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MODULI SPACES OF STABLE REAL ALGEBRAIC CURVES

BY M. SEPPÄLÄ (1)

ABSTRACT. — The construction that the Geometric Invariant Theory has developed to compactify the moduli spaces of smooth complex algebraic curves can be interpreted analytically. In that setting the same constructions can be applied to curves defined over the real numbers. That way we construct a topology for the space of real isomorphism classess of stable real algebraic curves of a given genus g, $g>1$. In that topology this moduli space is a connected and compact Hausdorff space.

1. Introduction

We study real algebraic curves. Locally they are defined as sets of zeros of a finite number of real polynomials in a real projective (or affine) space. This set of polynomials has to satisfy certain regularity conditions.

We may equally well consider the same set polynomials as polynomials defined in a complex projective (or affine) space. From this point of view real algebraic curves are complex algebraic curves defined by real polynomials. That is the usual way of looking at real algebraic curves (*cf.* [9] *or* [2]).

A complex projective curve is simply a compact Riemann surface. If the curve is defined by real polynomials, then the corresponding Riemann surface carries an antiholomorphic involution (which is induced by the complex conjugation). Conversely, any compact Riemann surface together with an antiholomorphic involution can be embedded in a complex projective space in such a way that its image is a curve defined by real polynomials. We conclude, therefore, that a projective real algebraic curve is simply a compact Riemann surface together with an antiholomorphic involution. We call such a Riemann surface *symmetric*.

The arithmetic genus of a real curve is the genus of the corresponding Riemann surface, *i.e.*, the genus of the corresponding complex curve.

Real algebraic curves of a given arithmetic genus g fall into $[(3g+4)/2]$ topologically different types. This was realized already by Felix Klein (*see* e.g. [10] and

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[15]). Consequently Klein started investigating the set of isomorphism classes of real algebraic curves *of a given topological type*. Analytically this can be viewed as the space $M(p, n, k)$ of isomorphism classes of dianalytic structures of a compact surface of genus p with n boundary components and k cross-caps. Here the topological invariants p, n and k satisfy $g=2p+n+k-1$. For more details *see* e. g. [2].

It follows from the classical Teichmüller theory that this moduli space $M(p, n, k)$ is connected. It carries also real analytic and semialgebraic structures [13], Theorem 2.2.

The disjoint union of the moduli spaces $M(p, n, k)$ forms the space of isomorphism classes of real algebraic curves of arithmetic genus g. Keeping this topological classification in mind it is clear that there is no natural topology on the space $M_{\mathbf{R}}^g$ of smooth real algebraic curves of arithmetic genus g which would make that space connected. One cannot change the topological type of a real algebraic curve without doing some violence to it.

The situation changes, however, if we extend our considerations to stable real algebraic curves. We say that a real algebraic curve (with double points) is *stable* if the corresponding complex curve is stable. It turns out that one can change the topological type of a real algebraic curve by a continuous deformation that first pinches some Jordan curve on the real algebraic curve to a point and then thickens it again to a Jordan curve.

To be more precise, a real algebraic curve is a complex algebraic curve C together with an antiholomorphic involution $\sigma: C \rightarrow C$. Two such curves (C_1, σ_1) and (C_2, σ_2) are *real isomorphic* if there exists an isomorphism $f: C_1 \rightarrow C_2$ of complex curves satisfying $f \circ \sigma_1 = \sigma_2 \circ f$.

A real curve (C, σ) is stable if the complex curve C is stable. We construct in this paper a topology for the space $\bar{M}_{\mathbf{R}}^g$ of real isomorphism classes of stable real algebraic curves of arithmetic genus g, $g>1$. We show that this space is a compact and connected topological Hausdorff space.

The case $g=1$ can be treated with explicit methods. There are, however, some unexpected technical complications here. One compactification of the moduli space of smooth genus 1 real algebraic curves is a circle. Robert Silhol has worked out the details in this case. The case $g=0$ is completely elementary. The moduli space $\bar{M}_{\mathbf{R}}^0$ of real algebraic curves reduces in this case to be a set consisting of two points.

Our problems and results can be best formulated in the language of real algebraic geometry. Our constructions and proofs are, however, quite geometrical. In all proofs we view a real algebraic curve as a compact Riemann surface with a symmetry.

Therefore it is necessary for our proofs to recall a number of well known results concerning real algebraic curves and Riemann surfaces. That is done in the subsequent chapter.

Certain proofs included here are technically quite complicated. It would be desirable to find shortcuts. Most of what is presented here was known already to Klein. For instance, the result which states that the moduli space of stable real curves is connected, was probably known to Klein. In [10]. p. 8, Klein writes:

> Wir könnten z. B. mehrere Züge unserer Curve gleichzeitig in isolierte Doppelpunkte überführen. Wir köntten auch jeden einzelnen der erhaltenen Doppelpunkte vollends verschwinden lassen, wodurch das Geschlecht wieder auf p steigt, die Zügerzahl und Art unserer Curve aber eine andere wird. Es ist besonderes intressant... alle diese Möglichkeiten ins Einzelne zu vervolgen... wir lassen also alle diese Entwicklungen hier der Kürze halber bei Seite.

Here the word "Züge" refers to the components of the fixed-point set of the complex conjugation of the real curve.

It is clear that Klein understood that one can pass with a continuous deformation from one topological type of real algebraic curves to any other type by letting the smooth curve degenerate to a curve having isolated double points or, more precisely, to a stable real algebraic curve.

Klein was mainly working with polynomials that define the real curve in question.

Using only the polynomials and the information one can derive from them it is probably impossible to give a precise proof to the connected-ness of the moduli space.

Here we start from this interesting observation of Klein and follow through all the possible configurations. That leads to proving that the moduli space of stable real curves is connected. Our methods are, however, completely different from those of Klein.

This paper is closely related with [12] and [13]. In [12] I studied the moduli space of stable complex curves of a given genus g, $g>3$, and the subset formed by real algebraic curves. It turned out that this subset is a connected semialgebraic subvariety and the quasiregular real part of the complex moduli space. This result has been improved recently. Robert Silhol and, independently of Silhol, Marc Coppens and Jan Denef have shown that the subset of real curves is a semialgebraic subvariety even in the case $g=3$.

In [12] I studied complex isomorphism classes of real algebraic curves. In the present paper we study real isomorphism classes of real algebraic curves. The latter moduli space is a covering of the former. This covering is generically one-to-one but it is not injective because some complex curves carry several different real structures.

The space of real isomorphism classes of real algebraic curves is, of course, the correct moduli space for real algebraic curves. Together with Robert Silhol I studied the real moduli space of smooth real algebraic curves first in [13]. There we showed that this moduli space is a semialgebraic variety (which is not connected). In this paper we compactify that moduli space by adding points corresponding to stable real curves. The construction is based on an extension (Theorem 4.8) of a result of Lipman Bers. This theorem is the key result which enables us to compactify the real moduli space of real algebraic curves. It also yields a new and direct proof for Theorem 7.1 in [12].

We observe finally that the methods of the present paper cannot be applied to the case of genus 1 curves. The reason is that real algebraic curves of genus 1 do not carry metrics of constant curvature -1. The methods of the present paper rely on Theorem 4.8 which does not make any sense in the case of genus 1 curves.

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Finally I dedicate this work to my father.

2. Real algebraic curves and symmetric Riemann surfaces

For the benefit of the reader we start with recalling a number of classical results in this and in the proceeding section. Everything here is well-known, but, for later applications, it is necessary at least to fix the notation as clearly as possible.

A smooth complex algebraic curve C of genus $g>1$ is a smooth Riemann surface X of genus g. C is isomorphic to a curve defined by real polynomial if and only if X carries an antiholomorphic involution $\sigma: X \to X$. The pair (X, σ) is *a symmetric Riemann surface*. The involution $\sigma: X \to X$ is *a symmetry of* X.

A symmetric Riemann surface (X, σ) is determined topologically by the following invariants:

1. The genus g of X.

2. The number $n=n(\sigma)$ of connected components of the fixed-point set X_σ of the mapping σ.

3. The index of orientability, $k=k(\sigma)$, which is defined setting $k=2-$the number of connected components of $X \setminus X_\sigma$.

These invariants satisfy:

1. $0 \leqq n \leqq g+1$.

2. For $k=0$, $n>0$ and $n \equiv g+1 \pmod 2$.

3. For $k=1$, $0 \leqq n \leqq g$.

These are the only restrictions for topological types of involutions of a genus g Riemann surface X. One computes that there are $\lfloor (3g+4)/2 \rfloor$ topological types of orientation reversing involutions of a genus g surface. This formula was shown by G. Weichhold, a student of F. Klein ([15], *see* also [9]).

For our applications it is necessary to get a concrete picture of the different topological involutions of a surface. Consider first involutions σ with $k(\sigma)=0$. Let n be an integer with $g-(n-1)=g+1-n$ even.

Take first a Riemann surface of genus $(g+1-n)/2$. Delete n open disks from it. Assume that the disks are chosen in such a manner that their closures are disjoint. Then one gets a Riemann surface Y of genus $(g+1-n)/2$ with n boundary components.

Let $\bar{Y}$ denote the Riemann surface obtained from Y by replacing the complex structure of Y with its conjugate structure, *i.e.* by replacing all local variables z with their complex conjugates $\bar{z}$. $\bar{Y}$ is simply the mirror image of Y. Glue the Riemann surfaces Y and $\bar{Y}$ together identifying the boundary points. In that way one gets a compact Riemann surface X of genus g. The identity mapping $Y \to \bar{Y}$ induces an antiholomorphic involution $\sigma: X \to X$ such that the curves of X corresponding to the boundary curves of Y remain point-wise fixed. Therefore $n(\sigma)=n$ and $k(\sigma)=0$ for this involution. This is how one can construct topologically all symmetries σ of a genus g Riemann surface X satisfying $k(\sigma)=0$ and $n(\sigma)\equiv g+1 \pmod 2$.

