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A. M. BLOKH
M. YU. LYUBICH

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# MEASURABLE DYNAMICS OF S-UNIMODAL MAPS OF THE INTERVAL

BY A. M. BLOKH AND M. YU. LYUBICH

ABSTRACT. — In this paper we sum up our results on one-dimensional measurable dynamics reducing them to the S-unimodal case (compare Appendix 2). Let $f$ be an S-unimodal map of the interval having no limit cycles. Then $f$ is ergodic with respect to the Lebesgue measure, and has a unique attractor A in the sense of Milnor. This attractor coincides with the conservative kernel of $f$. There are no strongly wandering sets of positive measure. If $f$ has a finite a. c. i. (absolutely continuous invariant) measure $\mu$, then it has positive entropy: $h_\mu(f)>0$. This result is closely related to the following: the measure of Feigenbaum-like attractors is equal to zero. Some extra topological properties of Cantor attractors are studied.

CONTENTS



## 1. Introduction

Let $f: [0, 1] \to [0, 1]$ be a map of the interval satisfying the following conditions:

S1. $f$ is a $C^3$-smooth map with a negative Schwarzian derivative:

$$Sf=\frac{f'''}{f'}-\frac{3}{2}\left(\frac{f''}{f'}\right)^2<0;$$

S2. $f$ has a unique critical point $c \in (0, 1)$; this point is non-degenerate: $f''(c) \neq 0$; so $c$ is the extremum.

In addition to these main properties we introduce at once the convenient normalization (the possibility of such a normalization is explained, for example, *in* [CE]):

S3. $f: c \mapsto 1 \mapsto 0$ (in particular, $c$ is the maximum point).

The class of maps satisfying S1-S3 we denote by $\mathscr{S}_1$ (the index "1" means the number of extrema). Throughout the paper we will assume (unless otherwise stated) that $f \in \mathscr{S}_1$ and call such maps S-unimodal.

S-unimodal maps are very interesting from the dynamical viewpoint and have been studied intensively since the 70's (*see* [CE]). However, many essential problems are still unsolved. In the present paper the measurable dynamics of such transformations (*i.e.*, the asymptotic behaviour of a.a. (almost all) trajectories with respect to the Lebesgue measure) is studied.

In what follows $f^n$ will denote the $n$-th iterate of $f$. The set $\operatorname{orb}(x) = \{f^n x\}_{n=0}^{\infty}$ is called the *trajectory* (or the *orbit*) of $x \in [0, 1]$. The limit set of $\operatorname{orb}(x)$ is denoted by $\omega(x)$. The further general concepts related to the measurable dynamics of endomorphisms (invariant and wandering sets, ergodicity, conservativity and dissipativity, etc.) can be found in Appendix 1.

Now let us introduce the important concept of the measure-theoretical attractor in the sense of Milnor [M]. Let A be a closed invariant set, and $\operatorname{rl}(A) = \{x : \omega(x) \subset A\}$ be its "realm of attraction". The set A is called a *measure-theoretical attractor* if

(i) $\lambda(\operatorname{rl}(A)) > 0$ (*i.e.* A attracts "many" orbits);

(ii) $\lambda(\operatorname{rl}(A) \setminus \operatorname{rl}(A')) > 0$ for any closed invariant proper subset $A' \subset A$ (so the realm of attraction of A′ is essentially less than that of A).

As a rule, we will call measure-theoretical attractors simply *attractors* (unlike topological attractors defined in Section 2).

In order to present the results of this paper we need to formulate the authors' earlier results. The terminology used in the following theorem (a limit cycle, a solenoid, a transitive interval) will be explained in detail in Section 2.

THE THEOREM ON THE ATTRACTOR [BL1, 2, 3]. A map $f \in \mathscr{S}_1$ has a unique measure-theoretical attractor A and $\omega(x) = A$ for a.a. $x \in [0, 1]$. The attractor A has the structure of one of the following four types:

A1: A is a limit cycle;

A2: A is a solenoid (or a Feigenbaum-like attractor);

A3: A is a cycle of transitive intervals;

A4. A is a "strange" attractor, *i.e.*, a Cantor set contained in the cycle of transitive intervals. Moreover, in case A4, A contains the critical point $c$ and $A = \omega(c)$.

These cases will be called *cyclic*, *solenoidal*, *interval*, and "*strange*," respectively.

THE MAIN PROBLEM (*cf.* [M]). – *Are there $f \in \mathscr{S}_1$ having "strange" attractors?*

The authors regard the results of the present paper as steps toward solving this problem.

THE THEOREM ON ERGODICITY [BL1], [BL3], [BL4], [BL6]). — In the non-cyclic case the map $f \in \mathcal{S}_1$ is ergodic with respect to the Lebesgue measure $\lambda$.

The following concepts of a strongly wandering set and the conservative kernel $C(f)$ (or the conservative part) of $f$ are defined in Appendix 1.

THE THEOREM ON STRONGLY WANDERING SETS. — In the non-cyclic case the map $f \in \mathcal{S}_1$ has no measurable strongly wandering sets of positive measure.

THE THEOREM ON THE CONSERVATIVE KERNEL. — The conservative kernel $C(f)$ of the map $f \in \mathcal{S}_1$ coincides (mod 0) with the attractor A. Moreover, $f$ is purely dissipative in the cyclic and solenoidal cases, and asymptotically conservative in the standard transitive case.

*Remark* 1.1. — The dissipativeness in the solenoidal case means that solenoids have zero measure (*see* the Theorem on solenoids' measure in Section 8). So, in this case we have the amusing example of a purely dissipative endomorphism without strongly wandering sets of positive measure.

*Remark* 1.2. — From the viewpoint of the Theorem on the Conservative Kernel, the Main Problem can be reformulated in the following way:

*Is it true that f is conservative on the cycle of transitive intervals?*

*Remark* 1.3. — The exponential map $z \mapsto e^z$ of the complex plane gives an example of a topologically transitive but purely dissipative and non-ergodic endomorphism ([L1], [R]) (on a non-compact phase space, though).

Let us pass now to the problem of a. c. i. measure. The properties of a. c. i. measures of positive entropy are well known in the one-dimensional case [Le]: they possess strong statistical properties of exactness, weak Bernoullity and exponential decreasing of correlations. The following result, concluding this paper, shows that actually every finite a. c. i. measures of $f \in \mathcal{S}_1$ has positive entropy (and hence possesses all the above properties).

THE THEOREM ON ENTROPY. — Let $\mu$ be a finite a.c.i. measure of $f \in \mathcal{S}_1$. Then $h_\mu(f) > 0$. In such a case the attractor A is the cycle of transitive intervals (case A3).

This theorem and the Theorem on solenoid's measure will be proved from a common viewpoint in Section 12.

Let us also mention some additional properties of "strange" attractors A, making them similar to solenoidal attractors (*see* § 11): $f|A$ is topologically minimal [*i.e.*, $\omega(x) = A$ for all $x \in A$] and the topological entropy $h(f|A)$ is equal to zero.

The structure of the present paper is clear from the Table of Contents. In order to make the picture complete, we present the proofs of the Theorems on the Attractor and

Ergodicity. These proofs have some advantages over those published earlier ([BL1]-[BL6]) due to the adaptation of them to the S-unimodal case as well as to the systematical use of the Koebe Principle instead of the Minimum Principle.

We would also like to draw the reader's attention to a number of technical results collected in this paper: the Expanding, the Distortion and the Density Lemmas. The most non-trivial ones are Lemmas 4.4 and 9.1.

Most of the results the authors can prove in a more general situation (polymodal and smooth). The survey of these generalizations will be given in Appendix 2.

The results of the present paper are announced in [BL1] and [BL5].


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SOME NOTATIONS AND CONVENTIONS:

$\mathbb{N}=\{1, 2, \ldots\}$ is the set of natural numbers;

$\mathrm{cl}\,X \equiv \bar{X}$ is the closure of the set X;

$\mathrm{int}\,X \equiv X^0$ is the interior of the set X;

$X^c=[0, 1]\setminus X$ is the complement of X;

$D(a, r)=\{x\in[0, 1] \,|\, |x-a|\leqq r\}$;

$[a, b]$ is the (closed) interval ended at $a$ and $b$ (without assuming that $a\leqq b$);

$c_k=f^k c$ where $c$ is the extremum; note that $c_1=1$, $c_2=0$ by the normalization S3.

*Note*: Recently we reveived G. Keller's preprint, which is closely related to the present paper.

## 2. Topological picture of the dynamics

In this section we have collected the well-known facts on the topological dynamics of S-unimodal maps, which will be systematically used (limit cycles, homtervals, spectral decomposition). We begin with some remarks about limit cycles of $f$.

Let $A=\{f^n a\}_{n=0}^{p-1}$ be the cycle of a periodic point $a$. It is said to be a *limit cycle* if $\mathrm{rl}^0(A)\neq\varnothing$. For maps with negative Schwarzian derivative it is equivalent for A to be attractive [*i.e.*, the modulus of the multiplier $\nu=(f^p)'(a)$ is less than 1] or neutral (*i.e.*, $|\nu|=1$) (*see* [CE]).

The role of the negative Schwarzian derivative condition was shown for the first time by D. Singer in 1978 by the following result which is the real analogue of the classical Fatou-Julia Theorem (1918-1920).

THEOREM A (on the Limit Cycle) [Si]. — *Let* A *be a limit cycle of the map* $f \in \mathcal{S}_1$. *Then* $c \in \mathrm{rl}^0(\mathrm{A})$. *Hence, f has at most one limit cycle.*

Notice that if the multiplier $\nu$ of the limit cycle A isn't equal to 1 then $\mathrm{A} \subset \mathrm{rl}^0(\mathrm{A})$. If $\nu = 1$ then for $f \in \mathcal{S}_1$ we have that $\mathrm{A} \subset \partial(\mathrm{rl}(\mathrm{A}))$. Moreover, $\mathrm{rl}(\mathrm{A}) = \mathrm{rl}^0(\mathrm{A}) \cup \bigcup_{n=0}^{\infty} f^{-n}\mathrm{A}$ and rl(A) contains some semi-neighborhood of A.

Let us pass now to Guckenheimer's important result on the absence of wandering intervals (1978). By *wandering intervals* we will always mean *strongly* wandering intervals, *i.e.*, such that $f^n \mathrm{J} \cap f^m \mathrm{J} = \varnothing$ $(n > m \geqq 0)$.

An interval J is called a *homterval* if all iterates $f^n$ are monotone on J. In other words, $\mathrm{int}(f^n \mathrm{J}) \not\ni c$ for $n \geqq 0$.

It is easy to understand the connection between wandering intervals and homtervals. If J is a homterval then either its orbit converges to a limit cycle, or J is wandering (perhaps, both).

