}s_{l-2}\ldots s_1(\lambda+\rho)\\&=\omega_1+\omega_2+\ldots+\omega_{l-2}+\frac{1}{2}\omega_{l-1}+\omega_l\end{aligned}$$

we thus obtain $J(\lambda)=J(\mu)$. Yet μ is dominant, so $J(\mu)$ is a maximal ideal. Let $\Delta(\mu)$ denote the subset of Δ of roots integral with respect to μ. Since μ is also regular we obtain from [19], 3.5, that

$$(*)\qquad d(U(\mathfrak{g})/J(\mu))=|\Delta|-|\Delta(\mu)|.$$

Now consider $L(\xi_2)$. By definition $\mathrm{Ann}_{U(\mathfrak{m})}L(\xi_2)=Q_2$. From EJ, Table, we have $\varepsilon_{\mathfrak{g},\alpha}=l-3/2$ and so

$$\xi_2+\rho=-\left(l-\frac{5}{2}\right)\omega_1+\omega_2+\ldots+\omega_l.$$

One may check that

$$\begin{aligned}\mu'+\rho:&=s_{l-2}s_{l-3}\ldots s_1(\xi_2+\rho)\\&=\omega_1+\omega_2+\ldots+\omega_{l-3}+\frac{1}{2}(\omega_{l-2}+\omega_{l-1})+\omega_l.\end{aligned}$$

As before $J(\xi_2)=J(\mu')$. The ideal is an old friend (*see* [17], Sect. 6, Table) being the unique completely prime, primitive ideal whose associated variety is **O**. Now obviously $\Delta(\lambda)=\Delta(\xi_2)$ whereas $|\Delta(\mu)|=|\Delta(\lambda)|$ and $|\Delta(\mu')|=|\Delta(\xi_2)|$. From $(*)$ we conclude that

$$d(U(\mathfrak{g})/J(\lambda))=d(U(\mathfrak{g})/J(\xi_2))<2\dim\mathfrak{m}$$

as required. By say 7.10 this further gives that $\mathscr{V}(L(\lambda))=\mathscr{V}_2$ and hence that $G\mathscr{V}_2$ is also the associated variety of $J(\lambda)$. Since $\mu+\rho$, $\mu'+\rho$ are both dominant, regular but distinct, so are $J(\mu)$, $J(\mu')$. By the above remarks $\mathrm{rk}(U(\mathfrak{g})/J(\lambda))>1$. On the other hand $J(\lambda)$ is maximal. Actually we can compute $\mathrm{rk}(U(\mathfrak{g})/J(\lambda))$ explicitly from [21], II, 6.1. Indeed this is given by a polynomial p which is exactly the product of the positive roots in $\Delta(\lambda)$, normalized to take the value 1 when $J(\lambda)$ is replaced by the completely prime ideal $J(\xi_2)$. Since $\Delta(\lambda)$ is generated by $\alpha_1+\alpha_2+\ldots+\alpha_l$ and the compact root system Δ_c we conclude that

$$\mathrm{rk}\,U(\mathfrak{g})/J(\lambda)=\frac{1}{2}\dim V(\omega_l)$$

where the (1/2) factor comes from the normalization. One should check that (1/2) dim $V(\omega_l)$ is an integer >1 for $l \geqq 3$. In fact its value is 2^{l-2}. We may recognize $V(\omega_l)$ as the spin representation of $\mathfrak{so}(2l-1)$.

It is perhaps worth mentioning that for $\lambda_{r,s} = -(l+s-(1/2))\,\omega_1 + r\,\omega_l$ with $s \geqq 0$ and $r-2s>0$ a similar reasoning proves that

$$\operatorname{rk}(U(\mathfrak{g})/J(\lambda_{r,s}) = \frac{1}{2}(r-2s)\dim V(r\,\omega_l).$$

Here $L(\lambda)$ is a non-trivial quotient of the induced module $N(\lambda)$ which has rank equal to dim $V(r\,\omega_l)$ as a free $U(\mathfrak{m})$ module. Hence the bound in 6.5 (iii) is not necessarily satisfied if $\lambda_{r,s}$ is not a unitary place. On the other hand we can choose say $s=1$, $r=3$ and then the bound of 6.5 (iii) is satisfied even though $\lambda_{r,s}$ is not a unitary place. This completes the analysis in type B_l.

