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SKEIN QUANTIZATION OF POISSON ALGEBRAS OF LOOPS ON SURFACES

BY VLADIMIR G. TURAEV

Introduction

0.1. The aim of the present paper is to establish a relationship between two previously unrelated subjects: the skein invariants of links in 3-manifolds generalizing the Jones-Conway polynomial and the Poisson algebras of loops on surfaces, due to W. Goldman [4]. The nature of the relationship may be expressed by the word "quantization". We show that skein algebras of links lying in the cylinder over an oriented surface F quantize Poisson algebras generated by the homotopy classes of loops on F.

The quantum nature of knots and links does not come as a surprise after the considerable work done on relationships between the knot theory and the quantum R-matrices (*see*, in particular, [12], [18]). It is more surprising that some deep algebraic notions (such as Lie bialgebra and Poisson-Lie group), introduced by V. Drinfeld ([1], [2]) in the frameworks of his algebraic formalization of the quantum inverse scattering method, naturally come up in the purely topological study of loops on surfaces. On the other hand, an adequate treatment of the geometric situation leads to new algebraic notions (such as "bi-Poisson bialgebra" and "biquantization"), interesting in themselves.

It should be emphasized that quantization is considered in this paper from a purely algebraic point of view, the analytical aspects of the notion being ignored.

0.2. Everywhere in the paper the symbol F denotes an oriented surface and K denotes a commutative associative ring, containing the field of rationals $\mathbb{Q}$.

0.3. The notion of algebraic quantization is suggested by the following well known construction. Let A be an algebra over the polynomial ring $K[h]$, which is free as the $K[h]$-module. Assume that the quotient algebra A/hA is commutative so that $ab-ba \in hA$ for any $a, b \in A$. The formula

$$(0.3.1) \qquad [a \bmod hA,\ b \bmod hA] = h^{-1}(ab-ba) \bmod hA$$

equips A/hA with a Lie bracket which satisfies the Leibniz rule

$$(0.3.2) \qquad [ab, c] = a[b, c] + [a, c]b.$$

The algebra A/hA with this bracket is a Poisson algebra. The inverse to this construction is called a quantization of the Poisson algebra (*see*, for instance, [2], [21]). In this paper we will use somewhat weaker notion of quantization, avoiding the freeness condition (*see* § 1).

Speaking informally, quantization is a non-commutative extension (or h-deformation) of the Poisson algebra so that the first approximation to non-commutativity is determined by the Lie bracket. Of course, h is the "Planck constant".

Dually, one may define co-Poisson coalgebras and their quantizations.

0.4. The main objects considered in the paper are the Lie algebra Z over K, generated by free homotopy classes of loops on F, and the skein algebra $\mathscr{A}$ over the polynomial ring $K[x, x^{-1}, h, \hbar]$. The Lie algebra Z was introduced by W. Goldman [4]. (A related Lie algebra $Z_\square$ generated by homotopy classes of non-oriented loops on F was implicit in the earlier paper of S. Wolpert [23]). The Lie algebras Z, $Z_\square$ are intimately related to Poisson algebras of smooth functions on the spaces of linear representations of $\pi_1(F)$ (*see* [4] or § 2).

The algebra $\mathscr{A}$ is additively generated by isotopy classes of oriented links in $F \times [0, 1]$ modulo a relation which imitates the famous Jones-Conway relation for links in S^3. The Jones-Conway relation says that if three oriented links L_+, L_-, L_0 coincide outside a

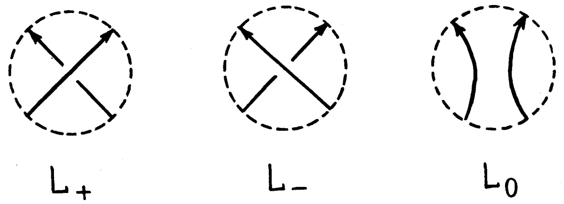

Fig. 1.

ball and look as in Figure 1 inside the ball then one must put (formally) $xL_+ - x^{-1}L_- - hL_0 = 0$. We use a refined form of the relation, depending on whether two branches of L_+ depicted on Figure 1 lie on different components of L_+ or they lie on the same component. In the first case we leave the relation as it is, whereas in the second case we replace h by another variable $\hbar$. The distinction between h and $\hbar$ plays an important role in the paper. Note, however, that for links in S^3 this distinction does not lead to new invariants of links (*cf.* § 5).

We establish three quantization theorems relating $\mathscr{A}$ and Z. First we show that $\mathscr{A}$ gives rise to a 1-parameter family of algebras $\mathscr{A}_k$, $k \in K$ which quantize certain Poisson brackets in the symmetric algebra $S(Z)$, induced by the Goldman-Lie bracket in Z and the homological intersection form in $H_1(F)$. This result admits a non-oriented version in which the role of $\mathscr{A}$ is played by an algebra of non-oriented link diagrams on F modulo Kauffman-type relations.

The other two quantization theorems deal with the structure of Lie bialgebra in Z introduced in this paper. We provide the $K[h, \hbar]$-algebra $A = \mathscr{A}/(x-1)\mathscr{A}$ with a structure of bialgebra (Hopf algebra) and show that this bialgebra quantizes the Lie bialgebra

Z in the sense close to the one discussed in [4]. Technically, this means that A quantizes certain co-Poisson bialgebra $V_h(Z)$ associated with Z. Here the role of the Planck constant is played by $\hbar$.

The third quantization theorem is formulated in terms of the Lie subbialgebra Z_0 of Z generated by non-contractible loops. We associated with Z_0 an "infinite dimensional Poisson-Lie group" $\mathrm{Exp}_\hbar(Z_0^*)$ which gives rise to a Poisson bialgebra $\varepsilon_\hbar(Z_0)$ consisting of "polynomial functions" on the group. We show that the quotient bialgebra $A_0 = A/\delta A$, where δ is the class of the trivial knot, quantizes $\varepsilon_\hbar(Z_0)$.

The quantization theorems mentioned above imply that the bialgebra A_0 quantizes both $\varepsilon_\hbar(Z_0)$ and $V_h(Z_0)$ with the Planck constants respectively h and $\hbar$. In particular, $A_0/h A_0 = \varepsilon_\hbar(Z_0)$ and $A_0/\hbar A_0 = V_h(Z_0)$. To formalize the situation we introduce the notion of biquantization and show that A_0 is a so-called normal biquantization of the Lie bialgebra Z_0.

The algebraic structure of the skein algebra $\mathscr{A}$ is rather mysterious. We compute the algebra in the simplest cases when F is the 2-disc, the 2-sphere or the annulus. For arbitrary F we compute the quotients $\mathscr{A}/\hbar\mathscr{A}$ and $\mathscr{A}/(\delta\mathscr{A} + h\mathscr{A})$ in terms of Z.

0.5. The main results of the present paper were announced in [20]. At the same time E. Witten [22] gave a treatment of the Jones-type invariants of links in 3-manifolds from the viewpoint of the quantum field theory defined by the non-abelian Chern-Simons action. For a mathematical treatment of similar invariants, *see* [8], [13]. The full relations between the constructions of [8], [13], [22] and the theory developed here are still to be explored. Note only that for $K = \mathbb{C}$ the skein algebras $\mathscr{A}_k$ canonically act in $\mathbb{C}[y, y^{-1}] \otimes_{\mathbb{C}} H$ where H is the finite dimensional Hilbert space associated with F in [8], [13], [22].

0.6. The paper consists of four chapters. The first three chapters are centered around the three quantization theorems mentioned above. The fourth chapter deals with biquantization. Each chapter includes a dose of Poisson Algebra which makes the paper essentially self-contained.

CHAPTER I

POISSON ALGEBRAS OF LOOPS AND THEIR TOPOLOGICAL QUANTIZATION

1. Preliminaries on Poisson algebras and quantization

1.1. Poisson algebras. — A Poisson algebra is a commutative associative algebra S equipped with a Lie bracket which satisfies the Leibniz rule (0.3.2) for all a, b, $c \in S$. Such a bracket is called a Poisson bracket in S.

The following examples of Poisson algebras play a central role in the paper. With each K-module $\mathfrak{g}$ one associates its symmetric (commutative and associative) algebra

$$S(\mathfrak{g}) = \bigoplus_{i \geqq 0} S^i(\mathfrak{g}),$$

where $S^0(\mathfrak{g})=K$, $S^1(\mathfrak{g})=\mathfrak{g}$ and $S^i(\mathfrak{g})$ is the i-th symmetric tensor power of $\mathfrak{g}$ for $i \geqq 2$. If $\mathfrak{g}$ is a Lie algebra then the Lie bracket in $\mathfrak{g}$ uniquely extends by the Leibniz rule to a Lie bracket in $S(\mathfrak{g})$. This makes $S(\mathfrak{g})$ a Poisson algebra which is called the symmetric Poisson algebra of $\mathfrak{g}$.

The category of Poisson K-algebras is a tensor category. The tensor product of two Poisson K-algebras S, T is defined to be the algebra $S \otimes T$ equipped with the Lie bracket

$$[a \otimes b, a' \otimes b'] = aa' \otimes [b, b'] + [a, a'] \otimes bb'. \tag{1.1.1}$$

(Unless the contrary is stated explicitly the symbol $\otimes$ denotes tensor product over K). One easily verifies that $S \otimes T$ is a Poisson algebra.

1.2. Quantization of Poisson algebras. — Let Q be a commutative associative K-algebra with unit. Let $\varphi : Q \to K$ be a unit preserving K-algebra homomorphism (an augmentation of Q). An additive homomorphism $p : A \to S$ of a Q-module A into a K-module S is called linear over φ (or φ-linear) if $p(qa) = \varphi(q) p(a)$ for any $q \in Q$, $a \in A$.

Let $h \in \operatorname{Ker} \varphi$. A *quantization* over (Q, φ, h) of a Poisson K-algebra S is a pair (a Q-algebra A, a surjective φ-linear ring homomorphism $p : A \to S$) such that for any $a, b \in A$

$$ab - ba = hp^{-1}([p(a), p(b)]) \bmod (h \operatorname{Ker} p) \tag{1.2.1}$$

where [,] is the Lie bracket in S. Note that the indeterminacy of $hp^{-1}([p(a), p(b)])$ is exactly $h \operatorname{Ker} p$ so that (1.2.1) makes sense.

For the sake of brevity the quantization $(A, p : A \to S)$ over (Q, φ, h) is also called a quantization over Q or over φ. The homomorphism p is called a quantization homomorphism. Clearly $\operatorname{Ker} p \supset \operatorname{Ker} \varphi \cdot A$ If this inclusion happens to be equality then the quantization (A, p) is said to be reduced.

Remarks. — 1. Our definition of quantization differs in certain details from the one given in [2]. In particular, we do not require the algebra A to be free as the Q-module.

2. It is a simple fact, often used below, that if (1.2.1) holds for generators of A then it holds for arbitrary $a, b \in A$.

3. Quantizations over the polynomial ring $K[h]$ (with the usual augmentation) are universal: each such quantization $p : A \to S$ induces a quantization $\varphi \otimes p : Q \otimes_{K[h]} A \to S$ over (Q, φ, h).

1.3. Algebra $V_h(\mathfrak{g})$. — Symmetric Poisson algebras of Lie K-algebras admit rather simple canonical quantization over $K[h]$. Recall that with each K-module $\mathfrak{g}$ one associates its tensor algebra $T(\mathfrak{g}) = \bigoplus_{m \geqq 0} \mathfrak{g}^{\otimes m}$ where $\mathfrak{g}^{\otimes m}$ is the tensor product over K of m copies of $\mathfrak{g}$. (In particular, $\mathfrak{g}^{\otimes 0} = K$.) The algebra multiplication in $T(\mathfrak{g})$ is defined by the rule

$$(a_1 \otimes \ldots \otimes a_m)(b_1 \otimes \ldots \otimes b_n) = a_1 \otimes \ldots \otimes a_m \otimes b_1 \otimes \ldots \otimes b_n.$$

Let $\mathfrak{g}$ be a Lie algebra over K. Consider the $K[h]$-module $\mathfrak{a}=K[h]\otimes\mathfrak{g}$ and its tensor $K[h]$-algebra $T(\mathfrak{a})$. Let $V=V_h(\mathfrak{g})$ be the quotient of this algebra by the two-sided ideal generated by $\{ab-ba-h[a, b] \mid a, b\in\mathfrak{g}\subset\mathfrak{a}\}$. Clearly V is an associative $K[h]$-algebra with unit, generated by $\mathfrak{g}$. Obviously, $V/hV=S(\mathfrak{g})$ and $V/(h-1)V$ is the universal enveloping algebra $U(\mathfrak{g})$ of $\mathfrak{g}$ (*see* [15]). Denote the projection $V\to V/hV=S(\mathfrak{g})$ by v.

1.4. THEOREM. — *(V, v) is a reduced quantization of* $S(\mathfrak{g})$.

Proof. — Clearly v is a surjective algebra homomorphism linear over the augmentation $\mathrm{aug}: K[h]\to K$ with $\mathrm{Ker}\, v=hV=\mathrm{Ker}(\mathrm{aug})V$. For generators $a, b\in\mathfrak{g}$ we have $v([a, b])=[v(a), v(b)]$ and therefore

$$ab-ba=h[a,b]=hv^{-1}([v(a),v(b)]) \bmod hV.$$

2. Poisson algebras of loops

2.1. LIE ALGEBRA Z. — Let F be an oriented surface. Denote by $\hat{\pi}$ the set of free homotopy classes of loops $S^1\to F$. Denote by Z the free K-module with the basis $\hat{\pi}$. Recall Goldman's definition of the Lie bracket in Z (*see* [4]).

For a loop $\alpha: S^1\to F$ we denote its class in $\hat{\pi}$ by $\langle\alpha\rangle$. Let α, β be two loops on F lying in general position. Denote the (finite) set $\alpha(S^1)\cap\beta(S^1)$ by $\alpha\#\beta$. For $q\in\alpha\#\beta$ denote by $\varepsilon(q; \alpha, \beta)=\pm 1$ the intersection index of α and β in q. Denote by $\alpha_q\beta_q$ the product of the loops α, β based in q. Up to homotopy the loop $(\alpha_q\beta_q)(S^1)$ is obtained from $\alpha(S^1)\cup\beta(S^1)$ by the orientation preserving smoothing of the crossing in the point q. Set

$$[\langle\alpha\rangle, \langle\beta\rangle]=\sum_{q\in\alpha\#\beta}\varepsilon(q; \alpha, \beta)\langle\alpha_q\beta_q\rangle \tag{2.1.1}$$

According to Goldman [4], Theorem 5.2, the bilinear pairing $[\ ,\ ]: Z\times Z\to Z$ given by (2.1.1) on the generators is well defined and makes Z a Lie algebra.

2.2. POISSON ALGEBRA S(Z) AND ITS DEFORMATIONS. — According to the results of Section 1.1 one associates with the Lie algebra Z its Poisson symmetric algebra S(Z).

The Lie bracket $[\ ,\ ]$ in S(Z) admits a canonical deformation depending on one parameter $k\in K$. (This deformation is implicit in [4], Theorem 3.12.) Namely for a, $b\in\hat{\pi}$ set

$$[a,b]_k=[a,b]-k(a.b)ab$$

where $a.b$ is the integer (homological) intersection index of a and b, and ab is the product of a and b in S(Z). The bracket $[\ ,\ ]_k$ may be extended by the Leibniz rule to a pairing $[\ ,\ ]_k: S(Z)\times S(Z)\to S(Z)$. This pairing is actually a Lie bracket: the skew-commutativity is obvious and the Jacobi identity follows from the Jacobi identity for the Goldman bracket. Indeed, for $a, b, c \in \hat{\pi}$ we have

$$[[a, b]_k, c]_k = [[a, b], c] - k\{(a.c)[a, b]c + (b.c)[a, b]c + (a.b)[a, c]b + (a.b)[b, c]a\} + k^2((a.c)+(b.c))(a.b)abc.$$

When we cyclically permute a, b, c and sum up, the coefficients at k and k^2 cancel; thus the sum does not depend on k and equals 0.

The algebra $S(Z)$ equipped with the bracket $[\ ,\ ]_k$ is a Poisson algebra. It is denoted by $S_k(Z)$. In particular, $S_0(Z) = S(Z)$.

2.3. Poisson algebra $S(Z_\square)$. According to [4] there is a Lie algebra similar to Z but based on homotopy classes of unoriented loops. Namely, the map $a \mapsto a^{-1} : \hat{\pi} \to \hat{\pi}$ which reverses the orientation of oriented loops extends to an automorphism, say j, of Z of order two. Denote its stationary set $\{x \in Z \mid j(x) = x\}$ by $Z_\square$. For $a \in \hat{\pi}$ put $a_\square = a + j(a)$. Clearly, $Z_\square$ is a Lie subalgebra of Z which is additively a free module based on the set of all $a_\square$ for $a \in \hat{\pi}$. These generators bijectively correspond to free homotopy classes of unoriented loops on F. The following formula given in [4] computes the bracket of generators $\langle \alpha \rangle_\square$, $\langle \beta \rangle_\square$

$$(2.4.1.) \qquad [\langle \alpha \rangle_\square, \langle \beta \rangle_\square] = \sum_{q \in \alpha \# \beta} \varepsilon(q; \alpha, \beta)(\langle \alpha_q \beta_q \rangle_\square - \langle \alpha_q \beta_q^{-1} \rangle_\square)$$

The Lie algebra $Z_\square$ is implicit in [23].

As above, one associates with $Z_\square$ its symmetric Poisson algebra $S(Z_\square)$.

2.4. Poisson actions of algebras of loops. — The Poisson algebras $S_k(Z)$, $S(Z_\square)$ are intimately connected with Poisson algebras of functions on the spaces of conjugacy classes of linear representations of $\pi_1(F)$ (*see* [4]). Though this connection is not used in the present paper, it establishes a proper framework for the results of the paper and a conceptual point of view on the results. I briefly describe here the relevant results of [4].

Recall that a Poisson manifold is a smooth finite-dimensional manifold N equipped with a Lie bracket in the algebra $C^\infty(N)$ of smooth real-valued functions on N so that $C^\infty(N)$ becomes a Poisson algebra. A Poisson action of a Poisson algebra S on a Poisson manifold N is a Poisson algebra homomorphism $S \to C^\infty(N)$ (*i. e.* an algebra homomorphism preserving the Lie bracket).

Assume that the surface F is connected and denote its fundamental group by π. As usual the basis $\hat{\pi}$ of Z is identified with the set of conjugacy classes of elements of π. Assume also that $K = \mathbb{R}$.

For a Lie group G satisfying fairly general conditions Goldman [3] constructs a symplectic structure on the smooth part N_G of the space $\mathrm{Hom}(\pi, G)/G$ of conjugacy classes of representations $\pi \to G$. This symplectic structure generalizes the Weil-Petersson Kähler form on the Teichmüller space and induces a Poisson structure on N_G. It turns out that for some classical Lie groups, one of the Poisson algebras of loops considered above canonically acts on N_G. In particular, $S(Z)$ acts on N_G when $G = GL(n, *)$, $* = \mathbb{R}, \mathbb{C}, \mathbb{H}$. The corresponding Poisson algebra homomorphism $S(Z) \to C^\infty(N_G)$ associates with a generator $a \in \hat{\pi}$ the function

$\psi \mapsto 2\,\mathrm{Re}\,\mathrm{tr}\,\psi(a): \mathrm{N_G} \to \mathbb{R}$. The same formula defines a Poisson action of $S_{1/n}(Z)$ on $\mathrm{N_G}$ for $G = SL(n, \mathbb{R})$. Similarly, $S(Z_\square)$ acts on $\mathrm{N_G}$ for $G = O(p, q)$, $O(n, \mathbb{C})$, $O(n, H)$, $U(p, q)$, $Sp(n, \mathbb{R})$, $Sp(p, q)$.

2.5. *Remark.* – With the Lie groups $G = SL(n, \mathbb{C})$, $SL(n, \mathbb{H})$, $SU(p, q)$ one may also associate somewhat more complicated Poisson algebras of loops and their Poisson actions on $\mathrm{N_G}$. Here the underlying commutative algebra is $S(Z) \otimes_{\mathbb{R}} S(Z)$ and the definition of the Lie bracket mimics the formulas given in [4], Theorems 3.16, 3.17. Note that the Poisson algebra of loops corresponding to $G = SL(n, \mathbb{C})$, $SL(n, \mathbb{H})$ is a real form of the complex Poisson algebra $S_{i/2n}(Z) \otimes_{\mathbb{C}} S_{-i/2n}(Z)$ where $K = \mathbb{C}$, $i = \sqrt{-1}$. The action of the latter algebra on $\mathrm{N_G}$ maps $a \otimes 1$ and $1 \otimes a$ with $a \in \hat{\pi}$ into the complex valued functions respectively $\psi \mapsto \mathrm{tr}\,\psi(a)$ and $\psi \mapsto \overline{\mathrm{tr}\,\psi(a)}$ on $\mathrm{N_G}$. (When $G = SL(n, H)$ the trace tr is defined via the inclusion $G \to GL(2n, \mathbb{C})$, *cf.* [4], § 1.)

3. Topological quantization of $S_k(Z)$

3.1. SKEIN MODULES $\mathscr{A}(M)$ AND $\mathscr{A}_k(M)$. – Let M be an oriented 3-manifold. By a link in M we mean a finite family of non-intersecting smooth imbedded circles in Int M. Two links in M are isotopic if they may be smoothly deformed into each other in the class of links. The empty set is considered as the unique (up to isotopy) link Φ in M with 0 components. The number of components of a link $L \subset M$ is denoted by $|L|$. The oriented trivial knot (*i. e.* the boundary of an imbedded 2-disc) will be denoted by $\mathcal{O}$.

A triple of non-empty oriented links L_+, L_-, L_0 in M is called a *Conway triple* if L_+, L_-, L_0 are identical outside some ball $B \subset M$ and look as in Figure 1 inside B. Clearly $|L_+| = |L_-| = |L_0| \pm 1$. We define the type of the Conway triple L_+, L_-, L_0 to be 1 if $|L_+| = |L_0| + 1$ and to be 2 if $|L_+| = |L_0| - 1$. For example if L is an arbitrary non-empty oriented link in M then $(L, L, L \amalg \mathcal{O})$ is a Conway triple of type 2 (see *Fig.* 2). In what follows the triple $(\Phi, \Phi, \mathcal{O})$ is considered as a Conway triple (of type 2).

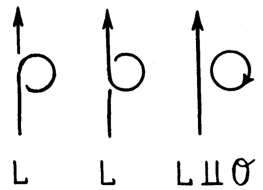

Fig. 2.

The *skein module* $\mathscr{A}(M)$ is a module over the commutative polynomial ring $K[x, x^{-1}, h, \hbar]$ defined as follows (*cf.* [11], [19]). Let $\mathscr{L}$ be the set of isotopy classes of oriented links in M. Then $\mathscr{A}(M)$ is the quotient of the free $K[x, x^{-1}, h, \hbar]$-module with basis $\mathscr{L}$ by the submodule generated by elements of 2 types: (i) the elements

$x\mathrm{L}_+ - x^{-1}\mathrm{L}_- - h\mathrm{L}_0$ corresponding to arbitrary Conway triples of type 1; (ii) the elements $x\mathrm{L}_+ - x^{-1}\mathrm{L}_- - \hbar\mathrm{L}_0$ corresponding to arbitrary Conway triples of type 2. The element of this quotient module represented by an oriented link L will be denoted by [L]. This element may be viewed as the universal Jones-Conway invariant of this link. Note the identity $(x - x^{-1})[\Phi] = \hbar[\mathcal{O}]$.

Consider the ring $\mathrm{K}[\hbar][[h]]$ whose elements are formal power series of h with coefficients in the polynomial ring $\mathrm{K}[\hbar]$. The skein module $\mathscr{A}(\mathrm{M})$ gives rise to a one parameter family $\mathscr{A}_k(\mathrm{M})$ of modules over $\mathrm{K}[\hbar][[h]]$ parametrized by $k \in \mathrm{K}$. The module $\mathscr{A}_k(\mathrm{M})$ is obtained from $\mathscr{A}(\mathrm{M})$ by the substitution $x = \exp(kh)$. More precisely,

$$\mathscr{A}_k(\mathrm{M}) = \bar{\mathscr{A}}/(x - \exp(kh))\bar{\mathscr{A}}$$

where

$$\bar{\mathscr{A}} = \mathrm{K}[x, x^{-1}, \hbar][[h]] \otimes_{\mathrm{K}[x, x^{-1}, h, \hbar]} \mathscr{A}(\mathrm{M}).$$

When one puts $h = \hbar = 0$, $x = 1$ in the defining relations of $\mathscr{A}(\mathrm{M})$ one gets the ordinary homotopy relation for families of loops in M. This observation relates the skein modules of M with $\pi_1(\mathrm{M})$ as follows. Let $\hat{\pi}$ be the set of free homotopy classes of loops in M. Let S be the symmetric algebra of the free K-module with basis $\hat{\pi}$. Each oriented link $\mathrm{L} \subset \mathrm{M}$ with components $\mathrm{L}_1, \ldots, \mathrm{L}_l$ gives rise to an element $\langle \mathrm{L} \rangle = \prod_{i=1}^{l} \langle \mathrm{L}_i \rangle \in \mathrm{S}$ where $\langle \mathrm{L}_i \rangle$ is the class of L_i in $\hat{\pi}$. In particular $\langle \Phi \rangle = 1$. The formula $[\mathrm{L}] \mapsto \langle \mathrm{L} \rangle$ determines an additive homomorphism $\mathscr{A}(\mathrm{M}) \to \mathrm{S}$ linear over the coefficient ring homomorphism

$$h \mapsto 0,\ \hbar \mapsto 0,\ x \mapsto 1 : \mathrm{K}[x, x^{-1}, h, \hbar] \to \mathrm{K}.$$

This homomorphism $\mathscr{A}(\mathrm{M}) \to \mathrm{S}$ induces an additive homomorphism $\mathscr{A}_k(\mathrm{M}) \to \mathrm{S}$ linear over the coefficient ring homomorphism

$$h \mapsto 0,\ \hbar \mapsto 0 : \mathrm{K}[\hbar][[h]] \to \mathrm{K}.$$

Denote this homomorphism $\mathscr{A}_k(\mathrm{M}) \to \mathrm{S}$ by $p(k)$. It is easy to see that

$$\operatorname{Ker} p(k) = h\mathscr{A}_k(\mathrm{M}) + \hbar\mathscr{A}_k(\mathrm{M}). \tag{3.1.1}$$

3.2. SKEIN ALGEBRAS $\mathscr{A}$ AND $\mathscr{A}_k$. — Consider the 3-manifold $\mathrm{F} \times [0, 1]$ where F is an oriented surface and provide $\mathrm{F} \times [0, 1]$ with the product orientation of the given orientation in F and the standard orientation in [0, 1]. For links $\mathrm{L}, \mathrm{L}' \subset \mathrm{F} \times [0, 1]$ define the product LL′ by the formula

$$\mathrm{LL}' = \{(a, t) \in \mathrm{F} \times [0, 1] \mid t \geqq 1/2 \text{ and } (a, 2t - 1) \in \mathrm{L}, \text{ or } t \leqq 1/2 \text{ and } (a, 2t) \in \mathrm{L}'\}.$$

Clearly LL′ is a link in $\mathrm{F} \times [0, 1]$. The formula $\mathrm{L}, \mathrm{L}' \mapsto \mathrm{LL}'$ defines a structure of associative algebra in $\mathscr{A}(\mathrm{F} \times [0, 1])$ with the unit $[\Phi]$ (*cf.* [19]). This structure descends to $\mathscr{A}_k(\mathrm{F} \times [0, 1])$ (with $k \in \mathrm{K}$) and makes $\mathscr{A}_k(\mathrm{F} \times [0, 1])$ an associative algebra with unit. Denote the algebras $\mathcal{A}(F \times [0, 1])$, $\mathcal{A}_k(F \times [0, 1])$ respectively by $\mathcal{A}(F)$, $\mathcal{A}_k(F)$ or simply by $\mathcal{A}$, $\mathcal{A}_k$.

If F is the 2-disc or the 2-sphere, or the annulus $S^1 \times [0, 1]$ then the algebra $\mathcal{A}$ is commutative. For other surfaces the skein algebra is non-commutative. Note, however, that the quotient algebra $\mathcal{A}/h\mathcal{A}$ is always commutative (*cf.* the proof of Theorem 3.3 below).

It is convenient to present links in $F \times [0, 1]$ by link diagrams on F in the same fashion in which links in $\mathbb{R}^3$ may be presented by plane link diagrams. A link diagram on F is a finite general position collection of loops on F provided with an additional structure: at each intersection or self-intersection point of the loops one branch is cut and considered as the lower one (the undercrossing), the second branch being considered as the upper one (the overcrossing). The empty link is presented by the empty diagram. The oriented links are of course presented by oriented diagrams. The product of two links is presented by the diagram obtained by placing the diagram of the first link over the diagram of the second link. Note that trading overcrossings for undercrossings one may change an arbitrary link diagram into a diagram of a product of several knots. This argument together with an induction on the number of self-crossing points of diagrams easily show that the algebras $\mathcal{A}$, $\mathcal{A}_k$ are generated by classes of knots.

According to the results of Section 3.1 we have a homomorphism $p(k): \mathcal{A}_k \to S(Z)$ where Z is the Goldman-Lie algebra of F. Clearly $p(k)$ is an algebra homomorphism. This homomorphism is an algebraic counterpart of the projection $\mathrm{proj}: F \times [0, 1] \to F$. Indeed, if one represents additive generators of $S(Z)$ by finite collections of loops on F considered up to homotopy then one observes that $p(k)$ maps the class of any link L into the generator of $S(Z)$ represented by $\mathrm{proj}(L)$.

3.3. THEOREM. — *For any oriented surface* F *and for any* $k \in K$ *the pair* $(\mathcal{A}_k(F), p(k): \mathcal{A}_k(F) \to S(Z))$ *is a reduced quantization of the Poisson algebra* $S_{2k}(Z)$ *over* $K[\hbar][[h]]$.

Proof. — Surjectivity of $p(k)$ is obvious, the kernel of $p(k)$ is given by (3.1.1). It remains to verify (1.2.1) where [,] is the bracket $[\ ,\ ]_{2k}$ in $S(Z)$. Since the algebra $\mathcal{A}_k$ is generated by the classes of knots it suffices to verify (1.2.1) in case where a, b are classes of oriented knots, say L, L′ in $F \times [0, 1]$. Let α and β be loops in F parametrizing the projections of L, L′ into F. It is obvious that L′L may be obtained from LL′ by moving L down through L′. During this process L intersects L′ $\mathrm{card}(\alpha \# \beta)$ times. For each such intersection we have the relation

$$\exp(kh)[L_+]_k - \exp(-kh)[L_-]_k = h[L_0]_k$$

where $[L]_k$ denotes the class of the link L in $\mathcal{A}_k$. Modulo h^2 the latter equality amounts to

$$[L_+]_k - [L_-]_k = h[L_0]_k - kh([L_+]_k + [L_-]_k).$$

Combining together such relations corresponding to all intersections we get $ab-ba=[\mathrm{LL'}]_k-[\mathrm{L'}\,\mathrm{L}]_k=hd$ where

$$d=\sum_{q\in\alpha\neq\beta}\varepsilon(q;\alpha,\beta)([\mathrm{L}_q]_k-k[\mathrm{L}_q^1]_k-k[\mathrm{L}_q^2]_k).$$

Here: L_q is an oriented knot in $\mathrm{F}\times[0, 1]$ whose projection into F is parametrized by the loop $\alpha_q\beta_q$ (*see* Section 2.1); L_q^1 and L_q^2 are 2-component links in $\mathrm{F}\times[0, 1]$ whose components are homotopic to L and L'. Therefore, $p(k)$ maps d into $[\langle\alpha\rangle, \langle\beta\rangle]_{2k}$. This finishes the proof.

3.4. SPECULATIONS. — Goldman's results quoted in Section 2.4 show that loops on F may be treated as functions on the representation spaces of $\pi_1(\mathrm{F})$. In the language of classical mechanics these functions are (classical) observables. Theorem 3.3 shows that knots and links in $\mathrm{F}\times[0, 1]$ may be treated as quantum observables. Projection to the surface or, what is the same, forgetting the under/over crossing information, is the usual degeneration of quantum objects into the classical ones.

Heuristically, a space is equivalent to the algebra of functions on it. Thus, on the heuristic level, the skein algebras quantize the representation spaces of $\pi_1(\mathrm{F})$. From this point of view the variable $\hbar$ is somewhat redundant. One may eliminate it, replacing everywhere $\hbar$ by a formal power series $a_1 h+a_2 h^2+\ldots\in \mathrm{K}[[h]]$ with $a_1\neq 0$. The condition $a_1\neq 0$ ensures that the relation $\exp(kh)-\exp(-kh)=\hbar[\mathcal{O}]$, which holds in $\mathscr{A}_k$, is non-singular.

Theorem 3.3 also suggests viewing the skein module $\mathscr{A}(\mathrm{M})$ of an arbitrary oriented 3-manifold M as "the quantization" of $\pi_1(\mathrm{M})$.

3.5. *Remark.*— The defining relations of the skein module $\mathscr{A}(\mathrm{M})$ considered in Section 3.1 may be presented in the following form. Set $h_1=h$ and $h_{-1}=\hbar$. Then to each Conway triple L_+, L_-, L_0 of oriented links in M there corresponds the relation

$$x[\mathrm{L}_+]-x^{-1}[\mathrm{L}_-]=h_{|\mathrm{L}_+|-|\mathrm{L}_0|}[\mathrm{L}_0]. \tag{3.5.1}$$

It is easy to verify the following consistency property of the relations. If an oriented link L' is obtained from an oriented link L by several crossing changes then one may employ (3.5.1) to compute [L'] via [L] and classes of "intermideate" links obtained by smoothing certain crossings. It turns out that the resulting expression does not depend on the order in which we apply (3.5.1) to the crossings. This consistency lies behind the theory of skein modules though formally we have not used it. The proof of this consistency property is quite similar to the proof of Proposition 1 (n) of [10].

