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CAUCHY-LERAY FORMS AND VECTOR BUNDLES

BY BO BERNDTSSON

1. Introduction

The main object of this paper is to obtain integral formulas of Cauchy-Leray type for (p, q)-forms on a complex manifold. We will make precise what we mean by such a formula in section 4, but for the time being it is enough to think of it as a generalization of the one-variable Cauchy formula. When the manifold is a domain in $\mathbb{C}^n$ (and appropriate choices are made), the kernels involved in our formula coincide with the Cauchy-Leray kernel (*see* e. g. [He-L] [O]). In the case of a Stein manifold and $p=0$, we find the formulas of Henkin and Leiterer [He-L]. The original motivation for our paper was to generalize the constructions of Henkin and Leiterer to forms of arbitrary bidegree. Such a generalization has already been fiven by Demailly and Laurent-Thiebaut [Dem-Lth], but they only give the leading terms in the expansion of the kernels. Still, the idea in [Dem-LTh], to use a connection on a bundle, is of fundamental importance in this paper as well.

Our construction uses heavily the formalism and ideas developed by Bott and Chern in two papers, [B-Ch 1] and [B-Ch 2]. To explain the relation we consider first the Cauchy-Leray formula for a domain D in $\mathbb{C}^n$. Let Δ be the diagonal in $D \times D$, and let $[\Delta]$ be the current defined by integration over Δ. The basic point in all constructions of integral formulas is to find a kernel, or a differential form on $\bar{D} \times \bar{D}$ with singularities on Δ, K, that satisfies the equation of currents

$$dK = [\Delta]. \tag{1}$$

If moreover K is of bidegree $(n, n-1)$ we must have $\partial K = 0$ (since $[\Delta]$ is of bidegree (n, n)), so $dK = \bar{\partial} K = [\Delta]$, and this gives n-dimensional Cauchy formulas.

For a general complex manifold, M we may still consider Δ as a submanifold of $M \times M$ and try to solve (1). This is however in general impossible since $[\Delta]$ may represent a non-trivial cohomology class in $M \times M$. Therefore we settle for the weaker

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equation

$$dK = [\Delta] - \alpha \tag{2}$$

where α is a smooth form. One way to solve this equation can be explained as follows.

Consider a general complex manifold X (e.g. $M \times M$) and a complex submanifold, Y, of X (e.g. $Y = \Delta$).

Suppose we have given a complex vector-bundle $E \xrightarrow{\pi} X$ of rank $p = \text{codim } Y$, and let η be a smooth section to E. We say that η defines Y if $Y = \{\eta = 0\}$ and moreover

$$d\eta_1 \wedge \ldots d\eta_p \wedge d\bar{\eta}_1 \wedge \ldots d\bar{\eta}_p \neq 0 \quad \text{on } Y$$

where the η_j:*s* are the coefficients of *n* w.r.t. some frame. Then the cohomology class defined by the current [Y] equals $c_p[\Theta]$ -the p:th Chern class of E (*see* [Hi]). As explained in section 3 this fact is almost equivalent to a general form of the Gauss-Bonnet theorem. Concretely it means that whenever Θ is the curvature form of some connection on E, we can solve the equation

$$dK = [Y] - c_p[\Theta] \quad \text{on } X, \tag{3}$$

where $c_p(\Theta)$ is the $p:th$ Chern form of Θ (*see* section 2 and [Gr-H]). The main part of this paper consists in finding a large family of explicit solutions K to (3) (section 2).

For this we first solve

$$dK = -c_p[\Theta] \quad \text{on } X \diagdown Y, \tag{4}$$

with a form K which is singular on Y. Notice that E′, the restriction of E to $X \diagdown Y$ has a global nonvanishing section (namely η). By the theory of Chern classes this implies that $c_p[E'] = 0$, which is the same as saying that we can solve (4). The method we use to do this is adapted from [B-Ch 1], where one actually treats the more refined equation $\partial\bar{\partial}L = c_p[\Theta]$. Our situation is therefore simpler than [B-Ch 1], but on the other hand our solution formulas have an additional degree of freedom. In the language of integral formulas, [B-Ch 1] is concerned with the Bochner-Martinelli kernel, whereas we are interested in general Cauchy-Leray forms.

Finally we have to verify that our solution to (4) actually solves (3) as well. To do this it is practical to compute K using the formalism of Bott and Chern [B-Ch 2]. It is then easy to see that K is a generalization of the Cauchy-Leray kernel in the flat case. It is somewhat surprising that our method gives a nontrivial result even if the *bundle* E is trivial.

The solution of (3) occupies section 2. In section 3 we show how our formulas imply some classical results in differential geometry, namely the Hopf index theorem, the Gauss-Bonnet theorem and the Bott residue theorem. Neither of these proofs are very much different from the ones in the literature (*see* [Gr-H] and [Ch]) but we have included them as an illustration. The first two theorems follow from (3), even without knowing the explicit form of K, whereas the last one relies more on the method to solve (3) from section 2.

Finally, in section 4, we get back to the subject of integral formulas. Here $Y = \Delta$, the diagonal in $M \times M$. We then prove a version of "Koppelman's" formula, assuming that we can find a holomorphic bundle E and a holomorphic section η to feed into the machinery we have just described. In which generality this assumption is fulfilled seems to be a difficult question, so we content ourselves with some examples.

As is clear from this introduction many of the ideas and methods in this paper are not new. The reason for writing it is the relation it shows between two areas that do not seem to have been previously connected. As a general background reference we quote [Gr-H], chapter 3.

Most of this work was done when I visited the Universitat Autònoma de Barcelona, and I want to express my sincere thanks to the members of the mathematics department there for making my stay both stimulating and enjoyable.