Let α be a closed curve left point-wise fixed under the above involution σ. Let A be a tubular neighborhood of α. Then the universal covering of A is the strip

$$\tilde{A}=\{z\in\mathbf{C} \mid -1<\operatorname{Im} z<1\}.$$

Furthermore we may suppose that σ maps A onto itself and that the complex conjugation is a lifting of $\sigma: A \to A$ onto $\tilde{A}$. Then the real axis covers the curve α.

Everything here is only topological. So assuming that the covering group of $\tilde{A} \to A$ is generated by $z \mapsto z+2$ we do not restrict the generality.

Define the function $H: \tilde{A} \to \tilde{A}$ setting $H(x+iy)=(x+1-y+iy)$. Then the complex conjugation $\tau(x+iy)=x-iy$ and $H\circ\tau$ are both self-mappings of $\tilde{A}$. Both of them map the real axis onto itself but only the complex conjugation keeps it point-wise fixed.

Let $f_\alpha: X \to X$ be defined setting $f_\alpha(p)=p$ for $p\in X\setminus A$. In A define f_α as the mapping induced by $H: \tilde{A} \to \tilde{A}$. Then $f_\alpha: X \to X$ is continuous and $f_\alpha\circ\sigma$ is also an involution

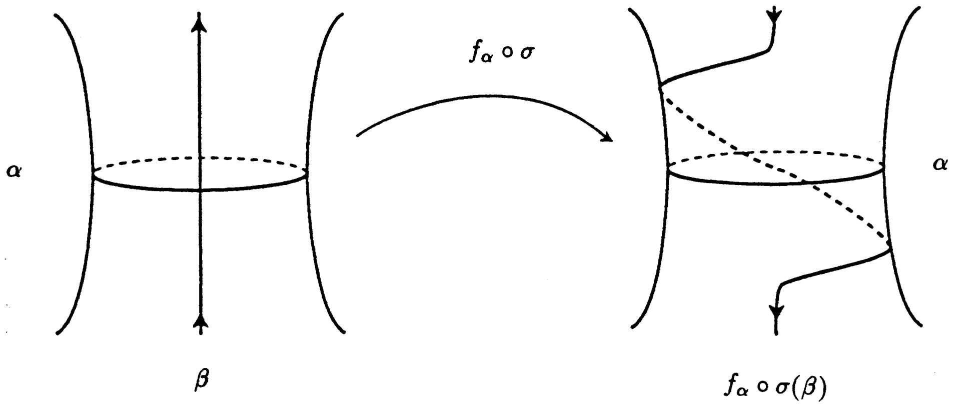

Fig. 1. – The twisted involution $f_\alpha\circ\sigma$ does not keep the curve α point-wise fixed.

of X. For this involution we have $k(f_\alpha\circ\sigma)=1$ and $n(f_\alpha\circ\sigma)=n(\sigma)-1$. Figure 1 illustrates how the involution $f_\alpha\circ\sigma$ maps a curve that intersects the curve α.

This is how one can construct topological models for symmetric Riemann surfaces.

3. Stable Riemann surfaces, Fenchel-Nielsen coordinates

Next we have to review certain definitions related to stable Riemann surfaces. Here we follow the presentation of Bers [3].

Recall first that *a surface with nodes* Σ is a Hausdorff space whose every point has a neighborhood homeomorphic either to the open disk in the complex plane or to

$$N = \{ (z, w) \in \mathbf{C}^2 \mid zw = 0, |z| < 1, |w| < 1 \}.$$

A point p of Σ is *a node* if every open neighborhood of p contains a open set homeomorphic to N. Component of the complement of the nodes of Σ is *a part* of Σ. *The genus* of a compact surface with nodes Σ is the genus of the compact smooth surface obtained by thickening each node of Σ.

A stable surface with nodes is a compact surface with nodes whose every part has a negative Euler characteristic. *A stable Riemann surface with nodes* is a stable surface Σ together with a complex structure X for which each component of the complement of the nodes of Σ is obtained deleting a certain number p_j points from a compact Riemann surface of genus g_j. The stability condition means that

$$2 - 2g_j - p_j < 0.$$

If X is a stable Riemann surface, then every part X_j of X is a hyperbolic Riemann surface, *i.e.*, every X_j carries a canonical metric of constant curvature -1. This metric is obtained from the non-euclidean metric of the upper half-plane (or the unit disk) via uniformization. When we later speak of lengths of curves on parts of a stable Riemann surface, we always refer to this canonical hyperbolic metric.

A stable surface Σ of genus g can have at most $3g-3$ nodes. We say that Σ is *terminal* if it has this maximal number of nodes.

A strong deformation of a surface with nodes Σ_1 onto a surface with nodes Σ_2 is a continuous surjection $\Sigma_1 \to \Sigma_2$ such that the following holds:

- the image of each node of Σ_1 is a node of Σ_2,
- the inverse image of a node of Σ_2 is either a node of Σ_1 or a simple closed curve on a part of Σ_1,
- the restriction of $\Sigma_1 \to \Sigma_2$ to the complement of the inverse image of the nodes of Σ_2 is an orientation preserving homeomorphism onto the complement of the nodes of Σ_2.

A pair of pants is a sphere from which three disjoint closed disks (or points) have been removed. A pair of pants P^2 has three boundary curves α_1, α_2 and α_3. A pair of pants can be equipped with a hyperbolic metric m for which the boundary curves are geodesic curves of a finite length l_1, l_2 and l_3. We call such a metric *intrinsic*. It is well known that the lengths of the boundary curves specify the metric m up to an isometry isotopic to the identity mapping (*cf.* e.g. [1], Theorem on page 82). We allow the boundary components to be of length 0. We say that the pair of pants with an intrinsic hyperbolic metric for which some boundary component(s) has (have) length 0 is *thight*.

Let Σ be a stable genus g surface. *A decomposition of Σ into pairs of pants* is an ordered collection

$$\mathscr{P}=(P_1, P_2, \ldots, P_{2g-2})$$

of disjoint pairs of pants on Σ such that:

- The union of the closures of the pairs of pants P_j covers the whole surface Σ.
- The intersection of the closures of any two pairs of pants P_i, P_j, $i \neq j$, is either empty or a union of nodes of Σ and of closed curves α on Σ.

It follows that all the nodes of Σ appear as boundary components of pairs of pants in any decomposition of Σ into pairs of pants.

If Σ is a terminal stable surface, then all the boundary components appearing in any decomposition of Σ into pairs of pants are nodes of Σ. If Σ is not terminal, then, in addition to the nodes, there will be a number of other boundary components which are simple closed curves on Σ. We call these nodes and curves *decomposing nodes and curves of $\mathscr{P}$*.

A decomposition $\mathscr{P}=(P_j)$ of Σ into pairs of pants is *oriented* if:

1. The set of boundary components of each pair of pants $P_j \in \mathscr{P}$ is ordered as well.
2. All decomposing curves are oriented.

If $\mathscr{P}$ is an oriented decomposition of Σ into pairs of pants, then we may speak of the first, second and third boundary component of any pair of pants belonging to $\mathscr{P}$. Furthermore, the ordering of the pairs of pants together with the ordering of the boundary components in the various pairs of pants induce an order in the set of boundary components of the individual pairs of pants in the decomposition $\mathscr{P}$. Observe that each decomposing curve appears twice in this ordered set of boundary components of the pairs of pants.

Let (P, d) be a pair of pants with ordered boundary components and with an intrinsic hyperbolic metric d. For a later construction it is necessary to associate *a base point* ζ_j to all boundary curves α_j of (P, d). That is, as usual, done in the following way. Let $\gamma_{i,j}$ be the geodesic curve in (P, d) which joins the ith boundary component to the jth boundary component, $i \neq j$, and is perpendicular to both of them. Such a geodesic curve is always uniquely defined. For a notational convenience, define $\gamma_{3,4}$ setting $\gamma_{3,4}=\gamma_{3,1}$. *The base point* ζ_j of the boundary component α_j is the starting point of $\gamma_{j,j+1}$ on α_j.

Let X be a complex structure of Σ. Then $X=(\Sigma, X)$ is a stable Riemann surface. Each part of X carries a canonical hyperbolic metric.

An oriented decomposition of X into pairs of pants is called *geodesic* if every boundary curve of that decomposition is a geodesic curve on X. If $\mathscr{P}$ is any decomposition of X into pairs of pants, then we get always a geodesic oriented decomposition of X into pairs of pants by replacing each decomposing curve by the geodesic curve in its homotopy class.

Let $\mathscr{P}$ be any geodesic and oriented decomposition of X into pairs of pants. Let $\alpha_1, \ldots, \alpha_{3g-3}$ be the decomposing curves or points. Each curve (or point) α_j is either a boundary component of two different pairs of pants or appears twice as a boundary component of a single pair of pants.

Let ζ_j^k be the distinguished boundary point of the jth boundary component of the kth pair of pants of the decomposition $\mathscr{P}$, $j=1, 2, 3$, $k=1, \ldots, 2g-2$. On each curve (or point) α_s, $s=1, \ldots, 3g-3$, there are exactly two points ζ_j^k. The ordering of the pairs of pants belonging to $\mathscr{P}$ and their respective boundary components gives us an ordering of these points ζ_j^k lying on one decomposing curve α_s. We conclude that on each α_s we have two distinguished points ξ_s^1 and ξ_s^2. These distinguished points are uniquely defined by the complex structure X.