4. Algebra A. Computation of $\mathscr{A}/\hbar\mathscr{A}$

4.1. ALGEBRA A. — We define $\mathrm{A}=\mathrm{A}(\mathrm{F})$ to be the quotient of $\mathscr{A}=\mathscr{A}(\mathrm{F})$ by $(x-1)\mathscr{A}$. Clearly A is an associative algebra over the polynomial ring $\mathrm{K}[h, \hbar]$. This algebra is a refined version of $\mathscr{A}_0$: it follows from definitions that

$$\mathscr{A}_0 = \mathrm{K}[\hbar][[h]] \otimes_{\mathrm{K}[h, \hbar]} \mathrm{A}.$$

The algebra A may be defined directly along the same lines as $\mathscr{A}$ though instead of $\mathrm{K}[x, x^{-1}, h, \hbar]$ one should use $\mathrm{K}[h, \hbar]$ and in the defining relations the factors x, x^1 should be omitted.

Substituting $x=1$ in the Jones-Conway polynomial one gets the Alexander-Conway polynomial. Thus the algebra A may be viewed as the Alexander-Conway reduction of $\mathscr{A}$.

4.2. THEOREM. — *There exists a canonical algebra homomorphism* $p: \mathrm{A}(\mathrm{F}) \to \mathrm{V}_h(\mathrm{Z})$ *linear over the projection* $\hbar \mapsto \mathrm{O}: \mathrm{K}[h, \hbar] \to \mathrm{K}[h]$ *and such that* $\mathrm{Im}\, p = \mathrm{V}_h(\mathrm{Z})$ *and* $\mathrm{Ker}\, p = \hbar \mathrm{A}$. *Thus* $\mathrm{A}/\hbar\mathrm{A} = \mathrm{V}_h(\mathrm{Z})$.

For the definition of the algebra $\mathrm{V}_h(\mathrm{Z})$, *see* Section 1.3. To construct p we need the following Lemma.

4.3. LEMMA. — *There exists a unique mapping* $\mathscr{P}$ *which associates with each oriented link* $\mathrm{L} \subset \mathrm{F} \times [0, 1]$ *an element* $\mathscr{P}(\mathrm{L})$ *of the algebra* $\mathrm{V}_h(\mathrm{Z})$ *such that:*

(i) $\mathscr{P}(\mathrm{L})$ *is an isotopy invariant of* L;

(ii) *if* L *is a knot then* $\mathscr{P}(\mathrm{L})$ *is the class* $\langle \alpha \rangle \in \mathrm{Z}$ *of the loop* α *on* F *parametrizing the projection of* L *into* F;

(iii) *for any Conway triple* L_+, L_-, L_0 *of type* 1 (*resp.* 2) *we have*

$$\mathscr{P}(\mathrm{L}_+) - \mathscr{P}(\mathrm{L}_-) = h\,\mathscr{P}(\mathrm{L}_0) \qquad [\textit{resp. } \mathscr{P}(\mathrm{L}_+) = \mathscr{P}(\mathrm{L}_-)];$$

(iv) *for any oriented links* L, L′ *we have* $\mathscr{P}(\mathrm{LL}') = \mathscr{P}(\mathrm{L})\,\mathscr{P}(\mathrm{L}')$.

Proof. — By a product link we shall mean an oriented link which is a product of knots. For product links the value of $\mathscr{P}$ is fixed by Conditions (ii), (iv). The diagram of any link may be transformed into a diagram of a product link by changing certain overcrossings to undercrossings. Thus applying (iii) and inducting on the number of crossing points we may reduce the calculation of $\mathscr{P}(\mathrm{L})$ to the case of product links. This shows that if there exists $\mathscr{P}$ satisfying (i-iv) then $\mathscr{P}$ is unique.

The arguments of the preceding paragraph being specialized actually give a construction of $\mathscr{P}$. It is quite similar to the construction of the 2-variable Jones-Conway polynomial of links in S^3 given in [10], § 1. I will point out the necessary changes (in fact, simplifications) in the arguments of [10], § 1, leaving the details to the reader. In the Inductive Hypothesis instead of ascending ordered diagrams one should use ordered diagrams, such that each component lies below the preceding ones and over the next ones. The Propositions 1 (*n*), 2 (*n*), 3 (*n*), 4 (*n*) and their proofs transfer directly to our setting whereas instead of the formula $l\mathrm{K}_+ + l^{-1}\mathrm{K}_- + m\mathrm{K}_0 = 0$ one should use the formula $\mathrm{L}_+ - \mathrm{L}_- - j_{|\mathrm{L}_+| - |\mathrm{L}_-|}\mathrm{L}_0 = 0$ where $j_1 = h$ and $j_{-1} = 0$. [In the proof of Proposition 2 (*n*) case (*b*) is obvious since the homotopy type of a knot projection in preserved under changing an overcrossing to undercrossing.] The analogue of Proposition 5 (*n*) is unnecessary. The analogue of Proposition 6 (*n*) (independence of the invariant on the ordering of components) is straightforward: one just notes that if L_1, L_2 are two knots and we calculate $\mathscr{P}(L_2 L_1)$ via the process of altering the crossings then we will get

$$\mathscr{P}(L_2 L_1)=\mathscr{P}(L_1 L_2)+h[\langle \alpha_2 \rangle, \langle \alpha_1 \rangle]=\langle \alpha_1 \rangle \langle \alpha_2 \rangle + h[\langle \alpha_2 \rangle, \langle \alpha_1 \rangle]$$

where α_1, α_2 are loops in F parametrizing the projections of L_1, L_2 and where [,] is the Goldman-Lie bracket in Z (*cf.* the proof of Theorem 3.3). In the algebra $V_h(Z)$ we have $h[\langle \alpha_2 \rangle, \langle \alpha_1 \rangle]=\langle \alpha_2 \rangle \langle \alpha_1 \rangle - \langle \alpha_1 \rangle \langle \alpha_2 \rangle$ and so $\mathscr{P}(L_2 L_1)=\langle \alpha_2 \rangle \langle \alpha_1 \rangle$ as it must be. Finally, the analogues of Propositions 1 (*n*)-4 (*n*), 6 (*n*) imply that $\mathscr{P}$ does exist.

4.4. *Proof of Theorem* 4.2. — Put $V=V_h(Z)$. The mapping $[L]\mapsto \mathscr{P}(L)$ extends by linearity to an additive homomorphism $A\to V_h(Z)$ linear over the projection $\hbar\mapsto 0: K[h, \hbar]\to K[h]$. Denote this homomorphism by p. Property (iv) of $\mathscr{P}$ implies that p is an algebra homomorphism.

Clearly, $\hbar A\subset \operatorname{Ker} p$. Therefore, p induces a $K[h]$-algebra homomorphism $A/\hbar A\to V$. Denote it by $\bar{p}$. To complete the proof it suffices to construct a two-sided inverse $r: V\to A/\hbar A$ to $\bar{p}$. We define r as follows. If α is a loop on F then we lift α to a knot, say, L_α in $F\times[0, 1]$. Because of the defining relations of A the class of L_α in $A/\hbar A$ does not depend on the lifting. Denote this class by $r(\langle \alpha \rangle)$. If α, β are two loops on F then

$$r(\langle \alpha \rangle) r(\langle \beta \rangle) - r(\langle \beta \rangle) r(\langle \alpha \rangle) = h \sum_{q\in \alpha \# \beta} \varepsilon(q; \alpha, \beta) r(\langle \alpha_q \beta_q \rangle)$$

(*cf.* the proof of Theorem 3.3). Therefore, r extends to a $K[h]$-algebra homomorphism $r: V\to A/\hbar A$. Both equalities $r\circ\bar{p}=\mathrm{id}$, $\bar{p}\circ r=\mathrm{id}$ follow from definitions.

4.5. COROLLARY. — *The algebra* $\mathscr{A}/((x-1)\mathscr{A}+(h-1)\mathscr{A}+\hbar\mathscr{A})$ *is isomorphic to the universal enveloping algebra of the Goldman-Lie algebra Z.*

4.6. *Remarks.* — 1. It is easy to see that the following diagram

$$\begin{array}{ccc} A & \xrightarrow{p} & V_h(Z) \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle v} \\ \mathscr{A}_0 & \xrightarrow{p(0)} & S(Z) \end{array}$$

(where i is the natural extension) is commutative. Thus $(\mathscr{A}_0, p(0))$ is an extension of the canonical quantization $(V_h(Z), v)$ of $S(Z)$.

2. In view of Theorem 4.2 it makes sense to discuss in more detail the structure of the algebra $V=V_h(\mathfrak{g})$ associated with a Lie algebra $\mathfrak{g}$ over K. Here are three observations to this effect.

The algebra V is isomorphic to the universal enveloping algebra of the Lie algebra over $K[h]$ obtained from the Lie elgebra $K[h]\otimes\mathfrak{g}$ via replacing the Lie bracket [,] by its multiple h[,]. In particular, this implies that if $\mathfrak{g}$ is free as the K-module then V is free as the $K[h]$-module.

The algebra V gives rise to a line of K-algebras $\{V/(h-k)V\}_{k\in K}$. This line traverses $S(\mathfrak{g})$ and $U(\mathfrak{g})$. If K is a field then each algebra $V/(h-k)V$ with $k\neq 0$ is isomorphic to $U(\mathfrak{g})$ via the isomorphism transforming $a\in\mathfrak{g}$ into ka. Thus V is a kind of cone over $U(\mathfrak{g})$ with the cone point $S(\mathfrak{g})$.

There is a neat description of V in terms of $U(\mathfrak{g})$ and its canonical filtration

$$K=U^0\subset U^1\subset\ldots \qquad \text{where} \qquad U^n=(K+\mathfrak{g})^n\subset U(\mathfrak{g}).$$

Namely, as it follows directly from definitions, the formula $a\mapsto h\otimes a$ where $a\in\mathfrak{g}$ defines an algebra homomorphism $V\to K[h]\otimes U(\mathfrak{g})$. Denote this homomorphism by w. Obviously, $w(V)=\bigoplus_{n\geqq 0}(h^n\otimes U^n)$. If $\mathfrak{g}$ is a free K-module then w is an injection. Indeed, if $\{x_i\}$ is a totally ordered basis of $\mathfrak{g}$ then the monomials $x_{i_1}\ldots x_{i_n}$ with $i_1\leqq\ldots\leqq i_n$ linearly generate V over $K[h]$. The Poincaré-Birkhoff theorem implies that the images of these monomials under w are linearly independent over $K[h]$. Hence, these monomials freely generate V and $\operatorname{Ker} w=0$.

4.7. Computation of $\mathscr{A}/\hbar\mathscr{A}$. — The relation $x-x^{-1}=\hbar[\mathcal{O}]$ which holds in $\mathscr{A}$ implies that both $\mathscr{A}/\hbar\mathscr{A}$ and $\mathscr{A}/[\mathcal{O}]\mathscr{A}$ are quotients of the algebra $\mathscr{A}/(x^2-1)\mathscr{A}$ over $K[x, h, \hbar]/(x^2-1)$. The latter algebra may be computed via A:

$$\mathscr{A}/(x^2-1)\mathscr{A}=(K[x]/(x^2-1))\otimes A. \tag{4.7.1}$$

Indeed, if in the definition of the skein module instead of the generators $\{L\}$ used in Section 3.1 one considers generators $\{x^{|L|}L\}$ then the defining relations (3.5.1) (written with respect to this new set of generators) will involve only degrees of x of the same parity. This directly implies (4.7.1). The formula (4.7.1) implies that

$$\mathscr{A}/\hbar\mathscr{A}=(K[x]/(x^2-1))\otimes(A/\hbar A)=(K[x]/(x^2-1))\otimes V_h(Z),$$
$$\mathscr{A}/[\mathcal{O}]\mathscr{A}=(K[x]/(x^2-1))\otimes(A/[\mathcal{O}]A).$$

5. Skein algebras of the 2-disc and the annulus. Quantum states

Denote the ring $K[x, x^{-1}, h, \hbar]$ by Λ.

5.1. Theorem. — (i) *The inclusion* $D^2\subsetneq S^2$ *induces an algebra isomorphism* $\mathscr{A}(D^2)\to\mathscr{A}(S^2)$. *(Here* D^2 *is the 2-disc.)*

(ii). *The algebra* $A(D^2)$ *is generated over* Λ *by the element* $\delta=[\mathcal{O}]$ *subject only to the relation* $\hbar\,\delta=x-x^{-1}$. *Thus* $\mathscr{A}(D^2)=\Lambda[\delta]/(\hbar\,\delta-x+x^{-1})$.

The first claim of the Theorem is obvious since both inclusions $D^2\times[0, 1]\to S^3$ and $S^2\times[0, 1]\to S^3$ induce isomorphisms of the skein modules. Claim (ii) is proven in Section 5.4.

5.2. THEOREM. — *Fix a generator of* $\pi_1(S^1)$. *Let* L_n *be an oriented knot in* $S^1 \times [0, 1]^2$ *homotopic to the n-th degree of the generator and presented by a diagram in* $S^1 \times [0, 1]$ *with the (minimal possible) number of crossings* $\max(|n|-1, 0)$, *where* $n \in \mathbb{Z}$. *Put* $l_n = [L_n] \in \mathscr{A}(S^1 \times [0, 1])$. *Then* $\mathscr{A}(S^1 \times [0, 1])$ *is the quotient of the polynomial ring* $\Lambda[\ldots, l_{-1}, l_0, l_1, l_2, \ldots]$ by the ideal generated by $\hbar l_0 - x + x^{-1}$.

This Theorem is proven in Section 5.5. The proof of both theorems is based on a construction presented in the next section.

5.3. HOMOMORPHISM ω. — In the study of the Jones-Conway polynomial of links in S^3 as well as in the previous papers [11], [19] concerned with the skein modules of 3-manifolds no distinction is made between the types of Conway triples, and also, the variable h entering the main relation is assumed to be invertible. Thus the module associated in [11], [19] with an oriented 3-manifold M is the $K[x, x^{-1}, h, h^{-1}]$-module

$$\mathscr{A}'(M) = K[x, x^{-1}, h, h^{-1}] \otimes_{K[x, x^{-1}, \hbar]} (\mathscr{A}(M)/(h-\hbar)\mathscr{A}(M)).$$

It is easier to compute $\mathscr{A}'(M)$ then $\mathscr{A}(M)$. For example, the Jones-Conway polynomial establishes an isomorphism $\mathscr{A}'(S^3) = K[x, x^{-1}, h, h^{-1}]$.

It turns out that the natural projection $\mathscr{A}(M) \to \mathscr{A}'(M)$ has a canonical 1-parameter deformation. More formally, there is a canonical additive homomorphism

$$(5.3.1) \qquad \omega: \mathscr{A}(M) \to K[y, y^{-1}] \otimes_K \mathscr{A}'(M)$$

linear over the coefficient ring homomorphism

$$(5.3.2) \qquad f(x, h, \hbar) \mapsto f(x, hy, hy^{-1}): \ K[x, x^{-1}, h, \hbar] \to K[x, x^{-1}, y, y^{-1}, h, h^{-1}].$$

The mapping ω is defined on the generators by the formula $\omega([L]) = y^{|L|}[L]'$ where $[L]'$ is the class of the link L in $\mathscr{A}'(M)$. It is easy to see that this mapping extends by linearity to a well defined homomorphism (5.3.1). In particular, substituting $y=1$ one gets the projection $\mathscr{A}(M) \to \mathscr{A}'(M)$.

For an oriented surface F set $\mathscr{A}'F) = \mathscr{A}'(F \times [0, 1])$. Clearly

$$(5.3.3) \qquad \omega: \mathscr{A}(F) \to K[y, y^{-1}] \otimes_K \mathscr{A}'(F)$$

is an algebra homomorphism.

5.4. *Proof of Theorem* 5.1. — The standard argument used to prove the uniqueness of the Jones-Conway polynomial (and based on the crossing changes and induction on the number of crossing points of a link diagram) shows that δ generates $\mathscr{A} = \mathscr{A}(D^2) = \mathscr{A}(S^3)$ over Λ. It remains to check that all relations in $\mathscr{A}$ follow from the relation $\hbar\delta = x - x^{-1}$. To this end we need the 2-variable Jones-Conway polynomial P of oriented links in S^3 (*see* [9]). Recall that P associates with each non-empty oriented link L in S^3 an isotopy invariant Laurent polynomial $P_L \in \mathbb{Z}[x, x^{-1}, h, h^{-1}]$ such that for any Conway triple L_+, L_-, L_0 in S^3

$$(5.4.1) \qquad x P_{L_+} - x^{-1} P_{L_-} = h P_{L_0}$$

and for the trivial knot $\mathcal{O} \subset S^3$ we have $P_{\mathcal{O}}=1$. We need a renormalized version P′ of the polynomial P defined by the formula $P'_L=(x-x^{-1})h^{-1}P_L$. For the empty link Φ put $P'_{\Phi}=1$. It is well known that the value of P′ for the trivial n-component link is $((x-x^{-1})h^{-1})^n$.

It follows from (5.4.1) that the mapping $L \mapsto P'_L$ induces a $K[x, x^{-1}, h, h^{-1}]$-linear homomorphism $\mathcal{A}'(S^3) \to K[x, x^{-1}, h, h^{-1}]$. Denote this homomorphism by q. Actually q is an isomorphism but we will not use this. The only thing we need is the fact that the homomorphism

$$(5.4.2) \qquad (1 \otimes q) \circ \omega : \mathcal{A} = \mathcal{A}(S^3) \to K[x, x^{-1}, y, y^{-1}, h, h^{-1}]$$

is linear over the ring homomorphism (5.3.2) and maps δ^n into $((x-x^{-1})h^{-1}y)^n$.

Suppose that we have in $\mathcal{A}$ an algebraic relation $\sum_{i=0}^{N} f_i \delta^i = 0$ where $f_0, f_1, \ldots, f_N \in \Lambda$. In view of the relation $\hbar\delta = x - x^{-1}$ we may assume that $f_1, \ldots, f_N \in K[x, x^{-1}, h]$. We claim that $f_0 = f_1 = \ldots = f_N = 0$. Indeed

$$(5.4.3) \qquad 0 = \omega\left(\sum_{i=0}^{N} f_i \delta^i\right) = f_0(x, hy, hy^{-1}) + \sum_{i=1}^{N} f_i(x, hy)(h^{-1}y)^i (x-x^{-1})^i.$$

Note that x, hy and hy^{-1} are algebraically independent in $K[x, x^{-1}, y, y^{-1}, h, h^{-1}]$. The polynomial $f_0(x, hy, hy^{-1})$ contains only monomials with non-negative powers of hy^{-1}, whereas the other terms of the R.H.S. of (5.4.3) involve strictly negative and mutually distinct powers of hy^{-1}. Therefore (5.4.3) implies that $f_0 = f_1 = \ldots = f_N = 0$. This shows that all relations in $\mathcal{A}$ follow from the relation $\hbar\delta = x - x^{-1}$.

5.5. *Proof of Theorem* 5.2. – Fix a point $a \in \partial(S^1 \times [0, 1])$. Let us call an oriented knot diagram on $S^1 \times [0, 1]$ ascending if the point a lies on the diagram and when traversing the diagram from a in the direction specified by the orientation, every crossing is first encountered as an under-crossing. It is easy to see that the isotopy type of a knot presented by an ascending link diagram is completely determined by the homotopy type of this knot, *i.e.* by the number of times the diagram winds around S^1.

Let X_n be the oriented knot presented by an ascending diagram in $S^1 \times [0, 1]$ which winds n times around S^1 and has $\max(|n|-1, 0)$ self-crossing points. It is obvious that an arbitrary knot diagram in $S^1 \times [0, 1]$ may be changed to an ascending diagram by an isotopy and by altering some of its crossings from overpasses to underpasses. As usual this implies that the classes $\{[X_n] \mid n \in \mathbb{Z}\}$ generate $\mathcal{A}(S^1 \times [0, 1])$. According to [19] (*see* also [5]) the algebra $\mathcal{A}'(S^1 \times [0, 1])$ is the (commutative) polynomial ring over $K[x, x^{-1}, h, h^{-1}]$ generated by the algebraically independent classes $[X_n]' \in \mathcal{A}'(S^1 \times [0, 1])$ where $n \in \mathbb{Z} \setminus \{0\}$. Now the same reasoning as in the proof of Theorem 5.1, based on the use of the homomorphism (5.3.3) with $F = S^1 \times [0, 1]$ proves the claim of Theorem 5.2 in the case where $L_n = X_n$ for all n. In the general case $L_0 = X_0 = \mathcal{O}$ and for $n \neq 0$ one may inductively prove that $[L_n]$ equals $[X_n]$ plus a certain polynomial of $[X_i]$, where $0 \leq i \leq n-1$ if $n > 0$ and $n+1 \leq i \leq 0$ if $n < 0$. This implies the result. (The minimality of the diagram of L_n enables to perform the latter induction: smoothing the diagram of L_n in an arbitrary crossing point gives two mutually disjoint minimal knot diagrams.)

5.6. QUANTUM STATES. — According to the general physical philosophy linear functionals on the skein algebra $\mathscr{A}(F)$ and algebras $\mathscr{A}_k(F)$, $k \in K$ should be treated as quantum states. A rich set of such functionals is provided by embeddings $F \to S^3$. Each embedding $i: F \to S^3$ extends to an orientation preserving embedding $\bar{i}: F \times [0, 1] \to S^3$ which induces a linear functional $\mathscr{A}(F) \to \mathscr{A}(S^3)$. We may combine it with the homomorphism (5.4.2) and pass to $\mathscr{A}_k$ to get a functional $i_*: \mathscr{A}_k(F) \to K[y, y^{-1}, h^{-1}][[hy]]$ linear over the ring homomorphism

$$(5.6.1) \qquad f(h, \hbar) \mapsto f(hy, hy^{-1}): \quad K[\hbar][[h]] \to K[y, y^{-1}, h^{-1}][[hy]].$$

For the class $[L]_k \in \mathscr{A}_k(F)$ of an oriented link $L \subset F \times [0, 1]$ we have

$$i_*([L]_k) = y^{|L|}(\exp(khy) - \exp(-khy))\, h^{-1}\, P_{\bar{i}(L)}(\exp(khy), h).$$

This construction has an important drawback: if $k=0$ then $i_*=0$. However, when one has two embeddings $i, j: F \to S^3$ one may define a "relative" state

$$(i, j)_*: \mathscr{A}_0(F) \to K[y, y^{-1}, h^{-1}][[hy]]$$

[again linear over (5.6.1)] by the formula

$$(i, j)_*([L]_0) = y^{|L|}\, h^{-1}(P_{\bar{i}(L)}(1, h) - P_{\bar{j}(L)}(1, h))$$

(When $L=\varnothing$ the R.H.S. is assumed to be 0). Note that $P(1, h)$ is the Alexander-Conway polynomial. Note also that the homomorphism $(i, j)_*$ annihilates the ideal generated by $[\mathscr{O}]_0$.

5.7. *Remark.* — The proof of Theorems 5.1, 5.2 shows that when F is the disk (or the 2-sphere, or the annulus) then the homomorphism (5.3.3) is injective. Is the same true for all surfaces?

6. Unoriented skein algebras and topological quantization of $S(Z_\square)$

6.1. THE RING R. — The theory of unoriented skein algebras runs parallel to the theory of oriented skein algebras presented in Sections 3-5. The major difference is that instead of the relation (3.5.1) we use modified Kauffman relations, introduced below. Instead of two Planck variables h, $\hbar$ we need three variables h_{-1}, h_0, h_1 subject to the equality $h_0^2 = h_1 h_{-1}$. The role of the ring $K[x, x^{-1}, h, \hbar]$ will be played by the commutative ring $K[x, x^{-1}, h_{-1}, h_0, h_1]/(h_0^2 - h_1 h_{-1})$. The latter ring will be denoted by R.

6.2. UNORIENTED SKEIN ALGEBRAS. — We proceed directly to the case of links in $F \times [0, 1]$ leaving aside unoriented skein modules of arbitrary 3-manifolds (for the latter *see* [19]). One may define unoriented skein modules of links in $F \times [0, 1]$ in terms of regular isotopy types of link diagrams. Two unoriented link diagrams on the (oriented)

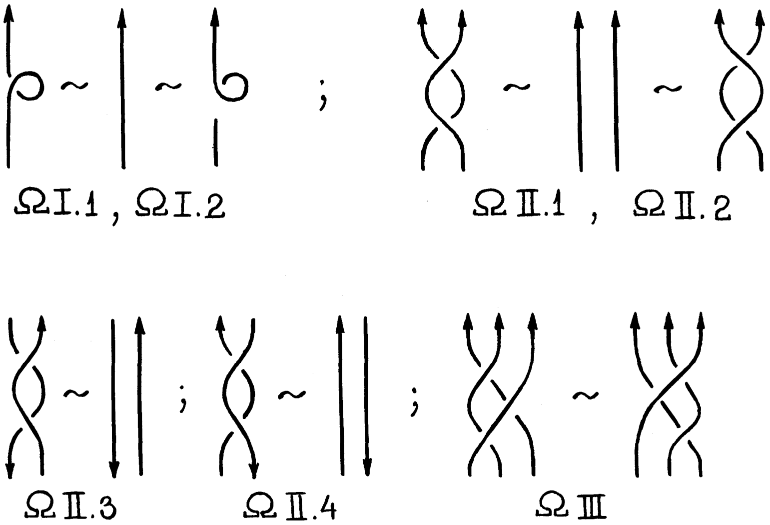

Fig. 3.

surface F are called regularly isotopic if they may be obtained from each other by Reidemeister moves of type II and III (see *Fig*. 3; here orientations of components should be ignored).

For a link diagram $\mathscr{D}$ denote by $|\mathscr{D}|$ the number of components of (the link presented by) $\mathscr{D}$. Denote by $\mathscr{L}_\square$ the set of regular isotopy types of unoriented link diagrams on F including the type Φ of the empty diagram and the type $\mathscr{O}$ of a small simple loop on F, presenting the trivial knot. The unoriented skein module $\mathscr{K}(\mathrm{F})$ is the quotient of the free R-module $\mathrm{R}\mathscr{L}_\square$ with basis $\mathscr{L}_\square$ by the submodle generated by elements of three kinds: (i) the elements

$$\mathscr{D}_+ - \mathscr{D}_- - h_{|\mathscr{D}_+|-|\mathscr{D}_0|}\mathscr{D}_0 + h_{|\mathscr{D}_+|-|\mathscr{D}_\infty|}\mathscr{D}_\infty \tag{6.2.1}$$

corresponding to arbitrary sets of four non-empty link diagrams $\mathscr{D}_+$, $\mathscr{D}_-$, $\mathscr{D}_0$, $\mathscr{D}_\infty$ which are exactly the same except near one point where they look as in Figure 4; (ii) the

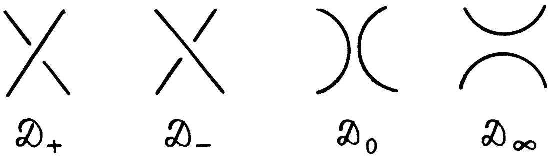

Fig. 4.

elements $\mathscr{D}' - x\mathscr{D}$ corresponding to arbitrary pairs of non-empty link diagrams $\mathscr{D}$, $\mathscr{D}'$ which are the same except near one point where $\mathscr{D}'$ is obtained from $\mathscr{D}$ by inserting a righthanded positive curl (as in the Reidemeister move ΩI.1, see *Fig*. 3); (iii) the element $h_{-1}\mathscr{O} - (x - x^{-1} + h_0)\Phi$. The reason for factorizing out the element (iii) is that for any non-empty link diagram $\mathscr{D}$ the element $h_{-1}(\mathscr{D} \amalg \mathscr{O}) - (x - x^{-1} + h_0)\mathscr{D} \in \mathrm{R}\mathscr{L}_\square$ is a linear

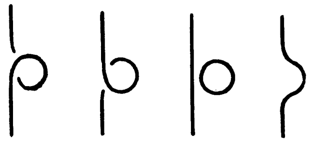

Fig. 5.

combination of elements of types (i), (ii) (see *Fig.* 5). Remark that the defining relations of $\mathscr{K}(F)$ are consistent in the same sense as in Section 3.5.

As in Section 3.2 placing one diagram over another makes $\mathscr{K}(F)$ an associative algebra with unit.

As in Section 3.1 substituting $x=\exp(kh_1)$ with $k\in K$ one gets a 1-parameter family $\mathscr{K}_k(F)$ of algebras over the ring

$$\tilde{R}=K[h_{-1}, h_0][[h_1]]/(h_0^2-h_1 h_{-1}).$$

Each diagram $\mathscr{D}$ on F with l components gives rise to the element

$$\langle \mathscr{D} \rangle_\square = \prod_{i=1}^{l} \langle \alpha_i \rangle_\square \in S(Z_\square)$$

(see sect. 2.3) where $\alpha_1, \ldots, \alpha_l$ are loops on F parametrizing the components of $\mathscr{D}$. The formula $\mathscr{D} \mapsto \langle \mathscr{D} \rangle_\square$ determines an algebra homomorphism $\mathscr{K}(F) \to S(Z_\square)$ linear over the coefficient ring homomorphism

$$h_{-1} \mapsto 0, \qquad h_0 \mapsto 0, \qquad h_1 \mapsto 0, \qquad x \mapsto 1 : R \to K.$$

This algebra homomorphism $\mathscr{K}(F) \to S(Z_\square)$ induces an algebra homomorphism $\mathscr{K}_k(F) \to S(Z_\square)$ linear over the coefficient homomorphism

$$h_{-1} \mapsto 0, \qquad h_0 \mapsto 0, \qquad h_1 \mapsto 0 : \tilde{R} \to K.$$

6.3. Theorem. — *For each* $k\in K$ *the algebra* $\mathscr{K}_k(F)$ *together with the homomorphism* $\mathscr{K}_k(F) \to S(Z_\square)$ *constructed above is a reduced quantization over* $(\tilde{R}, h_1)$ *of the symmetric Poisson algebra* $S(Z_\square)$.

Proof of this theorem is similar to the proof of Theorem 3.3: one uses the nullity of elements (6.2.1) in the case $|\mathscr{D}_+|-|\mathscr{D}_0|=|\mathscr{D}_+|-|\mathscr{D}_\infty|=1$ and the formula (2.4.1).

In contrast with what we have in the oriented set-up all algebras $\mathscr{K}_k(F)$ quantize the same Poisson algebra $S(Z_\square)$.

6.4. Theorem. — *Put* $\mathscr{K}=\mathscr{K}(F)$. *The algebra* $\mathscr{K}/((x-1)\mathscr{K}+h_{-1}\mathscr{K})$ *over* $K[h_1]$ *is canonically isomorphic to* $V_h(Z_\square)$, *where* $h=h_1$.

Note that $h_0=h_{-1}\mathscr{O}-x+x^{-1}$ so that $\mathscr{K}/(x-1)\mathscr{K}+h_{-1}\mathscr{K})$ is indeed an algebra over $K[h]$, $h=h_1$. The proof of Theorem 6.4 follows the same lines as the proof of Theorem 4.1.

We also have

$$\mathscr{K}/(h_{-1}\mathscr{K}+h_0\mathscr{K})=(\mathrm{K}[x]/(x^2-1))\otimes_{\mathrm{K}}\mathrm{V}_{h_1}(\mathrm{Z}_\square).$$

To show this it suffices to rewrite the defining relations of $\mathscr{K}$ with respect to the following set of generators: with a link diagram $\mathscr{D}$ associate the new generator $x^{|\mathscr{D}|+c(\mathscr{D})}\mathscr{D}$ where $c(\mathscr{D})$ is the number of self-crossings of $\mathscr{D}$.

The results of Section 5.3 may be transferred to the present setting. Let $\mathscr{K}'$ be the $\mathrm{K}[x, x^{-1}, h, h^{-1}]$-algebra $\mathrm{K}[x, x^{-1}, h, h^{-1}]\otimes_{\mathrm{R}}\mathscr{K}$ where R acts in $\mathrm{K}[x, x^{-1}, h, h^{-1}]$ via the projection sending x into x and all three variables h_{-1}, h_0, h_1 into h. The mapping $\mathscr{D}\mapsto y^{|\mathscr{D}|}\mathscr{D}'$, where $\mathscr{D}'$ is the class of the link diagram $\mathscr{D}$ in $\mathscr{K}'$, extends to an additive (and actually multiplicative) homomorphism

$$\mathscr{K}\to\mathrm{K}[y, y^{-1}]\otimes_{\mathrm{K}}\mathscr{K}' \tag{6.4.1}$$

linear over the coefficient ring homomorphism

$$f(x, h_{-1}, h_0, h_1)\mapsto f(x, hy^{-1}, h, hy):\quad \mathrm{R}\to\mathrm{K}[y, y^{-1}, h, h^{-1}].$$

Using the homomorphism (6.4.1), the calculation of $\mathscr{K}'(\mathrm{D}^2)$ due to Kauffman [7] and the calculation of $\mathscr{K}'(\mathrm{S}^1\times[0, 1])$ due to the author [19] it is easy to compute the unoriented skein algebras of D^2, S^2 and $\mathrm{S}^1\times[0, 1]$.

6.5. THEOREM. — *(i) The inclusion* $\mathrm{D}^2\subsetneq\mathrm{S}^2$ *induces an algebra isomorphism*

$$\mathscr{K}(\mathrm{D}^2)\to\mathscr{K}(\mathrm{S}^2).$$

(ii) *The algebra* $\mathscr{K}(\mathrm{D}^2)$ *is generated over* R *by the class* δ *of the trivial knot diagram* $\mathcal{O}$ *subject only to the relation* $h_{-1}\delta=x-x^{-1}+h_0$. *Thus*

$$\mathscr{K}(\mathrm{D}^2)=\mathrm{R}[\delta]/(h_{-1}\delta-x+x^{-1}-h_0).$$

6.6. THEOREM. — *For* $n\geqq 0$ *let* l_n *be the class of a knot diagram in* $\mathrm{S}^1\times[0, 1]$ *which has* $\max(|n|-1, 0)$ *self-crossings and which is homotopic to the n-th degree of a generator of* $\pi_1(\mathrm{S}^1)$. *Then* $\mathscr{K}(\mathrm{S}^1\times[0, 1])$ *is the quotient of the polynomial ring* $\mathrm{R}[l_0, l_1, \ldots]$ *by the ideal generated by* $h_{-1}l_0-x+x^{-1}-h_0$.

Proofs of both theorems follow the same lines as the proofs of Theorems 5.1, 5.2.

The material of Section 5.6 may be also transferred to the unoriented setting. This is left to the reader.