2. Chern forms and currents defined by varieties

Let X be a complex manifold and let Y be a complex submanifold of codimension p. Suppose that

$$E \xrightarrow{\pi} X$$

is a holomorphic vector-bundle of rank p, which has a holomorphic section

$$X \xrightarrow{\eta} E,$$

that defines Y in the sense explained in the introduction. Let D be any connection on E. Thus D is a map which sends sections of E to one-forms with values in E. D extends naturally to a map sending q-forms with values in E to $(q+1)$-forms with values in E and satisfies

$$D fe = df e + (-1)^s D e$$

if f is a q-form and e is a local section. The curvature operator of D is defined by

$$\Theta e = D^2 e$$

(*see* [Gr-H] Ch. 0 sec. 5). Θ is a 2-form with values in Hom(E, E), and is given by a matrix of 2-forms (Θ_{jk}) as soon as we fix a local frame. The $p:th$ Chern form of Θ is defined as

$$c_p[\Theta] = \left(\frac{i}{2\pi}\right)^p \det(\Theta_{jk})$$

The determinant of a matrix of 2-forms is defined just like the determinant of a matrix with complex entries. The matrix (Θ_{jk}) of course depends on our local frame, but $c_p[\Theta]$ does not. Thus $c_p[\Theta]$ is a global form on X and it turns out it is closed. Its cohomology class, $c_p[E]$, is the $p:th$ Chern class of the bundle E.

We want to solve the equation

$$dK = [Y] - c_p[\Theta] \quad \text{on } X \tag{1}$$

explicitly. This equation clearly implies

$$dK = -c_p[\Theta] \quad \text{on } X \setminus Y \tag{2}$$

and we shall start by solving (2). We know *a priori* that this is possible since $E' = E$ restricted to $X \setminus Y$ has a global non-vanishing section, which implies $c_p[E'] = 0$. To solve (2) we shall basically retrace the proof of this fact.

Let us start by explaining the case $p = 1$ since the formulas are very simple then. If e_1 is a local holomorphic frame then

$$D f_1 e_1 = (df_1 + \theta f_1) e_1$$

where θ is the connection form. Then the curvature form, Θ, is given by

$$\Theta = d\theta.$$

If D' is another holomorphic connection with connection form θ', then

$$\beta = \theta' - \theta$$

is a 1-form with values in Hom (E, E) so it is a scalar-valued one-form when $p = 1$. Hence, if $\Theta' = 0$

$$d \frac{i}{2\pi} \beta = c_1[\Theta'] - c_1[\Theta] = -c_1[\Theta].$$

We therefore want to define a connection, D', with vanishing curvature over $X \setminus Y$. This we achieve by requireing

$$D' \eta = 0$$

which uniquely determines D' when $p = 1$. Then $\Theta' \eta = 0$ so $\Theta' = 0$, and we are done. Explicitly, if e_1 is a local frame,

$$0 = D' \eta = D'(\eta_1 e_1) = (d\eta_1 + \theta' \eta_1) e_1$$

Hence

$$\theta' = -\frac{d\eta_1}{\eta_1}$$

so our solution is

$$\frac{i}{2\pi}(\theta'-\theta)=\frac{1}{2\pi i}\left(\frac{d\eta_1}{\eta_1}+\theta\right)=:\mathrm{K}.$$

Clearly we also have

$$d\mathrm{K}=[\{\eta=0\}]-c_1[\Theta]$$

and the relation to Cauchy's formula is evident.

Let now p be arbitrary and let D be an arbitrary connection on E. Again we want to define a new connection on E′, the restriction of E to $X\setminus Y$, so that

$$D'\eta=0.$$

When $p>1$, this does not determine D′, so we need to choose a rank $(p-1)$ subbundle F which together with η spans E at every point, and then decide how D′ should be defined on F. A choice of F is equivalent to a choice of a section ξ to the dual bundle E* satisfying $\langle\xi,\eta\rangle=1$ by the correspondence $F=\{e;\langle\xi,e\rangle=0\}$. For simplicity we let $D'=D$ on F, which implies

$$D'e=De-\langle\xi,e\rangle D\eta.$$

Clearly $D'\eta=0$ implies $\Theta'\eta=0$ which in turn implies

$$c_p[\Theta']=0$$

since Θ' has a non-trivial null-vector. By a theorem of Weil ([B-Ch 1]) the Chern class is independent of the choice of connection so there exists a form α on $X\setminus Y$ solving

$$d\alpha=c_p[\Theta']-c_p[\Theta]=-c_p[\Theta].$$

To find α explicitly we shall recall the usual proof of Weils theorem.

The difference between our two connections

$$\beta e=D'e-De=-\langle\xi,e\rangle D\eta \tag{3}$$

defines a 1-form with values in Hom(E, E) over $X\setminus Y$. Let

$$D_t e=:De-t\langle\xi,e\rangle D\eta \qquad \text{for} \quad 0\leqq t\leqq 1,$$

and let $\Theta_t=D_t^2$. If now $P(A_1,\ldots,A_p)$ is the polarization of the homogeneous matrix polynomial

$$\left(\frac{i}{2\pi}\right)^p\det(A),$$

the formula is

$$c_p[\Theta'] - c_p[\Theta_0] = d\left\{ p \int_0^1 \mathrm{P}(\beta, \Theta_t, \ldots, \Theta_t)\, dt \right\}$$

so the form α we are looking for is given by

$$\alpha = p \int_0^1 \mathrm{P}(\beta, \Theta_t, \ldots, \Theta_t)\, dt. \tag{4}$$

We shall now compute this integral.