Thus, the existence of wandering intervals is equivalent to the existence of homtervals.

THEOREM B (on Wandering Intervals) [G1]. — *Suppose* $f \in \mathcal{S}_1$ *has a wandering interval* J. *Then f has a limit cycle* A *and* $f^n \mathrm{J} \to \mathrm{A}$ *as* $n \to \infty$.

For a point $x \notin \{0, 1\} \cup \bigcup_{k=0}^{n-1} f^{-k} c$ denote by $\mathrm{H}_n = \mathrm{H}_n(x)$ the maximal interval on which $f^n$ is monotone. Set $\mathrm{M}_n \equiv \mathrm{M}_n(x) = f^n \mathrm{H}_n(x)$. The intervals $\mathrm{H}_n$ end at points $\{0, 1\} \cup \bigcup_{k=0}^{n-1} f^{-k} c$, and the intervals $\mathrm{M}_n$ end at points $\{c_k\}_{k=1}^{n}$ (recall that $c_1 = 1$, $c_2 = 0$). It follows from the Theorem on Wandering Intervals that if orb$(x)$ doesn't converge to a limit cycle, then

$$\lambda(\mathrm{H}_n(x)) \to 0 \qquad (n \to \infty). \tag{2.1}$$

Let us pass now to the description of the spectral decomposition of $f \in \mathcal{S}_1$. To this end we need a few concepts.

An interval $\mathrm{I} \subset [0, 1]$ will be called *periodic* with period $p$ if $\mathrm{int}(f^k \mathrm{I}) \cap \mathrm{int}(f^l \mathrm{I}) = \varnothing$ $(0 \leqq k < l \leqq p-1)$ and $f^p \mathrm{I} \subset \mathrm{I}$. In such a case the set $\mathcal{O} = \bigcup_{k=0}^{p-1} f^k \mathrm{I}$ is called a *cycle of intervals*.

An invariant compact set K will be called *transitive* if $f|\mathrm{K}$ is topologically transitive, *i.e.*, it has a dense orbit. The map $f|\mathrm{K}$ is called *topologically exact* if for any relative neighbourhood $\mathrm{U} \subset \mathrm{K}$ there exists an $n \in \mathbb{N}$ such that $f^n \mathrm{U} = \mathrm{K}$.

A periodic interval I of period $p$ will be called *transitive* (*exact*) if the restriction $f^p|I$ is topologically transitive (exact). Clearly, in such a case $c \in \operatorname{int} \mathcal{O}$, where $\mathcal{O}$ is the cycle of the interval I.

An invariant set S is called a *solenoid* (or a *solenoidal attractor* or a *Feigenbaum-like attractor*) of type $\{p\}_{n=1}^{\infty}$ if it has the following structure:

$$S = \bigcap_{n=1}^{\infty} \bigcup_{k=0}^{p_n - 1} f^k I_n$$

where $p_n \to \infty$ and where each $I_n$ is a periodic interval of period $p_n$, with $I_1 \supset I_2 \ldots$ It easily follows from the Theorem on Wandering Intervals that $\operatorname{int}(S) = \varnothing$ (*i.e.*, S is a Cantor set) and $f|S$ is topologically conjugate to a shift on a compact group (*see* [CE], [B1]).

Remark also that $c \in S$ and hence $S = \omega(c)$. Besides, it is easy to see that if $\omega(x) \supset S$ then $\omega(x) = S$. So, S is a maximal $\omega$-limit set.

If a set X isn't a set of first Baire category (*i.e.*, a countable union of nowhere dense sets) then X is said to be of 2nd category. Saying "*almost all* (*a.a.*) *in the sense of Baire*" we mean "on a set of 2nd category."

A closed invariant set $T \subset [0, 1]$ will be called a *topological attractor* of $f$ if

(i) $\operatorname{rl}(T)$ is a set of second category;

(ii) for any proper closed invariant subset $T' \subset T$, the set $\operatorname{rl}(T) \setminus \operatorname{rl}(T')$ is of 2nd category as well. (Compare with the concept of measure-theoretical attractor in Section 1.)

Denote by $\operatorname{Per}(f)$ the set of periodic points of $f$.

Let us state now the principal result on the topological dynamics of one-dimensional maps $f \in \mathcal{S}_1$. It is essentially based upon the Theorem on Wandering Intervals.

THEOREM C (on the Spectral Decomposition) [JR]. — *There is the following decomposition*:

$$\overline{\operatorname{Per}(f)} = T \cup \bigcup_j R_j$$

*where* T *is the unique topological attractor of* $f$, *and the* $R_j$ *are invariant transitive Cantor sets. Moreover*, T *has the structure of one of the following three types:*

T1. $T = \{f^k a\}_{k=0}^{\infty}$ *is a limit cycle;*

T2. $T = \bigcap_{n=1}^{\infty} \bigcup_{n=1}^{p_n - 1} f^k I_n$ *is a solenoid;*

T3. $T = \bigcup_{k=0}^{p-1} f^k I$ *is a cycle of exact intervals.*

In cases T1 and T3 there are only a finite number of repellers $R_j$, while in case T2 there are countably many of them. Any two sets of the decomposition have at most finitely many points in common. For any point $x \in [0, 1]$ either $\omega(x) = \mathrm{T}$ or $f^n x \in \mathrm{R}_j$ for some $n, j$.

*Remark.* — The restrictions $f | \mathrm{R}_j$ are topologically conjugate to One-Sided Topological Markov Chains.

Cases T3 and T2 sometimes are called *finitely* and *infinitely renormalizable* correspondingly. We usually will refer to case T2 as *solenoidal.* (The terminology agrees with that introduced in Section 1.)

## 3. Distortion lemmas

In this section we collect the principal analytical tools for studying the measurable dynamics of one-dimensional maps. Within it X will denote a measurable set and I will denote some interval. Set

$$\operatorname{dens}(\mathrm{X} \mid \mathrm{I}) = \lambda(\mathrm{X} \cap \mathrm{I}) / \lambda(\mathrm{I}).$$

Let us introduce at once all notation and terms related to the density notion that we will use throughout the paper. For a point $a \in [0, 1]$ set

$$\operatorname{dens}(\mathrm{X} \mid a) = \lim_{\varepsilon \to 0} \frac{1}{2\varepsilon} \operatorname{dens}(\mathrm{X} \mid [a-\varepsilon, a+\varepsilon])$$

if this limit exists. If $\operatorname{dens}(\mathrm{X} \mid a) = 1$ then $a$ is called a density point of X. In such a case we say also that "X is λ-dense at $a$" ("λ" is used in order to avoid the confusion with topological density). The Lebesgue Theorem on Density Points states that any measurable set X is λ-dense at a. a. its points. The expressions "X is λ-dense at a from the left" or "from the right" as well as the notations $\overline{\operatorname{dens}}(\mathrm{X} \mid a)$ and $\overline{\operatorname{dens}}(\mathrm{X} \mid a)$ for upper and lower density are clear without extra explanations.

Finally, let us introduce one non-standard notation. For an interval $\mathrm{I} = [a, b]$ set

$$\operatorname{Dens}_a(\mathrm{X} \mid \mathrm{I}) \equiv \operatorname{Dens}(\mathrm{X} \mid [a, b]) = \sup_{y \in \mathrm{I}^0} \operatorname{Dens}(\mathrm{X} \mid [a, y]).$$

Note that $\operatorname{Dens}(\mathrm{X} \mid [a, b]) \neq \operatorname{Dens}(\mathrm{X} \mid [b, a])$.

LEMMA 3.1 (the First Distortin Lemma) ([BL6], [BL8]). — *Let $f: [0, 1] \to [0, 1]$ be a $\mathrm{C}^1$-smooth map for which*

$$\mathrm{C}_1 |x - c|^{\nu} \leqq |f'(x)| \leqq \mathrm{C}_2 |x - c|^{\nu}$$

*in neighbourhoods of critical points (where* ν *depends on* $c$*). If* $\mathrm{I} \subset [0, 1]$ *and* $\operatorname{dens}(\mathrm{X} \mid \mathrm{I}) \leqq 1/4$ *then*

$$\operatorname{dens}(f\mathrm{X} \mid f\mathrm{I}) \leqq \mathrm{A} \operatorname{dens}(\mathrm{X} \mid \mathrm{I})$$

*where the constant* A *is independent of* I, X. ■

The following result establishes the main analytical property of functions with negative Schwarzian derivative. It is called the Koebe Principle because it is the exact analogue of the classical Koebe Theorem in geometric function theory.

Up to the end of this section $\varphi : I \to J$ will denote a diffeomorphism of open intervals with negative Schwarzian derivative.

THE KOEBE PRINCIPLE ([vS], [G2]). — Let $r \in (0, 1)$. Then there exists a constant $C_r$ independent of $\varphi$ such that for any points $x_1, x_2 \in I$ for which $\operatorname{dist}(\varphi(x_i), \partial J) \geqq r\lambda(J)$ the following estimate holds:

$$\left|\frac{\varphi'(x_1)}{\varphi'(x_2)}\right| \leqq C_r. \quad ■$$

LEMMA 3.4 (The Second Distortion Lemma) [BL6]. — *Divide the interval* I *into two intervals* L ∪ R *with a common endpoint a. Then*

$$\left.\begin{array}{r}\operatorname{Dens}_a(X \mid L) \leqq \delta \\ \lambda(\varphi L)/\lambda(\varphi R) \leqq K\end{array}\right\} \Rightarrow \operatorname{Dens}_{\varphi(a)}(\varphi X \mid \varphi L) \leqq \gamma(\delta, K)$$

*where the function* $\gamma(\delta, K)$ *is independent of* $\varphi$ *and* $\gamma(\delta, K) \to 0$ *as* $\delta \to 0$ *for any fixed* K.

*In other words, if the set* X *is thin in the interval* L *and the interval* $\varphi L$ *isn't too long compared with* $\varphi R$, *then* $\varphi X$ *is thin in* $\varphi L$.