Observe that strong deformations act on the set of oriented decompositions of a stable surface Σ into pairs of pants. More precisely, let $f:\Sigma'\to\Sigma$ be a strong deformation of stable surfaces and let $\mathscr{P}$ be an oriented decomposition of Σ into pairs of pants $P_1, \ldots, P_{2g-2}$. Then $f^*(\mathscr{P})$ is the decomposition of Σ' into pairs of pants $f^{-1}(P_j)$, $j=1, 2, \ldots, 2g-2$. This is *the pull back* of the pants decomposition $\mathscr{P}$.

If $f:\Sigma'\to\Sigma$ is a strong deformation and $\mathscr{P}'$ is an oriented decomposition of Σ into pairs of pants such that each curve $f^{-1}\{\text{a node of } \Sigma\}$ is a decomposing curve (or point) of $\mathscr{P}$, then we can define the induced decomposition $f(\mathscr{P}')$ of Σ into pairs of pants. The pairs of pants of $f(\mathscr{P}')$ are images of pairs of pants in $\mathscr{P}'$ under the strong deformation f. If f is a homeomorphism, then the induced decomposition $f(\mathscr{P}')$ is defined for any decomposition $\mathscr{P}$.

Let $\bar{M}^g$ be the set of isomorphism classes of stable genus g Riemann surfaces. We proceed and recall the definition of the Fenchel-Nielsen coordinates for (parts of) $\bar{M}^g$.

Let Σ be a stable topological surface of genus g. Fix first an oriented decomposition $\mathscr{P}$ of Σ into pairs of pants. Let X be a complex structure of Σ. Then there is always a mapping $f:\Sigma\to\Sigma$, homotopic to the identity mapping, such that the decomposition $\mathscr{P}$ is a geodesic decomposition for the pull back structure $f^*(X)$. Recall that the pull-back structure $f^*(X)$ is defined requiring the mapping $f:(\Sigma, f^*(X))\to(\Sigma, X)$ be holomorphic.

In $\bar{M}^g$ the complex structures $f^*(X)$ and X define the same point. Therefore, when defining parameters for subsets of $\bar{M}^g$, we may start with a *fixed* decomposition $\mathscr{P}$ of Σ into pairs of pants and restrict – without loss of generality – our considerations to those complex structures of Σ for which the given decomposition $\mathscr{P}$ is geodesic. Let $\mathscr{M}(\mathscr{P})$ be the set of these complex structures.

Let $\alpha_1, \ldots, \alpha_{3g-3}$ be the oriented decomposing curves of the pants decomposition $\mathscr{P}$. Recall that any $X\in\mathscr{M}(\mathscr{P})$ defines two distinguished points ξ_j^1 and ξ_j^2 on each α_j. Let x_j denote the distance from ξ_j^1 to ξ_j^2 measured to the positive direction of α_j.

In $\mathscr{M}(\mathscr{P})$ we can define the functions l_j, $j=1, 2, \ldots, 3g-3$, setting

$$l_j = \text{the length of } \alpha_j \tag{1}$$

$$\begin{cases} \theta_j = 2\pi x_j/l_j & \text{if } l_j>0 \\ \theta_j=0 & \text{if } l_j=0 \end{cases} \tag{2}$$

It is clear that X and $X'\in\mathscr{M}(\mathscr{P})$ are isomorphic complex structures if $l_j(X)=l_j(X')$ and $\theta_j(X)=\theta_j(X')$ for all $j=1, 2, \ldots, 3g-3$. A necessary condition for X and X′ to be isomorphic is that there exists a homeomorphism $f: X \to X'$ and a decomposition $\mathscr{P}'$ of X' into pairs of pants such that the following holds:

1. $\mathscr{P}=f^*(\mathscr{P}')$.

2. Let l'_j and θ'_j be the coordinates of X' with respect to $\mathscr{P}'$ which correspond to the coordinates l_j and θ_j of X with respect to $\mathscr{P}$. Then $l_j=l'_j$ and $\theta_j=\theta'_j$ for all $j=1, 2, \ldots, 3g-3$.

The coordinates l_j and θ_j are referred to as the *Fenchel-Nielsen coordinate*. The above definitions are quite classical. For a clearly written account of the Fenchel-Nielsen coordinates *see* [5].

4. Pants decomposition of symmetric Riemann surfaces

Let Σ be a fixed compact smooth topological surface of genus g, $g>1$, and let $\sigma: \Sigma \to \Sigma$ be a fixed orientation reversing involution. Let X be such a complex structure of Σ that the mapping σ: $(\Sigma, X) \to (\Sigma, X)$ is antiholomorphic. That is equivalent to saying that σ is an isometry of the hyperbolic metric of the Riemann surface (Σ, X).

It is well known that there exists a constant $M=M(g)$ – which depends only on the topological type of Σ – such that the Riemann surface (Σ, X) can be divided into pairs of pants in such a manner that the boundary curves of this pants decomposition are of length $<M$ (*see* e.g. [4], Theorem 2, p. 88).

For our applications it is necessary to see that the above pants decomposition can be chosen in such a way that it is invariant under the antiholomorphic involution $\sigma: (\Sigma, X) \to (\Sigma, X)$.

We shall prove that by symmetrizing the proof presented in [4] for this result in the case of Riemann surfaces (without a symmetry).

First of all we need a number of well known auxiliary results concerning the hyperbolic geometry of Riemann surfaces. For the convenience of the reader we shall here briefly recall these results.

Let W be a compact hyperbolic Riemann surface with boundary curves $\alpha_1, \ldots, \alpha_p$. On W we use *the intrinsic hyperbolic metric* in which the boundary components are geodesic curves of a finite length (*see* e.g. [1], p. 45). This metric can be obtained in the following way. First form the Schottky double of W by gluing W and its mirror image $\bar{W}$ together along the boundary components (for the definition of the mirror image of W *see* Section 2). *The intrinsic metric* of W is the restriction of the usual hyperbolic metric of the Schottky double of W to W itself.

Let $\varepsilon>0$. For an index j, $1 \leqq j \leqq p$, form the set

$$A_j=\{p \in W \mid \text{distance of } p \text{ from } \alpha_j<\varepsilon\}.$$

We say that A_j is *a collar at* α_j if it is homeomorphic to a ring domain. The number ε is *the width* of the collar A_j. A collar A_j of area μ will be called *a* μ-*collar*. *N.B.* We are here always using the intrinsic metric in which the boundary components α_j are geodesic curves of a finite length.

A collar at a boundary component α_j has itself two boundary components α_j and α_j^*. The latter is called *the inner boundary component*, while α_j is *the outer boundary component*. Let λ_j be the length of α_j and λ_j^* that of α_j^*.

LEMMA 4.1. — *The length of the inner boundary component α_j^* of a μ-collar A_j at α_j is*

$$\lambda_j^* = \sqrt{\lambda_j^2 + \mu^2}.$$

Any simple closed curve that is freely homotopic to α_j and lies outside of this collar has length at least λ_j^*.

Proof [4, Lemma 4, p. 89].

LEMMA 4.2. — *A μ-collar at a boundary component of length λ_0 contains a γ-collar for every γ, $0<\gamma<\mu$. Every point of the inner boundary component the former lies at the distance*

$$\log \frac{\mu + \sqrt{\mu^2 + \lambda_0^2}}{\gamma + \sqrt{\gamma^2 + \lambda_0^2}}$$

from the inner boundary component of the latter.

Proof [4, Lemma 5, p. 90].

LEMMA 4.3. — *There is positive continuous decreasing function $\alpha(t)$, $t \geqq 0$, such that about every boundary component of length λ_0 there is a μ-collar for every μ, $\mu < \alpha(\lambda_0)$. Two of these collars about two disjoint boundary component are disjoint.*

The best possible value of $\alpha(t)$ is

$$\alpha(t) = \frac{t}{\sinh(t/2)}.$$

Proof. — *See* [4], Lemma 6, p. 90, [8], [7] or [1], pp. 95-96.

We also need the following well known result:

LEMMA 4.4. — *There exists a universal constant η, $\eta > 0$, such that for any compact Riemann surface W (which may have boundary components) the following is true: Let α and β be closed geodesic curves on W with lengths $<\eta$. Then either $\alpha=\beta$ (as set of points) or the curves α and β do not intersect.*

This result is well known. For a best possible estimate *see* e. g. [14]. There we give an inequality for the lengths of intersecting geodesic curves on a Riemann surface. That inequality is the best possible and yields the constant of Lemma 4.4. Another inequality of the same type is given in [1], Lemma 1 on page 94.

Let Σ be a surface with boundary components and $\sigma:\Sigma\to\Sigma$ an orientation reversing involution. A pair (α_1, α_2) of simple closed curves α_1 and α_2 on Σ is called a σ-*pair* if

$\sigma(\alpha_1)=\alpha_2$ and if either $\alpha_1=\alpha_2$ or the curves are disjoint. A σ-pair (α_1, α_2) is called *essential* if α_1 is not freely homotopic to any boundary component of Σ. Observe that this condition implies that for an essential σ-pair (α_1, α_2) neither one of the curves α_j is freely homotopic to a boundary component.

A σ-pair (α_1, α_2) on a Riemann surface (Σ, X) is *geodesic* if both curves α_j, $j=1, 2$, are geodesic curves in the intrinsic hyperbolic metric of (Σ, X).

THEOREM 4.5. — *Let* (Σ, X) *be a hyperbolic Riemann surface with* (*or without*) *boundary components, and let* $\sigma:(\Sigma, X)\to(\Sigma, X)$ *be an antiholomorphic involution. There exists a constant* M *that depends only on the topological type of* Σ *and on the intrinsic lengths of the boundary curves of* (Σ, X) *such that the Riemann surface* (Σ, X) *has an essential geodesic* σ*-pair* (α_1, α_2) *such that both geodesic curves* α_j *are of length less than* M.