For this we will use the exterior algebra introduced in [F] and applied in [B-Ch 2]. Consider the Whitney sum

$$\mathrm{E}^* \oplus \mathrm{E}$$

and the exterior algebra $\wedge (\mathrm{E}^* \oplus \mathrm{E})$.

$\wedge (\mathrm{E}^* \oplus \mathrm{E})$ has a natural bigrading in the following way: If $e_1, \ldots, e_p$ is a local frame for E^* and $e_1^*, \ldots, e_p^*$ the dual frame for E^* we define

$$\Lambda^{r,s}(\mathrm{E}^* \oplus \mathrm{E}) = \{ \sum_{|\mathrm{I}|=r, |\mathrm{J}|=s} a_{\mathrm{IJ}}\, e_{\mathrm{I}}^* \wedge e_{\mathrm{J}};\, e_{\mathrm{I}}^* = e_{i_1}^* \wedge \ldots e_{i_p}^*, e_{\mathrm{J}} = e_{j_1} \wedge \ldots e_{j_s} \}$$

This is clearly invariant under change of frame and $\Lambda = \sum_{r, s \leqq p} \Lambda^{r,s}$.

(It should be noted that our definition differs from [B-Ch 2] in as much as we use the dual bundle instead of the conjugate bundle).

The exterior product on $\mathrm{E}^* \oplus \mathrm{E}$ extends to forms with values in this bundle. In terms of our local trivialization a form α with values in $\Lambda^{r,s}$ can be written

$$\alpha = \sum a_{\mathrm{IJ}}\, e_{\mathrm{I}}^* \wedge e_{\mathrm{J}}$$

where a_{IJ}, are forms. If

$$\beta = \sum b_{\mathrm{KL}}\, e_{\mathrm{K}}^* \wedge e_{\mathrm{L}}$$

is another such object we define

$$\alpha \wedge \beta = \sum a_{\mathrm{IJ}} \wedge b_{\mathrm{KL}}\, e_{\mathrm{I}}^* \wedge e_{\mathrm{J}} \wedge e_{\mathrm{K}}^* \wedge e_{\mathrm{L}}$$

In other words forms commute with vectors from $\mathrm{E}^* \oplus \mathrm{E}$ and the product between forms is the usual one.

Now suppose that A is an m-form with values in Hom (E, E) (for instance a curvature form or a difference between two connections). Given a local trivialization, A is given by a matrix A_{jk} of m-forms so that

$$\mathrm{A}\left(\sum y_j e_j\right) = \sum \mathrm{A}_{jk}\, y_j e_k.$$

We can associate to A the element

$$\tilde{A}=\sum A_{jk}e_j^* \wedge e_k$$

of $\wedge^{1,1}$. This correspondence is independent of the choice of frame.

LEMMA 2.1:

$$\tilde{A}^p =: \underbrace{\tilde{A} \wedge \ldots \tilde{A}}_{p \text{ times}} = = p! \det(A_{jk})\, e_1^* \wedge e_1 \wedge \ldots e_p^* \wedge e_p$$

if the A_{jk}*: s are forms of even degree.*

Here, as in the scalar-valued case,

$$\det(A_{jk})=\sum_{I} e_I A_{1i_1} \wedge A_{2i_2} \wedge \ldots A_{pi_p}$$

where the sum is over all permutations $I=(i_1 \ldots i_p)$ and e_I is the sign of the permutation. The proof is the same as in the scalar-valued case since forms of even degree commute.

In particular if Θ is a curvature form we have the formula

$$(5) \qquad \left(\frac{i}{2\pi}\right)^p \frac{1}{p!}\tilde{\Theta}^p = c_p[\Theta]\, e_1^* \wedge e_1 \wedge \ldots e_p^* \wedge e_p$$

Let us put $e_1^* \wedge e_1 \wedge \ldots e_p^* \wedge e_p = \Lambda$. Since it equals $\frac{1}{p!}\tilde{I}^p$ where $I \in \operatorname{Hom}(E, E)$ it is independent of frame.

We shall use this formalism in (4), so we need to compute the elements $\tilde{\beta}$ and $\tilde{\Theta}_t$ in $\wedge^{1,1}$ corresponding to β and Θ_t. First it is clear from (3) that

$$\tilde{\beta} = -\xi \wedge D\eta.$$

On the other hand

$$\Theta_t e =: D_t^2 e = D_t(De - t\langle \xi, e\rangle D\eta) = \Theta e - td\langle \xi, e\rangle \wedge D\eta - t\langle \xi, e\rangle \Theta\eta$$
$$- tD\eta \wedge \langle \xi, De\rangle + t^2\langle \xi, e\rangle D\eta \wedge \langle \xi, D\eta\rangle.$$

At this point we recall that the dual connection on E^* is defined by

$$d\langle \xi, \eta\rangle = \langle D^*\xi, \eta\rangle + \langle \xi, D\eta\rangle,$$

if ξ and η are arbitrary smooth sections to E^* and E, respectively. Hence

$$\Theta_t e = \Theta e - t\langle D^*\xi, e\rangle \wedge D\eta - t\langle \xi, e\rangle \Theta\eta + t^2\langle \xi, e\rangle D\eta \wedge \langle \xi, D\eta\rangle,$$

which gives

$$\tilde{\Theta}_t = \tilde{\Theta} - tD^*\xi \wedge D\eta - t\xi \wedge \Theta\eta + r^2\xi \wedge D\eta \wedge \langle \xi, D\eta\rangle.$$

Clearly

$$\tilde{\beta} \wedge \tilde{\Theta}_t^{p-1} = p!\, \mathrm{P}(\beta, \Theta_t, \ldots, \Theta_t)\, \Lambda.$$