*Proof.* — Suppose that for some interval $N = [a, b] \subset L$ we have: $\operatorname{dens}(\varphi X \mid \varphi N) \geqq \varepsilon$. Consider the point $\alpha \in N$ for which

$$|\varphi(\alpha) - \varphi(b)| = \frac{\varepsilon}{2}\lambda(\varphi N).$$

Then

$$\operatorname{dens}(\varphi X \mid [\varphi(\alpha), \varphi(a)]) \geqq \frac{\varepsilon}{2}.$$

Now apply the Koebe Principle to the map $\varphi \mid [\alpha, a]$. We obtain a constant $C = C(\varepsilon, K)$ such that for any $x_1, x_2 \in [\alpha, a]$ the following estimate is satisfied:

$$\left|\frac{\varphi'(x_1)}{\varphi'(x_2)}\right| \leqq C.$$

Consequently,

$$\frac{\varepsilon}{2} \leqq \operatorname{dens}(\varphi X \mid [\varphi(\alpha), \varphi(a)]) \leqq C \operatorname{dens}(X \mid [\alpha, a]) \leqq C\delta$$

(here we use the estimate for $\mathrm{Dens}_a$, not for dens). It follows that $\varepsilon \to 0$ as $\delta \to 0$, and the lemma is proved. ■

## 4. Expanding lemmas

The first three purely topological lemmas of this section are less well known than they deserve. In spite of their simplicity, they work very efficiently. The fourth lemma is much more complicated, but up to now we've been applying it only in the solenoidal case.

LEMMA 4.1 (On Non-contractability). — *Let* J *be an interval whose orbit doesn't converge to a limit cycle. Then*

$$\lim_{n \in \mathrm{N}} \lambda(f^n \mathrm{J}) > 0.$$

*Proof.* — Since $f$ has no wandering intervals, $f^p \mathrm{J} \cap \mathrm{J} \neq \varnothing$ for some $p \in \mathbb{N}$. Hence, the set $\mathrm{I} = \bigcup_{n=0}^{\infty} f^{pn} \mathrm{J}$ is an $f^p$-invariant (perhaps, non-closed) interval. If I contains two $f^p$-fixed points $\alpha$, $\beta$ then $f^{pn} \mathrm{J} \supset [\alpha, \beta]$ for all sufficiently large $n$ which implies the required inequality. The simple analysis of the case when I contains only one fixed point we leave to the reader. ■

Now let us consider the involution $\tau: x \mapsto x'$ where $f(x) = f(x')$ (it is well-defined on the interval $[0, c'_3]$). The points $x$ and $x' = \tau(x)$ will be called *τ-symmetric*. A set X is called τ-symmetric if $\tau(\mathrm{X}) = \mathrm{X}$ and *locally τ-symmetric* if it is τ-symmetric in a neighbourhood of $c$.

Say that a point $y$ lies τ-nearer to $c$ than $x$ if $y \in (x, x')$.

The following lemma is, in fact, contained in the paper by Guckenheimer [G1]. In an explicit form, it was stated in [BL3], [BL4]. In this lemma the main specificity of the unimodal case is concentrated.

LEMMA 4.2 (the First Expanding Lemma). — *Let* $\mathrm{I} = [a, a']$ *be a τ-symmetric interval, let* $x \notin (0, 1) \setminus \bigcup_{k=0}^{\infty} f^{-k} c$ *be a point whose orbit passes through* $\mathrm{I}^0$, *and let* $n$ *be the first moment for which* $f^n x \in \mathrm{I}^0$. *Then provided* $\lambda(\mathrm{I})$ *is sufficiently small, we have*

(i) $\mathrm{M}_n(x) \supset \mathrm{I}$ *in the finitely renormalizable case;*

(ii) $\mathrm{M}_n(x) \supset \mathrm{I}^{\vee}$ *in the solenoidal case, where* $\mathrm{I}^{\vee}$ *is that half-interval* $[a, c]$ *or* $[c, a']$ *which contains* $f^n x$.

*Proof.* — (i) We must show that none of the intervals $\mathrm{M}_n^{\pm}$ is contained in I. Fix $\sigma \in \{\pm 1\}$.

If $\lambda(I)$ is sufficiently small then $n$ is large enough, and by (2.1) the interval $H_n(x)$ doesn't end at the points $c$, 0, 1. Hence, it ends at some points of the set $\bigcup_{k=1}^{n} f^{-k}c$. Consequently, there is a $p\in[1, n-1]$ such that $M_n^\sigma = f^p[f^{n-p}x, c]$.

Let us consider the $\tau$-symmetric interval $K=[f^{n-p}x, \tau(f^{n-p}x)]$. By the definition of $n$ we have that $K\supset I$. If $M_n^\sigma\subset I$ then $f^p I\subset f^p K = M_n^\sigma\subset I$, so I is $f^p$-invariant. But as we consider the transitive case, I is contained in an exact periodic interval, which is, of course, impossible.

(ii) By the same argument as above one can be convinced that otherwise there is a $p$ such that $f^p I^v\subset I^v$ and $f^p|I^v$ is monotone. But then $I^v$ contains a limit cycle, which contradicts the assumption. ■

LEMMA 4.3 (the Second Expanding Lemma). — *Suppose $c$ is non-periodic. Let $I\subset[0, 1]$ be an arbitrary interval of sufficiently small length such that the orbit of $c_3$ passes through its interior. Let $n$ be the first moment for which $f^n c_3\in I^c$. Then $M_n(c_3)\supset I$.*

*Proof.* — The endpoints of the interval $M_n(c_3)$ belong to the set $\{c_k\}_{k=1}^n$. By assumption, they lie outside of I. ■

Remark now that by non-degenerateness of the extremum $c$, the involution $\tau$ is smooth and $\tau'(c)=-1$. Hence, in a sufficiently small neighbourhood $[c-\eta, c+\eta]$ of the extremum, the involution $\tau$ is Lipschitz with the constant 2.

Further, under the circumstances of Lemma 4.2 let $M_n^+$ be the component of $M_n\diagdown\{f^n x\}$ which contains $c$, and $M_n^-$ be the other component. Unlike the previous results of this section, the following lemma is of an analytical nature and uses the negative Schwarzian derivative condition in an essential manner.

LEMMA 4.4 (the Third Expanding Lemma). — *Let a map $f\in\mathscr{S}_1$ have no limit cycles. Then under the assumptions of Lemma 4.2 there exists a constant $K>0$ such that*

(i) $\lambda(M_n^+)\geqq K^{-1}\lambda(I)$;

(ii) *for* $x=c_3$, $\lambda(M_n^-)\geqq K^{-1}\lambda(I)$.

Part (ii) of the lemma was proved in [BL7] and was applied there for proving that $\lambda(S)=0$ for solenoidal attractors S (*see* § 8). We think that this lemma could be useful in plenty of other problems.

Here we confine ourselves to the proof of Part (i). Remark at once that in the finitely renormalizable case the statement trivially follows from Lemma 4.2 (i) and the Lipschitz property of $\tau$. Thus, Lemma 4.4 (i) is actually concerned with the solenoidal case only.

*Proof of Lemma* 4.4 (i). — Recall that $I=[a, a']$. Denote $x_k=f^k x$. Let, for definiteness, $x_n\in(a, c)$. By Lemma 4.2 $M_n^+\supset[x_n, c]$. If $M_n^+\supset[x_n, a']$ then as we have already noted the statement is trivial. So, from now on we assume that

$$M_n^+\subset[x_n, a')\subset I^0. \tag{4.1}$$

If $\lambda(I)$ is sufficiently small then $M_n^+=f^{n-s}[x_s, c]$ for some $s\in[0, n-1]$. By assumption, $x_s$ lies outside $I^0$. Let us show that it lies $\tau$-nearer to $c$ than all points

$x_l (l=s+1, \ldots, n-1)$. Otherwise denote by $\tilde{x}_s$ that point $x_s$, $x'_s$ for which $x_l \in (\tilde{x}_s, c)$. Note that $c_{l-s}$ lies on the same side of $c$ as $x_l$ (since $f^{n-l}$ is monotone on $[x, c_{l-s}] = f^l \mathrm{H}_n$).

Now consider two cases:

(*a*) If $c_{l-s} \in [x_s, c]$, then the map $f^{l-s}$ monotonically transforms the interval $[\tilde{x}_s, c]$ inside itself—on the interval $[x_l, c_{l-s}]$. In such a case $f$ would have a limit cycle, which contradicts our assumption.

(*b*) If $\tilde{x}_s \in [c_{l-s}, x_l]$ then by (4.1)

$$\mathrm{I}^0 \supset \mathrm{M}_n^+ = f^{n-l}[c_{l-s}, x_l] \ni x_{n-(l-s)},$$

despite $n$ being the first moment for which $f^n x \in \mathrm{I}^0$.

Thus, the points $x_{s+1}, \ldots, x_{n-1}$ lie $\tau$-farther from $c$ than $x_s$, while $x_n$ lies $\tau$-nearer.

Let now $\hat{x}_s$ denote that one of the points $x_s$, $x'_s$ that lies on the same side of $c$ as $x_n$. It is shown in the papers [Mi], [G1] that under such circumstances there exists an interval $(\alpha, \beta) \ni \hat{x}_s$ such that

$$f^{n-s} \alpha = \alpha, \qquad f^{n-s} \beta = \beta' \equiv \tau(\beta).$$

Moreover,

$$|(f^{n-s})'(y)| \geqq \gamma > 0 \tag{4.2}$$

for all $y \in [\alpha, \beta]$ where $\gamma$ does not depend on $x$, $n$.

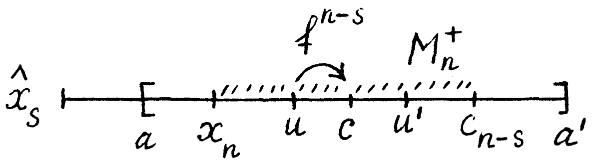


Fig. 1.

Consider the point $u \in [\hat{x}_s, \beta]$ for which $f^{n-s} u = c$. By (4.2)

$$|x_n - c| \geqq \gamma |\hat{x}_s - u|. \tag{4.3}$$

Remark now that

$$[x_n, c_{n-s}] \equiv \mathrm{M}_n^+ \supset [x_n, u'], \tag{4.4}$$

for otherwise the function $f^{n-s}$ would monotonically transform the interval $[c, u']$ into itself (despite the absence of limit cycles). Now consider two cases:

(*a*) $|u-c| \geqq (1/2)|a-c|$. Then by (4.4) and the 2-Lipschitz property of $\tau$, we have

$$\lambda(\mathrm{M}_n^+) \geqq |c-u'| \geqq \frac{1}{2}|c-u| \geqq \frac{1}{4}|a-c| \geqq \frac{1}{12}\lambda(\mathrm{I}).$$

(*b*) $|u-c|<(1/2)|a-c|$. In particular, $u\in(a,c)$. Then by (4.3) and (4.4) we get

$$\lambda(M_n^+)\geqq|x_n-c|\geqq\gamma|a-u|\geqq\frac{\gamma}{2}|a-c|\geqq\frac{\gamma}{6}\lambda(I).$$

The lemma is proved. ■

## 5. The measure-theoretical attractor

Here we will prove the Theorem on the Attractor stated in the Introduction. We are starting with the well-known lemma whose proof gives the simplest illustration of the self-similarity idea (passing from small scales to large ones controlling the distortion).

