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LOGARITHMIC DECAY AND OVERCONVERGENCE OF THE UNIT ROOT AND ASSOCIATED ZETA FUNCTIONS

BY B. DWORK AND S. SPERBER

Introduction

Let V_λ be a family of algebraic sets parametrized rationally by the points λ of an affine variety U defined over $\mathbf{F}_q$. Thus for each $\lambda_0 \in U$ algebraic over $\mathbf{F}_q$ we have an algebraic set V_{λ_0} defined over $\mathbf{F}_q(\lambda_0)$. Let $\zeta(V_{\lambda_0}, t)$ be the corresponding zeta function. We fix an embedding of the algebraic closure of $\mathbf{Q}$ into the algebraic closure of $\mathbf{Q}_p$ and define the unit (resp.: p^ν) root factor, $\zeta_u(V_{\lambda_0}, t)$, to be the product of factors $(1-t\alpha)^{\pm 1}$ of $\zeta(V_{\lambda_0}, t)$ such that $|\alpha|=1$ (resp. $|\alpha|=|p^\nu|$) for the indicated p-adic valuation. The unit root zeta function of the family is

$$\zeta_u(t)=\prod \zeta_u(V_\lambda, t^{\deg \lambda})^{1/\deg \lambda}$$

the product being over all $\lambda \in U$ algebraic over $\mathbf{F}_q$. The definition may be modified in a number of inessential ways. The product may be restricted to a subset U′ of U corresponding to non-supersingularity of V_λ. Furthermore the rational function $\zeta_u(V_\lambda, t^{\deg \lambda})$ may be replaced by the characteristic polynomial of the Frobenius acting on the cohomology of fixed dimension. A similar unit root object may also be assigned in the case of L-functions associated with character sums. The methods of this paper apply with minor modifications to such unit-root L-functions as well.

The first mention of such a zeta function seems to have been in [Dw 0] where ζ_u was shown to be meromorphic in a non-trivial example. Further examples were given in [Dw 2]. These functions have been examined by Katz [Ka 2], Crew [C] and more recently (under the hypotheses that U be smooth and proper) by Etesse, [E].

In the present article we provide a method which shows that starting with any over convergent cohomology theory one can show that the unit root zeta function is meromorphic on a disk $\operatorname{ord}_q t > (p-1)/(p+1)$. In fact our treatment seems to suggest the utility of the ring of functions having logarithmic decay (intermediate between rings of convergent and overconvergent functions). Like the convergent theory, in the logarithmic decay case, valuation-based "sub and quotient objects" again belong to the given category, and the associated L-functions over converge. To make our result more understandable we give (§ 12) a completely non-cohomological treatment of the generic family of hypersurfaces.

More generally we show that if A is a Frobenius matrix analytic on the set $\mathscr{D}_f = \{ |x| \leqq 1, |f| = 1 \}$ in n-space with logarithmic decay at the boundary in the sense of Section 6 below then the associated unit root zeta function is meromorphic on the disk indicated above. By induction one can then treat the p^{ν} root zeta function.

The main point of the present work is that the notion of logarithmic decay is made precise, that the logarithmic decay of the fixed point of the mapping θ [(7.2) below] is clearly established and that the relation between logarithmic decay and convergence of unit root zeta function is clarified (§ 11). This program has existed since the mapping θ was introduced by the first named author [Dw 1]. Sperber and Sibuya [S-S] demonstrated logarithmic decay for the logarithmic derivative of ${}_0F_N(1, \ldots, 1; \pi^{N+1} x)$ by a direct calculation and explicitly pointed out the application to the unit root zeta function. The present work is applicable to their example and more generally to solutions of matrix Riccati equations appearing in the theory of normalized period matrices ([Dw 1], Lemma 5.1 (ii)).

In particular the present results apply to the old favorite, the logarithmic derivative of ${}_2F_1(1/2, 1/2; 1; x)$ [as indeed was also conjectured by N. Koblitz (letter of 12/13/75)]. This case can also be treated by means of the excellent lifting of Frobenius, φ_0, so that θ [equation (7.2)] may in this case be replaced by the simpler mapping $\eta \mapsto h + \varphi_0' (\eta \circ \varphi_0)$, where h is overconvergent relative to the Hasse domain.

We observe that the example of [S−S] suggests that the condition $\alpha' < \operatorname{Inf}(\alpha, (p-1)/(p+1))$ of Section 7 might be replaceable by the condition $\alpha' < \operatorname{Inf}(\alpha, 1)$.

The following conjecture may clarify the relation between the present work and the general arithmetic theory of linear differential equations.

Let L be an n-th order linear differential operator defined over $\mathbf{C}_p(X)$. Let L_1 be monic operator of minimal order [defined over E, the completion of $\mathbf{C}_p(X)$ under the gauss norm] which annihilates those elements u analytic and *bounded* on the generic unit disk $D(t, 1^-)$ which lie in the kernel of L. We conjecture that the coefficients of L_1 have logarithmic decay in the sense of the present article.

1. Logarithmic type functions

Let $m \geqq 2$, $q \geqq 2$. Let $\rho = \rho_{m, q}$ be the piecewise linear function on the positive real line which takes the values

$$\rho(s) = 0 \qquad \textit{for} \quad 0 \leqq s \leqq m$$
$$\rho(m, q^t) = t \qquad \textit{for} \quad t = 0, 1, \ldots$$

Explicitly for $t = 0, 1, \ldots$

$$\rho(s) = t + \frac{s - m q^t}{m(q-1) q^t} \qquad \textit{for} \quad m q^t \leqq s \leqq m q^{t+1}$$

We shall use $\rho'(s)$ to denote the derivative (resp.: left hand and right hand derivatives) if s is not mq^t (resp.: is mq^t). For $s>m$, $\rho'(s)$, is monotonically decreasing and the right hand derivative never exceeds the left hand derivative. The purpose of this section is to describe the basic properties of ρ which we will need in subsequent sections.

LEMMA 1. (i) *For* $x \geqq 0$, $\rho(mq^t+x) \leqq t+x/m(q-1)q^t$

(ii) $(1/(q-1))+\rho(s_1)+\rho(s_2) \geqq \rho(s_1+s_2)$

(iii) $(1/(q-1)q^t)+\rho(s_1)+\rho(s_2) \geqq \rho(s_1+s_2)$, *if* $s_1+s_2 \geqq m+mq^t$

(iv) $\rho_{m_1}(s_1)+\rho_{m_2}(s_2) \geqq \rho_{m_1+m_2}(s_1+s_2)$

(v) *Let* $1 \leqq c \leqq q$. *For* $s \geqq cm$

$$\rho(s)-\rho(s/c) \geqq \frac{c-1}{q-1}.$$

(vi) *For* $s \geqq 0$.

$$\rho(sq) \leqq 1+\rho(s).$$

(vi') *For* $s \geqq m$, $\rho(sq)=1+\rho(s)$.

(vii) $\rho_{mq,q}(s)=\rho_m(s/q)$.

(viii) *For* $a \in \mathbf{N}$, $\rho_{m.q}(s) \geqq a\,\rho_{mq^a,\,q^a}(s)$.

For ease of exposition, we present the proof of this lemma in the appendix (§ 13).

2. Amice ring in N-space

Let K be a field of characteristic zero complete under a rank one ultrametric norm. Let $\mathcal{O}_K$ be the ring of integers and let f be an element of $\mathcal{O}_K[X_1, \ldots, X_n]$ regular in X_n of degree d, *i.e.* $\deg f=d$ and X_n^d has unit coefficient in f. We define the gauss norm on $\mathcal{O}_K[[X]]$,

$$\left|\sum a_u X^u\right|_{\text{gauss}}=\sup_u |a_u|.$$

Let

$$\hat{L}_f=\left\{\sum_{j=0}^{d-1}\sum_{s\in\mathbf{Z}}\xi_{j,s}X_n^j f^s \,\middle|\, \xi_{j,s}\in\mathcal{O}_K[[X_1, \ldots, X_{n-1}]], |\xi_{j,s}|_{\text{gauss}}\to 0 \text{ as } s\to-\infty\right\}.$$

To define multiplication in $\hat{L}_f$ we write

$$\sum_{j=0}^{d-1}\sum_{s\in\mathbf{Z}}\xi_{j,s}X_n^j f^s\cdot\sum_{j=0}^{d-1}\sum_{s\in\mathbf{Z}}\xi'_{j,s}X_n^j f^s=\sum_t f^t\sum_{\substack{s+s'=t\\0\leqq j,j'<d}}X_n^{j+j'}\xi_{j,s}\xi'_{j',s'}$$

which is formally well defined since

$$\sum_{s+s'=t} \xi_{j,s}\xi'_{j',s'} = \sum_{s \in \mathbf{Z}} \xi_{j,s}\xi'_{j',t-s}$$

which clearly converges in the gauss norm on $\mathcal{O}_K[[X_1, \ldots, X_{n-1}]]$. Denote this sum by $\tau_{j,j',t}$. To complete the calculation we must write for $\alpha \leqq 2(d-1)$

$$X_n^\alpha = \sum_{i=0}^{d-1} X_m^i \rho_{\alpha,i} + f \sum_{i=0}^{d-1} X_n^i \sigma_{\alpha,i}$$

where $\rho_{\alpha,j}$ and $\sigma_{\alpha,j} \in \mathcal{O}_K[X_1, \ldots, X_{n-1}]$. (Of course for $\alpha \leqq d-1$ we have $\rho_{\alpha,i} = \delta_{\alpha,i}$, $\sigma_{\alpha,i} = 0$, while for $\alpha \geqq d$, $\sigma_{\alpha,i} = \delta_{\alpha-d,i}$.) The product may be written

$$\sum_{t \in \mathbf{Z}} \sum_{i=0}^{d-1} f^t X_n^i \sum_{j,j'=0}^{d-1} (\tau_{j,j',t}\rho_{j+j',i} + \tau_{j,j',t-1}\sigma_{j+j',i}).$$

Thus $\hat{L}_f$ is a ring with a norm

$$\left|\sum \xi_{j,s} X_n^j f^s\right| = \operatorname{Sup}_{j,s} |\xi_{j,s}|_{\text{gauss}}.$$

Under this norm, $\hat{L}_f \otimes K$ is a banach space. We are unable to show that $\hat{L}_f$ may be identified with $\hat{L}_{f'}$ if f' is also regular in X_n and with the same image in $\bar{K}[X]$ as f. (Here $\bar{K}$ denotes the residue class field of K.)

Each element of $\hat{L}_f$ may be written uniquely in the form

$$\xi = \sum a_{u.s} X^u f^s \tag{2.1}$$

the sum being over all $s \in \mathbf{Z}$, all $u \in \mathbf{N}^n$, $u_n < d$. For $u \in \mathbf{N}^n$ we put $\|u\| = u_1 + \ldots + u_n$. Here all $a_{u,s} \in \mathcal{O}_K$ and $|a_{u,s}| \to 0$ uniformly (with respect to u) as $s \to -\infty$.