8.4. In the remaining cases λ is integral so we need a slightly finer comparison result. Fix $\lambda+\rho \in P^+$ regular. For all $w \in W$ set $\tau(w) = \{\gamma \in \pi \mid w s_\gamma < w\}$, where $<$ denotes the Bruhat order with the identity $e \in W$ being the unique smallest element. By [22], Thm. 15, one has

(∗) *Fix* $\gamma \in \tau(w^{-1})$. *Then* $J(s_\gamma w.\lambda) \supset J(w.\lambda)$ *with equality unless* $\tau((s_\gamma w)^{-1}) \subset \tau(w^{-1})$.

By the translation principle in [16], 5.16, this also holds if only $\lambda+\rho \in P^+$. A proof of (∗) for equal root lengths (which is all we need here) appears in [16], 5.18.

To begin with we use the following immediate consequence of (∗).

COROLLARY. — *Suppose* $\lambda+\rho \in P^+$. *Suppose* $w \in W$ *has a unique reduced decomposition* $s_{\alpha_1} s_{\alpha_1} \ldots s_{\alpha_l}$, $\alpha_i \in \pi$ (*with any ordering and repetitions of simple roots allowed*). *Then* $J(w.\lambda) = J(s_{\alpha_l}.\lambda)$.

Remarks. — Assume $\lambda+\rho \in P^+$ and regular. Then $J(\lambda)$ is maximal (and of finite codimension). By [16], 5.20, 5.21, one has $J(\lambda) \supsetneqq J(s_\gamma.\lambda)$, $\forall \gamma \in \pi$ with no primitive ideals between. Suppose γ, $\delta \in \pi$ do not commute. Then $J(s_\gamma s_\delta.\lambda) = J(s_\gamma.\lambda)$, whilst $J((s_\gamma s_\delta)^{-1}.\lambda) = J(s_\gamma.\lambda)$. From this last equality and (i), (ii) below we conclude that $\mathscr{V}(J(s_\gamma.\lambda)) = \mathscr{V}(J(s_\gamma s_\delta.\lambda)) = \mathscr{V}(J(s_\delta.\lambda))$. Recalling that $\mathfrak{g}$ is simple this checks the (known) fact that the $\mathscr{V}(J(s_\gamma.\lambda))$, $\gamma \in \pi$ all coincide. We have used [16], 17.12 (7), that

$$\text{(i)} \qquad \begin{cases} \mathscr{V}(L(x.\lambda)) \subset \mathscr{V}(L(y.\lambda)) \Leftrightarrow J(x^{-1}.\lambda) \supset J(y^{-1}.\lambda), \\ \qquad\qquad \forall x, y \in W, \end{cases}$$

and [3], 4.10, that

$$\text{(ii)} \qquad \mathscr{V}(J(x.\lambda)) = G\,\mathscr{V}(L(x.\lambda)), \qquad \forall x \in W.$$

All this may be put in the language of left, right and two-sided cells. Though we don't need this a complete description of cells are given implicitly by the Kazhdan-Lusztig polynomials and in almost all cases explicitly by the work of D. Barbasch and D. A. Vogan [2].

The above results do not quite go through if λ is replaced by μ where $\mu+\rho\in P^+$ is not regular owing to some subtleties in the translation principle. Yet by [16], 17.13 (4), one has $\mathscr{V}(L(x.\lambda))=\mathscr{V}(L(x.\mu))$ if x is maximal in the right coset $xW(\mu)$ where $W(\mu)$ denotes the stabilizer of $\mu+\rho$ in W. Since $\mu+\rho$ is dominant, this just means that $xs_\gamma<x$, $\forall\gamma\in\pi$ for which $(\gamma,\mu+\rho)=0$. In general we will have to check this stabilizer condition. Finally we remark that $J(\mu)$ is always the unique maximal ideal in the set $J(w.\mu)$, $w\in W$.

8.5. Now take $\mathfrak{g}$ simple of type D_l, $l\geqq 4$ with $\alpha=\alpha_1$. Then $t=2$. From the Table we see that $l(\tau)=2$ only if $\operatorname{Supp}\tau=\{\alpha_{l-1}\}$ or $\{\alpha_l\}$. Both cases are equivalent so we shall just take $\tau_k=k\omega_l$, $k\in\mathbb{N}^+$. Set $\lambda_k=\tau+\mu_1^{\tau_k}\omega$. By (*) of 7.3 we have $u_1^{\tau_k}=1+(l-2)-(2l-3)-k=2-l-k$. Thus in the Bourbaki notation ([5], *Pl. IV*) we obtain

$$\lambda_k+\rho=(3-k-l)\,\omega_1+\omega_2+\ldots+\omega_{l-1}+(k+1)\,\omega_l.$$