LEMMA 5.1 [G1]. — *Let* K *be an invariant compact set which does not contain the extremum c and* (*in the case when f has a limit cycle* $\alpha$) *does not intersect* $\mathrm{rl}^0(\alpha)$ (1). *Then* $\lambda(K)=0$.

*Proof.* — Let $x$ be an arbitrary point of K distinct from the endpoints 0, 1 and preimages of the neutral cycle $\alpha$ (if there is such an $\alpha$). As $K\cap \mathrm{rl}^0(\alpha)=\varnothing$, these assumptions exclude at most countably many points.

Since $\mathrm{orb}(x)$ does not converge to a limit cycle, by (2.1) the intervals $H_n$ for sufficiently large $n$ end at preimages of $c$, not at the endpoints 0, 1. Hence $M_n^\pm=f^{n-k\pm}[c,f^{k\pm}x]$ for some $k_\pm\in[0,n-1]$. But

$$|c-f^{k\pm}x|\geqq\mathrm{dist}(c,K)\equiv\delta>0.$$

By Lemma 4.1, $\lambda(M_n^\pm)\geqq\varepsilon>0$ [remark that $\varepsilon=\varepsilon(x)$ depends on $x$ since $x$ can be near to the neutral cycle; the independence of $\varepsilon$ from $n$ for fixed $x$ is important].

Thus, for any $\gamma\in(0,\varepsilon]$ there is an interval $H_n^\gamma\ni x$ which is monotonically transformed by $f^n$ onto $D(f^n x,\gamma)$.

Remark now that by the Theorem on Wandering Intervals the set K is nowhere dense (in fact, it follows from the easier fact: The orbit of a hypothetical homterval would have to approach the critical point, Schwartz, 1963). If follows from the compactness of K, that for all $y\in K$

$$\mathrm{dens}(K\,|\,D(y,\gamma))\leqq q(\gamma)<1.$$

Applying to $f^n: H_n^\varepsilon\to D(f^n x,\varepsilon)$ the Koebe Principle, we obtain

$$\mathrm{dens}(K\,|\,H_n^{\varepsilon/2})\leqq\rho(\varepsilon)<1.$$

As $\lambda(H_n^{\varepsilon/2})\to 0\ (n\to\infty)$, $\mathrm{dens}(K\,|\,x)<1$; so $x$ is not a density point of K. Since $x$ is an arbitrary point of K excluding at most countably many points, we conclude $\lambda(K)=0$. ■

(1) But it may happen that $K\supset\alpha$ where $\alpha$ is a neutral cycle.

*Remark*. – In [G1] a stronger statement is proved: if an invariant compact set K does not contain critical points and limit cycles, then $f|K$ is an expanding transformation, *i.e.*, $\exists C>0$, $\gamma>1$ such that

$$|(f^n)'(x)|\geqq C\gamma^n, \qquad x\in K, \quad n\in\mathbb{N}.$$

However, we will not make use of this.

COROLLARY 5.1. — *If a map f has no limit cycles, then* $\omega(x)\ni c$ *for a. a.* $x\in[0,1]$.

*Proof*. – Let us consider the following family of invariant compact sets:

$$K_n=\left\{x:|f^m x-c|\geqq\frac{1}{n}(m=0,1,\ldots)\right\}.$$

By the above lemma $\lambda(K_n)=0$. But $[0,1]\diagdown\bigcup_{n=1}^{\infty}K_n=\{x:\omega(x)\ni c\}$. ∎

Call a set E *nowhere λ-dense* if $\operatorname{dens}(E|I)<1$ for any interval I.

LEMMA 5.2. — *If* E *is a nowhere λ-dense invariant set of positive measure, then for a. a.* $x\in E$ *we have* $\omega(x)=\omega(c)\ni c$.

*Remark*. – So, $c$ is a recurrent point: $\omega(c)\ni c$.

*Proof*. – By Corollary 5.1, $\omega(x)\supset\omega(c)$ for a. a. $x\in E$.

In order to prove the inverse inclusion, let us consider a density point $x\in E$ and show that $\omega(x)\subset\omega(c)$. Indeed, otherwise there is a sequence $L\subset\mathbb{N}$ such that $\operatorname{dist}(f^n x,\operatorname{orb}(c))\geqq\varepsilon>0$ for all $n\in L$. Estimating the distortion of the map $f^n:H_n^{\varepsilon/2}\to D(f^n x,\varepsilon/2)$ in the same way as in the proof of Lemma 5.1, we will be convinced that $\operatorname{dens}(E|x)<1$, contradicting the assumption.

So, $\omega(x)=\omega(c)$ for a. a. $x\in E$. As $\omega(x)\ni c$ for *a. a.* $x\in E$, $\omega(c)\ni c$, and we are done. ∎

PROOF OF THE THEOREM ON THE ATTRACTOR. – Let us consider subsequently Cases T1-T3 of the Theorem on Spectral Decomposition (§ 2).

In case T1 the limit cycle T attracts the critical point $c$ (Theorem A). Hence, the invariant compact set $K=[0,1]\diagdown\mathrm{rl}^0(T)$ satisfies the assumptions of Lemma 5.1. We conclude that $\lambda(K)=0$ and hence T is the unique measure-theoretical attractor.

In case T2 we have

$$c\in\omega(c)=T=\bigcap_{n=1}^{\infty}\mathcal{O}_n$$

where $\mathcal{O}_n$ are the cycles of intervals of period $p_n\to\infty$. By Corollary 5.1 $\omega(x)\ni c$ for a. a. $x\in[0,1]$. For such an $x$ we have $\omega(x)\supset\omega(c)=T$ and hence $\omega(x)=T$, since T is the maximal ω-limit set.

Let us pass to the main case, T3, when there is the cycle $\mathcal{O}$ of transitive intervals. As $c \in \operatorname{int} \mathcal{O}$, by Corollary 5.1, a.a. orbits are absorbed by $\mathcal{O}$. Hence, it is sufficient to consider the restriction $f | \mathcal{O}$.

We intend to show that one of the following holds:

(i) $\omega(x) = \mathcal{O}$ for a.a. $x \in \mathcal{O}$;

(ii) $\omega(x) = \omega(c) \ni c$ for a.a. $x \in \mathcal{O}$.

In the first case we obtain the standard transitive attractor (of type A 3), in the second case we obtain the standard or the "strange" attractor (A3 or A4) depending on $\omega(c) = \mathcal{O}$ or $\omega(c) \neq \mathcal{O}$.

So, let us consider a countable base of intervals $J_n$ of the space $\mathcal{O}$ and construct for each of them the following invariant compact set:

$$K_n = \{ x \in \mathcal{O} : f^m x \notin J_n \ (m = 0, 1, \ldots) \}.$$

Set $K_\infty = \bigcup_{n=1}^{\infty} K_n$. Then $\mathcal{O} \setminus K_\infty = \{ x \in \mathcal{O} : \omega(x) = \mathcal{O} \}$. Hence, if $\lambda(K_\infty) = 0$ then case (i) holds. So, assume in what follows that $\lambda(K_\infty) > 0$ and hence $\lambda(K_n) > 0$ for some $n$.

All the sets $K_n$ are nowhere dense. Indeed, if $K_n$ contains an interval L then $K_n \supset \bigcup_{m=0}^{\infty} f^m L = \mathcal{O}$ which is not the case. Consequently, we can apply Lemma 5.2: $\omega(x) = \omega(c) \ni c$ for a.a. $x \in K_n$. Since $K_n$ is nowhere dense, $\omega(c)$ is nowhere dense as well.

Thus, $A = \omega(c)$ is the unique measure-theoretical attractor for the set $K_\infty$ possessing all the properties enumerated in the theorem.

It remains to show that $\lambda(\mathcal{O} \setminus K_\infty) = 0$. To this end remark that $\mathcal{O} \setminus K_\infty \equiv K_\infty^c$ is nowhere $\lambda$-dense. Indeed, otherwise there is an interval $I \subset \mathcal{O}$ such that $\operatorname{dens}(K_\infty^c | I) = 1$. As $\bigcup_{k=1}^{n} f^k I = \mathcal{O}$ for some $n \in \mathbb{N}$, $\operatorname{dens}(K_\infty^c | \mathcal{O}) = 1$. Hence, $\lambda(K_\infty) = 0$, contradicting the assumption. Consequently, by Lemma 5.2 $\omega(x) = \omega(c)$ for a.a. $x \in \mathcal{O} \setminus K_\infty$. But as we know, $\omega(x) = \mathcal{O}$ for all $x \in K_\infty$, while $\omega(c) \neq \mathcal{O}$. This contradiction completes the proof. ■

## 6. Ergodicity

In this section we will prove the Theorem on Ergodicity. It is based upon the following technical lemma which will be used throughout the paper.

LEMMA 6.1 (On $\lambda$-density at the extremum). — *Suppose f has no limit cycles. Let* X *be a measurable invariant locally* $\tau$*-symmetric set of positive measure. Then* $\operatorname{dens}(X | c) = 1$.

*Remark-term.* – Hence, any invariant set X of positive measure (perhaps, non-$\tau$-symmetric) is $\lambda$-dense at $c_n$ ($n = 1, 2, \ldots$) on that side of $c_n$ which is the $f^n$-image of a neighbourhood of $c$. Such a side of $c_n$ will be called *good*.

*Proof.* — Consider a $\tau$-symmetric neighbourhood $I=[a, a']$ of $c$ in which the involution $\tau$ is 2-Lipschitz and the set X is $\tau$-symmetric. By Corollary 5.1 $\omega(x)\ni c$ for a.a. $x\in X$. Let us consider a density point $x\in X$ with this property. Let $n$ be the first moment when $f^n x\in I^0$. Assume for definiteness that $x_n \equiv f^n x\in(a, c)$. By the First and Third Expanding Lemmas

$$M_n^- \supset [a, x_n], \qquad \text{and} \qquad \lambda(M_n^+)\geqq K^{-1}|a-x_n|.$$

Consequently, there exist intervals $R_n$, $L_n$ with the common endpoint $x$ for which

$$f^n R_n=[a, x_n], \qquad f^n L_n=M_n^+$$

and $f^n$ has no critical points inside $L_n$, $R_n$. Applying the Second Distortion Lemma, we obtain

$$\text{dens}(X^c\,|\,[a, x_n])\leqq\gamma(\delta, K) \tag{6.1}$$

where $\delta=\text{Dens}_a(X^c\,|\,R_n)$.

If the interval I is short, then by Lemma 4.1 the interval $R_n$ is short as well. As $x$ is a density point of X, $\delta$ is small and hence $\gamma(\delta, K)$ is small as well. Hence, by (6.1) the set X is thick in the interval $[a, x_n]$.