Since $\tilde{\beta}$ contains a factor ξ and since $\xi \wedge \xi = 0$ we can replace $\tilde{\Theta}$ by

$$\tilde{\Theta} - t\, \mathrm{D}^* \xi \wedge \mathrm{D}\eta$$

in this equation. The result is

$$\mathrm{P}(\beta, \Theta_t, \ldots, \Theta_t) \wedge = -\frac{1}{p!} \xi \wedge \mathrm{D}\eta \wedge \sum_{k=0}^{p-1} \binom{p-1}{k} (-t)^{p-1-k} (\mathrm{D}^* \xi \wedge \mathrm{D}\eta)^{p-1-k} \wedge \tilde{\Theta}^k$$

Thus

$$\mathrm{K} =: p \int_0^1 \mathrm{P}(\beta, \Theta_t, \ldots, \Theta_t)\, dt$$

is determined by

$$\text{(6)} \qquad \mathrm{K}\Lambda = \frac{1}{p!} \left(\frac{i}{2\pi} \right)^p \xi \wedge \mathrm{D}\eta \wedge \sum_{k=0}^{p-1} \binom{p}{k} (-1)^{p-k} (\mathrm{D}^* \xi \wedge \mathrm{D}\eta)^{p-1-k} \wedge \tilde{\Theta}^k.$$

Let us now also note that if we replace ξ by $f\xi$ in (6), the k:*th* term in K becomes multiplied by f^{p-k}, since again, $\xi \wedge \xi = 0$.

Therefore, if we replace ξ in (6) by

$$\frac{\xi}{\langle \xi, \eta \rangle},$$

where ξ is an arbitrary section to E^* we obtain

$$\text{(7)} \qquad \mathrm{K}\Lambda = \frac{1}{p!\,(2\pi i)^p} \xi \wedge \mathrm{D}\eta \sum_{k=0}^{p-1} \binom{p}{k} (-1)^k \frac{(\mathrm{D}^* \xi \wedge \mathrm{D}\eta)^{p-1-k}}{\langle \xi, \eta \rangle^{p-k}} \wedge \tilde{\Theta}^k.$$

Writing $\mathrm{K}[\xi, \eta]$ to indicate the dependence of K on ξ and η we have proved

PROPOSITION 2.2. — $d\mathrm{K}[\xi, \eta] = -c_p[\Theta]$ *where* $\langle \xi, \eta \rangle \neq 0$.

Before we go on, let us note that so far we have not used the holomorphicity of our bundle E, only its complex structure. Indeed, we have not even used that X is a complex manifold so Proposition 2.2 holds for complex bundles over arbitrary manifolds.

Let us also see what becomes of K in case our bundle and connection are trivial. Let $e_1 \ldots e_p$ be a global frame with respect to which

$$\mathrm{D} \sum \eta_j e_j = \sum d\eta_j e_j.$$

Then Θ is zero so

$$K\Lambda = \frac{1}{p!\,(2\pi i)^p}\xi \wedge D\eta \wedge (D^*\xi \wedge D\eta)^{p-1}$$
$$= \frac{1}{p!\,(2\pi i)^p}\xi\,\Lambda\,(D^*\xi)^{p-1} \wedge (D\eta)^p$$

since $D^*\xi$ and $D\eta$ commute.

LEMMA 2.3. — *Let* $D\eta = \sum (D\eta)_j e_j$ *and* $D^*\xi = \sum (D^*\xi)_j e_j^*$ *then*

$$(D\eta)^p = p!\,(D\eta)_1 \wedge \ldots (D\eta)_p e_1 \wedge \ldots e_p =: \omega(D\eta)\, e_1 \wedge \ldots e_p$$

and

$$\xi \wedge (D^*\xi)^{p-1} = p! \sum_0^p \xi_j \underset{k\neq j}{\wedge} (D^*\xi)_k e_1^* \wedge \ldots e_p^* =: \omega'(\xi, D^*\xi)\, e_1^* \wedge \ldots e_p^*.$$

Hence

$$K = \frac{p!}{(2\pi i)^p}(-1)^{p(p-1)/2}\frac{\omega'(D^*\xi) \wedge \omega(D\eta)}{\langle \xi, \eta\rangle^p} \tag{9}$$

if $\Theta = 0$.

Proof:

$$(D\eta)^p = \sum_{\text{permutations}} (D\eta)_{i_1} \wedge \ldots (D\eta)_{i_p} e_{i_1} \wedge \ldots e_{i_p}$$
$$= p!\,(D\eta)_1 \wedge \ldots (D\eta)_p e_1 \wedge \ldots e_p,$$

and the formula for ξ is proved similarily.

Let us now say that ξ is admissible for η if for any compact $K \subset X$

$$|\xi| \leqq C_K |\eta| \text{ and}$$
$$|\langle \xi, \eta\rangle| \geqq c_K |\eta|^2$$

This is for instance the case if ξ is the dual vector to η with respect to some metric.

We shall now compute the contribution to dK that comes from K:*s* singularity assuming that ξ is admissible for η and that X is orientable. First we need to discuss the orientation of the manifold $Y = Y_\eta$ where η vanishes. Recall that η is assumed to define Y_η, which we have taken to mean that if $\eta_1, \ldots, \eta_p$ are the coefficients of η with respect to some frame then there are, locally, complex valued functions $\eta_{p+1}, \ldots, \eta_n$ such that all the η_j:*s* together form a local coordinate system on X. Then

$$\operatorname{Re}\eta_{p+1}, \operatorname{Im}\eta_{p+1}, \ldots \operatorname{Re}\eta_n, \operatorname{Im}\eta_n$$

form a local coordinate system on Y_η. We declare this coordinate system oriented if the coordinate system

$$\operatorname{Re}\eta_1, \operatorname{Im}\eta_1, \ldots \operatorname{Re}\eta_n, \operatorname{Im}\eta_n$$

is oriented on X, and then we define the current of integration $[Y_\eta]$ using this orientation. In case $p=n$ this should mean that $[Y_\eta]$ is a sum of Dirac measures

$$\sum_{\eta(x)=0} \operatorname{ind}_x(\eta)\,\delta_x$$

where, by definition, $\operatorname{ind}_x(\eta)$ is $+1$ if η defines an oriented coordinate system at x_j, and -1 otherwise.

