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ON THE CHARACTERISTIC POLYNOMIALS OF ORBITAL VARIETIES

BY ANTHONY JOSEPH

1. Introduction

1.1. This paper is a sequel to [19] and we shall retain the same notation though with some modifications following ([7], [8]). Varieties and vector spaces will be nearly always defined over the complex field $\mathbb{C}$, and often considered as being over the real field $\mathbb{R}$.

1.2. Let G be a complex connected simply connected Lie group with $\mathfrak{g}$ its Lie algebra and having the triangular decomposition $\mathfrak{g}=\mathfrak{n}\oplus\mathfrak{h}\oplus\mathfrak{n}^-$. Set $\mathfrak{b}=\mathfrak{h}\oplus\mathfrak{n}$ and let B, N, H be the subgroups of G corresponding to $\mathfrak{b}$, $\mathfrak{n}$, $\mathfrak{h}$. Identify $\mathfrak{g}$ with $\mathfrak{g}^*$ through the Killing form. Choose $u\in\mathfrak{g}^*$ which is ad-nilpotent and let O_u (or simply, O) denote the G orbit in $\mathfrak{g}^*$ containing u. We call O a nilpotent orbit. We call a component C of $O\cap\mathfrak{n}$ an orbital variety.

1.3. Let L be a simple highest weight module. It is easy to show that the associated variety $\mathscr{V}(L)$ [resp. the vanishing cycle $\mathscr{S}(L)$] of L is a union (resp. sum) of orbital varieties. Calculating this sum turns out to be a difficult task especially when one discovers that $\mathscr{V}(L)$ is not necessarily irreducible ([19], 10.1 and Tanisaki [30] or [20], 8.6-8.8).

1.4. A basic construction in [19] assigned to each orbital variety C a polynomial p_C in $S(\mathfrak{h})$ called the characteristic polynomial of C. Let W denote the Weyl group for the pair $\mathfrak{g}$, $\mathfrak{h}$. It was shown ([19], 3.1) that the p_C, as C runs over the components of $O\cap\mathfrak{n}$, span a W submodule of $S(\mathfrak{h})$. It was conjectured ([19], 3.3) that this is just the representation of W assigned to O by the Springer correspondence. This would have had the important consequence that the p_C would be linearly independent, as C runs over all orbital varieties. Shortly afterwards this conjecture was proved by Hotta [14] but only by a tedious comparison of matrix coefficients and detailed knowledge of the Springer correspondence.

1.5. A factorization theorem which obtains from a basic result on Goldie rank polynomials ([19], Sect. 5) relates the characteristic polynomial $p_{\mathscr{S}(L)}$ assigned to $\mathscr{S}(L)$ to the Goldie rank polynomial associated to Ann L. The latter can be computed (up to a scalar) from the Jantzen matrix ([18], 5.1) which is of course now known to be given by the Kazhdan-Lusztig polynomials.

1.6. In [19], 9.8, we conjectured that p_C can be computed from an analogue of the Jantzen matrix introduced by Kazhdan and Lusztig ([22], Sect. 7) in their study of the Springer correspondence. This would have implied Hotta's result and indeed is practically equivalent to it. In any case combining Hotta's result with a knowledge of the p_C gives via the remarks in 1.5 an effective way of computing the associated varieties $\mathscr{V}(L)$.

1.7. Following these ideas Kashiwara and Tanisaki [21], and independently Ginsburg [11] showed that $\mathscr{V}(L)$ can be computed by directly combining these two not quite identical versions of the Jantzen matrix. Later Borho, Brylinski and MacPherson ([7], [8]) reproved Hotta's result by using and developing a further interpretation of characteristic polynomials which they called character polynomials of cone bundles (associated to orbital varieties). Finally Rossmann [25] gave a particularly nice interpretation of the (inverse) of the Kazhdan-Lusztig geometric analogue of the Jantzen matrix, obtaining a new and simpler proof of the Springer correspondence. He also established the conjecture for p_C discussed in 1.6 though here was constrained to use Hotta's result.

1.8. The above mentioned results might be thought to imply a thorough understanding of characteristic polynomials. Yet apart from Rossmann, the authors use a baggage of heavy machinery from sheaf theory and consequently depart far from the elementary approach which was a key point of [19]. Consequently one could well question, and as we shall see with good reason, if a full understanding had been reached.

1.9. Identify the flag variety $X := G/B$ with the set $\mathscr{B}$ of all Borel subgroups of G. Let $\mathscr{B}_u$ denote the variety of all Borel subgroups whose Lie algebra contains u, equivalently the fixed point variety for the action of one parameter group defined by u. Let A_u denote the component group of u, namely the quotient of $\mathrm{Stab}_G\, u$ by its connected component. Then A_u permutes the irreducible components of $\mathscr{B}_u$ and after Spaltenstein [28] the A_u orbits of components in $\mathscr{B}_u$ are in bijection with the components of $O_u \cap \mathfrak{n}$. When [19] was being written we had also considered a dimension function q_Z which could be assigned to a component of $\mathscr{B}_u$ and indeed to any subset Z of X. It was easily seen that q_Z was independent of the choice of an irreducible component of $\mathscr{B}_u$ in an A_u orbit and moreover it appeared to often coincide with a p_C. This was discussed with Bernstein and with Borho during their visit to the Weizmann Institute (June 1982). Bernstein immediately observed that q_Z is asymptotically a polynomial (as a consequence of coherent sheaf theory on the projective variety G/B). He further showed that the top degree part $\mathrm{gr}\, q_Z$ of q_Z is the image of the fundamental class of Z in the cohomology ring of $\mathscr{B}$ identified via Borel's construction [5] with the W harmonic polynomials on $\mathfrak{h}^*$ and hence coincides (Corollary 6.7) with the degree polynomial c_Z of Z. Via the very natural Hotta-Springer specialization theorem [15] it is then immediate that the $\mathrm{gr}\, q_Z$: Z a component of $\mathscr{B}_u$ span the Springer representation assigned to O_u (and by a counting argument using [28] that the distinct $\mathrm{gr}\, q_Z$ form a basis).

1.10. An obvious and important goal is to interrelate the q_Z and the p_C polynomials. One may interpret the Borho, Brylinski, MacPherson work ([7], [8]) as in particular achieving this goal. However, these authors use a very roundabout analysis using on the one hand the deep consequences of the decomposition theorem for perverse sheaves giving in particular the irreducibility of Springer's representation and on the other hand the rather messy arguments of [19] to relate these polynomials.

1.11. In this paper we prove a second factorization theorem which relates the p_C to degree polynomials c_Z for the components of the $\mathscr{B}_u$. This uses the Berline-Vergne integration method [4] which can be understood with only a knowledge of elementary differential geometry. Their method was used in a particularly imaginative way by Rossmann [25] to relate Harish-Chandra's character formula to the geometry of the flag variety obtaining notably an action of the Weyl group which recovers the Springer correspondence. For our convenience we use Rossman's formulation of the integration formula. The main distinction with Rossmann's work is that we have to make do with just one complex structure on G/B (whereas Rossmann uses $G/B \times G/B$) and this leads to a less natural theory. (A similar cutting down was also necessary in the proof of the first factorization theorem ([19], Sect. 5 and [18], Sect. 4) and this was particularly unnatural.) We also need to interpret (Sect. 5) our characteristic polynomials as weighted Lelong numbers. This immediately gave a result which we had long thought to be true, namely that the characteristic polynomials (and hence the Goldie rank polynomials) are sums with rational coefficients $\geqq 0$ of products of distinct positive roots. Actually (once one knows this to be true!) it turns out that a natural generalization of [19], 2.9, also gives this result using the original definition of characteristic polynomials (Sect. 8). Moreover the latter method is quite an effective computational tood in low rank. Again, it gives an important corollary concerning when certain completely prime ideals are induced (8.7). By a result of Vogan ([32], Prop. 7.12) this also gives information on the possible unitarity of representations of complex groups. Although this information is not new (for example, Enright lectured at the Weizmann Institute in June 1982 on a proof using the Dirac operator) the present analysis is particularly neat.

Acknowledgements

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2. Some basic constructions and notations

2.1. Let R denote the set of non-zero roots for the pair $(\mathfrak{g}, \mathfrak{h})$ and R^+ (resp. S) the set of positive (resp. simple) roots corresponding to the triangular decomposition defined in 1.2. Set $n=|\mathrm{R}^+|=\dim_{\mathbb{C}}\mathfrak{n}=\dim_{\mathbb{C}}\mathrm{G/B}$. Let P(R) [resp. $\mathrm{P(R^+)}$, $\mathrm{P(R)^{++}}$] denote the set of integral (resp. dominant, dominant and regular) weights. Let $\{e_\alpha : \alpha\in\mathrm{R}, h_\alpha : \alpha\in\mathrm{S}\}$ be a Chevalley basis for the complex Lie algebra $\mathfrak{g}$ and let $z\mapsto\bar{z}$ denote complex conjugation with respect to that basis. Define a Cartan involution θ for $\mathfrak{g}$ by composing complex conjugation with the product of the Chevalley and principal antiautomorphisms. Set $\mathfrak{k}=\{x\in\mathfrak{g}\,|\,\theta(x)=x\}$, $\mathfrak{p}=\{x\in\mathfrak{g}\,|\,\theta(x)=-x\}$, $\mathfrak{t}=\mathfrak{k}\cap\mathfrak{h}$, $\mathfrak{a}=\mathfrak{p}\cap\mathfrak{h}$ which of course are real subspaces of $\mathfrak{g}$. Let K, A, T be the real Lie subgroups of G corresponding to $\mathfrak{k}$, $\mathfrak{a}$, $\mathfrak{t}$. Set $\mathfrak{t}^\perp=\{x\in\mathfrak{k}^*\,|\,x(h)=0,\ \forall\, h\in\mathfrak{t}\}$ which we view as a subspace of $\mathfrak{k}$ via the Killing form on $\mathfrak{k}$ (which is a real compact semisimple Lie algebra). Similarly $\mathfrak{b}^\perp\subset\mathfrak{g}^*$ identifies with $\mathfrak{n}$.

2.2. Note that $\{ih_\alpha:\alpha\in\mathrm{S}\}$ and $\{i(e_\alpha+e_{-\alpha}), (e_\alpha-e_{-\alpha}):\alpha\in\mathrm{R}\}$ form $\mathbb{R}$ bases for $\mathfrak{t}$ and $\mathfrak{t}^\perp$ respectively. Define a linear map $(1+\theta):\mathfrak{n}\to\mathfrak{t}^\perp$ by $(1+\theta)(z)=z+\theta(z)$. Writing

$$z=\sum_{\alpha\in\mathrm{R}^+} z_\alpha e_\alpha:\qquad \alpha\in\mathrm{R}^+$$

we obtain

$$(1+\theta)(z)=\sum_{\alpha\in\mathrm{R}^+} x_\alpha(e_\alpha-e_{-\alpha})+iy_\alpha(e_\alpha+e_{-\alpha})$$

where $x_\alpha=\mathrm{re}\,z_\alpha$, $y_\alpha=\mathrm{im}\,z_\alpha$. Consequently $1+\theta$ is bijective. It translates a complex subvariety of $\mathfrak{n}$ into a real even dimensional subvariety of $\mathfrak{t}^\perp$ with no homological boundary. (Indeed the set of singular points is a closed subvariety of complex codimension $\geqq 1$ and any smooth point is interior.)