Proof. — We will show the above theorem by making the elegant arguments presented by Lipman Bers in [4] symmetric. Argumentation proceeds exactly as presented by Bers. We only have to check a number of possible different configurations related to the involution σ.

To start, let η, be the constant of Lemma 4.4.

Assume that there is a simple closed geodesic curve α on (Σ, X) such that the length of α is $<\eta$ and α is not freely homotopic to any boundary component of Σ. Then by Lemma 4.4 either $\sigma(\alpha)=\alpha$ or the curves α and $\sigma(\alpha)$ do not intersect. If that is the case, we are done because we may choose $M=\eta$ and $(\alpha_1, \alpha_2)=(\alpha, \sigma(\alpha))$.

Assume next that every simple closed geodesic curve that is not freely homotopic to a boundary component is of length at least η. Let L be the intrinsic length of the longest boundary curve of the hyperbolic Riemann surface (Σ, X). If (Σ, X) does not have any boundary curves, we set $L=0$.

Let

$$2\mu=3\tau=\alpha(L), \qquad \delta=\log\frac{\mu+\sqrt{\mu^2+L^2}}{\tau+\sqrt{\tau^2+L^2}},$$

where $\alpha(t)$ is the function of Lemma 4.3.

Let

$$4\varepsilon=\min(\tau, \delta, \eta), \tag{3}$$

where η is the constant of Lemma 4.4. Then $\varepsilon>0$ and it depends only on L.

Let $B_1, \ldots, B_p$ be the μ-collars and $B'_1, \ldots, B'_p$ the τ-collars at the boundary components $\alpha_1, \ldots, \alpha_p$ of the Riemann surface $X=(\Sigma, X)$. Then $B'_j\subset B_j$ and the collars B_j are disjoint.

Let X_{**} be the complement of $B_1\cup B_2\cup\ldots\cup B_p$ and X_* that of $B'_1\cup B'_2\cup\ldots\cup B'_p$. Since $\varepsilon<\delta$ Lemma 4.2 implies that every point of $B'_1\cup B'_2\cup\ldots\cup B'_p$ lies at a distance $>\varepsilon$ from X_{**}, *i.e.*, X_* contains the ε-neighborhood of X_{**}.

If the Riemann surface (Σ, X) does not have boundary components, we set $X_{**}=X_*=X$.

By the preceding considerations and by the choice of ε we are now left to consider the case in which the surface X_{**} does not have any simple closed geodesic curves of length less than 4ε. Considering Lemma 4.1 these choices imply also that, for any point $q \in X_{**}$, the set of points lying at a distance less than ε from the point q is a disk.

We have two cases to consider: let us first assume that the topological genus p of Σ is at least 1. That means that the surface Σ has at least one handle. The case of genus 0 surfaces Σ will be considered later.

Considering an explicit model for the action of the involution σ as described in Section 2 it is clear that we can always find simple closed curves α that are not freely homotopic to any product of the boundary curves and which satisfy $\sigma(\alpha)=\alpha$. Let β be one such simple closed geodesic curve having minimal length among all these curves.

Since $\sigma:(\Sigma, X)\to(\Sigma, X)$ is an isometry of the intrinsic metric, we have the following possibilities:

1. σ has exactly two fixed points on β.
2. σ has no fixed-points on β.
3. σ keeps all the points of β fixed.

Assume first that we have case (1), *i.e.*, that σ has exactly two fixed-points in β. Call these fixed points p_1 and p_2. They divide the curve β into two arcs β_1 and β_2 satisfying $\sigma(\beta_1)=\beta_2$.

Let L_β denote the length of β. Set $m=[L_\beta/6\varepsilon]$. On β_1 choose first $m-2$ inner points $q_1, \ldots, q_{m-2}$ of β_1 such that the arc of β_1 between any two of them is of length more than 2ε and that the arc of β connecting any one of them to either one of the end-points p_1 or p_2 of β_1 is of length more than 2ε.

This can be done, of course. On the arc β_2 consider the points

$$\sigma(q_1), \ldots, \sigma(q_{m-2}).$$

They satisfy the same condition as the points $q_1, \ldots, q_{m-2}$.

In this way we have chosen $2m$ points on the geodesic curve β and the set of these points is symmetric with respect to σ.

Let $D(p_1)$, $D(p_2)$, $D(q_1), \ldots, D(q_{m-2})$, $D(\sigma(q_1)), \ldots, D(\sigma(q_{m-2}))$ be disks of radius ε and centers at the respective points $p_1, p_2, q_1, \ldots$

The choice of ε implies first that the disks $D(p_1)$ and $D(p_2)$ are disjoint. For if $D(p_1)\cap D(p_2)\neq\varnothing$, then this intersection has two components which get mapped onto each other by the involution σ. Therefore the union $D(p_1)\cup D(p_2)$ would contain a geodesic curve of length less than 4ε. That is not possible by our present assumptions.

The area of a hyperbolic disk of radius ε is $4\pi\sinh^2(\varepsilon/2)$. If all of the above $2m$ disks are disjoint, then their the total area is

$$m\,8\pi\sinh^2(\varepsilon/2).$$

This number is, of course, bounded by the intrinsic area of (Σ, X) which equals

$$-2\pi\chi(\Sigma)$$

where $\chi(\Sigma)$ is the Euler characteristic of Σ.

We conclude therefore that

$$m \leqq -\frac{\chi(\Sigma)}{4\sinh^2(\varepsilon/2)}. \tag{4}$$

Since $L_\beta < 6\varepsilon(m+1)$, (4) gives an upper bound

$$M = -3\varepsilon\frac{\chi(\Sigma)}{2\sinh^2(\varepsilon/2)} \tag{5}$$

for the length of the curve β. This number M depends only on the topological type of Σ and on the lengths of the boundary curves of (Σ, X). The curve β is, furthermore, invariant under the involution σ. Therefore an essential σ-pair satisfying the conditions of the lemma is simply $(\beta, \sigma(\beta))$.

Assume next that the above disks $D(p_1)$, $D(p_2)$, $D(q_j)$, $D(\sigma(q_j))$, $j=1, 2, \ldots$, are not disjoint. If $D(p_j) \cap D(p_k) \neq \varnothing$ then also

$$\sigma(D(p_j)) \cap \sigma(D(p_k)) = D(\sigma(p_j)) \cap D(\sigma(p_k)) \neq \varnothing.$$

Using arcs lying in the unions of intersecting disks and arcs of β we can then find simple closed geodesic curves $\gamma_1, \ldots, \gamma_m, \sigma(\gamma_1) \ldots, \sigma(\gamma_m)$ such that the curve β is homologous to the product

$$\gamma_1 \gamma_2 \ldots \gamma_m \sigma(\gamma_1) \ldots \sigma(\gamma_m), \tag{6}$$

and the length of each curve γ_j is bounded by the number M of (5). Figure 2 illustrates this case.

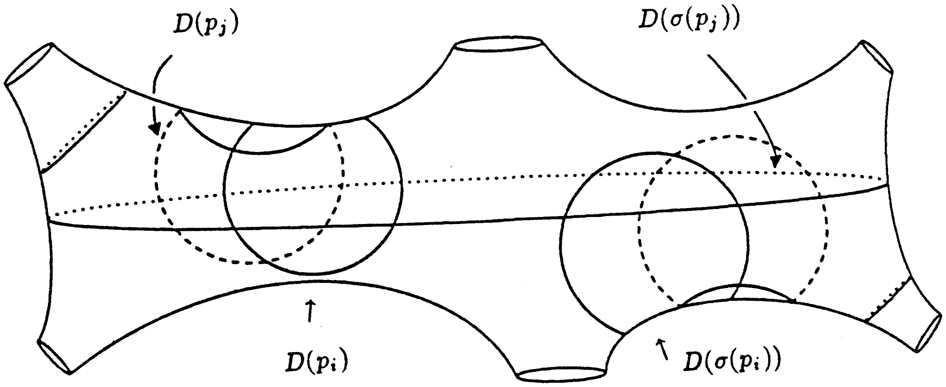

Fig. 2. — The disks $D(q_i)$, $D(q_j)$ and $D(\sigma(p_i))$, $D(\sigma(q_j))$ intersect.

If now the simple closed curve γ_j is homologous to 0 modulo the boundary of Σ then such is $\sigma(\gamma_j)$ as well. The product (6) is homologous to β which is not homologous to

0 modulo the boundary. We conclude therefore that among the above curves we may choose geodesic curves γ_j and $\sigma(\gamma_j)$ such that neither one of them is homologous to 0 modulo the boundary and both of them are of length less than the above number M. This concludes the proof in the subcase 1.

In subcase 2 the involution σ has no fixed points in β. Then choose first any point p_1 of β. Let $p_2=\sigma(p_1)$. Replace now, in the previous argument, the fixed points p_1 and p_2 of σ on β by these points p_1 and p_2. Then we may repeat the above argument word by word to prove the theorem also in subcase 2.

In subcase 3 the involution σ keeps the geodesic curve β point-wise fixed. In this case σ is a reflection in the curve β.

Let now $m'=[L_\beta/3\varepsilon]$. On the curve β choose m' points $q_1, \ldots, q_m$ such that the arcs of β connecting any two points q_i and q_j is of length more than 2ε. Let $D(q_j)$ be a disk of radius ε with center at the point q_j.

We next proceed to show that the disks $D(q_j)$ are disjoint. To that end assume that $D(q_j)\cap D(q_k)\neq\varnothing$. Then the sets

$$D(q_j)\setminus\beta \quad \text{and} \quad D(q_k)\setminus\beta$$

both have two component. Call them D_j^1, D_j^2 and D_k^1, D_k^2, respectively.

