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TAMELY RAMIFIED SUPERCUSPIDAL REPRESENTATIONS OF CLASSICAL GROUPS. I. FILTRATIONS

BY LAWRENCE MORRIS ⁽¹⁾

Introduction

Let G be a classical group defined over a local non-archimedean field k . In [M] we isolated a class of tamely ramified compact maximal tori which we called *principal*. Under additional assumptions, to each such principal torus T , and a *cuspidal datum* of T , we associated an irreducible supercuspidal representation of G . The question of equivalences is not discussed in [M]. As remarked in [M], the construction works best for the symplectic groups, and least well for orthogonal groups. Even in the most favorable case it does not produce all supercuspidal representations; indeed, the calculations in [Mo2] suggest that some new tools will be necessary.

In this paper, and a sequel, we shall construct irreducible supercuspidal representations associated to any tamely ramified compact maximal torus of any classical group (excluding those over quaternion algebras). The key step is to produce a parahoric subgroup P of G , together with a remarkable filtration of P by normal open subgroups of P which reflects the arithmetic properties of a given “ramified piece” of T . This filtration is not necessarily the natural lattice filtration of P , or the canonical height filtration (via affine roots); on the other hand it satisfies the properties FI-FIV of [M1]. It is the construction of P and the filtration, and an examination of some of the properties of this filtration that is the content of this paper.

To motivate the constructions that occur in this paper, consider first a field extension E of degree $2n$ over k . Let $\sigma \neq 1$ be an involution on E , and let $\mu \in E$ be such that $\sigma\mu = -\mu$. Define a non-degenerate alternating form f on the k -vector space E by the rule $f(x, y) = \text{trace}_{E/k}(\mu x \sigma y)$; then

$$T = \{ x \in E \mid x \sigma x = 1 \}$$

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is an anisotropic maximal torus in the symplectic group G associated to f . These is a natural parahoric subgroup P containing T which is important for the purposes of representation theory. Namely, the k -vector space E contains the \mathcal{O}_k -lattice chain $\{\mathcal{P}_E^n\}_{n \in \mathbf{Z}}$ where \mathcal{O}_k (resp. \mathcal{P}_k) denotes the ring of integers (resp. maximal ideal) in k (resp. \mathcal{O}_k). This lattice chain is "self dual" with respect to f ; its stabilizer in G is a parahoric subgroup P . If $n \geq 1$ is an integer, we set

$$P_n = \{x \in P \mid (x-1) \mathcal{P}_E^j \subseteq \mathcal{P}_E^{j+n}, \text{ all } j\}.$$

Then P_n is an open normal subgroup of P , and the family $\{P_n \cap T\}$ is then the restriction to T of the standard filtration of the multiplicative group E^* .

The above construction simply imitates what one does for a compact mod centre maximal torus in the general linear group, and in this context it is well known. The point is that there is a natural filtration of P by open normal subgroups which reflects the arithmetic of T by extending the natural filtration of T .

Unfortunately (or fortunately, depending on one's point of view) such T do not exhaust the compact maximal tori in G . Indeed, G is a symplectic group of dimension $2n$; let E_1, E_2 be two field extensions over k of respective degrees $2r, 2s$ such that $r+s=n$. Suppose that E_1, E_2 are equipped with involutions $\sigma_1, \sigma_2 \neq 1$, and let $\mu_i (i=1, 2)$ be elements of E_i such that $\sigma_i \mu_i = -\mu_i$. The group

$$T = \{(x_1, x_2) \in E_1 \times E_2 \mid x_1 \sigma_1 x_1 = x_2 \sigma_2 x_2 = 1\}$$

can be embedded in G , and is a compact maximal torus in that group. (The general situation is described in section 3 of this paper.) The group T is naturally a product; $T = T_1 \times T_2$, where T_1, T_2 are smaller compact tori (embedded in smaller symplectic groups) and each is equipped with a natural filtration. One would like to find a parahoric subgroup $P \subseteq G$ such that $T \subseteq P$, and a suitable filtration of P which reflects in some measure, the arithmetic of T . The trick is to find a parahoric subgroup $P \supset T$ and a filtration of P which reflects the filtration of *one* of T_1, T_2 . For many purposes this suffices because it allows one to work inductively.

How does one construct such a filtration? We know we have a self dual lattice chain \mathcal{L}_1 in E_1 and a self dual lattice chain \mathcal{L}_2 in E_2 from the earlier remarks. It is relatively easy to construct a new self dual lattice chain, denoted by $\mathcal{L}_1 \oplus \mathcal{L}_2$, in $E_1 \oplus E_2$, which will provide P . The putative filtration is to reflect the arithmetic of T_2 (say). Now, the elements of T_2 all lie in $\mathcal{O}_{E_2}^*$; the filtration is then obtained by considering the effect of $\mathbf{P}_{E_2}^n$ (not $1 + \mathbf{P}_{E_2}^n$) on a given lattice in the chain $\mathcal{L}_1 \oplus \mathcal{L}_2$. For more details, and perhaps more motivation, we refer the reader to example 3.9 (b) (i) in the paper (see the last remarks there), as well as Section 2.2.

There is an alternative heuristic way of regarding the problem, which the reader may find helpful. Suppose that one takes the viewpoint that "sufficiently regular" characters of $T \subset G$ (notation as above) ought to provide irreducible supercuspidal representations of G . A character \mathcal{X} of T is just a product $\mathcal{X} = \mathcal{X}_1 \times \mathcal{X}_2$ of characters on T_1, T_2 . Each of $\mathcal{X}_1, \mathcal{X}_2$ has a conductor on T_1, T_2 respectively. "Sufficiently regular" ought at least to mean that $\mathcal{X}_1, \mathcal{X}_2$ each have some sort of Howe factorization. The problem is then

to use this data to construct a supercuspidal representation; the formulation suggests that one might try to do this by beginning with T_1 or T_2 first, working with the data associated with \mathcal{X}_1 or \mathcal{X}_2 (respectively), and then using an inductive procedure. This paper is concerned with the families of filtrations that are necessary to carry out this idea. Having said this, we warn the reader that this alternative should only be used as a heuristic, and not in any way as an expectation that characters of compact maximal tori will parametrize supercuspidal representations of G . (They do not.)

We now sketch the contents of this paper. Section 1 is devoted to a description of properties of “self-dual” lattice chains, which are used later, notably in Section 2. The main results are propositions 1.4, 1.7, 1.10; they are undoubtedly known to many people, but I know of no convenient reference.

Section 2 is the heart of the paper. Let $V_i (i = 1, 2)$ be two finite dimensional k -vector spaces, equipped with non-degenerate sesquilinear forms f_i of the same type (*i.e.* both alternating, or bilinear or hermitian). Given self dual lattice chains \mathcal{L}_1 in V_i we show how to “sum” \mathcal{L}_1 and \mathcal{L}_2 to obtain a new self dual lattice chain in $V_1 \oplus V_2$ (equipped with the form $f = f_1 \oplus f_2$). From this chain we find (2.2) a (self dual) sub-chain which we denote by $\mathcal{L}_1 \oplus \mathcal{L}_2$, and an associated hereditary order \mathcal{A} . In fact, $\mathcal{L}_1 \oplus \mathcal{L}_2$ is defined as the union of two canonically chosen (in general) non self dual lattice chains $\mathcal{L}', \mathcal{L}''$ in $V = V_1 \oplus V_2$. Let σ be the involution on $\text{End}_k(V)$ associated to f , and let $\mathcal{A}', \mathcal{A}''$ denote the hereditary order associated to $\mathcal{L}', \mathcal{L}''$. Then $\sigma \mathcal{A}' = \mathcal{A}''$, and $\mathcal{A} = \mathcal{A}' \cap \mathcal{A}''$. To \mathcal{A} is naturally associated a parahoric subgroup P , and from $\mathcal{A}', \mathcal{A}''$ one can construct (2.3-2.13) a filtration of P by open normal subgroups which has some remarkable properties. Thus in theorem 2.13 we show that this filtration satisfies the axioms FI to FIV of [M1]. It is worth emphasizing that the constructions and properties of this section work so well because of the close relation between the chains $\mathcal{L}', \mathcal{L}''$; in particular see lemma 2.2.

The last part of Section 2 is concerned with the special case when one of the added lattice chains has period 1 and the other has period at most 2 (of a special form). It corresponds to the fact that if T is a compact maximal torus in G (the isometry group of f), such that all the corresponding field extensions are unramified over k , then one only ever recovers a maximal parahoric subgroup with its standard filtration (via height functions or lattice chains).

We remark that more general filtrations of the type constructed in Section 2 appear to play a role in the existence of fundamental G strata (*see* [M2]).

Given a tamely ramified compact maximal torus T in a classical group G (with ambient vector space V , and form f . . .), we apply the results of Section 2 inductively in Sections 3.1-3.8 to obtain a parahoric subgroup P , and a filtration of P by open normal subgroups $\{P_n\} n > 0$ such that $T \subset P$, and such that the family $\{P_n\}$ reflects the arithmetic of a prespecified ramified piece of T . The case when T is unramified is considered separately (which is the reason for the discussion of the special cases in Section 2). We illustrate this construction in 3.9-3.10 with some examples in small groups, to indicate how it covers known cases of small rank. The remainder of Section 3 (proposition 3.11 seq.) shows how nicely the construction above behaves with respect to block decomposition. This is important for applications.

Particular cases of the filtrations that we construct in Sections 2 and 3, occur in a nascent form in Allen Moy's work on Sp_4 and rank one unitary groups. It was our attempt to understand Moy's work that led to the results of Sections 2 and 3.

In Part II of this paper we shall apply the preceding constructions to obtain supercuspidal representations. For more details we refer the reader to the introduction of that paper.

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Notation and conventions

In general, the notation and conventions are as in [M] to which we shall refer the reader on occasion.

In particular:

k : non-archimedean local field with involution σ_0 , fixed field k_0 .

\mathcal{O} : ring of integers in k , with prime ideal \mathcal{P} , residue field $\mathbf{F}_q = \mathcal{O}/\mathcal{P}$.

π : generator for \mathcal{P} .

p : characteristic of \mathbf{F}_q ; we suppose $p > 2$, so that $2 \in \mathcal{O}^*$.

If \mathbf{H} is an algebraic group defined over k , we write $\mathbf{H} = \mathbf{H}(k)$.

In this paper, \mathbf{V} will denote a finite dimensional k -vector space, with a non degenerate (ε, σ_0) sesquilinear form $f: \mathbf{V} \times \mathbf{V} \rightarrow k$, and $\mathbf{G} = \mathbf{U}(f)$ will denote the corresponding classical group (defined over k_0). Thus, for all $u, v \in \mathbf{V}$,

$$f(u, v) = \varepsilon \sigma_0 f(v, u),$$

and

$$f(\lambda u, v) = \lambda f(u, v), \quad \text{all } \lambda \in k$$

where $\varepsilon = \varepsilon_f \in \{-1, +1\}$.

1. Generalities on self dual lattice chains

1.1. In this section we set down some elementary but useful facts about self-dual lattice chains. Thus, let \mathbf{V} be a finite dimensional k vector space which is equipped with a non-degenerate (skew)-hermitian form f with respect to an involution σ_0 of the field k . If \mathbf{L} is a lattice in \mathbf{V} we define its complementary (dual) lattice \mathbf{L}^* by

$$\mathbf{L}^* = \{x \in \mathbf{V} \mid f(x, \mathbf{L}) \subseteq \mathcal{O}\}.$$

1.2. Recall that a *lattice chain* in \mathbf{V} is a family $\mathcal{L} = \{\mathbf{L}_i\}_{i \in \mathbb{1}}$ of lattices in \mathbf{V} which is totally ordered with respect to inclusion, and stable under multiplication by elements

of k^* . It is well known (see [B-T3]) that it is equivalent to give a sequence of lattices

$$\dots \supseteq \pi^{-1} L_{e-1} \supseteq L_0 \supseteq \dots \supseteq L_{e-1} \pi L_0 \supseteq \dots$$

The integer e is unique, and is referred to as the *period* of the chain. We shall refer to $L_0 \supseteq L_1 \supseteq \dots \supseteq L_{e-1} \supseteq \pi L_0$ as a *slice* for \mathcal{L} , and we shall often denote \mathcal{L} , together with a slice by

$$\mathcal{L}: L_0 \supseteq \dots \supseteq L_{e-1} \supseteq \pi L_0.$$

By a *self dual lattice chain* \mathcal{L} we shall mean a lattice chain such that $L \in \mathcal{L}$ implies $L^* \in \mathcal{L}$. The first thing we want to do is show that for such a chain, one can always find a particular type of slice with nice properties. We begin with the following lemma.

LEMMA. — *There is an $L \in \mathcal{L}$ such that $L^* \supset L \supset \pi L^*$.*

Proof. — Choose any $L \in \mathcal{L}$, then either $L^* \supset L$ or $L \supset L^*$; without loss we may assume $L^* \supset L$. If $L \supset \pi L^*$ we are done. Otherwise, consider the largest integer $r \geq 0$ such that $\pi^r L^* \supset \pi^{-r} L$. Since \mathcal{L} is a lattice chain, we then have $\pi^{-(r+1)} L \supset \pi^{r+1} L^*$. If $\pi^r L^* \supset \pi^{-r} L \supset \pi^{r+1} L^*$ we are done, for we simply replace L by $\pi^{-r} L$. If not, we have

$$\pi^{r+1} L^* \supset \pi^{-r} L$$

which implies $\pi^{-r-1} L \supset \pi^{r+1} L^* \supset \pi^{-r} L$, and we finish by replacing L by $\pi^{r+1} L^*$.

1.3. Choose an $L \in \mathcal{L}$ with the property of lemma 1.2. By taking all the lattices in \mathcal{L} which lie between L^* and πL^* , we obtain a slice of \mathcal{L} .

LEMMA. — *Among all such $L \in \mathcal{L}$ such that $L^* \supset L \supset \pi L^*$, there is one with the property that there are no lattices $L' \in \mathcal{L}$ such that $L^* \supseteq L' \supseteq L$.*

Proof. — Choose any $L \in \mathcal{L}$ such that $L^* \supset L \supset \pi L^*$. Since \mathcal{L} is a lattice chain there can only be a finite number of $L' \in \mathcal{L}$ with $L^* \supset L' \supseteq L \supset \pi L^*$. (If there are none, we are done.) Moreover, if $L^* \supset L' \supset L$ then either $L'^* \supset L'$ or $L^* \supset L' \supset L'^*$. In the former case one has

$$L^* \supset L'^* \supset L' \supset L \supset \pi L^* \supset \pi L'^*,$$

and in the latter case

$$L^* \supset L' \supset L'^* \supset L \supset \pi L^* \supset \pi L'$$

It follows immediately from these observations that we can always find a pair $L^* \supset L \supset \pi L^*$ such that there are no lattices in \mathcal{L} between L^* and L , other than L^* and L .

1.4. Now fix an $L_0 \in \mathcal{L}$ with the property of lemma 1.3: we have

$$L_0^* \supset L_0 \supset \pi L_0^*$$

and the only lattices in \mathcal{L} which lie between L_0^* and L_0 are these two lattices.

Suppose L lies in the slice $L_0^* \supset \dots \supset \pi L_0^*$. Then in fact, we have $L_0 \supset L \supset \pi L_0^*$. Replacing L by πL^* if necessary, we can assume we have

$$L^* \supset L_0^* \supset L_0 \supset L \supset \pi L^* \supset \pi L_0^*.$$

It follows that we can find a lattice $L_1 \in \mathcal{L}$ with the property that $L_1^* \supset L_0^* \supset L_0 \supset L_1 \supset \pi L_1^* \supset \pi L_0^*$, and such that no other lattice in lies between L_0 and L_1 . If we continue in this way we see that the slice $L_0^* \supset \dots \supset \pi L_0^*$ can be put in the form

$$L_0^* \supset L_1 \supset L_2 \supset \dots \supset L_{r-1} \supset \pi L_{r-1}^* \supset \pi L_{r-2}^* \supset \dots \supset \pi L_1^* \supset \pi L_0^*$$

We now replace it by the slice

$$L_{r-1}^* \supset \dots \supset L_1^* \supset L_0^* \supset L_0 \supset \dots \supset L_{r-1} \supset \pi L_{r-1}^*$$

where possibly $L_0^* = L_0$, $L_{r-1} = \pi L_{r-1}^*$.

Summing up, we have the following proposition.