THEOREM 2.4. — *Suppose* E *is a complex vector bundle over the oriented manifold* X, *that* η *is a smooth section to* E *and that* ξ *is admissible for* η. *Suppose that* η *defines the oriented submanifold* Y_η. *Then*

$$dK[\xi, \eta] = [Y_\eta] - c_p[\Theta].$$

If moreover E, D *and* η *are holomorphic, then* $\partial K = 0$ *so*

$$\bar\partial K[\xi, \eta] = [Y] - c_p[\Theta].$$

Proof. — All that remains to be checked is that the contribution from K: s singularity is precisely $[Y_\eta]$. This is purely local so we may choose a frame. Moreover, we can assume that D is trivial with respect to this frame since all terms involving the curvature or the connection form have singularities of lower order. In other words we may assume that

$$K = K_0 = \frac{p!}{(2\pi i)^p}(-1)^{p(p-1)/2}\frac{\omega'(\xi, d\xi) \wedge \omega(d\eta)}{\langle \xi, \eta \rangle^p}.$$

But this is precisely the classical Cauchy-Leray form and a proof that

$$dK_0 = [\{\eta = 0\}]$$

can be found in [A-B].

The last assertion of the theorem follows since K can have no component of bidegree (m, l) if $m<p$ and D is holomorphic. Since [Y] and $c_p[\Theta]$ are of bidegree (p, p) in this case we must have $\partial K = 0$.

Remark. — For any choice of sections η and ξ we obtain a kernel $K[\xi, \eta]$. As in [B-Ch 1] we can consider all those forms as pullbacks of a universal form $\mathscr{K}$ defined on the set

$$F = \{(\xi, \eta) \in E^* \oplus E; \langle \xi, \eta \rangle \neq 0\}.$$

$\mathscr{K}$ can be defined in the following way: First we use the projection map

$$\Pi: \quad E^* \oplus E \to X$$

to pull back $E^* \oplus E$ to a bundle over itself, and then we take the restriction of that bundle to F. On F we can define ξ and η as the tautological sections taking the *point* (ξ, η) in F to the same *vector* in $E^* \oplus E$. Then $\mathscr{K}$ is the kernel $K[\xi, \eta]$ on F.

Introducing a metric on E we can choose ξ as the dual vector to η under this metric. Then K defines a form on E by a variant of the above argument. Since this ξ is clearly admissible for y we find

$$dK = [X] - c_p[\pi^*(\Theta)]$$

where X is considered as the submanifold of E defined by its zero-section.

Remark 2. – It is tempting to define a notion *generalized connection*, so that D′ becomes a connection on E and not just on E′.

A generalized connection would then be a map

$$D: \quad \Gamma(E) \to \mathrm{Curr}_1 \otimes \Gamma(E)$$

where Curr_1 is the space of currents of degrees 1. D should satisfy

$$D fe = dfe + f De$$

whenever e is a section of E and f is a C^∞-function.

3. Some geometric applications

Theorem 2.4 says that $[Y_\eta]$ and $c_p[\Theta]$ are cohomologous. The Hopf index theorem and the general Gauss-Bonnet theorem are easy consequences of this (together with Lemma 3.2), and do not depend on the explicit form of K or on the holomorphicity of any object involved. They are thus not related directly to the subject matter of this paper, but we want to give the proofs anyway since they require very little additional effort. However, in the proof of Bott's residue theorem we will use arguments similar to those in section 2.

First we write Theorem 2.4 in integrated form.

THEOREM 3.1. — *Under the same assumptions as in Theorem* 2.4 *we have, if f is a closed* $2(n-p)$*-form on the compact manifold* X

$$\int_Y f = \int_X c_p[\Theta] \wedge f$$

In particular, if $p=n$, we obtain

$$\sum_{\eta(x)=0} \mathrm{ind}_x(\eta)=\int_X c_n[\Theta].$$

Theorem 3.1 could have been obtained by combining the Hopf index theorem and the Gauss-Bonnet theorem, but here we shall follow the reverse path. The missing part is given by an alternate description of $[\Delta]$, where, again, Δ is the diagonal in $X=M\times M$. Suppose M is compact. Then its cohomology groups are finite so for each q we can choose a basis

$$f_1^q, \ldots f_{d_q}^q$$

for $H^q(M, \mathbb{C})$. Let

$$g_1^q, \ldots g_{d_q}^q$$

be the dual basis for $H^{2n-q}(M, \mathbb{C})$, so that

$$\int_M f_j^q \wedge g_k^q=\delta_{jk}.$$

Then it is well known that the set

$$f_j^q(\zeta) \wedge g_k^q(z), \qquad 1 \leqq j, k \leqq d_q, \quad 0 \leqq q \leqq 2n$$

form a basis for $H^{2n}(M_\zeta \times M_z, \mathbb{C})$

LEMMA 3.2. — *The cohomology class of* $[\Delta]$ *in* $H^{2n}(M\times M, \mathbb{C})$ *equals*

$$c=:\sum_{q=0}^{2n}(-1)^q \sum_{j=1}^{d_q} f_j^q(\zeta) \wedge g_j^q(z).$$

Proof. — We just need to check that

$$\int_{M_\zeta \times M_s} c \wedge g_l^s(\zeta) \wedge f_k^s(z)=\int_M g_l^s \wedge f_k^s.$$

The left hand side equals

$$(-1)^s \int_{M_\zeta \times M_z} f_l^s(\zeta) \wedge g_l^s(z) \wedge g_l^s(\zeta) \wedge f_k^s(z)=(-1)^{s+(2n-s)^2} \int_{M_z} g_l^s(z) \wedge f_k^s(z)$$

which is the same as the right hand side.