CHAPTER II

LIE BIALGEBRAS Z, Z_0 AND THEIR TOPOLOGICAL QUANTIZATION

7. Lie bialgebras and their quantization

7.1. LIE BIALGEBRAS (*cf*. [1], [2]). – Recall the notion of Lie coalgebra which is dual to the notion of Lie algebra.

A *Lie coalgebra* over K is a K-module $\mathfrak{g}$ provided with a linear homomorphism $\nu : \mathfrak{g} \to \mathfrak{g} \otimes \mathfrak{g}$ such that $\mathrm{Perm}_{\mathfrak{g}} \circ \nu = -\nu$ and

$$(7.1.1) \qquad (\tau^2 + \tau + 1) \circ (id_{\mathfrak{g}} \otimes \nu) \circ \nu = 0$$

where $\mathrm{Perm}_{\mathfrak{g}}$ is the permutation $a \otimes b \mapsto b \otimes a$ in $\mathfrak{g} \otimes \mathfrak{g}$ and τ is the permutation $a \otimes b \otimes c \mapsto c \otimes a \otimes b$ in $\mathfrak{g} \otimes \mathfrak{g} \otimes \mathfrak{g}$. The homomorphism ν is called a Lie cobracket in $\mathfrak{g}$. The dual homomorphism $\nu^* : \mathfrak{g}^* \otimes \mathfrak{g}^* \to \mathfrak{g}^*$, clearly, is a Lie bracket in $\mathfrak{g}^* = \mathrm{Hom}_K(\mathfrak{g}, K)$ (and vice versa if $\mathfrak{g}$ is free of finite rank over K). Indeed, (7.1.1) is dual to the Jacobi identity.

A *Lie bialgebra* over K is a K-module $\mathfrak{g}$ provided with a Lie bracket [,] and a Lie cobracket $\nu : \mathfrak{g} \to \mathfrak{g} \otimes \mathfrak{g}$ so that for any $a, b \in \mathfrak{g}$

$$(7.1.2) \qquad \nu([a, b]) = a\nu(b) - b\nu(a).$$

Here, as usual, $\mathfrak{g}$ acts in $\mathfrak{g} \otimes \mathfrak{g}$ by the rule $a(b \otimes c) = [a, b] \otimes c + b \otimes [a, c]$. Condition (7.1.2) means that ν is a 1-cocycle of $\mathfrak{g}$.

In the class of free modules of finite rank the notion of Lie bialgebra is self-dual: any Lie bialgebra structure in such a module $\mathfrak{g}$ induces a Lie bialgebra structure in $\mathfrak{g}^*$ and vice versa. Each Lie algebra may be considered as a Lie bialgebra with the zero Lie cobracket. Similarly, each Lie coalgebra may be considered as a Lie bialgebra with the zero Lie bracket.

The notion of Lie bialgebra is a kind of algebraic counterpart of the so-called classical r-matrices, *see* [2].

7.2. Poisson and co-Poisson bialgebras (*cf.* [2]). — Recall that a coalgebra over K is a K-module A equipped with a linear homomorphism (comultiplication) $\Delta : A \to A \otimes A$ which is coassociative, *i.e.*

$$(\mathrm{id}_A \otimes \Delta) \circ \Delta = (\Delta \otimes \mathrm{id}_A) \circ \Delta : A \to A^{\otimes 3}.$$

The coalgebra is called cocommutative if $\mathrm{Perm}_A \circ \Delta = \Delta$.

A bialgebra over K is a K-module equipped with a structure of associative algebra and coalgebra so that the comultiplication is an algebra homomorphism. Note that we do not require existence of counits and antipodal homomorphisms.

A Poisson bialgebra over K is a K-module A equipped with the structure of bialgebra and Poisson algebra with the same commutative multiplication so that the bialgebra comultiplication $\Delta : A \to A \otimes A$ preserves the Lie bracket:

$$(7.2.1) \qquad \Delta([a, b]) = [\Delta(a), \Delta(b)]$$

for any $a, b \in A$. [Here the bracket in the right hand side is given by (1.1.1).]

Non-trivial examples of Poisson bialgebras will be exhibited in Chapter III. In the present chapter we will use rather the dual notion of co-Poisson bialgebras. It is convenient to define first a weaker structure of co-Poisson coalgebra. A co-Poisson coalgebra over K is a cocommutative coalgebra A over K equipped with a Lie cobracket

$\nu: A \to A \otimes A$ which is related to the comultiplication $\Delta: A \to A \otimes A$ by the formula

$$(\mathrm{id}_A \otimes \Delta) \circ \nu = (\nu \otimes \mathrm{id}_A + (\mathrm{Perm}_A \otimes \mathrm{id}_A) \circ (\mathrm{id}_A \otimes \nu)) \circ \Delta. \tag{7.2.2}$$

The latter formula is dual to (0.3.2).

A co-Poisson bialgebra over K is a K-module A provided with the structure of bialgebra and co-Poisson coalgebra with one and the same cocommutative comultiplication $\Delta: A \to A \otimes A$ such that the Lie cobracket $\nu: A \to A \otimes A$ satisfies the identity

$$\nu(ab) = \nu(a) \Delta(b) + \Delta(a) \nu(b) \tag{7.2.3}$$

for all $a, b \in A$. This identity is dual to (7.2.1).

7.3. COQUANTIZATION OF CO-POISSON COALGEBRAS AND BIALGEBRAS. — Let Q be a commutative associative K-algebra with unit and with augmentation $\varphi: Q \to K$. Let $\hbar \in \mathrm{Ker}\, \varphi$. The following definition is dual to the definition of quantization of Poisson algebras given is Section 1.2. A coquantization over $(Q, \varphi, \hbar)$ of a co-Poisson K-coalgebra S is a pair (a Q-coalgebra A, a φ-linear coalgebra epimorphism $p: A \to S$) such that for any $a \in A$

$$\Delta(a) - \mathrm{Perm}_A(\Delta(a)) = \hbar (p \otimes p)^{-1}((\nu \circ p)(a)) \bmod \hbar\, \mathrm{Ker}(p \otimes p)$$

where Δ is the comultiplication in A and ν is the Lie cobracket in S. If, additionally, A is a bialgebra, S is a co-Poisson bialgebra and p is a bialgebra homomorphism then we say that (A, p) is a coquantization of the co-Poisson bialgebra S. The terminology which follows the definition of quantization in section 1.2 will be also applied to coquantizations with the obvious changes. In particular, (A, p) is called reduced if $\mathrm{Ker}\, p = (\mathrm{Ker}\, \varphi)\, A$

7.4. THEOREM. — *Let* $\mathfrak{g}$ *be a Lie bialgebra over* K. *Let* V *be the* $K[h]$*-algebra* $V_h(\mathfrak{g})$ *constructed in Section* 1.3. *The formula* $a \mapsto a \otimes 1 + 1 \otimes a$ *where* $a \in \mathfrak{g}$ *induces a* $K[h]$*-linear algebra homomorphism* $\Delta: V \to V \otimes_{K[h]} V$ *which makes* V *a bialgebra. The Lie cobracket* $\mathfrak{g} \to \mathfrak{g} \otimes \mathfrak{g}$ *extends by* (7.2.3) *to a* $K[h]$*-linear Lie cobracket* $V \to V \otimes_{K[h]} V$ *which makes* V *a co-Poisson bialgebra over* $K[h]$.

The co-Poisson bialgebra V constructed in the Theorem will be still denoted by $V_h(\mathfrak{g})$. The quotient co-Poisson bialgebra structure in $U(\mathfrak{g}) = V/(h-1)V$ was pointed out in [2]. Theorem 7.4 may be deduced from this result of [2]. For completeness I will give in Sections 7.5, 7.6 a direct proof of Theorem 7.4.

Theorem 7.4 enables us to define quantization of Lie bialgebras. By a quantization of a Lie K-bialgebra $\mathfrak{g}$ we shall mean a reduced coquantization of the co-Poisson bialgebra $V_h(\mathfrak{g})$ over the projection $\hbar \mapsto O: K[h, \hbar] \to K[h]$. In particular, for such a coquantization $p: A \to V_h(\mathfrak{g})$ we must have the equality $A/\hbar A = V_h(\mathfrak{g})$ in the category of bialgebras. Note that the homomorphism $p: A \to V_h(\mathfrak{g})$ induces a homomorphism $A/(h-1)A \to U(\mathfrak{g})$ which is a reduced coquantization of the co-Poisson bialgebra $U(\mathfrak{g})$ over $K[\hbar]$. This is essentially what Drinfeld [2] calls a quantization of $\mathfrak{g}$, though he considers the ring $K[[\hbar]]$ and requires the quantized bialgebra to be topologically free as the $K[[\hbar]]$-module. If the bialgebra $A/(h-1)A$ considered above is free as the $K[\hbar]$-module we may complete it via the inclusion $K[\hbar] \subsetneq K[[\hbar]]$ and get there by a quantization of $\mathfrak{g}$ in the Drinfeld sense.

7.5. LEMMA. — *Let* $\mathfrak{g}$ *be a Lie coalgebra and let* T *be the tensor algebra of* $\mathfrak{g}$ (*see Section* 1.3). *The formula* $a \mapsto a \otimes 1 + 1 \otimes a$ *induces an algebra homomorphism* $\Delta : T \to T \otimes T$ *which makes* T *a cocommutative bialgebra. The Lie cobracket* $\mathfrak{g} \to \mathfrak{g} \otimes \mathfrak{g}$ *extends by* (7.2.3) *to a Lie cobracket* $T \to T \otimes T$ *which makes* T *a co-Poisson bialgebra.*

Proof. — The first claim is easy and well known. Let us prove the second claim. Denote the Lie cobracket $\mathfrak{g} \to \mathfrak{g} \otimes \mathfrak{g}$ by ν. Denote by the same symbol ν the extension $T \to T \otimes T$ of the cobracket which transforms the monomial $a_1 a_2 \ldots a_n$ with $a_1, a_2, \ldots, a_n \in \mathfrak{g}$ into

$$\sum_{i=1}^{n} ((\prod_{j<i} \Delta(a_j))\, \nu(a_i) \prod_{j>i} \Delta(a_j)) \in T \otimes T.$$

In particular, $\nu(1) = 0$.

It is evident that (7.2.3) holds for any $a, b \in T$. Since $\mathrm{Perm}_T \circ (\nu|_{\mathfrak{g}}) = -\nu|_{\mathfrak{g}}$ and $\mathit{Perm}_T \circ \Delta = \Delta$ we have $\mathrm{Perm}_T \circ \nu = -\nu$. It remains to verify (7.2.2) with A replaced by T and the identity (7.1.1) with $\mathfrak{g}$ replaced by T.

Fix $a_1, \ldots, a_n \in \mathfrak{g}$. Put $b_i = \nu(a_i) \in \mathfrak{g} \otimes \mathfrak{g}$ for $i = 1, \ldots, n$. Apply both sides of (7.2.2) to $a_1 \otimes \ldots \otimes a_n$. The L.H.S. will be

$$\sum_{i=1}^{n} ((\prod_{j<i} \Delta^2(a_j))\, (I \otimes \Delta)(b_i) (\prod_{j>i} \Delta^2(a_j)))$$

where $I = \mathrm{id}_T$ and $\Delta^2 = (I \otimes \Delta)\Delta = (\Delta \otimes I)\Delta$. The R.H.S. will be

$$\sum_{i=1}^{n} [(\prod_{j<i} \Delta^2(a_j))(b_i \otimes 1)(\prod_{j>i} \Delta^2(a_j)) + (\mathrm{Perm}_T \otimes I)((\prod_{j<i} \Delta^2(a_j))(1 \otimes b_i)(\prod_{j>i} \Delta^2(a_j)))].$$

Clearly, $\mathrm{Perm}_T \otimes I$ is an automorphism of $T^{\otimes 3}$, preserving $\Delta^2(a)$ for each $a \in \mathfrak{g}$. Therefore it suffices to show that for each i

$$(I \otimes \Delta)(b_i) = b_i \otimes 1 + (\mathrm{Perm}_T \otimes I)(1 \otimes b_i).$$

This equality holds for an arbitrary $b_i \in \mathfrak{g} \otimes \mathfrak{g}$ as directly follows from the definition of Δ.

To prove (7.1.1) note that

$$(I \otimes \nu)(\nu(a_1 a_2 \ldots a_n)) = \sum_{i=1}^{n} \alpha_i + \sum_{1 \leq i < k \leq n} (\beta_{i,k} + \gamma_{i,k})$$

where

$$\alpha_i = (\prod_{j<i} \Delta^2(a_j))(I \otimes \nu)(\nu(a_i))(\prod_{j>i} \Delta^2(a_j)),$$

$$\beta_{i,k}=(\prod_{j<i}\Delta^2(a_j))(I\otimes\nu)(\Delta(a_i))(\prod_{i<j<k}\Delta^2(a_j))(I\otimes\Delta)(\nu(a_k))(\prod_{j>k}\Delta^2(a_j)),$$

$$\gamma_{i,k}=(\prod_{j<i}\Delta^2(a_j))(I\otimes\Delta)(\nu(a_i))(\prod_{i<j<k}\Delta^2(a_j))(I\otimes\nu)(\Delta(a_k))(\prod_{j>k}\Delta^2(a_j)).$$

Since $\tau\in \operatorname{End} T^{\otimes 3}$ preserves $\Delta^2(a)$ for each $a\in\mathfrak{g}$ and $\nu|_{\mathfrak{g}}$ is a Lie cobracket, $(\tau^2+\tau+I)(\alpha_i)=0$ for each i. Let us show that

$$(\tau^2+\tau+I)(\beta_{i,k}+\gamma_{i,k})=0$$

for all $i<k$. Fix i, k and put

$$b=\nu(a_i),\qquad \pi=\prod_{i<j<k}\Delta^2(a_j),\qquad b'=\nu(a_k).$$

Then $(I\otimes\nu)(\Delta(a_k))=1\otimes b'$ and $(I\otimes\nu)(\Delta(a_i))=1\otimes b$. Thus, it suffices to show that $\tau^2+\tau+I$ annihilates

$$(I\otimes\Delta)(b)\,\pi\,(1\otimes b')+(1\otimes b)\,\pi\,(I\otimes\Delta)(b').$$

If b is a finite sum $\sum_r d_r\otimes e_r$ and b' is a finite sum $\sum_s d'_s\otimes e'_s$ with $d_r, e_r, d'_s, e'_s\in\mathfrak{g}$ then $(I\otimes\Delta)(b)\ \pi\ (1\otimes b')=\sigma_1+\sigma_2$ and $(1\otimes b)\,\pi\,(I\otimes\Delta)(b')=\sigma_3+\sigma_4$ where

$$\sigma_1=\sum_r(d_r\otimes e_r\otimes 1)\cdot\pi\cdot\sum_s(1\otimes d'_s\otimes e'_s)$$

$$\sigma_2=\sum_r(d_r\otimes 1\otimes e_r)\cdot\pi\cdot\sum_s(1\otimes d'_s\otimes e'_s)$$

$$\sigma_2=\sum_r(1\otimes d_r\otimes e_r)\cdot\pi\cdot\sum_s(d'_s\otimes 1\otimes e'_s)$$

$$\sigma_4=\sum_r(1\otimes d_r\otimes e_r)\cdot\pi\cdot\sum_s(d'_s\otimes e'_s\otimes 1).$$

Using the equalities $\tau(\pi)=\pi$, $\operatorname{Perm}_{\mathfrak{g}}(b)=-b$, $\operatorname{Perm}_{\mathfrak{g}}(b')=-b'$ one easily shows that $\tau^2+\tau+I$ annihilates both $\sigma_1+\sigma_3$ and $\sigma_2+\sigma_4$.

7.6. *Proof of Theorem* 7.4. — Let T be the tensor algebra of the Lie bialgebra $K[h]\otimes\mathfrak{g}$. Lemma 7.5 provides us with a comultiplication Δ and a Lie cobracket ν in T. To prove Theorem 7.4 it suffices to show that both Δ and ν descends to V, *i. e.* that

(7.6.1) $$\Delta(\operatorname{Ker} g)\subset \operatorname{Ker}(g\otimes g),$$

(7.6.2) $$\nu(\operatorname{Ker} g)\subset \operatorname{Ker}(g\otimes g)$$

where g is the projection $T\to V_h(\mathfrak{g})$. The inclusion (7.6.1) is straightforward. To prove (7.6.2) we compute for $a, b\in\mathfrak{g}$

$$\begin{aligned}\nu(ab-ba)&=\nu(a)\Delta(b)+\Delta(a)\nu(b)-\nu(b)\Delta(a)-\Delta(b)\nu(a)\\&=\nu(a)(b\otimes 1+1\otimes b)-(b\otimes 1+1\otimes b)\nu(a)+(a\otimes 1-1\otimes a)\nu(b)\\&\qquad -\nu(b)(a\otimes 1+1\otimes a)\equiv h(a\,\nu(b)-b\,\nu(a)) \bmod \operatorname{Ker}(g\otimes g).\end{aligned}$$

Here we use the action of $\mathfrak{g}$ in $\mathfrak{g}\otimes\mathfrak{g}$ described in Section 7.1. Because of (7.1.2) we have

$$\nu(ab-ba-h[a, b])=0 \bmod \operatorname{Ker}(g\otimes g).$$

This formula together with (7.6.1), (7.2.3) imply (7.6.2).

8. Lie bialgebras Z and Z_0

8.1. COBRACKET ν. — Let F, $\hat{\pi}$, Z be the same objects as in Section 2.1. For a non-contractible loop $\alpha: S^1 \to F$ we shall denote by $\langle \alpha \rangle_0$ its class $\langle \alpha \rangle$ in $\hat{\pi} \subset Z$. For a contractible loop α put $\langle \alpha \rangle_0 = 0 \in Z$.

We shall provide Z with a Lie cobracket ν as follows. Let α be a generic loop on F. Denote by $\#\alpha$ its (finite) set of double points $\{q \in \alpha(S^1) \mid \operatorname{card} \alpha^{-1}(q) > 1\}$. Each point $q \in \#\alpha$ is traversed by α twice, the tangent vectors of α in q being linearly independent. Assume that these vectors are numerated u_1, u_2 so that the pair u_1, u_2 is positively oriented. For $i=1,2$ denote by α_q^i the loop wich starts from q in the direction u_i and goes along α till the first return to q. Clearly, up to a choice of parametrization $\alpha = \alpha_q^1 \alpha_q^2$. Set

$$(8.1.1) \qquad \nu(\langle \alpha \rangle) = \sum_{q \in \#\alpha} (\langle \alpha_q^1 \rangle_0 \otimes \langle \alpha_q^2 \rangle_0 - \langle \alpha_q^2 \rangle_0 \otimes \langle \alpha_q^1 \rangle_0)$$

8.2. LEMMA. — *For any loops* α, β *on* F

$$[\langle \alpha \rangle_0, \langle \beta \rangle_0] = [\langle \alpha \rangle, \langle \beta \rangle] = \sum_{q \in \alpha \# \beta} \varepsilon(q; \alpha, \beta) \langle \alpha_q \beta_q \rangle_0$$

where [,] *is the Goldman-Lie bracket.*

The first equality is obvious, since the class of contractible loops lies in the center of the Lie algebra Z. The second equality follows from Proposition 5.9 of [4].

8.3. THEOREM. — *The linear homomorphism* $\nu: Z \to Z \otimes Z$ *given on the generators of* Z *by the formula* (8.1.1) *is a Lie cobracket. The module* Z *provided with the Goldman-Lie bracket* [,] *and the Lie cobracket* ν *is a Lie bialgebra.*

Proof. — To show that ν is well-defined it suffices to show that $\nu(\langle \alpha \rangle)$ does not change when we apply to α "elementary homotopies" (see *Fig.* 6). This is straightforward.

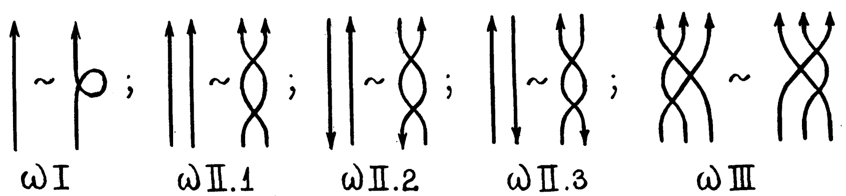

Fig. 6.

The equality $\mathrm{Perm}_Z \circ \nu = \nu$ is obvious. It follows from definitions that for a generic loop α $(1 \otimes \nu)(\nu(\langle \alpha \rangle))$ is a sum of certain expressions $X(p, q)$ associated with non-ordered pairs of (distinct) self-intersection points p, q of α. Namely, smoothing α in p and q we obtain three loops, say, β, γ, δ where δ hits both p and q, β hits p and γ hits q. As described above the orientation of F induces an order in each pair (β, δ), (γ, δ). Put $\varepsilon(p, q) = 1$ if δ is either minimal or maximal in both pairs simultaneously. Otherwise put $\varepsilon(p, q) = -1$. It is easy to compute that

$$X(p, q) = \varepsilon(p, q)[\langle \beta \rangle_0 \otimes \langle \gamma \rangle_0 \otimes \langle \delta \rangle_0 + \langle \gamma \rangle_0 \otimes \langle \beta \rangle_0 \otimes \langle \delta \rangle_0 - \langle \beta \rangle_0 \otimes \langle \delta \rangle_0 \otimes \langle \gamma \rangle_0 - \langle \gamma \rangle_0 \otimes \langle \delta \rangle_0 \otimes \langle \beta \rangle_0].$$

Therefore, $(\tau^2 + \tau + 1)(X(p, q)) = 0$ where τ is the permutation $a \otimes b \otimes c \mapsto c \otimes a \otimes b$ in $Z^{\otimes 3}$. Thus $\tau^2 + \tau + 1$ annihilates $(1 \otimes \nu)(\nu(\langle \alpha \rangle))$. This proves that ν is a Lie cobracket.

Let us show that for arbitrary loops α, β in F lying in general position we have (7.1.2) for $a = \langle \alpha \rangle$, $b = \langle \beta \rangle$. For any $p, q \in \alpha \# \beta$ denote by $(\alpha\beta)_p^q$ the loop in F which starts in p goes along α till q and then goes back to p along β. Clearly,

$$\langle (\alpha\beta)_p^q \rangle = \langle \beta\alpha)_q^p \rangle.$$

Note that for any $p \in \alpha \# \beta$ the self-intersection set of the loop $\alpha_p \beta_p$ splits into disjoint union of three subsets: $\# \alpha$, $\# \beta$, and $(\alpha \# \beta) \setminus \{p\}$. This implies that

$$\nu([a, b]) = \sum_{p \in \alpha \# \beta} \varepsilon(p; \alpha, \beta)\, \nu(\langle \alpha_p \beta_p \rangle) = \sigma - \mathrm{Perm}_Z(\sigma)$$

where

$$\sigma = \sigma_1 + \sigma_2 + \sigma_3 + \sigma_4 + \sigma_5$$

and

$$\sigma_1 = \sum_{\{q \in \# \alpha \mid p \in \alpha_q^1 \# \beta\}} \varepsilon(p; \alpha, \beta) \langle (\alpha_q^1)_p \beta_p \rangle_0 \otimes \langle \alpha_q^2 \rangle_0,$$

$$\sigma_2 = \sum_{\{q \in \# \alpha \mid p \in \alpha_q^2 \# \beta\}} \varepsilon(p; \alpha, \beta) \langle \alpha_q^1 \rangle_0 \otimes \langle (\alpha_q^2)_p \beta_p \rangle_0,$$

$$\sigma_3 = \sum_{\{q \in \# \beta \mid p \in \alpha \# \beta_q^1\}} \varepsilon(p; \alpha, \beta) \langle \alpha_p (\beta_q^1)_p \rangle_0 \otimes \langle \beta_q^2 \rangle_0,$$

$$\sigma_4 = \sum_{\{q \in \# \beta \mid p \in \alpha \# \beta_q^2\}} \varepsilon(p; \alpha, \beta) \langle \beta_q^1 \rangle_0 \otimes \langle \alpha_p (\beta_q^2)_p \rangle_0,$$

$$\sigma_5 = \sum_{\substack{p, q \in \alpha \# \beta \\ p \neq q}} \varepsilon(p; \alpha, \beta)\, \varepsilon(q; \alpha, \beta) \langle (\alpha\beta)_q^p \rangle_0 \otimes \langle (\beta\alpha)_q^p \rangle_0.$$

It follows directly from Lemma 8.2 and definitions that

$$\sigma_1 + \sigma_2 - \mathrm{Perm}_Z(\sigma_1 + \sigma_2) = -b\,\nu(a); \qquad \sigma_3 + \sigma_4 - \mathrm{Perm}_Z(\sigma_3 + \sigma_4) = a\,\nu(b)$$

and also $\mathrm{Perm}_Z(\sigma_5) = \sigma_5$. This implies (7.1.2).

8.4. LIE BIALGEBRA Z_0. — Denote by Z_0 the submodule of Z generated by the homotopy classes of non-contractible loops. Clearly Z_0 is the free K-module with the basis $\hat{\pi}\setminus\{1\}$ where $\{1\}$ is the homotopy class of contractible loops in F. It follows from [4], Prop. 5.9 that Z_0 is a Lie subalgebra of Z. It follows directly from definitions that the Lie cobracket ν in Z maps Z_0 into $Z_0 \otimes Z_0$. Thus Z_0 is a Lie subbialgebra of Z. Moreover the Lie bialgebra Z is the direct sum of Z_0 and the one-dimensional Lie bialgebra $K\{1\}$ with zero Lie bracket and zero Lie cobracket.

8.5. *Remarks.* — 1. If α is a simple loop then $\nu(\langle \alpha \rangle)=0$ and, moreover, $\nu(\langle \alpha^n \rangle)=0$ for any integer n. Conjecture: if $\nu(\langle \alpha \rangle)=0$ then α is homotopic to a power of a simple loop. A similar assertion for pairs of loops is proved in [4]: if α is a simple loop and β is a loop on F then α, β are homotopic to non-intersected loops iff $[\langle \alpha \rangle, \langle \beta \rangle]=0$.

2. The previous remark shows that if F is the annulus or the torus $S^1 \times S^1$ then $\nu=0$.

3. In general the Lie bialgebra, dual to Z does not exist: one may show that if the genus of F is non-zero then the image of the homomorphism $Z^* \to (Z \otimes Z)^*$, dual to the Goldman-Lie bracket, does not lie in $Z^* \otimes Z^*$. On the other hand the cobracket $\nu: Z \to Z \otimes Z$ always induces a Lie bracket in $Z^* = \mathrm{Hom}_K(Z, K)$. The Lie algebra Z^* may be shown to be a projective limit of nilpotent Lie algebras, whose underlying K-modules are free of finite rank.

4. The Goldman bracket and the cobracket ν may be computed purely algebraically from the operations λ, μ introduced in [16], supplement 2. Put $\pi=\pi_1(F, f)$ with $f \in F$ and denote by Y the group ring $K\pi$. With each pair $a, b \in \pi$ the "intersection" λ associates $\lambda(a, b) \in Y/((a-1)Y+Y(b^{-1}-1))$. With each $a \in \pi$ the "self-intersection" μ associates an element $\mu(a)$ of the quotient module of Y by the sumbodule generated over K by $\{1\}$ and the set $\{c+c^{-1}a^{-1} \mid c \in \pi\}$. Let $q: \pi\setminus\{1\} \to \hat{\pi}$ be the natural projection. Put $q(\{1\})=0 \in Z$. It is easy to compute that if $\lambda(a, b)$ is represented by $\sum_i k_i c_i \in Y$ with $k_i \in K$ and $c_i \in \pi$ then

$$[q(a), q(b)] = \sum_i k_i q(a c_i b c_i^{-1}).$$

If $\mu(a)$ is represented by $\sum_i k_i c_i \in Y$ then

$$\nu(q(a)) = \sum_i k_i (q(c_i^{-1}) \otimes q(c_i a) - q(c_i a) \otimes q(c_i^{-1})).$$

The study of intersections λ, μ in [16] was motivated by an important role played by λ in the theory of multivariable Seifert forms of knots (*see* [17]). Note also that if $\partial F \neq \varnothing$ and $f \in \partial F$ then there exist more precise versions of λ and μ which are respectively a bilinear pairing $Y \times Y \to Y$ and a mapping $\pi \to Y$ (*see* [16]).

9. Skein bialgebra A

The aim of the present Section 9 is to construct a canonical comultiplication in the $K[h, \hbar]$-algebra $A=\mathscr{A}/(x-1)\mathscr{A}$ (*see* Section 4.1). The construction of the comultiplication is partially inspired by the composition product of polynomial invariants of links in S^3 due to F. Jaeger [6]. The construction goes in terms of link diagrams.

9.1. LINK DIAGRAMS AND THEIR LABELLINGS. — For an oriented link diagram $\mathscr{D}$ on F we denote by $L(\mathscr{D})$ the oriented link in $F\times[0, 1]$ presented by $\mathscr{D}$. Recall that $|\mathscr{D}|=|L(\mathscr{D})|$ is the number of components of $L(\mathscr{D})$. Denote by $[\mathscr{D}]$ the element of A represented by $L(\mathscr{D})$.

The union of loops underlying the diagram $\mathscr{D}$ is a four-valent graph on F. Its edges and vertices are called edges and vertices (or self-crossing points) of $\mathscr{D}$. The set of edges is denoted by $\operatorname{Edg}\mathscr{D}$. Each vertex has a sign 1 or -1: see *Fig.* 1 where the depicted vertex of L_+ (resp. L_-) has sing 1 (resp. -1).

Let n be a positive integer. By a n-labelling of the oriented link diagram $\mathscr{D}$ we mean a function $f:\operatorname{Edg}\mathscr{D}\to\{1, 2, \ldots, n\}$ such that the following condition holds: For each vertex v of $\mathscr{D}$ if a, b are resp. upper and lower edges which look into v and c, d are resp. upper and lower edges which look out of v then either $f(a)=f(c)$ and $f(b)=f(d)$ or $f(a)=f(d)>f(b)=f(c)$. The vertices in which the last possibility occurs are called f-cutting vertices of $\mathscr{D}$. Denote the number of f-cutting vertices of sign 1 (resp. -1) by $|f|_+$ (resp. $|f|_-$). Set $|f|=|f|_+ +|f|_-$. Note the Kirchoff rule for labellings: in all vertices v of $\mathscr{D}$ we have $f(a)+f(b)=f(c)+f(d)$ where a, b, c, d are edges incident to v as above.

The set of n-labellings of $\mathscr{D}$ is denoted by $\operatorname{Lbl}_n(\mathscr{D})$. If $f\in\operatorname{Lbl}_n(\mathscr{D})$ then for each $i=1, \ldots, n$ the edges of $\mathscr{D}$ lying in $f^{-1}(i)$ make an oriented link diagram, denoted by $\mathscr{D}_{f,i}$. (It is understood that in the self-crossing points of $\mathscr{D}_{f,i}$ the choice of the lower/upper branches is the same as in $\mathscr{D}$.) Put

$$\|f\|=|\mathscr{D}|-\sum_{i=1}^{n}|\mathscr{D}_{f,i}|.$$

Clearly, $|f|\geqq\|f\|\geqq-|f|$ and $|f|\equiv\|f\| \bmod 2$. Therefore we may safely define

$$\langle\mathscr{D}|f\rangle=(-1)^{|f|_-}\,h^{|f|+\|f\|/2}\,\hbar^{|f|-\|f\|/2}.$$

Put

$$\Delta(\mathscr{D}, f)=\langle\mathscr{D}|f\rangle[\mathscr{D}_{f,1}]\otimes[\mathscr{D}_{f,2}]\otimes\ldots\otimes[\mathscr{D}_{f,n}]\in A^{\otimes n},$$

where $A^{\otimes n}$ is the tensor product over $K[h, \hbar]$ of n copies of A. Note that we do not exclude the possibility that certain link diagram $\{\mathscr{D}_{f,i}\}$ are empty so that $[\mathscr{D}_{f,i}]=[\Phi]=1\in A$.

The set $\mathrm{Lbl}_2(\mathscr{D})$ will be also denoted by $\mathrm{Lbl}(\mathscr{D})$. For a non-empty oriented link diagram $\mathscr{D}$ put

$$\Delta(\mathscr{D}) = \sum_{f \in \mathrm{Lbl}(\mathscr{D})} \Delta(\mathscr{D}, f) \in \mathrm{A}^{\otimes 2}.$$

For the empty link diagram Φ we define $\Delta(\Phi) = 1 \in \mathrm{A}^{\otimes 2}$.

9.2. THEOREM. — *There exists a unique* $\mathrm{K}[h, \hbar]$-*linear homomorphism* $\Delta : \mathrm{A} \to \mathrm{A} \otimes \mathrm{A}$ *such that for any oriented link diagram* $\mathscr{D}$ *we have* $\Delta([\mathscr{D}]) = \Delta(\mathscr{D})$. *The pair* (A, Δ) *is a bialgebra.*

Proof. — Uniqueness of Δ (modulo existence) is obvious. To prove existence we have to verify the following two claims: (i) Reidemeister moves on $\mathscr{D}$ (see *Fig.* 3) do not change $\Delta(\mathscr{D})$; (ii) if three diagrams $\mathscr{D}^+$, $\mathscr{D}^-$, $\mathscr{D}^0$ are identical outside a disk and look as in Figure 1 inside the disk then

$$(9.2.1) \qquad \Delta(\mathscr{D}^+) - \Delta(\mathscr{D}^-) = h_\varepsilon \Delta(\mathscr{D}^0)$$

where $\varepsilon = |\mathscr{D}^+| - |\mathscr{D}^0|$ and $h_1 = h$, $h_{-1} = \hbar$.

The proof of the claims, given below, is rather similar to the proof of Theorem 5.4 in [18] and Proposition 1 in [6].

We will use the following notation: for any subset B of $\mathrm{Lbl}(\mathscr{D})$ put

$$\Delta(\mathscr{D}, \mathrm{B}) = \sum_{f \in \mathrm{B}} \Delta(\mathscr{D}, f) \in \mathrm{A}^{\otimes 2}.$$

In particular, $\Delta(\mathscr{D}, \mathrm{Lbl}(\mathscr{D})) = \Delta(\mathscr{D})$.