PROPOSITION. — *Let \mathcal{L} be a self dual lattice chain in V . Then one can always find a slice of the form*

$$L_{r-1}^* \supset \dots \supset L_0^* \supset L_0 \supset \dots \supset L_{r-1} \supset \pi L_{r-1}^*$$

where possibly $L_0^* = L_0$, or $L_{r-1} = \pi L_{r-1}^*$. Moreover such a slice is unique.

The only comment that needs to be made here is that the lattice L_0 is unique: if M_0 were another such lattice we must have either $M_0 \supset L_0$ or $L_0 \supset M_0$ which implies $L_0^* \supset M_0^* \supset M_0 \supset L_0$ or $M_0^* \supset L_0^* \supset L_0 \supset M_0$ respectively.

1.5. DEFINITION. — By a *self dual slice* for \mathcal{L} we shall mean the slice given by proposition 1.4.

1.6. With the existence of self dual slices assured, we now proceed to describe self dual lattice chains by common Witt bases.

Suppose that \mathcal{L} is a self dual lattice chain with self dual slice $L_{r-1}^* \supset \dots \supset L_0^* \supset L_0 \supset \dots \supset L_{r-1} \supset \pi L_{r-1}^*$ as in 1.4. Consider the lattices $L_{r-1}^* \supset L_{r-1} \supset \pi L_{r-1}^*$, and drop subscripts for the time being.

LEMMA. — *Suppose that V has isotropic elements (with respect to f). Then there are isotropic elements $x \in L^*$, $x \notin \pi L^*$, such that $f(x, L^*) = \mathcal{O}$ or \mathcal{P}^{-1} .*

Suppose that $f(x, L^) = \mathcal{P}^{-1}$. Then there is an element $z \in \pi L^*$ (and $z \notin \pi^2 L^*$) such that $f(x, z) = 1$, together with a (split) short exact sequence of \mathcal{O} -modules.*

$$0 \rightarrow N^* \rightarrow L^* \rightarrow \mathcal{O}x \oplus \mathcal{P}^{-1}z \rightarrow 0$$

If $f(x, L^) = \mathcal{O}$, there is an element $z \in L^*$ ($z \notin L^*$) such that $f(x, z) = 1$, together with a (split) short exact sequence of \mathcal{O} -modules*

$$0 \rightarrow N^* \rightarrow L^* \rightarrow \mathcal{O}x \oplus \mathcal{O}z \rightarrow 0.$$

Proof. — If V has isotropic elements, then let $x \in L^*$ be any isotropic element. After multiplying by an appropriate power of π we can assume $x \in L^*$, $x \notin \pi L^*$. Then

$$f(x, L^*) \subseteq f(L^*, L^*) \subset f(L^*, \pi^{-1}L) = \mathcal{P}^{-1}.$$

Since $x \notin \pi L^*$ we see that $f(x, L^*)$ cannot be \mathcal{P} . Then as fractional ideal it is \mathcal{O} or \mathcal{P}^{-1} .

For the rest of the proof we shall treat the case $f(x, L^*) = \mathcal{P}^{-1}$, the remaining case being similar but easier. Then since $f(x, L^*) = \mathcal{P}^{-1}$, we can find $y \in L^*$ such that $f(x, y) = \pi^{-1}$. Put $z' = \pi y$; then $f(x, z') = 1$, $z \in \pi L^*$. Let $z = z' + ax$ where $a = -(1/2)f(z', z')$. Then z is isotropic. Also, one checks that $a \in \mathcal{P}$ so that $z \in \pi L^*$.

Now define a \mathcal{O} -module map

$$L^* \rightarrow \mathcal{O}x \oplus \mathcal{P}^{-1}z$$

by the rule $l \mapsto \sigma_0 f(z, l)x + \sigma_0 f(x, l)z$ and denote its kernel by N^* . This map is surjective, and the inclusion $\mathcal{P}^{-1}z \oplus \mathcal{O}x \rightarrow L$ provides a splitting.

1.7. Lemma 1.6 provides us with the tool we need to prove the following result.

PROPOSITION. — *Let L be a lattice (in V) such that $L^* \supset L \supset \pi L^*$. There is a Witt basis for L^* : $e_1, \dots, e_{l-1}, e_l, \dots, e_n, e_{-1}, \dots, e_{-(l-1)}, e_{-l}, \dots, e_{-n}$ such that*

$$\begin{aligned} L^* \cong \mathcal{P}^{-1}e_{-n} \oplus \dots \oplus \mathcal{P}^{-1}e_{-l} \oplus \mathcal{O}e_{-(l-1)} \oplus \dots \\ \oplus \mathcal{O}e_{-1} \oplus E^* \oplus \mathcal{O}e_1 \oplus \dots \oplus \mathcal{O}e_{l-1} \oplus \mathcal{O}e_l \oplus \dots \oplus \mathcal{O}e_n \\ L \simeq \mathcal{O}e_{-n} \oplus \dots \oplus \mathcal{O}e_{-1} \oplus E \oplus \mathcal{O}e_1 \oplus \dots \oplus \mathcal{O}e_{l-1} \oplus \mathcal{P}e_l \oplus \dots \oplus \mathcal{P}e_n \end{aligned}$$

where E^*, E are complementary lattices in the anisotropic part of V , such that $E^* \supset E \supset \pi E^*$.

Proof. — If V has no isotropic vectors, there is nothing to prove. Otherwise we apply lemma 1.6. Let x, z, N^* be as in that lemma. We treat the case $f(x, L^*) = \mathcal{P}^{-1}$. There is an exact sequence of \mathcal{O} -modules

$$0 \rightarrow N^* \rightarrow L^* \rightarrow \mathcal{O}x \oplus \mathcal{P}^{-1}z \rightarrow 0$$

which is split in the obvious way. Define

$$L \rightarrow \mathcal{P}x \oplus \mathcal{O}z$$

by

$$l \mapsto f(z, l)x + f(x, l)z$$

This defines an \mathcal{O} -module map, with a natural splitting. Let N denote its kernel. Using the natural inclusion maps $L \subset L^*$, $\mathcal{P}x \oplus \mathcal{O}z \subset \mathcal{O}z \oplus \mathcal{P}^{-1}z$ one finds an (induced) injection $N \hookrightarrow N^*$. Now form $N^* \otimes_{\mathcal{O}} k$ and the corresponding sesquilinear

form. One finds easily that N is the complementary lattice for N^* in this smaller vector space. Also $f(\mathcal{P}N^*, N^*) \subset f(\mathcal{P}L^*, L^*) \subseteq \mathcal{O}$ so that $N^* \supset N \supset \pi N^*$.

A similar argument works if $f(x, L^*) = \mathcal{O}$. In either case one can proceed by induction (on $\dim V$), using $N^*, N, N^* \otimes k, f|_{N^* \otimes k}$ to obtain the result.

1.8. Although we do not need it for the sequel, it is of interest to know the possibilities for $E^* \supset E \supset \pi E^*$ in proposition 1.7. We summarize these below.

Up to equivalence of isometry groups, the possibilities for anisotropic (skew) hermitian forms are well known when k is local. The Tables below list them.

In what follows c is a fixed element of \mathcal{O}_0^* which is not a square mod \mathcal{P} .

f symmetric bilinear.

dim V	form	E	E*
1	$x_1 x'_1$	\mathcal{O}	\mathcal{O}
2	$x_1 \bar{x}_1 - cx_2 x'_2$	$\mathcal{O}e_1 \oplus \mathcal{O}e_2$	$\mathcal{O}e_1 \oplus \mathcal{O}e_2$
	$x_1 x'_1 - \pi x_2 x'_2$	$\mathcal{O}e_1 \oplus \mathcal{O}e_2$	$\mathcal{O}e_1 \oplus \mathcal{P}^{-1}e_2$
3	$x_1 x'_1 - cx_2 x'_2$	$\mathcal{O}e_1 \oplus \mathcal{O}e_2$	$\mathcal{O}e_1 \oplus \mathcal{P}^{-1}e_2$
	$x_1 x'_1 - cx_2 x'_2 - \pi x_3 x'_3$	$\bigoplus_{i=1}^3 \mathcal{O}e_i$	$\mathcal{O}e_1 \oplus \mathcal{O}e_2 \oplus \mathcal{P}^{-1}e_3$
4	$x_1 x'_1 - cx_2 x'_2 - c\pi x_3 x'_3$	$\bigoplus_{i=1}^3 \mathcal{O}e_i$	$\mathcal{O}e_1 \oplus \mathcal{O}e_2 \oplus \mathcal{P}^{-1}e_3$
	$x_1 x'_1 - cx_2 x'_2 - \pi(x_3 x'_3 - cx_4 x'_4)$	$\bigoplus_{i=1}^4 \mathcal{O}e_i$	$\mathcal{O}e_1 \oplus \mathcal{O}e_2 \oplus \mathcal{P}^{-1}(\mathcal{O}e_3 \oplus \mathcal{O}e_4)$

f hermitian. - c as above, π_0 uniformizer in $k_0 : k = k_0(\sqrt{\delta})$ where

$$\delta = \begin{cases} c \\ \pi_0 \\ c\pi_0 \end{cases}$$

dim V	form	E	E*
1	$x_1 \bar{x}_1$	\mathcal{O}	\mathcal{O}
2	$x_1 \bar{x}_1 - \pi_0 x_2 \bar{x}_2$	$\mathcal{O}e_1 \oplus \mathcal{O}e_2$	$\mathcal{O}e_1 \oplus \mathcal{P}^{-1}e_2$
	$x_1 \bar{x}_1 - cx_2 \bar{x}_2$	$\mathcal{O}e_2 \oplus \mathcal{O}e_2$	$\mathcal{O}e_1 \oplus \mathcal{O}e_2$

1.9. Suppose for example that f is a quaternary bilinear form. Let $x \in V$ be such that $f(x, x) \in \mathcal{O}$. A calculation using the formula for f shows that $x \in \bigoplus_{i=1}^4 \mathcal{O}e_i = M$. It follows that M is the largest \mathcal{O} -module with the property that $x \in M$ implies $f(x, x) \in \mathcal{O}$. Moreover $M^* = \mathcal{O}e_1 \oplus \mathcal{O}e_2 \oplus \mathcal{P}^{-1}e_3 \oplus \mathcal{P}^{-1}e_4$.

If now $E^* \supset E \supset \pi E^*$, then $\mathcal{P} \subseteq f(E, E) \subseteq \mathcal{O}$, so that $E \subset M$.

In fact $E = M$. Indeed if $f(E, E) = \mathcal{O}$, then we have the F_q -spaces $E^*/\pi E^*$ and M^*/M in any case (note that $E^* \supset M^* \supset M \supset E \supset \pi E^*$). If $\pi E^* = E$ we would have $f(E, E) \subseteq \mathcal{P}$ which is impossible. Thus a dimension count shows that $E^* = M^*$.

On the other hand we cannot have $f(E, E) = \mathcal{P}$. For let $N = \pi M^*$: it is the largest \mathcal{O} -module with the property that $x \in N$ implies $f(x, x) \in \mathcal{P}$. In the case at hand we would have $E \subset N$ so $E \supset \pi E^* \supset \pi N^* = M$. This implies that $E = M$ and then that $f(E, E) = \mathcal{O}$, which is a contradiction.

The other cases in Table 1.8 can be handled in a similar fashion. In every case E is characterized as the largest \mathcal{O} -module with the property that $x \in E$ implies $f(x, x) \in \mathcal{O}$.

1.10. Now let \mathcal{L} be a self dual lattice chain with self dual slice

$$L_{r-1}^* \supseteq \dots \supseteq L_0^* \supseteq L_0 \dots \supseteq L_{r-1} \supseteq \pi L_{r-1}^*.$$

Define a sesquilinear form on the F_q -vector space $L_{r-1}^*/\pi L_{r-1}^*$ as follows

$$(\bar{x}, \bar{y}) \rightarrow \pi f(x, y) \text{ mod } \mathcal{P}.$$

Here, x, y are representatives for \bar{x}, \bar{y} in L_{r-1}^* . This map is well defined, and the radical is precisely $L_{r-1}/\pi L_{r-1}^*$.

We denote the resulting non-degenerate form on $\bar{L}_{r-1}^* = L_{r-1}^*/L_{r-1}$ by $\overline{\pi f}$. We then have a flag of F_q -vector spaces.

$$\bar{L}_{r-1}^* \supset \bar{L}_{r-2}^* \supset \dots \supset \bar{L}_0^* \supset \bar{L}_0 \supset \dots \supset \bar{L}_{r-2} \supset \{0\}$$

which is entirely determined by the F_q -isotropic flag $\bar{L}_{r-1}^* \supset \bar{L}_0 \supset \dots \supset \bar{L}_{r-2} \supset \{0\}$.

There is also a non-degenerate form \bar{f} , induced from f , on $L_{r-1}/\pi L_{r-1}^*$. Let $U = L_{r-1}^*/L_{r-1}$, $W = L_{r-1}/\pi L_{r-1}^*$, and endow $U \oplus W$ with the form $\overline{\pi f} \oplus \bar{f}$.

Now take $L = L_{r-1}$ in proposition 1.7. We obtain

$$\begin{aligned} L_{r-1}^*/L_{r-1} &\simeq F_1 e_{-n} \oplus \dots \oplus F_q e_{-l} \oplus E^*/E \oplus F_q e_l \oplus \dots \oplus F_q e_n \\ L_{r-1}/\pi L_{r-1}^* &\simeq F_q \bar{e}_{-(l-1)} \oplus \dots \oplus F_q \bar{e}_{-1} \oplus E/\pi E^* \oplus F_q \bar{e}_1 \oplus \dots \oplus F_q \bar{e}_{l-1} \end{aligned}$$

where $E^* \supset E$ are the anisotropic lattices occurring in 1.7. (The notation should be obvious.)

In each case the appropriate vector space is the direct sum of two maximal isotropic spaces (given by the chosen Witt basis), and a space of dimension at most 2. (In all cases, this last space is anisotropic with respect to the appropriate form.)

Now let us define a *standard* isotropic flag for L_{r-1}^*/L_{r-1} to be one, each of whose members is an isotropic subspace spanned by members of a given fixed Witt basis for L_{r-1}^*/L_{r-1} . In a similar way we can define a *standard* (self dual) lattice chain (cf. proposition 1.7).

In the unitary group $\text{Stab}(U, \overline{\pi f})$, we can always find an element \bar{g} which conjugates the flag $\bar{L}_{r-1}^* \supset \bar{L}_0 \supset \dots \supset \bar{L}_{r-2} \supset \{0\}$ into a standard one, with respect to the basis given by L_{r-1}^*/L_{r-1} . According to a well known approximation theorem (see e. g. [M], 2.12) \bar{g} lifts to an element $g \in \text{Stab}(L_{r-1}^*, \pi f)$: g moves the given chain into a standard one with respect to the Witt basis for L_{r-1}^* given above.

Combining these observations with proposition 1.7 yields the following result.

PROPOSITION. — *Every self dual lattice chain can be conjugated by an element of G into a standard self dual lattice chain.*

Proof. — Let $\mathcal{L} : L_{r-1}^* \supset \dots \supset L_0^* \supset L_0 \supset \dots \supset L_{r-1} \supset \pi L_{r-1}^*$ be a self dual lattice chain. By applying proposition 1.7 we can conjugate \mathcal{L} to a chain $\mathcal{L}' = g\mathcal{L}$ such that $gL_{r-1}^* = L_{r-1}'$ is in standard form with respect to given Witt basis.

Applying the remarks above enables us to move the new chain into one such that all the lattices in the slice are in the standard form with respect to a standard basis for L_{r-1}' .

1.11. *Remark.* — This proof amounts to moving a facet in the affine building into one which lies in the link of a vertex which is “standard” (with respect to some set of simple affine roots). The new facet can then be moved by an element of the stabilizer of the vertex into one which is also “standard”.

1.12. Suppose now that we are given a Witt basis for V so that

$$V = \bigoplus_{i=1}^n ke_{-i} \oplus V_0 \oplus \bigoplus_{i=1}^n ke_i$$

where $\dim V_0 \leq 4$ and $f|_{V_0 \times V_0}$ is anisotropic.

From proposition 1.10, any self dual lattice chain can be conjugated into one of standard form with respect to this basis.

Given $\mathcal{L} : L_{r-1}^* \supset \dots \supset L_{r-1} \supset \pi L_{r-1}^*$ as above, define

$$\begin{aligned} P = P(\mathcal{L}) &= \{g \in G \mid gL_i = L_i, \text{ each } i\} \\ &= \{g \in G \mid gL_i^* = L_i^*, \text{ each } i\} \end{aligned}$$

For the purposes of this paper, we shall refer to P as the *parahoric subgroup* determined by \mathcal{L} . We remark that the parahorics used here are typically slightly larger than those defined in [B-T2].