Assuming now that the numbering is propely chosen, we have $D_j^1\cap D_k^1\neq\varnothing$. Then $\sigma(D_j^1)=D_j^2$ and $\sigma(D_k^1)=D_k^2$. Therefore also $D_j^2\cap D_k^2\neq\varnothing$. Figure 3 illustrates this

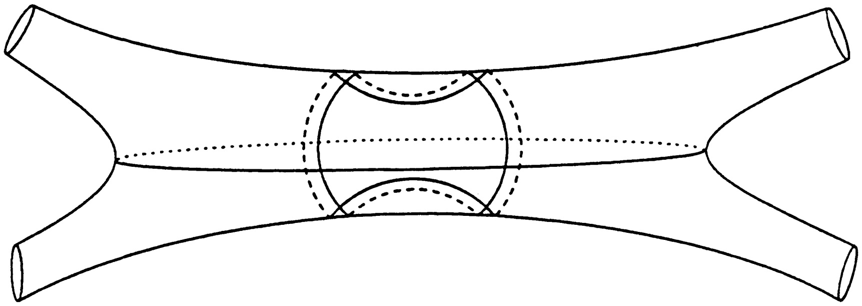

Fig. 3. — The intersection $D(p_i)\cap D(p_j)$ has two components.

case. This would imply that we can find, on the Riemann surface (Σ, X) a simple closed geodesic curve of length less than $4\varepsilon<\eta$ which is contained in the union $D(p_j)\cup D(p_k)$. By our initial choices this is not possible. We conclude that all the disks $D(p_i)$ are disjoint. As before this gives an upper bound M for the length of the geodesic curve β. This upper bound depends only on the topological type of Σ and on the intrinsic lengths of the boundary curves of (Σ, X). The pair $(\beta, \sigma(\beta))$ satisfies now the requirements of the theorem.

To complete the proof of the lemma we still have to consider the case where Σ is homeomorphic to a sphere from which a number ($\geqq 4$) of disks have been removed.

Let X_* and X_{**} be the surfaces obtained from the Riemann surface (Σ, X) by removing the collars B_j' and B_j, as explained in the beginning of this proof. Let ε be the number fiven by equation (3). Recall that we have used the notation $\alpha_1, \ldots, \alpha_p$ for the boundary curves of the Riemann surface (Σ, X). Let $\alpha_1^*, \alpha_2^*, \ldots, \alpha_p^*$ be the corresponding boundary curves of the surface X_{**}.

By Lemma 4.1, the intrinsic lengths of the curves α_j^* are bounded by a number depending only on the intrinsic lengths of the respective curves α_j.

We say that boundary components α_j^* and α_k^* are *adjacent* if, among all curves connecting α_j^* to α_k^* in X_{**} there is one which has minimal length and which does not intersect any other boundary curve α_i^*, $i \neq j, k$, of X_{**}.

LEMMA 4.6. — *The length of a minimal curve connecting two adjacent boundary components of* X_{**} *to each other is bounded by a number depending only on the intrinsic length of the longest boundary component of the Riemann surface* (Σ, X) *and on the total intrinsic area of* (Σ, X).

Proof. — Let α_j^* and α_k^* be adjacent boundary components of X_{**}, and let γ be a curve connecting them to each other, having minimal length and not intersecting any other boundary component. Let L_γ be the intrinsic length of γ. Then we proceed exactly as in the preceeding argumentation replacing the curve β by the curve γ and the number L_β by L_γ.

For any point $q \in \gamma$, the set of consisting of points lying at a distance less than ε from the point q is a disk. If q and q' are two points of γ such that the arc of γ connecting q to q' is of length more than 2ε, then the corresponding disks *are* disjoint. Repeating the above arguments we now get an upper bound for the intrinsic length of the curve γ.

LEMMA 4.7. — *There is a number* M *depending only on the intrinsic length of the longest boundary component* α_j *of* (Σ, X) *and on the number* p *of the boundary components of* Σ *such that the distance, in* X_{**}, *between any two boundary components* α_j^* *and* α_k^* *of* X_{**} *is bounded by* M.

Proof. — Lemma 4.6 proves this result for adjacent boundary components. Assume that α_j^* and α_k^* are arbitrary boundary components. Let γ be a curve of minimal length connecting α_j^* to α_k^* in X_{**}.

The curve γ can be divided into arcs $\gamma_1, \beta_1, \ldots, \gamma_{n-1}, \beta_{n-1}, \gamma_n$, $n \leqq p$, such that the arcs β_i are arcs on the boundary components α_i^* and the arcs γ_i are arcs connecting two adjacent boundary components to each other. Now the length of each arc γ_i is bounded by a number depending only on the length of the longest boundary component and the total area of the Riemann surface (Σ, X). This can be proved by repeating the previous arguments.

Each curve β_i is, on the other hand, shorter than the longest boundary component of X_{**}. There are less than $2p$ arcs γ_i and β_i. This proves the lemma.

In order to complete the proof of Theorem 4.5 we use the above lemmata to find a σ-pair $(\gamma, \sigma(\gamma))$ satisfying the conditions of the theorem.

Let β be a curve of minimal length connecting two adjacent boundary components α_j^* and α_k^* of X_{**} to each other in X_{**}. By Lemma 4.6 the length of β is bounded by a number depending only of the length of the longest boundary component and on the area of (Σ, X).

We conclude first that the curves β and $\sigma(\beta)$ do not intersect. For if they would intersect, then they would intersect at a positive angle at some point. That would allow us to find a shorter curve β' connecting two boundary components of X_{**} to each other. Since β has minimal length among all such curves this is not possible.

Assume that $\sigma(\alpha_i^*) \neq \alpha_l^*$, for $i, l=j, k$. In this case take γ to be the geodesic curve homotopic to the product $\alpha_j^* \beta \alpha_k^* \beta^{-1}$ (this is homotopic to a simple closed curve provided that the orientations of the curves α_i^* and β are properly chosen. It is obvious how to do this.

The length of this curve γ is bounded by a number depending only on the length of the longest boundary curve α_i and on the total area of (Σ, X). Therefore the σ-pair $(\gamma, \sigma(\gamma))$ satisfies the conditions of the theorem.

We still have to consider the following cases:

1. $\sigma(\alpha_j^*) = \alpha_j^*$ and $\sigma(\alpha_k^*) = \alpha_k^*$,
2. $\sigma(\alpha_j^*) = \alpha_j^*$ and $\sigma(\alpha_k^*) \neq \alpha_k^*$,
3. $\sigma(\alpha_j^*) = \alpha_k^*$.

All these cases can be treated repeating the above arguments in suitably modified forms. In these cases we construct a geodesic curve γ such that $\sigma(\gamma) = \gamma$ and whose length is bounded by a constant depending only on the length of the longest boundary curve and on the total area of the Riemann surface (Σ, X). This geodesic curve will be homotopic to a closed curve constructed by taking twice the curve β and suitable arcs on the boundary curves α_j^* and α_k^* of X_{**}. This finally proves the theorem. Details in this last part are easy and left to the reader.

Theorem 4.5 can be use inductively to prove the following result.

THEOREM 4.8. — *Let (Σ, X) be a Riemann surface of genus g, $g > 1$. Let $\sigma : (\Sigma, X) \to (\Sigma, X)$ be an antiholomorphic involution. There exists a constant* M *depending only on the topological type of Σ such that (Σ, X) has a decomposition $\mathscr{P}$ into pairs of pants such that*

- $\sigma(\mathscr{P}) = \mathscr{P}$, *and*
- *the decomposing curves of $\mathscr{P}$ are simple closed geodesic curves of length* $<$M.

The proof is an immediate consequence of Theorem 4.5. The proof is word-by-word same as the one presented by Lipman Bers in [4], § 5, pp. 92-93, and will be omitted here.

The above proof contains a number of rather straightforward details. That result is necessary for our applications. It would be desirable to find a shorter proof by using the corresponding lemma in the case of Riemann surfaces (without symmetries). Unfortunately I was not able to do this.

5. Moduli spaces of real curves

Recall that stable real algebraic curves are stable symmetric Riemann surfaces. Two such real curves (X, σ) and (Y, τ) are *real isomorphic* if there exists a holomorphic homeomorphism $f: X \to Y$ satisfying $f \circ \sigma = \tau \circ f$. Our basic object of study is the moduli space of real algebraic curves of a given genus. Here is a formal definition for it.

DEFINITION 5.1. — The set $\bar{M}^g_{\mathbf{R}}$ of real isomorphism classes of stable real curves of genus g is *the moduli space of real algebraic curves of genus g*.

We aim to show that the moduli space $\bar{M}^g_{\mathbf{R}}$ carries a natural topology in which $\bar{M}^g_{\mathbf{R}}$ is a compact and connected Hausdorff space. To that end we use the above defined Fenchel-Nielsen coordinates. We continue proceeding in the spirit of the exposition of Abikoff [1].

Let ε and δ be positive numbers. We say that the point $[(Y, \tau)] \in \bar{M}^g_{\mathbf{R}}$ belongs to the (ε, δ)-neighborhood $U_{\varepsilon, \delta}([(X, \sigma)])$ if and only if the following conditions are met:

1. There exists a decomposition $\mathscr{P}$ of X into pairs of pants which is invariant under the symmetry σ.

2. There exists a strong deformation $f: Y \to X$ such that $f \circ \tau = \sigma \circ f$.

3. The Fenchel-Nielsen coordinates $l'_1, \theta'_1, \ldots, l'_{3g-3}, \theta'_{3g-3}$ of Y with respect to $f^*(\mathscr{P})$ satisfy:

(*a*) $|l_j - l'_j| < \varepsilon$ for all $j = 1, 2, \ldots, 3g-3$.

(*b*) For all values of j, if $l_j > 0$ then $|\theta_j - \theta'_j| < \delta$.