2.3. Recall that the cotangent bundle $\mathrm{T^*(X)}$ of the flag variety $\mathrm{X=G/B}$ identifies with $\mathrm{G}\times_{\mathrm{B}}\mathfrak{n}$. Since $1+\theta$ is bijective and commutes with the action of T we obtain a map $\Theta:\mathrm{K}\times_{\mathrm{T}}\mathfrak{t}^\perp\to\mathrm{G}\times_{\mathrm{B}}\mathfrak{n}$ by setting $\Theta(k, x)=(k, (1+\theta)^{-1}(x))$. Via the Iwasawa decomposition G=KAN it follows that Θ is bijective and gives a further realization of $\mathrm{T^*(X)}$ (due to Borel). Writing $g=kan$, $x=(1+\theta)\,anz : z\in\mathfrak{n}$, we have $\Theta^{-1}(g, z)=(k, x)$ and moreover if $gz\in\mathfrak{n}$, then $(1+\theta)(gz)=kx$. The map $\pi:(g, z)\mapsto gz$ of $\mathrm{G}\times_{\mathrm{B}}\mathfrak{n}$ onto $\mathrm{G}\mathfrak{n}$ [resp. $\pi':(k, x)\mapsto kx$ of $\mathrm{K}\times_{\mathrm{T}}\mathfrak{t}^\perp$ onto $\mathrm{K}\mathfrak{t}^\perp$] is called the moment map. The above calculation shows for every subset $\mathrm{Y}\subset\mathrm{K}\times_{\mathrm{T}}\mathfrak{t}^\perp$ one has $\pi'(\mathrm{Y})\subset\mathfrak{t}^\perp$ if $\pi(\Theta(\mathrm{Y}))\subset\mathfrak{n}$. Of course we may also identify $\mathrm{G}\times_{\mathrm{B}}\mathfrak{n}$ (resp. $\mathrm{K}\times_{\mathrm{T}}\mathfrak{t}^\perp$) with the subvariety $\{g\mathrm{B}, g\mathfrak{n}\,|\,g\in\mathrm{G}\}$ of $\mathrm{G/B}\times\mathfrak{g}$ (resp. $\{k\mathrm{T}, k\mathfrak{t}^\perp\,|\,k\in\mathrm{K}\}$ of $\mathrm{K/T}\times\mathfrak{k}^*$) and then π (resp. π') is just projection onto the second factor so is a closed mapping as X is projective. However, whilst π is birational obviously π' is not. The latter will not cause difficulty here because we only apply π' to the images of conormals of B orbits.

2.4. Recall the Bruhat decomposition

$$\mathrm{X}=\coprod_{w\in\mathrm{W}}\mathrm{X}(w):\qquad \mathrm{X}(w)=\mathrm{B}\,w\,\mathrm{B/B}.$$

Let $Y(w)$ denote the conormal corresponding to the B orbit $X(w)$, that is

$$Y(w) := \{(bwB, b(\mathfrak{n} \cap w\mathfrak{n}) \mid b \in B\} \subset G/B \times \mathfrak{g}.$$

Let $O(w)$ denote the unique dense nilpotent orbit in the irreducible algebraic set $G(\mathfrak{n} \cap w\mathfrak{n})$ and let $C(w)$ denote the Zariski closure of $\pi(Y(w)) \cap O(w)$ in $O(w)$. By the remark in 2.3 one has $\pi(\overline{Y(w)}) = \overline{C(w)}$ where the latter denotes the Zariski closure in $\mathfrak{n}$. Also $\overline{C(w)} = \{\overline{b(\mathfrak{n} \cap w\mathfrak{n}) \mid b \in B}\}$ which is irreducible. The embedding $x \mapsto (x, 0)$ of $X(w)$ into $Y(w)$ carries $\overline{X(w)}$ into $\overline{Y(w)}$. In particular $(yB, 0) \in \overline{Y(w)}$ if and only if $y \leqq w$ (Bruhat order). We remark that the map $w \mapsto O(w)$ of W into the set of nilpotent G orbits in $\mathfrak{g}^*$ was introduced by Steinberg [29]. It is known to be surjective ([29]. Sect. 3); but except in $\mathfrak{gl}(n)$ and some other special cases its fibres are unknown. Again every orbital variety is some $C(w)$ ([19], Sect. 9); but it is not known which $C(w)$ are distinct.

2.5. Given M a complex (or real) smooth irreducible algebraic variety, viewed as a C^∞ manifold, let $\mathscr{F}(M)$ denote the space of complex valued C^∞ functions on M and $\mathscr{D}^1(M)$ the space of C^∞ vector fields on M. Set $\mathscr{D}_0(M) = \mathscr{F}(M)$, $\mathscr{D}_*(M) = \bigoplus_{m=0}^{\infty} \mathscr{D}_m(M)$, where for each $m = 1, 2, \ldots, \mathscr{D}_m(M)$ denotes the space of exterior differential m forms on M. Let $d : \mathscr{D}_m(M) \to \mathscr{D}_{m+1}((M)$ denote the usual exterior derivative and for each $x \in \mathscr{D}^1(M)$ define the contraction map $c(x) : \mathscr{D}_{m+1}(M) \to \mathscr{D}_m(M)$ by $(c(x)\omega)\,(x_1 \wedge x_2 \ldots \wedge x_m) = \omega(x \wedge x_1 \wedge \ldots \wedge x_m)$. Then the x-equivariant derivative $d_x : \mathscr{D}_*(M) \to \mathscr{D}_*(M)$ is defined as $d_x = d + c(x)$ and satisfies $d_x^2 = \mathscr{L}_x$ where $\mathscr{L}_x$ is the Lie derivative. Consequently we have a cohomology theory on x invariant forms, an important observation due to Berline and Vergne [4]. An element $\sigma \in \mathscr{D}_2(M)$ is called symplectic if it is closed and non-degenerate (forcing M to be even dimensional and in this respect all our varieties will be considered as being real). Given that $\sigma \in \mathscr{D}_2(M)$ is closed we say that $x \in \mathscr{D}^1(M)$ is Hamiltonian if there exists a function (called a Hamiltonian function) f_x such that $c(x)\sigma + df_x = 0$. Observe this implies that $d_x(\sigma + f_x) = 0$ and hence that $d_x(\sigma + f_x)^m = 0$, $\forall m = 1, 2, \ldots$, where the product is in the even part of $\mathscr{D}_*(M)$. Since this is a commutative subalgebra of $\mathscr{D}_*(M)$ we have omitted $\wedge$. Given $m \in M$ let $T_m(M)$ denote the tangent spane of M at m.

Given $\varphi : M \to N$ a C^∞ map of C^∞ manifolds, let $\varphi_* : T_m(M) \to T_{\varphi(m)}(N)$ denote the Jacobian of φ defined by $(\varphi_* x)(f) = x(f \circ \varphi)$ and φ^* the pull-back $\mathscr{D}_*(N) \to \mathscr{D}_*(M)$ defined through $\langle \varphi^*(\omega)_m, . \rangle = \langle \omega_{\varphi(m)}, \varphi_*(.) \rangle$. One checks that d commutes with the pull-back φ^* and so if $\sigma \in \mathscr{D}_*(N)$ is closed, then so is $\varphi^*(\sigma) \in \mathscr{D}_*(M)$. Again take $y \in \mathscr{D}^1(N)$ and suppose $x \in \mathscr{D}^1(M)$ satisfies $y_{\varphi(m)} = \varphi_*(x_m)$, $\forall m \in M$ (for example if $y \in \operatorname{Lie} H$, for an equivariant map φ of H manifolds) then $\varphi^*(c(y)\omega) = c(x)\varphi^*(\omega)$, for all $\omega \in \mathscr{D}_*(N)$. Consequently, if y is Hamiltonian relative to $\sigma \in \mathscr{D}_2(N)$ with Hamiltonian function f, then x is Hamiltonian relative to $\varphi^*(\sigma)$ with Hamiltonian function $f \circ \varphi$.

Finally, if M is a homogeneous space, say $M = K/T$ then $\mathscr{D}^1(M)$ is generated over $\mathscr{F}(M)$ by an image of $\mathfrak{k} = \operatorname{Lie} K$ (in this case $\mathfrak{k}$ itself).

2.6. Fix $i\lambda \in \mathfrak{t}^*$ regular [*i. e.* $(\lambda, \alpha) \neq 0$, $\forall \alpha \in R$] and extend $i\lambda$ to an element of $\mathfrak{k}^*$ via linearity and the condition $\lambda(\mathfrak{t}^\perp) = 0$. Following Borel [5] we define a K invariant closed

2-form σ_λ on K/T through

$$\sigma_\lambda(x, y)(k\,T) = \langle [x, y], k\lambda \rangle$$

$\forall x, y \in \mathfrak{k}, k \in K$. Moreover $x \in \mathfrak{k}$ is Hamiltonian with respect to σ_λ having Hamiltonian function $f^{(1)}_{\lambda, x}(k\,T) := \langle x, k\lambda \rangle$. The form σ_λ is non-degenerate for λ regular.

2.7. Consider the Cartesian co-ordinates $x_\alpha, y_\alpha : \alpha \in R^+$ on $\mathfrak{t}^\perp$ defined in 2.2. Consider $h \in \mathfrak{t}$ as a vector field on $\mathfrak{t}^\perp$ (or on $\mathfrak{k}$). Noting the relation $[h, e_\alpha] = \alpha(h) e_\alpha$ and recalling that $\alpha(h)$ is pure imaginary for all $\alpha \in R^+$ we obtain

$$\begin{aligned} h &= \sum_{\alpha \in R^+} \alpha(h)\left(z_\alpha \frac{\partial}{\partial z_\alpha} - \bar{z}_\alpha \frac{\partial}{\partial \bar{z}_\alpha} \right), \\ &= \sum_{\alpha \in R^+} i\,\alpha(h)\left(y_\alpha \frac{\partial}{\partial x_\alpha} - x_\alpha \frac{\partial}{\partial y_\alpha} \right). \end{aligned}$$

Set $\mathfrak{t}_r = \{ h \in \mathfrak{t} \mid i\,\alpha(h) \neq 0, \forall \alpha \in R \}$. From now on fix $h \in \mathfrak{t}_r$. Observe that

$$\tau_h := \sum_{\alpha \in R^+} (1/i\,\alpha(h))\, dx_\alpha \wedge dy_\alpha$$

is a T invariant symplectic form on $\mathfrak{t}^\perp$ which pulls back via the decomposition $\mathfrak{k} = \mathfrak{t} \oplus \mathfrak{t}^\perp$ to a T-invariant closed two-form on $\mathfrak{k}$ which we shall also denote by τ_h. Moreover, with respect to τ_h one checks that h is a Hamiltonian vector field on $\mathfrak{t}^\perp$ (or on $\mathfrak{k}$) with Hamiltonian function

$$f^{(2)} = -\frac{1}{2} \sum_{\alpha \in R^+} (x_\alpha^2 + y_\alpha^2).$$

2.8. Recall that we have an embedding $(k, x) \mapsto (k\,T, kx)$ of $K \times_T \mathfrak{t}^\perp$ into $K/T \times \mathfrak{k}$. Under this map the closed two-form $\Sigma_{\lambda, h} := \sigma_\lambda + \tau_h$ defined on $K/T \times \mathfrak{k}$ by 2.6, 2.7 pulls back to a closed two-form on the cotangent bundle $T^*(X) = K \times_T \mathfrak{t}^\perp$ and which we shall also denote by $\Sigma_{\lambda, h}$. Moreover with respect to the diagonal action of K on $K/T \times \mathfrak{k}$ the above map is K equivariant and so from 2.5 it easily follows that each $h \in \mathfrak{t}$ viewed as a vector field on $T^*(X)$ is Hamiltonian with respect to $\Sigma_{\lambda, h}$ and has Hamiltonian function

$$f_{\lambda, h}(k, a) = \langle h, k\lambda \rangle - \frac{1}{2} \sum_{\alpha \in R^+} (x_\alpha^2 + y_\alpha^2)$$

where $x_a, y_\alpha : \alpha \in R^+$ denote the co-ordinates of the image of the point $ka \in \mathfrak{k}$ under the projection onto $\mathfrak{t}^\perp$ defined by the decomposition $\mathfrak{k} = \mathfrak{t} \oplus \mathfrak{t}^\perp$.

2.9. Compared to Rossmann ([25], I, Sect. 2) we seem to be forced into less structure here. Thus $\Sigma_{\lambda, h}$ is only T invariant, we only claim the vector fields coming $\mathfrak{t}$ are Hamiltonian and that $\Sigma_{\lambda, h}$ is closed (but not necessarily symplectic).

3. Factorization via integration

3.1. Let $Y(w) \subset T^*(X)$ or simply, Y be the conormal of a B orbit (2.4) viewed as a real subvariety of $K \times_T t^\perp$. Then Y (or $\bar{Y}$) is $2n$ dimensional and has no boundary in homology. We wish to consider integrals of the form

$$I_Y^\lambda(h) := \frac{1}{n!} \int_Y e^{f_{\lambda, h}(k, a)} \Sigma_{\lambda, h}^n : \qquad \forall \lambda \in t^*, h \in t$$

where $f_{\lambda, h}$ and $\Sigma_{\lambda, h}$ are defined by 2.8. Since K/T is compact the only possible non-convergence comes from integration over $\mathfrak{k}^*$. However, here the convergence is ensured by the exponential decay implied by our choice of $f^{(2)}$. We conclude that this (and similar integrals below) converge.

To evaluate this and similar integrals we use an integration lemma of Berline and Vergne [4] in the form proposed by Rossmann ([25], 3.1). For completeness we state the result needed below.