Since we know that $[\Delta]$ is cohomologous to $c_n[\Theta]$, so is c. Hence

$$(8) \qquad \int_\Delta c_n[\Theta]=\int_\Delta c=\sum_0^n(-1)^q \dim H^q(M, \mathbb{C}).$$

All we know *a priori* about Θ is that it is the curvature of some bundle which has a smooth section that defines Δ. Such a bundle and section always exist, at least over a neighbourhood of Δ in $M \times M$. (To see this we can use the fact that Δ has a neighbourhood which is diffeomorphic to a neighbourhood of the zero section in the normal bundle of Δ.) From this (8) follows, and moreover it is easy to see that the restriction of any such bundle to Δ must be the normal bundle of Δ in $M \times M$. This bundle is in turn isomorphic to the tangent bundle of M so (8) implies:

THE GAUSS-BONNET THEOREM. — *If* M *is a compact complex manifold of dimension n then*

$$\chi(M) =: \sum_0^n (-1)^q \dim H^q(M, \mathbb{C}) = \int_M c_n [T_{1,0}]$$

Combining this with Theorem 3.1 we get

POINCARÉ-HOPF THEOREM. — *If* η *is any field of tangent vectors with only simple zeros on the compact complex manifold* M *then*

$$\sum_{\eta(x)=0} \operatorname{ind}_x(\eta) = \chi(M).$$

Let now $P(A_1, \ldots, A_p)$ be an arbitrary invariant polynomial, homogeneous of degree p in the matrix arguments A_k (*see* [Gr-H]). Here we assume that E is the complex tangent-bundle of X so that dim $X = p$. We assume η is a holomorphic section to E with isolated simple zeros, $x_1, \ldots, x_N$, on the compact manifold X. Near an x_j we can choose local holomorphic coordinates so that $z(x_j) = 0$. Define η_k by

$$\eta = \sum \eta_k \frac{\partial}{\partial z_k}$$

in a neighbourhood of x_j. By assumption, the matrix

$$A_j = \left(\frac{\partial \eta_k}{\partial z_l}(0) \right)$$

is nonsingular. Since P is invariant $P(A_j) =: P(A_j, \ldots, A_j)$ is well defined and independent of choice of local coordinates.

THEOREM (Bott residue theorem):

$$\sum_1^N \frac{P(A_j)}{\det A_j} = \left(\frac{i}{2\pi} \right)^p \int_X P(\Theta) \tag{1}$$

if Θ *is the curvature form of any holomorphic connection on* $E = T_{1,0}(X)$.

To prove this we shall again construct a new connection D′ so that its curvature form Θ' satisfies $P(\Theta') = 0$ and then apply Weils theorem. Let first ∇ be the connection defined in the previous section *i. e.*

$$\nabla e = \mathrm{D}e - \langle \xi, e \rangle \mathrm{D}\eta,$$

where ξ is a (1,0) form satisfying $\langle \xi, \eta \rangle = 1$ outside $\{\eta = 0\}$. As usual if Z is any vectorfield on X and D is an connection we let $\mathrm{D}_{\mathrm{Z}}\, e$ denote

$$(\mathrm{D}e, \mathrm{Z})$$

(the vector valued form $\mathrm{D}e$ acts on the vector Z, so $\mathrm{D}_{\mathrm{Z}} e$ is the covariant derivative of e in the direction Z). We now define, if Z and W are (1,0) vector-fields,

$$\mathrm{D}'_{\mathrm{Z}} \mathrm{W} = \nabla_{\mathrm{W}} \mathrm{Z} + [\mathrm{Z}, \mathrm{W}].$$

The basic idea is thus to change the role of Z and W and we have added the Lie bracket of Z and W to make sure D′ is a connection. If Z is of type (0, 1) we put

$$\mathrm{D}'_{\mathrm{Z}} = \mathrm{Z}(\mathrm{W}).$$

which means that Z acts on W componentwise in any holomorphic frame. With this definition D′ is a holomorphic connection on $\mathrm{T}^{(1,0)}(\mathrm{X})$.

Since $\nabla\eta = 0$ we get

$$\mathrm{D}'_{\eta} \mathrm{W} = [\eta, \mathrm{W}].$$

Assume that W is also holomorphic. Then

$$\mathrm{D}'_{\bar\eta} \mathrm{D}'_{\eta} \mathrm{W} = 0$$

and

$$\mathrm{D}'_{\eta} \mathrm{D}'_{\bar\eta} \mathrm{W} = 0.$$

Thus

$$(\Theta' \mathrm{W}, \eta \wedge \bar\eta) = 0. \tag{2}$$

But this implies immediately that $\mathrm{P}(\Theta') = 0$ for any invariant polynomial of degree p. If we think of $\eta = \partial/\partial\zeta_1$, in a local coordinate system, (2) means that (Θ_{jk}) has no component $d\zeta_1 \wedge d\bar\zeta_1$, so any product of p such forms is zero).