Given $P = P(\mathcal{L})$ as above, stabilizing L_i, L_i^* , we see that there is an induced map on the quotients

$$L_{i-1}/L_i, \quad L_i^*/L_{i-1}^* \quad (1 \leq i \leq r-1), \quad L_0^*/L_0, \quad L_{r-1}/\pi L_{r-1}^*.$$

Moreover, as we pointed out earlier there are non-degenerate pairings.

$$\begin{aligned} \overline{\pi f}_i &: L_{i-1}/L_i \times L_i^*/L_{i-1}^* \rightarrow \mathcal{O}/\mathcal{P}, \quad 1 \leq i \leq r-1 \\ \overline{\pi f}_0 &: L_0^*/L_0 \times L_0^*/L_0 \rightarrow \mathcal{O}/\mathcal{P} \\ \overline{f}_r &: L_{r-1}/L_{r-1}^* \times L_{r-1}/\pi L_{r-1}^* \rightarrow \mathcal{O}/\mathcal{P}. \end{aligned}$$

Set

$$\begin{aligned} U_i &= L_{i-1}/L_i \oplus L_i^*/L_{i-1}^*, \quad 1 \leq i \leq r-1 \\ U_0 &= L_0^*/L_0 \end{aligned}$$

$$U_r = L_{r-1}/\pi L_{r-1}^*$$

and extend $\overline{\pi f_i}$ on U_i by

$$\overline{\pi f_i}((\overline{l_1}, \overline{l_1}^*), (\overline{l_2}, \overline{l_2}^*)) = \overline{\pi f_i}(\overline{l_1}, l_2^*) + \overline{\pi f_i}(\overline{l_1}^*, \overline{l_2})$$

Then $\overline{\pi f_i}$ is a non degenerate (skew) hermitian form, and L_{i-1}/L_i is a maximal isotropic subspace for this form. Moreover P acts on this space, preserving the form and the spaces $L_{i-1}/L_i, L_i^*/L_{i-1}^*$. Let M_i denote the Levi component of the parabolic subgroup in $U(U_i, \overline{\pi f_i})$ which preserves the isotropic F_q -flag

$$U_i \supset L_{i-1}/L_i \supset \{0\}.$$

Thus M_i preserves each of the quotients in this flag.

Our discussion implies that the image of P in $U(U_i, \overline{\pi f_i})$ factors through the unipotent radical of this parabolic subgroup: there is a map

$$P \rightarrow U(U_0, \overline{\pi f_0}),$$

and a map

$$P \rightarrow U(U_r, \overline{f_r}).$$

Proposition 2.12 [M] implies that the homomorphism

$$(1) \quad P \rightarrow \prod_{i=1}^{r-1} M_i \times U(U_0, \overline{\pi f_0}) \times U(U_r, \overline{f_r})$$

is surjective. We denote the kernel by P_1 , and set $P/P_1 = \overline{P}$; we call \overline{P} the *Levi component* of P , and P_1 the *pro-unipotent radical* of P . Of course the exact sequence

$$0 \rightarrow P_1 \rightarrow P \rightarrow \overline{P} \rightarrow 0$$

does not split.

Now suppose that P, Q are parahorics with respective Levi components $\overline{P}, \overline{Q}$. We shall say that P and Q are *associate* if there is an element $g \in G$ such that ${}^gP, Q$ have the same Levi components (*i. e.* the factors in each product on the right of (1) are the same, up to permutations].

1.13. For more general reductive groups, an appropriate notion of associate would be that there is a $w \in W_{\text{aff}}$ such that $w\Theta = \Psi$ here $P = P_\Theta, Q = P_\Psi$ for $\Theta, \Psi \subseteq \Delta$ (a chosen basis of affine roots).

1.14. *Remark.* — The types of the Levi components that can arise can easily be determined by using the Tables in section 1.8 (which effectively determine the anisotropic kernels), and the discussion in 1.10-1.11 (which effectively determines the appropriate Witt indices).

2. Filtrations

2.1. As usual, let V be a finite dimensional k -vector space endowed with a non-degenerate σ_0 -(skew) hermitian form f :

$$\lambda f(v, w) = f(\lambda v, w) = \varepsilon \sigma_0 f(w, \lambda v) \quad \text{all } v, w \in V, \quad \lambda \in k.$$

Here $\varepsilon = \varepsilon(f)$ is always $+1$ or always -1 . We let $\mathcal{L}: L_{r-1}^* \supset \dots \supset L_{r-1} \supset \pi L_{r-1}^*$ be a self dual chain, and for brevity we denote all this by (V, f, \mathcal{L}) .

Suppose that (W, g, \mathcal{M}) is another triple, where g is an σ_0 -(skew) hermitian form on W , with $\varepsilon(g) = \varepsilon(f)$. We can then form the space $V \oplus W$ and equip it with the (ε, σ_0) -hermitian form $f \oplus g$. Furthermore, we can construct a (new) lattice chain on $V \oplus W$; namely we form the one with self dual slice

$$\begin{aligned} L_{r-1}^* \oplus M_{s-1}^* &\supset \dots \supset L_0^* \oplus M_0^* \\ &\supset L_0 \oplus M_0 \supset L_0 \oplus M_1 \supset L_0 \oplus M_2 \supset \dots \supset L_0 \oplus M_{s-1} \\ &\supset L_1 \oplus M_{s-1} \supset \dots \supset L_{r-1} \oplus M_{s-1} \supset \pi L_{r-1}^* \oplus \pi M_{s-1}^*. \end{aligned}$$

The omitted lattices in the leftmost set of dots are just the duals of the lattices on the right here. Also we have taken a self dual slice for $\mathcal{M}: M_{s-1}^* \supset \dots \supset M_{s-1} \supset \pi M_{s-1}^*$.

We denote this lattice chain by $\mathcal{L} \oplus' \mathcal{M}$. If we had formed the chain $\mathcal{M} \oplus' \mathcal{L}$ in $W \oplus V = V \oplus W$, so that the L_i 's varied first, the discussion in 1.12-1.14 tells us that $\mathcal{L} \oplus' \mathcal{M}$ and $\mathcal{M} \oplus' \mathcal{L}$ are associate (*i. e.* the corresponding parahoric subgroups are associate). Moreover if \mathcal{M}' is associate to \mathcal{M} , then $\mathcal{L} \oplus' \mathcal{M}'$ is associate to $\mathcal{L} \oplus' \mathcal{M}$, and similarly if we replace \mathcal{L} by an associate \mathcal{L}' .

Finally, if $M_{s-1} = \pi M_{s-1}^*$, the definition above still makes sense: we form $L_0 \oplus M_{s-1} = L_0 \oplus \pi M_{s-1}^*$, then $L_1 \oplus M_{s-1} = L_1 \oplus \pi M_{s-1}^*$ and so on.

2.2. Consider then the chain $\mathcal{L} \oplus' \mathcal{M}$ given by the self dual slice

$$\begin{aligned} L_{r-1}^* \oplus M_{s-1} &\supset L_{r-2}^* \oplus M_{s-1}^* \supset \dots \\ &\supset L_0^* \oplus M_{s-1}^* \supset L_0^* \oplus M_{s-2}^* \supset \dots \supset L_0^* \oplus M_0^* \supset L_0 \oplus M_0 \supseteq L_0 \oplus M_1 \supseteq \dots \\ &\supseteq L_0 \oplus M_{s-2} \supseteq L_0 \oplus M_{s-1} \supseteq L_1 \oplus M_{s-1} \supseteq \dots \\ &\supseteq L_{r-1} \oplus M_{s-1} \supset \pi L_{r-1}^* \oplus \pi M_{s-1}^*. \end{aligned}$$

Associated to this chain there is the hereditary order $\mathcal{A}_{\mathcal{L} \oplus' \mathcal{M}}$; there is an evident embedding $\mathcal{A}_{\mathcal{M}} \subset \mathcal{A}_{\mathcal{L} \oplus' \mathcal{M}}$. The idea behind the definition (of the following lattice chains) is to produce a pair of (non self dual in general) lattice chains whose union encapsulates how the Jacobson radical $\mathcal{B}_{\mathcal{M}}$ acts on $\mathcal{L} \oplus' \mathcal{M}$. In particular, we shall produce subchains $\mathcal{L}', \mathcal{L}''$ such that $\mathcal{B}_{\mathcal{M}} \subseteq \mathcal{B}_{\mathcal{L}'} \cap \mathcal{B}_{\mathcal{L}''}$.

Suppose initially that $M_{s-1} = \pi M_{s-1}^*$, $M_0^* \neq M_0$. We consider the following pair of lattice chains, *provided* $s \geq 2$.

$$\begin{aligned} \dots \supset L_0^* \oplus M_{s-1}^* &\supset L_0^* \oplus M_{s-2}^* \supset \dots \supset L_0 \oplus M_{s-2} \supset \pi L_0^* \oplus \pi M_{s-1}^* \\ &\supset L_0^* \oplus M_{s-2}^* \supset L_0^* \oplus M_{s-3}^* \supset \dots \supset L_0 \oplus M_{s-2} \supset L_0 \oplus M_{s-1} \supset \pi L_0^* \oplus \pi M_{s-2}^* \end{aligned}$$

which we denote by \mathcal{L}' , \mathcal{L}'' respectively. Each of these chains is the dual (complement) of the other; they are identical iff $L_0 = \pi L_0^*$.

On the other hand, suppose $M_0^* = M_0$. Then we consider instead the following pair of lattice chains, *provided* $s \geq 2$.

$$\begin{aligned} \mathcal{L}': \quad L_0^* \oplus M_{s-1}^* \supset L_0^* \oplus M_{s-2}^* \supset \dots \supset L_0^* \oplus M_1^* \supset L_0 \oplus M_0 \supset L_0 \oplus M_1 \supset \dots \\ \supset L_0 \oplus M_{s-2} \supset \pi L_0^* \oplus \pi M_{s-1}^*. \end{aligned}$$

$$\begin{aligned} \mathcal{L}'': \quad L_0^* \oplus M_{s-2}^* \supset L_0^* \oplus M_{s-3}^* \supset \dots \\ \supset L_0^* \oplus M_0 \supset L_0 \oplus M_1 \supset \dots \\ \supset L_0 \oplus M_{s-2} \supset L_0 \oplus M_{s-1} \supset \pi L_0^* \oplus \pi M_{s-2}^*. \end{aligned}$$

We note that these two chains would be identical iff $L_0 = L_0^*$ and $L_0 = \pi L_0^*$, which is impossible.

In each case we denote the corresponding hereditary orders by \mathcal{A}' , \mathcal{A}'' . If σ is the involution on $\text{End}_k(V)$ induced from the form f , then $\mathcal{A}'' = \sigma \mathcal{A}'$.

Next, suppose that $M_{s-1} \not\supseteq \pi M_{s-1}^*$, $M_0^* \not\supseteq M_0$. We consider instead the following pair of non self dual lattice chains

$$\begin{aligned} \mathcal{L}': \quad L_{r-1}^* \oplus M_{s-1}^* \supset L_0^* \oplus M_{s-2}^* \supset \dots \supset L_0 \oplus M_{s-2} \supset L_0 \oplus M_{s-1} \supset \pi L_{r-1}^* \oplus M_{s-1}^* \\ \mathcal{L}'': \quad L_0^* \oplus M_{s-1}^* \supset L_0^* \oplus M_{s-2}^* \supset \dots \supset L_0 \oplus M_{s-2} \supset L_{r-1} \oplus M_{s-1} \supset \pi L_0^* \oplus M_{s-1}^* \end{aligned}$$

If $M_{s-1} \not\supseteq \pi M_{s-1}^*$, $M_0^* = M_0$ we consider instead

$$\begin{aligned} \mathcal{L}': \quad L_{r-1}^* \oplus M_{s-1}^* \supset L_0^* \oplus M_{s-2}^* \supset \dots \\ \supset L_0^* \oplus M_1^* \supset L_0 \oplus M_0 \supset \dots \supset L_0 \oplus M_{s-2} \supset L_0 \oplus M_{s-1} \supset \pi L_{r-1}^* \oplus \pi M_{s-2}^* \\ \mathcal{L}'': \quad L_0^* \oplus M_{s-1}^* \supset L_0^* \oplus M_{s-2}^* \supset \dots \\ \supset L_0^* \oplus M_0 \supset L_0 \oplus M_1 \supset \dots \supset L_0 \oplus M_{s-2} \supset L_{r-1} \oplus M_{s-1} \supset \pi L_0^* \oplus \pi M_{s-1}^*. \end{aligned}$$

Again, we denote the corresponding hereditary orders by \mathcal{A}' , \mathcal{A}'' ; we have $\mathcal{A}'' = \sigma \mathcal{A}'$.

In all cases we define $\mathcal{L} \oplus \mathcal{M} = \mathcal{L}' \cup \mathcal{L}''$; we remind the reader that in the first two cases, $s \geq 2$.

The following lemma states two elementary properties of the chain $\mathcal{L}' \cup \mathcal{L}'' = \mathcal{L} \oplus \mathcal{M}$.

LEMMA. — (a) Let $L \in \mathcal{L}' \setminus \mathcal{L}''$ (respectively, $\mathcal{L}'' \setminus \mathcal{L}'$). Then its successor in $\mathcal{L} \oplus \mathcal{M}$ lies in \mathcal{L}'' (respectively, \mathcal{L}').

(b) Consider a sequence in $\mathcal{L} \oplus \mathcal{M} \supset \dots \supset L_i \supset L_{i+1} \supset \dots \supset L_{i+k} \supset \dots$ where $L_i, \dots, L_{i+k} \in \mathcal{L}' \cap \mathcal{L}''$, $L_{i-1}, L_{i+k+1} \notin \mathcal{L}' \cap \mathcal{L}''$. Then $L_{i-1} \in \mathcal{L}' \setminus \mathcal{L}''$, $L_{i+k+1} \in \mathcal{L}'' \setminus \mathcal{L}'$.

Proof. — Inspect the possibilities.

2.3. Let \mathcal{B}' (respectively \mathcal{B}'') denote the Jacobson radical of \mathcal{A}' (respectively \mathcal{A}''). We are going to consider the following sequence of $\mathcal{A}' \cap \mathcal{A}''$ bi-modules: $\mathcal{B}'^i \cap \mathcal{B}''^i$, $i \in \mathbf{Z}$; in particular $\mathcal{A}' \cap \mathcal{A}'' = \mathcal{B}'^0 \cap \mathcal{B}''^0$.

2.4. Set $U = V \oplus W$. Recall that there is a non-degenerate bilinear form on $\text{End}(U)$ given by

$$\langle X, Y \rangle = \text{trace}_{\text{End}_k(U)/k}(XY).$$

If J is an \mathcal{O} lattice in $\text{End}_k(U)$ we set

$$J^* = \{ X \in \text{End}_k(U) \mid \text{trace}(X \cdot J) \subseteq \mathcal{O} \}$$

Thus J^* is the complementary (dual) lattice of J with respect to trace. If J is an $\mathcal{A} - \mathcal{A}$ bimodule for some order $\mathcal{A} \subset \text{End}_k(U)$, then J^* is also an $\mathcal{A} - \mathcal{A}$ bimodule.

We remind the reader of the following lemma, which has been used implicitly in 2.1-2.2.

LEMMA. — *Let V be a finite dimension k -vector space endowed with a non-degenerate bilinear form*

$$f: V \times V \rightarrow k$$

If L is an \mathcal{O} lattice in V we write L^* for the complementary lattice $L^* = \{ v \in V \mid f(v, L) \subseteq \mathcal{O} \}$.

Then if L, M are lattices in V , we have

$$L^* + M^* = (L \cap M)^*$$

Proof. — First, note the following two facts:

- (i) If $L_1 \supset L_2$ are lattices in V , then $L_1^* \subset L_2^*$.
- (ii) $(L^*)^* = L$, as follows by taking dual bases.

Now, $L^* + M^*$ is the smallest \mathcal{O} -module containing L^*, M^* . From (i) we see that $(L \cap M)^* \supset L^*, M^*$ so that $(L \cap M)^* \supset L^* + M^*$. Let H be any lattice containing L^*, M^* . Then $H^* \subset L, H^* \subset M$ so that $H^* \subset L \cap M$ by (ii); this implies $(H^*)^* \supset (L \cap M)^*$. Thus H contains $(L \cap M)^*$. Hence $L^* + M^* = (L \cap M)^*$.