By the Lipschitz property of $\tau$, X is thick also in the symmetric interval $[x_n', a']$. Setting $I_1=[x_n, x_n']$, we obtain

$$\text{dens}(X\,|\,I\setminus I_1)\geqq 1-\varepsilon$$

for some small $\varepsilon>0$. Replacing $I\equiv I_0$ by $I_1$, we obtain

$$\text{dens}(X\,|\,I_1\setminus I_2)\geqq 1-\varepsilon$$

where $I_2=[x_{n(1)}, x'_{n(1)}]$, and $n(1)$ is the first moment for which $f^{n(1)}x\in I_1^0$. Continuing the process, we obtain the nested sequence of intervals $I_k$ shrinking to $c$ and such that

$$\text{dens}(X\,|\,I_k\setminus I_{k+1})\geqq 1-\varepsilon, \qquad k=0, 1, \ldots$$

Hence, $\text{dens}(X\,|\,I)\geqq 1-\varepsilon$. Since $\varepsilon\to 0$ as $\lambda(I)\to 0$, the lemma is proved. ■

PROOF OF THE THEOREM ON ERGODICITY. — Let $[0, 1]=X_1\cup X_2$ where $X_i$ are completely invariant sets of positive measure, $X_1\cap X_2=\varnothing$. As $\tau(X_i)\subset f^{-1}(fX_i)=X_i$, the sets $X_i$ are locally symmetric. By Lemma 6.1, $\text{dens}(X_i\,|\,c)=1$ which is impossible. ■

## 7. Absence of strongly wandering sets

Here we are proving the Theorem on Strongly Wandering Sets stated in the Introduction. It can be regarded as the strengthening of the Theorem on Wandering Intervals. It is also some sort of "conservativity" of $f$.

Let us note that the absence of strongly wandering sets X for which $f^n|X$ is monotone for $n\in\mathbb{N}$ (*cf.* [S]) immediately follows from the ergodicity. Indeed, if Y is an arbitrary non-trivial measurable subset of X, then $\bigcup_{n=0}^{\infty} f^{-n}\left(\bigcup_{m=0}^{\infty} f^m Y\right)$ is a non-trivial measurable completely invariant subset of [0, 1].

PROOF OF THE THEOREM ON STRONGLY WANDERING SETS. — Let X be a set of a positive measure, and $X_n=f^n X$. Take a density point $x\in X\setminus\bigcup_{n=0}^{\infty} f^{-j}c$ such that $\omega(x)\ni c$ and consider the moments $n(1)<n(2)<\ldots$ of the $\tau$-nearest approaches of orb$(x)$ to the extremum $c$ [this means that $x_{n(k)}\equiv f^{n(k)}x$ lies $\tau$-nearer to $c$ than all points $x_1$ $(0\leq l<n(k))$]. Moreover, let us start from the moment $n(1)$ for which $x_{n(1)}$ is close to $c$. Let

$u_{k-1}$ be that one of the points $x_{n(k-1)}$, $x'_{n(k-1)}$ which lies on the same side of $c$ as $x_{n(k)}$;

$v_{k+1}$ be that one of $x_{n(k+1)}$, $x'_{n(k+1)}$ which lies on the same side of $c$ as $x_{n(k)}$; and $x^{\#}_{n(k)}$ be that one of $x_{n(k)}$, $x'_{n(k)}$ which lies to the left of $c$.

By the First Expanding Lemma

$$M_{n(k)}\supset[u_{k-1}, v_{k+1}].$$

Denote by $L_k$ and $R_k$ the semi-neighbourhoods of $x$ which are monotonically mapped by $f^{n(k)}$ onto $[x_{n(k)}, v_{k+1}]$ and $[x_{n(k)}, u_{k-1}]$ correspondingly.

Fix $\varepsilon>0$. Proving Lemma 6.1, we have shown that $X_{n(l)}$ is thick in the interval $[u_{l-1}, x_{n(l)}]$ for all sufficiently large $l$:

$$\mathrm{dens}\,(X_{n(l)}\,|\,[u_{l-1}, x_{n(l)}])\geqq 1-\varepsilon.$$

Applying $f$ once more, we get, by the First Distortion Lemma,

$$\mathrm{dens}\,(X_{n(l)+1}\,|\,[x_{n(l-1)+1}, x_{n(l)+1}])\geqq 1-A\,\varepsilon. \tag{7.1}$$

On the other hand, as $\sum_{k=1}^{\infty}|x^{\#}_{n(k)}-x^{\#}_{n(k+1)}|<\infty$, there exist arbitrary large $k$ for which $|x^{\#}_{n(k+1)}-x^{\#}_{n(k)}|<|x^{\#}_{n(k)}-x^{\#}_{n(k-1)}|$. Then by the 2-Lipschitz property of $\tau$ we get

$$|x_{n(k)}-v_{k+1}|<4\,|x_{n(k)}-u_{k-1}|. \tag{7.2}$$

Applying to $f^{n(k)}|L_k\cup R_k$ the second Distortion Lemma and taking into account (7.2), we conclude

$$\mathrm{dens}\,(X_{n(k)}\,|\,[x_{n(k)}, v_{k+1}])\geqq 1-\gamma(\delta_k, 4)$$

where $\delta_k=\mathrm{Dens}_x(X^c\,|\,L_k)$. As $x$ is a density point of X, $\gamma(\delta_k, 4)<\varepsilon$ for sufficiently large $k$.

By the First Distortion Lemma we obtain

$$(7.3) \qquad \operatorname{dens}(X_{n(k)+1} \mid [x_{n(k)+1}, x_{n(k+1)+1}]) \geqq 1 - A\gamma(\delta_k, 4) \geqq 1 - A\varepsilon.$$

For $l=k+1$ and sufficiently small $\varepsilon>0$ the estimates (7.1), (7.3) yield $X_{n(k+1)+1} \cap X_{n(k)+1} \neq \varnothing$. The theorem is proved.

## 8. The solenoidal case: pure dissipativeness

In Sections 8-10 we will classify maps $f \in \mathscr{S}_1$ from the viewpoint of the Hopf decomposition. In this section we dwell on the solenoidal case which gives an amusing example of a purely dissipative endomorphism having no wandering sets of positive measure ([2]).

Let us start from the simple remark: *the conservative kernel* $C(f)$ (*see* Appendix 1) *is contained in the attractor* A. Indeed, otherwise there is a set $Y \subset C(f)$ of positive measure such that $\operatorname{dist}(Y, A) > 0$. Then the orbits of a. a. points $y \in Y$ must return to Y infinitely many times. But this is impossible since $f^n y \to A$ for a. a. $y$.

Thus, if $\lambda(A)=0$ then $f$ is purely dissipative (in Section 10 the converse will be proved). Clearly, it is held in the cyclic case. The next theorem shows that solenoidal maps are purely dissipative as well. It was proved in [G2] for dyadic solenoids (*i. e.*, $p_n = 2^n$) and in [BL7], [MMSS] for unimodal solenoids of any type.

THEOREM ON SOLENOID'S MEASURE. — Let a transformation $f \in \mathscr{S}_1$ have a solenoidal attractor A of type $\{p_n\}_{n=0}^{\infty}$. Then

(i) $\lambda(A)=0$;

(ii) If A is a solenoid of finite type (*i. e.*, $p_{n+1}/p_n \leqq c$), then $\dim A < 1$.

*Sketch of the proof.* — (i) By definition, $A = \bigcap_{n=1}^{\infty} \bigcup_{k=0}^{p_n - 1} I_k^{(n)}$, where $I_k^{(n)}$ are periodic intervals of period $p_n$, and $f: I_k^{(n)} \to I_{k+1}^{(n)}$, $I_0^{(n)} \ni c$. Evidently, $c_{p_n}$ is the $\tau$-nearest point to $c$ of the orbit $\{c_k\}_{k=1}^{p_n}$. Setting $K_n^{\pm} = M_{p_n - 3}^{\pm}(c_3)$ ([3]) we find from the Second and Third (ii) Expanding Lemmas:

$$(8.1) \qquad K_n^{+} \supset [c_{p_n}, c'_{p_n}] \equiv J_n, \qquad \lambda(K_n^{-}) \geqq K^1 \lambda(J_n).$$

([2]) In [He] a special construction of a strongly wandering set X for any purely dissipative endomorphism is given. But the meaning of this result is unclear as X can have zero measure.

([3]) It is easy to see that in the solenoidal case $c_3 \in \{0, 1\} \bigcup_{n=0}^{\infty} f^{-k} c$ and hence, the intervals $M_n^{\pm}(c_3)$ are well-defined. This holds in the transitive case as well if we exclude the well-known Ulam-Neumann map (when $c_3 = 0$).

Let $R_n$ and $L_n$ be the semi-neighbourhoods of $c_3$ which are mapped by $f^{p_n-3}$ onto $K_n^-$ and $J_n$ correspondingly. It easily follows from the absence of limit cycles that $J_n$ lies on the "good" side of $c_{p_n}$ (*see* the Remark in Section 6). Hence, $L_n$ lies on the "good" side of $c_3$. If $\lambda(A)>0$, then by Lemma 6.1 the attractor A is $\lambda$-dense on this side. Consequently, for sufficiently large $n$ the set A is thick in $L_n$, *i.e.*, $\mathrm{Dens}_{c_3}(A \mid L_n) \equiv 1-\delta_n$ is close to 1.

Applying the Second Distortion Lemma to the function $f^{p_n-3} \mid R_n \cup L_n$ and taking into account (8.1), we get

$$\mathrm{dens}(A \mid J_n) \geqq 1-\gamma(\delta_n, K) \geqq 1-\varepsilon$$

for sufficiently large $n$. Then $\mathrm{dens}(\tau A \mid J_n) \geqq 1-4\varepsilon$, and hence $\lambda(A \cap \tau A)>0$. This contradicts the injectivity of $f \mid A$.

(ii) In [G2], [BL7], [MMSS] more is proved: $\mathrm{dens}(Q^{(n+1)} \mid I_k^{(n)}) \leqq q<1$ where $Q^{(n)}=\bigcup_{k=0}^{p_n-1} I_k^{(n)}$, and $q$ is independent of $n$, $k$. This easily implies $\dim A<1$. ■

## 9. Density lemmas

By the relative length of an interval J in an interval I we mean $\mathrm{dens}(J \mid I)$.

LEMMA 9.1 (The Main Density Lemma). — *Suppose $f$ has no limit cycles. Let* X *be an invariant set of positive measure. Then* $\forall\, \varepsilon>0$, $\exists\, \delta>0$ *with the following property:*

*If* I *is any short interval* ($\lambda(I)<\delta$) *intersecting* A, *then there is an open interval* $J \subset I \setminus A$ (*perhaps empty*) *such that* $\mathrm{dens}(X \mid L)>1-\varepsilon$ *for any component* L *of* $I \setminus J$.