Here l_j and θ_j are the Fenchel-Nielsen coordinates of X with respect to $\mathscr{P}$.

The sets $U_{\varepsilon, \delta}([(X, \sigma)])$ form a basis for the topology of $\bar{M}^g_{\mathbf{R}}$. This definition for the topology of $\bar{M}^g_{\mathbf{R}}$ is an adaptation of the definition given in [1], p. 103, for the topology of the moduli space $\bar{M}^g$ of stable Riemann surfaces of genus g. The fact that this topology is a Hausdorff topology is rather obvious and can be shown by repeating the arguments that Abikoff has presented in [1], pp. 103-104.

We next proceed and show that $\bar{M}^g_{\mathbf{R}}$ together with this topology is a connected and compact space.

6. Compactness theorem

The moduli space $\bar{M}^g_{\mathbf{R}}$ can be divided in a natural way into parts that can be studied using the classical methods. These parts are very similar to charts of a manifold but they do not form an open covering of $\bar{M}^g_{\mathbf{R}}$.

Let

$$V(n, k) = \{[(X, \sigma)] \in \bar{M}^g_{\mathbf{R}} \mid X \text{ a smooth Riemann surface}, n(\sigma) = n, k(\sigma) = k\}.$$

We will first show that the closure of each $V(n, k)$ is compact in $\bar{M}^g_{\mathbf{R}}$.

The proof of the compactness is an extension of the arguments presented in [1], pp. 99-104, and relies on Theorem 4.8. We assume that the constructions and results concerning the moduli space $\bar{M}^g$ of stable Riemann surfaces of genus g, $g>1$, are known. For an analytic approach *see* e.g. [1]. Observe especially, that it follows immediately from the definitions of the topology of $\bar{M}^g_{\mathbf{R}}$ and that of $\bar{M}^g$ that the projection

$$\bar{M}^g_{\mathbf{R}} \to \bar{M}^g, \qquad [(X, \sigma)] \mapsto [X] \tag{7}$$

is continuous.

We will have to deal with several different symmetries of the surface Σ. To make this distinction clear we write sometimes (Σ, X, σ) to denote the symmetric Riemann surface $X=(\Sigma, X)$ together with the symmetry $\sigma:(\Sigma, X)\to(\Sigma, X)$ which is then an antiholomorphic involution.

THEOREM 6.1. — *The closure of* $V(n, k)$ *in* $\bar{M}^g_{\mathbf{R}}$ *is compact.*

Proof. — Let $\sigma:\Sigma\to\Sigma$ be an orientation reversing involution, $k(\sigma)=k$ and $n(\sigma)=n$. Let $([\Sigma, X_n, \sigma)])$ be an infinite sequence of points of $V(n, k)$ in $\bar{M}^g_{\mathbf{R}}$. It suffices to show that there exists a subsequence $([(\Sigma, X_{n_k}, \sigma)])$ that converges in $\bar{M}^g_{\mathbf{R}}$.

We shall, at various stages of the proof pass from a sequence to its subsequence. To keep the notation as simple as possible we use the same notation for a sequence and its suitable subsequence when there is no danger of confusion.

Let M be the constant of Theorem 4.8. Use first Theorem 4.8 to find, for each index n, a decomposition $\mathcal{P}_n$ of Σ into pairs of pants in such a way that each decomposing curve α^n_j of each pants decomposition $\mathcal{P}_n$ has length $<M$ on (Σ, X_n). By Theorem 4.8, we can choose these pants decompositions $\mathcal{P}_n$ in such a way that, for each n, $\sigma(\mathcal{P}_n)=\mathcal{P}_n$.

There are only finitely many topologically different decompositions of the surface Σ into pairs of pants. Therefore we may—by passing to a subsequence—assume that there is a fixed decomposition $\mathcal{P}$ of Σ into pairs of pants and orientation preserving homeomorphisms $f_n:\Sigma\to\Sigma$ such that $f_n(\mathcal{P}_n)=\mathcal{P}$ for each n.

Let Y_n be that complex structure of Σ for which the mapping $f_n:(\Sigma, X_n)\to(\Sigma, Y_n)$ is holomorphic. Let $\tau_n=f_n\circ\sigma\circ f_n^{-1}$. Each mapping $\tau_n:(\Sigma, Y_n)\to(\Sigma, Y_n)$ is then an antiholomorphic involution. Furthermore, $\tau_n(\mathcal{P})=\mathcal{P}$ for each n.

LEMMA 6.2. — *For infinitly many indices n we can choose a representative* (Σ, W_n, τ) *of the point* (Σ, Y_n, τ_n) *such that the following holds.*

1. $\tau:(\Sigma, W_n)\to(\Sigma, W_n)$ *is an antiholomorphic involution. The corresponding mapping* $\tau:\Sigma\to\Sigma$ *does not depend on the index n.*

2. τ *maps the pants decomposition* $\mathcal{P}$ *of* Σ *onto itself.*

3. *Lengths of each decomposing curve of* $\mathcal{P}$ *on hyperbolic Riemann surfaces* (Σ, W_n) *form a bounded sequence.*

Proof. — There are only $[(3g+4)/2]$ topologically different orientation reversing involutions $\tau_n:\Sigma\to\Sigma$ of a genus g surface Σ. Therefore, by passing to a subsequence, we may suppose that all the involutions τ_n are of the same topological type. This means that all the surfaces $\Sigma/\langle\tau_n\rangle$ are homeomorphic to each other. Here $\langle\tau_n\rangle$ is the group generated by the involution τ_n. Observe that these surfaces may have boundary and need not be orientable.

Since $\tau_n(\mathscr{P})=\mathscr{P}$ for each n, the decomposition $\mathscr{P}$ induces a generalized decomposition of the surface $\Sigma/\langle \tau_n \rangle$ into pairs of pants. This decomposition is *generalized* in the sense that it is a decomposition of $\Sigma/\langle \tau_n \rangle$ into pairs of pants and into parts that are quotients of pairs of pants modulo an orientation reversing involution of the pair of pants. In any case, each surface $\Sigma/\langle \tau_n \rangle$ has only finitely many topologically different such decompositions. This means that – by passing to a subsequence and changing the numeration again – we may, for $n>1$, find homeomorphisms $G_n : \Sigma/\langle \tau_1 \rangle \rightarrow \Sigma/\langle \sigma_n \rangle$ which map the induced decomposition of $\Sigma/\langle \tau_1 \rangle$ onto that of $\Sigma/\langle \tau_n \rangle$.

Let $g_n : \Sigma \rightarrow \Sigma$ be the orientation preserving lifting of $G_n : \Sigma/\langle \tau_1 \rangle \rightarrow \Sigma/\langle \tau_n \rangle$. Write $\tau=\tau_1$. Then $g_n \circ \tau_1=\tau_n \circ g_n$ and g_n maps the decomposition $\mathscr{P}$ of Σ into pairs of pants onto itself, *i.e.*, g_n maps each decomposing curve of $\mathscr{P}$ onto some decomposing curve of $\mathscr{P}$.

Let now W_n be that complex structure of Σ for which the mapping $g_n : (\Sigma, W_n) \rightarrow (\Sigma, Y_n)$ is holomorphic. Then, for each n, the mapping $\tau=\tau_1=g_n^{-1} \circ \tau_n \circ g_n$ is an antiholomorphic involution, $\tau(\mathscr{P})=\mathscr{P}$ and, in $\bar{M}_{\mathbf{R}}^g$, $[(\Sigma, W_n, \tau)]=[(\Sigma, Y_n, \tau_n)]$.

The decomposition $\mathscr{P}$ of (Σ, W_n) into pairs of plants satifies also the second condition of Theorem 4.8. This proves the lemma.

Let l_j^n, θ_j^n, $j=1, 2, \ldots, 3g-3$, be the Fenchel-Nielsen coordinates of W_n with respect to the pants decomposition $\mathscr{P}$. Then by statement 3 of Lemma 6.2, we find a constant M, $M<\infty$, such that $l_j^n<M$ for each j and n. Also we have $0 \leqq \theta_j^n < 2\pi$. Therefore – by passing again to a subsequence – we may assume that all the sequences $l_j^1, l_j^2, l_j^3, \ldots$ and $\theta_j^1, \theta_j^2, \theta_j^3, \ldots$ converge. Let $l_j=\lim_{n \rightarrow \infty} l_j^n$ and $\theta_j=\lim_{n \rightarrow \infty} \theta_j^n$, $j=1, 2, \ldots, 3g-3$.

We deform next the surface Σ in the following fashion. If $l_j=0$ then we replace the decomposing curve α_j of the pants decomposition $\mathscr{P}$ by a node. Do that for each j with $l_j=0$. That construction yields a stable surface Σ^* of genus g. The identity mapping $\Sigma \rightarrow \Sigma$ induces a strong deformation

$$f: \quad \Sigma \rightarrow \Sigma^*. \tag{8}$$

By Lemma 6.2 we deduce that τ induces an orientation reversing involution $\tau : \Sigma^* \rightarrow \Sigma^*$. Also the decomposition $\mathscr{P}$ of Σ into pairs of pants gives a similar decomposition $\mathscr{P}$ of Σ^* into pairs of pants. Here we actually need only the following observation. If the length of a decomposing curve α_j on (Σ, W_n) tends to 0 as $n \rightarrow \infty$ then also the length of $\tau(\alpha_j)$ on (Σ, W_n) tends to 0.

On Σ^* define a complex structure Y^* using the Fenchel-Nielsen coordinates l_j and θ_j with respect to the decomposition $\mathscr{P}$ of Σ^* into pairs of pants. We do that in such a fashion that the mapping $\tau : (\Sigma^*, Y^*) \rightarrow (\Sigma^*, Y^*)$ becomes antiholomorphic.