2.5. Let $e = e' = e''$ be the period of the orders $\mathcal{A}', \mathcal{A}''$. The complementary lattices $(\mathcal{B}'^i)^*, (\mathcal{B}''^i)^*$ are readily computed since $\mathcal{A}', \mathcal{A}''$ are hereditary orders. Indeed from [B], 1.11 we have $(\mathcal{B}'^i)^* = \mathcal{B}'^{1-e-i}, (\mathcal{B}''^i)^* = \mathcal{B}''^{1-e-i}$. Applying Lemma 2.4 we obtain the following result.

$$\text{LEMMA. — } (\mathcal{B}'^i \cap \mathcal{B}''^i)^* = \mathcal{B}'^{1-e-i} + \mathcal{B}''^{1-e-i}.$$

2.6 LEMMA. — *For each $i \in \mathbf{Z}$, $\mathcal{B}^i \supset \mathcal{B}''^{i+1}, \mathcal{B}''^i \supset \mathcal{B}^{i+1}$.*

Proof. — This result depends essentially on the properties described by lemma 2.2 and is proved in a more general situation in [M2]. For the sake of completeness we shall sketch a proof. By periodicity we can suppose $i \geq 0$.

Lemma 2.2 implies that we can label the chain $\mathcal{L} \oplus \mathcal{M} = \mathcal{L}' \cup \mathcal{L}''$ in the form $\dots \supset M_0 \supset L_0 \supseteq M_1 \supset L_1 \supseteq M_2 \supset L_2 \supseteq \dots$. Here, $M_i \in \mathcal{L}'$, $L_i \in \mathcal{L}''$ and possibly

$L_i = M_i$. Thus it follows that

$$\mathcal{B}''^i L_i \subseteq \mathcal{B}''^i M_i \subset M_{i+1}$$

and

$$\mathcal{B}'' M_{i+1} \subseteq M_{i+1+1} \subseteq L_{i+1}.$$

It follows immediately that $\mathcal{B}''^{i+1} \subseteq \mathcal{B}''^i$, all $i \in \mathbb{Z}$.

(In fact this argument shows that $\mathcal{B}''^i L_i \subseteq L_{i+1}$, unless $M_{i+1} \neq L_{i+1}$.)

2.7. For $i \geq 0$ we now have the following filtration.

$$\mathcal{A}' \cap \mathcal{A}'' \supset \mathcal{B}' + \mathcal{B}'' \supset \mathcal{B}' \cap \mathcal{B}'' \supset \mathcal{B}'^2 + \mathcal{B}''^2 \supset \dots \supset \mathcal{B}'^e + \mathcal{B}''^e \supset \mathcal{B}'^e \cap \mathcal{B}''^e \supset \dots$$

which extends by periodicity into an infinite filtration in both directions:

$$\begin{aligned} \dots \supset \mathcal{B}'^{-1} \cap \mathcal{B}''^{-1} \supset \mathcal{A}' + \mathcal{A}'' \supset \mathcal{A}' \cap \mathcal{A}'' \supset \mathcal{B}' + \mathcal{B}'' \supset \mathcal{B}' \cap \mathcal{B}'' \supset \dots \\ \supset \mathcal{B}'^e + \mathcal{B}''^e \supset \mathcal{B}'^e \cap \mathcal{B}''^e \supset \dots \end{aligned}$$

We summarize the observations of 2.5-2.6 in the following proposition.

PROPOSITION. — (a) $\mathcal{B}' + \mathcal{B}'' = \mathcal{B}_{\mathcal{A}' \cap \mathcal{A}''}$.

(b) $\mathcal{B}'^e \cap \mathcal{B}''^e = \pi(\mathcal{A}' \cap \mathcal{A}'')$.

(c) For each $i \in \mathbb{Z}$, $(\mathcal{B}''^i \cap \mathcal{B}''^i)^* = \mathcal{B}'^{1-e-i} + \mathcal{B}''^{1-e-i}$.

(d) For each $i \in \mathbb{Z}$, we have $\mathcal{B}''^i + \mathcal{B}''^i \supset \mathcal{B}''^i \cap \mathcal{B}''^i \supset \mathcal{B}''^{i+1} + \mathcal{B}''^{i+1}$.

Proof. — We proved (c) in Lemma 2.5. For any hereditary order \mathcal{A} one has $\mathcal{A}^* = \mathcal{B}_{\mathcal{A}}^{1-e} = \pi^{-1} \mathcal{B}_{\mathcal{A}}$ (e the period of \mathcal{A} ; see [B] 1.11). By lemma 2.5

$$\mathcal{A}^* = (\mathcal{A}' \cap \mathcal{A}'')^* = \mathcal{B}'^{1-e} + \mathcal{B}''^{1-e}.$$

Applying π we obtain $\mathcal{B}_{\mathcal{A}} = \mathcal{B}' + \mathcal{B}''$. This gives (a), and (b) is obvious since $\pi \mathcal{A}' = \mathcal{B}'^e$, $\pi \mathcal{A}'' = \mathcal{B}''^e$. The left most inclusion of (d) is obvious, and the second follows from 2.6.

2.8 COROLLARY. — *The sequence*

$$\dots \supset \mathcal{A}' + \mathcal{A}'' \supset \mathcal{A}' \cap \mathcal{A}'' \supset \mathcal{B}' + \mathcal{B}'' \supset \mathcal{B}' \cap \mathcal{B}'' \supset \dots \supset \mathcal{B}'^e + \mathcal{B}''^e \supset \mathcal{B}'^e \cap \mathcal{B}''^e \supset \dots$$

is a periodic lattice chain in $\text{End}_k(V)$, which is self dual with respect to trace, consisting of $\mathcal{A}' \cap \mathcal{A}''$ -bimodules which are σ -stable.

The only comment we need to make here is that $\mathcal{B}''^i = \sigma \mathcal{B}''^i$ for each $i \in \mathbb{Z}$.

2.9. *Remarks.* — (i) One can easily find a self dual slice for the above chain: if $e = -2i$ say, we take the slice centered around $\mathcal{B}''^{i+1} + \mathcal{B}''^{i+1}$ (with dual $\mathcal{B}''^i \cap \mathcal{B}''^i$), while if $e = -(2i-1)$ we take the slice centered around $\mathcal{B}''^i \cap \mathcal{B}''^i$ (with dual $\mathcal{B}''^i + \mathcal{B}''^i$).

(ii) The period of this filtration will in general be equal to $2e$. This number is frequently larger than that given by powers of the Jacobson radical.

(iii) The reader may ask if there are any collapsings in this filtration, e. g. if $\mathcal{B}^i \cap \mathcal{B}''^i = \mathcal{B}^{i+1} + \mathcal{B}''^{i+1}$ for some i . Provided the chains \mathcal{L}' , \mathcal{L}'' are distinct, there is not. This is shown implicitly in [M2] Section 5, where a more general type of filtration is considered. For the actual filtrations that we shall use in this paper, this will be apparent.

As a trivial but important example note that if $V = \{0\}$, we have trivially $\mathcal{L}' = \mathcal{L}''$, and the resulting filtration is simply that resulting from the powers of the Jacobson radical: one that has $\mathcal{B}^i + \mathcal{B}''^i = \mathcal{B}^i \cap \mathcal{B}''^i = \mathcal{B}^i$, all i .

2.10. We shall now suppose that \mathcal{L}' , \mathcal{L}'' are distinct [cf. remark 2.9 (iii) above] and label the filtration in Corollary 2.8 as follows.

$$\begin{aligned}\mathcal{B}_{2i} &= \mathcal{B}^i \cap \mathcal{B}''^i \\ \mathcal{B}_{2i+1} &= \mathcal{B}^{i+1} + \mathcal{B}''^{i+1}\end{aligned}$$

We then have

$$\begin{aligned}\dots \supset \mathcal{B}_{-2} \supset \mathcal{B}_{-1} \supset \mathcal{A}' \cap \mathcal{A}'' \supset \mathcal{B}_1 &= \mathcal{B}' + \mathcal{B}'' \\ &= \mathcal{B}_{\mathcal{A}' \cap \mathcal{A}''} \supset \mathcal{B}_2 \supset \dots \supset \mathcal{B}_{2e-1} = \mathcal{B}'^e + \mathcal{B}''^e \supset \pi(\mathcal{A}' \cap \mathcal{A}'') = \mathcal{B}_{2e} \supset \dots\end{aligned}$$

Our next result tells us how these $\mathcal{A}' \cap \mathcal{A}''$ -bimodules multiply together.

LEMMA. — $\mathcal{B}_m \cdot \mathcal{B}_n \subseteq \mathcal{B}_{m+n}$, for $m, n \in \mathbf{Z}$.

Proof. — (a) Suppose $m = 2i$, $n = 2j$. Then

$$\mathcal{B}_{2i} \cdot \mathcal{B}_{2j} \subseteq (\mathcal{B}^i \cap \mathcal{B}''^i) \cdot (\mathcal{B}^j \cap \mathcal{B}''^j) \subseteq \mathcal{B}^{i+j} \cap \mathcal{B}''^{i+j} = \mathcal{B}_{2(i+j)}$$

(b) Suppose $m = 2i$, $n = 2j+1$. Then

$$\mathcal{B}_{2i} \cdot \mathcal{B}_{2j+1} = (\mathcal{B}^i \cap \mathcal{B}''^i) \cdot (\mathcal{B}^{j+1} + \mathcal{B}''^{j+1}) \subseteq \mathcal{B}^{i+j+1} + \mathcal{B}''^{i+j+1} = \mathcal{B}_{2(i+j)+1}$$

(c) Suppose $m = 2i+1$, $n = 2j+1$. Then

$$\begin{aligned}\mathcal{B}_{2i+1} \cdot \mathcal{B}_{2j+1} &= (\mathcal{B}^{i+1} + \mathcal{B}''^{i+1}) \cdot (\mathcal{B}^{j+1} + \mathcal{B}''^{j+1}) \\ &\subseteq \mathcal{B}^{i+1} \cdot \mathcal{B}^{j+1} + \mathcal{B}^{i+1} \cdot \mathcal{B}''^{j+1} + \mathcal{B}''^{i+1} \cdot \mathcal{B}^{j+1} + \mathcal{B}''^{i+1} \cdot \mathcal{B}''^{j+1} \\ &\subseteq \mathcal{B}^{i+j+2} + \mathcal{B}''^{i+j+1} + \mathcal{B}^{i+j+1} + \mathcal{B}''^{i+j+2}\end{aligned}$$

Here we are using lemma 2.6. Since $\mathcal{B}''^{i+j+2} \subseteq \mathcal{B}^{i+j+1}$ by lemma 2.6 again, we see that

$$\mathcal{B}_{2i+1} \cdot \mathcal{B}_{2j+1} \subseteq \mathcal{B}^{i+j+1}$$

It follows also that $\mathcal{B}_{2i+1} \cdot \mathcal{B}_{2j+1} \subseteq \mathcal{B}''^{i+j+1}$, thus

$$\mathcal{B}_{2i+1} \cdot \mathcal{B}_{2j+1} \subseteq \mathcal{B}^{i+j+1} \cap \mathcal{B}''^{i+j+1} = \mathcal{B}_{2(i+j)+2}$$

2.11. Our filtration is σ -stable; thus we can define for each l

$$\mathcal{B}_l^- = \{x \in \mathcal{B}_l \mid x + \sigma x = 0\}.$$

(We shall refer to such elements as *skew-elements* in general.)

2.12. Now recall our lattice chain $\mathcal{L} \oplus \mathcal{M}$. The hereditary order \mathcal{A} that stabilizes it is precisely $\mathcal{A}' \cap \mathcal{A}''$, hence the filtration above consists of $\mathcal{A} - \mathcal{A}$ bimodules. This remark will be of some importance in the sequel.

2.13. Let P be the group $\mathcal{A} \cap G$, and for each integer $i > 0$, set $P_i = \{x \in P \mid x \equiv 1 \pmod{\mathcal{B}_i}\}$. Set $\mathcal{B}_1 = \mathcal{B} = \mathcal{B}_{\mathcal{A}}$: it is a two sided σ -stable ideal in \mathcal{A} . It follows that the F_q -algebra $\bar{\mathcal{A}} = \mathcal{A}/\mathcal{B}$ is naturally endowed with an involution $\bar{\sigma}$. Let $N(\bar{\mathcal{A}}, \bar{\sigma}) = \{x \in \bar{\mathcal{A}} \mid x \bar{\sigma} x = 1\}$.

THEOREM. — (a) For each $i > 0$, P_i is a normal subgroup of P , and the commutator subgroup (P_m, P_n) lies in P_{m+n} .

(b) The natural map $P \rightarrow N(\bar{\mathcal{A}}, \bar{\sigma})$ is surjective, and $P/P_1 \simeq N(\bar{\mathcal{A}}, \bar{\sigma})$.

(c) for each $i > 0$ there is a bijection

$$\begin{aligned} \mathcal{B}_i^- &\rightarrow P_i \\ x &\rightarrow (1-x)(1+x)^{-1} \end{aligned}$$

with inverse $P_i \rightarrow \mathcal{B}_i^-$ given by

$$p \rightarrow (1-p)(1+p)^{-1}$$

(d) If $2i \geq j \geq i \geq 1$ then there is an isomorphism of abelian groups induced by $x \rightarrow x - 1$:

$$P_i/P_j \xrightarrow{\sim} \mathcal{B}_i^-/\mathcal{B}_j^-$$

Proof. — Part (a) follows directly from the definitions and lemma 2.10. Part (b) is simply the approximation theorem 2.11, and part (a) of proposition 2.12 of [M].

As for part (c), we note that the ideals \mathcal{B}_i , $i \geq 0$ form a base of neighbourhoods of the identity in $\text{End}_k(V)$ (they contain among them all powers $\pi^n \mathcal{A}$, $n > 0$). Thus if $x \in \mathcal{B}_i^-$, $1+x$ is an invertible transformation with inverse given by the usual convergent expansion $1 - x + x^2 - \dots$. It follows that $(1-x)(1+x)^{-1}$ exists. (All this is saying, is that the Cayley transform is defined on \mathcal{B}_i^- , $i > 0$.) It is well known, and easy to check, that when this map is defined it takes a skew matrix x to one satisfying $y \sigma y = 1$.

Let $y \in P_i$. Solving for x in the equation $y = (1-x)(1+x)^{-1}$ we see that such an x must be given by the equation

$$x = (1-y)(1+y)^{-1}$$

Now by definition $y = 1 + b$, $b \in \mathcal{B}_i$, so that $1+y = 2+b = 2(1+b/2)$, where

$$2 \in \mathcal{O}^* \text{ (so } b/2 \in \mathcal{B}_i \text{)}.$$

It follows that $1+y$ is indeed invertible (although it does not necessarily belong to P_i). Then

$$\begin{aligned} (1-y)(1+y)^{-1} + \sigma(1-y)(1+y)^{-1} &= 0 \\ \leftrightarrow (1+\sigma y)(1-y) + (1-\sigma y)(1+y) &= 0 \\ \leftrightarrow 2 = 2(\sigma y)y \leftrightarrow 1 = y\sigma y. \end{aligned}$$

It follows that x as above is indeed skew. Finally, since $y=1+b, b \in \mathcal{B}_i$, we see that $x = -b/2(1-b/2+(b/2)^2-(b/2)^3+\dots)$ which evidently lies in \mathcal{B}_i . This completes the proof of part (c).

We turn now to part (d). For this we note that lemma 2.8 of [M] (the trace condition) is satisfied by the ideals $\mathcal{B}_i, i > 0$. Moreover, as mentioned above, the ideals $\mathcal{B}_i, i > 0$ define a base of neighbourhoods of the identity of $\text{End}_k(V)$. These observations enable us to copy the proof of proposition 2.12 (b) of [M]. We sketch this briefly.

First, we define a map

$$\begin{aligned} \rho: P_i &\rightarrow \mathcal{B}_i/\mathcal{B}_j \\ x &\rightarrow (x-1) + \mathcal{B}_j \end{aligned}$$

with the assumptions on i, j as in the assertion in part (d). With these assumptions in place, we see that the argument for 2.12 (b) of [M] shows that p has kernel P_j , and image in $\mathcal{B}_i^-/\mathcal{B}_j^-$. (One uses the trace condition for the last assertion.)