We first verify Claim (ii). Let α, β, γ, δ be the edges of $\mathscr{D}^+$, incident to the vertex of $\mathscr{D}^+$ which lies in the disk in question, as depicted in Figure 7. Denote by the same

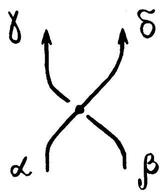

Fig. 7.

symbols α, β, γ, δ the corresponding edges of $\mathscr{D}^-$. For each of the three diagrams $\mathscr{D} \in \{\mathscr{D}^+, \mathscr{D}^-, \mathscr{D}^0\}$ the set $\mathrm{Lbl}(\mathscr{D})$ splits into disjoint union of six subsets $\mathrm{B}_r(\mathscr{D})$, $r = 1, \ldots, 6$ singled out by the values of labellings on the edges:

$f \in$	B_1	B_2	B_3	B_4	B_5	B_6
$f(\alpha) =$	1	2	2	1	1	2
$f(\beta) =$	1	2	1	2	2	1
$f(\gamma) =$	1	2	2	1	2	1
$f(\delta) =$	1	2	1	2	1	2

Here for $f \in \mathrm{Lbl}(\mathscr{D}^0)$ the expressions $f(\alpha)$, $f(\beta)$, $f(\gamma)$, $f(\delta)$ denote the values of f on the edges of $\mathscr{D}^0$ containing respectively α, β, γ, δ. Note that

$$\mathrm{B}_3(\mathscr{D}^-)=\mathrm{B}_4(\mathscr{D}^+)=\mathrm{B}_5(\mathscr{D}^0)=\mathrm{B}_6(\mathscr{D}^0)=\varnothing.$$

We shall prove that for each $r=1, \ldots, 6$

$$(9.2.2) \qquad \Delta(\mathscr{D}^+, \mathrm{B}_r(\mathscr{D}^+))-\Delta(\mathscr{D}^-, \mathrm{B}_r(\mathscr{D}^-))=h_\varepsilon \Delta(\mathscr{D}^0, \mathrm{B}_r(\mathscr{D}^0)).$$

This would imply (9.2.1). Let $r=1$. Each labelling $f \in \mathrm{B}_1(\mathscr{D}^+)$ in the evident fashion gives rise to labellings of $\mathscr{D}^-$, $\mathscr{D}^0$ belonging to $\mathrm{B}_1(\mathscr{D}^-)$, $\mathrm{B}_1(\mathscr{D}^0)$. Denote these latter labellings resp. by f^-, f^0 and put $f^+=f$. For $\chi \in \{+, -, 0\}$ we have

$$\Delta(\mathscr{D}^\chi, f^\chi)=\langle \mathscr{D}^\chi \,|\, f^\chi \rangle [\mathscr{D}_1^\chi] \otimes [\mathscr{D}_2^\chi]$$

where $\mathscr{D}_i^\chi=\mathscr{D}_{f^\chi, i}^\chi$ for $i=1, 2$.

Obviously, $\mathscr{D}_2^+=\mathscr{D}_2^-=\mathscr{D}_2^0$, and $\mathscr{D}_1^+, \mathscr{D}_1^-, \mathscr{D}_1^0$ make a Conway triple. Put $\mu=|\mathscr{D}_1^+|-|\mathscr{D}_1^0|=\pm 1$. One easily derives from definitions that

$$\langle \mathscr{D}^+ \,|\, f^+ \rangle=\langle \mathscr{D}^- \,|\, f^- \rangle \qquad \text{and} \qquad h_\varepsilon \langle \mathscr{D}^0 \,|\, f^0 \rangle=h_\mu \langle \mathscr{D}^+ \,|\, f^+ \rangle.$$

Therefore,

$$\Delta(\mathscr{D}^+, f^+)-\Delta(\mathscr{D}^-, f^-)-h_\varepsilon \Delta(\mathscr{D}^0, f^0)=\langle \mathscr{D}^+ \,|\, f^+ \rangle([\mathscr{D}_1^+]-[\mathscr{D}_1^-]-h_\mu[\mathscr{D}_1^0]) \otimes [\mathscr{D}_2^0]=0.$$

This implies (9.2.2) with $r=1$. The case $r=2$ is quite similar. If $r=3$ then (9.2.2) is equivalent to $\Delta(\mathscr{D}^+, \mathrm{B}_3(\mathscr{D}^+))=h_\varepsilon \Delta(\mathscr{D}^0, \mathrm{B}_3(\mathscr{D}^0))$. This equality follows from the fact that each labelling $f \in \mathrm{B}_3(\mathscr{D}^+)$ gives rise to a labelling f^0 of $\mathscr{D}^0$ so that

$$\mathscr{D}_{f, i}^+=\mathscr{D}_{f^0, i}^- \qquad \text{for} \quad i=1, 2;$$
$$\langle \mathscr{D}^+ \,|\, f \rangle=h^{(\varepsilon+1)/2}\, \hbar^{(1-\varepsilon)/2} \langle \mathscr{D}^0 \,|\, f^0 \rangle=h_\varepsilon \langle \mathscr{D}^0 \,|\, f^0 \rangle.$$

The cases $r=4, 5, 6$ are treated similarly. This finishes the proof of Claim (ii).

Let us verify Claim (i). Let a diagram $\mathscr{D}'$ be obtained from a diagram $\mathscr{D}$ by an application of Ω I.1 (see *Fig.* 3). Each set $\mathrm{Lbl}(\mathscr{D})$, $\mathrm{Lbl}(\mathscr{D}')$ splits into disjoint union of

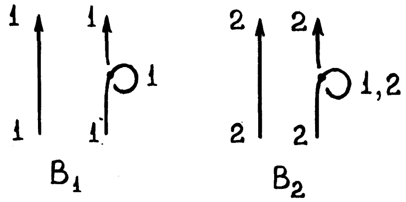

Fig. 8.

two subsets B_1, B_2 (cf. *Fig.* 8). The sets $\mathrm{B}_1(\mathscr{D})$, $\mathrm{B}_1(\mathscr{D}')$ bijectively correspond to each other and, obviously, $\Delta(\mathscr{D}, \mathrm{B}_1(\mathscr{D}))=\Delta(\mathscr{D}', \mathrm{B}_1(\mathscr{D}'))$. Each $f \in \mathrm{B}_2(\mathscr{D})$ gives rise to two labellings f_1 and f_2 of $\mathscr{D}'$ taking values resp. 1 and 2 on the small curl of $\mathscr{D}'$. Obviously, $|f_1| \geqq 1$ and $\|f_1\| \geqq 1$. Thus $\langle \mathscr{D}' \,|\, f_1 \rangle$ is divisible by $\hbar$. On the other hand the diagram

$\mathscr{D}'_{f_1,1}$ splits off a small simple circle $\mathcal{O}$. In A we have $\hbar[\mathcal{O}]=0$ and therefore $\Delta(\mathscr{D}', f_1)=0$. Clearly $\Delta(\mathscr{D}', f_2)=\Delta(\mathscr{D}, f)$. Thus $\Delta(\mathscr{D}, \mathrm{B}_2(\mathscr{D}))=\Delta(\mathscr{D}', \mathrm{B}_2(\mathscr{D}'))$. This shows that $\Delta(\mathscr{D})=\Delta(\mathscr{D}')$. The move Ω I.2 is considered similarly.

Let a diagram $\mathscr{D}'$ be obtained from a diagram $\mathscr{D}$ by an application of Ω II.1. As above the sets Lbl$(\mathscr{D})$, Lbl$(\mathscr{D}')$ split into disjoint union of six subsets $\mathrm{B}_1-\mathrm{B}_6$ in

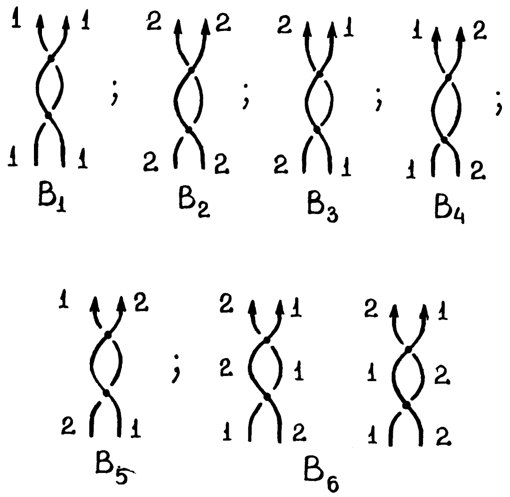

Fig. 9.

accordance with the values of labellings on the four "border" edges [see *Fig.* 9 for $\mathrm{B}_r(\mathscr{D}')$, $r=1, \ldots, 6$]. As above it is easy to show that $\Delta(\mathscr{D}, \mathrm{B}_r(\mathscr{D}))=\Delta(\mathscr{D}', \mathrm{B}_r(\mathscr{D}'))$ for all r. The cases $r=1, \ldots, 4$ are straightforward. In the case $r=5$ one should note that $\mathrm{B}_5(\mathscr{D})=\mathrm{B}_5(\mathscr{D}')=\varnothing$. In the case $r=6$ we have $\mathrm{B}_6(\mathscr{D})=\varnothing$ and the labellings from $\mathrm{B}_6(\mathscr{D}')$ are naturally divided into pairs (f, g) such that $\Delta(\mathscr{D}', f)=-\Delta(\mathscr{D}', g)$ (see *Fig.* 9).

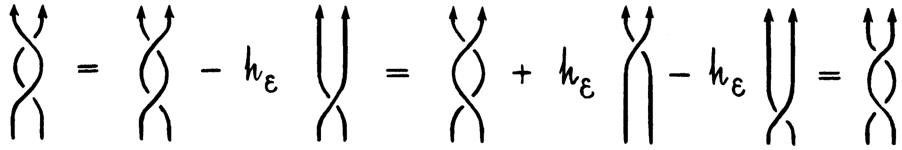

Fig. 10.

Invariance of Δ under the move Ω II.2 follows: see Figure 10 where $\varepsilon=\pm 1$ and where we have used Claim (ii) (*cf.* [6]). The moves Ω II.3, Ω II.4 are treated similarly.

Invariance of Δ under Ω III is proven in much the same way. The corresponding sets of labellings split into disjoint union of 20 subsets which should be considered separately. Two of these subsets are singled out by the condition that the values of labellings on the six "border" edges are equal to each other and equal to $i\in\{1, 2\}$. The other 18 subsets are numerated by triples $(r, s, i)\in\{1, 2, 3\}^2\times\{1, 2\}$, the subset corresponding to a triple (r, s, i) being specified by the condition that the r-th inlooking and the s-th outlooking border edges are labelled by i and the remaining four border edges are labelled by $3-i$. As above, the labellings of the border edges may be extended to the three "inner" edges in several ways, the corresponding invariants are to be summed up. Anyway, for each of these 20 subsets we do have the equality $\Delta(\mathscr{D}, \mathrm{B}(\mathscr{D}))=\Delta(\mathscr{D}', \mathrm{B}(\mathscr{D}'))$; *cf.* [6], where, in particular, a complete list of labellings of inner edges is given. This finishes the proof of Claim (i).

To show that (A, Δ) is a bialgebra we should check that $(1\otimes\Delta)\Delta=(\Delta\otimes 1)\Delta$ and that Δ is an algebra homomorphism. Note first for an arbitrary non-empty oriented link diagram $\mathscr{D}$

$$(9.2.3)\qquad (1\otimes\Delta)\,\Delta([\mathscr{D}])=\sum_{f\,\in\,\mathrm{Lbl}_3(\mathscr{D})}\Delta(\mathscr{D}, f).$$

Indeed, each 3-labelling $f\in\mathrm{Lbl}_3(\mathscr{D})$ gives rise to two 2-labellings $i=i(f)\in\mathrm{Lbl}(\mathscr{D})$ and $j=j(f)\in\mathrm{Lbl}(\mathscr{D}_{i,2})$. They are defined as follows: the value of i on an edge a of $\mathscr{D}$ equals 1 if $f(a)=1$ and equals 2 if $f(a)\in\{2, 3\}$; the value of j on an edge a of $\mathscr{D}_{i,2}$ is $f(b)-1\in\{1, 2\}$. It is easy to show that

$$\langle\mathscr{D}\,|\,f\rangle=\langle\mathscr{D}\,|\,i\rangle\langle\mathscr{D}_{i,2}\,|\,j\rangle.$$

Moreover, for each $i\in\mathrm{Lbl}(\mathscr{D})$ the formula $f\mapsto j(f)$ establishes a bijective correspondence between the sets $\{f\in\mathrm{Lbl}_3(\mathscr{D})\,|\,i(f)=i\}$ and $\mathrm{Lbl}(\mathscr{D}_{i,2})$. Therefore, the R.H.S. of (9.2.3) is equal to

$$\sum_{i\,\in\,\mathrm{Lbl}(\mathscr{D})}\langle\mathscr{D}\,|\,i\rangle[\mathscr{D}_{i,1}]\otimes\Delta(\mathscr{D}_{i,2})=(1\otimes\Delta)\,\Delta([\mathscr{D}]).$$

A similar argument shows that the R.H.S. of (9.2.3) is equal to $(\Delta\otimes 1)\,\Delta([\mathscr{D}])$. Thus $(1\otimes\Delta)\,\Delta=(\Delta\otimes 1)\,\Delta$.

To prove that Δ is an algebra homomorphism it suffices to show for any two link diagrams $\mathscr{D}$, $\mathscr{D}'$ that $\Delta(\mathscr{D})\,\Delta(\mathscr{D}')=\Delta(\mathscr{D}\mathscr{D}')$ where $\mathscr{D}\mathscr{D}'$ is the link diagram obtained by putting $\mathscr{D}$ over $\mathscr{D}'$. Each pair of labellings $f\in\mathrm{Lbl}(\mathscr{D})$, $f'\in\mathrm{Lbl}(\mathscr{D}')$ gives rise to a labelling $ff'\in\mathrm{Lbl}(\mathscr{D}\mathscr{D}')$: the value of ff' on an edge of $\mathscr{D}\mathscr{D}'$ contained in an edge a of $\mathscr{D}$ (resp. in an edge a' of $\mathscr{D}'$) is equal to $f(a)$ [resp. to $f'(a')$]. It is easy to see that

$$\Delta(\mathscr{D}\mathscr{D}', ff')=\Delta(\mathscr{D}, f)\,\Delta(\mathscr{D}', f).$$

It remains to show that the mapping

$$(9.2.4)\qquad (f, f')\mapsto ff' : \mathrm{Lbl}(\mathscr{D})\times\mathrm{Lbl}(\mathscr{D}')\to\mathrm{Lbl}(\mathscr{D}\mathscr{D}')$$

is bijective. Injectivity of this mapping is obvious, let us prove surjectivity. Let $v_1, \ldots, v_m$ be the self-crossing points of $\mathscr{D}\mathscr{D}'$ in which $\mathscr{D}$ interests $\mathscr{D}'$. Let a_i and b_i be the edges of $\mathscr{D}\mathscr{D}'$ looking into v_i. Let c_i and d_i be the edges of $\mathscr{D}\mathscr{D}'$ looking out of v_i. Assume that a_i, $c_i\subset\mathscr{D}$ and b_i, $d_i\subset\mathscr{D}'$ for all $i=1, \ldots, m$. Then a_i and c_i are contained in a common edge of $\mathscr{D}$ which lies over the edge of $\mathscr{D}'$ containing b_i and d_i. Let $j\in\mathrm{Lbl}(\mathscr{D}\mathscr{D}')$. To prove that j belongs to the image of (9.2.4) it suffices to show that $j(a_i)=j(c_i)$ and $j(b_i)=j(d_i)$ for all i. Consider the punctured diagram $\mathscr{D}\setminus\{v_1, \ldots, v_m\}$. The labelling j of $\mathscr{D}\mathscr{D}'$ induces a labelling of this punctured diagram. Its values on the inputs $c_1, \ldots, c_m$ are equal to $j(c_1), \ldots, j(c_m)$; its values on the outputs $a_1, \ldots, a_m$ are equal to $j(a_1), \ldots, j(a_m)$. Since labellings satisfy the Kirchoff rule in all self-crossing points, we have $j(a_1)+\ldots+j(a_m)=j(c_1)+\ldots+j(c_m)$. By the very definition of labelling $j(a_i)\geqq j(c_i)$ and $j(a_i)-j(c_i)=j(d_i)-j(b_i)$ for all i. Thus $j(a_i)=j(c_i)$ and $j(b_i)=j(d_i)$ for all i.

9.3. *Remarks.* — 1. For each $r\geqq 1$ the homomorphism

$$\Delta^n=(\mathrm{id}_{\mathrm{A}}^{\otimes(n-1)}\otimes\Delta)\circ\ldots\circ(\mathrm{id}_{\mathrm{A}}\otimes\Delta)\circ\Delta:\mathrm{A}\to\mathrm{A}^{\otimes(n+1)}$$

induced by Δ is given by the formulas: $\Delta^n([\Phi])=1$ and for a non-empty oriented link diagram $\mathscr{D}$

$$\Delta^n([\mathscr{D}])=\sum_{f\in\mathrm{Lbl}_{n+1}(\mathscr{D})}\Delta(\mathscr{D}, f).$$

2. The projection $\mathrm{A}\to\mathrm{K}[h, \hbar][\Phi]$ along the classes of all non-empty links is the counit of the bialgebra A. I believe that the $\mathrm{K}[h, \hbar]$-linear additive homomorphism $\mathrm{A}\to\mathrm{A}$ which transforms [L] into $(-1)^{|\mathrm{L}|}[\tilde{\mathrm{L}}]$, where $\tilde{\mathrm{L}}=\{(a, t)\in\mathrm{F}\times[0, 1]\,|\,(a, 1-t)\in\mathrm{L}\}$, is an antipode (a conjugation) of A.

3. It seems that there is no canonical lift of the comultiplication in A to a comultiplication in $\mathscr{A}$. However, if $\partial\mathrm{F}\neq\varnothing$, or F is non-compact, or $\mathrm{F}=\mathrm{S}^1\times\mathrm{S}^1$ then there is a comultiplication in $\mathscr{A}$ depending on the choice of parallelization in F. (All such F are parallelizable.) Namely, for an oriented link diagram $\mathscr{D}$ on F denote by $r(\mathscr{D})$ the total rotation angle of (the tangent vector of) $\mathscr{D}$ with respect to the given parallelization of F. We normalize $r(\mathscr{D})$ so that for a small simple circle $\mathscr{O}$ on F with the conterclockwise orientation we have $r(\mathscr{O})=1$. The writhe of the diagram $\mathscr{D}$, denoted by $w(\mathscr{D})$, is the sum of the signs of the vertices of $\mathscr{D}$. It turns out that there exists a $\mathrm{K}[h, \hbar]$-linear additive homomorphism $\nabla:\mathscr{A}\to\mathscr{A}\otimes_{\mathrm{K}[h, \hbar]}\mathscr{A}$ specified by three properties: $\nabla(1)=1\otimes 1$; $\nabla(xa)=(x\otimes x)\nabla(a)$ for all $a\in\mathscr{A}$; for any oriented link diagram $\mathscr{D}$

$$\nabla([\mathrm{L}(\mathscr{D})])=\sum_{f\in\mathrm{Lbl}(\mathscr{D})}\langle\mathscr{D}\,|\,f\rangle(x^{r_1(f)}[\mathrm{L}(\mathscr{D}_{f, 1})])\otimes(x^{r_2(f)}[\mathrm{L}(\mathscr{D}_{f, 2})])$$

where $r_i(f)=w(\mathscr{D}_{f, i})-w(\mathscr{D})+r(\mathscr{D}_{f, 3-i})$, $i=1, 2$. This mapping ∇ is coassociative and covers $\Delta:\mathrm{A}\to\mathrm{A}^{\otimes 2}$. However ∇ is not an algebra homomorphism. One may show that ∇ is a multiplicative homomorphism with respect to multiplication $*$ in $\mathscr{A}$ defined by the formula $[\mathrm{L}]*[\mathrm{L}']=x^{\mathrm{L}\,.\,\mathrm{L}'}[\mathrm{LL}']$ where $\mathrm{L}\,.\,\mathrm{L}'$ is the homological intersection index of the projections of L, L′ into F. Note that to check the properties of ∇ stated above it is convenient to use the Kauffman's language of regular isotopy types of link diagrams (*see* [7]). In this language one associates with a diagram $\mathscr{D}$ not the class $[\mathrm{L}(\mathscr{D})]\in\mathscr{A}$ but rather $x^{w(\mathscr{D})}[\mathrm{L}(\mathscr{D})]\in\mathscr{A}$. It should be mentioned that contrary to what was said in [20], the image of $(\mathrm{Perm}_{\mathscr{A}}\circ\nabla)-\nabla$ is contained in $\hbar\mathscr{A}\otimes_{\mathrm{K}[h, \hbar]}\mathscr{A}$.

Dualizing the comultiplications Δ, ∇ we obtain pairings which enable us to pair quantum states and produce thus some other quantum states. This procedure is especially useful in the case of ∇ since here one may pair linear functionals on $\mathscr{A}$ with different linear coefficients corresponding to multiplication by x in $\mathcal{A}$. In the case $F=D^2$, $h=\hbar$ the pairing dual to ∇ was introduced by F. Jaeger [6], who used it to derive the V. Jones state models for an infinite sequence of specializations of the Jones-Conway polynomial.

10. Quantization of Z and Z_0

10.1. THEOREM. — *The pair* (*bialgebra* A, *constructed in Sections* 4, 9; *homomorphism* $p: A \to V_h(Z)$, *constructed in Section* 4) *is a reduced coquantization of the co-Poisson bialgebra* $V_h(Z)$. *In other words*, (A, p) *is a quantization of the Lie bialgebra* Z.

Proof. — In view of Theorem 4.2 we have to check only that p is a coalgrebra homomorphism and a coquantization of the co-Poisson structure in $V_h(Z)$. Let $\mathcal{D}$ be an oriented knot diagram on F. For any 2-labelling $f \in \mathrm{Lbl}(\mathcal{D})$ either $\| f \| < | f |$ and $\Delta(\mathcal{D}, f) \in \hbar A^{\otimes 2}$ or $\| f \| = | f | = 0$ which means that f is a constant labelling taking the same value 1 or 2 on all edges. Thus, $\Delta(\mathcal{D}) = [\mathcal{D}] \otimes 1 + 1 \otimes [\mathcal{D}] \bmod \hbar A^{\otimes 2}$. Therefore

$$(p \otimes p)(\Delta(\mathcal{D})) = \Delta_V(p([\mathcal{D}]))$$

where Δ and Δ_V are the comultiplications resp. in A and $V_h(Z)$. Since classes of knots generate A, and p is an algebra homomorphism we see that p is also a coalgebra homomorphism.

To show that p is a co-quantization we will prove that for any $a \in A$

$$\Delta(a) - \mathrm{Perm}_A(\Delta(a)) = \hbar (p \otimes p)^{-1}(\nu(p(a))) \bmod \hbar^2 A^{\otimes 2}$$

where ν is the Lie cobracket in $V_h(Z)$. As above it suffices to consider the case $a = [\mathcal{D}]$ where $\mathcal{D}$ is an oriented knot diagram. To this end we compute $\Delta(\mathcal{D}) \bmod \hbar^2 A^{\otimes 2}$. Associate with each self-crossing point v of $\mathcal{D}$ the loop which starts off in v in the direction of the upper outlooking edge and goes along $\mathcal{D}$ until the first return to v. Assign to all edges of $\mathcal{D}$ lying on this loop the label 1 and assign 2 to all remaining edges of $\mathcal{D}$. Denote this labelling by $\bar{v}$. Clearly, $|\bar{v}| - \|\bar{v}\| = 1 - (-1) = 2$. This property characterizes such labellings. Indeed if $f \in \mathrm{Lbl}(\mathcal{D})$ and $|f| - \|f\| = 2$ then f is not a constant labelling and so $1 \leqq |f| = \|f\| + 2 \leqq 1$. Thus $|f| = 1$ which means that $f = \bar{v}$ for some crossing point v of D. Denote the set of self-crossing points of $\mathcal{D}$ by $\# \mathcal{D}$. Then modulo $\hbar^2 A^{\otimes 2}$

$$\Delta(a) \equiv a \otimes 1 + 1 \otimes a + \hbar \sum_{v \in \# \mathcal{D}} \varepsilon_v [\mathcal{D}_{\bar{v}, 1}] \otimes [\mathcal{D}_{\bar{v}, 2}],$$

where $\varepsilon_v = \pm 1$ is the sign of the crossing of $\mathcal{D}$ in v. Therefore, modulo $\hbar^2 A^{\otimes 2}$

$$(10.1.1) \qquad \Delta(a) - \mathrm{Perm}_A(\Delta(a)) \equiv \hbar \sum_{v \in \# \mathcal{D}} \varepsilon_v ([\mathcal{D}_{\bar{v}, 1}] \otimes [\mathcal{D}_{\bar{v}, 2}] - [\mathcal{D}_{\bar{v}, 2}] \otimes [\mathcal{D}_{\bar{v}, 1}]).$$

Note that if a knot diagram $\mathscr{D}'$ is homotopic to the trivial knot diagram $\mathcal{O}$ then the relation (3.5.1) implies that $[\mathscr{D}']-[\mathcal{O}]\in\hbar A$ and since $\hbar[\mathcal{O}]=0$ in A we have $\hbar[\mathscr{D}']\in\hbar^2 A$. Therefore we may ignore those terms in the R.H.S. of (10.1.1) which involve contractible knot diagrams. Thus the R.H.S. is mapped by $p\otimes p$ into $\nu(p(a))$. This finishes the proof.

10.2. Bialgebra A_0 and homomorphism $p_\hbar: A_0 \to V_h(Z_0)$. Denote by $A_0 = A_0(F)$ the quotient of the algebra $A = A(F)$ by the ideal generated by the class δ of the trivial knot. (Note that δ lies in the centre of A.) The comultiplication Δ in A introduced in Section 9 maps δ into $\delta\otimes 1 + 1\otimes\delta$. Therefore Δ induces a comultiplication in A_0 which makes A_0 a bialgebra over $K[h, \hbar]$.

The bialbegra homomorphism $p: A \to V_h(Z)$ maps δ into $\{1\}\in Z$ and therefore induces a bialgebra homomorphism $A_0 \to V_h(Z_0) = V_h(Z)/\{1\} V_h(Z)$. Denote the latter homomorphism by $p_\hbar$. Theorem 10.1 implies that $(A_0, p_\hbar)$ is a reduced coquantization of the co-Poisson bialgebra $V_h(Z_0)$. In other words, $(A_0, p_\hbar)$ is a quantization of the Lie bialgebra Z_0.

10.3. *Remarks*. — 1. The algebra A has a non-trivial $\hbar$-torsion: $\hbar\delta A = 0$ whereas $\delta A \neq 0$. This is an important drawback of the construction, as it is clear from the discussion in Section 7.4. On the other hand it seems reasonable to conjecture that A_0 is free as the $K[h, \hbar]$-module. When F is the 2-disc, or the 2-sphere, or the annulus this conjecture follows from the results of Section 5. For related conjectures see [11].

2. The quantization $(A_0, p_\hbar)$ of the Lie bialgebra Z_0 induces a quantization of the Lie bialgebra Z as follows.

Provide the polynomial ring $K[y]$ with a comultiplication by the formula $y \mapsto y\otimes 1 + 1\otimes y$. It is easy to show that the bialgebra $K[y]\otimes A_0$ and its homomorphism into $V_h(Z)$ sending $y^n a$, with $n \geqq 0$, $a\in A_0$, into $\{1\}^n p_\hbar(a)$ make a quantization of Z. Clearly if A_0 is freee as the $K[h, \hbar]$-module then $K[y]\otimes A_0$ is also free as the $K[h, \hbar]$-module.

CHAPTER III.

POISSON BIALGEBRA $\varepsilon_\hbar(Z_0)$ AND ITS QUANTIZATION

11. Spiral Lie bialgebras and associated Poisson-Lie groups

11.1. Spiral Lie coalgebras and bialgebras. — Let $\mathfrak{g}$ be a Lie coalgebra over K with the Lie bracket $\nu: \mathfrak{g} \to \mathfrak{g}\otimes\mathfrak{g}$. Put $\nu^1 = \nu$ and put for $n \geqq 2$

$$\nu^n = (\mathrm{id}_{\mathfrak{g}}^{\otimes(n-2)}\otimes\nu)\circ\ldots\circ(\mathrm{id}_{\mathfrak{g}}\otimes\nu)\circ\nu : \mathfrak{g} \to \mathfrak{g}^{\otimes(n+1)}.$$

We call $\mathfrak{g}$ spiral if $\mathfrak{g}$ is free as the K-module and the filtration $\mathrm{Ker}\,\nu^1 \subset \mathrm{Ker}\,\nu^2 \subset \ldots$ exhausts $\mathfrak{g}$, *i.e.* $\mathfrak{g} = \bigcup_{n\geqq 1} \mathrm{Ker}\,\nu^n$. For instance, if the underlying K-module of $\mathfrak{g}$ is free of finite rank then spiralness of $\mathfrak{g}$ is equivalent to the nilpotency of the dual Lie algebra $\mathfrak{g}^*$.

The dual Lie algebra $\mathfrak{g}^*$ of a spiral Lie coalgebra $\mathfrak{g}$ has the following completeness property. Let $\mathfrak{g}^*=\mathfrak{g}^1\supset\mathfrak{g}^2\supset\ldots$ be the lower central series of $\mathfrak{g}^*$; here $\mathfrak{g}^{n+1}=[\mathfrak{g}^*, \mathfrak{g}^n]$ for $n\geqq 1$. Let $z_1, z_2, \ldots\in\mathfrak{g}^*$ be a sequence such that for any $n\geqq 1$ all terms of the sequence starting from a certain place belong to $\mathfrak{g}^n$. Clearly, if $a\in\operatorname{Ker}\nu^n$, then $\mathfrak{g}^{n+1}(a)=0$. Since $\mathfrak{g}=\bigcup_n\operatorname{Ker}\nu^n$ the formula $a\mapsto z_1(a)+z_2(a)+\ldots$ determines a linear homomorphism $\mathfrak{g}\to K$, *i.e.* an element of $\mathfrak{g}^*$ which is the (infinite) sum $z_1+z_2+\ldots$ (A similar argument shows that $\bigcap_n\mathfrak{g}^n=0$.)

A Lie bialgebra is called spiral if it is spiral as the Lie coalgebra. For spiral Lie bialgebras we shall develop in Section 11.5 a construction dual to the construction of $V_h(\mathfrak{g})$.

11.2. Exponentiating spiral Lie bialgebras. Bialgebra $\varepsilon(\mathfrak{g})$. — Let $\mathfrak{g}$ be a spiral Lie coalgebra over K. In view of the results of Section 11.1 for any elements x, y of the dual Lie algebra $\mathfrak{g}^*$ we may take the infinite sum

$$\mu(x, y)=x+y+\frac{1}{2}[x, y]+\frac{1}{12}([x, [x, y]]+[y, [y, x]])+\ldots$$

where the R.H.S. is the classical Campbell-Hausdorff series for $\log(e^x e^y)$ (*see* [15]). As usual, the mapping $(x, y)\mapsto\mu(x, y):\mathfrak{g}^*\times\mathfrak{g}^*\to\mathfrak{g}^*$ is a group multiplication in $\mathfrak{g}^*$. Here $x^{-1}=-x$ and 0 is the group unit. The resulting group is denoted by $\operatorname{Exp}\mathfrak{g}^*$.

The group multiplication μ induces in the symmetric algebra $S=S(\mathfrak{g})$ a bialgebra structure as follows. Since $\mathfrak{g}$ is free as the K-module the natural imbedding $\mathfrak{g}\to(\mathfrak{g}^*)^*$ extends to an imbedding of S into the algebra of K-valued functions on $\mathfrak{g}^*$. Identify S with the corresponding algebra of functions. Similarly identify $S\otimes S$ with the corresponding algebra of K-valued functions on $\mathfrak{g}^*\times\mathfrak{g}^*$. It is easy to show that for any $a\in S$ we have $a\circ\mu\in S\otimes S$. (Indeed, it suffices to consider the case $a\in\mathfrak{g}$; in this case out claim follows directly from the duality between the Lie bracket in $\mathfrak{g}^*$ and the Lie cobracket $\nu:\mathfrak{g}\to\mathfrak{g}\otimes\mathfrak{g}$. For example, if $a\in\operatorname{Ker}\nu^3$ and if $\nu^2(a)=\sum_i\alpha_i\otimes\beta_i\otimes\gamma_i\in\mathfrak{g}^{\otimes 3}$ then

$$a\circ\mu=a\otimes 1+1\otimes a+\frac{\nu(a)}{2}+\frac{1}{12}\Big(\sum_i(\alpha_i\,\beta_i\otimes\gamma_i+\gamma_i\otimes\alpha_i\,\beta_i)\Big).$$

The algebra $S(\mathfrak{g})$ equipped with the comultiplication $a\mapsto a\circ\mu$ is clearly a bialgebra over K. It will be denoted by $\varepsilon(\mathfrak{g})$. Heuristically, $\varepsilon(\mathfrak{g})$ is the bialgebra dual to the universal enveloping bialgebra $U(\mathfrak{g}^*)$.

11.3. Theorem. — *Let* $\mathfrak{g}$ *be a spiral Lie bialgebra over* K *with the Lie bracket* $\langle\ ,\ \rangle$. *There exists a unique Lie bracket* $\{\ ,\ \}$ *in the bialgebra* $\varepsilon=\varepsilon(\mathfrak{g})$ *such that* (1) *the pair* $(\varepsilon, \{\ ,\ \})$ *is a Poisson bialgebra and* (2) *for any* $a, b\in\mathfrak{g}\subset\varepsilon$

(11.3.1) $$\{a, b\}\equiv\langle a, b\rangle \bmod \bigoplus_{n\geqq 2} S^n(\mathfrak{g}).$$

Theorem 11.3 should be considered from the viewpoint of the theory of Poisson-Lie groups (*see* [1], [2], [14]). Though we will not appeal to this point of view in the paper, I present it here in a concise form. A Poisson-Lie group is a finite dimensional real Lie group G provided with a Poisson bracket in $C^\infty(G)$ compatible with the group multiplication $G \times G \to G$ which means that the mapping $C^\infty(G) \to C^\infty(G \times G)$ induced by the group multiplication is a Lie algebra homomorphism. The notion of Poisson-Lie group was introduced by Drinfeld as a global version of the notion of Lie bialgebra. He proved that the category of connected simply-connected real Poisson-Lie groups is equivalent to the category of finite dimensional Lie bialgebras over $\mathbb{R}$. This equivalence associates with a Poisson-Lie group G its Lie algebra $\mathfrak{p}$ equipped with a Lie cobracket dual to the Lie bracket [,] in $\mathfrak{p}^*$ such that for any $a, b \in C^\infty(G)$

$$d\{a, b\} = [da, db] \tag{11.3.2}$$

where d is the differential in the unit of G and $\{\ ,\ \}$ is the Poisson bracket on G.