We define the *restricted Amice ring*, L_f, to consist of all ξ which satisfy the further condition that $a_{u,s} \to 0$ as $\|u\| + |s| \to \infty$. The norm of $\hat{L}_f$ induces a norm on L_f, $|\xi|_f$ and

$$|\xi|_f |\eta|_f \geqq |\xi\eta|_f.$$

3. Restricted Amice Ring (*cf.* [R], [B])

Let $f \in \mathcal{O}_K[X]$ be regular with respect to X_n as in Section 2. Let Ω be an extension field of K which is algebraically closed and complete under a valuation extending that of K. Let $\mathcal{O}_\Omega$ be the ring of integers of Ω.

Let $\mathcal{D}_f = \{x \in \mathcal{O}_\Omega^n \mid |f(x)| = 1\}$. Certainly $\mathcal{D}_f$ depends only on $\bar{f}$, the image of f in $\bar{K}[X]$.

Let

$$H_f = \{ g/h \in K(X) \mid h \text{ has no zero in } \mathcal{D}_f \}$$

By Reich [R], lemma A, the elements of H_f are bounded on $\mathcal{D}_f$ and we denote by $| \ |_{\mathcal{D}_f}$ the sup norm on $\mathcal{D}_f$. The space $H(\mathcal{D}_f)$ of analytic elements on $\mathcal{D}_f$ is defined to be the completion of H_f under the sup norm.

Clearly each element of L_f may be viewed as a function on $\mathcal{D}_f$ and in fact may be identified with an element of $H_0(\mathcal{D}_f)$, the set of all $\xi \in H(\mathcal{D}_f)$ bounded by unity on $\mathcal{D}_f$.

LEMMA. — *The natural mapping of* L_f *into* $H_0(\mathcal{D}_f)$ *is an isomorphism and an isometry.*

Proof. — We first show the isometry. It is clear that for $\xi \in \hat{L}_f$, $|\xi|_{\mathcal{D}_f} \leqq |\xi|_f$. To reverse the inequality let $|\xi|_f = 1$ but $|\xi|_{\mathcal{D}_f} = \varepsilon < 1$. By hypothesis in the representation (2.1), $|a_{u,s}| \leqq \varepsilon$ for all (u, s) outside of a finite set, B. The reduction modulo the prime ideal of $\mathcal{O}_K$ is thus

$$\bar{\xi} = \sum_{(u,s) \in B} \bar{a}_{u,s} X^u \bar{f}^s \in \bar{K}[X, \bar{f}^{-1}].$$

By hypothesis $\bar{\xi}(\bar{X}) = 0$ for all $\bar{X} \in \bar{K}^n$ such that $\bar{f}(\bar{X}) \neq 0$. Hence $\bar{\xi} = 0$ as element of $\bar{K}(X)$. It follows from the regularity of $\bar{f}$ that $\bar{a}_{u,s} = 0$ for all $(u, s) \in B$. This shows that $|a_{u,s}| < 1$ for all $(u, s) \in B$ and since B is finite there exists $\varepsilon' < 1$ such that $|a_{u,s}| \leqq \varepsilon'$ for all $(u, s) \in B$. This shows that $|a_{u,s}| \leqq \varepsilon'' = \sup(\varepsilon, \varepsilon') < 1$ for all (u, s), contradicting the hypothesis that $|\xi|_f = 1$.

Since the mapping is an isometry, it must be an injection. To show surjectivity it is enough to show that each element of H_f may be represented by an element of L_f. Since each element of H_f is a ratio g/h of polynomials and the representation of g is trivial, we may restrict our attention to $1/h$ where $h \in \mathcal{O}_K[X]$, $\bar{h} \neq 0$, and h has no zero in $\mathcal{D}_f$. We may conclude that each zero of $\bar{h}$ in $\bar{\Omega}^n$ is also a zero of $\bar{f}$ [as otherwise the zero in $\bar{\Omega}$ could be lifted to a zero of h in $\mathcal{O}_\Omega^n$ at which $|f(x)| = 1$]. ([R], lemma A). By the Hilbert nullstellensatz, a power of $\bar{f}$ lies in the ideal generated by $\bar{h}$, *i.e.* there exists $s \in N$ such that $\bar{f}^s = \overline{hk}$ for some $k \in \mathcal{O}_K[X]$. Thus putting $\rho = f^s - hk$, we conclude that $\rho \in \mathcal{O}_K[X]$, $|\rho|_{\text{gauss}} = \varepsilon < 1$, and so

$$1/h = kf^{-s} \sum_{j=0}^{\infty} (\rho/f^s)^j.$$

Certainly ρ/f^s is the image of an element η of L_f, $|\eta|_f = |\eta|_{\mathcal{D}_f} = \varepsilon < 1$ and so $\sum_{j=0}^{\infty} \eta^j \in L_f$ and has image $(1 + \rho/f^s)^{-1}$ in $H(\mathcal{D}_f)$. The assertion is now clear.

4. Tempered Amice spaces

Let g be an unbounded, monotonically increasing mapping of $\mathbf{R}_+$ into itself. Let $f \in \mathcal{O}_K[X]$ be regular relative to X_n. We define a subset $L_{g,f}$ of L_f

$$L_{g,f} = \{ \sum_{\substack{u_n < d \\ s \in \mathbf{Z}}} a_{u,s} X^u f^s \mid \operatorname{ord} a_{u,s} \geqq g(\|u\| + d|s|) \}.$$

We may view $L_{g,f} \otimes K$ as a banach space under the norm associated with g. The elements of $L_{g,f} \otimes K$ may be identified with a subspace of $H(\mathscr{D}_f)$.

LEMMA (*Simplified Criterion*):

$$L_{g,f} = \{ \sum_{\substack{v \in \mathbf{N}^n \\ s \in \mathbf{Z}}} a_{v,s} X^v f^s \mid \operatorname{ord} a_{v,s} \geqq g(\| v \| + d \mid s \mid) \}.$$

(The point here is that v_n may exceed d in this sum.)

Proof. – The indicated sum $\eta = \sum a_{v,s} X^v f^s$ obviously lies in $H(\mathscr{D}_f)$ and hence has a representation

$$\eta = \sum_{\substack{u_n < d \\ t \in \mathbf{Z}}} c_{u,t} X^u f^t$$

in terms of reduced monomials. To determine this representation we write

$$X^v = \sum_{\substack{i \geqq 0 \\ u_n < d \\ \| u \| + di \leqq \| v \|}} b_{v,u,i} X^u f^i, \qquad b_{v,u,i} \in \mathscr{O}_K$$

and so

$$c_{u,t} = \sum_{\substack{i \geqq 0 \\ \| v \| \geqq \| u \| + di}} b_{v,u,i} a_{v,t-i}$$

a sum which converges since $a_{v,t-i} \to 0$ as either $\| v \|$ or i goes to infinity. Now in the region of this sum

$$\| v \| + d \mid t - i \mid \geqq \| u \| + di + d \mid t - i \mid \geqq \| u \| + d \mid t \mid$$

and so

$$\operatorname{ord} a_{v,t-i} \geqq g(\| v \| + d \mid t - i \mid) \geqq g(\| u \| + d \mid t \mid)$$

which shows that $c_{u,t}$ satisfies the required condition.

PROPOSITION. — $L_{g,f}$ *is a ring if and only if*

(4.1) $$g(t_1) + g(t_2) \geqq g(t_1 + t_2)$$

for all $t_1, t_2 \in \mathbf{N}$.

Proof. – The example $X_1^{t_1} . X_1^{t_2} = X_1^{t_1 + t_2}$ shows the condition to be necessary. To show sufficiency it is enough to check that if $a X^u f^s$, $b X^v f^t$ lie in $L_{g,f}$ with $\sup(u_n, v_n) < d$, then $\operatorname{ord}(ab) \geqq g(\| u + v \| + d \mid s + t \mid)$, *i.e.*

$$g(\| u \| + d \mid s \mid) + g(\| v \| + d \mid t \mid) \geqq g(\| u + v \| + d \mid s + t \mid)$$

which is an immediate consequence of the hypothesis.

NOTATION. — For $\varepsilon > 0$, let

$$L_{g,f}(\varepsilon) = \{ \xi = \sum_{u_n < d} a_{u,s} X^u f^s \mid \operatorname{ord} a_{u,s} \geqq g(\|u\| + d|s|) + \varepsilon.$$

5. Units

LEMMA. — *The units of the restricted Amice ring are characterized by the condition*

(5.1) $$|\xi(x)| = 1, \qquad \forall x \in \mathcal{D}_f.$$

Proof. — If ξ is a unit of $H_0(\mathcal{D}_f)$ then both ξ and ξ^{-1} are bounded by unity on $\mathcal{D}_f$ which implies (5.1). Conversely if $\xi \in H(\mathcal{D}_f)$ is bounded away from zero on $\mathcal{D}_f$ then ξ^{-1} lies in $H(\mathcal{D}_f)$. Thus if $\xi \in H(\mathcal{D}_f)$ satisfies (5.1) then it and its reciprocal lie in $H_0(\mathcal{D}_f)$ as asserted.

Let g satisfy (4.1) so that $L_{g,f}$ is a ring. Certainly the units of $L_{g,f}$ lie in $L_f^\times$ the group of units of L_f. We cannot conclude that $L_{g,f} \cap L_f^\times$ lies in the group of units of $L_{g,f}$.

In fact in our application we must consider the case in which g does not satisfy (4.1). We will need a weak statement about $L_{g,f} \cap L_f^\times$ lies in the group of units of $L_{g,f}$.

In fact in our application we must consider the case in which g does not satisfy (4.1). We will need a weak statement about $L_{g,f} \cap L_f^\times$. This will appear in the next section.

6. Logarithmic modules

We assume that $f \in \mathcal{O}_K[X]$ and is regular with respect to X_n. For $\alpha \geqq 0$ we set (*cf.* § 1)

$$g_{\alpha,m,q} = \alpha \rho_{m,q}$$

$$G_{\alpha,m,q} = \alpha \left(\frac{1}{q-1} + \rho_{m,q} \right).$$

To simplify the notation we write

$$L_{\alpha,m} \text{ (or occasionally } L_{\alpha,m,q}) \quad \textit{for} \quad L_{g_{\alpha,m,q},f}$$

$$\tilde{L}_{\alpha,m} \text{ (or occasionally } \tilde{L}_{\alpha,m,q}) \quad \textit{for} \quad L_{G_{\alpha,m,q},f}$$

Note $\tilde{L}_{\alpha,m} \subseteq L_{\alpha,m}$. By condition (4.1), $\tilde{L}_{\alpha,m}$ is a ring but $1 \notin \tilde{L}_{\alpha,m}$. Furthermore, by Lemma 1 (ii), $L_{\alpha,m}$ is an $\tilde{L}_{\alpha,m}$-module. We would like to describe $L_{\alpha,m} \cap L_f^\times$. We achieve something less which is adequate for our purposes.

LEMMA. — *Let* $\alpha' \in \{0, \alpha)$. *For each* $\xi \in L_f^\times \cap L_{\alpha,m}$ *there exists* m' *in* $\mathbf{N}$ *such that* $\xi^{-1} \in L_{\alpha',m'}$.