On the other hand by EJ, Table, we have $\varepsilon_{\mathfrak{g},\alpha}=l-2$ in this case, so

$$\xi_2+\rho=(3-l)\,\omega_1+\omega_2+\ldots+\omega_{l-1}+\omega_l.$$

Set $w=s_1s_2\ldots s_{l-2}s_{l-1}$. Then

$$\mu_k+\rho:=w^{-1}(\lambda_k+\rho)=\omega_1+\omega_2+\ldots+\omega_{l-2}+(k-1)\,\omega_{l-1}+\omega_l.$$

By 8.4 we conclude that $J(\lambda_k)=J(w.\mu_k)=J(s_{l-1}.\mu_k)$. If $k>1$, then μ_k is regular and so $J(\lambda_k)$ is *not* maximal. When $k=1$, then $J(\lambda_k)=J(\mu_k)$ is maximal. Again set $y=s_1s_2\ldots s_{l-2}$. Then

$$\mu'+\rho:=y^{-1}(\xi_2+\rho)=\omega_1+\omega_2+\ldots+\omega_{l-3}+\omega_{l-1}+\omega_l.$$

By 8.4 we obtain $J(\xi_2)=J(y.\mu')=J(s_{l-2}.\mu')=J(\mu')$. From the corollary and (i), (ii) above we obtain $\mathscr{V}(L(w.\mu_k))=\mathscr{V}(L(w.\rho))=\mathscr{V}(L(y.\rho))=\mathscr{V}(L(y.\mu'))$, where the middle equality holds because $J(w^{-1}.\rho)=J(y^{-1}.\rho)$ and the extreme equalities because the stabilizer condition is satisfied. Hence $\mathscr{V}(L(\lambda_k))=\mathscr{V}(L(\xi_2))=\mathscr{V}_2$, for all $k\in\mathbb{N}^+$ as required. Again $J(\xi_2)=J(\mu')$ is completely prime with associated variety $G\mathscr{V}_2=\bar{O}$ and by [17] is the unique ideal with these properties. Consequently $\operatorname{rk}(U(\mathfrak{g})/J(\lambda_k))>1$, $\forall k\in\mathbb{N}^+$.

Notice the above gives examples of unitary modules $L(\lambda_k)$ satisfying $\operatorname{Ann}_{U(\mathfrak{m})}L(\lambda_k)\neq 0$, yet do not (for $k>1$) have maximal annihilators.

8.6. We digress slightly to consider an example in type E_6. Take $\alpha=\alpha_1$. Then $t=2$. From the Table we see that $l(\tau)=2$, $\tau\in P_c^+$ has no solution. Consider nevertheless $\tau_k:=k\omega_6$ with $k\in\mathbb{N}^+$. Set $\lambda_k=\tau_k+u_1^{\tau_k}\omega$. One finds that $u_1^{\tau_k}=1+6-11-k=-4-k$. Now τ_k has level 2 so λ_k is *not* a first reduction point. We can nevertheless compute the highest weight $\lambda_{k,2}$ of $\overline{N(\lambda_k)}$ from EJ, 7.2. One finds that

$$\lambda_{k,2}=(k-1)\,\omega_6+\omega_2-(k+5)\,\omega_1.$$

This is not a unitary place, because by EJ, 4.2

$$\lambda'_{k,2} = (k-1)\,\omega_6 + \omega_2 - (k+11)\,\omega_1$$

is a last place of unitarity. This establishes the remark made in 5.11.

This example also illustrates nicely why $\mathrm{Ann}_{\mathrm{U}(\mathfrak{m})}\,\mathrm{L}(\lambda_k)=0$. By the above

$$\lambda_k + \rho = -(3+k)\,\omega_1 + \omega_2 + \omega_3 + \omega_4 + \omega_5 + (k+1)\,\omega_6.$$

Yet $\varepsilon_{\mathfrak{g},\alpha}=3$ by EJ, Table, so

$$\xi_2 + \rho = -2\,\omega_1 + \omega_2 + \omega_3 + \omega_5 + \omega_6.$$

Set $w_0 = s_1 s_3 s_4$, $w_1 = w_0 s_5 s_2 s_4$, $w_2 = w_1 s_3$, $w_k = w_2 s_1$, $\forall k \geqq 3$. Then

$$\mu_0 + \rho := w_0^{-1}(\xi_2 + \rho) = \omega_1 + \omega_2 + \omega_3 + \omega_5 + \omega_6$$