Hence

$$-\mathrm{P}(\Theta) = \mathrm{P}(\Theta') - \mathrm{P}(\Theta) = d\alpha$$

where

$$\alpha = p \int_0^1 \mathrm{P}(\beta_1 \Theta_t, \ldots, \Theta_t)\, dt.$$

As before $\beta = \mathrm{D}' - \mathrm{D}$, $\mathrm{D}_t = \mathrm{D} + t\beta$, and $\Theta_t = \mathrm{D}_t^2$.

We need to compute α only near a zero x_j of η. Here we may assume we have chosen a connection D which is flat. Hence if

$$e=\sum e_k \frac{\partial}{\partial z_k}$$

then

$$\mathrm{D}\,e=\sum de_k \frac{\partial}{\partial z_k}$$

and

$$\nabla e=\mathrm{D}\,e-\langle \xi, e\rangle \sum d\eta_k \frac{\partial}{\partial z_k}.$$

Therefore

$$\mathrm{D}'_{\partial/\partial z_j}\frac{\partial}{\partial z_l}=\nabla_{\partial/\partial z_l}\frac{\partial}{\partial z_j}=\xi_j\sum_k \frac{\partial \eta_k}{\partial z_l}\,\frac{\partial}{\partial z_k}.$$

In other words

$$\mathrm{D}'\frac{\partial}{\partial z_l}=\xi\sum_k \frac{\partial \eta_k}{\partial z_l}\,\frac{\partial}{\partial z_k}$$

so

$$\beta\,e=\mathrm{D}'\,e-\mathrm{D}\,e=\xi\sum_{k,\,l}\frac{\partial \eta_k}{\partial z_l}e_l\frac{\partial}{\partial z_k}$$

and the matrix corresponding to β in our local frame is $\xi\left(\dfrac{\partial \eta_k}{\partial z_l}\right)=:\mathrm{A}$. Now

$$\mathrm{D}_t\,e=de+t\,\mathrm{A}\,e$$

so

$$\Theta_t\,e=(d+t\,\mathrm{A})\,(de+t\,\mathrm{A}\,e)=td(\mathrm{A}\,e)+t\,\mathrm{A}\,de++t^2\,\mathrm{A}^2\,e=t\,(d\mathrm{A})\,e,$$

since $\mathrm{A}^2=0$. Hence the matrix of Θ_t is

$$td\xi\left(\frac{\partial \eta_k}{\partial z_l}\right)+t\,\xi\wedge d\left(\frac{\partial \eta_k}{\partial z_l}\right)$$

Since $\xi\wedge\xi=0$ the second term gives no contribution so we find

$$\mathrm{P}(\beta,\Theta_t,\ldots,\Theta_t)=t^{p-1}\,\xi\wedge(d\xi)^{p-1}\,\mathrm{P}\left(\frac{\partial \eta_k}{\partial z_l}\right)$$

and

$$\alpha=\xi\wedge(d\xi)^{p-1}\,P\left(\frac{\partial\eta_k}{\partial z_l}\right).$$

Thus $\alpha=\dfrac{P(\partial\eta_k/\partial z_l)}{\det(\partial\eta_k/\partial z_l)}\,.\,\alpha'$ where α' is the form α we get when P = determinant. Since

$$d\alpha'=\sum_1^N\delta_{xj}-c_p[\Theta]$$

in that case (*see* Theorem 2.4), it must hold

$$d\alpha=\sum\frac{P(A_j)}{\det(A_j)}\delta_{x_j}-\left(\frac{i}{2\pi}\right)^p P(\theta)$$

in general. This completes the proof.

4. The Cauchy-Leray-Koppelman formula

We now return to the situation in section 2, in the special case which is relevant to integral formulas. Let M be a complex manifold of dimension n and let X be $M\times M$. E is now supposed to be a holomorphic vectorbundle of rank n over X and η is a holomorphic section to E such that

$$\{\eta=0\}=\Delta=\{(\zeta,z)\in M\times M,\zeta=z\}$$

If finally ξ is any smooth section to E^* which is admissible for η, we have by Theorem 2.4

$$(1)\qquad dK=[\Delta]-c_n[\Theta].$$

This means that if ψ is a $2n$-form on X with compact support then

$$(2)\qquad \int_X K\wedge d\psi=\int_\Delta\psi-\int_X c_n[\Theta]\wedge\psi.$$

Let in particular ψ be of the form

$$\psi=f(\zeta)\wedge\varphi(z)$$

where f is a k-form on M_ζ and φ is a $(2n-k)$-form on M_z. Then

$$(3)\quad \int_X K\wedge df(\zeta)\wedge\varphi(z)+(-1)^k\int_X K\wedge f(\zeta)\wedge d\varphi(z)=\int_\Delta f\wedge\varphi-\int_X c_n[\Theta]\wedge f\wedge\varphi.$$

Now, if α is a form or current on $M_\zeta \times M_z$ of degree $(2n-s)$, then α defines an operator which sends compactly supported $(k+s)$-forms on M_ζ to k-currents on M_z, in the following way:

Let $g(\zeta) \in C^\infty_{(k+s),\,c}(M_\zeta)$. Then $\alpha(g)$ is defined by

$$\int_{M_z} \alpha(g) \wedge \varphi = \int_X \alpha \wedge g \wedge \varphi$$

for any test form $\varphi \in C^\infty_{(2n-k),\,c}(M_z)$. If α has coefficients that are locally integrable we can write

$$\alpha = \sum_{|I|+|J|=2n-s} \alpha_{IJ}\, a_I \wedge b_J$$

where a_{IJ} are locally integrable functions and a_I and b_J are forms in ζ and z respectively, then

$$(4)\quad \int_X \alpha \wedge f \wedge \varphi = \sum_{|I|=2n-k-s,\,|J|=k} \int_X \alpha_{IJ}\, a_I \wedge b_J \wedge f \wedge \varphi$$

$$= (-1)^{k(k+s)} \sum \int \alpha_{IJ}\, a_I \wedge f \wedge b_J \wedge \varphi.$$