Proof. — Let $\varepsilon > \alpha\alpha'/(\alpha-\alpha')$. We choose B_ε, a finite subset of all (u,s) appearing in the representation (2.1) of ξ such that

$$\operatorname{ord} a_{u,s} > \varepsilon \quad \text{for all } (u,s) \notin B_\varepsilon.$$

Let $\xi_1 = \sum a_{u,s} X^u f^s$, the sum being over all $(u,s) \in B_\varepsilon$. The image of ξ in $\bar{K}(X)$ may be written

$$\bar{\xi} = \sum_{(u,s) \in B_\varepsilon} \bar{a}_{u,s} X^u \bar{f}^s$$

and so $\bar{\xi}\bar{f}^N = \bar{h} \in \bar{K}[X]$ for some $N \in \mathbf{N}$. Since $\bar{f} = 0$ on the variety $\bar{h}(X) = 0$, we conclude that $\bar{f}^s = \bar{h}\bar{k}$ for some $\bar{k} \in \bar{K}[X]$. Thus $\bar{\xi}\bar{k}\bar{f}^{N-s} = 1$. Let $y_1 \in \mathcal{O}_K[X, f^{-1}]$ be a lifting of $\bar{k}\bar{f}^{N-s}$. We write $y_1 \xi_1 = 1 - z_1$, where $z_1 \in \mathcal{O}_K[X, f^{-1}]$. Certainly $\operatorname{ord} z_1 > 0$ in the topology of L_f. We choose $e \in \mathbf{N}$ such that $e \operatorname{ord} z_1 > \varepsilon$. Let $y = y_1(1 + z_1 + \ldots + z_1^{e-1})$ and so $y\xi_1 = 1 - z'$, $\operatorname{ord} z' > \varepsilon$. By definition

$$\operatorname{ord} y \sum_{(u,s) \notin B_\varepsilon} a_{u,s} X^u f^s > \varepsilon$$

and so $y\xi = 1 - z$, $\operatorname{ord} z > \varepsilon$.

Certainly y as element of $\mathcal{O}_K[X, f^{-1}]$ lies in L_{α,m_1} for some $m_1 \in \mathbf{N}$ and hence by lemma 1 (iv), $y\xi \in L_{\alpha,m_2}$ where $m_2 = m + m_1$.

The graph in the (t_1, t_2) plane, $t_1 \geqq 0$

$$t_2 = \sup(\varepsilon, \alpha t_1)$$

lies above the graph of

$$t_2 = \alpha'(1 + t_1) + \delta, \qquad \delta = \varepsilon \frac{\alpha - \alpha'}{\alpha} - \alpha' > 0.$$

Indeed the critical point is at $t_1 = \varepsilon/\alpha$ where the two graphs intersect.

We conclude that for $t \geqq 0$

$$\operatorname{Sup}(\varepsilon, \alpha, \rho_{m_2}(t)) \geqq \delta + \alpha'(1 + \rho_{m_2}(t))$$

and so $z \in \tilde{L}_{\alpha',m_2}(\delta)$, *i.e.* $\operatorname{ord}_{\tilde{L}_{\alpha',m_2}} z \geqq \delta > 0$. Thus $\sum_{j=1}^{\infty} z^j$ converges in the ring $\tilde{L}_{\alpha',m_2}$ while $y \in L_{\alpha,m_1}$ a subset of L_{α',m_1} and since $L_{\alpha',m_1} \subseteq L_{\alpha',m_2}$ and L_{α',m_2} is an $\tilde{L}_{\alpha',m_2}$-module, we find $\xi^{-1} = y(1-z)^{-1} \in L_{\alpha',m'}$ with $m' = m_2$. This concludes the proof.

Note. — We have no method for bounding m_1 and hence there is no point in trying to determine m' more precisely by these methods.

7. The fixed point (*cf.* [Dw 1], [Dw 2])

Let K_0 be a complete subfield of K which is unramified over $\mathbf{Q}_p$ and let σ be an automorphism of K extending the frobenius automorphism of $K_0/\mathbf{Q}_p$. Let $f \in \mathcal{O}_{K_0}[X]$, regular with respect to X_n. Let

$$H_0(\mathcal{D}_f) = \{ \xi \in H(\mathcal{D}_f) \mid |\xi|_f \leqq 1 \}.$$

Let ν_1, ν_2 be positive integers and for $i.j \in \{1, 2\}$ let B_{ij} be a $\nu_i \times \nu_j$ matrix with coefficients in $H_0(\mathcal{D}_f)$. We further assume

$$(7.1) \qquad |\det B_{11}(x)| = 1 \quad \text{for all } x \in \mathcal{D}_f.$$

For $h \in K(X)$ we define h^σ to be the element of $K(X)$ obtained by applying σ to the coefficients. Trivially σ maps H_f into H_{f^σ} and the mapping is an isometry relative to the sup norms on $\mathcal{D}_f$ and on $\mathcal{D}_{f^\sigma}$. Taking limits we obtain an isometry of $H(\mathcal{D}_f)$ with $H(\mathcal{D}_{f^\sigma})$. On the other hand $\xi \mapsto \xi^\varphi = \xi(X^p)$ maps $H(\mathcal{D}_{f^\sigma})$ into $H(\mathcal{D}_f)$ again an isometry since $\mathcal{D}_f = \mathcal{D}_{f^\sigma(X^p)}$. We consider a free module of rank $\nu_1 + \nu_2$ defined over $H(\mathcal{D}_f)$ with an action of Frobenius

$$\mathcal{Y} \to \mathcal{Y}^{\sigma\varphi} A, \qquad \text{for} \quad \mathcal{Y} \in H(\mathcal{D}_f)^{\nu_1 + \nu_2}$$

where A has the block form

$$A = \begin{pmatrix} B_{11} & B_{12} \\ p B_{21} & p B_{22} \end{pmatrix}.$$

We define a map θ

$$(7.2) \qquad \theta: \quad \eta \mapsto (B_{11} + p\,\eta^{\sigma\varphi} B_{21})^{-1} (B_{12} + p\,\eta^{\sigma\varphi} B_{22})$$

from $\mathcal{M}_{\nu_1, \nu_2}(H_0(\mathcal{D}_f))$ into itself. The purpose of the map θ is to identify the unit-root part of the given F-crystal. A fixed point $\eta \in \mathcal{M}_{\nu_1, \nu_2}(H_0(\mathcal{D}_f))$ of the map θ corresponds to a matrix $(I, \eta) \in \mathcal{M}_{\nu_1, \nu_1 + \nu_2}(H_0(\mathcal{D}_f))$ which is fixed under the Frobenius map above. Using η the unit-root sub-crystal is computed in Section 8 below.

LEMMA. — *The mapping θ above is a contractive map of $\mathcal{M}_{\nu_1, \nu_2}(H_0(\mathcal{D}_f))$ into itself and hence has a unique fixed point*, η_0.

(*Note:* We use $\bar{f} \neq 0$, but $\bar{f}$ need not be regular.)

Proof. — That θ is stable is quite clear. We may write

$$(7.3) \qquad \theta_0(\eta) = (I + p\,\eta^{\sigma\varphi} B_{21} B_{11}^{-1})^{-1}$$

$$(7.4) \qquad \theta_1(\eta) = B_{11}^{-1}\, \theta_0(\eta)\, B_{12}$$

(7.5) $$\theta_2(\eta)=p\,\theta_0(\eta)\,\eta^{\sigma\varphi}$$

(7.6) $$\theta_3(\eta)=B_{11}^{-1}\,\theta_2(\eta)\,B_{22}$$

$$\theta=\theta_1+\theta_3$$

It is enough to show θ_1 and θ_3 contractive and hence it is enough to show θ_0 and θ_2 contractive. Let $\eta_1, \eta_2 \in \mathcal{M}_{\nu_1, \nu_2}(H_0(\mathcal{D}_f))$. We write

(7.7) $$\theta_0(\eta_1)-\theta_0(\eta_2)=p\,\theta_0(\eta_1)(\eta_2-\eta_1)^{\sigma\varphi}B_{21}B_{11}^{-1}\theta_0(\eta_2)$$

(7.8) $$\theta_2(\eta_1)-\theta_2(\eta_2)=p(\theta_0(\eta_1)-\theta_0(\eta_2))\,\eta_1^{\sigma\varphi}+\theta_0(\eta_2)\,p(\eta_1-\eta_2)^{\sigma\varphi}$$

from which the assertion follows without difficulty.

PROPOSITION. — *Let g be a monotonic real valued function as in Section 4 satisfying the further condition that*

(7.9) $$j+g(t)\geqq g(t+2jd)$$

for all $j\in\mathbf{N}$, $t\in\mathbf{R}_+$. Let $\tilde g$ denote $t\to g(t/p)$. Then $\eta\mapsto\eta^{\sigma\varphi}$ maps $L_{g,f}$ into $L_{\tilde g, f}$.

Proof. — Let $\eta=\sum_{\substack{u\in\mathbf{N}^n\\ s\in\mathbf{Z}}} a_{u,s}X^u f^s$, ord $a_{u,s}\geqq g(\|u\|+d|s|)$. The main point is that

$$f^\sigma(X^p)=f^p+pk=f^p(1+pk/f^p)$$

where $k\in\mathcal{O}_K[X]$, deg $k\leqq pd$. Thus

$$\eta^{\sigma\varphi}=\sum_{\substack{u\in\mathbf{N}^n\\ s\in\mathbf{Z}\\ j\geqq 0}}\binom{s}{j}a_{u,s}^\sigma p^j X^{pu}k^j f^{p(s-j)}.$$

By hypothesis

$$\begin{aligned}\operatorname{ord}(a_{u,s}^\sigma p^j)&\geqq j+g(\|u\|+d|s|)\geqq g(\|u\|+d|s|+2jd)\\ &\geqq g(\|u\|+jd+d(|s|+j))\geqq g(\|u\|+jd+d|s-j|)\\ &=\tilde g(p\|u\|+jpd+dp|s-j|)\\ &\geqq \tilde g(\deg(X^{pu}k^j)+dp|s-j|)\end{aligned}$$

which by the simplified criterion of Section 4 completes the proof.

PROPOSITION. — *The hypothesis* (7.9) *of the preceding proposition is satisfied by $g=\alpha\rho_{m,p}$ provided $\alpha\leqq p-1$, $m\geqq 2d$.*

Proof. — By Lemma 1 (ii)

$$\begin{aligned}\alpha\rho(t+2dj)&\leqq\alpha j(p-1)^{-1}+\alpha\rho(t)+\alpha j\,\rho(2d)\\ &\leqq j+\alpha\rho(t).\end{aligned}$$

THEOREM. — *Let* $B \in \mathscr{M}_{v_1+v_2, v_1+v_2}(L_{\alpha, m, p})$. *We retain hypothesis* (7.1). *Let* $\alpha' \in (0, \alpha)$, $\alpha' < 1/(1+(2/(p-1)))$. *We assert that* η_0, *the fixed point of* (7.2), *lies in* $\mathscr{M}_{v_1, v_2}(L_{\alpha', m'})$ *for suitable* m'.