At the end of the section we will dwell in more detail on the case $\lambda(A)>0$. And now let us formulate one corollary. The notation used for it has the same meaning as in the above lemma.

COROLLARY 9.1. — *Under the assumptions of Lemma* 9.1, *for any point* $a \in A$ *the following holds:*

$$\max\{\mathrm{dens}(X \mid [a-\varepsilon, a]), \mathrm{dens}(X \mid [a, a+\varepsilon])\} \to 1 \ (\varepsilon \to 0).$$

*Hence* $\mathrm{dens}(X \mid a) \geqq 1/2$.

*Proof.* — Let $\sigma_a: x \mapsto 2a-x$ denote the usual central symmetry with respect to $a$. For $b \in \mathbb{R}$ set $\bar{b}=\sigma_a(b)$.

Given $\varepsilon>0$, find a $\delta>0$ by Lemma 9.1. Let $I=[b, \bar{b}]$ be an arbitrary interval symmetric with respect to a with $\lambda(I)<\delta$. We want to show that X is thick either in $[b, a]$ or in $[a, \bar{b}]$.

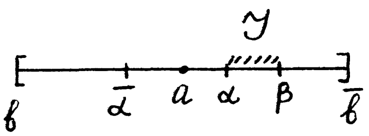

Fig. 2.

To this end consider an interval $J=(\alpha, \beta) \subset I$ with the properties described in Lemma 9.1. Let, for definiteness, $J \subset [a, \bar{b}]$ and $\alpha$ lies nearer to $a$ than $\beta$. By Lemma 9.1, $\operatorname{dens}(X \mid [b, \alpha]) > 1-\varepsilon$ and, hence, $\operatorname{dens}(X \mid [b, a]) > 1-2\varepsilon$. This proves the corollary. ■

*Remark.* — If $a=c$ then instead of the intervals $[a-\varepsilon, a]$ and $[a, a+\varepsilon]$ we can consider $[b, a]$ and $[a, \tau(b)]$ correspondingly. So, $\operatorname{dens}(X \mid A)$ can be understood in the sense of $\tau$-symmetric intervals.

PROOF OF THE MAIN DENSITY LEMMA. — Let first $A=\mathcal{O}$ be a cycle of intervals (case A3) and $\omega(c) \neq \mathcal{O}$. Then $X=A \bmod 0$. Indeed, otherwise X is nowhere $\lambda$-dense (due to exactness) and by Lemma 5.2 $\omega(x)=\omega(c) \neq A$ for a.a. $x \in X$ — a contradiction. So, $\operatorname{dens}(X \mid I)=1$ for any interval $I \subset A$.

Now, let $A=\omega(c)$. Consider an arbitrary interval $I=[a, b]$ such that $I^0 \cap A \neq \varnothing$. Then $\operatorname{orb}(c_3)$ passes through $I^0$ infinitely many times. Consider, as usual, the first moment $n$ for which $c_{3+n} \equiv f^n c_3 \in I^0$. Let us show that there is an interval $V \subset I$ containing $c_{3+n}$ and having a common endpoint with I, in which X is thick:

$$\operatorname{dens}(X \mid V) \geqq 1-\varepsilon \qquad \text{if} \quad \lambda(I) < \delta = \delta(\varepsilon).$$

By the Second Expanding Lemma, $M_n(c_3) \supset I$. Consequently, there exist intervals L and R ending at $c_3$ which are diffeomorphically mapped by $f^n$ onto $[a, c_{n+3}]$ and $[c_{n+3}, b]$ correspondingly.

Suppose for definiteness that L lies on the "good" side of $c_3$. By Lemma 4.1, L is a short interval if $\lambda(I) < \delta$ is sufficiently small. Hence by Lemma 6.1, the set X is thick in $L: \eta \equiv \operatorname{Dens}_{c_3}(X^c \mid L)$ is small.

Fix now a constant $K > 2/\varepsilon$ and consider two cases:

(*a*) $|a-c_{n+3}| \leqq K |c_{n+3}-b|$. Then by the Second Distortion Lemma we have

$$\operatorname{dens}(X^c \mid [a, c_{n+3}]) < \gamma(\eta, K).$$

For sufficiently small $\eta$ we have $\gamma(\eta, K) < \varepsilon$ and, hence, we can set $V=[a, c_{n+3}]$.

(*b*) $|a-c_{n+3}] > K |c_{n+3}-b|$. Then let us consider the point $d \in (a, c_{n+3})$ such that $|d-c_{n+3}| = K |c_{n+3}-b|$.

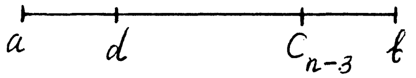


Fig. 3.

Let us show that one can set $V=[d, b]$. Indeed, in the same way as in (*a*) we get for sufficiently small $\eta$:

$$\operatorname{dens}(X^c \mid [d, c_{n+3}]) \leqq \gamma(\eta, K) < \varepsilon/2.$$

Consequently,

$$\operatorname{dens}(X^c \mid [d, b]) \leqq \frac{\lambda(X^c \cap [d, c_{n+3}]) + |c_{n+3} - b|}{|d-b|}$$

$$\leqq \operatorname{dens}(X^c \mid [d, c_{n+3}]) + \frac{|c_{n+3} - b|}{|d-b|} < \frac{\varepsilon}{2} + \frac{1}{K+1} < \varepsilon.$$

So, the existence of an interval V with the required properties is proved. Now consider the interval $I_1 = I \setminus V$. If $I_1^0 \cap A \neq \varnothing$ then we make the above construction again replacing I by $I_1$. More precisely, we consider the first moment $n(1) > n$ when $f^{n(1)} c_3 \in I_1^0$ and find the interval $V_1 \subset I_1$, containing $c_{n(1)+3}$ and having a common endpoint with $I_1$, such that $\operatorname{dens}(X \mid V_1) \geqq 1 - \varepsilon$.

Continuing in such a manner, we will construct a sequence (finite or infinite) of intervals $V = V_0, V_1, \ldots$ with disjoint interiors, and a decreasing sequence of intervals $I = I_0 \supset I_1 \supset \ldots$ such that $\bigcup_{i=0}^{m} V_i = I \setminus I_{m+1}$; $\operatorname{dens}(X \mid V_i) \geqq 1 - \varepsilon$. Moreover, $V_i \ni c_{n(i)+3}$, where $n(i)$ is the first moment for which $f^m c_3 \in I_i$.

Let us consider the interval $J = \bigcap I_i$. Then the set $I \setminus J$ is covered by the intervals $V_i$. Hence, for any component L of this set we have

$$\operatorname{dens}(X \mid L) \geqq 1 - \varepsilon. \tag{9.1}$$

It remains to show that $J^0 \cap A = \varnothing$. Indeed, if the process above was finite, then it was stopped at the moment $m$ for which $I_m^0 \cap A = \varnothing$.

Suppose the process was infinite. If $J^0 \cap A \neq \varnothing$, we can consider the first moment $l$ for which $f^l c_3 \in J^0$. As $c_{n(m)} \notin J^0$, $l \neq n(m)$ for $m = 0, 1, \ldots$ Find an $m$ such that $n(m) < l < n(m+1)$. Then $I_{m+1}^0 \supset J^0 \ni c_l$ contradicting the choice of $n(m+1)$ as the first moment $k$ for which $f^k c_3 \in I_{m+1}^0$. The lemma is proved. ■

For an interval $I \subset [0, 1]$, components of $I \setminus A$ will be called *gaps in* I. Gaps in [0, 1] will be called simply "*gaps*".

The statement of Lemma 9.1 can be made more accurate in the case $\lambda(A) > 0$.

LEMMA 9.2. — *Under the conditions of Lemma* 9.1 *suppose also that* $\lambda(A) > 0$. *Then one of the following holds:*

(i) *all gaps in* I *have the relative length* $< \varepsilon$; *in such a case* $\operatorname{dens}(X \mid I) > 1 - 2\varepsilon$;

(ii) *there is a unique gap* J *in* I *of the relative length* $\geqq \varepsilon$; *in such a case* $\operatorname{dens}(X \mid I) > 1 - \varepsilon$ *for any component* L *of* $I \setminus J$.

*Proof.* — By ergodicity, $\lambda(A \cap X) > 0$. So, replacing X by $X \cap A$, we can assume $X \subset A$. Then, clearly, the interval J in Lemma 9.1 can be chosen as a gap in I.

As $\operatorname{dens}(X \mid I \setminus J) > 1 - \varepsilon$, J can be the only gap in I of relative length $> \varepsilon$. If J is such a gap, case (ii) holds, otherwise we have case (i). ■

COROLLARY 9.2. — *Under the conditions of Lemma* 9.2 *we have*

(i) *if a is a boundary point of some gap, then* $\operatorname{dens}(X \mid a) = 1/2$, i.e., X *is* $\lambda$*-dense at a on the* A*-side*;

(ii) *if a is not a boundary point of any gap, then* $\overline{\operatorname{dens}}(X \mid a) = 1$.

*Proof.* — (i) is the immediate consequence of Corollary 9.1 and the convention $X \subset A$ (which, as we have remarked, can be accepted without loss of generality).

(ii) Let us consider any short interval $I = [\bar{b}, b]$ symmetric with respect to $a$. We want to show that X is thick in some symmetric interval $K \subset I$. In case (i) of Lemma 9.2 we can set $K = I$. In case (ii) let us consider the maximal gap $J = [\alpha, \beta]$ (see *Fig.* 2) and note that $\alpha \neq a$, as $a$ is not a boundary point of any gap. Set $T = [\bar{\alpha}, \beta]$. Then $T^0 \cap A \ni a$ and, thus, we can apply Lemma 9.2 to T. It gives $\operatorname{dens}(X \mid [\bar{\alpha}, \alpha]) \geqq 1 - \varepsilon$, and we are done. ■

## 10. The conservative kernel

The Theorem on the Conservative Kernel has been stated in the Introduction. It has been explained in Section 8 that the Conservative Kernel $C(f)$ is contained in the attractor A. The solenoidal case has been studied there as well. The following result completes the proof of the Theorem.