Let M be a real C^∞ manifold [eventually either $T^*(X)$ or $t^\perp$ viewed as real manifolds] and Γ an irreducible $2m$ dimensional real algebraic subvariety (eventually always coming from a complex algebraic variety) with no boundary in homology; but possibly with singularities. Fix $h \in \mathscr{D}^1(M)$, $\omega \in \mathscr{D}_*(M)$, $\varphi \in \mathscr{D}_1(M)$. Assume

(1) h has finitely many zeros on M.

(2) Γ is tangential to h.

(3) $d_h \omega = 0$.

(4) φ is h invariant and $\varphi(h)$ vanishes only on the zeros of h.

Take any zero p of h. Assume there exists a co-ordinate system (x_j) on M around p such that

(5) $$\varphi(h) = \sum_j x_j^2 + o\left(\sum_j x_j^2\right)$$

and

(6) The ball $B_\varepsilon(p)$ of radius $\varepsilon > 0$ around p is h invariant.

Condition (3) generally forces ω to be inhomogeneous. Indeed it relates in particular the $2m$ component ω_{2m} to its zero component ω_0. Here (and elsewhere) we shall use

$$\int_\Gamma \omega \quad \text{to mean} \int_\Gamma \omega_{2m}.$$

Finally to ensure convergence we assume that M admits an increasing sequence of compact submanifolds $M_s : s \in \mathbb{N}$ stable under h whose union is M. [In $T^*(X)$ take the inverse images under π of closed balls in $\mathfrak{k}$ of radius s centered at the origin.] Writing $\theta = \varphi/\varphi(h)$ we further assume that

(7) $$\lim_{s \to \infty} \int_{\partial(\Gamma \cap M_s)} \theta (d\theta)^l \omega = 0, \qquad \forall l = 0, 1, 2, \ldots$$

The integration lemma asserts the truth of the following formula

$$\int_{\Gamma} \omega = (-1)^m \sum_p \omega_0(p) \lim_{\varepsilon \to 0} \frac{1}{\varepsilon^{2m}} \int_{\Gamma \cap B_\varepsilon(p)} (d\varphi)^m.$$

We sketch briefly the proof. Let M′ denote the complement in M of the zeros of h. On M′ the one-form $\theta = \varphi/\varphi(h)$ is defined and satisfies $d_h \theta = 1 + d\theta$ by definition of d_h and $d_h^2 \theta = 0$ by (4) and 2.5. Using (3) one checks the key identity $\omega = d_h(\theta(d_h\theta)^{-1}\omega)$ defined on M′. Set $\Gamma_s = \Gamma \cap M_s$ and choose s sufficiently large to ensure that $B_\varepsilon(p) \subset M_s$, $\forall p$. Then by (1)

$$\begin{aligned}\int_{\Gamma_s} \omega &= \lim_{\varepsilon \to 0} \sum_p \int_{\Gamma_s \setminus \Gamma_s \cap B_\varepsilon(p)} d_h(\theta(d_h\theta)^{-1}\omega) \\ &= -\lim_{\varepsilon \to 0} \sum_p \int_{\partial(\Gamma \cap B_\varepsilon(p))} \theta(d_h\theta)^{-1}\omega + \int_{\partial\Gamma_s} \theta(d_h\theta)^{-1}\omega\end{aligned}$$

by the equivariant Stokes' lemma ([4], 1.4) using that Γ has no boundary in homology and (2), (6) to ensure that $B_\varepsilon(p) \cap \Gamma$ and Γ_s are tangential to h needed for the hypothesis of that lemma. Expanding $(d_h\theta)^{-1}$, using (5) to replace $\varphi(h)$ by ε^2 and applying Stokes' lemma we obtain

$$\int_{\partial(\Gamma \cap B_\varepsilon(p))} \theta(d_h\theta)^{-1}\omega = \sum_{j=0}^{m} \frac{(-1)^{j-1}}{\varepsilon^{2j}} \left(\int_{\Gamma \cap B_\varepsilon(p)} (d\varphi)^j \omega_{2(m-j)} \right),$$

up to terms that can be ignored in the limit $\varepsilon \to 0$. As noted in Rossmann ([25], Sect. 3) only the term with $j = m$ is non-vanishing in the limit [essentially because the volume of $\Gamma \cap B_\varepsilon(p)$ goes like ε^{2m}]. *By* (7) the integral over $\partial\Gamma_s$ vanishes in the limit $s \to \infty$. This proves the lemma.

3.2. Let C be a (complex) quasi-affine irreducible subvariety of $\mathfrak{n}$ viewed as a real irreducible subvariety of $\mathfrak{t}^\perp$ (via 2.2) of dimension $2m$: $m = \dim_{\mathbb{C}} C$. Assume that C is T invariant.

Fix $h \in \mathfrak{t}_r$ and consider the 1 form

$$\varphi_h := \sum_{\alpha \in R^+} \frac{1}{\alpha(h)} (x_\alpha \, dy_\alpha - y_\alpha \, dx_\alpha)$$

on $\mathfrak{t}^\perp$. From the formula (2.7) for h viewed as a vector field we conclude that φ_h is h invariant and

$$\varphi_h(h) = \sum_{\alpha \in R^+} (x_\alpha^2 + y_\alpha^2).$$

For $\varepsilon > 0$, let B_ε denote the closed ball in $\mathfrak{t}^\perp$ of radius ε centered at the origin (with respect to the above Euclidean co-ordinates).

Viewed as a function on $\mathfrak{t}_r$, it is clear that φ_h and hence $(d\varphi_h)^m$ are rational in h.

LEMMA. — *The integral*

$$J_C(h) := \frac{1}{m!}\int_C \exp\left(-\frac{1}{2}\sum_{\alpha\in R^+}(x_\alpha^2+y_\alpha^2)\right)\tau_h^m$$

converges and coincides with the rational function

$$\lim_{\varepsilon\to 0}\frac{(-1)^m}{\varepsilon^{2m}}\int_{C\cap B_\varepsilon}(d\varphi_h)^m.$$

on $\mathfrak{t}_r$.

The above integral is computed by the method described in 3.1. First we check conditions (1)-(7). Since h has only a zero at the origin condition (1) holds. Since C is T invariant any $h\in\mathfrak{t}$ is tangential to C so condition (2) holds. Let $\omega_h\in\mathscr{D}_*(\mathfrak{t}^\perp)$ be the form

$$\omega_h=\exp(f^{(2)}+\tau_h).$$

Then by 2.7 one has $d_h\omega_h=0$ so condition (3) holds. Condition (4) with φ_h above have already been verified. In the coordinates (x_α, y_α) condition (5) holds trivially and condition (6) is easily checked. Condition (7) is a trivial consequence of the exponential factor $\exp f^{(2)}$ in ω_h, taking M_s to be the closed ball of radius s with respect to the Euclidean metric defined by $x_\alpha, y_\alpha : \alpha\in R^+$.

Since

$$J_C(h)=\int_C\omega_h$$

the conclusion of 3.1 gives for all $h\in\mathfrak{t}_r$

$$J_C(h)=\lim_{\varepsilon\to 0}\frac{(-1)^m}{\varepsilon^{2m}}\int_{C\cap B_\varepsilon}(d\varphi_h)^m$$

as required.

3.3. Recall that the Euclidean volume Vol B^m of the $2m$ ball B_ε^m of radius ε is just $(\varepsilon^{2m}\pi^m)/m!$ Thus had we been able to replace $i/\alpha(h)$ by $-(1/\pi)$, the rational function in the conclusion of 3.2 would take the form

$$\lim_{\varepsilon\to 0}\frac{\operatorname{Vol}(C\cap B_\varepsilon)}{\operatorname{Vol}B_\varepsilon^m}$$

which is just the Lelong number of C at the origin. It obviously vanishes if $0\notin\bar{C}$ and takes the value 1 if $\bar{C}$ is non-singular at 0. More generally it is determined by the multiplicity of an appropriate Hilbert-Samuel polynomial. (*See* 5.4.)

In section 4 we obtain an expression for the above rational function which may be viewed as a weighted Lelong number. For the moment we observe that up to the factor $(-i\pi)^m$ it may be written in the form

$$(\bigstar) \qquad \frac{1}{\prod_{\alpha\in R}\alpha(h)}\left(\sum_{U\subset R^+}\prod_U \alpha(h)\, a_U\right)$$

where $\prod_U$ is the product of the roots in $R^+\setminus U$ and

$$a_U=\lim_{\varepsilon\to 0}\frac{1}{\mathrm{Vol}\, B_\varepsilon^m}\int_{C\cap B_\varepsilon}\prod_{\alpha\in U}(dx_\alpha\wedge dy_\alpha).$$

Despite the similarity with our previous expression, a_U is not itself a Lelong number and in fact does not even take integer values (*see* Sect. 8). Again there does not seem to be any way of computing a_U from Hilbert-Samuel polynomials.

Nevertheless $a_U\geqq 0$, so we conclude that the numerator in ($\bigstar$) is a positive sum of products of distinct positive roots.

3.4. Fix $w\in W$ and set $C^0(w)=\pi(Y(w))\cap O(w)$. Since $\pi(Y(w))=\{b(\mathfrak{n}\cap w\mathfrak{n})\,|\,b\in B\}$ and has the same closure as $C(w)$ it follows from ([27], Chap. 1.5, thm. 6) that $C^0(w)$ contains an open subset of $C(w)$. For each $u\in C^0(w)$ set $Z_u^0(w)=\pi^{-1}(u)\cap Y(w)$ which we may identify with a subvariety of X and hence of $\mathscr{B}$. Under the second identification one has $Z_u^0(w)\subset\mathscr{B}_u=\pi^{-1}(u)$. Let $Z_u(w)$ denote the closure of $Z_u^0(w)$ in $\mathscr{B}_u$. Let $(\mathrm{Stab}_G\, u)_0$ denote the identity component of $\mathrm{Stab}_G\, u$ and set $A_u=(\mathrm{Stab}_G\, u)/(\mathrm{Stab}_G\, u)_0$ which is a finite group acting on the set of irreducible components of $\mathscr{B}_u$. In what follows we shall sometimes omit w and we use a left superscript to denote conjugation of B by an element of G.

LEMMA. — *For all $u\in C^0(w)$,*

(i) $\dim_{\mathbb{C}} Z_u^0(w)=\dim_{\mathbb{C}}\mathscr{B}_u=\dim_{\mathbb{C}} X-(1/2)\dim_{\mathbb{C}} O(w)$.

(ii) *$Z_u(w)$ is a union of irreducible components of $\mathscr{B}_u$ belonging to a single A_u orbit.*

There exists $C^{00}(w)\subset C^0(w)$ Zariski open in $C(w)$ such that

(iii) *$Z_u(w)$ is irreducible, for all $u\in C^{00}(w)$.*

(iv) *The image of the fundamental class of $[Z_u(w)]$ of $Z_u(w)$ in $H_*(\mathscr{B},\mathbb{C})$ is independent of the choice of $u\in C^{00}(w)$.*

(i) One has (dropping $\mathbb{C}$) that

$$\dim\mathscr{B}_u\geqq\dim Z_u\geqq\dim Y-\dim C,\quad\text{by [27], Chap. 1.6, thm. 7}$$

$$=n-\dim C=n-\frac{1}{2}\dim O=\dim_{\mathbb{C}}\mathscr{B}_u,\quad\text{by [28].}$$

(ii) Observe that $Y\subset G\times_B\bar{C}$ and so $Z_u\subset G\times_B C\cap\pi^{-1}(u)$. Yet

$$G\times_B C\cap\pi^{-1}(u)=\{{}^gB\,|\,g^{-1}u\in C\}$$

which from the discussion in the fourth paragraph of [28] is exactly an A_u orbit of an irreducible component of $\mathscr{B}_u$. By (i), this proves (ii).

For (iii) observe that

$$Z_u^0 = \{ B' \in \mathscr{B}_u \mid B' = {}^{bw}B,\ b \in B \}.$$

Following Steinberg we define

$$S_w(O) = \{ (v, {}^g B, {}^{gw}B) \mid v \in O \cap g\mathfrak{n} \cap gw\mathfrak{n},\ g \in G \} \subset O \times \mathscr{B} \times \mathscr{B}.$$

Consider the projection of $S_w(O)$ onto O defined by the first factor. By [19], 9.3 (ii) the inverse image M of u is an A_u orbit (for the diagonal action) of an irreducible component of $\mathscr{B}_u \times \mathscr{B}_u$. Take $g \in G/B$ such that $u \in g(\mathfrak{n} \cap w\mathfrak{n})$. Then the inverse image of ${}^gB$ in M under the first projection equals $\{ {}^{gbw}B \mid u \in gbw\mathfrak{n} \} = {}^g Z^0_{g^{-1}u}$. (Note here that $g^{-1}u \in C^0$.) If ${}^gB$ belongs to only one component of $\mathscr{B}_u$ which holds on an open set Ω of $\mathscr{B}_u$ then the asserted property of M implies that ${}^gZ_{g^{-1}u}$ and hence $Z_{g^{-1}u}$ is irreducible. It follows that Z_v is irreducible on the set obtained from $\{ gB \in \Omega \mid g^{-1}u \in \mathfrak{n} \cap w\mathfrak{n} \}$ by taking the inverse image of $p_1 : G \to G/B$ and applying the quotient map $g \overset{p_2}{\mapsto} g^{-1}u$. This is just $p_2(p_1^{-1}(\Omega)) \cap C^0$ and is open in C^0. Hence (iii).