Proof. — It follows from Lemma 1 (iv) that det B_{11} lies in $L_{\alpha, v_1 m, p}$ and hence by Section 6 $(\det B_{11})^{-1}$ lies in L_{α', m_2} for suitable m_2.

Rechoosing m we may assume that the coefficients of B_{11}^{-1} and of B all lie in $L_{\alpha', m}$. Also with no loss in generality we may assume $m > 2d$. We put $\varepsilon = 1 - \alpha'(1+(2/(p-1)))$.

We will show that θ is a contractive mapping of $\mathscr{M}_{v_1, v_2}(L_{\alpha', 4m})$ into itself. Let η lie in this set. We first show θ is stable on this space. We assert that the coefficients of $p\eta^{\sigma\varphi} B_{21} B_{11}^{-1}$ lie in $\tilde{L}_{\alpha', 4m}(\varepsilon)$, more precisely we assert

(7.10) $$p\, L_{\alpha', 4m}^{\sigma\varphi} L_{\alpha', 4m} \subset \tilde{L}_{\alpha', 4m}(\varepsilon).$$

This is a consequence of

(7.10.1) $$1 + \alpha'(\rho_{4m}(t_1/p) + \rho_{4m}(t_2)) \geqq \varepsilon + \alpha'\left(\frac{1}{(p-1)} + \rho_{4m}(t_1+t_2)\right).$$

This is verified by using Lemma 1 (ii), (vi) (with $q = p$) to compute

(7.10.2) $$\frac{1}{p-1} + \rho_{4m}(t_1+t_2) \leqq \frac{2}{p-1} + \rho_{4m}(t_1) + \rho_{4m}(t_2)$$
$$\leqq 1 + \frac{2}{p-1} + \rho_{4m}(t_1/p) + \rho_{4m}(t_2)$$

and so (7.10.1) follows from the choice of ε.

Since $\tilde{L}_{\alpha', 4m}$ is a ring we conclude that for $y = -p\eta^{\sigma\varphi} B_{21} B_{11}^{-1}$, the series $\sum_{j=1}^{\infty} y^j$ converges to a matrix with coefficients in $\tilde{L}_{\alpha', 4m}(\varepsilon)$. We conclude that

(7.11) $$\theta_0(\eta) \in I_{v_1} + \mathscr{M}_{v_1, v_2}(\tilde{L}_{\alpha', 4m}(\varepsilon)).$$

We assert

$$\theta_1(\eta) \in \mathscr{M}_{v_1, v_2}(L_{\alpha', 4m}).$$

Since $B_{11}^{-1} B_{12}$ lies in this set, it is enough by (7.11) to show that

$$L_{\alpha', m}^2 \tilde{L}_{\alpha', 4m}(\varepsilon) \subset L_{\alpha', 4m}.$$

(Here $L_{\alpha', m}^2$ denotes the additive group generated by products of pairs of elements of $L_{\alpha', m}$.) By Lemma 1 (iv) $L_{\alpha', m}^2 \subset L_{\alpha', 2m} \subset L_{\alpha', 4m}$ and so it is enough to show

(7.12.1) $$L_{\alpha', 4m} \tilde{L}_{\alpha', 4m} \subset L_{\alpha', 4m}$$

which is a consequence of Lemma 1 (ii).

We assert

$$(7.13) \qquad \theta_3(\eta) \in \mathscr{M}_{\nu_1, \nu_2}(L_{\alpha', 4m}).$$

Using (7.11) it is enough to show

$$(7.13.1) \qquad p\, B_{11}^{-1}\, \eta^{\sigma\varphi}\, B_{22} \in \mathscr{M}_{\nu_1, \nu_2}(L_{\alpha', 4m})$$

$$(7.13.2) \qquad p\, L_{\alpha', m}^2\, \tilde{L}_{\alpha', 4m}\, L_{\alpha', 4m}^{\sigma\varphi} \subset L_{\alpha', 4m}.$$

Assertion (7.13.1) follows from $p\ L_{\alpha', m}^2\ L_{\alpha', 4m}^{\sigma\varphi} \subset p\, L_{\alpha', 4m}\ L_{\alpha', 4m}^{\sigma\varphi}$ which by (7.10) lies in $L_{\alpha', 4m}$. Assertion (7.13.2) follows by using Lemma 1 (ii), (iv) to obtain

$$L_{\alpha', m}^2\, \tilde{L}_{\alpha', 4m} \subset L_{\alpha', 4m}$$

and then using (7.10). This completes the proof of stability.

To check the contractive property let η_1, η_2 be elements of $\mathscr{M}_{\nu_1, \nu_2}(L_{\alpha', 4m})$' We assert

$$(7.14) \qquad \operatorname{ord}_{g_{\alpha', 4m}}(\theta_1(\eta_1) - \theta_1(\eta_2)) \geqq \varepsilon + \operatorname{ord}_{g_{\alpha', 4m}}(\eta_1 - \eta_2).$$

By (7.4), (7.7) and applying (7.11) to $\theta_0(\eta_1)$ and to $\theta_0(\eta_2)$ it is more than enough to show

$$(7.14.1) \qquad p\, L_{\alpha', m}^4\ L_{\alpha', 4m}^{\sigma\varphi} \subset \tilde{L}_{\alpha', 4m}(\varepsilon)$$

$$(7.14.2) \qquad p\, L_{\alpha', m}^4\, L_{\alpha', 4m}^{\sigma\varphi}\, \tilde{L}_{\alpha', 4m} \subset L_{\alpha', 4m}$$

Since $L_{\alpha', m}^4 \subset L_{\alpha', 4m}$, (7.14.1) is implied by (7.10). Since $\tilde{L}_{\alpha', 4m}$ is a ring (7.14.2) is now clear.

To prove θ_3 contractive we show

$$(7.15.1) \qquad \operatorname{ord}_{g_{\alpha', 4m}} p\, B_{11}^{-1}(\theta_0(\eta_1) - \theta_0(\eta_2))\, \eta_1^{\sigma\varphi}\, B_{22} \geqq \varepsilon + \operatorname{ord}_{g_{\alpha', 4m}}(\eta_1 - \eta_2)$$

$$(7.15.2) \qquad \operatorname{ord}_{g_{\alpha', 4m}} B_{11}^{-1}\, p\, \theta_0(\eta_2)(\eta_1 - \eta_2)^{\sigma\varphi}\, B_{22} \geqq \varepsilon + \operatorname{ord}_{g_{\alpha', 4m}}(\eta_1 - \eta_2).$$

For the first assertion we use (7.7) and (7.11) to remark that it is enough to show

$$(7.15.3) \qquad p^2\, L_{\alpha', m}^4\, (L_{\alpha', 4m}^{\sigma\varphi})^2 \subset \tilde{L}_{\alpha', 4m}(\varepsilon)$$

$$(7.15.4) \qquad p^2\, L_{\alpha', m}^4\, (L_{\alpha, 4m}^{\sigma\varphi})^2\, \tilde{L}_{\alpha', 4m} \subset L_{\alpha', 4m}.$$

Now (7.15.3) is a consequence of (7.14.1) and so (7.15.4) follows from the fact that $\tilde{L}_{\alpha', 4m}$ is a ring. Finally (7.15.2) follows from (7.11), $p\, L_{\alpha', m}^2\, L_{\alpha', 4m}^{\sigma\varphi} \subset \tilde{L}_{\alpha', 4m}(\varepsilon)$ and $p\, L_{\alpha', m}^2\, L_{\alpha', 4m}^{\sigma\varphi}\, \tilde{L}_{\alpha', 4m} \subset L_{\alpha', 4m}$ which in turn follows from (7.14.1).

8. The Matrix A

Let B be the matrix of Section 7. We consider $A=\begin{pmatrix} B_{11} & B_{12} \\ p\,B_{21} & p\,B_{22} \end{pmatrix}$. Let η be the fixed point of the last section. We compute the transform

$$\tilde{A}=\begin{pmatrix} I_{v_1} & \eta^{\sigma\varphi} \\ 0 & I_{v_2} \end{pmatrix} A \begin{pmatrix} I_{v_1} & \eta \\ 0 & I_{v_2} \end{pmatrix}^{-1}$$

$$=\begin{pmatrix} \tilde{A}_{11} & 0 \\ p\,B_{21} & p\,\tilde{B}_{22} \end{pmatrix}$$

where

$$\tilde{A}_{11}=B_{11}+p\,\eta^{\sigma\varphi}\,B_{21}$$

$$\tilde{B}_{22}=B_{22}-B_{21}\,\eta.$$

Suppose that $\bar{K}=\mathbf{F}_q$, $q=p^s$. We define

$$A_q(X)=A^{\sigma^{s-1}}(X^{p^{s-1}})\,A^{\sigma^{s-2}}(X^{p^{s-2}})\ldots A(X).$$

For each $\bar{x}\in\bar{\Omega}^n$ algebraic over $\mathbf{F}_q$ such that $\bar{f}(\bar{x})\neq 0$ we have an associated polynomial

$$\Delta_{A_q,\bar{x}}(t)=\det(I-t\,A_q(x^{q^{\deg\bar{x}-1}})\ldots A_q(x))$$

where x is the Teichmüller lifting of $\bar{x}$ and $\deg\bar{x}=\deg\mathbf{F}_q(\bar{x})/\mathbf{F}_q$. For such x, the reduction (modulo the maximal ideal) of $A_q(x^{q^{\deg\bar{x}-1}})\ldots A_q(x)$ has the form

$$\begin{pmatrix} \hat{B}_{11} & \hat{B}_{12} \\ 0 & 0 \end{pmatrix}$$

and $\det\hat{B}_{11}\neq 0$. Thus $\Delta_{A_q,\bar{x}}(t)$ has exactly v_1 roots which are the reciprocals of units and these are precisely the reciprocal roots of $\Delta_{\tilde{A}_q,\bar{x}}(t)$ where

$$\tilde{A}_q(X)=\tilde{A}_{11}^{\sigma^{s-1}}(X^{p^{s-1}})\ldots\tilde{A}_{11}(X).$$

We define the corresponding unit root zeta function to be

$$\zeta(t)=\prod_{\substack{\bar{x}\ \text{alg over}\ \mathbf{F}_q \\ \bar{f}(\bar{x})\neq 0}}\frac{1}{\Delta_{\tilde{A}_q,\bar{x}}(t^{\deg\bar{x}})^{1/\deg\bar{x}}}.$$

It follows from Section 7 that if the coefficients of B lie in $L_{\alpha,m,p}$ and if $\alpha'<\operatorname{Sup}(\alpha,(p-1)/(p+1))$ then there exists m' such that the coefficients of η lie in $L_{\alpha',m,p}$ and hence by (7.10) the coefficients of $\tilde{A}_{11}$ lie in the same set. Since by Lemma 1 (viii)

$$\sum_{i=0}^{s-1}\rho_{m,p}(t_i/p^i)=\sum_{i=0}^{s-1}\rho_{mp^i,p}(t_i)\geqq\rho_{m(1+p+\ldots+p^{s-1}),p}(t_0+\ldots+t_{s-1}),$$

we conclude that the coefficients of $\tilde{A}_q$ lie in $L_{\alpha', m'q, p}$ which by Lemma 1 lies in $L_{s\alpha', m'', q}$, where $m''=m' q^2$.