which is dominant. Again one checks that $\mu_k + \rho := w_k^{-1}(\lambda_k + \rho)$ is dominant and also regular if $k>3$. Yet w_k does not have a unique reduced decomposition for $k \geqq 1$. Thus from 8.4 we only obtain an inclusion $\mathscr{V}(\mathrm{L}(\lambda_k)) \supset \mathscr{V}(\mathrm{L}(\xi_2)) = \mathscr{V}_2$. In fact this inclusion is strict. This is obtained by the following reasoning. For simplicity we assume $k>3$. Set $y = s_5 s_2 s_4 s_3 s_1 . \mu = \mu_k$ which is regular. Then by 8.4 (*) one has $\mathrm{J}(w_k . \mu) = \mathrm{J}(y . \mu)$, whilst again by 8.4 (*) one has $\mathrm{J}(y^{-1} . \mu) = \mathrm{J}(s_2 s_5 . \mu)$. By [16], 5.7, the inclusions $\mathrm{J}(\mu) \supset \mathrm{J}(s_5 . \mu) \supset \mathrm{J}(s_2 s_5 . \mu)$ are strict. Then by [4], 3.6, the dimensions of their associated varieties decrease strictly. This implies $\dim \mathfrak{m} \geqq \dim \mathscr{V}(\mathrm{L}(\lambda_k)) > \dim \mathscr{V}(\mathrm{L}(\xi_2)) = \dim \mathfrak{m} - 1$. Hence $\mathrm{Ann}_{\mathrm{U}(\mathfrak{m})}\,\mathrm{L}(\lambda_k)=0$ as required.

8.7. Now assume $\mathfrak{g}$ of type E_7. We have $\alpha=\alpha_7$ and $t=3$. From the Table we see that $l(\tau)=3$ only if $\tau_k = k\,\omega_6$ with $k \in \mathbb{N}^+$. Set $\lambda_k = \tau_k + u_1^{\tau_k}\omega$. One finds that $u_1^{\tau_1} = 1+8-2k-17 = -2k-8$. Thus

$$\lambda_k + \rho = \omega_1 + \omega_2 + \omega_3 + \omega_4 + \omega_5 + (k+1)\,\omega_6 - (2k+7)\,\omega_7.$$

By EJ, Table, we have $\varepsilon_{\mathfrak{g},\alpha}=4$ so

$$\xi_3 + \rho = \omega_1 + \omega_2 + \omega_3 + \omega_4 + \omega_5 + \omega_6 - 7\,\omega_7.$$

Set $w_{-2} = s_7 s_6 s_5 s_4 s_3 s_1 s_2$, $w_{-1} = w_{-2} s_4 s_3$ which we remark have unique reduced decompositions. Set $w_0 = w_{-1} s_5 s_6 s_4$ which no longer has a unique reduced decomposition because s_5, s_3 can be interchanged. Thus w_{-1}, w_{-2} belong to the unique sub-maximal two-sided cell, whereas w_0 need not and does not already by the reasoning of 8.6. One has

$$\mu_0 + \rho := w_0^{-1}(\xi_3 + \rho) = \omega_1 + \omega_2 + \omega_3 + \omega_5 + \omega_7$$

which is dominant. Again

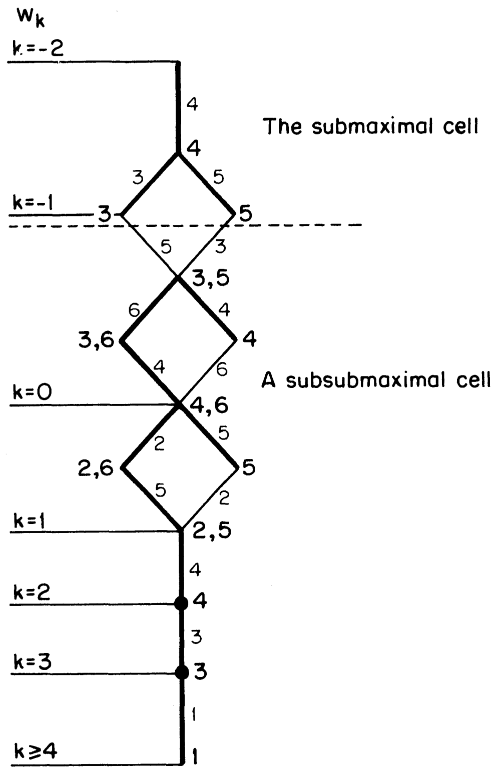
$$w_{-2}^{-1}(\xi_2 + \rho) = \omega_1 + \omega_2 + \omega_3 + \omega_5 + \omega_6 + \omega_7$$

is also dominant. Since the associated varieties of $J(\xi_2)$, $J(\xi_3)$ being respectively $G\mathscr{V}_2$, $G\mathscr{V}_3$ do *not* coincide we see again that w_0 is not in the submaximal two-sided cell. Now set $w_1 = w_0 s_2 s_5$, $w_2 = w_1 s_4$, $w_3 = w_2 s_3$, $w_k = w_3 s_1$, $\forall k \geqq 4$. One checks that

$$\mu_k + \rho := w_k^{-1}(\lambda_k + \rho)$$

is dominant. It is regular if and only if $k \geqq 5$.