THEOREM ON CONSERVATIVITY. — *Let* $\lambda(A) > 0$. *Then* $f \mid A$ *is conservative.*

*Proof.* — Let $X \subset A$ be an invariant set of positive measure. We have to show that $X = A$ mod 0 (*see* Appendix 1). By Corollary 9.1 $\underline{\operatorname{dens}}(X \mid a) \geqq 1/2$ for any point $a \in A$. So, the set $A \setminus X$ has no density points and hence $\lambda(A \setminus X) = 0$. ■

REMARK ON THE HOFBAUER-KELLER EXAMPLE [HK]. — In this amusing example the averages of the Lebesgue measure $\lambda$ converge to the Dirac measure on the repelling fixed point $b$:

$$\frac{1}{n} \sum_{k=0}^{n-1} f_*^k \lambda \to \delta_b. \tag{10.1}$$

We will show in the next section that in such a case, A is a standard transitive attractor (since "strange" attractors don't contain periodic points). Consequently, here we have a conservative map of the interval without finite a.c.i. measure.

## 11. Further topological properties of Cantor attractors

The present section is concerned with Cantor attractors, *i.e.*, solenoidal and "strange" attractors. Certainly, we do not get any new information about solenoids whose topological structure is completely clear.

LEMMA 11.1 (On Inverse Branches). — *Let* A *be a Cantor attractor. Then for any* $\varepsilon>0$ *there exists an* N *such that for* $n \geqq N$ *there are no single-valued inverse branches* $f_i^{-n}$ *defined in neighbourhoods* $D(a, \varepsilon)$ *of* $a \in A$ *and such that* $f_i^{-n} a \in A$.

THE EQUIVALENT STATEMENT. — Let $r_n(x)$ be the distance from $f^n x$ to the nearest endpoint of the interval $M_n(x)$. Then

$$\sup_{x \in A} r_n(x) \to 0 \qquad (n \to \infty). \tag{11.0}$$

*Proof.* — Suppose it is not the case: let $\varepsilon>0$, $a_n \in A$ and suppose there are inverse branches $f_i^{-n}$ defined on the intervals $D(a_n, \varepsilon)$ such that $f_i^{-n} a_n \equiv b_n \in A$. Then there are intervals $K_n^{\pm}$ ended at $b_n$ and such that

$$f^n K_n^{\pm} = D_n^{\pm}\left(a, \frac{\varepsilon}{2}\right) \equiv D_n^{\pm},$$

where $D_n^{\pm}(a, \delta) = \{x : 0 \leqq \pm(x-a) \leqq \delta\}$.

Further, consider an invariant set X of positive measure satisfying the following property: There is an $\eta>0$ such that for any interval I of length $\varepsilon/2$ the inequality $\mathrm{dens}(X \mid I) \leqq 1-\eta$ is valid. If $\lambda(A)>0$ then we can set $X=A$. Otherwise take a small $\xi>0$ and set

$$X = \{x : \mathrm{dist}(f^m x, A) \leqq \xi \ (m=0, 1, \ldots)\}.$$

Then $\mathrm{dens}(X^c \mid D_n^{\pm}) \geqq \eta > 0$. Applying the Koebe Principle to $f^n : K_n^{\pm} \to D_n^{\pm}$ we get

$$\mathrm{dens}(X^c \mid K_n^{\pm}) \geqq \kappa > 0, \tag{11.1}$$

$$C^{-1} \leqq \lambda(K_n^+)/\lambda(K_n^-) \leqq C. \tag{11.2}$$

Let $\sigma_n : x \mapsto 2b_n - x$ be the symmetry with respect to $b_n$. Suppose for definiteness that $\lambda(K_n^-) \geqq \lambda(K_n^+)$ and consider the interval $\sigma_n K_n^-$ containing $K_n^+$. Then (11.1) and (11.2) yield

$$\mathrm{dens}(X^c \mid \sigma_n K_n^-) \geqq \mathrm{dens}(X^c \mid K_n^+) \frac{\lambda(K_n^+)}{\lambda(K_n^-)} \geqq C^{-1} \kappa. \tag{11.3}$$

The estimates (11.1) and (11.3) contradict Corollary 9.1. ■

THE THEOREM ON CANTOR ATTRACTORS. — If A is a Cantor attractor then

(i) the transformation $f \mid A$ is minimal, *i.e.*, $\omega(x) = A$ for any point $x \in A$ (in particular, A does not contain periodic points);

(ii) the topological entropy $h(f \mid A)$ is equal to zero;

(iii) $\tau(c_n) \notin A$ for any $n \in \mathbb{N}$ (so, $c_n$ are boundary points of gaps).

*Proof.* — (i) Let us show that if $\omega(x) \not\ni c$ then $r_n(x) \geqq \varepsilon > 0$ [which contradicts (10.0)]. Indeed, $r_n(x) = \min_{\pm} \lambda(M_n^{\pm}(x))$. For sufficiently large $n$ we have

$$M_n^{\pm}(x) = f^{n-k_{\pm}}[c, f^{k_{\pm}} x] \qquad \text{where} \quad 0 \leqq k_{\pm} \leqq n-1.$$

If $\omega(x) \not\ni c$ then $|f^{k\pm} x - c| \geqq \delta > 0$ and the required condition follows from Lemma 4.1.

(ii) We will make use of some standard facts of the entropy theory of dynamical systems. Suppose $h(f|A) > 0$. Then by the Variational Principle there is an invariant measure $\mu$ of a positive entropy: $h_\mu(f) > 0$, with supp $\mu \subset A$.

Now make use of the Pesin-Ledrappier theory of unstable manifolds (*see* [Le]). It yields that for almost each point $\bar{x} = (x, x_{-1}, x_2, \ldots)$ of the natural extension of $(f, \mu)$ the series of the inverse branches $f_i^{-n}$ is well-defined in $\varepsilon = \varepsilon(x)$-neighbourhood of $\bar{x}$, $f_i^{-n} x = x_{-n}$. This contradicts Lemma 11.1.

(iii) Suppose the opposite is valid: $a_m = \tau(c_m) \in A$, $m \geqq 2$. Set $C_n = \{c_k\}_{k=0}^n$. Take an $\varepsilon > 0$ such that there is the inverse branch $f_0^{-1} : D(c_{m+1}, \varepsilon) \to L \ni a$ and $L \cap C_{m+1} = \varnothing$. By Lemma 4.1 there is a $\delta > 0$ such that for any component $K_{n,i}$ of $f^{-n} D(c_{m+1}, \delta)$ we have

$$\text{diam}\, K_{n,i} < \varepsilon. \tag{11.4}$$

Let us construct a chain of intervals in the following way: $K_1 = D(c_{m+1}, \delta)$, $K_0 = f_0^{-1} K_1$, $K_{-1}$-some component of $f^{-1} K_0$ intersecting A and so on. Let $i > 0$ be the first moment for which $K_{-i} \cap C_{m+1} \neq \varnothing$. Clearly $K_{-i} \ni c_{m+1}$.

By (11.4), the branch $f_0^{-1}$ is well-defined on $K_{-i}$. Set $K_{-(i+1)} = f_0^{-1} K_i \subset K_0$ and go on with the construction. In such a manner we will get a chain of intervals $\{K_{-i}\}_{i=0}^\infty$ such that $K_{-i}$ is a component of the inverse image $f^{-i} K_0$ and $K_{-i} \cap C_m = \varnothing$. Hence, there exist inverse branches $f_0^{-i} : K_0 \to K_{-i}$, contradicting Lemma 11.1. ∎

Let us mention a corollary of the above theorem which will be used in the next section.

Denote by $\mathscr{L}$ the family of gaps L for which $\tau L \cap A \neq \varnothing$ or, equivalently, $f L \cap A \neq \varnothing$.

COROLLARY 11.1. — *The family $\mathscr{L}$ is infinite.*

*Proof.* — If a gap L does not belong to $\mathscr{L}$, then $fL$ is a gap as well. So, if $f^n L \notin \mathscr{L}$ for all $n \in \mathbb{N}$, then L is a homterval. As there are no homtervals, there is an $n \in \mathbb{N}$ for which $f^n L \in \mathscr{L}$.

By the above theorem, the points $c_n$ lie on the boundary of some gaps $L_n (n \in \mathbb{N})$. If $\mathscr{L}$ were finite, the extremum $c$ would be preperiodic, despite the property $\omega(c) = A$. ∎

## 12. The finite a.c.i. measure has a positive entropy

This section is devoted to the proof of the Theorem on Entropy stated in the Introduction and at the same time to the new proof of the Theorem on Solenoid's Measure (*see* § 8).

LEMMA 12.1 (Injective Scheme of Gluing). — *Let the attractor* A *contain an invariant set* X *of positive measure such that* $f : X \to X$ *is an invertible map. Then*

(i) *the extremum c lies on the boundary of a gap;*

(ii) *if $f(a) = f(b)$ for some $a, b \in A$, then a and b lie on the boundary of some gaps* L *and* M. *Moreover,* $\tau L \cap M = \varnothing$. *In particular,* A *is a Cantor attractor.*

*Proof.* — (i) By Corollary 9.2 (ii), if $c$ is not a boundary point of any gap, then there exist arbitrary short $\tau$-symmetric intervals $I=[a, a']$ for which $\operatorname{dens}(X \mid I) \geqq 1-\varepsilon$. By the 2-Lipschitz property of $\tau$ near $c$, we get $\operatorname{dens}(\tau X \mid I) \geqq 1-2\varepsilon$. Consequently, $X \cap \tau X \neq \varnothing$, despite the assumption.

(ii) Suppose $a$ is not a boundary point of any gap. Then by Corollary 9.2 (ii) there is an interval $I=[d, \bar{d}]$ symmetric with respect to a and such that

$$\operatorname{dens}(X \mid I)>1-\varepsilon. \tag{12.1}$$

By the Main Density Lemma there exists an interval $K \subset \tau I$ having a common endpoint with $\tau I$ (say, $\alpha=\tau(d)$), containing $[\alpha, b]$ and such that $\operatorname{dens}(X \mid K) \geqq 1-\varepsilon$. Hence

$$\operatorname{dens}(\tau X \mid \tau K) \geqq 1-L^2 \varepsilon, \tag{12.2}$$

where L is the Lipschitz constant of the involution $\tau$ (on the whole interval $[0, c_3']$ where $\tau$ is defined).

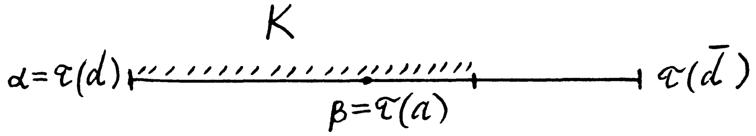


Fig. 4.

On the other hand, as $\tau K \supset [d, a]$, (12.1) implies

$$\operatorname{dens}(X \mid \tau K) \geqq 1-2\varepsilon. \tag{12.3}$$

It follows from (12.2) and (12.3) that $X \cap \tau X \neq \varnothing$ — a contradiction.