For the approximation argument which shows that the image is all of $\mathcal{B}_i^-/\mathcal{B}_j^-$, we can again copy the corresponding argument of proposition 2.12 (b) of [M]. Indeed, one checks that the induction step in 2.12 (b) of [M] constructs a_2 such that $a_2 + \sigma a_2 + a_2 \sigma a_2 \in \mathcal{B}_{2i+k}$. This lies in \mathcal{B}_{i+k+1} provided $2i+k \geq i+k+1$, i.e. provided $i \geq 1$. Thus the argument in *loc. cit.* is applicable.

2.14. We now consider the special situation of chains $\mathcal{L}, \mathcal{M}, \mathcal{L} \oplus' \mathcal{M}$ constructed in 2.1, in case $e(\mathcal{L}) \leq 2, e(\mathcal{M}) = 1$.

LEMMA. — Suppose \mathcal{L} has period 2, $\mathcal{L}: L_0^* \cong L_0 \cong \pi L_0^*$, and \mathcal{M} has period 1. Then $\mathcal{L} \oplus' \mathcal{M}$ has period 2, with a slice of the same type as that of \mathcal{L} .

Proof. — We can always find a self dual slice for \mathcal{L} of the form $\mathcal{L}: L_0 \cong L_0 \cong \pi L_0^*$. There are two possibilities for \mathcal{M} :

- (1) $\mathcal{M}: M_0^* = M_0 \cong \pi M_0^* = \pi M_0$.
- (2) $\mathcal{M}: M_0^* \cong M_0 = \pi M_0^*$.

In case (1) we see that $\mathcal{L} \oplus' \mathcal{M}$ is given by the self dual slice

$$\mathcal{L} \oplus' \mathcal{M} = \mathcal{L} \oplus \mathcal{M}: L_0^* \oplus M_0 \cong \pi L_0^* \oplus \pi M_0,$$

and with the notation of 2.2, $\mathcal{L}' = \mathcal{L}''$ while for case (2) we obtain

$$\mathcal{L} \oplus' \mathcal{M}: L_0^* \oplus M_0^* \cong L_0 \oplus M_0 \cong \pi L_0^* \oplus \pi M_0^* = \pi L_0^* \oplus M_0$$

In either case the resulting chain has period 2, as claimed.

Remark. — If $\mathcal{L} : L_1^* \supseteq L_0 = L^0 \supseteq L_1 = \pi L_1^*$, \mathcal{M} as above, then $\mathcal{L} \oplus' \mathcal{M}$ has period 3.

2.15. Recall that in 2.2, when $M_{s-1} = \pi M_{s-1}^*$, it was necessary to assume that $s \geq 2$. Henceforth, if the chain $\mathcal{L} \oplus' \mathcal{M}$ is constructed as in 2.14, when \mathcal{M} has period 1, we shall only ever consider the filtration on $\mathcal{A}_{\mathcal{L} \oplus' \mathcal{M}}$ given by powers of the Jacobson radical, where we define $\mathcal{L} \oplus \mathcal{M} = \mathcal{L} \oplus' \mathcal{M}$.

2.16. PROPOSITION. — *Suppose \mathcal{L} is a self dual lattice chain as in 2.14 with $e(\mathcal{L}) \leq 2$, and \mathcal{M} is a self dual lattice chain with $e(\mathcal{M}) = 1$. Then the lattice chains $\mathcal{L} \oplus \mathcal{M}$, $\mathcal{M} \oplus \mathcal{L}$ are the same, following the natural identification of $(V \oplus W, f \oplus g)$, with $(W \oplus V, g \oplus f)$, and are of period 2, with slice that of type $N_0^* \supseteq N_0 \supseteq \pi N_0^*$. Moreover the filtration in each case is simply that arising from powers of the Jacobson radical [cf. remark 2.9 (iii)].*

Proof. — Suppose that \mathcal{L} has period 1, \mathcal{M} has period 1. As in the proof of Lemma 2.14, there are only two possibilities for a lattice chain \mathcal{N} which is self dual of period 1:

- (1) $\mathcal{N} : N_0^* = N_0 \supseteq \pi N_0^* = \pi N_0$.
- (2) $\mathcal{N} : N_0^* \supseteq N_0 = \pi N_0^*$.

If \mathcal{L} , \mathcal{M} are both of type (1) we see that

$$\mathcal{L} \oplus' \mathcal{M} = \mathcal{L} \oplus \mathcal{M} : \dots \supset L_0 \oplus M_0 \supset \pi L_0 \oplus \pi M_0 \supset \dots,$$

with $\mathcal{L}' = \mathcal{L}''$ (with the obvious notation). Moreover,

$$\mathcal{M} \oplus' \mathcal{L} = \mathcal{M} \oplus \mathcal{L} : \dots \supset M_0 \oplus L_0 \supset \pi M_0 \oplus \pi L_0 \supset \dots$$

It follows that $\mathcal{L} \oplus \mathcal{M} = \mathcal{M} \oplus \mathcal{L}$ i.e. it makes no difference whether we change the elements in \mathcal{L} first, or whether we change the elements in \mathcal{M} first, after we have identified $(V \oplus W, f \oplus g)$ with $(W \oplus V, g \oplus f)$.

Suppose \mathcal{L} is of type (1), \mathcal{M} is of type (2), then we obtain (cf. 2.15)

$$\begin{aligned} \mathcal{L} \oplus' \mathcal{M} = \mathcal{L} \oplus \mathcal{M} : L_0^* \oplus M_0^* \supseteq L_0 \oplus M_0 \supseteq \pi L_0 \oplus M_0 = \pi L_0^* \oplus \pi M_0^* \\ \mathcal{M} \oplus \mathcal{L} : M_0^* \oplus L_0^* \supset M_0 \oplus L_0 \supset M_0 \oplus \pi L_0 = \pi M_0^* \oplus \pi L_0^*. \end{aligned}$$

Thus $\mathcal{L} \oplus \mathcal{M} = \mathcal{M} \oplus \mathcal{L}$ after the appropriate identification, and has period 2.

Finally, suppose \mathcal{L} , \mathcal{M} are both of type (2); we see that

$$\mathcal{L} \oplus' \mathcal{M} = \mathcal{L} \oplus \mathcal{M} : \dots \supset L_0^* \oplus M_0^* \supset L_0 \oplus M_0 = \pi L_0^* \oplus \pi M_0^* \supset \dots,$$

which is the same as $\mathcal{M} \oplus \mathcal{L}$, after identification.

Next, suppose that \mathcal{L} has period 2, with self dual slice given by

$$\mathcal{L} : L_0^* \supseteq L_0 \supseteq \pi L_0^*$$

while \mathcal{M} has period 1, given by case (1) and (2) in the proof of Lemma 2.14.

In case (1) we find

$$\begin{aligned}\mathcal{L} \oplus' \mathcal{M} = \mathcal{L} \oplus \mathcal{M} : L_0^* \oplus M_0 \supset L_0 \oplus M_0 \supset \pi L_0^* \oplus \pi M_0 & \quad \text{and} \quad \mathcal{L}' = \mathcal{L}'' \\ \mathcal{M} \oplus \mathcal{L} : M_0 \oplus L_0^* \supset M_0 \oplus L_0 \supset \pi M_0 \oplus \pi L_0^*\end{aligned}$$

and in case (2) we find

$$\begin{aligned}\mathcal{L} \oplus \mathcal{M} : L_0^* \oplus M_0^* \supset L_0 \oplus M_0 \supset \pi L_0^* \oplus \pi M_0^* = \pi L_0^* \oplus M_0 \\ \mathcal{M} \oplus \mathcal{L} : M_0^* \oplus L_0^* \supset M_0 \oplus L_0 \supset \pi M_0^* \oplus \pi L_0^*.\end{aligned}$$

Thus in each case we find $\mathcal{L} \oplus \mathcal{M} = \mathcal{M} \oplus \mathcal{L}$ after the appropriate identifications have been made.

This proves proposition 2.16.

2.17. We make one more observation in this connection which will be useful later. Recall that a *principal order* is one whose Jacobson radical is a principal ideal. (See [B-F] Theorem 1.3.2 for some equivalent formulations.)

LEMMA. — *Let \mathcal{L} be any self dual lattice chain with $e(\mathcal{L}) \leq 2$. Then either $\mathcal{A}_{\mathcal{L}}$ is principal ($e(\mathcal{L}) = 1$) or $\mathcal{A}_{\mathcal{L}} = \Lambda \cap \sigma \Lambda$ where Λ is principal ($e(\mathcal{L}) = 2$), or $\mathcal{A}_{\mathcal{L}} = \Lambda_1 \cap \Lambda_2$, where $\Lambda_i (i = 1, 2)$ is principal and σ -stable ($e(\mathcal{L}) = 2$).*

Proof. — Inspect the possibilities.

3. Filtrations associated to compact maximal tori

3.1. Let A be a commutative semisimple algebra which is finite dimensional over k , equipped with an involution σ such that $\sigma|_k = \sigma_0$. Thus A is a direct sum of separable field extensions E_i over k ; we suppose that $\sigma(E_i) = E_i$ and that $\sigma|_{E_i} \neq 1$, with the possible exception that for exactly one i , $E_i = k$ and $\sigma|_k = 1$. (This is the odd dimensional split orthogonal case, referred to as Case (b) in [M].) Let $\mu = (\mu_1, \dots, \mu_r) \in A$ where $0 \neq \mu_i \in E_i$ and define a form on A by

$$\begin{aligned}f_A : A \times A &\rightarrow A \rightarrow k \\ (x, y) &\rightarrow \text{trace}_{A/k}(\mu x \sigma y).\end{aligned}$$

Assume $\sigma\mu = \varepsilon\mu$, $\varepsilon = \pm 1$. Then f_A is a non-degenerate $\varepsilon - \sigma_0$ hermitian form on A . We shall suppose that $U(f, V) \simeq U(f_A, A)$; then

$$T = \{a \in A \mid a \sigma a = 1\}$$

is a compact maximal torus in $U(f_A, A)$, which we henceforth identify with $U(f, V) = G$.

Let $A = E_1 \oplus \dots \oplus E_r$, and write $\mu = (\mu_1, \dots, \mu_r)$. Then the form $f = \sum_{i=1}^r f_i$ where $f_i : E_i \times E_i \rightarrow k$ is an $\varepsilon - \sigma_0$ -hermitian form on E_i , $f_i = \text{trace}_{E_i/k} \circ F_{E_i}$ and

$$F_{E_i}(x, y) = \mu_i x \sigma_i y, \sigma_i = \sigma|_{E_i}.$$

3.2. The complementary lattice $(\mathcal{P}_i^{n_i})^*$ for a power of the prime ideal in E_i with respect to f_i is easily computed. If the E_i -valuation of μ_i is v_i , the complementary lattice $(\mathcal{P}_i^{n_i})^*$ is just $\mathcal{P}_i^{1 - e_i - n_i - v_i}$, where e_i is the ramification degree of E_i over k .

3.3. In this number, we suppose $A = E_1 = E$; we drop the subscript “1” in what follows. The remarks above tell us that $(\mathcal{P}_E^n)^* = \mathcal{P}_E^{1 - e - n - v}$ with respect to the form f . It follows that we obtain a self-dual lattice chain \mathcal{L} in V by taking the family of ideals $\{\mathcal{P}_E^n\}_{n \in \mathbb{Z}}$. In turn, this gives rise to an hereditary order \mathcal{A} , a parahoric subgroup $P \subseteq G$, and a filtration $\{P_i\}_{i \in \mathbb{N}}$ of P by open normal subgroups P_i of P , cf. section 2.9 remark (iii), and Theorem 2.13. (In this situation we have $\mathcal{A}' = \mathcal{A} = \mathcal{A}''$, using the notation 2.2.)

More precisely, we have

$$\begin{aligned} \mathcal{A} &= \{x \in \text{End}_k(E) \mid x \cdot \mathcal{P}_E^i \subseteq \mathcal{P}_E^i, \text{ each } i \in \mathbb{Z}\} \\ \mathcal{B} &= \{x \in \mathcal{A} \mid x \cdot \mathcal{P}_E^i \subseteq \mathcal{P}_E^{i+1}, \text{ each } i \in \mathbb{Z}\} \end{aligned}$$

$$P = \mathcal{A} \cap G; P_i = \{x \in G \mid x \equiv 1 \pmod{\mathcal{B}^i}, i \geq 1.$$

3.4. Next, suppose that each field E_i in the sum $A = \bigoplus_{i=1}^r E_i$ is unramified over k . Put $V' = E_1 \oplus \dots \oplus E_{r-1}$, $V'' = E_r$, with the corresponding forms f', f'' . In V'' we have a self dual lattice chain \mathcal{M} of period 1 via the construction in 3.3. Then, using induction, proposition 2.16 and lemma 2.14, we see that we can construct a self dual lattice chain \mathcal{L} in V' of period at most 2 (starting with the self dual chain of period 1 in E_1).

In fact, by using proposition 2.16 and an induction argument, we see that any lattice chain we construct in V' by summing the canonical ones (of period 1) in the E_i , will be identical to the one we start with (that is, the order in which we sum won't matter).

Again, using proposition 2.16, we see that $\mathcal{L} \oplus \mathcal{M} = \mathcal{M} \oplus \mathcal{L}$ after the obvious identification.

Let us call this lattice chain \mathcal{L}_u , with associated hereditary order \mathcal{A} , Jacobson radical \mathcal{B} . Following our convention in 2.15 and proposition 2.16, we give \mathcal{A} the filtration arising from powers of the Jacobson radical, and set

$$P = \mathcal{A} \cap G, \quad P_n = \{x \in \mathcal{A} \mid x - 1 \in \mathcal{B}^n\} \cap G, n > 0.$$

Let $A_j = \mathcal{B}^j \cap A$.

LEMMA (cf. [M] 2.28). — (a) $A_{2j-2} \supseteq A_{2j-1} = A_{2j}, j \in \mathbb{Z}$, if $e(L_u) = 2$.

(b) $A_{2j-2} \supseteq A_{2j-1} \supseteq A_{2j} = \pi A_{2j-1}, j \in \mathbb{Z}$ if $e(L_u) = 1$.

Proof. — (a) The chain \mathcal{L}_u has period 2, and $A_0 = \bigoplus_{i=1}^r \mathcal{O}_{E_i}$ as follows by inspection of the possibilities. It follows that $\mathcal{B}^2 \cap A = \bigoplus_{i=1}^r \mathcal{P}_i$. On the other hand, inspection also shows that $\mathcal{B} \cap \mathcal{A} = \bigoplus_{i=1}^r \mathcal{P}'_i$. The result now follows by periodicity, and the fact that $\mathcal{B} \cap A = \mathcal{B}^2 \cap A$ is principal.

Case (b) is similar but easier.

3.5. Now suppose that $A = E_1 \oplus \dots \oplus E_r$ and that at least one of the E_i is ramified over k . By reordering the summands of A (and thus replacing f by an equivalent form) if necessary, we may suppose that $A = E_1 \oplus \dots \oplus E_l \oplus \dots \oplus E_r$, where $E_i (1 \leq i \leq l)$ is unramified, and $E_i (l < i)$ is ramified over k . Set $A_u = E_1 \oplus \dots \oplus E_l$, so that

$$A = A_u \oplus E_{l+1} \oplus \dots \oplus E_r.$$

In A_u we have seen that there is a (unique up to equivalence) lattice chain \mathcal{L}_u obtained by summing the canonical chains (3.3) in the $E_i (1 \leq i \leq l)$ as in 3.4. Let $\mathcal{M}_{l+1}, \dots, \mathcal{M}_r$ denote the self dual lattice chains in E_{l+1}, \dots, E_r as in 3.3. Set

$$\mathcal{L}_0 = \dots = \mathcal{L}_l = \mathcal{L}_u, \quad \mathcal{L}_j = \mathcal{L}_{j-1} \oplus \mathcal{M}_j, \quad \text{if } j \geq l+1,$$

and

$$\mathcal{L}_r = \mathcal{L}_{r-1} \oplus \mathcal{M}_r = \mathcal{L}.$$

3.6. WARNING. — Unlike the situation for \mathcal{L}_u , this construction most decidedly depends on the ordering of the E_{l+1}, \dots, E_r : if we rearrange these and sum in the corresponding order, we obtain a different lattice chain (which will, however, be associate to the original one).