Set $\lambda_0 = \xi_3$. To show that $\mathscr{V}(L(\lambda_k)) = \mathscr{V}(L(\xi_3))$, $\forall k \in \mathbb{N}$ it is enough by the same reasoning in 8.5 to show that $J(w_k^{-1}.\mu)$ is independent of k for some and hence all $\mu + \rho \in P^+$ (and to check the stabilizer condition, for example; s_4, s_6 stabilize $\mu_0 + \rho$ yet $w_0 s_4 < w_0$ and $w_0 s_2 < w_0$ as required). Fortunately this can be done by just using



Section of the Weyl group in type E_7. Each vertex represents an element $w \in W$ and is joined by a vertex with label i to a vertex representating ws_i with the longer element below. Labels on the vertices determine $\tau(w)$. The element w_{-2} is $s_7 s_6 s_5 s_4 s_3 s_1 s_2$. The thick lines give the identifications prescribed by 8.4 (∗). In particular $J(w_k^{-1}.\rho)$ is independent of the choice of $k \in \mathbb{N}$.

8.4 (∗). The calculation is indicated in the Figure. The vertices correspond to elements of W, with length increasing downwards. The top vertex is w_{-2} and subsequent vertices are computed by right multiplication by s_i where i labels the corresponding edge. The labels on the vertex corresponding to y designate $\tau(y)$. The thick lines join a pair of vertices x, y when $J(x.\mu)=J(y.\mu)$ by 8.4 (*). The fact that there is an unbroken thick chain from w_0 to w_k for all k proves the required assertion. We remark that $J(\lambda_k)$ for $k \geqq 5$ is two steps away from being maximal.

8.8. Now assume $\mathfrak{g}$ of type D_l, $l \geqq 4$ with $\alpha=\alpha_l$ and Supp $\tau=\{\alpha_1\}$. Then $t=[l/2]$, whereas $l(\tau)=[(l+1)/2]$. This case is rather delicate and we should even like to see why $\mathrm{Ann}_{U(\mathfrak{m})} L(\tau+u_1^\tau \omega) \neq 0$ exactly for l even. The analysis of the previous sections becomes extremely messy though could probably be carried out with the complete description of cells for the classical groups due to D. Barbasch and D. A. Vogan [2], but such calculations are only for masochists. We shall use a different approach. The first step is the following combinatorial lemma.

Fix $i, k, r \in \mathbb{N}$ with $0 \leqq i \leqq r \leqq k$. Let $V_r^{k,i}$ denote the simple finite dimensional module for $\mathfrak{g}=\mathfrak{gl}(r+1)$ with highest weight $(k-i)\omega_1+\omega_{r-i+1}$ with the convention that $\omega_j=0$ for $j>r$.

A straightforward application of Weyl's dimension formula gives

$$\dim V_r^{k,i}=\frac{1}{(k-2i+r+1)}\cdot\frac{(k-i+r+1)!}{(k-i)!(r-i)!i!}=:d_r^{k,i}.$$

Set

$$e_r^k=\sum_{i=0}^{r}(-1)^i d_r^{k,i}.$$

LEMMA. — *One has* $e_r^k=(1/2)(1+(-1)^r)$.

Writing $k-i+r+1$ as $k-2i+r+1+i$ we obtain

$$e_r^k=\sum_{i=0}^{r}\frac{(-1)^i(k-i+r)!}{(k-i)!(r-i)!i!}+\sum_{i=1}^{r}\frac{(-1)^i}{(k-2i+r+1)}\,\frac{(k-i+r)!}{(k-i)!(r-i)!(i-1)!}.$$

The first term is a standard binomial sum and can for example be identified with

$$\frac{1}{r!}\left[\frac{d^r}{dy^r}y^k(x-y)^r\right]_{x=y=1}=1.$$

The second term is just $-e_{r-1}^{k-1}$, so we have the recurrence relation $e_r^k=1-e_{r-1}^{k-1}$. Finally one observes that $e_0^{k-r}=1$.

Remark. — Given that $0 \leqq i \leqq r$ one may note that $d_r^{k,i}$ is defined for all $k \in \mathbb{Z}$ and the above result is also valid for such k.

8.9. We take $\tau_k=k\omega_1$ in 8.8, $\lambda_k=\tau_k+u_1^{\tau_k}\omega$. Let $r(\lambda_k)$ denote the rank of $L(\lambda_k)$ considered as a $U(\mathfrak{m})$ module.