(iv) Since ${}^g\pi^{-1} = \pi^{-1}(gu)$ we have ${}^g(G \times_B C \cap \pi^{-1}(u)) = G \times_B C \cap \pi^{-1}(gu)$. It follows that g takes the irreducible components of $G \times_B C \cap \pi^{-1}(u)$ (which are also irreducible components of $\mathscr{B}_u$) to the irreducible components of $G \times_B C \cap \pi^{-1}(gu)$. Since G is connected this action does not change the images of their fundamental classes in $H_*(\mathscr{B}, \mathbb{C})$. Moreover since $G \times_B C \cap \pi^{-1}(u)$ is a single A_u orbit of irreducible components, and of course A_u is a subquotient of G, all these images are the same. Combined with (ii) and (iii) this proves (iv).

3.5. To apply 3.4 to the computation of $I_Y^\lambda(h)$ we first note that it is possible to interchange freely our two pictures concerning $T^*(X)$ and the moment map. This is because $\pi(Y) \subset \mathfrak{n}$ and we have a commutative diagram of maps

$$\begin{array}{ccc} Y & \overset{\pi}{\to} & C \\ \Theta \uparrow & & \downarrow (1+\theta) \\ Y' & \overset{\pi'}{\to} & C' \end{array}$$

defined in 2.2, 2.3.

3.6. Let $\mathscr{H}$ denote the set of all W harmonic polynomials on $S(\mathfrak{h})$. We recall that $\mathscr{H}$ is spanned by homogeneous polynomials on $\mathfrak{h}^*$ and that one has a direct sum decomposition $S(\mathfrak{h}) = \mathscr{H} \oplus I_+ S(\mathfrak{h})$ where I_+ denotes the augmentation ideal of $S(\mathfrak{h})^W$.

Let Z be a closed irreducible subvariety of K/T. After Borel ([5] *see* also [25], II, Sect. 7) the integral

$$c_Z(\lambda) := \int_Z \exp \sigma_\lambda$$

depends only on the image of the fundamental class [Z] of Z in $H_*(\mathscr{B}, \mathbb{C})$ and the map

$$[Z] \mapsto \left(\lambda \mapsto \int_Z \exp \sigma_\lambda\right)$$

extends linearly to an isomorphism of $H_*(\mathscr{B}, \mathbb{C})$ onto $\mathscr{H}$.

Now fix $w \in W$. Set $r = \dim_{\mathbb{C}} X - (1/2) \dim_{\mathbb{C}} O(w)$. From the above and 3.4 we obtain

COROLLARY. — *The function*

$$\lambda \mapsto c_w(\lambda) := \int_{Z_u(w)} \exp \sigma_\lambda = \frac{1}{r!} \int_{Z_u(w)} \sigma_\lambda^r$$

is independent of the choice of $u \in C^{00}(w)$ *and is a* W *harmonic polynomial of degree* r.

3.7. We may now prove the main result of this section. Fix $w \in W$ and set $Y = Y(w)$, $C = C(w)$, $Z_u = Z_u(w)$. Define I_Y^λ, J_C as in 3.1, 3.2. Set $r = \dim_{\mathbb{C}} Z_u$.

PROPOSITION. — *The integral* $I_{Y(w)}^\lambda$ *converges and as a function of* λ *it satisfies*

$$I_{Y(w)}^\lambda(h) = c_w(\lambda) J_{C(w)}(h) + O(\lambda^{r+1}).$$

One has

$$\begin{aligned} I_Y^\lambda(h) &:= \frac{1}{n!} \int_Y e^{f_{\lambda, h}(k, a)} \Sigma_{\lambda, h}^n \\ &= \sum_{s=0}^n \frac{1}{(n-s)!\, s!} \int_{\pi(Y)} \left[\int_{Z_u} e^{f_{\lambda, h}(k, a)} \sigma_\lambda^s\right] \tau_h^{n-s} \\ &= \sum_{s=0}^n \frac{1}{(n-s)!\, s!} \int_{\pi(Y)} \left[e^{f^{(2)}(u)} \int_{Z_u} e^{f_{\lambda, h}^{(1)}(kT)} \sigma_\lambda^s\right] \tau_h^{n-s} \end{aligned}$$

We claim that in the above sum only the term $s = r$ contributes to order $\leqq r$ in λ. Indeed for a term to be non-vanishing we must have $n - s \leqq n - r$ because $n - r$ is half the real dimension of $\pi(Y) = C^0$ and terms not satisfying this will make no contribution because of integration with respect to τ_h. To show that $s \leqq r$, it is enough to recall that σ_λ and $f_{\lambda, h}^{(1)}$ are linear in λ.

Finally the only dependence on λ comes from the integral over Z_u. Hence we can take $u \in C^{00}$ since the complement of C^{00} in $\pi(Y) = C^0$ is of measure zero (because C^{00} is Zariski open in the irreducible algebraic set C^0). Since $\exp f_{\lambda, h}^{(1)}(kT) = \exp \langle h, k\lambda \rangle$ and σ_λ^r is homogeneous of degree r in λ, expanding the exponential we conclude by 3.6 that

$$I_Y^\lambda(h) = c_w(\lambda) J_{\pi(Y)}(h) + O(\lambda^{r+1}).$$

Since $\pi(Y)$ and C differ only by sets of measure zero, the required conclusion is obtained.

3.8. The proof of the above factorization result follows closely the analysis of Rossmann ([25], Sect. II, 7), with some technical differences arising because here we are dealing with conormals of B orbits in X rather as in Rossmann with conormals of G orbits in X × X.

4. Comparison of factors

4.1 (Notation 3.7). We now compute $I_Y^\lambda(h)$ in another way. This will allow us to compare the factors c_w, $J_{C(w)}$. A similar comparison technique occurred in the thesis work of D. King [23]. It was used in an essential way in the work of D. Barbasch and D. Vogan [3] in [19], Sect. 5, in [19], 5.1 and in Rossmann ([25], Sects II, 7-9). Our method of computation follows Rossmann using RL (notation 3.2).

4.2 First we construct a K invariant Riemannian metric $\mathscr{G}$ on $K \times_T \mathfrak{t}^\perp$ as the pull-back of a K × K invariant Riemannian metric on $K/T \times \mathfrak{k}$ under the K equivariant map defined in 2.8. The latter will just be the product of metrics defined on K/T and $\mathfrak{k}$. The existence of such a metric is pretty obvious; but we shall describe it anyway.

Given $\lambda_0 \in i\mathfrak{t}^*$ a dominant integral weight, let V_{λ_0} denote the simple $\mathfrak{g}$ module with highest weight vector v_{λ_0} of weight λ_0. The map $g \mapsto [g v \lambda_0]$ of G into $\mathbb{P} V_{\lambda_0}$ factors to a map of G/B into $\mathbb{P} V_{\lambda_0}$ which is an embedding if λ_0 is regular. Take λ_0 regular. The pull-back of the Fubini-Study metric on $\mathbb{P} V_{\lambda_0}$ which is Riemannian and $GL(V_{\lambda_0})$ invariant ([12], pp. 30, 31) gives the required metric on $K/T = G/B$. It is the metric which is the real part (see [12], p. 28) for the Kähler form on K/T for which σ_0 is the imaginary part. It is obviously non-degenerate at the base point T (for λ_0 regular).

On $\mathfrak{k}$ we take the constant metric defined by the negative of the Killing form (which is negative definite on $\mathfrak{k}$ and $\mathfrak{k}$ invariant). The pull-back of the product metric (which is non-degenerate) is K invariant, positive and non-degenerate at any point $p \in (T, \mathfrak{t}^\perp)$.

4.3. Fix $h \in \mathfrak{t}_r$ considered as a vector field on $T^*(X)$. Then $(k, a) \in K \times_T \mathfrak{t}^\perp$ is a zero of h if and only if $(ad\, k^{-1}) h \in \mathfrak{t}$ and $[(\mathrm{ad}\, k^{-1}) h, a] = 0$. Since h is regular, so is $(\mathrm{ad}\, k^{-1}) h$ and thus the latter condition implies that $a = 0$. Again since h is regular one has $K h \cap \mathfrak{t} = W h$ (where the groups act by the adjoint action). We conclude that the zeros of h on $T^*(X)$ form the finite set $\{(y, 0) \mid y \in W\}$. Observe also that $f_{\lambda, h}(k, a)$ takes the value $\exp < h, y \lambda >$ at $(y, 0)$.

4.4. Fix $h \in \mathfrak{t}_r$. The function $x \mapsto \mathscr{G}(h, h)(x)$ on $K \times_T \mathfrak{t}^\perp$ is h invariant, takes positive values and vanishes exactly on the finitely many zeros of h. Thus for $\varepsilon > 0$ the set $P_\varepsilon := \{ x \in K \times_T \mathfrak{t}^\perp \mid \mathscr{G}(h, h)(x) < \varepsilon^2 \}$ is h (even T) invariant and for ε sufficiently small decomposes into a disjoint union of h-invariant open neighbourhoods $B_\varepsilon(y) : y \in W$ of the zeros of h in $K \times_T \mathfrak{t}^\perp$ viewed as a real $4n$ dimensional C^∞ manifold. Take the h invariant partition of unity defined by these neighbourhoods and patch together an h invariant 1 form φ_h on $T^*(X)$ by taking $\varphi_h(\xi) = \mathscr{G}(h, \xi)$ outside P_ε and on the above neighbourhood of $(y, 0)$ an h invariant 1 form to be constructed later (4.6); but which we assume for the moment satifies

$$(\bigstar) \qquad \varphi_h(h)(x)=\|x\|^2+o(\|x\|^2)$$

in terms of local (Euclidean) co-ordinates around $(y, 0)$. We can of course assume that around $(y, 0)$ that

$$\mathcal{G}(h, h)(x)=\|x\|^2+o(\|x\|^2)$$

and so we could have taken $\varphi_h(\xi)=\mathcal{G}(h, \xi)$ everywhere. However, this is not the most convenient choice.

4.5. Proposition. — *For each* $w\in W$ *the integral* $I^\lambda_{Y(w)}$ *converges on* $\mathfrak{h}_r$ *to the function*

$$h\mapsto \sum_{y\in W}(-1)^n \exp\langle h, y\lambda\rangle \lim_{\varepsilon\to 0}\frac{1}{\varepsilon^{2n}}\int_{Y(w)\cap B_\varepsilon(y)}(d\varphi_h)^n.$$

We apply 3.1 to the quadruple $(h, Y(w), \omega_h, \varphi_h)$ where $h\in t_r$, $\omega^\lambda_h=\exp(f_{\lambda,h}+\Sigma_{\lambda,h})$ and φ_h is defined in 4.4. Condition (1) of 3.1 was verified in 4.3. Condition (2) holds because $Y(w)$ is T invariant hence tangential to h. Moreover, $Y(w)$ has no homological boundary. Again $d_h\omega^\lambda_h=0$ by the construction of 2.6-2.8. This verifies condition (3). Condition (4) holds by construction. Condition (5) is just $(\bigstar)$ of 4.4. Condition (6) follows from the h invariance of $Y(w)$ and the $B_\varepsilon(y): y\in W$. Condition (7) follows from the exponential factor in ω_h and the specific choice of φ_h outside P_ε. Since

$$I^\lambda_{Y(w)}(h)=\int_{Y(w)}\omega^\lambda_h$$

the conclusion of 3.1 gives

$$I^\lambda_{Y(w)}(h)=\sum_{y\in W}(-1)^n \exp\langle h, y\lambda\rangle \lim_{\varepsilon\to 0}\frac{1}{\varepsilon^{2n}}\int_{Y(w)\cap B_\varepsilon(y)}(d\varphi_h)^n,$$

as required.