9. Basic trace formula for $H(\mathscr{D}_f)$ (*cf.* [B])

Let f, K, K_0, be as in Section 7 but let $\bar{K}_0=\mathbf{F}_q$. We define ψ_q a continuous endomorphism of $H(\mathscr{D}_f)$

$$(\psi_q \xi)(x)=q^{-n} \sum_{y^q=x} \xi(y).$$

The point here is that if $y^q=x\in\mathscr{D}_f$ then $y\in\mathscr{D}_f$ as certainly $y\in\mathscr{O}_\Omega^n$ and if $y\notin\mathscr{D}_f$ then $\bar{f}(\bar{y})=0$ which implies that $0=\bar{f}(\bar{y})^q=\bar{f}(\bar{x})$. Thus ψ_q is stable on H_f and as operator bounded by q^n. By taking limits ψ_q is extended to $H(\mathscr{D}_f)$

Let $F\in H(\mathscr{D}_f)$. Then $\alpha=\psi_q\circ F$, *i.e.* $\xi\mapsto\psi_q(\xi . F)$, is again an endomorphism of $H(\mathscr{D}_f)$. Using our basis $\{X^u f^s\}_{s\in\mathbf{Z},\, u_n<d}$ we associate a matrix $[\alpha]$ with α. We assert that $[\alpha]$ has a trace.

PROPOSITION. — *As operator on* $H(\mathscr{D}_f)$, $\|\psi\|=1$.

Proof. – The assertion is trivial for ψ_q as operator on polynomials. Furthermore we have the trivial estimate $\|\psi_q\|\leqq q^n$. By continuity it is enough to check the norm as map of $K[X, f^{-1}]$ into $H(\mathscr{D}_f)$. Let $\xi\in K[X, f^{-1}]$, $|\xi|_{\mathscr{D}_f}=1$. Choose $r\in\mathbf{N}$ such that $f^{qp^r}\xi\in K[X]$ and such that $|p^{r+1}/q^n|_p<1$. We know that $f(X^q)^{p^r}\equiv f(X)^{qp^r} \bmod p^{r+1}$ in the metric of $H(\mathscr{D}_f)$. Thus

$$f^{p^r}\psi_q(\xi)=\psi_q(f(X^q)^{p^r}\xi)\equiv\psi_q(\xi f^{qp^r}) \bmod p^{r+1}/q^n.$$

Since $|\psi_q(\xi f^{qp^r})|_{\mathscr{D}_f}\leqq 1$, the assertion is now clear.

The following trace formula is closely related to the trace formulas of Reich and of Monsky. The adaptation here is useful in the present context.

THEOREM. — *Let* $F\in H(\mathscr{D}_f)$. *For* $u_n<d$, $s\in\mathbf{Z}$ *we write*

$$\psi(FX^u f^s)=\sum_{\substack{u_n<d\\ s'\in\mathbf{Z}}} C_{u,s;u',s'} X^{u'} f^{s'}.$$

We assert that $\sum C_{u,s;u,s}$ *converges and that*

$$(q-1)^n \operatorname{trace} C=\sum_{\substack{X^{q-1}=(1,\ldots,1)\\ |f(X)|=1}} F(X).$$

Proof. — By continuity we may assume $F \in K[X, f^{-1}]$, $|F|_{\mathscr{D}_f} = 1$. Let $\alpha = \psi \circ F$, $\beta_r = \psi \circ F \cdot (f(X^q)/f(x))^{p^r}$. Trivially

$$\begin{aligned}\beta_r(X^u f^s) &= f(X)^{p^r} \alpha(X^u f^{s-p^r}) = \sum C_{u,\, s-p^r;u',\, s'} X^{u'} f^{s'+p^r}\\ &= \sum C_{u,\, s-p^r;u',\, s'-p^r} X^{u'} f^{s'}.\end{aligned}$$

Let $\alpha_r = \psi \circ F f^{(q-1)\,p^r}$. We write

$$\alpha_r(X^u f^s) = \sum A^{(r)}_{u,\, s;u',\, s'} X^{u'} f^{s'}.$$

Since

$$\frac{f(X^q)}{f(X)} \equiv f(X)^{q-1} \operatorname{mod} p,$$
$$\left(\frac{f(X^q)}{f(X)}\right)^{p^r} \equiv f(X)^{(q-1)\,p^r} \operatorname{mod} p^{r+1}$$

as elements of $H(\mathscr{D}_f)$. Thus as operators

$$|\alpha_r - \beta_r|_{\mathscr{D}_f} \leqq |p^{r+1}|$$

and so

$$A^{(r)}_{u,\, s;u',\, s'} \equiv C_{u,\, s-p^r;u',\, s'-p^r} \operatorname{mod} p^{r+1}$$

i.e.

$$C_{u,\, s;\, u',\, s'} \equiv A^{(r)}_{u,\, s+p^r;\, u',\, s'+p^r} \operatorname{mod} p^{r+1}.$$

Let $F = kf^l$, $k \in K[X]$, $l \in \mathbf{Z}$, $\deg k = D$. We restrict r to be so large that $l + (q-1)\,p^r \geqq 0$.

Case I. — Either $l \in \mathbf{N}$ or $q \geqq p^2$.

If $(q-1)\,p^r + s + l \geqq 0$ then $\alpha_r(X^u f^s)$ is a polynomial of degree bounded by $(1/q)(\|u\| + D + d(l + s + (q-1)\,p^r))$ and so $A^{(r)}_{u,\, s,\, u',\, s'} = 0$ if $s' < 0$ or if

$$\|u'\| + ds' > \frac{1}{q}(\|u\| + D + d(l + s + (q-1)\,p^r)).$$

Thus

$$C_{u,\, s;\, u',\, s'} \equiv 0 \operatorname{mod} p^{r+1}$$

if

$$qp^r + s + l \geqq 0$$

and either

$$s' + p^r < 0$$

or

$$\| u' \| + d(s' + p^r) > \frac{1}{q}(\| u \| + \mathrm{D} + d(l + s + p^r + (q-1)p^r))$$

i. e. if either $s' < -p^r$ or

$$(\gamma) \qquad \| u' \| + ds' > \frac{\| u \| + ds + (\mathrm{D} + dl)}{q}$$

Letting $r \to \infty$ we conclude that

$$\mathrm{C}_{u, s, u', s'} = 0 \qquad \text{if} \quad (\gamma) \text{ holds.}$$

We have also shown

$$\mathrm{C}_{u, s, u', s'} \equiv 0 \bmod p^{r+1} \qquad \text{if} \quad \begin{cases} s + l \geqq -qp^r \\ s' < -p^r \end{cases}.$$

In particular

$$\begin{aligned} \mathrm{C}_{u, s; u, s} &= 0 \qquad \text{if} \quad \| u \| + ds > (\mathrm{D} + dl)/(q-1) \\ &\equiv 0 \bmod p^{r+1} \qquad \text{if} \quad s \in [-l - qp^r, -p^r). \end{aligned}$$

Hence, if either $l \geqq 0$ or $q \geqq p^2$, then $-qp^{r+t} - l < -p^{r+t+1}$. Thus $(-\infty, -p^r) = \bigcup_{t \geqq 0} [-l - qp^{r+t}, -p^{r+t})$. Therefore, $\mathrm{C}_{u, s; u, s} \equiv 0 \bmod p^{r+1}$, if $s < -p^r$.

Thus we may compute trace C $\bmod p^{r+1}$ by summing over $s \geqq -p^r$, $\| u \| + ds < (\mathrm{D} + ld)/(q-1)$, a finite set. This demonstrates the existence of the trace of C in Case I but it also shows the existence of trace $\mathrm{A}^{(r)}$. Furthermore mod p^{r+1}

$$\begin{aligned} \text{Trace C} &\equiv \sum_{s \geqq -p^r,\, u_n < d} \mathrm{A}^{(r)}_{u, s+p^r; u, s+p^r} \\ &= \sum_{s \geqq 0,\, u_n < d} \mathrm{A}^{(r)}_{u, s; u, s} \\ &= \operatorname{trace}(\alpha_r \mid \mathrm{K}[\mathrm{X}]) \\ &= \frac{1}{(q-1)^n} \sum_{x^{q-1} = 1} \mathrm{F}(x) f(x)^{(q-1)p^r} \\ &\equiv \frac{1}{(q-1)^n} \sum_{\substack{|f(x)| = 1 \\ x^{q-1} = 1}} \mathrm{F}(x) \bmod p^{r+1}. \end{aligned}$$

This completes the proof in Case I.

Case II. $l<0$, $q=p$.

$F f^{(q-1)p^r}$ satisfies the condition of Case I [with l replaced by $l+(q-1)p^r$] and hence we may conclude that

$$\begin{aligned} A^{(r)}_{u,s;u,s} &= 0 \quad \text{if} \quad \|u\|+ds>(D+d(l+(q-1)p^r))/(q-1) \\ &\equiv 0 \bmod p^{r+1} \quad \text{if} \quad s<-p^r. \end{aligned}$$

Let $N_r=\{(u,s)\mid \|u\|+ds\leqq (D+d\,(l+(q-1)p^r))/(q-1),\ s\geqq -p^r\}$. Thus

$$\text{Trace } A^{(r)} = \sum_{(u,s)\in V} A^{(r)}_{u,s;u,s} \bmod p^{r+1}$$

where V is any finite set containing N_r. But then

$$\begin{aligned} \text{Trace } A^{(r)} &\equiv \sum_{(u,s)\in V} C_{u,s-p^r;u,s-p^r} \bmod p^{r+1} \\ &\equiv \sum_{(u,s)\in V'} C_{u,s;u,s} \bmod p^{r+1} \end{aligned}$$

where V′ is any finite set containing the set

$$N'_r=\{(u,s)\mid \|u\|+ds\leqq (D+dl)/(q-1),\ s\geqq -2p^r\}.$$

This shows that $\text{Trace } A^{(r)}\equiv \text{Trace } A^{(r-1)} \bmod p^{r+1}$ and so the limit as $r\to\infty$ exists. It is clear that the Trace of C is well defined, and is congruent mod p^{r+1} to the Trace of $A^{(r)}$. The formula for Trace C now follows by the same proof as in Case I. We extend this result to matrices.

THEOREM. — *Let* $F\in\mathcal{M}_{\nu,\nu}(H(\mathcal{D}_f))$, *then* $\psi\circ F$ *is an endomorphism of* $H(\mathcal{D}_f)^\nu$, *whose matrix relative to the basis* $\{X^u f^s \varepsilon_i\}_{u_n<d,\, s\in\mathbf{Z},\, 1\leqq i\leqq\nu}$ (*where* ε_i *is the standard i-th unit vector in* ν *space*) *has a characteristic series*, $\det(I-t\psi\circ F)$ *and*

$$\det(I-t\psi_q\circ F)^{-(\Phi-1)^n}=\frac{1}{\displaystyle\prod_{\substack{\bar{x}\text{ alg over }\mathbf{F}_q\\ \bar{f}(\bar{x}).\Pi\,\bar{x}_i\neq 0}} \Delta_{F,\bar{x}}(t^{\deg\bar{x}})^{1/\deg\bar{x}}}$$

where

$$\Delta_{F,\bar{x}}(t)=\det\left(I-t\prod_{i=0}^{\deg\bar{x}-1} F(\text{Teich }\bar{x}^{q^i})\right)$$

and Φ *maps* $1+tK[[t]]$ *into itself by* $t\mapsto qt$.