Theorem 11.3 establishes a similar correspondence for duals of spiral Lie bialgebras (not necessarily finite dimensional). Here Exp $\mathfrak{g}^*$ plays the role of the simply-connected Lie group associated with the Lie algebra $\mathfrak{g}^*$. The algebra $S(\mathfrak{g})$ plays the role of the algebra of smooth functions on the Lie group. Condition (1) of Theorem 11.3 means that the Poisson bracket $\{\ ,\ \}$ is compatible with the group multiplication in Exp $\mathfrak{g}^*$. Condition (2) is a version of (11.3.2). Of course, if $\mathfrak{g}$ is a finite-dimensional spiral Lie bialgebra over $\mathbb{R}$ then the bracket $\{\ ,\ \}$ in $\varepsilon(\mathfrak{g})$ is the restriction of the Poisson bracket in C^∞(Exp $\mathfrak{g}^*$) given by the Drinfeld theory.

We will need the following slight generalization of Theorem 11.3. Let Q be a commutative associative K-algebra with unit and with a preferred element $\hbar$. Let $\mathfrak{a}$ be a Lie bialgebra over Q with the Lie cobracket ν and Lie bracket $\langle\ ,\ \rangle$. Let $\mathfrak{a}_\hbar$ be the same module $\mathfrak{a}$ provided with the same Lie bracket $\langle\ ,\ \rangle$ and the Lie cobracket $\hbar\nu$. The group Exp $\mathfrak{a}_\hbar^*$ coincides as a set with Exp $\mathfrak{a}^*$ but has a different multiplication

$$xy = x + y + \frac{\hbar}{2}[x, y] + \frac{\hbar^2}{12}([x, [x, y]] + [y, [y, x]]) + \dots \tag{11.3.3}$$

where [,] is the Lie bracket in $\mathfrak{a}$ dual to ν and $x, y \in \mathfrak{a}^*$. The bialgebra $\varepsilon(\mathfrak{a}_\hbar)$ coincides as an algebra with $\varepsilon(\mathfrak{a})$ [and with $S(\mathfrak{a})$] but has a different comultiplication $\nabla_\hbar$. Note that for each $a \in \mathfrak{a}$ we have

$$\nabla_\hbar(a) = a \otimes 1 + 1 \otimes a + (\hbar/2)\,\nu(a) \bmod \bigoplus_{\substack{i, j \geqq 1\\ i+j \geqq 3}} \hbar^2 (S^i(\mathfrak{a}) \otimes S^j(\mathfrak{a})). \tag{11.3.4}$$

11.4. THEOREM. — *If $\mathfrak{a}$ is spiral Lie bialgebra over Q then there exists a unique Lie bracket $\{\ ,\ \}$ in $\varepsilon = \varepsilon(\mathfrak{a}_\hbar)$ such that* (1) *the pair* $(\varepsilon, \{\ ,\ \})$ *is a Poisson bialgebra and* (2) *for any $a, b \in \mathfrak{a} \subset \varepsilon$*

$$\{a, b\} \equiv \langle a, b\rangle \bmod \hbar \bigoplus_{n \geqq 2} S^n(\mathfrak{a}). \tag{11.4.1}$$

One may directly construct the bracket $\{\ ,\ \}$ in $\varepsilon(\mathfrak{a}_\hbar)$ from the bracket $\{\ ,\ \}$ in $\varepsilon(\mathfrak{a})$ given by Theorem 11.3; however we will not pursue this line here (*cf.* [20]).

Theorem 11.4 implies Theorem 11.3: one should take $Q=K$, $\mathfrak{a}=\mathfrak{g}$ and $\hbar=1$. The existence part of Theorem 11.4 wille be proven in the next Section 12; uniqueness is easier and will be proven in Section 11.6.

11.5. POISSON BIALGEBRA $\varepsilon_\hbar(\mathfrak{g})$. — Let $\mathfrak{g}$ be a spiral Lie bialgebra over K. Consider the Lie bialgebra $\mathfrak{a}=K[\hbar]\otimes\mathfrak{g}$ over $K[\hbar]$. Clearly $\mathfrak{a}$ is spiral. Denote by $\varepsilon_\hbar(\mathfrak{g})$ the Poisson bialgebra given by Theorem 11.4. Note that as the algebra $\varepsilon_\hbar(\mathfrak{g})$ is just $K[\hbar]\otimes S(\mathfrak{g})$.

The augmentation $\mathrm{aug}: K[\hbar]\to K$ induces an algebra homomorphism $\varepsilon_\hbar(\mathfrak{g})\to S(\mathfrak{g})$ linear over aug. Denote this algebra homomorphism by e. Formulas (11.3.4), (11.4.1) imply that e is a Poisson bialgebra homomorphism, where the Poisson bracket in $S(\mathfrak{g})$ is induced by the Lie bracket in $\mathfrak{g}$. Moreover, the formula (11.3.4) implies that $(\varepsilon_\hbar(\mathfrak{g}), e)$ is a coquantization of the co-Poisson bracket in $S(\mathfrak{g})$ induced by the Lie cobracket in $\mathfrak{g}$. Clearly $\operatorname{Ker} e=\hbar\,\varepsilon_\hbar(\mathfrak{g})$, so the coquantization is reduced.

Note that the Poisson bialgebra $\varepsilon_\hbar(\mathfrak{g})/(\hbar-1)\,\varepsilon_\hbar(\mathfrak{g})$ coincides with the Poisson bialgebra $\varepsilon(\mathfrak{g})$ described in Theorem 11.3.

11.6. *Proof of Theorem* 11.4: *uniqueness of the bracket.* — Assume that the symmetric algebra $S=\bigoplus_{n\geq 0} S^n(\mathfrak{a})$ is provided with a comultiplication Δ and with two Lie brackets $\{\ ,\ \}_1$, $\{\ ,\ \}_2$ so that the following conditions hold: (i) the triple S, Δ, $\{\ ,\ \}_i$ is a Poisson bialgebra for $i=1, 2$; (ii) for each $a\in\mathfrak{a}$

$$\Delta(a)=a\otimes 1+1\otimes a \bmod \bigoplus_{j,\,k\geq 1}(S^j\otimes S^k) \tag{11.6.1}$$

where

$$S^j=S^j(\mathfrak{a})\qquad\text{for}\quad j\geq 0;$$

(iii) for each $a, b\in\mathfrak{a}$

$$\{a, b\}_1-\{a, b\}_2\in\bigotimes_{j\geq 2} S^j$$

We shall show that $\{\ ,\ \}_1=\{\ ,\ \}_2$. This will imply the uniqueness part of Theorem 11.4.

For $n\geq 1$ put $I^n=\bigoplus_{j\geq n} S^j$. For $a, b\in S$ put $d(a, b)=\{a, b\}_1-\{a, b\}_2$. We shall prove inductively the following claim: $(*)_n$. For any $a, b\in\mathfrak{a}$ we have $d(a, b)\in I^n$.

Claim $(*)_2$ follows from our assumptions. Assume that Claim $(*)_{n-1}$ holds true and check Claim $(*)_n$. Let $a, b\in\mathfrak{a}$. We have

$$\Delta(a)=a\otimes 1+1\otimes a+\sum_q a_q\otimes a^q;$$

$$\Delta(b)=b\otimes 1+1\otimes b+\sum_r b_r\otimes b^r$$

where $\{ a_q, a^q, b_r, b^r \}_{q,r}$ is a finite collection of certain elements of $\mathfrak{a}$. Then

$$(11.6.2)\quad \Delta(d(a, b)) = \{ \Delta(a), \Delta(b) \}_1 - \{ \Delta(a), \Delta(b) \}_2$$
$$= d(a, b) \otimes 1 + 1 \otimes d(a, b) + \sum_q (d(a_q, b) \otimes a^q + a_q \otimes d(a^q, b))$$
$$+ \sum_r (d(a, b_r) \otimes b^r + b_r \otimes d(a, b^r))$$
$$+ \sum_{q,r} (d(a_q, b_r) \otimes a^q b^r + a_q b_r \otimes d(a^q, b^r)).$$

Claim $(*)_{n-1}$ implies that

$$(11.6.3)\qquad \Delta(d(a, b)) = d(a, b) \otimes 1 + 1 \otimes d(a, b) \bmod \bigoplus_{j+k \geqq n} (S^j \otimes S^k).$$

On the other hand, since $d(a, b) \in I^{n-1}$ we may compute $\Delta(d(a, b)) \bmod \bigoplus_{j+k \geqq n} (S^j \otimes S^k)$ knowing only the $(n-1)$-th homogeneous part of $d(a, b)$ and using (11.6.1). This homogeneous part is a degree $n-1$ polynomial over a set of free generators of the module $\mathfrak{a}$. Applying to these generators the formula (11.6.1) we easily see that (11.6.3) may hold only if this polynomial is equal to zero. Thus $d(a, b) \in I^n$, which proves Claim $(*)_n$. This implies that $\{\ ,\ \}_1 = \{\ ,\ \}_2$.

11.7. *Remarks.* — 1. For a non-spiral Lie bialgebra $\mathfrak{g}$ the group $\operatorname{Exp} \mathfrak{g}^*$ generally speaking does not exist. However, one may define an analogue of the bialgebra $\varepsilon(\mathfrak{g})$ if one passes to the category of topological bialgebras and considers instead of $S(\mathfrak{g})$ its completion $\times_{n \geqq 0} S^n(\mathfrak{g})$. Theorems 11.3, 11.4 and their proof may be easily generalized to the case of arbitrary Lie bialgebras.

2. I briefly describe an approach to the construction of the bracket $\{\ ,\ \}$ in Theorem 11.3 distinct from the one used in Section 12 and closer to the original approach of Drinfeld [1] to globalization of finite dimensional Lie bialgebras. Let $\mathfrak{g}$ be a spiral Lie bialgebra over K with the Lie cobracket ν. Note that the adjoint action of the Lie algebra $\mathfrak{g}^*$ induces a left $\mathfrak{g}^*$-module structure in $(\mathfrak{g} \otimes \mathfrak{g})^*$. The Lie bracket in $\mathfrak{g}$ dualizes to a 1-cocycle, say, $\psi : \mathfrak{g}^* \to (\mathfrak{g} \otimes \mathfrak{g})^*$. The formula

$$\Psi(x) = \frac{e^x - 1}{x} \psi(x) \qquad \left(\text{where } \frac{e^x - 1}{x} = \sum_{j \geqq 0} \frac{x^j}{(j+1)!} \right)$$

defines a mapping $\Psi : \mathfrak{g}^* \to (\mathfrak{g} \otimes \mathfrak{g})^*$ which is a 1-cocycle of the group $\operatorname{Exp} \mathfrak{g}^*$, corresponding to ψ. (Note that $\operatorname{Exp} \mathfrak{g}^*$ and $\mathfrak{g}^*$ coincide as sets.) In particular, $\Psi(xy) = \Psi(x) + x \Psi(y)$ for all $x, y \in \mathfrak{g}^*$. For $x \in \mathfrak{g}^*$ denote by R_x the right multiplication $y \mapsto yx : \mathfrak{g}^* \to \mathfrak{g}^*$. Then for any $a, b \in \mathfrak{g}$ the value of $\{ a, b \} \in S(\mathfrak{g})$ on any $x \in \mathfrak{g}^*$ is given by the formula

$$(11.7.3)\qquad \{ a, b \}(x) = \Psi(x)(d_1(a \circ R_x) \otimes d_1(b \circ R_x)),$$

where d_1 is the formal differential in $0 \in \mathfrak{g}^*$ so that $d_1(a \circ R_x)$, $d_1(b \circ R_x) \in \mathfrak{g}$. The R.H.S. of (11.7.3) may be computed as follows. Let $\sum_{i \geq 0} r_i (\operatorname{ad} y)^i (x)$ be the sum of those terms of the Campbell-Hausdorff series which include x in degree 1. In particular $r_0 = 1$, $r_1 = -1/2$, $r_2 = 1/12$. Then the R.H.S. of (11.7.3) equals

$$\psi(x)\left(\sum_{i, j \geq 0} \frac{(d_1(a \circ R_x) \circ (\operatorname{ad} x)^i) \otimes (d_1(b \circ R_x) \circ (\operatorname{ad} x)^j)}{i!\, j!\, (i+j+1)!}\right)$$

$$= \psi(x)\left(\sum_{i, j, k, l \geq 0} r_k r_l \frac{(a \circ (\operatorname{ad} x)^{i+k}) \otimes (b \circ (\operatorname{ad} x)^{j+l})}{i!\, j!\, (i+j+1)!}\right)$$

These calculations lead to an explicit though rather complicated formula for $\{a, b\}$. For each $m, n \geq 0$ present $\nu^m(a)$ and $\nu^n(b)$ as finite sums

$$\sum_s a_s^1 \otimes \ldots \otimes a_s^{m+1} \quad \text{and} \quad \sum_t b_t^1 \otimes \ldots \otimes b_t^{n+1}$$

where $a_s^\alpha, b_t^\beta \in \mathfrak{g}$. (Here $\nu^0 = \mathrm{id}_{\mathfrak{g}}$.) Put

$$u_{m, n}(a, b) = \sum_{s, t} a_s^1 a_s^2 \ldots a_s^m b_t^1 b_t^2 \ldots b_t^n [a_s^{m+1}, b_t^{n+1}] \in S(\mathfrak{g})$$

where [,] is the Lie bracket in $\mathfrak{g}$. (In particular, $u_{0,0}(a, b) = [a, b]$.) Then

$$\{a, b\} = \sum_{m, n \geq 0} \left(\sum_{i=0}^{m} \sum_{j=0}^{n} \frac{r_{m-i} r_{n-j}}{i!\, j!\, (i+j+1)!}\right) u_{m, n}(a, b).$$

Here are the terms of this expansion with $m, n \leq 2$:

(11.7.4) $$[a, b] + (1/12)\, u_{1,1}(a, b) + (1/720)\, u_{2,2}(a, b)$$

If $\nu^3(a) = \nu^3(b) = 0$ then $\{a, b\}$ equals (11.7.4).

12. Poisson bialgebras $\mathcal{T}$ and E. Proof of Theorem 11.4

In this section the symbol Q denotes a commutative associative K-algebra with unit and with a preferred element $\hbar$.

12.1. Poisson bialgebra $\mathcal{T}$. – Let $\mathfrak{a}$ be a module over Q. Set $\mathcal{T} = \bigoplus_{m \geq 0} \mathfrak{a}^{\otimes m}$. The aim of this subsection is to provide $\mathcal{T}$ with a canonical bialgebra structure. This structure is dual to the bialgebra structure in the tensor algebra $T(\mathfrak{a}^*)$ described in Sections 1.3, 7.5. If $\mathfrak{a}$ is a Lie algebra then there is a natural Lie bracket in $\mathcal{T}$ making $\mathcal{T}$ a Poisson bialgebra. This bracket is dual to the Lie cobracket in $T(\mathfrak{a}^*)$ constructed in Lemma 7.5.

Provide $\mathscr{T}$ with a multiplication as follows. The product of $a_1 \otimes \ldots \otimes a_m \in \mathfrak{a}^{\otimes m}$ and $b_1 \otimes \ldots \otimes b_n \in \mathfrak{a}^{\otimes n}$ is defined to be $\sum_\sigma c_1^\sigma \otimes c_2^\sigma \otimes \ldots \otimes c_{m+n}^\sigma$ where σ runs over permutations of the set $\{1, 2, \ldots, m+n\}$ such that

$$\sigma(1) < \sigma(2) < \ldots < \sigma(m); \sigma(m+1) < \sigma(m+2) < \ldots < \sigma(m+n),$$

and where $c_i = a_{\sigma^{-1}(i)}$ if $\sigma^{-1}(i) < m$, otherwise $c_i = b_{\sigma^{-1}(i)-m}$. (The sequence $c_1^\sigma, \ldots, c_{m+n}^\sigma$ is called a mixture of $a_1, \ldots, a_m$ and $b_1, \ldots, b_n$.) Denote this multiplication by $\circ$. Clearly, $(\mathscr{T}, \circ)$ is a commutative algebra. Note that the rule

$$a_1 \otimes \ldots \otimes a_m \mapsto (m!)^{-1} \prod_{i=1}^{m} a_i : \mathfrak{a}^{\otimes m} \to S^m(\mathfrak{a})$$

defines an algebra homomorphism $\mathscr{T} \to S(\mathfrak{a})$. It is denoted by symm.

Provide $\mathscr{T}$ with a comultiplication $\operatorname{diag}: \mathscr{T} \to \mathscr{T} \otimes_{\mathbf{Q}} \mathscr{T}$ by the formula

$$\operatorname{diag}(a_1 \otimes \ldots \otimes a_m) = \sum_{i=0}^{m} (a_1 \otimes \ldots \otimes a_i) \otimes (a_{i+1} \otimes \ldots \otimes a_m).$$

Here it is understood that for $i=0$, $a_1 \otimes \ldots \otimes a_i = 1$, and, similarly for $i=m$, $a_{i+1} \otimes \ldots \otimes a_m = 1$. It is straightforward to verify that $\mathscr{T}$ is a bialgebra.

We need the following notation. For a sequence $a_1, \ldots, a_m \in \mathfrak{a}$ and for integers i, j with $1 \leqq i \leqq j \leqq m$ put $a_{i,j} = a_i \otimes a_{i+1} \otimes \ldots \otimes a_j$. In particular, $a_{i,i} = a_i$.

Assume now that $\mathfrak{a}$ is a Lie algebra. We provide $\mathscr{T}$ with a bracket $\{\ ,\ \}: \mathscr{T} \otimes_{\mathbf{Q}} \mathscr{T} \to \mathscr{T}$ by the formula

$$(12.1.1) \quad \{a_1 \otimes \ldots \otimes a_m, b_1 \otimes \ldots \otimes b_n\} = \sum_{\substack{1 \leqq i \leqq m \\ 1 \leqq j \leqq n}} (a_{1,i-1} \circ b_{1,j-1}) \otimes [a_i, b_j] \otimes (a_{i+1,m} \circ b_{j+1,n})$$

where $[\ ,\ ]$ is the Lie bracket in $\mathfrak{a}$. In particular, $\{\mathfrak{a}^{\otimes 0}, \mathscr{T}\} = \{\mathbf{Q}, \mathscr{T}\} = 0$ and for $a, b \in \mathfrak{a} \subset \mathscr{T}$ we have $\{a, b\} = [a, b]$.

12.2. Lemma. — *$\mathscr{T}$ with the braket $\{\ ,\ \}$ is a Lie algebra. The bialgebra $\mathscr{T}$ with this bracket is a Poisson bialgebra.*

Proof. – The skew-commutativity of $\{\ ,\ \}$ follows from the skew-commutativity of $[\ ,\ ]$ and commutativity of $\circ$. Let us check the Jacobi identity

$$(12.2.1) \qquad [[a, b], c] + [[b, c], a] + [[c, a], b] = 0$$

for $a = a_1 \otimes \ldots \otimes a_m$, $b = b_1 \otimes \ldots \otimes b_n$, $c = c_1 \otimes \ldots \otimes c_q$. Note that

$$(12.2.2) \qquad [[a, b], c] = \Sigma_0 + \sum(a; b, c) + \sum(a; c, b) - \sum(b; a, c) - \sum(b; c, a)$$

where

$$\Sigma_0=\sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{k=1}^{q}(a_{1,i-1}\circ b_{1,j-1}\circ c_{1,k-1})\otimes[[a_i, b_j], c_k]\otimes(a_{i+1,m}\circ b_{j+1,n}\circ c_{k+1,q}),$$

$$\sum(a; b, c)=\sum_{\substack{1\leqq i<i'\leqq m\\ 1\leqq j<j'\leqq n\\ 1\leqq k<k'\leqq q}}(a_{1,i-1}\circ b_{1,j-1}\circ c_{1,k})$$

$$\otimes[a_i, b_j]\otimes(a_{i+1,i'-1}\circ b_{j+1,j'}\circ c_{k+1,k'-1})$$

$$\otimes[a_{i'}, c_{k'}]\otimes(a_{i'+1,m}\circ b_{j'+1,n}\circ c_{k'+1,q}).$$

The formula (12.2.2) implies (12.2.1). The identities (0.3.2), (7.2.1) follow directly from definitions.

12.3. Submodule E of $\mathscr{F}$. — Let $\mathfrak{a}$ be a Lie bialgebra over Q with the Lie cobracket $\nu:\mathfrak{a}\rightarrow\mathfrak{a}\otimes_{\mathrm{Q}}\mathfrak{a}$. For integers $n>i\geqq 1$ denote by σ_i^n the automorphism of $\mathfrak{a}^{\otimes n}$ permuting the i-th and the $(i+1)$-th tensor multiples so that

$$\sigma_i^n(a_1\otimes\ldots\otimes a_n)=a_{1,i-1}\otimes a_{i+1}\otimes a_i\otimes a_{i+2,n}.$$

Denote by ν_i the homomorphism $\mathfrak{a}^{\otimes(n-1)}\rightarrow\mathfrak{a}^{\otimes n}$ defined by the formula

$$\nu_i(a_1\otimes\ldots\otimes a_{n-1})=a_{1,i-1}\otimes\nu(a_i)\otimes a_{i+1,n-1}.$$

For an element a of $\mathscr{T}$ we shall denote by $^n a$ its n-th homogeneous part so that $a={}^0a+{}^1a+\ldots$ and $^n a\in\mathfrak{a}^{\otimes n}$ for $n\geqq 0$. Put

$$\mathrm{E}=\{a\in\mathscr{T}\mid {}^n a-\sigma_i^n({}^n a)=\hbar\,\nu_i({}^{n-1}a)\text{ for all }n>i\geqq 1\}.$$

Clearly E is a submodule of $\mathscr{T}$ and $\mathrm{Q}=\mathfrak{a}^{\otimes 0}\subset\mathrm{E}$.

12.4. Lemma. — *Let* $\mathfrak{a}$ *be a spiral Lie bialgebra over* Q. *Then* $\mathrm{E}=\mathrm{E}(\mathfrak{a})$ *is a Poisson subbialgebra of* $\mathscr{T}$. *The homomorphism* $\mathrm{symm}|_{\mathrm{E}}:\mathrm{E}\rightarrow\mathrm{S}(\mathfrak{a})$ *is an algebra isomorphism which transforms the comultiplication in* E *in the comultiplication* $\nabla_\hbar$ *in* $\mathrm{S}(\mathfrak{a})$ (see Section 11). *If* $a\in\mathfrak{a}$ *then* $(\mathrm{symm}|_{\mathrm{E}})^{-1}(a)=a+\hbar a'$ *with* $a'\in\bigoplus_{n\geqq 2}\mathfrak{a}^{\otimes n}$.

Lemma 12.4 shows that $\mathrm{symm}|_{\mathrm{E}}$ is a bialgebra isomorphism $\mathrm{E}\rightarrow\varepsilon_{\hbar}(\mathfrak{a})$. Lemma 12.4 implies Theorem 11.4. Indeed, the reduction of the Lie bracket in $\mathscr{T}$ to E may be transferred to $\varepsilon_\hbar(\mathfrak{a})$ by means of symm. This makes $\varepsilon_{\hbar}(\mathfrak{a})$ a Poisson bialgebra. The condition (2) of Theorem 11.4 follows from the last claim of Lemma 12.4.

Lemma 12.4 will be proven in Section 12.17 using preliminary constructions and results of Sections 12.5-12.16. In these sections $\mathfrak{a}$ denotes a spiral Lie bialgebra over Q with the Lie cobracket ν. Symbols $\mathscr{T}$ and E denote the objects associated with $\mathfrak{a}$ as above. The symbol $\otimes$ denotes tensor product over Q. We denote by S the symmetric algebra $\mathrm{S}(\mathfrak{a})$ and by ∇ the comultiplication $\nabla_\hbar$ in S constructed in Section 11.3. Denote by T the tensor algebra $\mathrm{T}(\mathfrak{a}^*)$ over Q (*cf.* Section 1.3).

For any Q-module W and any $a \in W$, $x \in W^* = \mathrm{Hom}_Q(W, Q)$ we put $\langle a | x \rangle = x(a) \in Q$. We have the pairing $a, x \mapsto \langle a, x \rangle : \mathscr{T} \times T \to Q$ such that if $a_1, \ldots, a_m \in \mathfrak{a}$, $x_1, \ldots, x_n \in \mathfrak{a}^*$ then the expression $\langle a_1 \otimes \ldots \otimes a_m | x_1 \otimes \ldots \otimes x_n \rangle$ is equal to 0 if $m \neq n$ and to $\prod_{i=1}^{m} \langle a_i | x_i \rangle$ if $m = n$.

12.5. Algebras $S^\vee$ and $(S \otimes S)^\vee$. – Put $S^n = S^n(\mathfrak{a})$ so that $S = S^0 \oplus S^1 \oplus \ldots$ Put $S^\vee = \bigoplus_{n \geqq 0} (S^n)^*$. The Q-module $S^\vee$ is the submodule of S^* consisting of homomorphisms $S \to Q$ which are non-zero only on a finite number of S^n.

The comultiplication $\nabla = \nabla_\hbar$ in S satisfies the following condition: for each $n \geqq 0$

$$\nabla(S^n) \subset \bigoplus_{i+j \geqq n} (S^i \otimes S^j). \tag{12.5.1}$$

Indeed, for $n=1$ this is equivalent to the absence of free term in the Campbell-Hausdorf series; the case $n>1$ follows by multiplicativity of ∇. The inclusion (12.5.1) implies that for any $x, y \in S^\vee$ the homomorphism $(x \otimes y)\nabla : S \to Q$ also belongs to $S^\vee$. Therefore, the formula $(x, y) \mapsto (x \otimes y)\nabla$ provides $S^\vee$ with a structure of associative algebra. The projection $S \to S^0 = Q$ is the unit of this algebra.

The comultiplication ∇ in S induces a comultiplication $\nabla \otimes \nabla$ in $S \otimes S$ by the usual formula $(\nabla \otimes \nabla)(a \otimes b) = P_{2,3}(\nabla(a) \otimes \nabla(b))$ where $a, b \in S$ and $P_{2,3}$ is the permutation homomorphism $a_1 \otimes a_2 \otimes a_3 \otimes a_4 \mapsto a_1 \otimes a_3 \otimes a_2 \otimes a_4$. As usual $\nabla \otimes \nabla$ makes the algebra $S \otimes S$ a bialgebra. The inclusion (12.5.1) implies that for any m, n

$$(\nabla \otimes \nabla)(S^m \otimes S^n) \subset \bigotimes_{\substack{i, j, k, l \\ i+j+k+l \geqq m+n}} (S^i \otimes S^j \otimes S^k \otimes S^l). \tag{12.5.2}$$

Set $(S \otimes S)^\vee = \bigoplus_{m, n \geqq 0} (S^m \otimes S^n)^*$. The inclusion (12.5.2) implies that the comultiplication $\nabla \otimes \nabla$ induces a multiplication in $(S \otimes S)^\vee$ which makes $(S \otimes S)^\vee$ an associative algebra with unit.

12.6. Lemma. — *The natural homomorphism* $i : S^\vee \otimes S^\vee \to (S \otimes S)^\vee$ *is an algebra homomorphism. The algebra multiplication* $m : S \otimes S \to S$ *induces an algebra homomorphism* $m^\vee : S^\vee \to (S \otimes S)^\vee$.

Proof. – Let $a, b \in S$. Then $\nabla(a)$ and $\nabla(b)$ are certain finite sums, say, $\sum_i a_i \otimes a^i$ and $\sum_j b_j \otimes b^j$ with $a_i, a^i, b_j, b^j \in S$. If $x, y, z, t \in S^\vee$ then

$$\langle a \otimes b | i(x \otimes y) i(z \otimes t) \rangle = \sum_{i,j} \langle a_i \otimes a^i \otimes b_j \otimes b^j | i(x \otimes y) \otimes i(z \otimes t) \rangle$$
$$= \sum_{i,j} x(a_i) y(a^i) z(b_j) t(b^j) = \langle a | xy \rangle \langle b | zt \rangle = \langle a \otimes b | i(xy \otimes zt) \rangle.$$

This implies the first claim.

It is clear that for any $x \in S^\vee$ we have $x \circ m \in (S \otimes S)^\vee$. According to definitions

$$\begin{aligned}\langle a\otimes b \,|\, m^\vee (xy) \rangle &= (xy)(ab) = \langle \nabla (ab) \,|\, x\otimes y \rangle \\ &= \langle \sum_{i,j} a_i b_j \otimes a^i b^j \,|\, x\otimes y \rangle = \sum_{i,j} \langle a_i \otimes b_j \,|\, m^\vee (x) \rangle \langle a^i \otimes b^j \,|\, m^\vee (y) \rangle \\ &= \langle P_{2,3} (\nabla (a) \otimes \nabla (b)) \,|\, m^\vee (x) \otimes m^\vee (y) \rangle \\ &= \langle (\nabla \otimes \nabla)(a\otimes b) \,|\, m^\vee (x) \otimes m^\vee (y) \rangle \\ &= \langle a\otimes b \,|\, m^\vee (x)\, m^\vee (y) \rangle.\end{aligned}$$

Thus, $m^\vee (xy) = m^\vee (x)\, m^\vee (y)$.

12.7. HOMOMORPHISM $g : T \to S^\vee$. — For $x \in \mathfrak{a}^*$ denote by $g(x)$ the Q-linear additive homomorphism $S \to Q$ which sends all S^n with $n \neq 1$ into 0 and sends $a \in \mathfrak{a}$ into $\langle a \,|\, x \rangle \in Q$. Clearly $g(x) \in S^\vee$. This mapping $x \mapsto g(x) : \mathfrak{a}^* \to S^\vee$ uniquely extends to an algebra homomorphism $T \to S^\vee$ denoted by g.

12.8. LEMMA. — *The following diagram* (*in which* Δ *is the canonical comultiplication in* T *sending* $x \in \mathfrak{a}^*$ *into* $x \otimes 1 + 1 \otimes x$) *is commutative*

$$\begin{array}{ccccc} T & \xrightarrow{g} & S^\vee & \overset{m^\vee}{\searrow} & \\ {\scriptstyle\Delta}\downarrow & & & & (S\otimes S)^\vee. \\ T\otimes T & \xrightarrow{g\otimes g} & S^\vee \otimes S^\vee & \underset{i}{\nearrow} & \end{array}$$

Proof. — In view of Lemma 12.6 it suffices to check commutativity for generators $x \in \mathfrak{g}^*$ of T. Let $a, b \in S$. According to definitions

$$\begin{aligned}\langle a\otimes b \,|\, m^\vee (g(x)) \rangle &= \langle ab \,|\, g(x) \rangle = \langle a \,|\, x \rangle\, {}^0 b + \langle b \,|\, y \rangle\, {}^0 a \\ &= \langle a\otimes b \,|\, g(x)\otimes 1 + 1 \otimes g(x) \rangle = \langle a\otimes b \,|\, i(g\otimes g)\Delta (x) \rangle.\end{aligned}$$

(Recall that ${}^0 a$, ${}^0 b$ is the 0-th homogeneous part of a, b.) Thus $m^\vee g = i(g\otimes g)\Delta$.

12.9. THE HOMOMORPHISM $g^\vee : S \to \bar{\mathcal{T}}$. — Put $\bar{\mathcal{T}} = \prod_{n=0}^{\infty} \mathfrak{a}^{\otimes n}$. The elements of $\bar{\mathcal{T}}$ are series ${}^0 a + {}^1 a + \ldots$ where ${}^n a \in \mathfrak{a}^{\otimes n}$. Clearly, $\bar{\mathcal{T}}$ contains $\mathcal{T}$ as the subset of finite series.

Note that the multiplication in $\mathcal{T}$ introduced in Section 12.1 extends to $\bar{\mathcal{T}}$ in the obvious fashion and make $\bar{\mathcal{T}}$ an algebra over Q. The pairing $\langle \;|\; \rangle : \mathcal{T} \times T \to Q$ also extends to the pairing $\langle \;|\; \rangle : \bar{\mathcal{T}} \times T \to Q$ defined by the formula

$$\langle {}^0 a + {}^1 a + \ldots \,|\, x \rangle = \sum_{n \geq 0} \langle {}^n a \,|\, x \rangle.$$

(The latter sum is always finite.)

The comultiplication ∇ in S gives rise to algebra homomorphisms $\nabla^n : S \to S^{\otimes (n+1)}$, $n \geqq 1$ where $\nabla^1 = \nabla$ (*cf.* Section 9.3.1). Denote by q the projection $S \to \mathfrak{a}$ along $\bigoplus_{n \neq 1} S^n$.

For $a \in S$ put

(12.9.1) $$g^{\vee}(a) = {}^0 a + q(a) + \sum_{n \geqq 2} q^{\otimes n}(\nabla^{n-1}(a)) \in \mathscr{T}.$$

Here ${}^0 a \in Q$ and $q^{\otimes n}(\nabla^{n-1}(a)) \in \mathfrak{a}^{\otimes n}$ for $n \geqq 2$.

12.10. LEMMA. — *For any* $a \in S$, $x \in T$

$$\langle g^{\vee}(a) \mid x \rangle = \langle a \mid g(x) \rangle.$$

Proof. – Let $x = x_1 \otimes \ldots \otimes x_n$ where $x_1, \ldots, x_n \in \mathfrak{a}^*$. Then

$$\begin{aligned} \langle a \mid g(x) \rangle &= \langle a \mid g(x_1) g(x_2) \ldots g(x_n) \rangle \\ &= \langle \nabla^{n-1}(a) \mid g(x_1) \otimes \ldots \otimes g(x_n) \rangle \\ &= \langle q^{\otimes n}(\nabla^{n-1}(a)) \mid g(x_1) \otimes \ldots \otimes g(x_n) \rangle \\ &= \langle q^{\otimes n}(\nabla^{n-1}(a)) \mid x_1 \otimes \ldots \otimes x_n \rangle = \langle g^{\vee}(a) \mid x \rangle. \end{aligned}$$

12.11. LEMMA. — $g^{\vee}$ *is an algebra homomorphism.*

Proof. – Let $a, b \in S$. The equality $g^{\vee}(ab) = g^{\vee}(a) g^{\vee}(b)$ is equivalent to the assertion that for any $x_1, \ldots, x_n \in \mathfrak{a}^*$

(12.11.1) $$\langle g^{\vee}(ab) \mid x_1 \ldots x_n \rangle = \langle g^{\vee}(a) g^{\vee}(b) \mid x_1 \ldots x_n \rangle.$$

In view of Lemmas 12.10, 12.8 and the definition of $m^{\vee}$ the L.H.S. of (12.11.1) equals

$$\begin{aligned} \langle ab \mid g(x_1 \ldots x_n) \rangle &= \langle a \otimes b \mid m^{\vee}(g(x_1 \ldots x_n)) \rangle \\ &= \langle a \otimes b \mid (g \otimes g) \Delta (x_1 \ldots x_n) \rangle \\ &= \sum_{\substack{i = (i_1, \ldots, i_r) \\ \hat{\imath} = (j_1, \ldots, j_{n-r})}} \langle a \mid g(x_{i_1} \ldots x_{i_r}) \rangle . \langle b \mid g(x_{j_1} \ldots x_{j_{n-r}}) \rangle \end{aligned}$$

where i runs over all subsequences of the sequence $1, 2, \ldots, n$ and $\hat{\imath}$ denotes the complementary subsequence. The latter expression equals the R.H.S. of (12.11.1) because of the definition of multiplication in $\mathscr{T}$ and Lemma 12.10.