We have shown that $a$ and $b$ are endpoints of some gaps L and M. Suppose $\tau L \cap M \neq \varnothing$. Consider then short semi-neighbourhoods U and V of $a$ and $b$ correspondingly such that $\tau U=V$. By Corollary 9.2 (i)

$$\operatorname{dens}(X \mid U) \geqq 1-\varepsilon, \qquad \operatorname{dens}(X \mid V) \geqq 1-\varepsilon.$$

Using the Lipschitz property of $\tau$ once more, we conclude $\tau X \cap X \neq \varnothing$. This contradiction completes the proof. ■

LEMMA 12.2. — *There are no attractors* A *possessing the Injective Gluing Scheme described in Lemma* 12.1.

*Proof.* — Let $\mathscr{L}$ denote the family of gaps L for which $\tau L \cap A \neq \varnothing$ (as at the end of Section 11). For each $L \in \mathscr{L}$ find a point $x_L \in A \cap \tau L$.

By Corollary 11.1 the family $\mathscr{L}$ is infinite. Hence, we can extract a sequence $L_i \in \mathscr{L}$ converging to some point $a \in A$. Then $x_{L_i} \rightarrow \tau(a)$. As points of A can approach $c$ only from one side (the Injective Gluing Scheme), we have $a \neq c$.

Further, as $a$ and $\tau(a)$ lie on A, by the Injective Gluing Scheme, they are the endpoints of some gaps L and M such that $\tau L \cap M=\varnothing$. Consequently, $L_i$ and L lie on the different sides of $a$ and, hence, $\tau L_i \subset M$ for sufficiently large $i$. This contradicts the property $\tau L_i \cap A \neq \varnothing$. ■

Now we get an immediate corollary from the above lemmas:

COROLLARY 12.1. — *An attractor* A *contains no measurable invariant sets* X *of positive measure for which* $f: X \to X$ *is invertible.* ■

THE SECOND PROOF OF THE THEOREM ON SOLENOID'S MEASURE (§ 8). — The theorem immediately follows from the above Corollary as $f|A$ is injective for a solenoidal A. ■

PROOF OF THE THEOREM ON ENTROPY. — Let $\mu$ be an a.c.i. probability measure, with supp $\mu = A$. If $h_\mu(f)=0$ then $(f, \mu)$ is invertible as the transformation with the invariant measure [Ro]. In other words, there is a measurable invariant set $X \subset A$ such that $\mu(X)=1$ and $f: X \to X$ is a one-to-one transformation. As $\mu$ is absolutely continuous, $\lambda(X)>0$. So, we have arrived at a contradiction with Corollary 12.1. ■

## APPENDIX 1

### Measurable endomorphisms with a quasi-invariant measure

Let X be a space with a finite measure $\nu$, and $g: X \to X$ be a measurable endomorphism. The measure $\nu$ is called *quasi-invariant* if $\nu(Y)=0 \Rightarrow \nu(g^{-1}Y)=0$ for any measurable set $Y \subset X$. If as well $\nu(Y)=0 \Rightarrow \nu(gY)=0$, then the transformation $g$ is called *non-singular*. *In what follows we will assume that g is non-singular.*

They say that some property is *valid mod* 0 if it is valid outside some null-set.

A set Y is called *invariant* if $gY \subset Y$ and *completely invariant* if also $g^{-1}Y \subset Y$.

A map $g$ is called *ergodic* if one of the following equivalent properties holds:

E1. There are no partitions $X = X_1 \cup X_2$ of X into two invariant measurable sets of positive measure.

E2. There are no non-trivial completely invariant subsets $Y \subset X$ [*i.e.*, such that $0<\nu(Y)<\nu(X)$].

A set X is called *weakly wandering* if $f^n X \cap X = \varnothing$ $(n \geqq 1)$ and *strongly wandering* if $f^n X \cap f^m X = \varnothing$ $(n>m\geqq 0)$. Of course, in the invertible case these notions are equivalent.

The transformation $g$ is called *conservative* if it satisfies one of the following equivalent conditions:

C1. $g$ has no weakly wandering sets of a positive measure.

C2. Any invariant measurable set Y is completely invariant mod 0, *i.e.*, $\nu(g^{-1}Y \setminus Y)=0$.

C3. The Poincaré Return Theorem holds: if $\nu(Y)>0$, then orbits of a.a. points $y \in Y$ return to Y infinitely many times.

Further, the transformation $g$ is *ergodic and conservative simultaneously* if one of the following properties is valid:

EC1. There are no non-trivial measurable invariant subsets $Y \subset X$ (exactly this property is useful for us).

EC2. Let $Y \subset X$ be any set of positive measure. Then a. a. orbits pass through it infinitely many times.

We call the endomorphism $g$ *asymptotically conservative* if $X = \bigcup_{n=0}^{\infty} g^{-n} C \bmod 0$ where C is an invariant set on which $g$ is conservative.

A non-conservative map is called *dissipative*. It is called *purely dissipative* if there are no invariant sets $Y \subset X$ of positive measure on which $g$ is conservative.

THEOREM D (The Hopf Decomposition). — *Let* $g: X \to X$ *be a non-singular transformation. Then there is the following decomposition*:

$$X = AC(g) \cup D(g)$$

*where* AC *and* D *are completely invariant sets such that* $g \mid AC$ *is asymptotically conservative, while* $g \mid D$ *is dissipative. Moreover,* $AC = \bigcup_{n=0}^{\infty} f^{-n} C$ *where* $C = C(g)$ *is the maximal invariant subset on which g is conservative. We call it* the Conservative Kernel *of* $g$. *The sets* AC, D *and* C *are uniquely defined mod* 0.

COROLLARY. — *An ergodic endomorphism g is either asymptotically conservative or purely dissipative.*

Remark that if an invertible transformation $g$ is purely dissipative then $X = \bigcup_{n=0}^{\infty} f^n Y$ where Y is a strongly wandering set. This statement fails for non-invertible transformations as consideration of solenoidal maps of the interval shows (*see* § 8).

## APPENDIX 2

### Polymodal and smooth generalizations: survey of the results

Let us introduce some classes of transformations of the interval or the circle.

$\mathscr{S}_d - C^3$-maps with a negative Schwarzian derivative and $d$ non-flat critical points each of which is an extremum.

$\mathscr{A}_d - C^2$-maps with $d$ non-flat critical points each of which is an extremum.

$\mathfrak{R}_d - C^2$-maps with $d$ non-flat critical points.

$$\mathscr{S} = \bigcup_{d=0}^{\infty} \mathscr{S}_d, \qquad \mathscr{A} = \bigcup_{d=0}^{\infty} \mathscr{A}_d, \qquad \mathfrak{R} = \bigcup_{d=0}^{\infty} \mathfrak{R}_d.$$

*Remark.* — For $C^\infty$ maps non-flatness of critical points means that $f^{(n)}(c) \neq 0$ for some $n \in \mathbb{N}$. For lower smoothness this term needs extra explanation of the sort: "$|f'|$ is of a power order near critical points" (*see* [BL9], [MMS]).

The widest reasonable class for which the results of the present paper should be valid is the class $\mathfrak{R}$. At the present time the authors can prove *all* the results for $f \in \mathscr{S}$ — and part of them in wider classes. Let us present these generalizations in more detail (we are dwelling on all, not only our own, results).

Section 2. SINGER'S THEOREM ON LIMIT CYCLES is valid for arbitrary maps with negative Schwarzian derivative [Si]. Consequently, such maps with $d$ critical points have at most $d+2$ limit cycles. In the recent paper [MMS] it was proved that any map $f \in \mathfrak{R}$ has finitely many limit cycles.

*The Theorem on Wandering Intervals* has been generalized subsequently to the following classes: homeomorphisms of the circle of class $\mathfrak{R}$ [Y]; class $\mathscr{A}_1$ [MS]; class $\mathscr{S}$ [L2]; class $\mathscr{A}$ [BL9]; class $\mathfrak{R}$ [MMS]. So by the present time it has been proved in the maximal sensible generality. The analytical tools for smooth generalizations were developed in [Y], [MS], while the principle step toward the polymodal case was made in [L2].

*The Theorem on the Spectral Decomposition* is of a purely topological nature and goes back to Sharkovskii's papers (1960's). The complete picture for arbitrary continuous maps of the interval is described in [B1] and for maps of one-dimensional branched manifolds in [B2]. For piecewise monotone maps it was described as well by many other authors (Z. Nitecki, F. Hofbauer, Preston...). F. Hofbauer treated also the discontinuous case.

Section 4. THE LEMMA ON NON-CONTRACTABILITY is valid for arbitrary continuous maps under the extra assumption that J is non-wandering.

In *The First Expanding Lemma* $f$ can be non-smooth, but still unimodal. In the polymodal case this lemma should be changed by the technique of unimodal decompositions [L2].

*The Second Expanding Lemma* clearly holds for arbitrary piecewise monotone maps.

For *the Third Expanding Lemma* the condition of unimodalness is essential. The proof uses the condition $Sf<0$ as well, but probably it is extra.

Section 5. *Lemma* 5.1 was proven by Mañé [Ma] for arbitrary $C^2$-maps.

*The Theorem on Attractors* was proven in [BL1], [BL2], [BL3] for arbitrary maps with $Sf<0$ and finitely many critical points (perhaps, flat). The authors also can prove it for $f \in \mathscr{A}$. For this class the way should be another: one must start from the Decomposition into Ergodic Components as in [BL6] and then construct the attractor for each ergodic component of positive measure.

Section 6. In the polymodal case *the Theorem on Ergodicity* must be replaced by the Decomposition into Ergodic Components. As we have just mentioned, it was realised in [BL6] for $f \in \mathscr{S}$.

Note, however, that in the polymodal solenoidal case *Lemma* 6.1 is proven in a slightly weaker form, since we have no exact polymodal version of Lemma 4.4.

Section 7. Using the technique of Unimodal Decompositions, *the Theorem on Strongly Wandering Sets* can be proven for $f \in \mathscr{S}$.

Section 8. *The Theorem on Solenoid's Measure* is proven in [MMSS] for $f \in \mathscr{A}_1$ and in [BL8] – for the widest class $\mathfrak{R}$.

Sections 9-11. The results of these sections are generalized to the class $\mathscr{S}$ without essential changes. Actually, we can prove them for $f \in \mathscr{A}$.

Section 12. These results we can prove for $f \in \mathscr{S}$.

*Note added in proof.* – We draw the reader's attention to a paper by J. Guckenheimer and S. Johnson, *Distortion of* S-*unimodal maps* closely related to the present one, and to a paper by M. Martens, *Cantor attractors of unimodal maps* proving that strange attractors have zero measure.

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A. M. BLOKH and M. YU. LYUBICH,
SUNY at Stony Brook,
Institute for Mathematical Sciences,
Stony Brook,
New York 11794, U.S.A.