To this end, let $P_1, P_2, \ldots, P_{2g-2}$ be the pairs of pants of the surface Σ^*. It is probably best to think of these pairs of pants as a collection of separate pairs of pants and not as subsets of the surface Σ^*. We start with defining the complex structure of each P_j first.

The limits l_j, $j=1, 2, \ldots, 3g-3$, of the Fenchel-Nielsen length coordinates l_j^n associated to W_n give the lengths of the decomposing curves of the pants decomposition $\mathscr{P}$ of (Σ^*, Y^*).

Start with defining on P_1 any complex structure for which the boundary curves have the lengths given by these limits l_j. If $\tau(P_1)=P_1$ then we require, in addition, that the mapping τ restricted to P_1 is antiholomorphic. Otherwise we define the complex structure of $\tau(P_1)$ by requiring the mapping $\tau : P_1 \to \tau(P_1)$ to be antiholomorphic.

If $\tau(P_1)\neq P_2$ then repeat the same for P_2. Otherwise we continue with P_3. We repeat this construction until every pair of plants P_j of the decomposition $\mathscr{P}$ of Σ^* into pairs of pants gets a complex structure. This can obviously be done without any problems.

Next we have to glue these complex structures of the various pairs of pants together to form a global complex structure of the surface Σ^*. That we do by the identification pattern given by the original decomposition $\mathscr{P}$ of Σ^* and the gluing angles given by the limits $\theta_j = \lim_{n \to \infty} \theta_j^n$ of the gluing angles associated to the complex structures W_n. That gives us the complex structure Y^* of Σ^*.

For each n, the diagram

$$(9) \qquad \left\{ \begin{array}{ccc} (\Sigma, W_n) & \xrightarrow{f} & (E^*, Y^*) \\ \downarrow \tau & & \downarrow \tau \\ (\Sigma, W_n) & \xrightarrow{f} & (E^*, Y^*) \end{array} \right.$$

commutes.

Here $f : (\Sigma, W_n) \to (\Sigma^*, Y^*)$ is the strong deformation induced by the identity mapping [*cf.* (8)]. Each mapping $\tau : (\Sigma, W_n) \to (\Sigma, W_n)$ is an antiholomorphic involution.

The restrictions of the mapping $\tau : (\Sigma^*, Y^*) \to (\Sigma^*, Y^*)$ to the pairs of pants P_j are antiholomorphic by the construction. The commutative diagram (9) together with the definition of $\tau : (\Sigma^*, Y^*) \to (\Sigma^*, Y^*)$ and the fact that each $\tau : (\Sigma, W_n) \to (\Sigma, W_n)$ is antiholomorphic then finally assures that the mapping $\tau : (\Sigma^*, Y^*) \to (\Sigma^*, Y^*)$ is globally antiholomorphic. Therefore (Σ^*, Y^*) defines a point in $\bar{M}_{\mathbf{R}}^g$.

From the construction it follows then immediately that

$$[(\Sigma, X_n)] = [(\Sigma, W_n)] \to [(\Sigma^*, Y^*)] \quad \text{as } n \to \infty$$

in $\bar{M}_{\mathbf{R}}^g$. This proves the theorem.

7. Connectedness theorem

Next we consider various points where the closures of the sets $V(n, k)$ intersect. We will show that there are enough intersection points so that the union of the closures is actually connected. We start with showing that the union of the closures of the sets $V(n, k)$ covers the whole moduli space $\bar{M}_{\mathbf{R}}^g$. To that end we need a continuous way of thickening the nodes of a stable symmetric Riemann surface. That is, we need to make the deliberations of Klein (*see* Introduction) precise.

Let (Σ, X) be a stable Riemann surface that represents a point $p \in \bar{M}^g$. Assume that the surface Σ has nodes, let them be $N_1, N_2, \ldots, N_m$. For our purposes it is necessary to obtain a concrete and continuous way to thicken these nodes. One such thickening has been given by Fay (*cf.* e. g. [6]).

Let us first describe this thickening at one node $N \in \Sigma$. Since N is a node of the stable Riemann surface X, we can take a neighborhood U of N such that $U \setminus \{N\}$ consists of two open sets U_1 and U_2 that are both holomorphically homeomorphic to the unit disk $\Delta^* \subset \mathbf{C}$ which is punctured at the origin. Let $\alpha_j : U_j \to \Delta^*$ be holomorphic homeomorphisms.

The thickening of the node N that we are presently describing depends on the sets U_j and on the conformal mappings α_j. So we have to fix them first. Assume that they are now fixed.

The thickening of the node N will depend on one complex parameter $z \in \Delta = \{z \in \mathbf{C} \mid |z| < 1\}$. Let now $z \in \Delta$ be fixed. Let $\Delta_z^* = \{\zeta \in \mathbf{C} \mid 0 < |\zeta| < |z|\}$ be the punctured disk of radius $|z|$, $1 > |z| > 0$, with center at the origin.

Delete, from the Riemann surface (Σ, X), the punctured disks $\alpha_j^{-1}(\Delta_z^*)$, $j = 1, 2$, and the node N. In that way get a Riemann surface with two boundary components γ_1 and γ_2 which correspond to the deleted punctured disks $\alpha_1^{-1}(\Delta_z^*)$ and $\alpha_2^{-1}(\Delta_z^*)$, respectively.

Let $\xi_z(w) = z|z|/w$. The mapping ξ_z is conformal and maps the inside of the punctured disk $\{w \mid |w| < |z|\}$ onto its outside. Therefore we obtain, by identifying the point $p \in \gamma_1$ with the point $\alpha_2^{-1} \circ \xi_z \circ \alpha_1(p)$, a stable Riemann surface X_z such that the node N has been replaced by a closed curve. Rotating the point $z \in \Delta^*$ means a partial Dehn twist on the surface X_z around the simple closed curve of X_z that corresponds to the curves γ_1 and γ_2.

Important is that the mapping

$$\Delta \to \bar{M}^g, \qquad [z] \mapsto [X_z] \tag{10}$$

is continuous. This is essentially the construction presented already by Fay [6] and then used by many other authors.

Actually the above construction could easily be modified to get a holomorphic thickening of the type (10). That is, actually, the way this is usually done. For our computations this mapping (10) is, however, more convenient.

The mapping (10) thickens only one node. If the stable surface X has m nodes $N_1, \ldots, N_m$, then we can repeat this construction and obtain a continuous mapping

$$\Delta^m \to \bar{M}^g, \qquad \zeta \mapsto [X_\zeta] \tag{11}$$

to get a continuous mapping that thickens all the nodes simultaneously.

We use this thickening of the nodes to prove the following result (*cf.* [11], Proposition 3.1, p. 90).

THEOREM 7.1. — $\bar{M}_{\mathbf{R}}^{g}$ *is the union of courses of the sets* $V(n, k)$.

Proof. – We have to show that any neighborhood of a point in $\bar{M}_{\mathbf{R}}^{g}$ defined by a stable Riemann surface (Σ, X, σ), with symmetry σ, contains points belonging to some $V(n, k)$.

Let $N_1, \ldots, N_m$, $m>0$, be the nodes of (Σ, X). To this end we have to thicken the nodes in a fashion which is compatible with the involution σ.

The mapping σ must map the set of nodes of (Σ, X) onto itself. To thicken the nodes of (Σ, X) we use the continuous thickening (11) with suitably chosen parameters $\zeta=(z_1, \ldots, z_m)\in\Delta^m$.

In order to see how we have to choose these parameters we need to divide the nodes N_j into different classes according to the behaviour of the mapping σ. So we assume that the nodes are numbered in such a fashion that the following holds.

1. Nodes $N_1, \ldots, N_{m_1}$ are kept pointwise fixed by σ in such a way that each node N_{m_j} has a neighborhood consisting of the node itself and of two punctured disks $U_{m_j}^1$ and $U_{m_j}^2$ which are both kept fixed by α (as sets).

2. Nodes $N_{m_1+1}, \ldots, N_{m_2}$ are kept pointwise fixed by σ in such a way that each N_{m_k} has a neighborhood consisting of the node itself and of punctured disks $U_{m_k}^1$ and $U_{m_k}^2$ around the node such that $\sigma(U_{m_k}^1)=U_{m_k}^2$ and $\sigma(U_{m_k}^2)=U_{m_k}^1$.

3. Nodes $N_{m_2+1}, \ldots, N_{m_3}$ are mapped by σ onto the nodes $N_{m_3+1}, \ldots, N_m$ which are numbered in such a way that $\sigma(N_{m_2+k})=N_{m_3+k}$ for all values of k.

The type of the node N_j determines how we have to choose the thickening parameter z_j in order to ensure that the involution σ induces an involution of the thickened surface $X_\zeta=X_{(z_j)}$.

First we choose disjoint neighborhoods $\{N_j\}\cup U_j^1\cup U_j^2$ of the nodes N_j such that each U_j^1 and each U_j^2 is holomorphically homeomorphic to the punctured disk, $U_j^1\cap U_j^2=\varnothing$ for each index j, and if $\sigma(N_j)=N_k$, then either $\sigma(U_j^1)=U_k^1$ and $\sigma(U_j^2)=U_k^2$ or $\sigma(U_j^1)=U_k^2$ and $\sigma(U_j^2)=U_k^1$.

Next we choose holomorphic homeomorphisms

$$\alpha_j^1:\quad U_j^1\to\Delta^*,\qquad \alpha_j^2:\quad U_j^2\to\Delta^*$$

such that

$$\alpha_j^t\circ\sigma\circ(\alpha_k^s)^{-1}(z)=\bar{z} \tag{12}$$

whenever defined.