4.6. We now compute the limits occuring in 4.5. This follows Rossmann ([25], I, Sect. 3) with some slight differences coming from our use of conormals of B orbits. Using Rossmann's notation we let $Eu_y(X(w))$ denote the Euler number of the variety $X(w)$ at the point y. This is defined as follows. As it is a local concept it is enough to consider an irreducible variety U of $\mathbb{C}^n$ of complex dimension d and that the point in question to be the origin 0. Let $V\subset(\mathbb{C}^n)^*$ be the corresponding conormal variety. Let $q=(q_1, \ldots, q_n)$, $p=(p_1, \ldots, p_n)$ denote co-ordinates in $\mathbb{C}^n$ and $(\mathbb{C}^n)^*$ respectively, and set $B_\varepsilon=\{(q, p)\mid |q|^2+|p|^2\leqq\varepsilon^2\}$. Then

$$Eu_0(U):=(-1)^{n-d}\lim_{\varepsilon\to 0}\frac{1}{\varepsilon^{2n}}\int_{V\cap B_\varepsilon}\left[\frac{1}{2\pi i}\sum_{j=1}^n(d\bar{q}_j\wedge dq_j-d\bar{p}_j\wedge dp_j)\right]^n.$$

As in the case of the Lelong number (*see* 5.3) a limiting argument shows that $E u_0(U)$ may be expressed as an integral over the tangent cone to V at (0, 0) and is an integer, vanishes if $0 \notin \bar{U}$, equals 1 if $0 \in \bar{U}$ is a non-singular point. All this is discussed in Rossmann ([25], I, Appendix). We expect the $E u_y(X(w))$ to be non-negative. For the moment one knows that the inverse of the A matrix (*see* 4.7) has entries $\geqq 0$. This is because by [25], II, Section 12 and [21] these determine multiplicities of certain characteristic cycles. Then a Kazhdan-Lusztig type inversion property should relate these two sets of integers. In any case positivity is not needed here.

Let $l(w)$ denote the reduced length of $w \in W$. We have the

LEMMA. — *For all* $y, w \in W$ *one has*

$$(\bigstar\bigstar) \qquad \lim_{\varepsilon \to 0} \frac{1}{\varepsilon^{2n}} \int_{Y(w) \cap B_\varepsilon(y)} (d\varphi_h)^n = \frac{(2\pi i)^n (-1)^{l(w)+l(y)}}{\prod_{\alpha \in R^+} (\alpha, h)} E u_y(X(w)).$$

Fix $y \in W$ such that $(y, 0) \in Y(w)$. Choose local (complex) co-ordinates $q_\alpha : \alpha \in -y^{-1} R^-$ around $y B \in G/B$ and extend these to local canonical co-ordinates q_α, $p_\alpha : \alpha \in -y^{-1} R^+$ around $z_y : =(y, 0) \in T^*(G/B)$ exactly as in Rossmann ([25], I, Sect. 3). One has $(\exp h)(q_\alpha, p_\alpha) = (e^{\alpha(h)} q_\alpha, e^{-\alpha(h)} p_\alpha)$ and so $(\exp h)(\bar{q}_\alpha, \bar{p}_\alpha) = (e^{-\alpha(h)} \bar{q}_\alpha, e^{\alpha(h)} \bar{p}_\alpha)$ since $\alpha(h)$ is pure imaginary.

We should eventually like to apply ([25], I, Lemma 3.4). For this we must show that for any differentiable arc $(q(t), p(t))$ on $\bar{Y}$ contained in Y for $t>0$ and passing through $z_y = (q(0), p(0)) = (0, 0)$ at $t=0$ one has $p'(0).q'(0)=0$ (where prime denotes differentiation and dot the obvious scalar product obtained by summing over $-y^{-1} R^+$).

Consider the canonical one-form

$$\psi = p . dq$$

on $T^*(G/B)$. The one-form ψ vanishes on any conormal, hence on Y. This translates to $p(t).q'(t)=0$. Differentiating with respect to t and setting $t=0$ gives $p'(0).q'(0)=0$ as required.

Now in the neighbourhood $B_\varepsilon(y)$ *of* z_y in $T^*(X)$ defined in 4.4 we take

$$\varphi_h = \sum_{\alpha \in -y^{-1} R^+} \frac{1}{\alpha(h)} (\bar{q}_\alpha + p_\alpha)(dq_\alpha + d\bar{p}_\alpha).$$

(Here we recall that $h \in t_r$.) Then on Y

$$\begin{aligned} \varphi_h(h) &= \sum (\bar{q}_\alpha + p_\alpha)(q_\alpha + \bar{p}_\alpha) \\ &= \| q + \bar{p} \|^2 = \| q \|^2 + \| p \|^2 + o(\| q \|^2 + \| p \|^2) \end{aligned}$$

by [25], I, Lemma 3.4, which we can now apply. Hence φ_h satisfies ($\bigstar$) of 4.4. From the above transformation properties of q_α, p_α we also see that φ_h is h invariant. Thus the above formula for φ_h can then be used in computing the left hand side of ($\bigstar\bigstar$). Since

$d\psi=\sum dp_\alpha \wedge dq_\alpha$ vanishes on Y we have

$$(d\varphi_h)^n=\frac{(-1)^{l(y)+n}}{\prod_{\alpha\in R^+}\alpha(h)}\Big(\sum_{\alpha\in -y^{-1}R^+}(d\bar{q}_\alpha\wedge dq_\alpha-d\bar{p}_\alpha\wedge dp_\alpha)\Big)^n.$$

Recalling that $X(w)$ has complex dimension $l(w)$, substitution gives the required conclusion

4.7. Let us define the matrix A with entries

$$A(w, y)=(-1)^{l(w)-l(y)}\, E\,u_y(X(w)).$$

Combining 4.5 and 4.6 we obtain

COROLLARY. — *For each* $w\in W$, *one has*

$$I^{\lambda}_{Y(w)}(h)=\frac{(-2\pi i)^n}{\prod_{\alpha\in R^+}\alpha(h)}\sum_{y\in W} A(w, y)\, e^{\langle h, y\lambda\rangle}$$

on $\mathfrak{t}_r$.

4.8. We need one further observation to apply this to 3.7 to obtain the result announced in 4.1. Namely that the Euler numbers $E\,u_y(X(w))$ coincide with those computed by conormals of G orbits in $X\times X$. This is because of the well-known relation between B orbits in X and G orbits in $X\times X$. (It is also noted in [25], I, Sect. 4.) Now the involution $(x, y)\mapsto(y, x)$ on $X\times X$ carries the G conormal defined by w to the G conormal defined by w^{-1} and the point defined by y to that defined by y^{-1}. Consequently $E\,u_{y^{-1}}(X(w^{-1}))=E\,u_y(X(w))$ and so $A(w, y)=A(w^{-1}, y^{-1})$. [We remark that a similar property holds for Verma module multiplicities for an essentially similar reason ([16], 5.4).] Set $m=\dim_{\mathbb{C}} C(w)$. We have already seen (3.2) that $J_{C(w)}(h)$ can be written in the form

$$(\bigstar)\qquad J_{C(w)}(h)=\frac{(-2\pi)^m/m!}{\prod_{\alpha\in R^+}\alpha(h)}\,p_{C(w)}(h)$$

for some homogenous polynomial $p_{C(w)}$ of degree $n-m$. Comparison of 3.7 and 4.7 shows (as in say [19], 5.1) that

THEOREM. — *For all* $w\in W$ *one has*

$$p_{C(w)}=c_{w^{-1}}$$

up to a non-zero scalar.

4.9. It follows from the work of Rossmann (explicitly [25], II, Cor. 5.2, Sect. 10, eq. (9)) that $A(w, y)$ is exactly the entries of the matrix used in [19], Conj. 9.8. It implies as in Rossmann ([25], II, Sect. 8) that c_w and $p_{C(w)}$ are determined (up to scalars) by the formula given in [19], Conj. 9.8. Of course we already know this for c_w from Rossmann ([25], II, Thm. 8.2).

Finally, we remark that because of the irreducibility of Springer's representation, Schur's lemma implies that the scalar in the conclusion of the theorem depends only on $O(w)$. Here we must also use the fact that $C(w)=C(w')$ implies the equality $p_{C(w)}=p_{C(w')}$ (and not just their proportionality as in the case of the Goldie rank polynomials q_w occuring in [18], Sect. 5.5, Remark 1).

The fact that the above scalar depends only on the orbit $O(w)$ was also noted by Rossmann ([25], II, Sect. 10, eq. (13)) in an equivalent form. At first this may seen an innocuous fact hardly worthy of special note, in fact it is extremely curious and the demanding reader should really question its validity. The point is that the c_w (or the $p_{C(w)}$) are overdetermined by the $A(w, y)$ coefficients (they have to coincide for several different w) and so this independence reflects a property of these coefficients. This was already true for the Goldie rank polynomials ([18], Sect. 5) and reflected a property of the entries $a(w, y)$ of the inverse Jantzen matrix. However, in the latter case the resulting polynomials q_w which are directly determined by this matrix can be proportional and *not equal.* This fact *forces* the A and a matrices to be distinct [outside $\mathfrak{sl}(n)$]. For example in type B_2 there is one left-cell $\mathscr{C}$ of W for which the $q_w : w \in \mathscr{C}$ (must be proportional but) differ by a factor of 2. Exactly here we get a distinction in the A and a matrices and which eventually leads to an extra nilpotent orbit not occurring as an associated variety of the integral fibre of primitive ideals ([17], Sect. 9).

5. Characteristic polynomials and weighted Lelong numbers

5.1. We now relate the rational function $J_C(h) : h \in \mathfrak{t}_r$ obtained from the conclusion of 3.2 to the rational function r_C defined in [19], 2.4. Such a relation may already be anticipated from the discussion in 3.3.

5.2. Our stategy is to recompute $J_C(h)$ using again 3.1; but this time by first introducing new variables. Here we identify $\mathfrak{t}$, $\mathfrak{t}^*$ through the Killing form and make the substitution $h=-i\mu$ with μ a dominant integral regular weight. Thus $\mu(\alpha)$ is a positive integer for all $\alpha \in R^+$. Now introduce new (complex) co-ordinate functions $\xi_\alpha : \alpha \in R^+$ on $\mathfrak{n}$ by taking ξ_α to be an $\mu(\alpha)$ th root of z_α. As we shall see we can apply 3.1 as previously.

5.3. Before going further let us recall the way to compute the Lelong number at a point w in a complex irreducible affine algebraic variety C viewed as real variety of twice its complex dimension.

Let $I(C)$ denote the ideal of definition of $\bar{C} \subset \mathbb{A}^n(\mathbb{C})$. Let $A(C)$ (or simply, A) denote the algebra of regular functions on C which of course identifies with $\mathbb{C}[z_1, z_2, \ldots, z^n]/I(C)$. Given $w \in C$, let $\mathfrak{m}_w$ (or simply, $\mathfrak{m}$) denote the maximal ideal corresponding to w. For all $s \in \mathbb{N}$ set $\mathrm{gr}^{\mathfrak{m}}(A)_s=(\mathfrak{m}^s+I(C))/(\mathfrak{m}^{s+1}+I(C))$, where

$\mathfrak{m}^0 = A$. We define

$$\mathrm{gr}_{\mathfrak{m}}(A) := \bigoplus_{s=0}^{\infty} \mathrm{gr}_{\mathfrak{m}}(A)_s$$

which is a graded ring.

We can describe $\mathrm{gr}_{\mathfrak{m}}(A)$ in a second way which makes clear that it is finitely generated. Namely for each $f \in I(C) \setminus \{0\}$ we let $\mathrm{gr}\, f$ denote the lowest degree term in the expansion of f as a homogeneous polynomial in the variables $z_j - w_j : j = 1, 2, \ldots, n$ and set $\mathrm{gr}\, I(C) = \{\mathrm{gr}\, f \mid f \in I(C)\}$. Then $\mathbb{C}[z_1, z_2, \ldots, z^n]/\mathrm{gr}\, I(C)$ is isomorphic as a graded ring to $\mathrm{gr}_{\mathfrak{m}}(A)$. This construction also makes it evident that Spec $\mathrm{gr}_{\mathfrak{m}}(A)$ is just the tangent cone $T_{\mathfrak{m}}(C)$ at $\mathfrak{m}$ ([27], p. 79).

Since $\mathrm{gr}_{\mathfrak{m}}(A)$ is finitely generated the function

$$s \overset{f}{\mapsto} \dim_{\mathbb{C}} \mathrm{gr}_{\mathfrak{m}}(A)_s$$

is polynomial for all s sufficiently large (the Hilbert-Samuel polynomial, [2], 11.2). Furthermore this function takes the form

$$f(s) = \frac{e s^{m-1}}{(m-1)!} + \text{lower order terms in } s,$$

where $m = \dim_{\mathbb{C}} C$ and e is a positive integer, depending on w and C, called the multiplicity of f. We denote it by $\mathrm{mult}_w(C)$ and call it the multiplicity of C at w.