LEMMA. — *Suppose* $k \geqq l-1$. *Then*

$$r(\lambda_k) = \begin{cases} 0, & l \text{ even} \\ 1, & l \text{ odd.} \end{cases}$$

In particular $\mathrm{Ann}_{U(\mathfrak{m})} L(\lambda_k) \neq 0$ *exactly when* l *is even. Moreover*

$$\mathrm{rk}(U(\mathfrak{g})/\mathrm{Ann}_{U(\mathfrak{m})} L(\lambda_k)) = 1,$$

if l *is odd.*

One checks that

$$\lambda_{k,0} := \lambda_k = k\omega_1 - (k+l-2)\omega_l.$$

Now for $1 \leqq i < l$, set

$$\lambda_{k,i} = (k-i)\omega_1 + \omega_{l-i} - (k+l-\delta_{i,l-1})\omega_l,$$

where δ is the Kronecker delta. As above one checks that $\lambda_{k,i}$, $1 \leqq i < l-1$ is a last place of unitarity. Yet by the hypothesis on k, we find that $(k-i)\omega_1 + \omega_{l-i}$ is of level 1. Hence each such $\lambda_{k,i}$ is a first reduction point. Then $\overline{N(\lambda_{k,i})}$ is unitary by 5.11, and from EJ, 7.2, one checks that its highest weight is just $\lambda_{k,i+1}$. Similarly $N(\lambda_{k,l-1})$ is unitary. Now $N(\lambda_{k,i})$ is a free $U(\mathfrak{m})$ module of rank $\dim V((k-i)\omega_1 + \omega_{l-i})$ defined with respect to Δ_c which is of type A_{l-1}. Hence the assertion results from 8.8 taking $r = l-1$ and using additivity of rank.

8.10. Assume l even. We now extend the conclusion $\mathrm{Ann}_{U(\mathfrak{m})} L(\lambda_k) \neq 0$ for all $k \in \mathbb{N}^+$. As we note below the hypothesis $k \geqq l-1$ implies that $\lambda_k + \rho$ is regular and so there exists a unique $w \in W$ such that $w(\lambda_k + \rho)$ is dominant. Now fix $1 \leqq k < l-1$ and assume we can find $w', w'' \in W$ such that $w = w'' w'$ where lengths add, $\mu_k + \rho := w'(\lambda_k + \rho)$ is dominant and $s_\gamma w' < w'$, whenever $\gamma \in \pi$ satisfies $(\gamma, \mu_k + \rho) = 0$. Then by 8.4 we obtain

$$\mathcal{V}(L(\lambda_k)) = \mathcal{V}(L(w'^{-1} . \mu_k)) = \mathcal{V}(L(w'^{-1} . \rho)) \subset \mathcal{V}(L(w^{-1} . \rho)) = \mathcal{V}(L(w^{-1} . \mu_{l-1})) = \mathcal{V}_{l/2},$$

where the last step follows by 8.9. Then the opposite inclusion follows from 7.12 and proves the required assertion.

Set $\varepsilon_i = (1/2)(1 + (-1)^i)$, $x_i = s_{i+1} s_{i+2} \dots s_{l-\varepsilon_i}$, $i = 1, 2, \dots, l-3$ with $x_{l-2} = s_{l-1}$. Set $w_i = x_i x_{i-1} \dots x_1$. One checks that

$$w_i(\lambda_k + \rho) = \omega_1 + \omega_2 + \dots + \omega_i + (k+1-i)\omega_{i+1} + \omega_{i+2} + \dots + \omega_{l-2} + \omega_{l-\varepsilon_i} - (k+l-3-2i)\omega_{l+\varepsilon_i-1}$$

for all $i \leqq \min\{k+1, l-3\}$. Thus if $k-l+3 \geqq 0$ we obtain

$$w_{l-2}(\lambda_k + \rho) = \omega_1 + \omega_2 + \dots + \omega_{l-1} + (k-l+3)\omega_l$$

which is dominant and also regular if $k-l+2 \geqq 0$. Observe that the expression for $w := w_{l-2}$ is reduced. Suppose now that $k-l+3 < 0$. Then $k+1-i = 1$ for some i,