10. ψ_q **as operator on** $H(\mathscr{D}_f)$ (*Cf.* [Dw 3], Chapter 5, for the case of $n=1$)

Let f, K_0, K satisfy the conditions of Section 9. We have shown $|\psi|_{\mathscr{D}_f}\leqq 1$ (§ 9). Thus if we write

$$\psi_q X^u f^s = \sum_{\substack{v_n<d\\ t\in\mathbf{Z}}} B_{u,s;v,t} X^v f^t$$

we have $|B_{u,s;v,t}|\leqq 1$.

For $s\geqq 0$ we know $B_{u,s;v,t}=0$ unless

(10.1) $$\frac{\|u\|+ds}{q}-\|v\|\geqq dt\geqq 0$$

To treat the case $s<0$ we change notation and for $s\geqq 1$ write

(10.2) $$\psi_q\frac{X^u}{f^s}=\sum_{\substack{v_n<d\\ t\in\mathbf{Z}}} B'_{u,s;v,t}\frac{X^v}{f^t}.$$

Let $q=p^a$

LEMMA: $B'_{u,s;v,t}=0$ *unless*

(10.1′)

$$dt-\|v\|\geqq\frac{1}{q}(ds-\|u\|)$$ (*hence the number of non-zero entries with* $t\leqq 0$ *is finite*).

Furthermore

$$\operatorname{ord} B'_{u,s;v,t}\geqq\sup\left(0,\,-a+\frac{qt-s}{pq}(p-1)+\operatorname{ord}\frac{(t-1)!}{\pi^{t-1}}\right)$$

where for $t\leqq 0$ *the symbol* $(t-1)!$ must be replaced by 1.

Proof. — We define a subspace, L, of $zK[[z, X_1, \ldots, X_n]]$.

$$L=\{\sum_{\substack{s\geqq 1\\ u\in\mathbf{N}^n}} a_{u,s}X^u z^s \mid \operatorname{ord} a_{u,s}>\alpha s+O(1)\text{ for some }\alpha>0,$$
$$\operatorname{ord} a_{u,s}\to+\infty\text{ as }\|u\|+ds\to+\infty\}.$$

Let $\pi^{p-1}=-p$. We define the (Laplace) map T_f of L into $H(\mathscr{D}_f)$ by linearity, continuity and the condition

$$T_f X^u z^s=\frac{X^u}{f^s}\,\frac{(s-1)!}{(-\pi)^{s-1}}\qquad(\text{note: }s\geqq 1).$$

Let $F=\exp \pi(zf(X)-z^q f(X^q))$. We assume (as in Section 7) that the coefficients of f lie in an unramified extension of $\mathbf{Q}_p$ of degree a, and that $\bar{f}\neq 0$. It is known [Dw 3], p. 242, that

$$F=\sum_{\substack{\| v \|\leqq ds\\ s\geqq 0}} c_{v,s} X^v z^s$$

$$\operatorname{ord} c_{v,s}\geqq s\frac{p-1}{pq}.$$

The operator $\psi_q\circ F:\xi\to\psi_q(\xi F)$ is a well-defined map of L into itself. Katz [Ka 1] showed that the diagram

$$\begin{array}{ccc} L & \xrightarrow{T_f} & H(\mathscr{D}_f) \\ {\scriptstyle \psi_q\circ F}\downarrow & & \downarrow{\scriptstyle q\psi_q} \\ L & \xrightarrow{T_f} & H(\mathscr{D}_f) \end{array}$$

commutes. A simpler proof may be found in [Dw 4], Chapter 10. Thus

$$\begin{aligned} q\psi_q\frac{(s-1)!}{(-\pi)^{s-1}}\frac{X^u}{f^s} &= T_f(\psi_q(X^u z^s F) \\ &= T_f(\sum_{v,\, t\geqq s/q} C_{qv-u,\, qt-s} X^v z^t) \\ &= \sum_{v,\, t\geqq 1} C_{qv-u,\, qt-s} X^v (t-1)!/f^t(-\pi)^{t-1}. \end{aligned}$$

We may assume that $u_n<d$ but in the sum we may have $v_n\geqq d$, and so we must eventually transform into the reduced form. We first observe that

$$\begin{aligned} \operatorname{ord}(C_{qv-u,\, qt-s}(t-1)!/\pi^{t-1}) &\geqq (qt-s)\frac{p-1}{pq}-\frac{t-1}{p-1}+\operatorname{ord}(t-1)! \\ &\geqq \left(\frac{p-1}{p}-\frac{1}{p-1}\right)t-s\frac{p-1}{pq}+\operatorname{ord}(t-1)!+\frac{1}{p-1}, \end{aligned}$$

a monotonically increasing function of t (for $p\geqq 3$). Under reduction this estimate [with $(t-1)!$ replaced by 1 if $t\leqq 0$] will be maintained (*cf.* simplified criterion of Section 4). We conclude that

$$\operatorname{ord} B'_{u,\, s;\, v,\, t}\geqq -a+\operatorname{ord}\left(\frac{\pi^{s-1}}{(s-1)!}\right)+(qt-s)\frac{p-1}{pq}+\operatorname{ord}\frac{(t-1)!}{\pi^{t-1}}.$$

We recall that $C_{qv-u,\, qt-s}=0$ unless

(10.1″) $$\| qv-u\|\leqq d(qt-s)$$

i.e.

$$dt - \| v \| \geqq (ds - \| u \|)/q.$$

To prove (10.1′) it remains to verify that if $dt - \| v \| \geqq \alpha$ then under reduction the same inequality holds, *i.e.* if $v_n \geqq d$ then writing $f = X_n^d + \sum_{w_n < d, \| w \| \leqq d} \gamma_w X^w$ we have

$$\frac{X^v}{f^t} = \frac{X^v}{X_n^d} \frac{1}{f^{t-1}} - \sum_{w_n < d} \gamma_w \frac{X^{w+v}/X_n^d}{f^t}.$$

Trivially (letting ε_n be the n-th unit vector),

$$d(t-1) - \| v - d\varepsilon_n \| \geqq \alpha,$$

$$dt - \| w + v - d\varepsilon_n \| \geqq \alpha - \| w \| + d \geqq \alpha,$$

$$w_n + v_n - d < v_n.$$

This completes the proof.

11. Frobenius on Logarithmic space

Let $H \in L_{\alpha, n, q}$. We write for $u_n < d$

$$\psi_q (X^u f^s H) = \sum_{\substack{v_n < d \\ t \in \mathbf{Z}}} \gamma_{u, s; v, t} X^v f^t.$$

THEOREM. — *Given* $\varepsilon < 1$ *there exists* $T_\varepsilon > 0$ *independent of* (u, s) *such that (for* $\rho = \rho_{m, q}$*) we have*

(11.1) $$\alpha\rho (\| u \| + d | s |) + \operatorname{ord} \gamma_{u, s; v, t} \geqq \varepsilon\alpha + \alpha\rho (\| v \| + d | t |)$$

for all

(11.1.1) $$\| v \| + d | t | > T_\varepsilon.$$

Proof. — Let us write $H = \sum_{\substack{w_n < d \\ z \in \mathbf{Z}}} a_{w, z} X^w f^z$. Then in the notation of Section 10,

(11.2) $$\gamma_{u, s; v, t} = \sum_{\substack{z \in \mathbf{Z} \\ w_n < d}} a_{w, z} B_{u+w, z+s; v, t}.$$

Thus it is enough to show existence of Γ_ε such that

(11.3) $$\alpha\rho (\| u \| + d | s |) + \alpha\rho (\| w \| + d | z |) + \operatorname{ord} B_{u+w, z+s; v, t} \geqq \alpha\varepsilon + \alpha\rho (\| v \| + d | t |)$$

for all (v, t) satisfying (11.1.1).

Case 1. $z+s \geqq 0$. — Here $B_{u+w,\, z+s;\, v,\, t}=0$ unless

$$(11.4) \qquad \begin{cases} t \geqq 0 \\ \| u+w \| + d(z+s) \geqq q(\| v \| + dt). \end{cases}$$

Thus putting $\tau_1 = \| u \| + d|s|$, $\tau_2 = \| w \| + d|z|$, $\tau = \| v \| + dt$ we have

$$(11.4.1) \qquad \tau_1 + \tau_2 \geqq q\tau.$$

By Lemma 1 (iii), for $q\tau \geqq m + mq^{e+1}$, $(e>0)$,

$$\rho(\tau_1) + \rho(\tau_2) - \rho(q\tau) \geqq -q^{-e}$$

and since $\tau > m$,

$$(11.4.2) \qquad \rho(\tau_1) + \rho(\tau_2) - \rho(\tau) \geqq 1 - q^{-e}.$$

Choose $e \in \mathbf{N}$ such that $\varepsilon < 1 - q^{-e}$. Then (11.3) is satisfied for $T_\varepsilon = m + mq^{e+1}/q$.

Case 2: $z+s<0$, $t \geqq 0$.

By (10.1′) we may assume

$$-dt - \| v \| \geqq \frac{1}{q}(-d(z+s) - \| u+w \|)$$

i.e.

$$d(z+s) + \| u+w \| \geqq q(dt + \| v \|).$$

Thus for τ, τ_1, τ_2 as in Case 1, (11.4.1) is again valid and so if $q\tau \geqq m + mq^{e+1}$ we have (11.4.2). We choose e as in Case 1 and obtain the inequality (11.3) with $T_\varepsilon = (m + mq^{e+1})/q$.

Case 3. $t<0$. — In this case we may assume $z+s<0$ in (11.2), (11.3). We change notation and rewrite (11.3) in the form: there exists T_ε such that

$$(11.5) \qquad \alpha\rho(\tau_1) + \alpha\rho(\tau_2) + \operatorname{ord} B'_{w+u,\, z+s;\, v,\, t} \geqq \alpha\varepsilon + \alpha\rho(\tau)$$

for all

$$t \geqq 1, \quad z+s \geqq 1, \quad v, u, w \in \mathbf{N}^n \qquad (v_n, u_n, w_n < d)$$

such that

$$\tau > T_\varepsilon$$

$$(11.5.1) \qquad \| u+w \| - d(z+s) \geqq q(\| v \| - dt).$$

Here once again

$$\tau_1 = \| u \| + d|s|, \qquad \tau_2 = \| w \| + d|z|, \qquad \tau = \| v \| + dt.$$

Condition (11.5.1) implies

$$\tau \leqq 2\,dt + q^{-1}(\| u+w \| - d(z+s)) \tag{11.5.2}$$

and so (11.5) is implied by

$$\alpha\rho(\tau_1) + \alpha\rho(\tau_2) + \operatorname{ord} B'_{w+u,\, z+s;\, v,\, t} \geqq \alpha\varepsilon + \alpha\rho(2\,dt + q^{-1}(\| u+w \| - d(z+s)) \tag{11.5'}$$

subject to (11.5.1).