3.7. We have $\mathcal{L} = \mathcal{L}_r = \mathcal{L}_{r-1} \oplus \mathcal{M}_r$. We now apply the construction of section 2.7. There are hereditary orders $\mathcal{A}', \mathcal{A}''$ with Jacobson radicals $\mathcal{B}', \mathcal{B}''$, and a two sided infinite filtration (which is σ -stable) given by

$$\begin{aligned} \dots \supset \mathcal{B}'^{-1} + \mathcal{B}''^{-1} \supset \mathcal{A}' + \mathcal{A}'' \supset \mathcal{A}' \cap \mathcal{A}'' \supset \mathcal{B}' + \mathcal{B}'' \supset \mathcal{B}' \cap \mathcal{B}'' \\ \supset \dots \mathcal{B}'^e + \mathcal{B}''^e \supset \mathcal{B}'^e \cap \mathcal{B}''^e. \end{aligned}$$

As in 2.10, we put

$$\begin{aligned} \mathcal{B}_{2i} &= \mathcal{B}'^i \cap \mathcal{B}''^i \\ \mathcal{B}_{2i+1} &= \mathcal{B}'^{i-1} + \mathcal{B}''^{i+1} \end{aligned}$$

From section 2.13 we obtain a parahoric subgroup P , with associated σ -stable hereditary order $\mathcal{A} = \mathcal{A}' \cap \mathcal{A}''$, and a filtration by open normal subgroups P_i of P , $i \in \mathbb{N}$.

3.8. Before proceeding further, we give some examples for groups of small rank which we hope will make this construction more palatable.

3.9. *Example* $G = \mathbf{Sp}_4$. – We consider some representative anisotropic (maximal) tori in \mathbf{Sp}_4 and the resulting groups $\{P_i\}_{i \in \mathbf{N}}$.

If we apply the machinery of [M] section 1 we see that such tori can arise from 4-dimensional commutative semisimple algebras with involution σ , each of whose field components is stable by σ , and not fixed pointwise by σ . The basic possibilities are listed below.

(a) **A:** separable field extension E of degree 4.

(b) **A:** product of two separable field extensions E_1, E_2 each of degree 2 over k .

Case (a). – Consider for example the situation where E is a Galois extension with an unramified subfield E_u such that σ fixes E_u , and E is totally ramified over E_u .

Suppose that $E_u = k(\sqrt{\tau})$ where $\tau \in \mathcal{O}^*$ such that τ is not a square mod \mathcal{P} . Suppose also that $E = E_u(\sqrt{\pi})$ where π is a given uniformizer in k , and hence E_u .

From [M], 2.4, we obtain a skew form on the k -space E via $(x, y) \rightarrow \text{trace}_{E/k}(\mu x \sigma y)$, and we identify \mathbf{Sp}_4 with the group preserving this form, with $\mu = \sqrt{\pi}$.

We obtain a Witt basis $\{e_i\}$ (cf. [M], 2.14) for $1 \leq i \leq 4$ by taking $e_1 = 1, e_4 = \sqrt{\pi}/4\pi, e_2 = \sqrt{\tau}, e_3 = \sqrt{\tau}\sqrt{\pi}/4\pi\tau$. Then $f(e_i, e_{5-j}) = \delta_{i,j}$.

The lattice chain in this case is given by 3.3, i.e. it is $\{\mathcal{P}_E^n\}_{n \in \mathbf{Z}}$. By using the basis above we see that

$$\begin{aligned} \mathcal{O}_E &= \mathcal{O}e_1 \oplus \mathcal{O}e_1 \oplus \pi\mathcal{O}e_3 \oplus \pi\mathcal{O}e_4 \\ \mathcal{P}_E &= \pi\mathcal{O}e_1 \oplus \pi\mathcal{O}e_2 \oplus \pi\mathcal{O}e_3 \oplus \pi\mathcal{O}e_4 \end{aligned}$$

Moreover \mathcal{P}_E^{-1} is self dual with respect to f ; and the lattice chain has period 2 since E has ramification degree 2. It follows that the parahoric P is given in 2×2 block matrix form as $\begin{pmatrix} \mathcal{O} & \mathcal{O} \\ \pi & \mathcal{O} \end{pmatrix}$ and the filtration is that coming from powers of Jacobson radical, or equivalently, from the canonical affine height function (cf. 5.16 of [Mo 2], II). This torus, and the filtration $\{P_i\}$, occur in [M]. (If one takes E such that E/E^σ is unramified, E^σ/k quadratic ramified, one obtains the non-standard maximal parahoric.)

Case (b). – (i) We consider first the situation where E_1 is unramified, E_2 is ramified, and take the form

$$\text{trace}_{E_1/k}(\sqrt{\tau_1}x_1\sigma_1y_1) + \text{trace}_{E_2/k}(\sqrt{\pi_2}x_2\sigma_2y_2).$$

Here we have taken $E_1 = k(\sqrt{\tau})$, $E_2 = k(\sqrt{\pi})$ and we use subscripts to denote the field we are considering.

For a Witt basis we take

$$e_1 = 1_2, \quad e_4 = \sqrt{\pi_2}/\pi_2, \quad e_2 = 1_1, \quad e_3 = \sqrt{\tau_1}/\tau_1$$

Now we apply the recipe of sections 3.5-3.7. We have $V = V_1 \oplus V_2$ where $V_i = E_i (i=1, 2)$. The lattice chain we take in E_1 is given by the self dual slice

$$\mathcal{L}: \dots \supset \mathcal{O}_{E_1} \supset \pi \mathcal{O}_{E_1} = \mathcal{P}_{E_1} \supset \dots$$

With respect to the basis e_2, e_3 above we have

$$\mathcal{O}_{E_1} = \mathcal{O} e_2 \oplus \mathcal{O} e_3, \quad \text{etc.}$$

In $E_2 = V_2$ the lattice chain \mathcal{M} is also $\{\mathcal{P}_{E_2}^n\}$, with $(\mathcal{P}_2^{-1})^* = \mathcal{P}_2^{-1}$, $\mathcal{O}_2^* = \mathcal{P}_2^{-1}$, etc. With respect to the basis e_2, e_4 the slice for this chain is as follows

$$\begin{array}{ccccccc} \mathcal{P}_2^{-2} & \supset & \mathcal{P}_2^{-1} & \supset & \mathcal{O}_2 & \supset & \mathcal{P}_2 \\ \mathbf{M}^* & & \mathbf{M}_0 & & \mathbf{M}_1 = \pi \mathbf{M}_1^* & & \\ \pi^{-1} \mathcal{O} e_1 \oplus \mathcal{O} e_4 & \supset & \mathcal{O} e_1 \oplus \mathcal{O} e_4 & \supset & \mathcal{O} e_1 \oplus \mathcal{O} e_4 & \supset & \end{array}$$

The lattice chain $\mathcal{L} \oplus \mathcal{M}$ is given by the self dual slice

$$\mathbf{L}_0 \oplus \mathbf{M}_1^* \supset \mathbf{L}_0 \oplus \mathbf{M}_0 \supset \mathbf{L}_0 \oplus \mathbf{M}_1 \supset \pi \mathbf{L}_0 \oplus \mathbf{M}_1 = \pi \mathbf{L}_0 \oplus \pi \mathbf{M}_1^*.$$

This chain has period 3. The orders \mathcal{A}' , \mathcal{A}'' we defined by the (non self dual) chains

$$\mathcal{L}': \dots \supset \mathbf{L}_0 \oplus \mathbf{M}_1^* \supset \mathbf{L}_0 \oplus \mathbf{M}_0 \supset \pi \mathbf{L}_0 \oplus \pi \mathbf{M}_1^* \supset \dots$$

$$\mathcal{L}'': \dots \supset \mathbf{L}_0 \oplus \mathbf{M}_0 \supset \mathbf{L}_0 \oplus \mathbf{M}_1 \supset \pi \mathbf{L}_0 \oplus \pi \mathbf{M}_0 \supset \dots$$

respectively. With respect to the basis above, they have the matrix form

$$\begin{array}{l} \mathcal{A}': \begin{bmatrix} \mathcal{O} & \mathcal{O} & \mathcal{O} & \mathcal{O} \\ \pi & \mathcal{O} & \mathcal{O} & \mathcal{O} \\ \pi & \mathcal{O} & \mathcal{O} & \mathcal{O} \\ \pi & \mathcal{O} & \mathcal{O} & \mathcal{O} \end{bmatrix}, \quad \mathcal{B}': \begin{bmatrix} \pi & \mathcal{O} & \mathcal{O} & \mathcal{O} \\ \pi & \pi & \pi & \pi \\ \pi & \pi & \pi & \pi \\ \pi & \pi & \pi & \pi \end{bmatrix}, \\ \mathcal{A}'': \begin{bmatrix} \mathcal{O} & \mathcal{O} & \mathcal{O} & \mathcal{O} \\ \mathcal{O} & \mathcal{O} & \mathcal{O} & \mathcal{O} \\ \mathcal{O} & \mathcal{O} & \mathcal{O} & \mathcal{O} \\ \pi & \pi & \pi & \mathcal{O} \end{bmatrix}, \quad \mathcal{B}'': \begin{bmatrix} \pi & \pi & \pi & \mathcal{O} \\ \pi & \pi & \pi & \mathcal{O} \\ \pi & \pi & \pi & \mathcal{O} \\ \pi & \pi & \pi & \pi \end{bmatrix}. \end{array}$$

From this we see that $\mathcal{A} = \mathcal{A}' \cap \mathcal{A}''$ is the hereditary order

$$\begin{array}{l} \begin{bmatrix} \mathcal{O} & \mathcal{O} & \mathcal{O} & \mathcal{O} \\ \pi & \mathcal{O} & \mathcal{O} & \mathcal{O} \\ \pi & \mathcal{O} & \mathcal{O} & \mathcal{O} \\ \pi & \pi & \pi & \mathcal{O} \end{bmatrix}, \quad \mathcal{B}: \begin{bmatrix} \pi & \mathcal{O} & \mathcal{O} & \mathcal{O} \\ \pi & \pi & \pi & \mathcal{O} \\ \pi & \pi & \pi & \mathcal{O} \\ \pi & \pi & \pi & \pi \end{bmatrix}, \end{array}$$

and

$$\mathcal{B}' \cap \mathcal{B}'' = \begin{bmatrix} \pi & \pi & \pi & \mathcal{O} \\ \pi & \pi & \pi & \pi \\ \pi & \pi & \pi & \pi \\ \pi & \pi & \pi & \pi \end{bmatrix},$$

$$\mathcal{B}'^2 + \mathcal{B}''^2 = \pi^2 (\mathcal{B}' \cap \mathcal{B}'')^* = \begin{bmatrix} \pi & \pi & \pi & \pi \\ \pi & \pi & \pi & \pi \\ \pi & \pi & \pi & \pi \\ \pi^2 & \pi & \pi & \pi \end{bmatrix},$$

$\mathcal{B}'^2 \cap \mathcal{B}''^2 = \pi (\mathcal{A}' \cap \mathcal{A}'')$. This filtration has period 4, and cannot be (visibly it is not) the standard filtration of period 3; it is the filtration (5.21) which occurs in [Mo 2], II.

With this all said, one can make a few comments which perhaps motivate the filtration (2.8) a little more.

Consider the lattice chain $\mathcal{L} \oplus \mathcal{M}$ constructed above, and imagine that we are considering elements in \mathcal{A} which only “see” \mathcal{M} (cf. the elements of \mathcal{O}_{E_2}). More precisely, consider the two sided ideal \mathcal{B}_2 in \mathcal{A} which consists of elements which send

$$(2) \quad \left\{ \begin{array}{l} L_0 \oplus M_1^* \text{ inside } L_0 \oplus M_0 \\ L_0 \oplus M_0 \text{ inside } \pi L_0 \oplus M_1 \\ L_0 \oplus M_1 \text{ inside } \pi L_0 \oplus \pi M_0 \end{array} \right.$$

(This imitates the action of \mathcal{P}_2 on $\mathcal{L} \oplus \mathcal{M}$.) A calculation shows that \mathcal{B}_2 consists of matrices of the form

$$\begin{bmatrix} \pi & \pi & \pi & \mathcal{O} \\ \pi & \pi & \pi & \pi \\ \pi & \pi & \pi & \pi \\ \pi & \pi & \pi & \pi \end{bmatrix}.$$

Thus it is the \mathcal{B}_2 considered above. One then finds that $\pi^2 \mathcal{B}_2^*$ consists of matrices of the form

$$\begin{bmatrix} \pi & \pi & \pi & \pi \\ \pi & \pi & \pi & \pi \\ \pi & \pi & \pi & \pi \\ \pi^2 & \pi & \pi & \pi \end{bmatrix}$$

Thus the “smallest” filtration which contains \mathcal{A} , \mathcal{B}_2 and is self dual with respect to trace is the one above.

In general one cannot define ideals (congruence subgroups) by requirements like (2). Instead, one uses the orders \mathcal{A}' , \mathcal{A}'' , \mathcal{B}' , \mathcal{B}'' , etc.

(ii) Next, consider the case where $E_1 = E_2 = k(\sqrt{\pi})$ and the form is given by

$$\text{trace}_{E_1/k}(\sqrt{\pi} x_1 \sigma y_1) + \text{trace}_{E_2/k}(\sqrt{\pi} x_2 \sigma y_2).$$

The respective slices are given by

$$\begin{aligned} \mathcal{M} : \mathcal{P}_2^{-2} \supset \mathcal{P}_2^{-1} \supset \mathcal{O}_2 \\ \quad \quad \quad \mathbf{M}_1^* \quad \mathbf{M}_0 \quad \mathbf{M}_1 = \pi \mathbf{M}_0^* \\ \mathcal{L} : \mathcal{P}_1^{-2} \supset \mathcal{P}_1^{-1} \supset \mathcal{O}_1 \\ \quad \quad \quad \mathbf{L}_1^* \quad \mathbf{L}_0 \quad \mathbf{L}_1 = \pi \mathbf{L}_1^* \end{aligned}$$

The self dual slice for $\mathcal{L} \oplus \mathcal{M}$ is given by

$$\mathbf{L}_1^* \oplus \mathbf{M}_1^* \supset \mathbf{L}_0 \oplus \mathbf{M}_1^* \supset \mathbf{L}_0 \oplus \mathbf{M}_0 \supset \mathbf{L}_0 \oplus \mathbf{M}_1 \supset \mathbf{L}_1 \oplus \mathbf{M}_1 = \pi \mathbf{L}_1^* \oplus \pi \mathbf{M}_1^*$$

For a Witt basis we take

$$e_1 = 1_2, \quad e_4 = \sqrt{\pi_2}/\pi, \quad e_2 = 1_1, \quad e_3 = \sqrt{\pi_1}/\pi$$

(as usual, the subscripts on the right are used to distinguish the fields.) Then

$$\begin{aligned} \mathcal{O}_1 \oplus \mathcal{O}_2 &= \mathcal{O} e_1 \oplus \mathcal{O} e_2 \oplus \pi \mathcal{O} e_3 \oplus \pi \mathcal{O} e_4 = \mathbf{L}_1 \oplus \mathbf{M}_1 \\ \mathcal{P}_1^{-1} \oplus \mathcal{O}_2 &= \mathcal{O} e_1 \oplus \mathcal{O} e_2 \oplus \mathcal{O} e_3 \oplus \pi \mathcal{O} e_4 = \mathbf{L}_0 \oplus \mathbf{M}_1 \\ \mathcal{P}_1^{-1} \oplus \mathcal{P}_2^{-1} &= \mathcal{O} e_1 \oplus \mathcal{O} e_2 \oplus \mathcal{O} e_3 \oplus \mathcal{O} e_4 = \mathbf{L}_0 \oplus \mathbf{M}_0 \\ \mathcal{P}_1^{-1} \oplus \mathcal{P}_2^{-2} &= \pi^{-1} \mathcal{O} e_1 \oplus \mathcal{O} e_2 \oplus \mathcal{O} e_3 \oplus \mathcal{O} e_4 = \mathbf{L}_0 \oplus \mathbf{M}_1^* \\ \mathcal{P}_1^{-2} \oplus \mathcal{P}_2^{-2} &= \pi^{-1} \mathcal{O} e_1 \oplus \pi^{-1} \mathcal{O} e_2 \oplus \mathcal{O} e_3 \oplus \mathcal{O} e_4 = \mathbf{L}_1^* \oplus \mathbf{M}_1^* \end{aligned}$$

We find $\mathcal{A}'(\mathcal{A}'')$ is given by the chain

$$\begin{aligned} \mathcal{L}' : \mathbf{L}_0 \oplus \mathbf{M}_1^* \supset \mathbf{L}_0 \oplus \mathbf{M}_0 \supset \pi \mathbf{L}_0 \oplus \pi \mathbf{M}_1^* \\ (\mathcal{L}'' : \mathbf{L}_0 \oplus \mathbf{M}_0 \supset \mathbf{L}_0 \oplus \mathbf{M}_1 \supset \pi \mathbf{L}_0 \oplus \pi \mathbf{M}_0) \end{aligned}$$

and in matrix form we obtain the same $\mathcal{A}' \cap \mathcal{A}''$, and filtration as in the preceding example.