Now view C as a real $2m$ dimensional variety by setting $x_j = \mathrm{re}\, z_j$, $y_j = \mathrm{im}\, z_j : j = 1, 2, \ldots, n$ and substituting for x_j, y_j, in the expression for $I(C)$. For each $\varepsilon > 0$ let $B_\varepsilon(w)$ denote the closed $(2n)$ ball centered at $w \in C$ of radius ε (with respect to the Euclidean norm on $\mathbb{R}^{2n}$ defined by the above co-ordinates). Let B_ε^m denote the closed $2m$ ball of radius ε (centered at the origin). Set

$$\tau_1 = \sum_{j=1}^{n} dx_j \wedge dy_j.$$

Then $(1/m!)\,\tau_1^m$ is the volume element for $2m$ dimensional analytic subvariety D of $\mathbb{R}^{2n}$ and we may define

$$\mathrm{Vol}\, D = \frac{1}{m!} \int_D \tau_1^m = \int_D \exp \tau_1.$$

(For more details *see* [12], p. 32.)

Consider the function

$$L(C, w, \varepsilon) = \frac{\mathrm{Vol}(C \cap B_\varepsilon(w))}{\mathrm{Vol}\, B_\varepsilon^m}.$$

THEOREM. — *The function* $L(C, w, \varepsilon)$ *takes positive values and is increasing in* ε. *Furthermore*

$$\lim_{\varepsilon \to 0} L(C, w, \varepsilon)=\operatorname{mult}_w(C).$$

It is called the Lelong number of C *at* $w \in \bar{C}$.

It is difficult to find a clear and simple proof of this remarkable result in the literature, let alone to find out to whom it is due. A rough proof occurs in [12], pp. 390-391. For the purposes of this section we only need the case when $w=0$ and I(C) is already graded [equivalently with A(C) and $T_0(C)$ are isomorphic as graded rings]. Then the limiting process is trivial and we may also view C as a projective variety of dimension $m-1$. In this case the result can be read off from [24], 5.22 and 6.25. The limiting process is also considered in Thie ([31], Sect. 2, 3) though in more generality than needed here or in 4.6.

5.4. From now on we shall assume that w is the origin (which we can do without loss of generality *except* with respect to the remarks concerning whether A(C) is a graded ring). Introduce an n tuple $k=(k_1, k_2, \ldots, k_n)$ of positive integers. A monomial $z_1^{l_1} z_2^{l_2} \ldots z_n^{l_n}$ will be said to have k-degree s if $\sum k_i l_i = s$. A polynomial will be said to be k-homogeneous if it is a sum of monomials of the same k-degree. An ideal $I \subset \mathbb{C}[z_1, z_2, \ldots, z_n]$ will be said to be k-homogeneous if it is generated by k-homogeneous polynomials.

We make a brief digression to discuss a generalization of the Lelong number which does *not* work. For each s let $\mathfrak{m}^{(s)}$ denote the ideal generated by the k-homogeneous polynomials of k-degree $\geqq s$. Exactly as in 5.3 we may introduce a k-graded ring by setting $\operatorname{gr}(A)_{(s)}=(\mathfrak{m}^{(s)}+I(C))/\mathfrak{m}^{(s+1)}+I(C))$ and

$$\operatorname{gr}(A)=\bigoplus_{s=0}^{\infty} \operatorname{gr}(A)_{(s)}.$$

Exactly as before gr (A) is finitely generated and we let $\operatorname{mult}_0^k(C)$ denote the multiplicity of the associated Hilbert-Samuel polynomial. Now set

$$\tau_k=\sum_{j=1}^{n} \frac{1}{k_j}\, dx_j \wedge dy_j$$

and define $L(C, 0, \varepsilon, k)$ as before; but replacing τ_1 by τ_k. One can show that $L(C, 0, \varepsilon, k)$ satisfies the first conclusion of theorem 5.3 and one can ask if

$$(\bigstar) \qquad \lim_{\varepsilon \to 0} L(C, 0, \varepsilon, k)=\operatorname{mult}_0^k(C)\,?$$

Actually, this fails, already for the variety defined by the equation $z_1^2=z_2^3$. In this case $\operatorname{mult}_0(C)=2$. Close to the origin C ressembles 2 discs approaching the z_2-plane $z_1=0$. It follows that only the integration over $dx_2 \wedge dy_2$ contributes and so we conclude that left hand side of ($\bigstar$) is just $2/k_2$. This equals the right hand side if $2k_1 \leqq 3k_2$; but otherwise the right hand side equals $3/k_1$. However we shall show that (★) holds if I(C) is k-graded. In this case the limiting process is trivial; but otherwise it does not seem that this more general result is on obvious consequence of theorem 5.3 or can be proved in a similar way to that result.

5.5. Now return to the situation described in 5.2. Recall in particular that C is a complex T stable irreductible quasi-affine subvariety of $\mathfrak{n}$ and that $ih=\mu\in P(R)^{++}$.

PROPOSITION. — *The integral*

$$J_C^\mu=\int_C\left[\exp\frac{-1}{2}\sum_{\alpha\in R^+}(x_\alpha^2+y_\alpha^2)\right]\tau_h^m$$

converges to $(-2\pi i)^m\,\mathrm{mult}_0^k(C)$, *where k is the n-tuple defined by* $k_\alpha=\mu(\alpha):\alpha\in R^+$.

To compute the integral we substitute for the co-ordinates $\xi_\alpha=z_\alpha^{1/\mu(\alpha)}$ discussed in 5.2. This must be done (as usual) in three places. First in C which means that in the ideal of definition $I\subset\mathbb{C}[z]:=\mathbb{C}[z_\alpha:\alpha\in R^+]$ of $\bar{C}$ we must replace z_α by $\xi_\alpha^{\mu(\alpha)}$ to obtain a new ideal $I'\subset\mathbb{C}[w]$. However, doing this leads to overcounting. Indeed for each point $w\in C$ we obtain

$$\prod_{\alpha\in R^+}\mu(\alpha)$$

solutions in the ξ_α variables (since $\mathbb{C}$ is algebraically closed). This introduces the above factor as a denominator in the expression for J_C^μ.

Second and third, we must introduce this substitution in the integrand and in τ_k. However, instead we can simply recompute h as a vector field in these new co-ordinates. A brief calculation gives

$$h=\sum_{\alpha\in R^+}\left(\xi_\alpha\frac{\partial}{\partial\xi_\alpha}-\bar{\xi}_\alpha\frac{\partial}{\partial\bar{\xi}_\alpha}\right)$$

$$=\sum_{\alpha\in R^+}\left(b_\alpha\frac{\partial}{\partial a_\alpha}-a_\alpha\frac{\partial}{\partial b_\alpha}\right)$$

where $\xi_\alpha=a_\alpha+ib_\alpha$. Now we can transport the above change of variables into the relation $d_h\omega_h=0$ (with ω_h as in 3.2) without upsetting its validity. Now consider the 1-form

$$\varphi=i\sum_{\alpha\in R^+}(a_\alpha\,db_\alpha-b_\alpha\,da_\alpha)$$

which is obviously h invariant. Furthermore

$$\varphi(h)=\sum_{\alpha\in R^+}(a_\alpha^2+b_\alpha^2).$$

Let B_ε denote the ball centered at the origin of radius ε with respect to the co-ordinates a_α, b_α.

As in 3.2 we can now check that RL applies. By its conclusion we obtain

$$\Big(\prod_{\alpha\in R^+}\mu(\alpha)\Big)\, J_C^\mu=\lim_{\varepsilon\to 0}\frac{(-1)^m}{\varepsilon^{2m}}\int_{C\cap B_\varepsilon}(d\varphi)^m.$$

Since

$$d\varphi=2i\sum_{\alpha\in R^+}(da_\alpha\wedge db_\alpha)=2i\tau_1$$

we conclude from 5.3 that

$$J_C^\mu=\frac{(-2\pi i)^m}{\prod_{\alpha\in R^+}\mu(\alpha)}\,\mathrm{mult}_0(C)$$

where $\mathrm{mult}_0(C)$ is computed with respect to the ξ_α variables. It remains to show that

$$(\bigstar)\qquad \frac{\mathrm{mult}_0(C)}{\prod_{\alpha\in R^+}\mu(\alpha)}=\mathrm{mult}_0^k(C)$$

where the right hand side is computed with respect to the z_α variables. Let p denote the denominator of the left hand side.

It is clear that $\mathbb{C}[w]$ is a free rank p module over $\mathbb{C}[z]$ with generators

$$g_l:=\prod_{\alpha\in R^+}\xi_\alpha^{l_\alpha}:\qquad l_\alpha\in\{1,\ldots,k_\alpha\}.$$

Set $A=\mathbb{C}[z]/I$, $A'=\mathbb{C}[\xi]/I'$. Then the image of the g_l generate A' over A. If this were not a free generation, we should have

$$(\bigstar\bigstar)\qquad \sum_l g_l P_l\in I'$$

for some P_l not all in I. Given $z\in C$ let $\xi_{l'}:l'_\alpha\in\{1,2,\ldots,k_\alpha\}$ denote the p possible coordinates in the ξ variables corresponding to z. Now $\det g_l(\xi_{l'})\neq 0$ by the linear independence of the characters of the appropriate product of cyclic groups. Then from $(\bigstar\bigstar)$ we conclude that $P_l(z)=0$, $\forall\, l$ and since z was arbitrary $P_l\in I$, $\forall\, l$ which is a contradiction. Hence, A' is a free rank p module over A.

Since $I(C)$ is k-homogeneous we may identify $\mathrm{gr}\,A_{(s)}$ with the k-homogeneous polynomials of k-degree s in the z_α variables and $\mathrm{gr}\,A'_s$ with the homogeneous polynomials of degree s in the ξ_α variables. Taking account of our first observation we conclude that

$$p\sum_{j=1}^{s+p}\dim\mathrm{gr}\,A_{(j)}\geqq\sum_{j=1}^{s}\dim\mathrm{gr}\,A'_j\geqq p\sum_{j=1}^{s}\dim\mathrm{gr}\,A_{(j)}$$

which implies $(\bigstar)$.

5.6. Comparison of 3.2 and 5.5 shows that the above analysis establishes (★) of 5.4 when I(C) is k-graded.

5.7. We now obtain the main result of this section.

THEOREM. — *For each orbital variety* $C \cap \mathfrak{n}$ *the polynomial* p_C *defined by* 4.8 (★) *and* 3.2 *coincides with the characteristic polynomial of* C *defined in* [19], 2.4.

From the definition of r_C in [19], 2.4 it is immediate that

$$r_C(\mu) = m!\ \mathrm{mult}_0^k(C)$$

where k is the n-tuple defined in 5.5. Recalling that the characteristic polynomial of C is just r_C times the product of the positive roots, the result obtains from 5.5 and substitution in 5.4 (★) using (as usual) that $P(R)^{++}$ is Zariski dense in $\mathfrak{h}^*$.

5.8. Combining 4.8, 5.7 and the remarks in 4.9 we obtain a proof of [19], Conj. 9.8. This implies Hotta's theorem [14] and avoids the use of any sheaf theory which one should always try to do.

6. Dimension polynomials

Section 6.1-6.6 are derived from very rough notes taken in discussions with Bernstein during June 1982. The first part 6.1-6.5 would probably be considered quite standard, whilst the second 6.6 is a little less so. The results are of computational interest and go beyond the considerations of fixed point varieties.

6.1. (Notation 2.1, 4.2). Let Z be a subset of the flag variety X. Given $\lambda \in P(R)^+$, we have a map $g \mapsto g v_\lambda$ of G into V_λ. Set $V_Z = \{ z v_\lambda : z \in Z \}$ and $q_Z(\lambda) := \dim_{\mathbb{C}} \mathbb{C} V_Z$. We call q_Z the dimension function of Z. Here $\mathbb{C} V_Z$ denotes the $\mathbb{C}$-linear span of V_Z which is obviously a bizarre thing to consider. Yet $V_Z^\perp := \{ f \in V_\lambda^* \mid f(a) = 0,\ \forall a \in V_Z \}$ is a subspace of V_λ^* so we way anticipate one should consider $V_\lambda^*/V_Z^\perp$.