We shall prove either (11.5) or (11.5′) by considering three situations covering all possibilities. We choose $c \in (1, q)$ such that $(c-1)/(q-1) > \varepsilon$. We fix $\kappa \in (0, q/3)$.

The three subcases of case 3 are

I. $\| w+u \| + d(|z| + |s|) \leqq \kappa\, T_\varepsilon$.

II. $\| u+w \| + d(|z| + |s|) \leqq c(2\,dt + ((\| u+w \| - d(z+s))/q))$.

III. Both inequalities I and II hold in the reverse directions.

Case I. — Certainly $|z| + |s| \geqq -(z+s)$ and so

$$\| w+u \| - d(z+s) \leqq \kappa\, T_\varepsilon. \tag{11.5.3}$$

Thus (11.5′) holds if for $\tau > T_\varepsilon$,

$$\operatorname{ord} B'_{w+u,\, z+s;\, v,\, t} \geqq \alpha\varepsilon + \alpha\rho\left(2\,dt + \frac{\kappa\, T_\varepsilon}{q}\right). \tag{11.6}$$

By (11.5.2) and (11.5.3),

$$\tau \leqq 2\,dt + \kappa\, T_\varepsilon\, q^{-1}; \tag{11.5.3'}$$

by I, $(z+s) \leqq \kappa\, T_\varepsilon/d$; while by Section 10,

$$\begin{aligned}\operatorname{ord} B'_{u+w,\, z+s;\, v,\, t} &\geqq \frac{p-1}{pq}(qt - (z+s)) - \operatorname{ord}\frac{(t-1)!}{\pi^{t-1}} - a \\ &\geqq -1 - a + \frac{p-1}{pq}(qt - \kappa\, T_\varepsilon/d) - \log t/\log p.\end{aligned}$$

So by (11.5.3′), for $\tau > T_\varepsilon$ we have

$$2\,dt \geqq (1 - \kappa/q)\, T_\varepsilon. \tag{11.5.3''}$$

Thus it is enough to choose T_ε such that subject to (11.5.3″) we have

$$\frac{p-1}{pq}(qt - \kappa\, T_\varepsilon/d) - 1 - a - \log t/\log p > \varepsilon\alpha + \alpha\rho(2\,dt + \kappa\, T_\varepsilon/q). \tag{11.7}$$

The function

$$h(t)=\frac{p-1}{p}t-\frac{\log t}{\log p}-\alpha\rho(2dt+\kappa T_{\varepsilon}/q)$$

has derivative

$$h'(t)=\frac{p-1}{p}-\frac{t^{-1}}{\log p}-2d\alpha\rho'(2dt+\kappa T_{\varepsilon}/q)$$

which is positive for T_{ε} large enough. Thus subject to (11.5.3''), h takes the minimal value at the extreme point $t_0=(1-\kappa/q)T_{\varepsilon}/2d$. Thus it is enough to subject T_{ε} to the further condition that

$$h(t_0)-\frac{p-1}{pq}\frac{\kappa T_{\varepsilon}}{d}>1+\varepsilon\alpha+a. \tag{11.7.1}$$

The left side is $\gamma T_{\varepsilon}-\log T_{\varepsilon}/\log p-\alpha\rho(T_{\varepsilon})-\delta$ where $\gamma=(p-1)(q-3\kappa)/(2dpq)$, $\delta\log p=\log((q-\kappa)/2dq)$. Since $q>3\kappa$, and since ρ grows logarithmically, it is clear that the left side of (11.7.1) goes to $+\infty$ as $T_{\varepsilon}\to+\infty$ which demonstrates the existence of T_{ε} satisfying (11.7.1).

Case II. We first observe that by Lemma 1 (ii)

$$\begin{aligned}2+\rho(2dt)+\rho(\tau_1)+\rho(\tau_2)&\geqq\rho(2dt+\tau_1+\tau_2)\\&\geqq\rho(2dt+q^{-1}(\|u+w\|-d(z+s))).\end{aligned}$$

Thus (11.5') holds if there exists T_{ε} such that

$$\operatorname{ord}B'_{w+u,\,z+s;\,v,\,t}\geqq\alpha(2+\varepsilon)+\alpha\rho(2dt) \tag{11.8}$$

subject to (11.5.1) and II.

By condition II

$$\left(1-\frac{c}{q}\right)(\|u+w\|-d(z+s))\leqq 2cdt$$

and so

$$\|u+w\|-d(z+s)\leqq 2cqdt/(q-c). \tag{11.9}$$

Thus by (11.5.2)

$$\tau\leqq 2dtq/(q-c). \tag{11.10}$$

and so it is enough to choose T_{ε} such that (11.8) holds subject to

$$2dt\geqq(1-c/q)T_{\varepsilon}. \tag{11.10'}$$

Again by II,

$$\begin{aligned} 2\,dct &\geqq \|w+u\|(1-c/q)+d(|z|+|s|)+dc(z+s)/q \\ &\geqq d(1+c/q)(z+s) \end{aligned}$$

and so

$$z+s \leqq 2\,tcq/(c+q). \tag{11.11}$$

Again by Section 10 and using (11.11)

$$\begin{aligned} \operatorname{ord} B'_{u+w,\,z+s;\,v,\,t} &\geqq \frac{qt-(z+s)}{pq}(p-1)-\operatorname{ord}\frac{\pi^t}{(t-1)!}-a \\ &\geqq \delta t-1-a-\log t/\log p \end{aligned}$$

where $\delta = p^{-1}(p-1)(q-c)/(q+c)$.

Thus it is enough to choose T_ε such that subject to (11.10′) we have

$$\delta t - \log t/\log p - 1 - a \geqq \alpha(2+\varepsilon)+\alpha\rho(2\,dt)$$

but it is clear as $\delta t - \alpha\rho\,(2\,dt) - \log t/\log p \to +\infty$ as $t\to\infty$.

Case III. Since $\operatorname{ord} B'_{u+w,\,s+z;\,v,\,t} \geqq 0$ it is enough by (11.5′) to find T_ε such that

$$\alpha\rho(\tau_1)+\alpha\rho(\tau_2) \geqq \alpha\varepsilon+\alpha\rho\left(2\,dt+\frac{\|u+w\|-d(s+z)}{q}\right) \tag{11.12}$$

for $\tau > T_\varepsilon$. By Lemma 1 (v), if $\tau_1+\tau_2 > cm$ then

$$\rho(\tau_1+\tau_2) \geqq \rho((\tau_1+\tau_2)/c)+(c-1)/(q-1)$$

and so by the falsity of II

$$\rho(\tau_1+\tau_2) \geqq \rho(2\,dt+q^{-1}(\|u+w\|-d(z+s)))+(c-1)/(q-1) \tag{11.13}$$

subject to $\tau_1+\tau_2 > cm$ which is precisely the consequence of the falsity of I if $T_\varepsilon > cm/\kappa$. By Lemma 1 (iii)

$$\rho(\tau_1)+\rho(\tau_2) \geqq \rho(\tau_1+\tau_2)-q^{-e} \tag{11.14}$$

if $\tau_1+\tau_2 \geqq m+mq^e$ which by not I holds if $T_\varepsilon > m(1+q^e)/\kappa$. For T_ε so large then, (11.12) holds provided

$$(c-1)/(q-1)-q^{-e} > \varepsilon.$$

By the choice of c, we may choose $e\in\mathbf{N}$ satisfying this condition.

THEOREM. — *Let* $F\in\mathscr{M}_{v,v}(L_{\alpha,m,q})$ *then the characteristic series* $\det(I-t\psi_q\circ F)$ (*as defined in Section* 9) *converges for* $\operatorname{ord}_p t > -\alpha$.

Proof. — Equation (11.1) implies that (given $\varepsilon<1$) the matrix of $\psi_q \circ F$ relative to the bases $\{\varepsilon_j p^{\rho(\|u\|+d|s|)} X^u f^s\}_{u_n<d,\, s\in \mathbf{Z},\, 1\leqq j\leqq \nu}$ of $(L_{\alpha,m,q,f}\otimes K)^\nu$ has all columns divisible by $p^{\varepsilon\alpha}$ for $\|v\|+d|s|>T_\varepsilon$. Thus the coefficient of t^λ in the characteristic series is divisible by $(p^{\varepsilon\alpha})^{\lambda-N}$ where N/ν is the cardinality of the set of (u,s) such that $\|u\|+d|s|\leqq T_\varepsilon$.

COROLLARY. — *If the matrix* B *of Section* 8 *has coefficients in* $L_{\alpha,m,p}$ *with* α *arbitrarily large then the coefficients of* $\tilde{A}_q\,(q=p^s)$ *lie in* $L_{s\alpha',m'',q}$ *for suitable* m'' *and* α' *arbitrary in* $(0,(p-1)/(p+1))$. *The unit root zeta function is meromorphic for* $\operatorname{ord}_q t>(p-1)/(p+1)$.

Proof. — The first sentence follows from Section 8. The unit root zeta function is by Section 9 essentially of the form $\det(I-t\,\psi_q\circ\tilde{A}_q)^{(-\Phi-1)^n}$ and the point is that the condition $\operatorname{ord}_p t>-s\alpha'$ is equivalent to $\operatorname{ord}_q t>-\alpha'$. However this determinant does not give the contribution coming from $\bar{X}$ with some zero coordinates. These may be restored by a combinatorial argument using the restriction of $\tilde{A}_q$ to sets of the form $\{X_i=0\}_{i\notin C}$ where C is a subset of $\{1,2,\ldots,n\}$.

Remark. — We have tacitly assumed that condition (7.1) is satisfied with f regular relative to X_n. Indeed the combinatorial argument of the preceding proof requires that f be regular with respect to each X_i. This need not hold over $\mathbf{F}_q$ but will hold over $\mathbf{F}_{q^s}$ for all $s>s_0$. By well known methods this shows meromorphy of $\prod_{v^s=1}\zeta(vt)$ for $\operatorname{ord}_q t>-(p-1)/(p+1)$ and all $s>s_0$.

12. Family of hypersurfaces

To illustrate our theory we will consider the generic hypersurface of degree d in $n+1$ variables which we write in the form

$$h(X,\lambda)=\sum_{i=1}^{n+1} X_i^d+\sum_{j=1}^{l}\lambda_j X^{w^{(j)}}$$

where $l=\binom{n+d}{d}-(n+1)$ and $\{w^{(j)}\}$ is the set of all solutions in $\mathbf{N}^{n+1}$ of the equation $u_1+\ldots+u_{n+1}=d$, the solutions $u_i=d\delta_{i,s}$ with fixed s being excluded.

We observe that the theorem of Section 7 and the discussion of Section 8 remains valid with ν_2 infinite provided that matrix B satisfies the further condition that in the $L_{\alpha,m,p}$ metric the s-th row of B approaches zero as $s\to\infty$.