We remark that the torus arising in this example is *principal* in the sense of [M] 1.4, but the filtration is not the one used in [M].

3.10. *Example.* — Let k be quadratic unramified over k_0 , with involution σ_0 , and let

\mathbf{G} be the unitary group preserving the hermitian form $\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$, where a Witt basis

is provided by the standard column vectors e_1, e_2, e_3 .

Let E be a quadratic ramified extension over k , say $E = k(\sqrt{\pi}) = E_0(\sqrt{\tau})$ where $E_0 = k_0(\sqrt{\pi})$, $k = k_0(\sqrt{\tau})$, $\tau \in \mathcal{O}_0^*$ a nonsquare mod \mathcal{P} . We define an involution σ on E by the rules $\sigma|_{E_0} \equiv 1$, $\sigma(\sqrt{\tau}) = -\sqrt{\tau}$, so that σ extends σ_0 . Let $A = E \oplus k$, and endow

it with the hermitian form

$$\text{trace}_{E/k}(\sqrt{\pi} x \sigma y) + x' \sigma_0 y'$$

where $x, y \in E, x', y' \in k$. By taking a basis of E consisting of $1, \sqrt{\pi/\pi}$ we see that this form is equivalent to the one above: we obtain a compact maximal torus T in G . Namely, $T = \{(x, y) \in E \oplus k \mid x \sigma x = 1, y \sigma_0 y = 1\}$.

We take $\mathcal{M} = \{\mathcal{P}_E^n\}, \mathcal{L} = \{\mathcal{P}_k^n\}$ (as we must) and obtain

$$\begin{aligned} \mathcal{L} \oplus \mathcal{M} : L_0 \oplus M_1^* &\supset L_0 \oplus M_0 \supset L_0 \oplus M_1 \supset \pi L_0 \oplus M_1 = \pi L_0 \oplus \pi M_1^* \\ \mathcal{O}_k \oplus \mathcal{P}_E^{-2} &\supset \mathcal{O}_k \oplus \mathcal{P}_E^{-1} \supset \mathcal{O}_k \oplus \mathcal{O}_E \supset \mathcal{P}_k \oplus \mathcal{O}_E. \end{aligned}$$

Then $\mathcal{A} = \mathcal{A}' \cap \mathcal{A}'' = \mathcal{A}_{\mathcal{L} \oplus \mathcal{M}}$ and is an Iwahori order.

With respect to the basis above (for $E \oplus k$) we have

$$\begin{aligned} \mathcal{A} &= \begin{bmatrix} \mathcal{O} & \mathcal{O} & \mathcal{O} \\ \mathcal{P} & \mathcal{O} & \mathcal{O} \\ \mathcal{P} & \mathcal{P} & \mathcal{O} \end{bmatrix}, & \mathcal{B}' \cap \mathcal{B}'' &= \begin{bmatrix} \mathcal{P} & \mathcal{P} & \mathcal{O} \\ \mathcal{P} & \mathcal{P} & \mathcal{P} \\ \mathcal{P} & \mathcal{P} & \mathcal{P} \end{bmatrix} \\ \pi^2 (\mathcal{B}' \cap \mathcal{B}'')^* &= \mathcal{B}'^2 + \mathcal{B}''^2 = \begin{bmatrix} \mathcal{P} & \mathcal{P} & \mathcal{P} \\ \mathcal{P} & \mathcal{P} & \mathcal{P} \\ \mathcal{P}^2 & \mathcal{P} & \mathcal{P} \end{bmatrix} \end{aligned}$$

In other words we obtain the filtration given by

$$\begin{aligned} \dots \supseteq \begin{bmatrix} \mathcal{O} & \mathcal{O} & \mathcal{O} \\ \pi & \mathcal{O} & \mathcal{O} \\ \pi & \pi & \mathcal{O} \end{bmatrix} &\supseteq \begin{bmatrix} \pi & \mathcal{O} & \mathcal{O} \\ \pi & \pi & \mathcal{O} \\ \pi & \pi & \pi \end{bmatrix} \supseteq \begin{bmatrix} \pi & \pi & \mathcal{O} \\ \pi & \pi & \pi \\ \pi & \pi & \pi \end{bmatrix} \\ &\supseteq \begin{bmatrix} \pi & \pi & \pi \\ \pi & \pi & \pi \\ \pi^2 & \pi & \pi \end{bmatrix} \supseteq \begin{bmatrix} \pi & \pi & \pi \\ \pi^2 & \pi & \pi \\ \pi^2 & \pi^2 & \pi \end{bmatrix} \supseteq \dots \end{aligned}$$

which occurs in [Mol], 3.5 (c).

3.11. Write $V = V_1 \oplus V_2, V_2 = E_r, V_1 = E_1 \oplus \dots \oplus E_{r-1}$.

We assume of course that the ramification degree of E_r is at least 2, so that the constructions of 3.5-3.7 apply, in particular $f = f_1 \oplus f_2, \mathcal{L} = \mathcal{L}_{r-1} \oplus \mathcal{M}_r$. We then have $U(f_1) \times U(f_2) \subseteq \text{End}_k(V_1) \oplus \text{End}_k(V_2)$. The next result explains the relationship between \mathcal{A} and $\text{End}_k(V_1) \oplus \text{End}_k(V_2)$.

PROPOSITION. — (a) $\mathcal{A} \cap (\text{End}_k(V_1) \oplus \text{End}_k(V_2)) = \mathcal{A} \cap \text{End}_k(V_1) \oplus \mathcal{A} \cap \text{End}_k(V_2)$.

(b) $\mathcal{A}_2 = \mathcal{A} \cap \text{End}_k(V_2)$ is a hereditary order, with lattice chain $\{\mathcal{P}_r^n\}_{n \in \mathbb{Z}}$. In particular, the Jacobson radical \mathfrak{J}_2 of \mathcal{A}_2 is principal, generated by π_r (cf. 2.17 for the definition of principal).

(c) With the notation of 2.10, 3.7, we have for each i

$$\mathcal{Q}_2^i = \mathcal{B}_{2i} \cap \text{End}_k(V_2) = \mathcal{B}_{2i-1} \cap \text{End}_k(V_2)$$

and moreover

$$\mathcal{B}_i \cap (\text{End}_k(V_1) \oplus \text{End}_k(V_2)) = \mathcal{B}_i \cap \text{End}_k(V_1) \oplus \mathcal{B}_i \cap \text{End}_k(V_2)$$

Proof. — (a) We evidently have

$$\mathcal{A} \cap (\text{End}_k(V_1) \oplus \text{End}_k(V_2)) \supseteq \mathcal{A} \cap \text{End}_k(V_1) \oplus \mathcal{A} \cap \text{End}_k(V_2).$$

Suppose that $(x, y) \in \mathcal{A}$, $x \in \text{End}_k(V_1)$, $y \in \text{End}_k(V_2)$. The self dual slice for \mathcal{A} is then given by

$$(1) \quad L_0^* \oplus M_{s-1}^* \supseteq L_0^* \oplus M_{s-2}^* \supseteq \dots \supseteq L_0 \oplus M_{s-1} \supseteq \pi L_0^* \oplus \pi M_{s-1}^*$$

or by

$$(2) \quad L_{r-1}^* \oplus M_{s-1}^* \supseteq L_0^* \oplus M_{s-1}^* \supseteq \dots \supseteq L_0 \oplus M_{s-1} \supseteq L_{r-1} \oplus M_{s-1} \supseteq \pi L_{r-1}^* \oplus \pi M_{s-1}^*$$

(we are using the notation of 2.2).

The result is now obvious: the point is that the lattice chain is split into direct summands in the respective vector spaces.

(b) This is now clear: $\mathcal{A}_2 = \mathcal{A} \cap \text{End}_k(V_2)$ is the stabilizer of the flag $\mathcal{M} = \{\mathcal{P}_r^n\}_{n \in \mathbb{Z}}$. It also follows from this that \mathcal{Q}_2 is a principal 2-sided ideal, generated by π_r .

(c) Consider \mathcal{B}^i , \mathcal{B}''^i ; in case (1) above in (a) we have:

$$\begin{aligned} \mathcal{L}' : \quad & L_0^* \oplus M_{s-1}^* \supseteq L_0^* \oplus M_{s-2}^* \supseteq \dots \supseteq L_0 \oplus M_{s-2} \supseteq \pi L_0^* \oplus \pi M_{s-1}^* \\ \mathcal{L}'' : \quad & L_0^* \oplus M_{s-2}^* \supseteq \dots \supseteq L_0 \oplus M_{s-2} \supseteq L_0 \oplus M_{s-1} \supseteq \pi L_0^* \oplus \pi M_{s-2}^* \end{aligned}$$

while in case (2) we have

$$\begin{aligned} \mathcal{L}' : \quad & L_{r-1}^* \oplus M_{s-1}^* \supseteq L_0^* \oplus M_{s-2}^* \supseteq \dots \supseteq L_0 \oplus M_{s-2} \supseteq L_0 \oplus M_{s-1} \supseteq \pi L_{r-1}^* \oplus \pi M_{s-1}^* \\ \mathcal{L}'' : \quad & L_0^* \oplus M_{s-1}^* \supseteq L_0^* \oplus M_{s-2}^* \supseteq \dots \supseteq L_0 \oplus M_{s-2} \supseteq L_{r-1} \oplus M_{s-1} \supseteq \pi L_0^* \oplus \pi M_{s-1}^* \end{aligned}$$

with the appropriate modifications if $M_0^* = M_0$ (cf. 2.2).

From this we see that $\pi_r^i \in \mathcal{B}^i \cap \mathcal{B}''^i$ [remember that in case (1) $M_{s-1} = \pi M_{s-1}^*$]. Furthermore if $(0, x) \in \mathcal{B}' \cap \mathcal{B}'' \cap \text{End}_k(V_2)$ we see by inspection that $x \in \mathcal{Q}_2$.

Now \mathcal{Q}_2 is principal, generated by π_r : $\mathcal{Q}_2 = \pi_r \mathcal{A}_2 = \mathcal{A}_2 \pi_r$, and we evidently have $\mathcal{Q}_2^i \subseteq (\mathcal{B}^i \cap \mathcal{B}''^i) \cap \text{End}_k(V_2)$. But again, if $(0, x) \in \mathcal{B}^i \cap \mathcal{B}''^i \cap \text{End}_k(V_2)$, we see that x sends every lattice M_j in \mathcal{M}_r into M_{j+i} (again by inspection: this was how $\mathcal{L} \oplus \mathcal{M}$ was constructed). This means that $x \in \mathcal{Q}_2^i$, by definition, and

$$\mathcal{Q}_2^i = \mathcal{B}_{2i} \cap \text{End}_k(V_2) \subseteq \mathcal{B}_{2i-1} \cap \text{End}_k(V_2).$$

Next, suppose that $(x, y) \in \mathcal{B}_{2i} \cap (\text{End}_k(V_1) \oplus \text{End}_k(V_2))$. Then just as above we see that $(0, y) \in \mathcal{B}_{2i} \cap \text{End}_k(V_2) = \mathcal{Q}_2^i$. It then follows that $(x, 0) = (x, y) - (0, y) \in \mathcal{B}_{2i}$. Thus

$$\mathcal{B}_{2i} \cap (\text{End}_k(V_1) \oplus \text{End}_k(V_2)) = \mathcal{B}_{2i} \cap \text{End}_k(V_1) \oplus \mathcal{B}_{2i} \cap \text{End}_k(V_2).$$

Taking complementary lattices with respect to trace in $\text{End}_k(V_1) \oplus \text{End}_k(V_2)$, we obtain

$$(\mathcal{B}_{2i} \cap (\text{End}_k(V_1) \oplus \text{End}_k(V_2)))^* = (\mathcal{B}_{2i} \cap \text{End}_k(V_1))^* \oplus (\mathcal{B}_{2i} \cap \text{End}_k(V_2))^*$$

Now Lemma 3.12 below tells us that

$$\mathcal{B}_{2i} = \bigoplus_{l, m} \mathcal{B}_{2i}(l, m), \quad 1 \leq l, m \leq 2$$

where $\mathcal{B}_{2i}(l, m) = \mathcal{B}_{2i} \cap \text{Hom}_k(V_m, V_l)$, whence $\mathcal{B}_{2i}^* = \bigoplus_{l, m} (\mathcal{B}_{2i})^*(l, m)$ where we put

$$(\mathcal{B}_{2i}(l, m))^* = \mathcal{B}_{2i} \cap \text{Hom}_k(V_m, V_l) \stackrel{\text{defn}}{=} \mathcal{B}_{2i}^*(l, m)$$

Thus

$$(\mathcal{B}_{2i} \cap (\text{End}_k(V_1) \oplus \text{End}_k(V_2)))^* = \mathcal{B}_{2i}^* \cap (\text{End}_k(V_1) \oplus \text{End}_k(V_2))$$

and

$$\mathcal{B}_{2i}^* \cap \text{End}_k(V_l) = (\mathcal{B}_{2i} \cap \text{End}_k(V_l))^*, \quad l = 1, 2$$

where, on the right hand side we mean complementary lattices with respect to $\text{trace}_{\text{End}_k(V_l)}$ (see Lemma 3.12 below).

We see from this that

$$\mathcal{B}_i \cap (\text{End}_k(V_1) \oplus \text{End}_k(V_2)) = \mathcal{B}_i \cap \text{End}_k(V_1) \oplus \mathcal{B}_i \cap \text{End}_k(V_2)$$

for all i in fact.

Now

$$(\mathcal{Q}_2^i)^* = \mathcal{Q}_2^{1-e-i} = \mathcal{B}_{2(1-e-i)} \cap \text{End}_k(V_2) = \mathcal{B}_{2i-1}^* \cap \text{End}_k(V_2) = (\mathcal{B}_{2i-1} \cap \text{End}_k(V_2))^*$$

where $e = e_r =$ ramification degree of $E_r =$ period of $\mathcal{M}_r =$ period of \mathcal{L}' or \mathcal{L}'' . Here the first and last “*”s are with respect to $\text{trace}_{\text{End}_k(V_2)}$. It follows that $\mathcal{Q}_2^i = \mathcal{B}_{2i-1} \cap \text{End}_k(V_2)$ as claimed.

Example. — We illustrate part of the previous proposition with a rather extreme example, which does not arise however in the context of supercuspidal representations.