12.12. LEMMA. — $g^{\vee}(S) \subset \mathscr{T}$.

Proof. – Let $\xi_1, \xi_2, \ldots, \xi_n$ be n independent variables. Applying (11.3.3) in the iterative manner we get a formal Lie series $(\ldots((\xi_1 \xi_2) \xi_3) \ldots) \xi_n$. Consider the degree n monomials of this series which contain each of variables $\xi_1, \ldots, \xi_n$ exactly ones. Let the sum of these monomials be

$$\sum_{\sigma} \alpha_{\sigma} [\xi_{\sigma(1)}, [\xi_{\sigma(2)}, \ldots, [\xi_{\sigma(n-1)}, \xi_{\sigma(n)}] \ldots]]$$

where σ runs through all permutations of the set $\{1, 2, \ldots, n\}$ and $\alpha_{\sigma} \in \mathbb{Q}[\hbar]$. The definition of $\nabla : \mathrm{S} \to \mathrm{S} \otimes \mathrm{S}$ implies that for any $a \in \mathfrak{a}$ and any $n \geqq 2$

(12.12.1) $$q^{\otimes n} (\nabla^{n-1}(a)) = \sum_{\sigma} \alpha_{\sigma} \sigma_{*} (\nu^{n-1}(a))$$

where σ_{*} is the permutation $a_1 \otimes \ldots \otimes a_n \mapsto a_{\sigma(1)} \otimes \ldots \otimes a_{\sigma(n)}$ in $\mathrm{S}^{\otimes n}$. Since $\mathfrak{a}$ is spiral the formula (12.12.1) implies that the sum in (12.9.1) is actually finite for any $a \in \mathfrak{a}$. Thus $g^{\vee}(\mathfrak{a}) \subset \mathcal{T}$. Since $g^{\vee}$ is an algebra homomorphism and $\mathfrak{a}$ generates S, we have $g^{\vee}(\mathrm{S}) \subset \mathcal{T}$.

12.13. LEMMA. — *The homomorphism $g^{\vee}$ is a bialgebra homomorphism.*

Proof. — Since $g^{\vee}$ is an algebra homomorphism we have to check only that $(g^{\vee} \otimes g^{\vee}) \nabla(a) = \mathrm{diag}(g^{\vee}(a))$ for any $a \in \mathfrak{a}$. This is equivalent to the equality

(12.13.1) $$\langle (g^{\vee} \otimes g^{\vee}) \nabla(a) \,|\, x \otimes y \rangle = \langle \mathrm{diag}(g^{\vee}(a)) \,|\, x \otimes y \rangle$$

which should hold for arbitrary $x, y \in \mathrm{T}$. Lemma 12.10 implies that the L.H.S. of (12.13.1) equals

$$\langle \nabla(a) \,|\, g(x) \otimes g(y) \rangle = \langle a \,|\, g(x) g(y) \rangle = \langle a \,|\, g(xy) \rangle = \langle g^{\vee}(a) \,|\, xy \rangle$$

Since the multiplication in T is dual to the comultiplication diag in $\mathcal{T}$, the latter expression is equal to the R.H.S. of (12.13.1).

12.14. LEMMA. — $g^{\vee}(\mathrm{S}) \subset \mathrm{E}$.

Proof. — Let us show first that for any $x, y \in \mathfrak{a}^*$

(12.14.1) $$g(xy - yx - \hbar [x, y]) = 0.$$

For this it suffices to show that for each $a \in \mathrm{S}$

(12.14.2) $$\langle a \,|\, g(x) g(y) - g(y) g(x) \rangle = \hbar \langle a \,|\, g([x, y]) \rangle.$$

For $a \in \mathrm{S}^n$ with $n \neq 1$ the R.H.S. of (12.14.2) is zero by the definition of g. The L.H.S. is also zero: for $n = 0$ this follows from the equality $\nabla(a) = a \otimes 1$; for $n \geqq 3$ this follows from the inclusion $\nabla(a) \in \oplus(\mathrm{S}^i \otimes \mathrm{S}^j)$ where $i + j \geqq 3$, and for $n = 2$ this follows from the equality $\nabla(a_1 a_2) = a_1 \otimes a_2 + a_2 \otimes a_1 + b$ where $a_1, a_2 \in \mathfrak{a}$ and $b \in \oplus(\mathrm{S}^i \otimes \mathrm{S}^j)$ with $i + j \geqq 3$. If

$a \in S^1 = \mathfrak{a}$ then (11.3.4) implies that

$$\begin{aligned}\langle a \,|\, g(x)g(y)-g(y)g(x)\rangle &= \langle \nabla(a) \,|\, g(x)\otimes g(y)-g(y)\otimes g(x)\rangle\\ &=(\hbar/2)\langle \nu(a) \,|\, g(x)\otimes g(y)-g(y)\otimes g(x)\rangle\\ &=(\hbar/2)\langle \nu(a)-\mathrm{Perm}_S(\nu(a)) \,|\, g(x)\otimes g(y)\rangle\\ &=\hbar\langle \nu(a) \,|\, g(x)\otimes g(y)\rangle=\hbar\langle \nu(a)\,|\,x\otimes y\rangle\\ &=\hbar\langle a\,|\,[x,y]\rangle=\hbar\langle a\,|\,g([x,y])\rangle.\end{aligned}$$

Since g is an algebra homomorphism, (12.14.1) implies that g annihilates the two-sided ideal of T generated by $\{xy-yx-h[x, y] \,|\, x, y \in \mathfrak{a}^*\}$. In view of Lemma 12.10 for any $a \in S$, any $x_1, x_2, \ldots, x_n \in \mathfrak{a}^*$ and any $i \leqq n-1$ we have

$$\langle g^{\vee}(a) \,|\, x_{1,n}-x_{1,i-1}\otimes x_{i+1}\otimes x_i\otimes x_{i+2,n}-\hbar x_{1,i-1}\otimes [x_i, x_{i+1}]\otimes x_{i+2,n}\rangle=0.$$

This is equivalent to the inclusion $g^{\vee}(S) \subset E$.

12.15. Lemma. — *If* $a \in \mathfrak{a}$ *then* $g^{\vee}(a)=a+\hbar a'$ *with* $a' \in \bigoplus_{n \geqq 2} \mathfrak{a}^{\otimes n}$ *and* symm $(a')=0$.

Proof. – It follows from the definition of the polynomials α_σ which enter into (12.12.1) that they are divisible by $\hbar$ in $Q[\hbar]$. A comparison of (12.9.1) and (12.12.1) shows that to prove the Lemma it suffices to prove that symm $(\nu^{n-1}(a))=0$ for all $n \geqq 2$. This is, however, obvious, since ν is skew-symmetric.

12.16. Lemma. — *The homomorphims* $g^{\vee}: S \to E$ *and* symm$|_E$ *are mutually inverse.*

Proof. – Lemmas 12.14 and 12.15 show that symm $\circ g^{\vee}=\mathrm{id}$. Thus it suffices to prove that symm$|_E$ is injective. Let $a \in E \cap \mathrm{Ker}(\mathrm{symm})$. Clearly, ${}^0a=0$. Assume that ${}^ja=0$ for $j<n$ and prove that ${}^na=0$. Since for all $i<n$

$${}^na-\sigma_i^n({}^na)=\hbar\,\nu_i({}^{n-1}a)=0,$$

${}^na$ is a symmetric element of $\mathfrak{a}^{\otimes n}$. Since symm $({}^na)=0$, ${}^na=0$. Thus, $a=0$.

12.17. *Proof of Lemma* 12.4. – Lemmas 12.13 and 12.16 imply that E is a subbialgebra of $\mathscr{F}$. Lemmas 12.15 and 12.16 imply all remaining claims of Lemma 12.4 except that E is a Poisson subbialgebra, *i. e.* that $[E, E] \subset E$. Let us prove this. Let a, $b \in E$. The bracket $[a, b]$ is calculated as follows. Both a and b should be presented as sums of certain elements of type $a_1\otimes \ldots \otimes a_m$ with $a_1, \ldots, a_m \in \mathfrak{a}$ (resp. $b_1 \otimes \ldots \otimes b_n$ with $b_1, \ldots, b_n \in \mathfrak{a}$) and then one should apply (12.1.1). Thus for each $r \geqq 0$ the r-th homogeneous term ${}^r[a, b]$ of $[a, b]$ is a sum of several expressions of type (12.1.1) where $m+n=r+1$. For integer $s \geqq 1$ denote by $\mu(r, s)$ the sum of those expressions which entry this decomposition of ${}^r[a, b]$ and which have $i+j=s+1$ so that the term $[a_i, b_j]$ stays on the s-th place. Thus, ${}^r[a, b]=\sum_{s=1}^{r} \mu(r, s)$. Fix integers $r>k \geqq 1$. We shall prove that

$$(12.17.1) \qquad (1-\sigma_k^r)({}^r[a, b]) = \hbar\, \nu_k({}^{r-1}[a, b]).$$

It follows from the inclusion $a, b \in E$ and from the definition of the multiplication $\circ$ in $\mathscr{F}$ that for any s with $r \geqq s \geqq k+2$

$$(12.17.2) \qquad (1-\sigma_k^r)(\mu(r, s)) = \hbar\, \nu_k(\mu(r-1, s-1)).$$

Similarly, for any s with $k-1 \geqq s \geqq 1$

$$(12.17.3) \qquad (1-\sigma_k^r)(\mu(r, s)) = \hbar\, \nu_k(\mu(r-1, s)).$$

Finally,

$$(12.17.4) \qquad (1-\sigma_k^r)(\mu(r, k)+\mu(r, k+1)) = \hbar\, \nu_k(\mu(r-1, k)).$$

This follows from (7.1.2) and the obvious fact that if $\nu(b) = c - \mathrm{Perm}_{\mathfrak{a}}(c)$ then $a\nu(b) = ac - \mathrm{Perm}_{\mathfrak{a}}(ac)$, where $a, b \in \mathfrak{a}$ and $c \in \mathfrak{a} \otimes \mathfrak{a}$. Summing up (12.17.2-12.17.4) for $s = 1, \ldots, r$ we get (12.17.1).

13. Topological quantization of $\varepsilon_\hbar(Z_0)$

13.1. THEOREM. — *The Lie bialgebras* Z *and* Z_0 *are spiral.*

Proof. — We apply the following obvious criteria: if a Lie coalgebra $\mathfrak{g}$ with the Lie cobracket ν has an increasing filtration $f_0 \subset f_1 \subset \ldots$ such that $\mathfrak{g} = \bigcup_n f_n$ and $\nu(f_n) \subset \sum_{i=0}^{n-1} f_i \otimes f_{n-i-1}$ for all $n \geqq 0$ then $\mathfrak{g} = \bigcup_n \mathrm{Ker}(\nu^n)$. This implies that Z and Z_0 are spiral: take f_n to be the submodule of Z (resp. of Z_0) generated by the homotopy classes of loops with $\leqq n$ self-intersections.

13.2. THEOREM. — *Put* $A_0 = A_0(F)$ (*see Section* 10.2) *and* $\varepsilon = \varepsilon_\hbar(Z_0)$. *There exists a canonical bialgebra homomorphism* $p_h : A_0 \to \varepsilon$ *which is linear over the projection* $K[h, \hbar] \to K[\hbar]$ *with kernel generated by* h *and which is a reduced quantization of the Poisson bialgebra* ε.

In particular, this Theorem gives a geometric interpretation of the bialgebras $\varepsilon_\hbar(Z_0)$ and $\varepsilon(Z_0)$ in terms of links: in the category of $K[\hbar]$-bialgebras $\varepsilon_\hbar(Z_0) = A_0/h A_0$ and in the category of K-bialgebras $\varepsilon(Z_0) = A_0/(h A_0 + (\hbar - 1) A_0)$.

In the remaining part of Section 13 we construct p_h and show that p_h is a quantization homomorphism in the category of bialgebras. In the next Section 14 we introduce an invariant of oriented trees which is used in Section 15 to show that $\mathrm{Ker}\, p_h = h A_0$. This will finish the proof of Theorem 13.2.

The class in A_0 of an oriented link $L \subset F \times [0, 1]$ will be denoted by $[L]_0$. The comultiplication in A_0 constructed in Section 10.2 will be denoted by Δ.

13.3. HOMOMORPHISM $\mathcal{J}: A_0 \to \mathcal{F}$. — Let $\mathcal{F}$ be the K[$\hbar$]-module

$$\bigoplus_{n \geqq 0} (K[\hbar] \otimes Z_0)^{\otimes n} = K[\hbar] \otimes \bigoplus_{n \geqq 0} Z_0^{\otimes n}$$

provided with the bialgebra structure as in Section 12.1. We first construct a homomorphism $\mathcal{J}: A_0 \to \mathcal{F}$ linear over the ring homomorphism $h \mapsto 0: K[h, \hbar] \to K[\hbar]$. This will enable us to define p_h to be symm $\circ\, \mathcal{J}: A_0 \to \varepsilon$.

To construct $\mathcal{J}$ we need an auxiliary additive homomorphism $q: A_0 \to K[\hbar] \otimes Z_0$ linear over the projection $h \mapsto 0: K[h, \hbar] \to K[\hbar]$. For an oriented link $L \subset F \times [0, 1]$ with $\geqq 2$ components or for $L = \varnothing$ we put $q([L]_0) = 0$. For an oriented knot $L \subset F \times [0, 1]$ we define $q([L]_0)$ to be the class $\langle \alpha \rangle_0$ of the loop α parametrizing the projection of L into F. It is easy to verify that the formula $[L]_0 \mapsto q([L]_0)$ does define a linear homomorphism $q: A_0 \to K[\hbar] \otimes Z_0$. This homomorphism induces for each $n \geqq 0$ an additive homomorphism $q^{\otimes n}: A_0^{\otimes n} \to (K[\hbar] \otimes Z_0)^{\otimes n}$ linear over the projection $K[h, \hbar] \to K[\hbar]$. In particular $q^{\otimes 0}$ is just this projection.

The comultiplication $\Delta: A_0 \to A_0^{\otimes 2}$ gives rise to an iterated homomorphism $\Delta^n: A_0 \to A_0^{\otimes (n+1)}$ for each $n \geqq 1$ (*see* Section 9.3.1). Denote by Δ^0 the identity mapping $A_0 \to A_0$ and by Δ^{-1} the $K[h, \hbar]$-linear homomorphism $A_0 \to K[h, \hbar]$ sending $[L]_0$ into 0 if $L \neq \varnothing$ and sending $[\varnothing]_0$ into 1. (It is easy to see that Δ^{-1} is a counit of the bialgebra A_0.)

Put

$$\mathcal{J} = \bigoplus_{n \geqq 0} (q^{\otimes n} \circ \Delta^{n-1}): A_0 \to \mathcal{F}.$$

In particular, $\mathcal{J}([\varnothing]_0) = 1$. Since $q(h A_0) = 0$ we have $\mathcal{J}(h A_0) = 0$.

We will use the following explicit formula for $\mathcal{J}$. Introduce for each non-empty oriented link diagram $\mathcal{D}$ and for each $n \geqq 1$ a subset $\mathrm{Lbl}_n^0(\mathcal{D})$ of $\mathrm{Lbl}_n(\mathcal{D})$ consisting of n-labellings f such that $|f| = -\|f\|$ and all the link diagrams $\mathcal{D}_{f,1}, \ldots, \mathcal{D}_{f,n}$ are non-empty knot diagrams, *i. e.* each of them has exactly 1 component. Put $[\mathcal{D}]_0 = [L(\mathcal{D})]_0 \in A_0$ where $L(\mathcal{D})$ is the link presented by $\mathcal{D}$. Then

$$\mathcal{J}([\mathcal{D}]_0) = \sum_{n \geqq 1,\ f \in \mathrm{Lbl}_n^0(\mathcal{D})} q^{\otimes n}(\Delta(\mathcal{D}, f)).$$

More explicitly,

$$(13.3.1) \quad \mathcal{J}([\mathcal{D}]_0) = \sum_{n \geqq 1} \sum_{f \in \mathrm{Lbl}_n^0(\mathcal{D})} (-1)^{|f|_-} \hbar^{|f|} \langle \mathcal{D}_{f,1} \rangle_0 \otimes \ldots \otimes \langle \mathcal{D}_{f,n} \rangle_0$$

where for a knot diagram d we denote by $\langle d \rangle_0$ the class in Z_0 of the underlying loop in F.

Note that the minimal n with $\mathrm{Lbl}_n^0(\mathcal{D}) \neq \varnothing$ is just the number $l = |\mathcal{D}|$ of components of $\mathcal{D}$. The set $\mathrm{Lbl}_l^0(\mathcal{D})$ consists of $l!$ labellings which are constant along the components of $\mathcal{D}$ and take different values on different components.

13.4. LEMMA. — *$\mathcal{J}$ is an algebra homomorphism.*

Proof. — Clearly Δ^n is an algebra homomorphism for all n. The homomorphism q is not multiplicative. Indeed, for any oriented links L_1, $L_2 \subset F \times [0, 1]$ the value $q([L_1]_0 [L_2]_0)$ is non-zero if and only if one of these two links, say, L_i is empty and the second one, *i. e.* L_{3-i} is a non-contractible knot in which case $q([L_1]_0 [L_2]_0) = q([L_{3-i}]_0)$. Note also that each n-labelling f of a link diagram $\mathcal{D}$ which does not attain certain values $i_1, \ldots, i_r \in \{1, 2, \ldots, n\}$ naturally gives rise to a $(n-r)$-labelling f of $\mathcal{D}$ with $\langle \mathcal{D} \mid f \rangle = \langle \mathcal{D} \mid f' \rangle$ (and vice versa). Combining these observations together one easily computes that for any non-empty oriented links L, L′ and any $n \geqq 1$

$$q^{\otimes n}(\Delta^{n-1}([L]_0 [L']_0)) = \sum_{r=1}^{n-1} q^{\otimes r}(\Delta^{r-1}([L]_0)) \circ q^{\otimes (n-r)}(\Delta^{n-r-1}([L']_0))$$

where $\circ$ denotes the multiplication in $\mathcal{F}$. Therefore, $\mathcal{J}([L]_0 [L']_0) = \mathcal{J}([L]_0) \circ \mathcal{J}([L']_0)$.

In case where L or L′ is empty the latter equality is obvious. Thus, $\mathcal{J}$ is an algebra homomorphism.

13.5. LEMMA. — *$\mathcal{J}$ is a bialgebra homomorphism.*

Proof. — We must show that for any oriented link diagram $\mathcal{D}$

$$\text{diag}(\mathcal{J}([\mathcal{D}]_0)) = (\mathcal{J} \otimes \mathcal{J})(\Delta([\mathcal{D}]_0)) \tag{13.5.1}$$

where diag is the comultiplication in $\mathcal{F}$. If $\mathcal{D} = \varnothing$ then both sides of (13.5.1) are equal to 1. Let $\mathcal{D} \neq \varnothing$. Since $\mathcal{J}([\varnothing]_0) = 1$, $\mathcal{J}(h A_0) = 0$, we have

$$\begin{aligned}(13.5.2)\quad (\mathcal{J} \otimes \mathcal{J})(\Delta([\mathcal{D}]_0)) &= \sum_{f \in \mathrm{Lbl}(\mathcal{D})} \langle \mathcal{D} \mid f \rangle \mathcal{J}([\mathcal{D}_{f,1}]_0) \otimes \mathcal{J}([\mathcal{D}_{f,2}]_0) \\ &= \sum_{\substack{f \in \mathrm{Lbl}(\mathcal{D}) \\ |f| = -\|f\| \neq 0}} \langle \mathcal{D} \mid f \rangle \mathcal{J}([\mathcal{D}_{f,1}]_0) \otimes \mathcal{J}([\mathcal{D}_{f,2}]_0) + \mathcal{J}([\mathcal{D}]_0) \otimes 1 + 1 \otimes \mathcal{J}([\mathcal{D}]_0).\end{aligned}$$

Each pair of labellings $g_1 \in \mathrm{Lbl}_m(\mathcal{D}_{f,1})$, $g_2 \in \mathrm{Lbl}_n(\mathcal{D}_{f,2})$ gives rise to a labelling $g \in \mathrm{Lbl}_{m+n}(\mathcal{D})$ whose value on an edge a of $\mathcal{D}$ equals $g_1(a)$ if $f(a) = 1$ and equals $m + g_2(a)$ if $f(a) = 2$. Clearly $|g|_\varepsilon = |g_1|_\varepsilon + |g_2|_\varepsilon + |f|_\varepsilon$ for $\varepsilon = +, -$ and

$$\|g\| = \|g_1\| + \|g_2\| + \|f\|.$$

Therefore

$$\langle \mathcal{D} \mid g \rangle = \langle \mathcal{D} \mid g_1 \rangle \langle \mathcal{D} \mid g_2 \rangle \langle \mathcal{D} \mid f \rangle.$$

It is easy to see that for any m, $n \geqq 1$ the rule $(g_1, g_2) \mapsto g$ establishes a bijective correspondence between the disjoint union of sets $\mathrm{Lbl}_m(\mathcal{D}_{f,1}) \times \mathrm{Lbl}_n(\mathcal{D}_{f,2})$ where f varies through all elements of $\mathrm{Lbl}_2(\mathcal{D})$, and the set $\mathrm{Lbl}_{m+n}(\mathcal{D})$. The subset $\mathrm{Lbl}^0_{m+n}(\mathcal{D})$ corresponds to the disjoint union of the subsets $\mathrm{Lbl}^0_m(\mathcal{D}_{f,1}) \times \mathrm{Lbl}^0_n(\mathcal{D}_{f,2})$ where f varies through elements of $\mathrm{Lbl}(\mathcal{D})$ with $|f| = -\|f\| \neq 0$. This implies that computing the

R.H.S. of (13.5.2) via the formula (13.3.1) we get

$$\sum_{m,n\geqq 1}\ \sum_{g\in \mathrm{Lbl}^{0}_{m+n}(\mathscr{D})} \langle \mathscr{D}\,|\,g\rangle \{\langle \mathscr{D}_{g,1}\rangle_0 \otimes \ldots \otimes \langle \mathscr{D}_{g,m}\rangle_0\} \otimes \{\langle \mathscr{D}_{g,m+1}\rangle_0 \otimes \ldots \otimes \langle \mathscr{D}_{g,m+n}\rangle_0\} + \mathscr{J}([\mathscr{D}]_0)\otimes 1 + 1 \otimes \mathscr{J}([\mathscr{D}]_0).$$

The latter expression is obviously equal to $\mathrm{diag}(\mathscr{J}([\mathscr{D}]_0))$.

13.6. Definition. — Put $\varepsilon=\varepsilon_{\hbar}(\mathrm{Z}_0)$ and define p_h to be $\mathrm{symm}\circ\mathscr{J}:\mathrm{A}_0\to\varepsilon$.

13.7. Lemma. — $p_h(\mathrm{A}_0)=\varepsilon$.

Proof. — Let α be a generic non-contractible loop on F with n self-intersections. Choosing in an arbitrary way an upper branch in each self-intersection we obtain a diagram of a knot L_α with n self-crossings. It is easy to see that $p_h([\mathrm{L}_\alpha]_0)=\langle\alpha\rangle_0+u$ where $u\in \mathrm{S}(\mathrm{K}[\hbar]\otimes\mathrm{Z}_0)$ is a polynomial over certain $\langle\alpha_1\rangle_0$, $\langle\alpha_2\rangle_0, \ldots$ where $\alpha_1, \alpha_2, \ldots$ are loops on F with $\leqq(n-1)$ self-intersections. In particular, if $n=0$ then $p_h([\mathrm{L}_\alpha]_0)=\langle\alpha\rangle_0$. Inducting on n and using the multiplicativity of p_h we get $\langle\alpha\rangle_0\in \mathrm{Im}\,p_h$. Thus, $p_h(\mathrm{A}_0)=\varepsilon$.

13.8. Lemma. — *Let* $\mathrm{E}=\mathrm{E}(\mathrm{K}[\hbar]\otimes\mathrm{Z}_0)\subset\mathscr{F}$ (see *Section* 12.3). *Then:* (i) $\mathscr{J}(\mathrm{A}_0)=\mathrm{E}$; (ii) $\mathscr{J}:\mathrm{A}_0\to\mathrm{E}$ *is a quantization of the Poisson bialgebra* E.

Since $\mathrm{symm}:\mathrm{E}\to\varepsilon$ is an isomorphism of Poisson bialgebras, Lemma 13.8 implies that p_h is a quantization of ε.

We will prove Claim (i) of the Lemma immediately and Claim (ii) in Section 13.11 using preliminary results of Section 13.9, 13.10.

Proof of Lemma 13.8, *Claim* (i). — Let L be an oriented link in $\mathrm{F}\times[0, 1]$ presented by a diagram $\mathscr{D}$. Put $a=\mathscr{J}([\mathrm{L}]_0)$. If $\mathrm{L}=\varnothing$ then $a=1\in\mathrm{E}$. Let $\mathrm{L}\neq\varnothing$. We must show that ${}^n a-\sigma^n_i({}^n a)=\hbar\,\nu_i({}^{n-1}a)$ for all $n>i\geqq 1$ where ν is the $\mathrm{K}[\hbar]$-linear extension of the Lie cobracket in Z_0. Fix the numbers $n>i\geqq 1$.

Note that each cutting point of a labelling $f\in\mathrm{Lbl}_n(\mathscr{D})$ is incident to four edges of $\mathscr{D}$ on which f takes two distinct values. Let G be the subset of $\mathrm{Lbl}_n(\mathscr{D})$ consisting of n-labellings f such that there is no f-cutting point for which the pair of values of f on the four incident edges would be $\{i, i+1\}$. Put $\mathrm{G}^0=G\cap\mathrm{Lbl}^0_n(\mathscr{D})$.

For each $f\in\mathrm{Lbl}_n(\mathscr{D})$ we define a mapping $f':\mathrm{Edg}(\mathscr{D})\to\{1,\ldots,n\}$ to be the composition of $f:\mathrm{Edg}(\mathscr{D})\to\{1,\ldots,n\}$ and the transposition permuting i with $i+1$. If $f\in\mathrm{G}$ then $f'\in\mathrm{G}$ and $\Delta(\mathscr{D},f')=\sigma^n_i(\Delta(\mathscr{D},f))$. Also, if $f\in\mathrm{G}^0$ then $f'\in\mathrm{G}^0$. Therefore

$$\sum_{f\in\mathrm{G}^0}\Delta(\mathscr{D},f)-\sigma^n_i\Big(\sum_{f\in\mathrm{G}^0}\Delta(\mathscr{D},f)\Big)=0. \tag{13.8.1}$$

Put $\mathrm{H}=\mathrm{Lbl}^0_n(\mathscr{D})\setminus\mathrm{G}^0$. Thus, $f\in\mathrm{H}$ iff $f\in\mathrm{Lbl}^0_n(\mathscr{D})$ and there exists at least one cutting vertex $v(f)$ of $\mathscr{D}$ such that f takes the values $\{i, i+1\}$ on the incident edges. Because of the condition $|f|=-\|f\|$ such a vertex $v(f)$ is unique. Denote its sign ± 1 by ε_f.

With each $g \in \mathrm{Lbl}^0_{n-1}(\mathscr{D})$ we associate a subset $\mathrm{H}(g)$ of H. It consists of $f \in \mathrm{H}$ such that for any edge d of $\mathscr{D}$ either $f(d)=g(d)<i$, or $g(d)=i$ and $f(d)\in\{i, i+1\}$, or $f(d)=g(d)+1\geqq i+2$. Since $\mathscr{D}_{g,i}$ is a knot diagram the formula $f \mapsto v(f)$ establishes a bijective correspondence between $\mathrm{H}(g)$ and the set of self-crossing points of $\mathscr{D}_{g,i}$. Moreover, according to the definition of the Lie cobracket ν

$$\nu(\langle \mathscr{D}_{g,i}\rangle_0)=\sum_{f\in \mathrm{H}(g)} \varepsilon_f(\langle \mathscr{D}_{f,i}\rangle_0 \otimes \langle \mathscr{D}_{f,i+1}\rangle_0 - \langle \mathscr{D}_{f,i+1}\rangle_0 \otimes \langle \mathscr{D}_{f,i}\rangle_0).$$

For any $f \in \mathrm{H}(g)$ we have $|f|=|g|+1$ and $(-1)^{|f|_-}=\varepsilon_f(-1)^{|g|_-}$. Therefore

$$(13.8.2) \qquad q^{\otimes n}\Big(\sum_{f\in \mathrm{H}(g)} (\Delta(\mathscr{D}, f)-\sigma_i^n(\Delta(\mathscr{D}, f)))\Big)=\hbar\, \nu_i(q^{\otimes(n-1)}(\Delta(\mathscr{D}, g))).$$

Note finally that the sets $\mathrm{H}(g)$ corresponding to various $g\in \mathrm{Lbl}^0_{n-1}(\mathscr{D})$ do not overlap and cover the whole set H. Therefore, summing up (13.8.1) and (13.8.2) where g runs over $\mathrm{Lbl}^0_{n-1}(\mathscr{D})$ we get ${}^n a-\sigma_i^n({}^n a)=\hbar\,\nu_i({}^{n-1}a)$. Thus $\mathscr{J}(\mathrm{A}_0)\subset \mathrm{E}$. Since $\mathrm{symm}|_{\mathrm{E}}:\mathrm{E}\to\varepsilon$ is a bijection, Lemma 13.7 implies that $\mathscr{J}(\mathrm{A}_0)=\mathrm{E}$.

13.9. Auxiliary definitions. – Let Γ be a finite oriented graph. The set of vertices of Γ will be denoted by $v(\Gamma)$. Each oriented edge of Γ leading from a vertex a to a vertex b will be somewhat abusively denoted by ab. By an *order* in Γ we mean a partial order in $v(\Gamma)$. An order $>$ in Γ is said to be compatible with the orientation of the edge ab if $a>b$. Denote by $\mathrm{Ord}(\Gamma)$ the set of total orders in Γ compatible with the orientations of all edges. [Note that a total order in Γ is essentially a bijective mapping $v(\Gamma)\to\{1, 2, \ldots, n\}$ where $n=\mathrm{card}\, v(\Gamma)$.]

By a *tree* we shall mean a connected finite graph without cycles.

13.10. Lemma. — *Let Γ be a finite oriented graph. Let $n\geqq 2$ and $a_1, \ldots, a_n$ be certain (distinct) vertices of Γ cyclically connected by oriented edges $a_1a_2, a_2a_3, \ldots, a_na_1$. Let $r_1, \ldots, r_n$ be the same edges numerated in an arbitrary way. Let for $i=1, \ldots, n$, Γ_i^r be the oriented graph obtained from Γ by eliminating the (open) edge r_i and inversing orientations in $r_1, \ldots, r_{i-1}$. If Γ_1^r is a tree then*

$$(13.10.1) \qquad \sum_{i=1}^{n} (-1)^i \,\mathrm{card}\,(\mathrm{Ord}\,(\Gamma_i^r))=0.$$

Proof. – Since Γ_1^r is a tree the initial graph Γ is the cycle $a_1a_2, a_2a_3, \ldots, a_na_1$ with some disjoint trees attached to the vertices $a_1, \ldots, a_n$. Denote the tree attached to a_i by T_i. More precisely, T_i is the connected component of a_i in the graph obtained from Γ by eliminating the edges $a_{i-1}a_i$ and a_ia_{i+1}. Note that if Γ_1^r is a tree then Γ_i^r is a tree for all i.

For $i=1, \ldots, n$ consider a total order ω_i in T_i compatible with the orientations of all edges of T_i. Then $\omega=(\omega_1, \ldots, \omega_n)$ is a partial order in Γ_i^r for each i. Let $\mathrm{Ord}_\omega(\Gamma_i^r)$ be the set of total orders in Γ_i^r compatible with ω and with the orientations of edges. Put

$$e(\Gamma, \omega, r) = \sum_{i=1}^{n} (-1)^i \operatorname{card}(\operatorname{Ord}_\omega(\Gamma_i^r)).$$

It is evident that the L.H.S. of (13.10.1) is equal to the sum of $e(\Gamma, \omega, r)$ over all possible sequences $\omega = (\omega_1, \ldots, \omega_n)$. We shall prove that $e(\Gamma, \omega, r) = 0$ for all ω. This will imply (13.10.1).

Show first that $e(\Gamma, \omega, r)$ does not depend on the choice of r, *i. e.* does not depend on the numeration of edges $a_1 a_2, \ldots, a_n a_1$. It suffices to show that for the sequence t obtained from $r_1, \ldots, r_n$ by the exchange of r_j and r_{j+1} we have $e(\Gamma, \omega, r) = e(\Gamma, \omega, t)$. Clearly, for $i \neq j, j+1$ $\Gamma_i^r = \Gamma_i^t$. Hence we have to show that

$$\operatorname{card}\operatorname{Ord}_\omega(\Gamma_j^r) + \operatorname{card}\operatorname{Ord}_\omega(\Gamma_{j+1}^t) = \operatorname{card}\operatorname{Ord}_\omega(\Gamma_j^t) + \operatorname{card}\operatorname{Ord}_\omega(\Gamma_{j+1}^r).$$

Both sides are easily seen to be equal to the number of total orders in $v(\Gamma)$ compatible with $\omega_1, \ldots, \omega_n$, with orientations of the edges $r_{j+2}, \ldots, r_n$ and with reversed orientations in $r_1, \ldots, r_j$. Thus $e(\Gamma, \omega, r) = e(\Gamma, \omega, t)$.