$0 \leqq i < l-3$. Inspection of the above formula shows that we may (by a relabelling) assume $k=i=1$ without loss of generality. Set $y_i = s_{2i} s_{2i+1} \dots s_{l-\varepsilon_i}$, $i=1, 2, \dots, (l/2)-1$, $w'_i = y_i y_{i-1} \dots y_1$. Setting $w' = w_{l/2-1}$ one checks that $\mu_0 + \rho := w'(\lambda_0 + \rho)$ is dominant. Moreover $(\gamma, \mu_0 + \rho) = 0$ if and only if $\gamma = \alpha_{2i}$, $1 \leqq i \leqq l/2 - 1$. Yet $s_{2i} w' < w'$, $\forall\, 1 \leqq i \leqq l/2 - 1$ so the stabilizer condition is satisfied. Finally set $y_i = 1$ for $i \geqq l/2$ and $z_i = x_i y_i^{-1}$. One checks that $y_j z_i = z_i y_j$ for $j > i$. Hence

$$\begin{aligned} w &= x_{l-2} \dots x_1 = z_{l-2} y_{l-2} \dots z_1 y_1 \\ &= z_{l-1} z_{l-2} \dots z_1 y_{l-1} \dots y_1 \\ &= w'' w', \quad \text{where } w'' = z_{l-2} \dots z_1. \end{aligned}$$

Since the expression for w was reduced, the lengths add. This completes the proof.

Index of notation

(*see* also EJ, index of notation)

Symbols appearing frequently throughout the text are listed below where they are defined.

1.1. $\mathfrak{g}$, $\mathfrak{n}^+$, $\mathfrak{h}$, $\mathfrak{n}$, $\mathfrak{g}_0$, $\mathfrak{k}$, $\mathfrak{p}^+$, $\mathfrak{m}^+$, α, ω, P_c^+, $V(\tau)$, $N(\lambda)$, $L(\lambda)$, s, u_i^τ, $\varepsilon_{\mathfrak{g},\alpha}$, $\overline{N(\lambda)}$.

1.2. u_i, $\mathscr{V}_i$, $\mathscr{D}_i$.

1.3 $\mathscr{V}_i^\tau$, $\mathfrak{m}$, t, $l(\tau)$.

1.4. Q_i.

2.2. β_i, μ_i.

2.3. $\mathrm{Spec}_{\mathfrak{k}}\, S(\mathfrak{m})$, v_i.

2.5. λ_i^τ, ξ_i.

3.2. Δ, Δ^+, Δ^-, Γ^i, Δ^i.

3.3. $\mathfrak{a}^i$, $\mathfrak{a}_n^i$, $\mathfrak{b}^i$, $\mathfrak{g}^i$, $\mathfrak{n}^i$, $\mathfrak{k}^i$, $\mathfrak{p}^i$, $\mathfrak{m}^i$, $\mathfrak{c}^i$.

3.4. y_i, Y_i, Θ^i, θ^i.

3.5. $\mathscr{R}(\mathscr{V}_i)$, $\mathscr{D}(\mathscr{V}_i)$.

4.2. $v_j^{(i)}$.

4.4. $F(\lambda)$, $A(\lambda)$.

5.1. $\mathscr{O}$.

5.3. W_c^i, w_c^i.

5.4. $\tilde{\sigma}$, q_γ, p_γ.

5.8. σ.

6.11. $d_A(M)$.

7.10. $l(\tau)$.

8.2. $w.\mu$.

We recall again that the subscripts c and n mean *compact* and *non-compact* respectively.

REFERENCES

[1] D. BARBASCH, *The Unitary Dual for Complex Classical Lie Groups* (*Inv. Math.*, Vol. 96, 1989, pp. 103-176).

[2] D. BARBASCH and D. A. VOGAN, *Primitive Ideals and Orbital Integrals in Complex Classical Groups* (*Math. Ann.*, Vol. 259, 1982, pp. 153-199).

[3] W. BORHO and J.-L. BRYLINSKI, *Differential Operators on Homogeneous Spaces III* (*Invent. Math.*, Vol. 80, 1985, pp. 1-68).

[4] W. BORHO and H. KRAFT, *Über die Gelfand-Kirillov Dimension* (*Math. Ann.*, Vol. 220, 1976, pp. 1-24).

[5] N. BOURBAKI, *Groupes et Algèbres de Lie*, Chaps. IV-VI, Act. Sci. Ind., 1337, Hermann, Paris, 1968.

[6] M. G. DAVIDSON, T. J. ENRIGHT and R. J. STANKE, *Differential Operators and Highest Weight Representations*, preprint, 1990.

[7] J. DIXMIER, *Algèbres enveloppantes* (Cahiers scientifiques, No. 37, Gauthier-Villars, Paris, 1974).

[8] T. J. ENRIGHT, *Analogues of Kostant's* $\mathfrak{u}$ *Cohomology Formulas for Unitary Highest Weight Modules* (*J. reine angew. Math.*, Vol. 392, 1988, pp. 27-36).