6.2. Let $\pi_0 : G \to G/B$ denote the natural projection. For each subvariety V of G, let A(V) denote the ring of regular functions on V. We view each $\lambda \in P(R)$ as a linear function on $\mathfrak{b}$ by setting $\lambda(\mathfrak{n}) = 0$ and we let χ_λ denote the character on B obtained from λ via the exponential map. We then have a G-equivariant sheaf $\mathcal{O}(\lambda)$ on X defined by local sections through

$$\Gamma(U, \mathcal{O}(\lambda)) = \{ f \in A(\pi_0^{-1}(U)) \mid f(gb) = \chi_\lambda(b) f(g),\ \forall b \in B,\ g \in G \}.$$

The structure sheaf $\mathcal{O}_X$ on X identifies with $\mathcal{O}(0)$. The sheaf $\mathcal{O}(\lambda)$ is a locally-free $\mathcal{O}_X$ module of rank 1 and may be identified with a G-equivariant line bundle on X.

Suppose $\lambda \in P(R)^+$. Then each $f \in V_\lambda^*$ defines a global section on $\mathcal{O}(\lambda)$ through the map $g \mapsto \langle g v_\lambda, f \rangle$ of G into $\mathbb{C}$. The Bott-Borel-Weil theorem [35], a very simple proof of which can be found in [9], asserts that the map $V_\lambda^* \to \Gamma(X, \mathcal{O}(\lambda))$ so obtained is bijective.

For any subset $Z \subset X$ we have $q_Z = \dim V_\lambda^* / V_Z^\perp$. It follows that $q_Z = q_{\bar{Z}}$ and so we can assume that Z is a closed subvariety of X without loss of generality. This will be done in the sequel.

6.3. Let $Z \subset X$ be a closed subvariety. The ideal sheaf $\mathscr{I}_Z$ of Z is defined by local sections though

$$\Gamma(U, \mathscr{I}_Z) = \{ f \in \Gamma(U, \mathscr{O}_X) \mid f(Z \cap U) = 0 \}.$$

Then the structure sheaf $\mathscr{O}_Z$ of Z identifies with the quotient sheaf $\mathscr{O}_X / \mathscr{I}_Z$ and so is obviously a coherent $\mathscr{O}_X$ module.

For each $\lambda \in P(R)$ define the sheaf $\mathscr{O}_Z(\lambda) := \mathscr{O}(\lambda) \otimes_{\mathscr{O}_X} \mathscr{O}_Z$ on Z. Right exactness of the tensor product gives a surjective map $\mathscr{O}(\lambda) \twoheadrightarrow \mathscr{O}_Z(\lambda)$.

Since $\mathscr{O}(\lambda)$ is ample for $\lambda \in P(R)^{++}$ and since the m fold tensor product $\mathscr{O}(\lambda)^m$ of $\mathscr{O}(\lambda)$ over $\mathscr{O}_X$ identifies with $\mathscr{O}(m\lambda)$, we conclude from Serre's theorem (concerning the Hilbert-Samuel polynomial [31], p. 15.3) applied to the projective variety X that the map $\Gamma(X, \mathscr{O}(\lambda)) \to \Gamma(Z, \mathscr{O}_Z(\lambda))$ with kernel $\{ f \in \Gamma(X, \mathscr{O}(\lambda)) \mid f(Z) = 0 \}$ induced on global sections is surjective for λ very dominant. This gives the

Lemma. — *For $\lambda \in P(R)$ very dominant*

$$\Gamma(Z, \mathscr{O}_Z(\lambda)) = \Gamma(X, \mathscr{O}(\lambda) / \{ f \in \Gamma(X, \mathscr{O}(\lambda)) \mid f(Z) = 0 \} \xrightarrow{\sim} V_\lambda^* / V_Z^\perp.$$

In particular $q_Z(\lambda) = \dim_{\mathbb{C}} \Gamma(Z, \mathscr{O}_Z(\lambda))$ for λ very dominant.

6.4. Let $K(X)$ [resp. $K_G(X)$] denote the Grothendieck group of locally free (resp. and G equivariant) sheaves (or vector bundles) on X. Let $R(B)$ denote the Grothendieck group generated by the algebraic B modules over $\mathbb{C}$. We have an isomorphism $\mathbb{Z} P(R) \xrightarrow{\sim} R(B)$ via the map $\lambda \mapsto \chi_\lambda$ and an isomorphism $R(B) \to K_G(X)$ via the map $F \to G \times_B F$. Also (for the flag variety X) the forgetful homomorphism $K_G(X) \to K(X)$ is surjective. (For more details on this, *see* [8], Sects. 2, 4.)

Finally, by [6], p. 106, every coherent sheaf on the projective variety X admits a finite free resolution by locally-free sheaves of length $\leqq n = \dim_{\mathbb{C}} X$.

6.5. Let $\mathscr{F}$ be a coherent sheaf on X. Define the Euler characteristic $\chi(\mathscr{F})$ of $\mathscr{F}$ by

$$\chi(\mathscr{F}) = \sum_{i=0}^{\infty} (-1)^i \dim_{\mathbb{C}} H^i(X, \mathscr{F}).$$

Let ρ denote the half sum of the positive roots and let Π denote Weyl's dimension function on $\mathfrak{h}^*$ namely

$$\Pi(\lambda) = \prod_{\alpha \in R^+} \frac{\langle \lambda, \alpha \rangle}{\langle \rho, \alpha \rangle},$$

which we recall is W harmonic. It is immediate from the Bott-Borel-Weil theorem [9] that

$$\chi(\mathscr{O}(\lambda)) = \Pi(\lambda + \rho), \qquad \forall \lambda \in P(R).$$

Set $\mathscr{F}(\lambda)=\mathcal{O}(\lambda)\otimes_{\mathcal{O}_X}\mathscr{F}$. Since $\mathcal{O}(\lambda)$ is locally free, the functor $\mathscr{F}\rightarrow\mathscr{F}(\lambda)$ is eaxct. So from a locally free resolution of $\mathscr{F}$ we obtain locally free resolution of $\mathscr{F}(\lambda)$. Let $[\mathscr{F}]$ denote the image of $\mathscr{F}$ in the Grothendieck group of coherent sheaves on X. By 6.4 there exists a finite set $I(\mathscr{F})\subset P(R)$ and integers $n_\mu^{\mathscr{F}}$ such that

$$[\mathscr{F}]=\sum_{\mu\in I(\mathscr{F})} n_\mu^{\mathscr{F}}[\mathcal{O}(\mu)]$$

and by tensoring with $\mathcal{O}(\lambda)$ that

$$[\mathscr{F}(\lambda)]=\sum_{\mu\in I(\mathscr{F})} n_\mu^{\mathscr{F}}[\mathcal{O}(\mu+\lambda)],\qquad \forall\lambda\in P(R).$$

Finally we remark that as in 6.3 one has $H^i(X, \mathscr{F}(\lambda))=0$, $\forall i>0$ for $\lambda\in P(R)$ very dominant.

Given Z as in 6.3 we use n_μ^Z [resp. I(Z)] to denote $n_\mu^{\mathscr{F}}$ [resp. $I(\mathscr{F})$] with $\mathscr{F}=\mathcal{O}_Z$. In view of 6.3 we obtain the

PROPOSITION. — *Let* $Z\subset X$ *be a closed subvariety. Then*

$$q_Z(\lambda)=\sum_{\mu\in I(Z)} n_\mu^Z\,\Pi(\mu+\lambda+\rho),$$

for all $\lambda\in P(R)$ *very dominant. The latter extends to a* W *harmonic polynomial on* Z.

6.6. Recall that K(X) inherits a ring structure by tensoring over $\mathcal{O}_X$ and that $H^*(X):=H^*(X, \mathbb{C})$ admits a ring structure via the cup product. After Borel [5] the first Chern class $c_1(\mathcal{O}(\lambda))\in H^2(X)$ of $\mathcal{O}(\lambda)$ identifies with σ_λ and the map $\lambda\mapsto\sigma_\lambda$ extends to ring homomorphism of $S(\mathfrak{h})$ onto $H^*(X)$ with kernel I_+ (notation, 3.6). Here the cup product is just the exterior or wedge product on exterior forms. Since there are no terms in odd dimension, $H^*(X)$ is commutative and we omit the wedge. We also recall that the isomorphism $\mathscr{H}=S(\mathfrak{h})/I_+\xrightarrow{\sim}H^*(X)$ which results, is an isomorphism of graded rings up to doubling of dimensions. Given $a\in H^*(X)$ non-zero, let gr a denote its lowest degree component.

Let $\eta: H_*(X)\rightarrow H^*(X)$ denote the isomorphism induced by Poincare duality. If Z is a closed subvariety in X of complex dimension m we denote by [Z] its fundamental class in $H_{2m}(X)$. One has

$$\eta[Z_1\cap Z_2]=\eta[Z_1]\,\eta[Z_2]$$

where $[Z_1\cap Z_2]$ is viewed as an algebraic cycle (*i. e.* multiplicities are counted).

Let ch : $K(X)\rightarrow H^*(X)$ denote the functorial homomorphism defined by the Chern character. For the flag variety ch is an isomorphism (*see* [8], 4.1 for example). The following result was pointed out to me by J. Bernstein.

THEOREM. — *Let* Z *be a closed subvariety of* X. *Then*

$$\text{gr ch}[\mathcal{O}_Z]=\eta[Z].$$

We first establish the assertion when Z is irreducible and of complex codimension 1. Then Z is given by a single homogeneous equation which for λ sufficiently dominant can be assumed to define a global section f of $\mathcal{O}(\lambda)$. This gives an exact sequence

$$0 \to \mathcal{O}_X \xrightarrow{a \mapsto fa} \mathcal{O}(\lambda) \to \mathcal{O}_Z \to 0.$$

We conclude that $[\mathcal{O}_Z] = [\mathcal{O}(\lambda)] - [\mathcal{O}_X]$ and so applying the Chern character map we obtain

$$\begin{aligned}\mathrm{ch}[\mathcal{O}_Z] &= \mathrm{ch}[\mathcal{O}(\lambda)] - \mathrm{ch}[\mathcal{O}_X] \\ &= e^{c_1(\mathcal{O}(\lambda))} - e^{c_1(\mathcal{O}(0))}, \quad \text{by say ([10], p. 56)}\end{aligned}$$

and so gr ch$[\mathcal{O}_Z] = c_1(\mathcal{O}(\lambda)) = \eta \,|\, (Z)$ by say ([10], Prop. 2, p. 141).

For the general case it suffices to show that

$$\mathrm{gr\ ch}[\mathcal{O}_{Z_1 \cap Z_2}] = \mathrm{gr\ ch}[\mathcal{O}_{Z_1}]\ \mathrm{gr\ ch}[\mathcal{O}_{Z_2}]$$

for closed subvarieties Z_1, Z_2 of X intersecting properly with Z_1 a divisor, as the result then obtains by induction on codimension and primary decomposition. (The result for $Z = X$ is trivial.)

Now recall that X is smooth and assume that

$$m := \mathrm{codim}\,(Z_1 \cap Z_2) = \mathrm{codim}\ Z_1 + \mathrm{codim}\ Z_2$$

(proper intersection) and that codim $Z_1 = 1$ (Z_1 is a divisor). Let $\mathrm{Tor}_i^{\mathcal{O}_X}(\mathcal{F}, .)$ denote the ith derived functor of the tensor product functor $\mathcal{F} \otimes_{\mathcal{O}_X} -$. By Krull's theorem ([27], Thm. 5, p. 58) we have the primary decomposition

$$(1) \qquad [Z_1 \cap Z_2] = \sum_{j=1}^{s} n_j [V_j]$$

where the V_j are irreducible, dim $V_j = \dim(Z_1 \cap Z_2) = \dim Z_2 - 1$. Then by Serre ([26], p. 145-146, Prop. 1 (3) and remarque 2) and because Z_1 is a divisor, which implies the vanishing of the higher Tors, one has

$$(2) \qquad \sum_{i=0}^{\infty} (-1)^i [\mathrm{Tor}_i^{\mathcal{O}_X}(\mathcal{O}_{Z_1}, \mathcal{O}_{Z_2})] = \sum_{j=1}^{s} n_j \mathcal{O}_{V_j}.$$

Now take resolutions of $\mathcal{O}_{Z_i} : i = 1, 2$ by locally free sheaves $\mathcal{F}^j$. Then we can write $[\mathcal{O}_{Z_i}] = \Sigma\, n_j^{(i)} [\mathcal{F}^j] : i = 1, 2$, $n_j^{(i)} \in \mathbb{Z}$, and we obtain by the bi-additivity of the alternating sum of the $\mathrm{Tor}_i^{\mathcal{O}_X}(.,\ .)$ that

$$\begin{aligned}\sum_{i=0}^{\infty} (-1)^i [\mathrm{Tor}_i^{\mathcal{O}_X}(\mathcal{O}_{Z_1}, \mathcal{O}_{Z_2})], \\ = \sum_{i,\ j,\ k} (-1)^i n_j^{(1)} n_k^{(2)} [\mathrm{Tor}_i^{\mathcal{O}_X}(\mathcal{F}^j, \mathcal{F}^k)], \\ = \sum_{j,\ k} n_j^{(1)} n_k^{(2)} [\mathcal{F}^j \otimes_{\mathcal{O}_X} \mathcal{F}^k].\end{aligned}$$

Applying the Chern character map this and (1) and (2) give

$$\begin{aligned}\mathrm{ch}[\mathcal{O}_{Z_1 \cap Z_2}] &= \Sigma\, n_j^{(1)} n_k^{(2)}\, \mathrm{ch}[\mathcal{F}^j]\, \mathrm{ch}[\mathcal{F}^k] \\ &= \mathrm{ch}[\mathcal{O}_{Z_1}]\, \mathrm{ch}[\mathcal{O}_{Z_2}].\end{aligned}$$

Applying gr proves the required assertion.