We show that $e(\Gamma, \omega, r) = 0$ by the induction on $\operatorname{card} v(\Gamma) - n$. If $\operatorname{card} v(\Gamma) = n$ then Γ is the cycle $a_1 a_2, \ldots, a_n a_1$ without any additional edges or vertices. (Of course, ω is irrelevant here.) To compute $e(\Gamma, \omega, r)$ we take

$$r_1 = a_1 a_2, r_2 = a_2 a_3, \ldots, r_n = a_n a_1.$$

Orientations of edges induce the following partial order in $v(\Gamma_i^r) = \{a_1, \ldots, a_n\}$:

(13.10.2) $$a_i > a_{i-1} > \ldots > a_1; \qquad a_{i+1} > a_{i+2} > \ldots > a_n > a_1.$$

The number of total orders compatible with this partial order is clearly $\binom{n-1}{i-1}$. Thus

$$e(\Gamma, \omega, r) = \sum_{i=1}^{n} (-1)^i \binom{n-1}{i-1} = -(1-1)^{n-1} = 0.$$

Now we proceed to the induction step. Note that for given $\omega = (\omega_1, \ldots, \omega_n)$ the set $\operatorname{Ord}_\omega(\Gamma_i^r)$ does not depend on the way in which edges of T_i connect the vertices of T_i. Therefore, $e(\Gamma, \omega, r)$ will not be changed if we rearrange the edges of T_i, $i = 1, \ldots, n$ so that each T_i is a segment, *i. e.* edges of T_i connect vertices successively in accordance with the total order ω_i. The initial and final vertices of this segment-type tree are respectively the maximal and minimal elements of $v(T_i)$ with respect to ω_i. Denote these vertices of T_i by B_i and b_i. Clearly, $B_i \geqq_{\omega_i} a_i \geqq_{\omega_i} b_i$.

If $B_i = a_i = b_i$ for all i then $\operatorname{card} v(\Gamma) = n$ and we are done. Assume that $b_j \neq a_j$ for some j. Denote the set $\{j = 1, \ldots, n \mid b_j \neq a_j\}$ by G. For $j \in G$ denote by R_j the oriented graph obtained from Γ by eliminating the vertex b_j and the only edge incident to b_j. This graph R_j still contains the cycle $a_1 a_2, \ldots, a_n a_1$ and the partial order ω induces an order in R_j denoted by ϖ. We shall prove that

$$e(\Gamma, \omega, r) = \sum_{j \in G} e(R_j, \varpi, r). \tag{13.10.3}$$

Inductive assumptions would imply that $e(\Gamma, \omega, r) = 0$.

To prove (13.10.3) we first cyclically renumber $a_1, \ldots, a_n$ so that $1 \in G$. As above take $r_1 = a_1 a_2, \ldots, r_n = a_n a_1$. Each total order in $(R_j)_i^r$ uniquely extends to a total order in Γ_i^r such that b_j is the minimal element of the latter order. Thus for each $i = 1, \ldots, n$ we have an inclusion of the disjoint union $\coprod_{j \in G} \mathrm{Ord}_\varpi((R_j)_i^r)$ into $\mathrm{Ord}_\omega(\Gamma_i^r)$. This inclusion is actually a bijection. Indeed the inclusion $1 \in G$ and the inequalities (13.10.2) imply that for any order $f \in \mathrm{Ord}_\omega(\Gamma_i^r)$ the f-minimal element of $v(\Gamma_i^r)$ must be a vertex b_j with $j \in G$. Therefore for all $i = 1, \ldots, n$

$$\mathrm{card}\, \mathrm{Ord}_\omega(\Gamma_i^r) = \sum_{j \in G} \mathrm{card}\, \mathrm{Ord}_\varpi((R_j)_i^r).$$

This implies (13.10.3).

If $b_i = a_i$ for all i then $B_j \neq a_j$ for some j, and the argument goes similarly with the following modifications: instead of "minimal" one should say "maximal"; in the role of r one should take the sequence $r_1 = a_n a_1, r_2 = a_{n-1} a_n, \ldots, r_n = a_1 a_2$.

13.11. *Proof of Lemma* 13.8, *Claim* (ii). — Let L, M be two oriented knots in $F \times [0, 1]$ presented respectively by diagrams C, $\mathscr{D}$ on F lying in general position. Let $v_1, \ldots, v_\alpha$ be the points of $C \cap \mathscr{D}$ numerated in an arbitrary way. Let $\varepsilon_i = \pm 1$ be the sign of the local intersection of (the underlying loops of) C, $\mathscr{D}$ in v_i, $1 \leqq i \leqq \alpha$. Denote by $C\mathscr{D}$ the link diagram obtained by placing C over $\mathscr{D}$. Denote by N^i the oriented knot diagram on F obtained from $C\mathscr{D}$ by changing overcrossings to undercrossings in

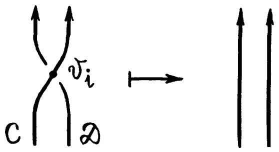

Fig. 11.

$v_1, \ldots, v_{i-1}$ and smoothing in v_i (as in Figure 11). It follows from the Conway relations of type 1 that

$$[L]_0 [M]_0 - [M]_0 [L]_0 = h \sum_{i=1}^{\alpha} \varepsilon_i [N^i]_0 \tag{13.11.1}$$

(*cf.* the proof of Theorem 3.3). We shall prove that

$$\mathscr{I}\left(\sum_{i=1}^{\alpha} \varepsilon_i [N^i]_0\right) = \{\mathscr{I}([L]_0), \mathscr{I}([M]_0)\} \tag{13.11.2}$$

where $\{\ ,\ \}$ is the Lie bracket in $\mathcal{F}$. Since classes of knots generate the algebra A_0, equalities (13.11.1), (13.11.2) imply that $\mathcal{J}: A_0 \to E$ is a quantization homomorphism.

We first compute $\{\mathcal{J}([L]_0), \mathcal{J}([M]_0)\}$ following the definitions. Choose integers $c \geqq m \geqq 1$, $d \geqq n \geqq 1$ and labellings $f \in \mathrm{Lbl}_c^0(C)$, $g \in \mathrm{Lbl}_d^0(\mathcal{D})$. Choose a mixture $x_1, \ldots, x_{m+n-2}$ of the sequences $C_{f,1}, \ldots, C_{f,m-1}$ and $\mathcal{D}_{g,1}, \ldots, \mathcal{D}_{g,n-1}$. Choose a mixture $y_1, \ldots, y_{c+d-m-n}$ of the sequences $C_{f,m+1}, \ldots, C_{f,c}$ and

$$\mathcal{D}_{g,n+1}, \ldots, \mathcal{D}_{g,d}.$$

Choose a crossing point $v_i \in C_{f,m} \cap \mathcal{D}_{g,n}$. Consider the following element of $K[\hbar] \otimes_K Z_0^{\otimes(c+d-1)}$:

$$(13.11.3)\quad (-1)^{|f|_- + |g|_-} \varepsilon_i \hbar^{|f|+|g|} \langle x_1 \rangle_0 \otimes \ldots \otimes \langle x_{m+n-2} \rangle_0 \\ \otimes \langle (C_{f,m})_{v_i} (\mathcal{D}_{g,n})_{v_i} \rangle_0 \otimes \langle y_1 \rangle_0 \otimes \ldots \otimes \langle y_{c+d-m-n} \rangle_0$$

where $(C_{f,m})_v (\mathcal{D}_{g,n})_v$ is the product of loops underlying $C_{f,m}$, $\mathcal{D}_{g,n}$ and based in v. Summing up such elements over all possible choices we get $\{\mathcal{J}([L]_0), \mathcal{J}([M]_0)\}$.

The L.H.S. of (13.11.2) is equal to

$$(13.11.4)\qquad \sum_{i=1}^{\alpha} \sum_{\mu \geqq 1} \sum_{j \in \mathrm{Lbl}_\mu^0(N^i)} (-1)^{|j|_-} \varepsilon_i \hbar^{|j|} \langle N_{j,1}^i \rangle_0 \otimes \ldots \otimes \langle N_{j,\mu}^i \rangle_0.$$

Let us call a labelling j of N^i primitive if all j-cutting vertices of N^i are either self-crossings of C or self-crossings of $\mathcal{D}$. Each element (13.11.3) is equal to the summand of (13.11.4) with the same i, specified as follows: $\mu = c+d-1$; the value of j on an edge a of N^i equals β if $a \subset x_\beta$, equals $m+n-1$ if $a \subset C_{f,m} \cup \mathcal{D}_{g,n}$, and equals $m+n-1+\beta$ if $a \subset y_\beta$. Note that j is a primitive labelling of N^i. Conversely, each element (13.11.3) may be uniquely reconstructed from the corresponding i, μ and the primitive labelling $j \in \mathrm{Lbl}_\mu^0(N^i)$. Therefore, the subsum of (13.11.4) over all primitive labellings is equal to $\{\mathcal{J}([L]_0), \mathcal{J}([M]_0)\}$. To finish the proof of the Lemma we have to show that the remaining subsum of (13.11.4) equals 0. Denote this latter subsum by Σ. Since $\mathcal{J}(A_0) \subset E$ and $\{E, E\} \subset E$ we have $\Sigma \in E$. Therefore it suffices to show that $\mathrm{symm}(\Sigma) = 0$.

For a non-primitive labelling $j \in \mathrm{Lbl}_\mu^0(N^i)$ we denote by $I(j)$ the set consisting of j-cutting vertices of N^i and the point v_i. Thus, $I(j)$ is a subset of the set of crossing points of the diagram $C\mathcal{D}$. Clearly, $\mathrm{card}\, I(j) = |j| + 1 = \mu$. Since j is non-primitive $I(j)$ contains at least two points from the set $C \cap \mathcal{D} = \{v_1, \ldots, v_\alpha\}$. We shall prove that for any set I of crossing points of $C\mathcal{D}$ containing at least two points of $C \cap \mathcal{D}$.

$$(13.11.5)\quad \mathrm{symm}\Big(\sum_{\substack{1 \leqq i \leqq \alpha \\ v_i \in I}} \sum_{\substack{j \in \mathrm{Lbl}_\mu^0(N^i) \\ I(j) = I}} (-1)^{|j|_-} \varepsilon_i \hbar^{|j|} \langle N_{j,1}^i \rangle_0 \otimes \ldots \otimes \langle N_{j,\mu}^i \rangle_0 \Big) = 0$$

where $\mu = \mathrm{card}\, I$. This will imply the desired equality $\mathrm{symm}(\Sigma) = 0$.

Note that the unordered set of knot diagrams $N^i_{j,1}, \ldots, N^i_{j,\mu}$ depends only on I and does not depend on the choice of i with $v_i \in I$ or on the choice of $j \in \mathrm{Lbl}^0_\mu(N^i)$ with $I(j) = I$. The number $|j|$ is also determined solely by I since $|j| = \mathrm{card}\, I - 1$. Let $v_{r_1}, \ldots, v_{r_n}$ be the point of $I \cap \{v_1, \ldots, v_\alpha\}$ where $1 \leqq r_1 < r_2 < \ldots < r_n \leqq \alpha$. Then the L.H.S. of (13.11.5) equals the product of a certain monomial and the integer

$$(13.11.6) \qquad \sum_{i=1}^{n} \sum_{j \in \mathrm{Lbl}^0_\mu (N^{p_i})} (-1)^{|j|} \cdot \varepsilon_{r_i}$$

To compute the summands of (13.11.6) we associate with each crossing point v of the diagram $C\mathscr{D}$ its sign $\varepsilon(v) = \pm 1$. In particular, $\varepsilon(v_r) = \varepsilon_r$. Crossings of the diagram N^r differ in their sign from the crossings of $C\mathscr{D}$ exactly in $r-1$ points $v_1, \ldots, v_{r-1}$. If $r = r_i$ and $j \in \mathrm{Lbl}^0_\mu(N^r)$ with $I(j) = I$, then exactly $i-1$ points $v_{r_1}, \ldots, v_{r_{i-1}}$ from the set $\{v_1, \ldots, v_{r-1}\}$ belong to the set of j-cutting points. Therefore

$$(-1)^{|j|} \cdot \varepsilon_r = \prod_{v \in I} \varepsilon(v) \,.\, (-1)^{i-1}.$$

This shows that up to sign the sum (13.11.6) equals

$$(13.11.7) \qquad \sum_{i=1}^{n} (-1)^{i-1} \,\mathrm{card}\, \{ j \in \mathrm{Lbl}^0_\mu (N^{r_i}) \,|\, I(j) = I \}.$$

We shall prove that the sum (13.11.7) equals 0. Assume that for some $k = 1, \ldots, n$ there exists $j \in \mathrm{Lbl}^0_\mu(N^{r_k})$ with $I(j) = I$. [Otherwise (13.11.7) is identically 0]. Existence of such j ensures that smoothing of N^{r_k} in the points of $I \setminus \{v_{r_k}\}$ produces a link diagram with $\mu = \mathrm{card}\, I$ components. Hence, when we successively smooth N^{r_k} in the points of $I \setminus \{v_{r_k}\}$ we each time increase the number of components by 1. Let $z_1, \ldots, z_n$ be the points of $I \cap \{v_1, \ldots, v_\alpha\}$ numerated so that traversing along C we meet $z_1, \ldots, z_n$ in this particular order (up to a cyclic permutation). By the remarks above when we smooth $C\mathscr{D}$ in the n vertices $z_1, \ldots, z_n$ we receive an n component link. Thus the arcs $z_1 z_2, \ldots, z_n z_1$ in which C is divided by $z_1, \ldots, z_n$ lie in different components of this link. This is possible if and only if $\mathscr{D}$ traverses $z_1, \ldots, z_n$ in the opposite cyclic order $z_n, z_{n-1}, \ldots, z_1$. In this case smoothing $C\mathscr{D}$ in $z_1, \ldots, z_n$ we obtain n knot diagrams each consisting of an arc $z_l z_{l+1}$ of C and the arc $z_{l+1} z_l$ of $\mathscr{D}$. The points of $I \setminus \{z_1, \ldots, z_n\}$ are certain self-crossings of these $2n$ arcs.

We associate with I an oriented graph Γ. The vertices of Γ bijectively correspond to the components of the link diagram obtained from $C\mathscr{D}$ by smoothing in all points of I. The edges of Γ bijectively correspond to elements of I: two vertices u, w of Γ corresponding to components U, W are connected by the oriented edge uw associated with $v \in I$ iff the incident to v in-looking upper and low edges of $C\mathscr{D}$ after smoothing lie respectively on U and W. In particular, the points $z_1, \ldots, z_n$ give rise to edges, say, $a_1 a_2, a_2 a_3, \ldots, a_n a_1$ which make a cycle in Γ. Consider the same oriented edges counted in the order r corresponding to the order $v_{r_1} < v_{r_2} < \ldots < v_{r_n}$ in the set

$I \cap \{v_1, \ldots, v_\alpha\} = \{z_1, \ldots, z_n\}$. It is easy to see that (in the notation of Lemma 13.10) each labelling $j \in \mathrm{Lbl}_\mu^0(\mathrm{N}^{r_i})$ with $\mathrm{I}(j) = \mathrm{I}$ gives rise to a bijective function $v(\Gamma_i^r) \to \{1, 2, \ldots, \mu\}$ which establishes a total order in Γ_i^r compatible with the orientation of all eges. This produces for each $i = 1, \ldots, n$ a bijective mapping $\{j \in \mathrm{Lbl}_\mu^0(\mathrm{N}^{r_i}) \mid \mathrm{I}(j) = \mathrm{I}\} \to \mathrm{Ord}\, \Gamma_i^r$. Therefore, the sum (13.11.7) is equal to the L.H.S. of (13.10.1). Hence, this sum is equal to 0.

14. An invariant of oriented trees

14.1. THEOREM. — *There exists a unique function* η *on the set of isomorphism types of (finite) oriented trees with values in* $\mathbb{Q}$ *such that the following three conditions hold:*

(i) *if* T *is the tree having one vertex (and no edges) then* $\eta(\mathrm{T}) = 1$;

(ii) *if the oriented tree* T′ *(resp.* U*) is obtained from an oriented tree T by reversing orientation in an edge e (resp. by contracting e to a point) then*

$$\eta(\mathrm{T}) + \eta(\mathrm{T}') + \eta(\mathrm{U}) = 0; \tag{14.1.1}$$

(iii) *if the oriented tree* T′ *(resp.* T″*) is obtained from an oriented tree* T *via replacing two dinstinct edges with a common origin ab, ac by ab, bc (resp. by ac, cb) and if* U *is obtained from* T *by identifying vertices b, c and edges ab, ac then*

$$\eta(\mathrm{T}) = \eta(\mathrm{T}') + \eta(\mathrm{T}'') + \eta(\mathrm{U}). \tag{14.1.2}$$

This Theorem is proven in Sections 14.3-14.7. We remark that the proof does not provide an explicit formula for η; it would be nice to have such a formula.

In Section 15 we will use the following property of η.

14.2. COROLLARY. — *Let ab, ac be two edges of an oriented tree* T *with the common origin a. Let* $\mathrm{T}_1, \mathrm{T}_2, \mathrm{T}_3$ *be oriented trees obtained from* T *via replacing ab, ac by ab, cb, resp. by ac, cb, resp. by ca, ab. Let* W *be obtained from* T *by contracting ac to a point. Then*

$$-\eta(\mathrm{T}_1) + \eta(\mathrm{T}_2) + \eta(\mathrm{T}_3) + \eta(\mathrm{W}) = 0. \tag{14.2.1}$$

Proof. — Let T′, T″, U be the same trees as in item (iii) of Theorem 14.1. It follows from (14.1.1) that $-\eta(\mathrm{T}_1) = \eta(\mathrm{T}') + \eta(\mathrm{U})$, $\eta(\mathrm{T}_3) + \eta(\mathrm{W}) = -\eta(\mathrm{T})$. Since $\mathrm{T}_2 = \mathrm{T}''$, (14.2.1) follows from (14.1.2).

14.3. A REFORMULATION OF THEOREM 14.1. — We reformulate Theorem 14.1 using the following "Fourier transform" on invariants of trees. Let T be an oriented tree. Denote by Edg T the set of edges of T deprived of their orientation. For a subset $\mathrm{H} \subset \mathrm{Edg}\,\mathrm{T}$ we put $|\mathrm{H}| = \mathrm{card}\,\mathrm{H}$ and denote by T/H the oriented tree obtained from T by contracting each edge of T belonging to H into a point. Let η be an arbitrary $\mathbb{Q}$-valued function defined on the set of isomorphism types of oriented trees. For $r \in \mathbb{Q}$ put

$$\eta_r(T) = \sum_{H \subset \mathrm{Edg}\, T} r^{|H|} \eta(T/H) = \eta(T) + \sum_{\varnothing \neq H \subset \mathrm{Edg}\, T} r^{|H|} \eta(T/H).$$

Clearly, η_r is a $\mathbb{Q}$-valued function on the set of isomorphism types of oriented trees. It is easy to check that $(\eta_r)_{-r} = \eta$. Put

$$\tilde{\eta}(T) = 2^{|\mathrm{Edg}\, T|} \eta_{1/2}(T).$$

14.3.1. LEMMA. — *A $\mathbb{Q}$-valued function η on the set of isomorphism types of oriented trees satisfies conditions* (i)-(iii) *of Theorem* 14.1 *if and only if $\rho = \tilde{\eta}$ satisfies the following conditons:*

(i)′ *if* T *is the* 1-*vertex tree then* $\rho(T) = 1$;

(ii)′ *if* T, T′ *are the same trees as in Condition* (ii) *then*

$$\rho(T) + \rho(T') = 0;$$

(iii)′ *if* T, T′, T″ *are the same trees as in Condition* (iii) *and if* V *is the oriented tree obtained from* T *by contracting both edges ab, ac to a point then*

$$\rho(T) = \rho(T') + \rho(T'') - \rho(V). \tag{14.3.2}$$

Proof. — Conditions (i) and (i)′ are clearly equivalent. Let us show that (ii) implies (ii)′. Let H run over all subsets of $\mathrm{Edg}\, T = \mathrm{Edg}\, T'$. Then

$$\begin{aligned}\rho(T) + \rho(T') &= 2^{|\mathrm{Edg}\, T|} \Big\{ \sum_{\substack{H \\ e \in H}} (1/2)^{|H|} (\eta(T/H) + \eta(T'/H)) \\ &\quad + \sum_{\substack{H \\ e \notin H}} (1/2)^{|H|} (\eta(T/H) + \eta(T'/H)) \Big\} \\ &= 2^{|\mathrm{Edg}\, T|} \Big\{ \sum_{\substack{H \\ e \in H}} (1/2)^{|H|-1} \eta(T/H) \\ &\quad + \sum_{\substack{H \\ e \notin H}} (1/2)^{|H|} (-\eta(T/(H \cup \{e\})) \Big\} = 0.\end{aligned}$$

Similar computations show that (ii)′ implies (ii) and that (iii) is equivalent to (iii)′.

14.3.3. LEMMA. — *There exists a unique $\mathbb{Q}$-valued function ρ on the set of isomorphism types of oriented trees which satisfies Conditions* (i)′-(iii)′ *of Lemma* 14.3.1.

Lemmas 14.3.1 and 14.3.3 imply Theorem 14.1. Lemma 14.3.3 will be proven in Section 14.7 basing on the results of Section 14.4-14.6.

14.4. MODULE OF ROOTED TREES. — By a rooted tree we mean a non-oriented tree provided with a preferred vertex (the root). Let X_n be the set of isomorphism types of rooted trees with $\leq n$ vertices. Denote by Y_n the $\mathbb{Q}$-module generated by elements of

X_n subject to the following relations: if rooted trees T', T'' are obtained from a rooted tree T by replacing two distinct edges ab, ac (incident to a vertex a) by ab, bc resp. by ac, cb, if a rooted tree V is obtained from T by collapsing ab, ac to a point, and if the root of T and the vertex a lie in the same component of $T \setminus \{b, c\}$ then $T - T' - T'' + V = 0$. Here it is understood that the roots of T, T', T'' are the same and project to the root of V. The class in Y_n of a rooted tree T with $\leqq n$ vertices will be denoted by $[T]^n$.

Denote by R_n the rooted tree having n vertices and $n-1$ (unoriented) edges $a_1 a_2$, $a_2 a_3, \ldots, a_{n-1} a_n$, the root being a_1. It is obvious that Y_1 is the 1-dimensional vector space over $\mathbb{Q}$ with basis $[R_1]^1$.

14.5. Lemma. — *For each* $n \geqq 0$, Y_n *is an n-dimensional vector space over* $\mathbb{Q}$ *with basis* $[R_1]^n, [R_2]^n, \ldots, [R_n]^n$. *The natural mapping* $Y_n \to Y_{n+1}$ *is an imbedding.*

Proof. — Let T be a rooted tree with n vertices. By a trunk of T we mean a sequence of vertices $a_1, \ldots, a_i$ of T such that: a_1 is the root of T; T includes the edges $a_1 a_2$, $a_2 a_3, \ldots, a_{i-1} a_i$; the vertices $a_1, \ldots, a_{i-1}$ are not incident to other edges of T; either $i = n$ or a_i is incident to at least two edges of T besides $a_{i-1} a_i$. Obviously, T has a unique trunk $a_1, \ldots, a_i$. (It may well happen that $i = 1$.) Let j be the number of edges of T incident to a_i. Clearly $j \geqq 3$. Let us call the pair $(-i, j)$ the complexity of T and denote it by comp (T). The set of complexities of rooted trees with n vertices is finite. Provide this set with the order induced by the lexicographic order in $\mathbb{Z} \times \mathbb{Z}$. The set has a unique minimal element $(-n, 1)$ which is the complexity of R_n. Note also that if T, T', T'' are rooted trees described in Section 14.4 then the complexities of T', T'' are strictly smaller than the complexity of T.

We shall construct a mapping $\mu: X_n \to Y_{n-1} \oplus \mathbb{Q}$ as follows. If T is a rooted tree with $\leqq (n-1)$ verices we put $\mu(T) = [T]^{n-1}$. For rooted trees with n vertices we define $\mu(T)$ by the induction on the complexity. For the minimal complexity tree R_n we put $\mu(R_n) = 1 \in \mathbb{Q}$. Assume that for each rooted tree T with n vertices and the complexity strictly smaller than $(-i, j)$ we have already defined $\mu(T)$ so that whenever T, T', T'', V are rooted trees described in Section 14.4, T has n vertices and $\text{comp}(T) < (-i, j)$ we have

$$\mu(T) = \mu(T') + \mu(T'') - \mu(V). \tag{14.5.1}$$

Take a rooted tree T with n vertices and $\text{comp}(T) = (-i, j) > (-n, 1)$. Since $T \neq R_n$ we have $j \geqq 3$. Let $a_1, \ldots, a_i$ be the trunk of T. Let $b_1, \ldots, b_k$ be all vertices of T distinct from a_{i-1} and such that T has edges $a_i b_1, \ldots, a_i b_k$. Here $k = j-1$ if $i \geqq 2$, and $k = j$ if $i = 1$. For $1 \leqq q$, $r \leqq k$ with $q \neq r$ denote by $T_{q, r}$ the rooted tree obtained from T via replacing the edge $a_i b_r$ by the new edge $b_q b_r$. Note that $\text{comp}(T_{q, r}) < \text{comp}(T)$. Denote by $T/b_q a_i b_r$ the rooted tree with $n-2$ vertices obtained from T by collapsing both edges $a_i b_q$, $a_i b_r$ to a point. Put

$$\mu_{q, r} = \mu(T_{q, r}) + \mu(T_{r, q}) - \mu(T/b_q a_i b_r). \tag{14.5.2}$$

Let us show that $\mu_{q, r}$ does not depend on the choice of q, r. Since $\mu_{q, r} = \mu_{r, q}$ it suffices to check that $\mu_{q, r} = \mu_{l, r}$ for any $l = 1, \ldots, k$; $l \neq q, r$. A schematic calculation of $\mu_{q, r}$

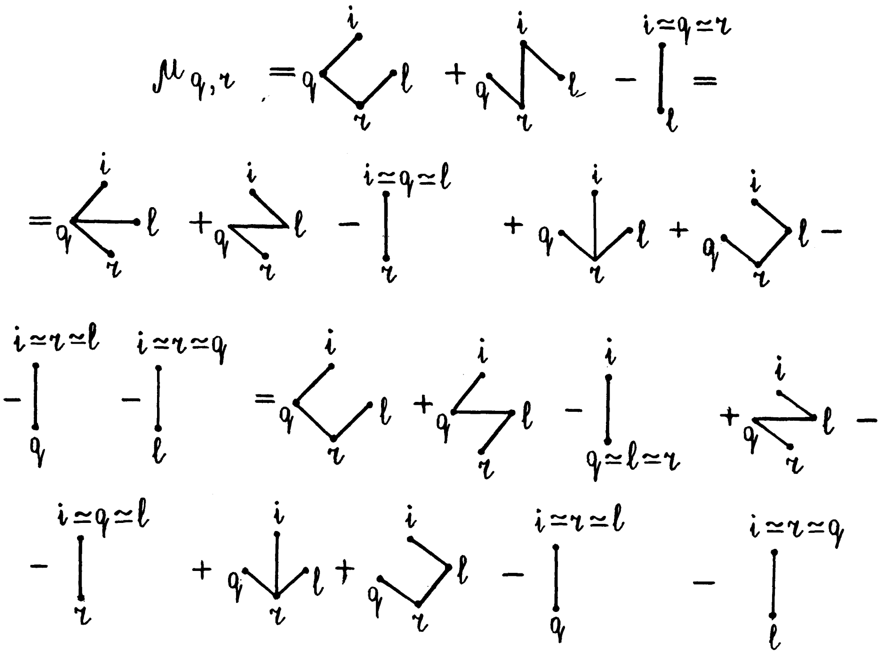

Fig. 12.

based on the definition (14.5.2) and the inductive assumption (14.5.1) is presented on Figure 12 where to save space we denote the vertices a_i, b_q, b_r, b_l simply by i, q, r, l and where the symbol $\simeq$ denotes identification of the vertices and contraction of edges between them. The final expression is symmetric in q, l and so $\mu_{q,r}=\mu_{l,r}$. Now we define $\mu(T)=\mu_{q,r}$ for arbitrary $q \neq r$. It remains to verify (14.5.1) in the case $\operatorname{comp}(T)=(-i, j)$. If the vertex a of T (involved in the construction of T′, T″) coincides with the vertex a_i then (14.5.1) is just the definition of $\mu(T)$. If $a \neq a_i$ then (14.5.1) easily follows from the definition of $\mu(T)$ and the inductive assumption. This finishes the construction of the mapping $\mu: X_n \to Y_{n-1} \oplus \mathbb{Q}$. Because of (14.5.1) this mapping induces a linear homomorphism $Y_n \to Y_{n-1} \oplus \mathbb{Q}$. It is easy to see that this homomorphism and the sum $Y_{n-1} \oplus \mathbb{Q} \to Y_n$ of the natural homomorphism $Y_{n-1} \to Y_n$ and the homomorphism $1 \mapsto [R_n]^n : \mathbb{Q} \to Y_n$ are mutual inverses. Thus $Y_n = Y_{n-1} \oplus \mathbb{Q}\,[R_n]^n$ which implies both claims of the Lemma.

14.6. Corollary. — *Let* $t=(t_0, t_1, \ldots)$ *be a sequence of rational numbers. There exists a unique function* θ_t *on the set of isomorphism classes of rooted trees such that:* $\theta_t(R_n)=t_n$ *for each* $n \geqq 0$*; for any rooted trees* T, T′, T″, V *satisfying conditions of Section* 14.4

(14.6.1) $$\theta_t(T)=\theta_t(T')+\theta_t(T'')-\theta_t(V).$$

14.7. *Proof of Lemma* 14.3.3. – Let G_n be the oriented tree having n vertices $a_1, \ldots, a_n$ and $n-1$ edges $a_1 a_2$, $a_2 a_3, \ldots, a_{n-1} a_n$ (here we use the same convention regarding orientation of edges as in Section 13.9). We first compute $\rho(G_n)$ basing solely on Conditions (i)′ – (iii)′. If one changes orientations of all edges of G_n one gets a tree isomorphic to G_n. Condition (ii)′ implies that $\rho(G_n)=(-1)^{n-1}\rho(G_n)$. Thus, for even n we have $\rho(G_n)=0$. Let n be odd. For $i=1, \ldots, n-1$ denote by G_n^i the oriented tree with vertices $a_1, \ldots, a_n$ and edges $a_1 a_2, a_2 a_3, \ldots, a_{n-2} a_{n-1}, a_i a_n$. In particular, $G_n^{n-1}=G_n$. Condition (iii)′ implies that for each $i=1, \ldots, n-2$

$$\rho(G_n^i)=\rho(G_n^{i+1})+\rho(G_n)-\rho(G_{n-2}).$$

Therefore

$$\rho(G_n^1)=\rho(G_n)+(n-2)(\rho(G_n)-\rho(G_{n-2})).$$

On the other hand, G_n^1 with the reversed orientation in $a_1 a_n$ is just G_n and so $\rho(G_n^1)=-\rho(G_n)$. Thus, $\rho(G_n)=(n-2)n^{-1}\rho(G_{n-2})$. Since $\rho(G_1)=1$ we have $\rho(G_n)=n^{-1}$ for any odd n. Now the same argument as in the proof of Lemma 14.5 shows that the value of ρ on any oriented tree may be computed from $\rho(G_n)$, $n=0, 1, \ldots$ This shows uniqueness of ρ. (This shows also that ρ must be identically zero on the trees with even number of vertices.)

Denote by ρ' the $\mathbb{Q}$-valued function on the set of isomorphism types of rooted trees which corresponds by Corollary 14.6 to the sequence $t=(t_0, t_1, \ldots)$ where $t_n=0$ for even n and $t_n=n^{-1}$ for odd n.

Let T be an oriented rooted tree. Denote the underlying rooted tree by T_0. An (oriented) edge ab of T is called negative if the root of T and the vertex a lie in different components of $T\setminus\{b\}$. Let $\langle T \rangle$ be the number of negative edges of T. Put $\rho(T)=(-1)^{\langle T\rangle}\rho'(T_0)$. We shall show that $\rho(T)$ does not depend on the choice of root of T and satisfies Conditions (i)′-(iii)′ of the Lemma. Conditions (i)′, (ii)′ are straightforward. Let us check (iii)′. There are three cases to consider depending on whether the root of T and the vertex a lie in the same component of $T\setminus\{b, c\}$, or the root of T and b lie in the same component of $T\setminus\{a, c\}$, or the root of T and c lie in the same component of $T\setminus\{a, b\}$. In the first case $\langle T \rangle=\langle T' \rangle=\langle T'' \rangle=\langle V \rangle$ and (14.3.2) follows from (14.6.1). In the second case $\langle T \rangle=\langle T' \rangle=\langle T'' \rangle-1=\langle V \rangle+1$ and because of (14.6.1)

$$\rho(T)-\rho(T')-\rho(T'')+\rho(V)=\pm[\rho'(T_0')-\rho'(T_0)-\rho'(T_0'')+\rho'(V_0)]=0.$$

The third case is similar. It remains to show that $\rho(T)$ does not depend on the choice of the root of T. Let T_a denote the oriented tree T rooted in a vertex a. It suffices to show that for any edge ab of T we have $\rho(T_a)=\rho(T_b)$. Since ρ satisfies Conditions (ii)′, (iii)′ we may reduce the general case to the case $T=G_n$, $a=a_1$ and $b=a_2$. Clearly, $\langle T_a \rangle=0$, $\langle T_b \rangle=1$. By the very definitions $(T_a)_0=R_n$. Thus $\rho(T_a)=0$ if n is even, and $\rho(T_a)=n^{-1}$ if n is odd. A computation similar to the one in the beginning of the proof (but proceeding in the category of unoriented rooted trees) shows that

$$\rho'((T_b)_0)=\rho'(R_n)+(n-2)(\rho'(R_n)-\rho'(R_{n-2})).$$

If n is even then $\rho(T_b)=0$. If n is odd then

$$\rho(T_b)=-\rho'((T_b)_0)=-[n^{-1}+(n-2)(n^{-1}-(n-2)^{-1})]=n^{-1}.$$

Thus in both cases $\rho(T_a)=\rho(T_b)$.