Following the combinatorial methods of [Dw 5], Section 4, we construct an infinite matrix A which may be used to compute the zeta function of the hypersurface defined by the specialization of the reduction of h. Specifically we choose γ algebraic over $\mathbf{Q}_p$

such that

$$\sum_{s=0}^{\infty} \frac{\gamma^{p^s}}{p^s} = 0$$

$$\operatorname{ord} \gamma = \frac{1}{p-1}.$$

We put $\theta(t) = \mathrm{E}(\gamma t)$ where E is the Artin-Hasse exponential function,

$$\mathrm{E}(t) = \exp\left(\sum_{s=0}^{\infty} t^{p^s}/p^s\right).$$

We define

$$\mathrm{F}(\mathrm{X}, \lambda) = \prod_{i=1}^{n+1} \theta(\mathrm{X}_0 \mathrm{X}_i^d) \cdot \prod_{j=1}^{l} \theta(\lambda_j \mathrm{X}_0 \mathrm{X}^{w(j)})$$

Let the λ_j be bounded by unity.

We define $\mathrm{A}(\lambda)$ to be the matrix of $(1/p)\psi \circ \mathrm{F}(\mathrm{X}, \lambda)$ (relative to the basis $\{\gamma^{u_0} \mathrm{X}^u\}_{du_0 = u_1 + \ldots + u_{n+1},\, u_i \geqq 1,\, \forall_i}$) as operator on the space of power series ξ with the properties

(i) ξ is divisible by $\mathrm{X}_1 \mathrm{X}_2 \ldots \mathrm{X}_{n+1}$;

(ii) $\xi(\mathrm{X}) = \sum_{du_0 = u_1 + \ldots + u_{n+1}} a_u \mathrm{X}^u$;

(iii) ξ converges on the closed unit ball.

Thus writing $\mathrm{F} = \sum \mathrm{C}_u(\lambda) \mathrm{X}^u$ we have

$$\mathrm{A}_{u, v} = \gamma^{v_0 - u_0} p^{-1} \mathrm{C}_{pu - v}$$

and so $\operatorname{ord} \mathrm{A}_{u, v} \geqq u_0 - 1$. (If we view the λ_i as variables then $\mathrm{A}_{u, v}$ is a polynomial in the λ_i of degree bounded by $pu_0 - v_0$.)

Let λ be specialized in $\mathbf{C}_p^l$ so that $\lambda^q = \lambda$, $q = p^a$. For each subset B of $\{1, 2, \ldots, n+1\}$, $\mathrm{B} \neq \Phi$, let V_B denote the projective hypersurface (in the projective space whose variables are the $\{\mathrm{X}_i\}_{i \in \mathrm{B}}$) defined by the vanishing of the reduction of $h_\mathrm{B}(x, \lambda)$, it being understood that h_B is obtained from h by setting $\mathrm{X}_i = 0$ for all $i \notin \mathrm{B}$. We view V_B as defined over $\mathbf{F}_q$ and let $\zeta(\mathrm{V}_\mathrm{B}, t)$ denote the corresponding zeta function. Let $\mathrm{A}^{(a)} = \prod_{j=0}^{a-1} \mathrm{A}(\lambda^{p^j})$. Then by a minor modification of equation (4.33) [Dw 5] we obtain

$$\det(\mathrm{I} - t\,\mathrm{A}^{(a)})^{-(\Phi - 1)^{n+1}} = (1 - t)^{(1 - \Phi)^n} \prod_{\varnothing \neq \mathrm{B} \subset \mathrm{S}} \zeta(\mathrm{V}_\mathrm{B}, t)^{(-\Phi)^{n+1-\mathrm{Card\,B}}}$$

where $\Phi : \mathbf{C}_p[[t]] \to \mathbf{C}_p[[t]]$ by $t \to qt$. It follows that the unit root factor of $(1-t)\zeta(\mathrm{V}, t)$, $(\mathrm{V} = \mathrm{V}_\mathrm{S})$ coincides with the $(-1)^n$ power of the unit root factor of $\det(\mathrm{I} - t\,\mathrm{A}^{(a)})$. It follows that the unit root zeta function of this family of hypersurfaces is given by the methods of Section 8. Here we must take $f(\lambda)$ to be det B_{11} where

$$B_{11}=(A_{u,v})_{u_0=1,\, v_0=1} \qquad (du_0=u_1+\ldots+u_{n+1})$$

$\nu_1=\binom{d-1}{n}$, $\nu_2=\infty$, and for example $p\,B_{22}$ is the matrix $(A_{u,v})_{u_0\geqq 2,\, v_0\geqq 2}$.

The advantage of this approach (based on cochains) is that while $\nu_2=\infty$, we avoid all necessity of reducing to finite dimensional cohomology spaces. Of course the matrix $\tilde{A}_{11}$ is a $\nu_1\times\nu_1$ matrix and the unit root zeta function of the family is given by $\det(I-t\psi\circ\tilde{A}_{11})^{-(\Phi-1)^l}$ as in the second theorem of Section 9.

13. Appendix

In this section, we provide the proof of Lemma 1.

Proof. — (i) This follows directly from the fact that for $s>mq^t$

$$\rho'(s)\leqq 1/m(q-1)q^t.$$

(ii) Let $\delta(s_1,s_2)=\rho(s_1+s_2)-\rho(s_1)-\rho(s_2)$.

Case 1. $s_1>m$. — Here $\rho'(s_1+s_2)-\rho'(s_1)\leqq 0$ and hence for fixed s_2, $\rho(s_1+s_2)-\rho(s_1)$ is maximum at $s_1=m$. Thus

$$\delta(s_1,s_2)\leqq\rho(m+s_2)-\rho(s_2).$$

The same argument shows that the right side assumes its maximum at $s_2\in[0,m]$ and so

$$\delta(s_1,s_2)\leqq \operatorname*{Sup}_{s_2\in[0,m]}\rho(m+s_2)=\rho(2m)=\frac{1}{q-1}.$$

Case 2. — $\operatorname{Sup}(s_1,s_2)\leqq m$ but then $\delta(s_1,s_2)=\rho(s_1+s_2)\leqq\rho(2m)$ as before.

(iii) Let $z=s_1+s_2$, $s_1\geqq s_2$, and so $s_2\leqq z/2$. For fixed z,

$$\delta(s_1,s_2)=\rho(z)-\rho(s_2)-\rho(z-s_2)$$

and so

$$\frac{d}{ds_2}\delta(s_1,s_2)=-\rho'(s_2)+\rho'(s_1)\leqq 0 \qquad \text{if} \quad s_2\geqq m.$$

Thus $\delta(s_1,s_2)$ assumes its maximal value when $s_2\in[0,m]$. In that range $\delta(s_1,s_2)=\rho(z)-\rho(z-s_2)$ which clearly is maximal at $s_2=m$. Trivially the difference is bounded from above by m.(right hand derivative of ρ at $z-m$). The assertion is now clear for $z\geqq m+mq^t$.

(iv) *Case* 1.

$$t_1 \geqq t_2 \geqq 0,$$
$$s_1 = m_1 q^{t_1} + x_1 < m_1 q^{t_1+1}$$
$$s_2 = m_2 q^{t_2} + x_2 < m_2 q^{t_2+1}.$$

Then trivially

$$s_1 + s_2 \leqq (m_1 + m_2) q^{t_1+t_2} + x_1 + x_2$$

and so by (i)

$$\rho_{m_1+m_2}(s_1+s_2) \leqq t_1 + t_2 + \frac{x_1+x_2}{(m_1+m_2) q^{t_1+t_2}(q-1)}$$

while

$$\rho_{m_1}(s_1) + \rho_{m_2}(s_2) = t_1 + t_2 + \frac{x_1}{m_1 q^{t_1}(q-1)} + \frac{x_2}{m_2 q^{t_2}(q-1)}.$$

The assertion is now clear in this case.

Case 2.

$$m_1 \leqq s_1 = m_1 q^{t_1} + x_1 < m_1 q^{1+t_1}$$
$$s_2 \leqq m_2$$

so

$$s_1 + s_2 = (m_1 + m_2) q^{t_1} + \delta$$

where $\delta = x_1 + s_2 - m_2 q^{t_1} \leqq x_1$. If $\delta \leqq 0$ then $\rho_{m_1+m_2}(s_1+s_2) \leqq t_1 \leqq \rho_{m_1}(s_1)$. Hence we may assume $\delta > 0$ and so

$$\rho_{m_1+m_2}(s_1+s_2) \leqq t_1 + \frac{\delta}{(m_1+m_2) q^{t_1}(q-1)}$$

while

$$\rho_{m_1}(s_1) = t_1 + \frac{x_1}{m_1 q^{t_1}(q-1)} \geqq \rho_{m_1+m_2}(s_1+s_2)$$

since $x_1 \geqq \delta$.

Case 3. $s_1 \leqq m_1$, $s_2 \leqq m_2$. — Here $0 = \rho_{m_1}(s_1) = \rho_{m_2}(s_2) = \rho_{m_1+m_2}(s_1+s_2)$.

(v) Let $h(s)=\rho(s)-\rho(s/c)$. For $t\geqq 0$, $cmq^t<s<mq^{t+1}$ we have

$$h'(s)=\left(1-\frac{1}{c}\right)(mq^t(q-1))^{-1}>0$$

and hence h is minimal at $s=cmq^t$.

For $mq^t<s<cmq^t$, $t\geqq 1$, we have

$$h'(s)=\frac{1}{m(q-1)q^t}-\frac{1}{c}\,\frac{1}{m(q-1)q^{t-1}}<0$$

and so again the minimum occurs at $s=cmq^t$. We compute

$$h(cmq^t)=\rho(cmq^t)-\rho(mq^t)=\frac{c-1}{q-1}$$

which gives the lower bound for h on $[cm, \infty)$.

(vi) The assertion is trivial for $s\leqq m$. For $s=mq^t+x<mq^{t+1}$, we have $qs=mq^{t+1}+xq<mq^{t+2}$ and so $\rho(qs)=1+t+xq/(q-1)mq^{t+1}=1+\rho(s)$. This proves both (vi) and (vi′).

(vii) If $s=mq\,.\,q^t+x<mq\,.\,q^{t+1}$, $t\geqq 0$, then $s/q=mq^t+x/q<mq^{t+1}$ and so

$$\rho_{mq}=t+x/mq^{t+1}(q-1)=\rho_m(s/q).$$

If $s<mq$ then $s/q<m$ and hence $\rho_m(s/q)=0=\rho_{mq}(s)$.

(viii) If $mq^a q^{at}\leqq s<mq^a\,.\,q^{a(t+1)}$ then

$$\rho_{m,q}(s)\geqq\rho_{m,q}(mq^{a(1+t)})=a(1+t)$$

$$a\,\rho_{mq^a,\,q^a}(s)<a\,\rho_{mq^a,\,q^a}(mq^a\,.\,q^{a(t+1)})=a\,.\,(1+t).$$

This proves

$$\rho_{m,q}(s)\geqq a\,\rho_{mq^a,\,q^a}(s)$$

if $s\geqq mq^a$, while the assertion is trivial if $s<mq^a$.

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