It is a simple matter to see that we can choose the mappings $\alpha_j^t: U_j^t\to\Delta^*$ in such a manner that equations (12) are satisfied. For any choice of holomorphic homeomorphisms α_j^t the mapping $\alpha_j^t\circ\sigma\circ(\alpha_k^s)^{-1}(z)$ is an antiholomorphic self-mapping of the unit disk which keeps the origin fixed whenever defined. Such a mapping is always conjugate to the complex conjugation.

After all these choices we can start thickening the nodes N_j. We have to perform this thickening in a way that is compatible with the involution σ. That imposes certain conditions on the coordinates z_j of the thickening parameter $\zeta \in \Delta^m$.

To shorten the notation, let X_{z_j} denote the deformed surface $X_{(0, \ldots, z_j, \ldots, 0)}$ where we have thickened only the node N_j.

The nodes $N_1, \ldots, N_{m_1}$ of the type 1 impose conditions on the coordinates $z_1, \ldots, z_{m_1}$. A straightforward verification shows that, for $j = 1, \ldots, m_1$, the involution $\sigma : X \to X$ induces an antiholomorphic involution of X_{z_j} if and only if z_j is real.

In the same way the nodes $N_{m_1+1}, \ldots, N_{m_2}$ impose conditions on the thickening coordinates $z_{m_1+1}, \ldots, z_{m_2}$. If these coordinates z_k are real then the involution $\sigma : X \to X$ induces an antiholomorphic $X_{z_k} \to X_{z_k}$.

At this point of the proof we make an observation that will be used later.

OBSERVATION 7.2. — *Let j be any of the indices $m_1, m_1+1, \ldots, m_2$. Then the nodes N_j are of type 2. In the construction of this proof, the curves*

$$\gamma_j = (\alpha_j^1)^{-1}\{z \mid |z| = |z_j|\} = (\alpha_j^2)^{-1}\{z \mid |z| = |z_j|\}$$

remain point-wise fixed under the involution $\sigma : X_{z_j} \to X_{z_j}$ if and only if $z_j > 0$. For $z_j < 0$, the involution σ maps the curve γ_j onto itself mapping each point $p \in \gamma$ onto its antipodal point.

To prove the above observation is a simple matter of checking all the definitions. This is also the observation that Klein made in [10], p. 8.

The remaining nodes impose a slightly different condition. Remember that for all values of k, $\sigma(N_{m_2+k}) = N_{m_3+k}$. Therefore, if we want to thicken these nodes in a way that the involution σ induces an antiholomorphic self-mapping of the deformed surface, we have to thicken the nodes N_{m_2+k} and N_{m_3+k} simultaneously for each value of k. Denote by $X_{z_{m_2+k} z_{m_3+k}}$ the surface obtained from X by thickening the nodes N_{m_2+k} and N_{m_3+k} according to the parameters z_{m_2+k} and z_{m_3+k}, respectively. Then a straightforward verification shows again that the involution $\sigma : X \to X$ induces an antiholomorphic mapping

$$\sigma : \quad X_{z_{m_2+k} z_{m_3+k}} \to X_{z_{m_2+k} z_{m_3+k}}$$

if and only if $z_{m_2+k} = \bar{z}_{m_3+k}$. This condition is always verified if the parameters z_{m_2+k} and z_{m_3+k} are real and $z_{m_2+k} = z_{m_3+k}$.

The mapping (11) is continuous into the complex moduli space of stable Riemann surface of genus g.

Let

$$(\Delta')^m = \{ r \in \Delta^m \mid r = (r_1, \ldots, r_{m_2}, r_{m_2+1}, r_{m_2+1}, \ldots, r_{m_2+1}) \}.$$

On basis of the above construction, it is now clear that we get a continuous mapping

$$(13) \qquad (\Delta')^m(\mathbf{R}) = \mathbf{R}^m \cap (\Delta')^m \to \bar{\mathrm{M}}_{\mathbf{R}}^g, \qquad r \mapsto [X_r, \sigma].$$

For a real number e, $-1<e<1$, let $\varepsilon=(e, \ldots, e)\in(\Delta')^m(\mathbf{R})$. Provided that $e\neq 0$, the surface X_ε is a smooth Riemann symmetric surface of genus g. Therefore, for any such ε with $e\neq 0$, the point $[(X_\varepsilon, \sigma)]$ belongs to the union of the sets $V(n, k)$. This completes the proof.

COROLLARY 7.3. — *The moduli space* $\bar{M}^g_{\mathbf{R}}$ *of stable and symmetric Riemann surface of genus* g, $g>1$, *is compact.*

Proof. — By Theorem 7.1 $\bar{M}^g_{\mathbf{R}}$ is the union of the closures of the sets $V(n, k)$. By Theorem 6.1 each of these closures is compact. For a fixed genus g there are only $[(3g+4)/2]$ sets $V(n, k)$. Therefore

$$\bar{M}^g_{\mathbf{R}}=\bigcup_{n,\,k}\overline{V(n, k)}$$

is compact.

We can draw another conclusion from the above theorem. From the continuity of the mapping $\bar{M}^g_{\mathbf{R}}\to\bar{M}^g$ and from the compactness of $\bar{M}^g_{\mathbf{R}}$ we deduce immediately:

COROLLARY 7.4. — *Stable real algebraic curves of genus* g, $g>1$, *form a compact subset in the moduli space* $\bar{M}^g$ *of stable complex algebraic curves of genus* g.

The moduli space of stable complex algebraic curves is a projective variety. In [12] I have shown that real algebraic curves form a real analytic and semialgebraic subset of the moduli space $\bar{M}^g$. The compactness of the set of real algebraic curves in $\bar{M}^g$ was a crucial part in that construction. The above Corollary gives an alternative proof for that fact.

We conclude this paper by the following result:

THEOREM 7.5. — *The moduli space* $\bar{M}^g_{\mathbf{R}}$ *of stable real algebraic curves of genus* g, $g>1$, *is connected.*

Proof. — In [13], Theorem 2.2, we have shown that each set $V(n, k)$ is connected. (This is actually a simple consequence of the connectedness of the corresponding Teichmüller spaces.) Therefore also the closures of the sets $V(n, k)$ are connected. It suffices to show that there are enough points where the closures of the sets $V(n, k)$ intersect. To that end we will use Observation 7.2 and the construction of the related proof.

To construct first such point, let $g>1$. Take two copies of the Riemann sphere $\bar{\mathbf{C}}$ and identity the points $0, 2, 4, \ldots, 2g$. In this way we get a stable Riemann surface X of genus g with $g+1$ nodes $N_1, N_2, \ldots, N_{g+1}$ at the points $0, 2, 4, \ldots, 2g$.

The mapping, which takes the point z of the first copy of the Riemann sphere onto the point $\bar{z}$ in the second copy, is an antiholomorphic involution $\sigma : X\to X$. Therefore (X, σ) defines a point in $\bar{M}^g_{\mathbf{R}}$.

When defining the thickening (11) we had first to choose punctured disks U_j^1 and U_j^2, which form a neighborhood of the node N_j plus holomorphic mappings $\alpha_j^1 : U_j^1\to\Delta^*$ and $\alpha_j^2 : U_j^2\to\Delta^*$. In the present construction we choose the punctured disk D_j^1 to be the punctured disk of radius 1 with center at the point $2(j-1)$ in the first copy of the Riemann sphere. Similarly, U_j^2 is the corresponding punctured disk in the second copy of the Riemann sphere. The mappings α_j^i are translations $\alpha_j^i(z)=z-2(j-1)$, $i=1, 2$.

Choose any number e, $0<e<1$. Let $\varepsilon(n)$ denote that $(g+1)$-tuple

$$\varepsilon(n)=(e, \ldots, e, -e, \ldots, -e),$$

where e appears n times and $-e$ appears $g+1-n$ times.

Then, by Observation 7.2 and by the considerations in Section 2, for each n, $0\leqq n\leqq g$,

$$[X_{\varepsilon(n)}, \sigma]\in V(n, 1) \quad \text{and} \quad [X_{\varepsilon(g+1)}, \sigma]\in V(g+1, 0). \tag{14}$$

From the continuity of the mapping (13) we conclude then that the set

$$\overline{V(g+1, 0)}\cup\left(\bigcup_{n=0}^{g}\overline{V(n, 1)}\right)$$

is connected in $\bar{M}_{\mathbf{R}}^g$.

In the same fashion and using the concrete construction of Section 2 we can construct points where the sets $\overline{V(n, 0)}$ and $\overline{V(n-1, 1)}$ intersect. That is done by replacing the Riemann sphere in the above example by a Riemann surface of genus $(g+1-n)/2$. [Recall that, by the considerations of Section 2, for a component $V(n, 0)$ the number $g+1-n$ is always even.]

Let X be this Riemann surface. Select n points $p_1, \ldots, p_n$ on X. Let $\bar{X}$ be the mirror image of X. Then the identity mapping induces an antiholomorphic mapping $X\to\bar{X}$. Glue X and $\bar{X}$ together identifying the points $p_1, \ldots, p_n$. In this manner one obtains a stable Riemann surface Y of genus g. The antiholomorphic mapping $X\to\bar{X}$ determines an antiholomorphic involution $\sigma: Y\to Y$. Then (Y, σ) is a symmetric stable Riemann surface of genus g, *i.e.*, (Y, σ) is a stable real algebraic curve.

Repeating the previous construction word by word we observe that the point $[(Y, \sigma)]$ lies in the closure of $V(n, 0)$ and in the closure of $V(p, 1)$ for any p, $p<n$. This concludes the proof.

Here we have considered only real algebraic curves of genus >1. Similar constructions can be carried over to real algebraic curves of genus 1. There are, however, unexpected technical complications. In this case a compactification of the moduli space of smooth real curves is a circle. Robert Silhol has worked out all the details in this case.

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