15. The equality $\operatorname{Ker} p_h = h A_0$

15.1. AUXILIARY DEFINITION: HOMOMORPHISM $\xi:\varepsilon \to A_0/h A_0$. — Let α be a non-contractible generic loop on F. Let $\mathscr{D}=\mathscr{D}(\alpha)$ be the oriented knot diagram obtained by replacing each self-intersection point of α by the positive self-crossing (see

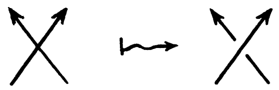

Fig. 13.

Fig. 13). Denote the set of self-crossing points of $\mathscr{D}$ by $\#\mathscr{D}$. With each subset H of $\#\mathscr{D}$ we associate an oriented link diagram $\mathscr{D}_H$ and an oriented graph Γ_H. The diagram $\mathscr{D}_H$ is obtained from $\mathscr{D}$ by smoothing in all self-crossing points belonging to H (cf. *Fig.* 11). The vertices of Γ_H bijectively correspond to components of $\mathscr{D}_H$. Two vertices a, b of Γ_H are connected by an oriented edge ab (leading from a to b) if there exists a self-crossing point of $\mathscr{D}$ belonging to H such that the adjacent inlooking upper and lower edges of $\mathscr{D}$ lie on the components of $\mathscr{D}_H$ corresponding respectively to a and b. (A similar construction was used in Section 13.11.) Notice that the number of components of $\mathscr{D}_H$ does not exceed card (H) + 1. We call H special (or $\mathscr{D}$-special) if $\mathscr{D}_H$ has card (H) + 1 components. A set $H\subset\#\mathscr{D}$ is special iff when we successively smooth $\mathscr{D}$ in the points of H we each time smooth a self-crossing of a certain component obtained at the previous step. This observation shows that if H is special then Γ_H is an oriented tree. Recall the invariant η of oriented trees introduced in Section 14. For a special $H\subset\#\mathscr{D}$ put

$$\langle \mathscr{D}\,|\,H\rangle = \hbar^{\operatorname{card}(H)}\,\eta(\Gamma_H)\,[\mathscr{D}_H]_0 \bmod h A_0 \in A_0/h A_0.$$

For a non-special $H\subset\#\mathscr{D}$ put $\langle\mathscr{D}\,|\,H\rangle=0$. Finally, put

$$\xi(\alpha)=\sum_{H\subset\#\mathscr{D}}\langle\mathscr{D}\,|\,H\rangle\in A_0/h A_0.$$

Note in particular the entry $\langle\mathscr{D}\,|\,\varnothing\rangle=[\mathscr{D}(\alpha)]_0 \bmod h A_0$ of $\xi(\alpha)$.

15.1.1. LEMMA. — *$\xi(\alpha)$ depends only on the free homotopy class of α.*

This Lemma will be proven in Section 15.3.

In view of the Lemma the formula $\langle\alpha\rangle_0\mapsto\xi(\alpha)$ defines a K-linear homomorphism $Z_0\to A_0/h A_0$. The Conway relations of type 1 guarantee that the algebra $A_0/h A_0$ is commutative. Therefore this homomorphism extends to a $K[\hbar]$-linear algebra homomorphis $\varepsilon=S(K[\hbar]\otimes Z_0)\to A_0/h A_0$. Denote the latter homomorphism also by ξ.

15.2. *Proof of the equality* Ker $p_h = h\,A_0$. The inclusion $h\,A_0 \subset \operatorname{Ker} p_h$ is obvious since p_h is linear over the projection $K[h, \hbar] \to K[\hbar]$. Denote by q the algebra homomorphism $A_0/h\,A_0 \to \varepsilon$ induced by p_h. For $n \geqq 0$ denote by B_n the $K[\hbar]$-submodule of $A_0/h\,A_0$ additively generated by classes of oriented links which may be represented by diagrams with $\leqq n$ self-crossings. Clearly $B_0 \subset B_1 \subset \ldots$ and $\bigcup_n B_n = A_0/h\,A_0$. It is easy to deduce from definitions that $(\xi q - 1)(B_0) = 0$ and $(\xi q - 1)\,(B_n) \subset B_{n-1}$ for each $n \geqq 1$. Therefore $(\xi q - 1)^{n+1}(B_n) = 0$ for all $n \geqq 0$. This implies that for any $a \in A_0/h\,A_0$ there exists n such that $(\xi q - 1)^n(a) = 0$. Therefore $\operatorname{Ker} q = 0$ and $\operatorname{Ker} p_h = h\,A_0$.

15.3. *Proof of Lemma* 15.1.1. — We must check that $\xi(\alpha)$ is preserved under homotopy of α. It suffices to consider the local moves presented on Figure 6. The case of ωI is straightforward since the link diagrams having a separate small simple circle represent 0 in $A_0 = A/[\mathcal{O}]A$. Let a loop α' be obtained from α by an application of ωII.1. Let $\mathcal{D}$, $\mathcal{D}'$ be the corresponding positive knot diagrams. Let a, b be the additional crossing points of $\mathcal{D}'$ so that $\#\mathcal{D}' = \#\mathcal{D} \cup \{a, b\}$. Let H be a subset of $\#\mathcal{D}$. We claim that the sum of expressions $\langle \mathcal{D}' \,|\, H \cup J \rangle$ over $J \subset \{a, b\}$ equals $\langle \mathcal{D} \,|\, H \rangle$. This claim, proven below, implies the equality $\xi(\alpha') = \xi(\alpha)$.

If H is not $\mathcal{D}$-special then neither of four sets $H \cup J$ with $J \subset \{a, b\}$ is $\mathcal{D}'$-special. Thus all these sets contribute 0 to $\xi(\alpha)$, $\xi(\alpha')$. If H is $\mathcal{D}$-special then we consider two cases: (i) the components of $\mathcal{D}'_H$ traversing a, b are distinct; (ii) a and b are self-crossings of a certain component of $\mathcal{D}'_H$. In case (i) only the set H from the four sets mentioned above is $\mathcal{D}'$-special and $\langle \mathcal{D} \,|\, H \rangle = \langle \mathcal{D}' \,|\, H \rangle \in A_0/h\,A_0$ (*cf.* the first equality presented on Figure 10). This implies our claim. In case (ii) we have three $\mathcal{D}'$-special sets H, $H \cup \{a\}$ and $H \cup \{b\}$. Clearly

$$\mathcal{D}'_{H \cup \{a\}} = \mathcal{D}'_{H \cup \{b\}} \qquad \text{and} \qquad [\mathcal{D}'_H]_0 = [\mathcal{D}_H]_0 + \hbar\,[\mathcal{D}'_{H \cup \{a\}}]_0.$$

Therefore the special sets H, $H \cup \{a\}$, $H \cup \{b\}$ contribute to $\xi(\alpha')$ the following:

$$(15.3.1) \qquad \hbar^{\operatorname{card}(H)}\,(\eta(\Gamma_H)\,[\mathcal{D}_H]_0 + \hbar\,\bar{\eta}\,[\mathcal{D}_{H \cup \{a\}}]_0)$$

where

$$\bar{\eta} = \eta(\Gamma_H) + \eta(\Gamma_{H \cup \{a\}}) + \eta(\Gamma_{H \cup \{b\}}).$$

The trees $\Gamma_{H \cup \{a\}}, \Gamma_{H \cup \{b\}}$ are obtained from each other by inversion of the orientation of an edge. The tree Γ_H is obtained from $\Gamma_{H \cup \{a\}}$ by collapsing this edge to a point. In view of Condition (ii) of Theorem 14.1, $\bar{\eta} = 0$. Thus (15.3.1) equals $\langle \mathcal{D} \,|\, H \rangle$. This proves our claim in case (ii). The move ωII.2 is considered similarly.

Let a loop α' be obtained from α by an application of ωIII. The corresponding positive diagrams $\mathcal{D}$, $\mathcal{D}'$ are obtained from each other by an application of the Reidemeister move ΩIII. Let a, b, c (resp. a', b', c') be the crossing points of $\mathcal{D}$ (resp. of $\mathcal{D}'$) presented on

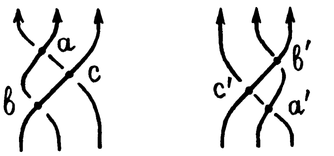

Fig. 14.

Figure 14. Let H be a subset of $\#\mathscr{D}\setminus\{a, b, c\} = \#\mathscr{D}'\setminus\{a', b', c'\}$. We shall show that the sum of expressions $\langle \mathscr{D} | H \cup J \rangle$ over $J \subset \{a, b, c\}$ equals the sum of expressions $\langle \mathscr{D}' | H \cup J' \rangle$ over $J' \subset \{a', b', c'\}$. This will imply the equality $\xi(\alpha) = \xi(\alpha')$.

If H is not $\mathscr{D}$-special then neither of sets $H \cup J$, $H \cup J'$ is special and our claim is obvious. Assume that H is $\mathscr{D}$-special. Of course, H is automatically $\mathscr{D}'$-special. Obviously,

$$\langle \mathscr{D} | H \rangle = \langle \mathscr{D}' | H \rangle; \qquad \langle \mathscr{D} | H \cup \{a\} \rangle = \langle \mathscr{D}' | H \cup \{a'\} \rangle;$$
$$\langle \mathscr{D} | H \cup \{b\} \rangle = \langle \mathscr{D}' | H \cup \{b'\} \rangle;$$
$$\langle \mathscr{D} | H \cup \{a, b, c\} \rangle = \langle \mathscr{D}' | H \cup \{a', b', c'\} \rangle.$$

To finish the proof of our claim it suffices to verify the following equality:

$$(15.3.2) \quad \mathscr{D} | H \cup \{c\} \rangle + \langle \mathscr{D} | H \cup \{a, c\} \rangle + \langle \mathscr{D} | H \cup \{b, c\} \rangle - \langle \mathscr{D}' | H \cup \{a', b'\} \rangle = \langle \mathscr{D}' | H \cup \{c'\} \rangle + \langle \mathscr{D}' | H \cup \{a', c'\} \rangle + \langle \mathscr{D}' | H \cup \{b', c'\} \rangle - \langle \mathscr{D} | H \cup \{a, b\} \rangle.$$

Note that the diagrams $\mathscr{D}_{H \cup \{c\}}$ and $\mathscr{D}_{H \cup \{c'\}}$ have the same number of components. Thus the sets $H \cup \{c\}$ and $H \cup \{c'\}$ are special or nonspecial simultaneously. If they are not special then neither of sets $H \cup \{\ldots\}$ entering (15.3.2) is special and so both parts of (15.3.2) are equal to 0. Assume that $H \cup \{c\}$ and $H \cup \{c'\}$ are special. We shall show that the L.H.S. of (15.3.2) equals

$$(15.3.3) \qquad \hbar^{\operatorname{card}(H)+1} \eta(\Gamma_{H \cup \{c\}}) [\mathscr{D}_{H \cup \{a, b, c\}}]_0 \bmod h A_0.$$

There are two cases to regard: (i) two branches of $\mathscr{D}_{H \cup \{c\}}$ traversing a, b lie on different components of $\mathscr{D}_{H \cup \{c\}}$; (ii) these two branches lie on the same component of $\mathscr{D}_{H \cup \{c\}}$.

In case (i) the sets $H \cup J$ with $J = \{a, c\}$, $\{b, c\}$, $\{a', b'\}$ are not special and the L.H.S. of (15.3.2) equals $\langle \mathscr{D} | H \cup \{c\} \rangle$. This is equal to (15.3.3). In case (ii) four sets $H \cup \{\ldots\}$ entering the L.H.S. of (15.3.2) are special. It is obvious that

$$\mathscr{D}_{H \cup \{a, c\}} = \mathscr{D}_{H \cup \{b, c\}} = \mathscr{D}'_{H \cup \{a', b'\}}.$$

It follows from the first equality presented on Figure 10 that $\langle \mathscr{D} | H \cup \{c\} \rangle$ equals the sum of (15.3.3) and

$$\hbar^{\operatorname{card}(H)+2} \eta(\Gamma_{H \cup \{c\}}) [\mathscr{D}_{H \cup \{a, c\}}]_0.$$

This shows that the L.H.S. of (15.3.2) equals the sum of (15.3.3) and

$$\hbar^{\mathrm{card}\,(H)+2}\,(\eta\,(\Gamma_{H\cup\{c\}})+\eta\,(\Gamma_{H\cup\{a,c\}})+\eta\,(\Gamma_{H\cup\{b,c\}})-\eta\,(\Gamma_{H\cup\{a',b'\}}))\,[\mathscr{D}_{H\cup\{a,c\}}]_0.$$

The four trees entering the latter expression are related exactly as the trees W, T_2, T_3, T_1 from Corollary 14.2. Therefore this expression equals 0 and the L.H.S. of (15.3.2) equals (15.3.3). A similar computation shows that the R.H.S. of (15.3.2) also equals (15.3.3). (One should note that $\Gamma_{H\cup\{c\}}=\Gamma_{H\cup\{c'\}}$.) This completes the proof of the equality $\xi(\alpha)=\xi(\alpha')$.

CHAPTER IV.

TOPOLOGICAL BIQUANTIZATION OF Z_0

16. Bi-Poisson bialgebras and their biquantization over K $[h, \hbar]$

16.1. Bi-poisson bialgebras. — A *bi-Poisson bialgebra* over K is a K-module S equipped with the structure of Poisson bialgebra and of co-Poisson bialgebra with the same underlying commutative and cocommutative bialgebra, so that the Lie cobracket $\nu: S\to S^{\otimes 2}$, the comultiplication $\Delta: S\to S^{\otimes 2}$, the Lie bracket [,] in S (and the induced Lie bracket in $S^{\otimes 2}$) are related by the formula

(16.1.1) $$\nu([a, b])=[\Delta(a), \nu(b)]+[\nu(a), \Delta(b)]$$

for all $a, b\in S$. This formula and the notion of bi-Poisson bialgebra are self-dual.

16.2. Biquantization of bi-poisson bialgebras. — It is convenient to describe first a construction which suggests the notion of biquantization. Let A be a bialgebra over the polynomial ring $K[h, \hbar]$, free as the $K[h, \hbar]$-module. Assume that for any $a, b\in A$

(16.2.1) $$ab-ba\in hA \quad\text{and}\quad \Delta(a)-\mathrm{Perm}_A(\Delta(a))\in\hbar A^{\otimes 2}$$

where Δ denotes the comultiplication in A. Put $S=A/(hA+\hbar A)$. The bialgebra structure of A factorizes to a bialgebra structure in S. Denote the projection $A\to S$ by p. Introduce a Lie bracket in A by the formula

(16.2.2) $$[p(a), p(b)]=p(h^{-1}(ab-ba))$$

where $a, b\in A$. Introduce a Lie cobracket ν in S by the formula

(16.2.3) $$\nu(p(a))=(p\otimes p)(\hbar^{-1}(\Delta(a)-\mathrm{Perm}_A(\Delta(a)))).$$

16.2.4. Theorem. — *S is a bi-Poisson bialgebra over* K.

The proof of the Theorem is deferred to Section 16.3.

Similarly, $A_{\hbar}=A/hA$ is provided with a Poisson bialgebra structure over $K[\hbar]$ and $A_h=A/\hbar A$ is provided with a co-Poisson bialgebra structure over $K[h]$. The homomorphism p is included in the commutative diagram of projections

(16.2.5)
$$\begin{array}{ccc} A & \overset{p_{\hbar}}{\rightarrow} & A_h \\ {\scriptstyle p_h}\downarrow & \searrow^{p} & \downarrow{\scriptstyle q_h} \\ A_{\hbar} & \overset{q_{\hbar}}{\rightarrow} & S \end{array}$$

The homomorphisms of the diagram have the following properties: (i) they are surjective bialgebra homomorphisms linear over the corresponding homomorphisms of the diagram of ring projections

$$\begin{array}{ccc} K[h,\hbar] & \overset{\hbar\mapsto 0}{\rightarrow} & K[h] \\ {\scriptstyle h\mapsto 0}\downarrow & \searrow & \downarrow{\scriptstyle h\mapsto 0} \\ K[\hbar] & \overset{\hbar\mapsto 0}{\rightarrow} & K \end{array};$$

(ii) p_h and q_h are quantization homomorphisms for Poisson algebras resp. $A_{\hbar}$ and S; (iii) $p_{\hbar}$ and $q_{\hbar}$ are coquantization homomorphisms for co-Poisson coalgebras resp. A_h and S; (iv) q_h is a co-Poisson bialgebra homomorphism and q is a Poisson bialgebra homomorphism (*i.e.* q_h and $q_{\hbar}$ preserve respectively the Lie cobracket and the Lie bracket).

Let now S be an arbitrary bi-Poisson bialgebra over K. A *biquantization* over $K[h,\hbar]$ of S is a commutative diagram (16.2.5) in the category of bialgebras where: A is a bialgebra over $K[h,\hbar]$ satisfying (16.2.1) for all $a, b\in A$; A_h is a co-Poisson bialgebra over $K[h]$; $A_{\hbar}$ is a Poisson bialgebra over $K[\hbar]$; the homomorphisms of the diagram satisfy the conditions (i)-(iv) formulated above.

Note that the conditions (i)-(iv) imply that the homomorphism $p:A\to S$ is simultaneously a quantization of the Poisson bracket in S and a coquantization of the co-Poisson cobracket in S.

A biquantization (16.2.5) is called reduced if $\operatorname{Ker} p_h=hA$, $\operatorname{Ker} p_{\hbar}=\hbar A$, $\operatorname{Ker} p=hA+\hbar A$. All ingredients of a reduced biquantization are determined by $p:A\to S$ except the Lie bracket in $A_{\hbar}$ and the Lie cobracket in A_h. If A is a free $K[h,\hbar]$-module (or at least no element of A is annihilated by h or $\hbar$) then these two brackets can be also reconstructed from p as explained in the beginning of this section.

Remarks. — 1. Every biquantization (16.2.5) gives rise to a 2-parameter family of K-bialgebras

$$A(k,k')=A/((h-k)A+(\hbar-k')A)$$

where $k, k'\in K$. This shows in a sense that to construct a biquantization of S one should construct bialgebras A_h, $A_{\hbar}$ and connect them by such a 2-parameter family. Note that if $A_h=A/\hbar A$ then the bialgebras $A(k,0)$, $k\in K$ are completely determined by A_h; indeed $A(k,0)=A_h/(h-k)A_h$. In this case $A(k,0)$ inherits a co-Poisson bialgebra structure from A_h and the homomorphism $p_\hbar$ induces a coquantization homomorphism $A/(h-k)A \to A(k, 0)$. Similar remarks apply to $A(0, k)$ and p_h in case $A_\hbar = A/hA$.

2. A somewhat more general notion of biquantization is given in [20].

16.3. *Proof of Theorem* 16.2.4. — It is easy to check up that the homomorphisms $[\ ,\ ]: S \times S \to S$ and $\nu: S \to S^{\otimes 2}$ given by (16.2.2), (16.2.3) are well-defined. It is also easy to see that the algebra S with the Lie bracket $[\ ,\ ]$ is a Poisson algebra and bialgebra with the same multiplication. To show that S is a Poisson bialgebra it remains to prove that the comultiplication $\Delta_0: S \to S^{\otimes 2}$ induced by $\Delta: A \to A^{\otimes 2}$ is a Poisson algebra homomorphism. For $a \in A$ put $a_0 = p(a)$ where p is the projection $A \to S$. For $u \in A^{\otimes 2}$ put $u_0 = (p \otimes p)(u) \in S^{\otimes 2}$. It is easy to deduce from definitions that for any $u, v \in A^{\otimes 2}$

$$[u_0, v_0] = \{h^{-1}(uv - vu)\}_0.$$

This implies that for any $a, b \in A$

$$\begin{aligned}\Delta_0([a_0, b_0]) = \{\Delta(h^{-1}(ab - ba))\}_0 = \{h^{-1}(\Delta(a)\Delta(b) - \Delta(b)\Delta(a))\}_0 \\ = [\Delta(a)_0, \Delta(b)_0] = [\Delta_0(a_0), \Delta_0(b_0)].\end{aligned}$$

Thus, Δ_0 is a Poisson algebra homomorphism.

Let us show that (S, ν) is a co-Poisson bialgebra. The equality $\mathrm{Perm}_S \circ \nu = -\nu$ is immediate. Put $\Delta^2 = (1 \otimes \Delta)\Delta = (\Delta \otimes 1)\Delta$. A direct computation shows that for $a \in A$

$$(1 \otimes \nu)\nu(a_0) = \{h^{-2}(1-\sigma)(1-\tau)(\Delta^2(a))\}_0 \in S^{\otimes 3}$$

where σ and τ are automorphisms of $A^{\otimes 3}$ sending $\alpha \otimes \beta \otimes \gamma$ resp. in $\alpha \otimes \gamma \otimes \beta$ and $\gamma \otimes \alpha \otimes \beta$. The operator $\tau^2 + \tau + 1$ annihilates $(1-\sigma)(1-\tau)$ which implies (7.1.1). An easy calculation shows that both sides of (7.2.2) applied to $a_0 \in S$ give

$$\{h^{-1}(1-\tau)(\Delta^2(a))\}_0.$$

To finish the proof we should check (16.1.1) and (7.2.3). For $a \in A$ put

$$\bar{a} = \hbar^{-1}(\Delta(a) - \mathrm{Perm}_A(\Delta(a))) \in A^{\otimes 2}.$$

A direct computation shows for all $a, b \in A$

$$\nu([a_0, b_0]) - [\Delta_0(a_0), \nu(b_0)] - [\nu(a_0), \Delta_0(b_0)] = \{\hbar h^{-1}(\bar{b}\bar{a} - \bar{a}\bar{b})\}_0 = 0.$$

Similarly, (7.2.3) follows from the obvious equality $(\hbar\,\bar{a}\,\bar{b})_0 = 0$.

17. Biquantization of Z_0

17.1. Biquantization of Lie bialgebras. — With each Lie bialgebra $\mathfrak{g}$ over K we associate a bi-Poisson bialgebra $S(\mathfrak{g})$ over K. As bialgebra $S(\mathfrak{g})$ is just the symmetric algebra of the K-module $\mathfrak{g}$ equipped with the comultiplication which sends $a \in \mathfrak{g}$ into

$a\otimes 1+1\otimes a$. According to results of Section 1.1 the Lie bracket of $\mathfrak{g}$ extends to Poisson bracket in $S(\mathfrak{g})$. Thus $S(\mathfrak{g})$ becomes a Poisson bialgebra. [It suffices to check (7.2.1) for a, $b\in\mathfrak{g}$ which is straighforward). On the other hand $S(\mathfrak{g})=V_h(\mathfrak{g})/hV_h(\mathfrak{g})$ and therefore the co-Poisson bialgebra structure in $V_h(\mathfrak{g})$ produced by Theorem 7.4 induces a co-Poisson bialgebra structure in $S(\mathfrak{g})$. This makes $S(\mathfrak{g})$ a bi-Poisson bialgebra. Indeed, it suffices to check (16.1.1) for a, $b\in\mathfrak{g}$ when it is equivalent to (7.1.2).

Note that the projection $v: V_h(\mathfrak{g})\to S(\mathfrak{g})$ is a co-Poisson bialgebra homomorphism and simultaneously a quantization of the Poisson bracket in $S(\mathfrak{g})$.

By a biquantization of a Lie bialgebra $\mathfrak{g}$ over K we shall mean a reduced biquantization (16.2.5) of the bi-Poisson bialgebra $S=S(\mathfrak{g})$ such that $A_h=V_h(\mathfrak{g})$ and $q_h=v$. Note that each biquantization of $\mathfrak{g}$ includes a quantization $p_\hbar: A\to V_h(\mathfrak{g})$ of $\mathfrak{g}$, *cf*. Section 7.4.

By a normal biquantization of a spiral Lie bialgebra $\mathfrak{g}$ over K we mean a reduced biquantization (16.2.5) of the bi-Poisson bialgebra $S=S(\mathfrak{g})$ such that $A_h=V_h(\mathfrak{g})$, $q_h=v$ and $A_\hbar=\varepsilon_\hbar(\mathfrak{g})$, $q_\hbar=e$ (*see* Section 11.5).

A purely algebraic example of a (non-normal) biquantization of a Lie bialgebra is given in the Appendix to [20].

17.2. THEOREM. — *Let $p_0: A_0(F)\to S(Z)$ be the homomorphism which is linear over the augmentation $K[h,\hbar]\to K$ and which transforms the class of an oriented l-component link $L\subset F\times[0,1]$ into $\prod_{i=1}^{l}\langle\alpha_i\rangle_0$, where $\alpha_1,\ldots,\alpha_l$ are loops parametrizing the projections of components of* L *into* F. *The following diagram is commutative and presents a normal biquantization of* Z_0:

(17.2.1)
$$\begin{array}{ccc} A_0(F) & \xrightarrow{p_\hbar} & V_h(Z_0) \\ {\scriptstyle p_h}\downarrow & \searrow{\scriptstyle p_0} & \downarrow{\scriptstyle v} \\ \varepsilon_\hbar(Z_0) & \xrightarrow{e} & S(Z_0) \end{array}$$

(*for definitions of the homomorphisms v, $p_\hbar$, e, p_h see resp. Sections* 1, 10, 11, 13).

Proof. — The homomorphism p_0, $v\circ p_\hbar$ and $e\circ p_h$ are linear over the augmentation $K[h,\hbar]\to K$. It is easy to see that they coincide on the classes of knots in $A_0(F)$. This implies commutativity of the diagram. Other claims of the Theorem follow from the results of Chapters I-III and Section 17.1.

17.3. *Remark*. — The same trick as in Remark 10.3.2 shows that the biquantization (17.2.1) of Z_0 induces a normal biquantization of Z.

18. An extension of Z_0 and its biquantization

18.1. LIE BIALGEBRAS $\tilde{Z}$ AND $\tilde{Z}_0$. — The Lie bialgebra $\tilde{Z}$ is defined along the same lines as Z though instead of free homotopy types of loops in F one uses regular homotopy types of immersed loops in F. By an immersed loop in F we mean an immersion of the oriented circle in F. Two immersed loops are regularly homotopic if they may be smoothly deformed into each other in the class of immersed loops. Let $\tilde{Z}$ be the free K-module freely generated by the regular homotopy classes of immersed loops in F. The infinite cyclic group $\{t^n \mid n \in \mathbb{Z}\}$ acts in $\tilde{Z}$ as follows: the action of t (resp. t^{-1}) adds one small curl to the right (resp. to the left) of immersed loops. Thus, $\tilde{Z}$ acquires a structure of $K[t, t]^{-1}$-module. The same constructions as in Section 8 make $\tilde{Z}$ a Lie bialgebra over $K[t, t^{-1}]$. Clearly, $Z=\tilde{Z}/(t-1)\tilde{Z}$. In the case of parallelizable F a study of $\tilde{Z}$ may be comletely reduced to a study of Z. Indeed, each parallelization of F induces a Lie bialgebra isomorphism $K[t, t^{-1}]\otimes_K Z \to \tilde{Z}$ which transforms $t^n \otimes \langle \alpha \rangle$, where α is a loop on F, into the class of an immersed loop on F which is freely homotopic to α and which has the total rotation angle $2\pi n$. The equality $\tilde{Z}=K[t, t^{-1}]\otimes Z$ implies that $\tilde{Z}$ is a free $K[t, t^{-1}]$-module. The latter holds also for closed F, distinct from S^2. This may be deduced purely algebraically from the following easy assertion which holds true for an arbitrary oriented F. Let $M \to F$ be the bundle of unitary tangent vectors of F. Clearly, each immersed loop $\alpha: S^1 \to F$ lifts to a loop $\tilde{\alpha}(x)=(\alpha(x), \alpha'(x)/\|\alpha'(x)\|)$ in M, where $x \in S^1$. Then the formula $\alpha \mapsto \tilde{\alpha}$ defines a bijection of the set of regular homotopy classes of immersed loop in F onto the set of conjugacy classes in $\pi_1(M)$.

The same argument as in Section 13 shows that the Lie bialgebra $\tilde{Z}$ is spiral for $F \neq S^2$. The Lie subbialgebra $\tilde{Z}_0$ of $\tilde{Z}$ generated by regular homotopy types of non-contractible immersed loops is spiral for all F. If $F \neq S^2$ then $\tilde{Z}=\tilde{Z}_0 \oplus K[t, t^{-1}]$.

18.2. Bialgebra $\tilde{A}$. — The bialgebra A(F) introduced in Sections 4, 9 has a canonical bialgebra extension $\tilde{A}$ which, however, is not a quotient of $\mathscr{A}(F)$. As the module $\tilde{A}$ is the quotient of the $K[h, \hbar, t, t^{-1}]$-module freely generated by the regular isotopy classes of oriented link diagrams on F, by the submodule generated by elements of two types: (i) the elements

$$\mathscr{D}_+ - \mathscr{D}_- - h_{|\mathscr{D}_+|-|\mathscr{D}_0|}\mathscr{D}_0$$

where $h_1=h$, $h_{-1}=\hbar$ and $\mathscr{D}_+$, $\mathscr{D}_-$, $\mathscr{D}_0$ run over all triples of non-empty oriented link diagrams on F which coincide outside a disk in F and look as in Figure 1 inside the disk; (ii) the elements $\mathscr{D}'-t\mathscr{D}$ (resp. $\mathscr{D}'-t^{-1}\mathscr{D}$), where $\mathscr{D}'$ is obtained from the non-empty diagram $\mathscr{D}$ by inserting a small either positive or negative curl to the right (resp. to the left) of $\mathscr{D}$.

Clearly $\tilde{A}/(t-1)\tilde{A}=A$. Placing one diagram over the other defines an associative multiplication in $\tilde{A}$. The class of the empty diagram is the unit of $\tilde{A}$. The same construction as in Section 9 gives a comultiplication in $\tilde{A}$, which makes $\tilde{A}$ a bialgebra over $K[h, \hbar, t, t^{-1}]$.

If δ is the class in $\tilde{A}$ of a trivial knot diagram then the quotient bialgebra $\tilde{A}_0=\tilde{A}/\delta\tilde{A}$ biquantizes the Lie bialgebra $\tilde{Z}_0$. All the relevant constructions and proofs follow the lines of Sections 1-17. Note that if F is parallelizable then $\tilde{A}_0=K[t, t^{-1}]\otimes_K A_0$, so the only interesting case here is the case of closed F.

REFERENCES

[1] V. G. DRINFELD, *Hamiltonian Structures on Lie Groups, Lie Bialgebras and the Meaning of the classical Yang-Baxter Equation* (*Doklady A.N. S.S.S.R.*, Vol. 268, 1982, pp. 285-287; English translation: *Soviet Math. Dokl.*, Vol. 27, No. 1, 1983).

[2] V. G. DRINFELD, *Quantum Groups* (*Proc. Internat. Congress Math.*, Berkeley, 1986, Amer. Math. Soc., Providence, R.I., 1987, pp. 798-820).

[3] W. M. GOLDMAN, *The Symplectic Nature of Fundamental Groups of Surfaces* (*Adv. Math.*, Vol. 54, 1984, pp. 200-225).

[4] W. M. GOLDMAN, *Invariant Functions on Lie Groups and Hamiltonian Flows of Surface Group Representations* (*Invent. Math.*, Vol. 85, 1986, pp. 263-302).

[5] J. HOSTE and M. K. KIDWELL, *Dichromatic Link Invariants*, (*Trans. Amer. Math. Soc.*, Vol. 321, 1990, pp. 197-229).

[6] F. JAEGER, *Composition Products and Models for the Homfly Polynomial, preprint, Grenoble*, 1988.

[7] L. H. KAUFFMAN, *New Invariants in the Theory of Knots* (*Am. Math. Mon.*, Vol. 95, 1988, pp. 195-242).

[8] M. KONTSEVICH, *Rational Conformal Field Theory and Invariants of* 3-*Manifolds*, preprint, 1989.

[9] W. B. R. LICKORISH, *Polynomials for Links* (*Bull. London Math. Soc.*, Vol. 20, 1988, pp. 558-588).

[10] W. B. R. LICKORISH and K. C. MILLETT, *A Polynomial Invariant of Oriented Links* (*Topology*, Vol. 26, 1987, pp. 107-141).

[11] J. H. PRZYTYCKI, *Skein Modules of* 3-*manifolds*, preprint, 1987.

[12] N. Y. RESHETIKHIN, *Quantized Universal Enveloping Algebras, the Yang-Baxter Equation and Invariants of Links*, I and II, L.O.M.I. preprints E-4-87 and E-17-87, Leningrad, 1988.

[13] N. Y. RESHETIKHIN and V. G. TURAEV, *Invariants of* 3-*manifolds via Link Polynomials and Quantum Groups*, (*Invent. Math.*, Vol. 103, 1991, pp. 547-597).

[14] M. A. SEMENOV-TIAN-SHANSKY, *Dressing Transformations and Poisson Group Actions* (*Publ. Res. Inst. Math. Sci. Kyoto Univ.*, Vol. 21, 1985, pp. 1237-1260).

[15] J.-P. SERRE, *Lie Algebras and Lie Groups*, Benjamin, New York-Amsterdam, 1965.

[16] V. G. TURAEV, *Intersections of Loops in Two-Dimensional Manifolds* (*Matem. Sbornik*, Vol. 106, 1978, pp. 566-588; English translation: *Math. U.S.S.R. Sbornik*, Vol. 35, 1979, pp. 229-250).

[17] V. G. TURAEV, *Multivariable Generalizations of the Seifert form of Classical Knot* (*Matem. Sbornik*, Vol. 116, 1981, pp. 370-397; English translation: *Math. U.S.S.R. Sbornik*, (Vol. 44, 1983, pp. 335-361).

[18] V. G. TURAEV, *The Yang-Baxter Equation and Invariants of Links* (*Invent. Math.*, Vol. 92, 1988, pp. 527-553).

[19] V. G. TURAEV, *The Conway and Kauffman Modules of the Solid Torus* (*Zap. nauchn. sem. L.O.M.I.*, Vol. 167, 1988, pp. 79-89; English translation: *J. Soviet. Math.*).

[20] V. G. TURAEV, *Algebras of Loops on Surfaces, Algebras of Knots and Quantization*, preprint L.O.M.I., Leningrad, 1988.

[21] J.-L. VERDIER, *Groupes quantiques* (*d'après V. G. Drinfel'd*) (*Astérisque*, Vol. 152-153, 1987, pp. 305-319).

[22] E. Witten, *Quantum Field Theory and the Jones Polynomial*, (*Comm. Math. Phys.*, Vol. 121, 1989, pp. 351-399).
[23] S. Wolpert, *On the Symplectic Geometry of Deformations of Hyperbolic Surface* (*Ann. Math.*, Vol. 117, 1983, pp. 207-234).

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