[9] T. J. ENRIGHT, R. HOWE and N. R. WALLACH, *A Classification of Unitary Highest Weight Modules*, in *Representation Theory of Reductive Groups*, P. C. TROMBI Ed., Boston, 1983, pp. 97-143.

[10] T. J. ENRIGHT and A. JOSEPH, *An Intrinsic Analysis of Unitarizable Highest Weight Modules* (*Math. Ann.*, Vol. 288, 1990, pp. 571-594).

[11] A. FREUDENTHAL, *Zur eben Oktavengeometrie* (*Indag. Math.*, Vol. 15, 1953, pp. 195-200).

[12] M. HARRIS and H. P. JAKOBSEN, *Singular Holomorphic Representations and Singular Modular Forms* (*Math. Ann.*, Vol. 259, 1982, pp. 227-244).

[13] H. P. JAKOBSEN, *On Singular Holomorphic Representations* (*Invent. Math.*, Vol. 62, 1980, pp. 67-78).

[14] H. P. JAKOBSEN, *Hermitian Symmetric Spaces and Their Unitary Highest Weight Modules* (*J. Funct. Anal.*, Vol. 52, 1983, pp. 385-412).

[15] H. P. JAKOBSEN and M. VERGNE, *Restrictions and Expansions of Holomorphic Representations* (*J. Funct. Anal.*, Vol. 34, 1979, pp. 29-53).

[16] J. C. JANTZEN, *Einhüllenden Algebren halbeinfacher Lie-Algebren* (*Ergebnisse der Mathematik*, Springer-Verlag, Berlin, 1983).

[17] A. JOSEPH, *The Minimal Orbit in a Simple Lie Algebra and Its Associated Maximal Ideal* (*Ann. Ec. Norm. Sup.*, Vol. 9, 1976, pp. 1-30).

[18] A. JOSEPH, *A Preparation Theorem for the Prime Spectrum of a Semisimple Lie Algebra* (*J. Algebra*, Vol. 48, 1977, pp. 241-289).

[19] A. JOSEPH, *Gelfand-Kirillov Dimension for the Annihilators of Simple Quotients of Verma Modules* (*J. Lond. Math. Soc.*, Vol. 18, 1978, pp. 50-60).

[20] A. JOSEPH, *Kostant's Problem, Goldie Rank and the Gelfand-Kirillov Conjecture* (*Invent. Math.*, Vol. 56, 1980, pp. 191-213).

[21] A. JOSEPH, *Goldie Rank in the Enveloping Algebra of a Semisimple Lie Algebra I-III* (*J. Algebra*, Vol. 65, 1980, pp. 269-283 and 284-306, Vol. 73, 1981, pp. 295-326).

[22] A. JOSEPH, *A Characteristic Variety for the Primitive Spectrum of a Semisimple Lie Algebra*, in *Non-Commutative Harmonic Analysis* (*Lectures Notes*, No. 587, Berlin, 1977, pp. 102-118).

[23] A. JOSEPH, *On the Variety of a Highest Weight Module* (*J. Algebra*, Vol. 88, 1984, pp. 238-278).

[24] A. JOSEPH, *A Surjectivity Theorem for rigid Highest Weight Modules* (*Invent. Math.*, Vol. 92, 1988, pp. 567-596).

[25] T. LEVASSEUR, S. P. SMITH and J. T. STAFFORD, *The Minimal Nilpotent Orbit, the Joseph Ideal and Differential Operators* (*J. Algebra*, Vol. 116, 1988, pp. 480-501).

[26] T. LEVASSEUR and J. T. STAFFORD, *Rings of Differential Operators on Classical Rings of Invariants* (*Mem. Am. Math. Soc.*, Vol. 412, 1989).

[27] J. C. MCCONNELL and J. C. ROBSON, *Non-Commutative Noetherian Rings*, Wiley-Interscience, New York, 1987.

[28] W. M. MCGOVERN, *Quantization of Nilpotent Orbit Covers in Complex Classical Groups*, preprint, 1989.

[29] D. A. VOGAN, *The Orbit Method and Primitive Ideals for Semisimple Lie Algebras*, (*Canad. Math. Soc. Conference Proceedings*, Vol. 5, 1986, pp. 281-316).

(Manuscript received February 9, 1990;
revised November 8, 1990).

A. JOSEPH,
The Donald Frey Professorial Chair,
Department of Theoretical Mathematics,
The Weizmann Institute of Science,
Rehovot 76100, Israel
and
Laboratoire de Mathématiques fondamentales,
Équipe de Recherche associée au C.N.R.S.,
Université Pierre-et-Marie-Curie,
4, place Jussieu,
75252 Paris Cedex 05, France.