For this we take V_1 to be a two-dimensional vector space with non degenerate alternating form f_1 , Witt basis e_1, e_6 . For our lattice chain \mathcal{L} we take (the slice).

$$\dots \supset \mathcal{P}^{-1} e_1 \oplus \mathcal{O} e_6 \supset \mathcal{O} e_1 \oplus \mathcal{P} e_6 \supset \dots \supset (\mathcal{L}_0^* \supset \mathcal{L}_0 = \pi \mathcal{L}_0^*)$$

We also take a four dimensional vector space V_2 , with non degenerate alternating form f_2 , Witt basis e_2, e_3, e_4, e_5 . For \mathcal{M} we take the chain given by the slice

$$\begin{aligned} \mathcal{P}^{-1} e_2 \oplus \mathcal{P}^{-1} e_3 \oplus \mathcal{O} e_4 \oplus \mathcal{O} e_5 &\supset \mathcal{P}^{-1} e_2 \oplus \mathcal{O} e_3 \oplus \mathcal{O} e_4 \oplus \mathcal{O} e_5 \\ &\supset \mathcal{O} e_2 \oplus \dots \oplus \mathcal{O} e_4 \oplus \mathcal{P} e_5 \\ &\supset \mathcal{O} e_2 \oplus \mathcal{O} e_3 \oplus \mathcal{P} e_4 \oplus \mathcal{P} e_5 \supset \dots \end{aligned}$$

which we label as

$$\supset M_1^* \supset M_0^* \supset M_0 \supset M_1 = \pi M_1^*.$$

We see that we are in the case 2.2 where $M_{s-1} = \pi M_{s-1}^*$, $L_0 = \pi L_0^*$. The lattice chains \mathcal{L}' , \mathcal{L}'' are equal to $\mathcal{L} \oplus \mathcal{M}$, we obtain as slice

$$L_0^* \oplus M_1^* \supset L_0^* \oplus M_0^* \supset L_0 \oplus M_0 \supset L_0 \oplus M_1 = \pi L_0^* \oplus \pi M_1^*.$$

The reader will find by a straightforward calculation that the order \mathcal{A} consists of matrices of the form

$$\begin{bmatrix} \mathcal{O} & \mathcal{O} & \mathcal{O} & \mathcal{O} & \pi^{-1} & \pi^{-1} \\ \mathcal{O} & \mathcal{O} & \mathcal{O} & \mathcal{O} & \pi^{-1} & \pi^{-1} \\ \pi & \pi & \mathcal{O} & \mathcal{O} & \mathcal{O} & \mathcal{O} \\ \pi & \pi & \pi & \mathcal{O} & \mathcal{O} & \mathcal{O} \\ \pi & \pi & \pi & \pi & \mathcal{O} & \mathcal{O} \\ \pi & \pi & \pi & \pi & \mathcal{O} & \mathcal{O} \end{bmatrix}$$

while \mathcal{B} is given by matrices of the form

$$\begin{bmatrix} \pi & \pi & \mathcal{O} & \mathcal{O} & \mathcal{O} & \mathcal{O} \\ \pi & \pi & \mathcal{O} & \mathcal{O} & \mathcal{O} & \mathcal{O} \\ \pi & \pi & \pi & \mathcal{O} & \mathcal{O} & \mathcal{O} \\ \pi & \pi & \pi & \pi & \mathcal{O} & \mathcal{O} \\ \pi^2 & \pi^2 & \pi & \pi & \pi & \pi \\ \pi^2 & \pi^2 & \pi & \pi & \pi & \pi \end{bmatrix}$$

Then $\mathcal{A} \cap \text{End}_k(V_1)$ is given by the outer four corners of the matrices above, while $\mathcal{A} \cap \text{End}_k(V_2)$ is given by the inner 4×4 square. Evidently $\mathcal{B}^3 = \pi \mathcal{A}$, $\mathcal{B}_{2i} = \mathcal{B}_{2i-1}$ while the resulting filtration on the 4×4 order

$$\begin{bmatrix} \mathcal{O} & \mathcal{O} & \mathcal{O} & \pi^{-1} \\ \pi & \mathcal{O} & \mathcal{O} & \mathcal{O} \\ \pi & \pi & \mathcal{O} & \mathcal{O} \\ \pi & \pi & \pi & \mathcal{O} \end{bmatrix}$$

is the standard one given by the radical $\mathcal{Q}_2 = \mathcal{B} \cap \text{End}_k(V_2)$ and it also has length 3.

If one instead inserts into \mathcal{M} the lattice $\mathcal{O}e_2 \oplus \dots \oplus \mathcal{O}e_5$ and repeats these calculations one sees that in this case \mathcal{L}' and \mathcal{L}'' are distinct so that the filtration $\{\mathcal{B}_i\}$ does not collapse to the standard lattice filtration. This example arises from a compact maximal torus, and in this case one finds that \mathcal{Q}_2 is principal, giving rise to the standard Iwahori filtration.

3.12. For each $1 \leq l, m \leq 2$, set

$$\mathcal{B}_i(l, m) = \mathcal{B}_i \cap \text{Hom}_k(V_m, V_l), \quad \mathcal{B}'^i(l, m) = \mathcal{B}'^i \cap \text{Hom}_k(V_m, V_l)$$

LEMMA. — $\mathcal{B}_i = \bigoplus_{1 \leq l, m \leq 2} \mathcal{B}_i(l, m)$.

Proof. — We begin by proving this when i is even. Write $V = V_1 \oplus V_2$, as usual. Then if $x \in \text{End}_k(V)$ we may represent it by a 2×2 block matrix $\begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix}$ where $x_{ij} \in \text{Hom}_k(V_j, V_i)$ for $1 \leq i, j \leq 2$. Let $L_a \oplus M_b$ be a lattice in, say, \mathcal{L}' . Then if $x \in \mathcal{B}'^i$, we have

$$x(L_a \oplus M_b) = (x_{11}L_a + x_{12}M_b) \oplus (x_{21}L_a + x_{22}M_b) \subseteq L_{f_L(a,b)} \oplus M_{f_M(a,b)}$$

where f_L, f_M are integer valued functions whose domains and ranges can be given explicitly depending on the lattices in \mathcal{L}, \mathcal{M} which occur as summands in \mathcal{L}' . We remark that we are using L_a to denote a generic lattice in \mathcal{L} (including dual lattices).

If we take the M_b -“coordinate” to be zero, we see that $x_{11}L_a \subseteq L_{f_L(a,b)}$, $x_{21}L_a \subseteq M_{f_M(a,b)}$.

Similarly, we find

$$x_{12}M_b \subseteq L_{f_L(a,b)}, \quad x_{22}M_b \subseteq M_{f_M(a,b)}$$

Putting all this together we see that

$$\mathcal{B}'^i \subseteq \bigoplus_{l,m} \mathcal{B}'^i(l, m), \quad \text{whence } \mathcal{B}'^i = \bigoplus_{l,m} \mathcal{B}'^i(l, m)$$

The same argument tells us that

$$\mathcal{B}''^i(l, m) = \bigoplus_{l,m} \mathcal{B}''^i(l, m)$$

It follows that if $x \in \mathcal{B}'^i = \bigoplus \mathcal{B}'^i(l, m)$ and $x \in \mathcal{B}''^i = \bigoplus \mathcal{B}''^i(l, m)$ then for each l, m , $x'_{l,m} = x''_{l,m}$ whence

$$\mathcal{B}_{2i} = \bigoplus \mathcal{B}_{2i}(l, m)$$

We treat the case i odd, by taking complementary lattices with respect to trace. Indeed from 2.7 we know that

$$\mathcal{B}_{2(-e-i)+1} = \mathcal{B}_{2i}^* = (\bigoplus \mathcal{B}_{2i}(l, m))^*$$

Let $x = \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix} \in \mathcal{B}_{2i}^*$, $b = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} \in \mathcal{B}_{2i}$.

Then

$$\text{trace}_{\text{End}_k(V_1)}(x_{11}b_{11} + x_{12}b_{21}) + \text{trace}_{\text{End}_k(V_2)}(x_{21}b_{21} + x_{22}b_{22}) \in \mathcal{O}$$

for any such b . Taking coordinates (with respect to b) as we may, the previous argument, we see that

$$\begin{aligned} x_{l,m} &\in \mathcal{B}_{2i}^*(l, m) \\ &= \mathcal{B}_{2i}^* \cap \text{Hom}_k(V_m, V_l) \\ &= \mathcal{B}_{2(-e-i)+1}(l, m) \end{aligned}$$

3.13. We keep the assumptions and notation of 3.11 and 3.12; we have $\mathcal{A} = \mathcal{B}_0$, $\mathcal{B} = \mathcal{B}_1 = \mathcal{B}' + \mathcal{B}''$. We write \mathcal{A}_{ij} for $\mathcal{A}(i, j)$, and \mathcal{B}_{ij} for $\mathcal{B}(i, j) = \mathcal{B}_1(i, j)$, when there is no risk of confusion with other notation. We then have the following result.

LEMMA. — (a) $\mathcal{A}/\mathcal{B} = \mathcal{A}_{11}/\mathcal{B}_{11} \oplus \mathcal{A}_{22}/\mathcal{B}_{22}$.

(b) \mathcal{A}_{ii} is the stabilizer of a lattice chain in V_i , with Jacobson radical \mathcal{B}_{ii} ($i = 1, 2$).

Proof. — The easiest way to see these assertions is to examine the lattice chain describing \mathcal{A} . Consider for example the case

$$\begin{aligned} \dots \subset L_{r-1}^* \oplus M_{s-1}^* \subset L_0^* \oplus M_{s-1}^* \supset L_0^* \oplus M_{s-2}^* \supset \dots \\ \supset L_0 \oplus M_{s-1} \supset L_{r-1} \oplus M_{s-1} \supset \pi L_{s-1}^* \oplus M_{s-1}^* \supset \dots \end{aligned}$$

Then \mathcal{A}/\mathcal{B} is the subring which preserves each of the quotients in

$$\begin{aligned} L_{r-1}^*/L_0^* \oplus M_{s-1}^*/M_{s-2}^* \oplus \dots \\ \oplus M_1^*/M_0^* \oplus L_0^*/L_0 \oplus M_0^*/M_0 \oplus M_0/M_1 \oplus \dots \\ \dots \oplus M_{s-2}/M_{s-1} \oplus L_0/L_{r-1} \oplus L_{r-1}/\pi L_{r-1}^*. \end{aligned}$$

We already know that \mathcal{A}_{22} is the stabilizer of

$$\supset M_{s-1}^* \supset \dots \supset M_{s-1} \supset \pi M_{s-1}^* \supset \dots$$

with radical \mathcal{B}_{22} (3.11). Suppose we know that \mathcal{A}_{11} is the stabilizer of $\dots \supset L_{r-1}^* \supset L_0^* \supset L_0 \supset \pi L_{r-1} \supset \dots$, with radical \mathcal{B}_{11} . Then part (a) follows immediately from the assertion above involving quotients.

As for part (b) (still for the same kind of lattice chain), we have just remarked that we only need to see it for \mathcal{A}_{11} , \mathcal{B}_{11} . Now $\mathcal{A}_{11} = \mathcal{A} \cap \text{End}_k(V_1)$ so that elements of \mathcal{A}_{11} preserve each lattice in the chain describing \mathcal{A} , and preserve V_1 as well. It follows that elements in \mathcal{A}_{11} must preserve $\dots \supset L_{r-1}^* \supset L_0^* \supset L_0 \supset L_{r-1} \supset \pi L_{r-1}^* \dots$, and it is clear that the stabilizer of this chain lies in \mathcal{A}_{11} . Consider again

$$\begin{aligned} L_{r-1}^* \oplus M_{s-1}^* \supset L_0^* \oplus M_{s-1}^* \supset L_0^* \oplus M_{s-2}^* \supset \dots \\ \supset L_{r-1} \oplus M_{s-1} \supset \pi L_{r-1}^* \oplus \pi M_{s-1}^* \supset \dots \end{aligned}$$

Let $x \in \mathcal{B} \cap \text{End}_k(V_1)$. Then we must have

$$x L_{r-1}^* \subset L_0^*, \quad x L_0^* \subset L_0^* \dots, \quad x L_0^* \subset L_0, \quad \dots, \quad x L_0 \subset L_{r-1}, \quad x L_{r-1} \subset \pi L_{r-1}^*$$

so that in fact $x \in \text{radical of } \mathcal{A}_{11}$. The reverse inclusion is also evident.

The arguments for the other kind of lattice chains are the same; thus if $M_{s-1} = \pi M_{s-1}^{\#}$, \mathcal{A}_{11} will be the stabilizer of

$$\dots \supset L_0^{\#} \supset L_0 \supset \pi L_0^{\#} \supset \dots$$

3.14. We conclude with a result which could have been stated and proved in the context of section 2. To state it we return to the framework of section 3.5. Thus we have $\mathcal{L}_0 = \mathcal{L}_u$, $\mathcal{L}_j = \mathcal{L}_{j-1} \oplus \mathcal{M}_j$ if $j > l$, and $\mathcal{L} = \mathcal{L}_r = \mathcal{L}_{r-1} \oplus \mathcal{M}_r$.

Applying section 3.7 we obtain orders $\mathcal{A}^{(0)}$, $\mathcal{A}'''^{(0)}$, $\mathcal{A}^{(0)} = \mathcal{A}^{(0)} \cap \mathcal{A}'''^{(0)}$, and parahoric subgroup $P = P^{(0)}$.

Similarly, we can apply section 3.7 to the chain \mathcal{L}_{r-1} if $\mathcal{L}_{r-1} \neq \mathcal{L}_u$, while if $\mathcal{L}_{r-1} = \mathcal{L}_u$ we apply the construction 3.4 to \mathcal{L}_u (in the vector space $E_1 \oplus \dots \oplus E_{r-1}$) to obtain $\mathcal{A}^{(1)}$, $\mathcal{A}'''^{(1)}$, $\mathcal{A}^{(1)} = \mathcal{A}^{(1)} \cap \mathcal{A}'''^{(1)}$, $P^{(1)}$ if $\mathcal{L}_{r-1} \neq \mathcal{L}_u$, and \mathcal{A}_u , P_u if $\mathcal{L}_{r-1} = \mathcal{L}_u$.

PROPOSITION. — (a) $\mathcal{A}^{(1)}$ (resp. \mathcal{A}_u) $\subseteq \mathcal{A}^{(0)} \cap \text{End}_k(E_1 \oplus \dots \oplus E_{r-1})$.

(b) $P^{(1)}$ (resp. P_u) $\subseteq P^{(0)} \cap U(f_1 \oplus \dots \oplus f_{r-1}, E_1 \oplus \dots \oplus E_{r-1})$.

Proof. — To prove both of these assertions, we must look at the various lattice chains. Let

$$\mathcal{L}_{r-1}: L_{s-1}^{\#} \supset \dots \supset L_{s-1} \supset \pi L_{s-1}^{\#}$$

$$\mathcal{M}_r: M_{t-1}^{\#} \supset \dots \supset M_{t-1} \supset \pi M_{t-1}^{\#}$$

and suppose initially that $M_{t-1} \not\subseteq \pi M_{t-1}^{\#}$. In this case $\mathcal{A}^{(0)}$ is the stabilizer of the chain $\mathcal{L}_{r-1} \oplus \mathcal{M}_r$:

$$L_{s-1}^{\#} \oplus M_{t-1}^{\#} \supset L_0^{\#} \oplus M_{t-1}^{\#} \supset \dots \supset L_0 \oplus M_{t-1} \supset L_{s-1} \oplus M_{t-1} \supset \pi L_{s-1}^{\#} \oplus \pi M_{t-1}^{\#}$$

so that as we have seen above in 3.13, $\mathcal{A}^{(0)} \cap \text{End}_k(E_1 \oplus \dots \oplus E_{r-1})$ is the stabilizer of

$$L_{s-1}^{\#} \supset L_0^{\#} \supset L_0 \supset L_{s-1} \supset \pi L_{s-1}^{\#}.$$

Now write $\mathcal{L}_{r-1} = \mathcal{L}_{r-2} \oplus \mathcal{M}_{r-1}$ if $\mathcal{L}_u \neq \mathcal{L}_{r-1}$, where

$$\mathcal{L}_{r-2}: P_{l-1}^{\#} \supset \dots \supset P_{l-1} \supset \pi P_{l-1}^{\#}$$

$$\mathcal{M}_{r-1}: Q_{m-1}^{\#} \supset \dots \supset Q_{m-1} \supset \pi Q_{m-1}^{\#}.$$

If $Q_{m-1} \not\subseteq \pi Q_{m-1}^{\#}$ then by definition (2.2)

$$\mathcal{L}_{r-2} \oplus \mathcal{M}_{r-1}: P_{l-1}^{\#} \oplus Q_{m-1}^{\#} \supset P_0^{\#} \oplus Q_{m-1}^{\#} \supset \dots$$

$$\supset P_0 \oplus Q_{m-1} \supset P_{l-1} \oplus Q_{m-1} \supset \pi P_{l-1}^{\#} \oplus \pi Q_{m-1}^{\#}$$

otherwise we have

$$\mathcal{L}_{r-2} \oplus \mathcal{M}_{r-1}: P_0^{\#} \oplus Q_{m-1}^{\#} \supset \dots \supset P_0 \oplus Q_{m-1} \supset \pi P_0^{\#} \oplus \pi Q_{m-1}^{\#}.$$

It follows that $L_{s-1}^* = P_{l-1}^* \oplus Q_{m-1}^*$ or $P_0^* \oplus Q_{m-1}^*$, as the case may be, while $L_0^* = P_0^* \oplus Q_0^*$ in either case. Moreover $\mathcal{A}^{(1)}$ is the stabilizer of $\mathcal{L}_{r-2} \oplus \mathcal{M}_{r-1}$, hence stabilizes these two lattices and their duals, so that we have

$$\mathcal{A}^{(1)} \subseteq \mathcal{A}^{(0)} \cap \text{End}_k(E_1 \oplus \dots \oplus E_{r-1}).$$

If $\mathcal{L}_{r-1} = \mathcal{L}_u$, the argument is similar, while assertion (b) follows from (a) and the fact that $P^{(0)} = \mathcal{A}^{(0)} \cap G$.

3.15. *Remark.* – In general the inclusions in proposition 3.14 are strict.

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