Therefore, in this case we can realize $\alpha(f)$ concretely as

$$\alpha(f) = (-1)^{k(k+s)} \sum \left(\int_{M_\zeta} \alpha_{IJ}\, a_J \wedge f \right) b_J =: \int_{M_\zeta} \alpha \wedge f$$

Using this definition we can rewrite (3) as

$$\int_{M_z} K(df) \wedge \varphi + (-1)^k \int_{M_z} K(f) \wedge d\varphi = \int_{M_z} f \wedge \varphi - \int_X c_n[\Theta] \wedge f \wedge \varphi.$$

Applying Stokes' Theorem to the second term on the left hand side we find

$$f = K(df) + dK(f) + c_n[\Theta](f).$$

This holds under the assumption that f has compact support in M. Suppose now that D is a smoothly bounded domain in M and that $f \in C^\infty(\bar{D})$. If $\chi \in C^\infty_c(D)$ we can apply (4) to χf and obtain

$$\chi f = K(d\chi f) + K(\chi df) + dK(\chi f) + c_p[\Theta](\chi f)$$

$$= -d\chi \wedge K(f) + K(\chi df) + dK(\chi f) + c_p[\Theta](\chi f).$$

Letting χ tend to the characteristic function, χ_D of D we get for $z \in D$ (recall that $-d\chi_D = [\partial D]$)

$$f = [\partial D] \wedge K(f) + K(\chi_D\, df) + dK(\chi_D f) + c_n[\Theta](\chi_D f)$$

THEOREM 4.1 (Koppelmans formula). — *Suppose that f is a smooth form of degree k on $\bar{\mathrm{D}}$. Then, for $z \in \mathrm{D}$*

$$f(z)=\int_{\partial \mathrm{D}} \mathrm{K} \wedge f+\int_{\mathrm{D}} \mathrm{K} \wedge df+d \int_{\mathrm{D}} \mathrm{K} \wedge f+\int_{\mathrm{D}} c_n[\Theta] \wedge f. \tag{6}$$

If E *is a holomorphic bundle and* η *and* D *are holomorphic then*

$$f(z)=\int_{\partial \mathrm{D}} \mathrm{K}_{p, q} \wedge f+\int_{\mathrm{D}} \mathrm{K}_{p, q} \wedge \bar{\partial} f+\bar{\partial} \int_{\mathrm{D}} \mathrm{K}_{p, q-1} \wedge f+\int_{\mathrm{D}} c_n[\Theta]_{p, q} \wedge f \tag{7}$$

if f is a (p, q)-form and $\mathrm{K}_{p, q}$ denotes the component of K *of bidegree (p, q) in z.*

Proof. — All that remains to prove is (7). This follows by the previous argument if we note that K is of degree at least $2n$ in the holomorphic differentials and if we choose the test form φ to be of bidegree $(n-p, n-q)$.

Example 1. — If $\mathrm{M}=\mathbb{C}^n$, the canonical choice of bundle E is of course the trivial bundle of rank n, and the usual choice of η is

$$\eta=\zeta-z.$$

The choice of ξ is then precisely equivalent to the choice of section to the Cauchy-Leray bundle in the classical construction (*see* e. g. [O]).

Example 2. — If M is a Stein manifold we can, following Henkin and Leiterer [He-L] find a bundle E whose transition functions depend only on z. More precisely, E is the pullback of the holomorphic tangent bundle on M_z under the projection map

$$\pi: \quad \mathrm{M}_\zeta \times \mathrm{M}_z \rightarrow \mathrm{M}_z.$$

This bundle has a holomorphic section η such that

$$\{\eta=0\}=\Delta \cup \Omega,$$

where Ω is a closed set disjoint from Δ. The extra zeros of η introduce a new difficulty, which however can be handled in the same way as in [He-L]. We may thus assume that the connection and curvature forms depend only on z and are of degree 1 in dz. Therefore they give no contribution to $\mathrm{K}_{0, q}$, so our form K is the kernel of Henkin and Leiterer in this case. Notice also that the first term in our expansion of K is precisely the kernel of Demailly and Laurent-Thiebaut [Dem-LTh], for arbitrary (p, q).

Finally, let us notice that if we reverse the roles of z and ζ then we obtain a curvature form that depends only on ζ, so that

$$c_n[\Theta]_{p, q}=0 \qquad \text{if} \quad (p, q) \neq(0,0).$$

Therefore, the last term in (7) represents no obstruction to solving the $\bar{\partial}$-equation in this case.

Example 3. — If $M = \mathbb{P}^n$ is the n-dimensional projective space, then we can choose E as follows. Let first E′ be the bundle over $\mathbb{P}^n_\zeta \times \mathbb{P}^n_z$ whose fiber over the point (ζ, z) is

$$E'_{(\zeta, z)} = \mathbb{C}^{n+1}/\mathbb{C}\,\zeta.$$

Then we let

$$E = \mathcal{O}_z(1) \otimes E'$$

where $\mathcal{O}_z(1)$ is the line bundle whose sections are 1-homogeneous in z and 0-homogenuous in ζ. If now

$$p_\zeta = \mathbb{C}^{n+1} \to \mathbb{C}^{n+1}/\mathbb{C}\,\zeta$$

is the natural projection we can define a section to E by

$$\eta(\zeta, z) = p_\zeta(z).$$

Then η is a holomorphic section to E that defines the diagonal.

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