6.7. We can now relate the dimension q_Z and degree

$$\lambda \mapsto c_Z(\lambda) = \int_Z \exp \sigma_\lambda$$

polynomials for any subvariety Z of X. Here Z can be assumed closed without loss of generality. Both are W harmonic polynomials on $\mathfrak{h}^*$. For any polynomial $p \in S(\mathfrak{h})$ let gr p denotes its *leading* homogeneous part.

COROLLARY. — *Let* Z *be a subvariety of* X. *Then* $c_Z = \mathrm{gr}\ q_Z$.
Set $m = \dim_{\mathbb{C}} Z$. Then

(1) $$c_Z = \frac{1}{m!} \int_Z \sigma_\lambda^m = \frac{1}{m!} \int_X \sigma_\lambda^m\, \eta[Z],$$

by the definition of Poincaré duality. By 6.5 we can write

$$[\mathcal{O}_Z] = \sum_{\mu \in I(Z)} n_\mu^Z [\mathcal{O}(\mu)]$$

and then by 6.6,

$$\begin{aligned}\eta[Z] &= \mathrm{gr}\Big(\sum_{\mu \in I(Z)} n_\mu^Z e^{\sigma_\mu}\Big), \\ &= \sum_{\mu \in I(Z)} n_\mu^Z \frac{\sigma_\mu^{n-m}}{(n-m)!}.\end{aligned}$$

Here we have noted that terms of degree $< n-m$ must cancel. Now by [18], 2.3 (i) the first non-vanishing term in the expansion

$$\sum_{\mu \in I(Z)} n_\mu^Z e^\mu$$

is always W harmonic. The remarks in 6.6 concerning the map $\mu \mapsto \sigma_\mu$ then imply that

(2) $$\sum_{\mu \in I(Z)} n_\mu^Z \mu^l = 0, \qquad \forall\, l < n-m.$$

Now we may substitute the expression for $\eta[Z]$ in (1) since higher order terms are eliminated by the definition of the integral. Given $p \in S(\mathfrak{h})$, let $p^\natural \in \mathscr{H}$ denote its harmonic part defined by the direct sum decomposition in 3.6. Let (notation 6.5) ω_X denote the image of Π in $H^*(X)$. Then identifying $\mathfrak{h}$, $\mathfrak{h}^*$ through the Cartan inner product gives

$$c_Z(\lambda) = \frac{1}{m!\,(n-m)!} \sum_{\mu \in I(Z)} n_\mu^Z \int_X \sigma_\lambda^m \sigma_\mu^{n-m},$$

and so

(3) $$c_Z(\lambda) = \frac{1}{m!\,(n-m)!} \Big(\sum_{\mu \in I(Z)} n_\mu^Z \lambda^m \mu^{n-m} \Big)^\natural (\rho) \int_X \omega_X.$$

[Here we have used that $\Pi(\rho) = 1$ to obtain the correct normalization.) Now for all $\nu \in \mathfrak{h}^*$ one knows that

(4) $$\sum_{w \in W} (-1)^{l(w)}\, w \nu^n \in \mathscr{H}$$

and so

(5) $$(\nu^n)^\natural (\rho) = \frac{1}{|W|} \sum_{w \in W} (-1)^{l(w)} (w\nu, \rho)^n = (\rho^n)^\natural (\nu).$$

Taking $\nu = \rho$ in (4) and use of the Weyl denominator formula gives that

(6) $$(\rho^n)^\natural = \frac{n!}{|W|} \prod_{\alpha \in R^+} \alpha.$$

From 6.5 and (5), (6) we obtain

$$q_Z(\lambda) = \frac{|W|}{n! \prod_{\alpha \in R^+} (\alpha, \rho)} \Big(\sum_{\mu \in I(Z)} n_\mu^Z ((\mu + \lambda + \rho)^n)^\natural \Big) (\rho)$$

and by (2) (using that $\mathscr{H}$ is graded, so gr and $\natural$ commute) we conclude that as a polynomial in λ

$$\operatorname{gr} q_Z(\lambda) = \frac{|W|}{(n-m)!\, m!\, \prod_{\alpha \in R^+} (\alpha, \rho)} \Big(\sum_{\mu \in I(Z)} n_\mu^Z \lambda^m \mu^{n-m} \Big)^{\natural} (\rho)$$

$$= \frac{|W|}{\prod_{\alpha \in R^+} (\alpha, \rho)} \cdot \frac{1}{\int_X \omega_X} \cdot c_Z(\lambda).$$

Consider the special case when Z is reduced to the base point $B \in G/B$. For any embedding $i_\lambda : X \subsetneq \mathbb{P} V_\lambda : \lambda \in P(R)^{++}$ one has deg $B=1$, whilst $q_B(\lambda)=1$, $\forall \lambda \in P(R)^{++}$ trivially. We conclude that the scale factor in the above expression equals 1. This establishes the corollary.

6.8. Combined 4.8, 5.7 and 6.7 achieve the goal described in the introduction. Indeed fix $w \in W$ and let $p_{C(w)}$ denote the characteristic polynomial of the orbital variety $C(w)$. Choose $u \in C(w)$ such that $Z_u(w) := \pi^{-1}(u)$ is irreducible [which is possible by 3.4 (iii)] and let $q_{Z_u(w)}$ denote the dimension polynomial of $Z_u(w) \subset G/B$.

THEOREM. — *Up to non-zero scalar depending only on* $O(w)$, *one has*

$$p_{C(w)} = \operatorname{gr} q_{Z_u(w^{-1})}, \qquad \forall w \in W.$$

We remark that by 6.6, the right hand side above represents the image of $[Z_u(w^{-1})]$ in $H_{2m}(X) : m = \dim Z_u(w^{-1})$. Consequently by the Hotta-Springer specialization theorem [15] these polynomials span the Springer module $H_{2m}(\mathscr{B}_u)^{A_u}$ associated to the nilpotent orbit $O(w)$ containing u. Of course this is also true of the degree polynomial; but we had felt that the dimension polynomial is easier to visualize and to compute. In particular when Z is a Schubert variety the dimension polynomial is given by the Demazure character formula [1]. However, as the referee was quick to point out, these degree polynomials can be deduced from ([34], 4.1) or by setting $Y=0$ in [33], p. 51, proposition. Moreover, at least for the moment, these calculations are easier than the proof of the Demazure character formula. By Rossmann ([25], II, Thm. 8.2) the degree polynomials for the irreducible components can also be computed from the matrix A with entries $A(w, y) : w, y \in W$ of Euler numbers. (For the moment the latter are not known completely.)

7. Comparison with the results of Borho, Brylinski, MacPherson ([7], [8])

7.1. Fix $w \in W$ and define $O(w)$, $C(w)$ as before. Fix $u \in C(w)$ and set $D_u(w) = \pi^{-1}(u) \cap G \times_B C(w)$. One easily checks that $GD_u(w) = G \times_B C(w)$. The latter is referred to as an orbital cone bundle in [7], 2.1, and it is shown that the characteristic classes of the distinct orbital cone bundles for $O(w)$ form a basis for Springer's representation in $H_{2m}(\mathscr{B}_u)^{A_u} : m=n-\dim_{\mathbb{C}} C(w)$. We shall in particular obtain another proof of this assertion. The key observation is the following curious fact.

7.2. LEMMA. — *For all* $w \in W$ *and all* $u \in C^0(w)$ *one has* $A_u Z_u(w^{-1}) = \overline{D_u(w)}$.

Set $E_u(w) = \{ g \in G \mid g^{-1} u \in \overline{B(\mathfrak{n} \cap w\mathfrak{n})} \}$. This is just the inverse image in G of the closed subvariety $\overline{B(\mathfrak{n} \cap w\mathfrak{n})} = \overline{C(w)} = \overline{C^0(w)}$ under the map $g \mapsto g^{-1} u$. Under the natural projection $\pi_0 : G \to G/B$ one has $\pi_0(E_u(w)) = \overline{D_u(w)}$. From Spaltenstein ([28], fourth paragraph) we further see that $\overline{D_u(w)}$ is a single A_u orbit of components of $\mathscr{B}_u$. By 3.4 (ii) this also holds for $A_u Z_u(w^{-1})$ so it suffices to show that $Z_u^0(w^{-1}) \subset \overline{D_u(w)}$. In fact

$$\begin{aligned} Z_u^0(w^{-1}) :&= \pi^{-1}(u) \cap Y(w^{-1}) \\ &= \{ bw^{-1}B \mid b^{-1}u \in \mathfrak{n} \cap w^{-1}\mathfrak{n} \} \\ &= \{ bw^{-1}B \mid (bw^{-1})^{-1} u \in \mathfrak{n} \cap w\mathfrak{n} \} \\ &\subset \pi_0(E_u(w)) = \overline{D_u(w)}, \end{aligned}$$

as required.

7.3. Fix $w \in W$ and let p_C denote the characteristic polynomial of $C(w)$. Let c_D denote the degree polynomial of $D_u(w)$.

COROLLARY. — *For each orbital variety* C, *one has* $c_D = p_C$, *up to a non-zero scalar, depending only on* O.

This follows from 7.1, 4.8 and 5.7 if we recall that by 3.4 and 3.5 $c_x = c_y$ if $Z_u(x)$ and $Z_u(y)$ lie in the same A_u orbit. This latter fact can also be seen using the dimensional polynomial and 6.7.

7.4. The above result can be read off from [8], 4.7 and [7], Theorem 3.2. It implies the result we wished to prove in 7.1. Although it would seem to be more natural than 4.8 and 5.7 where we are obliged to make explicit reference to $w \in W$, this is not really so. Indeed the proof by Borho, Brylinski and MacPherson of 7.3 is extremely roundabout as we explained in the introduction. A main difficulty is that p_C is not obviously W harmonic (and need not be if C is not orbital) whereas the degree polynomial c_D is always W harmonic for any closed subvariety of G/B.

8. A positivity lemma and some examples

8.1. It follows from 5.7 and 3.3 that the characteristic polynomial of an orbital variety C is a sum with positive coefficients of products of distinct positive roots. Moreover by [19], 5.2, this also holds for the Goldie rank polynomials, a fact which we had long since thought to be true; but which is completely unobvious from say [18], 5.1 even knowing explicitly the Jantzen matrix via the Kazhdan-Lusztig polynomials. [Similar remarks apply to the degree polynomials which by 4.8, 5.7 also have this property when Z is an irreducible component of a fixed point set $\mathscr{B}_u$ (however it is not known if this holds for more general Z).] Here we give a second proof of this positivity property of p_C.

8.2. Let C be any closed irreducible H stable subvariety of $\mathfrak{n}$ and define its characteristic polynomial p_C as in [19], 2.3. Take any $\alpha \in R^+$ and let $\mathfrak{m}_\alpha$ denote the subspace of $\mathfrak{n}$ generated by the $e_\beta : \beta \in R^+ \setminus \{\alpha\}$. Consider the intersection $C \cap \mathfrak{m}_\alpha$ as an algebraic cycle.

LEMMA:

$$p_{C \cap \mathfrak{m}_\alpha} = \begin{cases} p_C & \text{if } C \subset \mathfrak{m}_\alpha, \\ \alpha p_C & \text{otherwise.} \end{cases}$$

This is proved exactly as in [19], 2.9; apart from the sign error occurring there!

8.3. Retain the above notation. Fix any ordering $\alpha_1, \alpha_2, \ldots, \alpha_n$ of the positive roots. Set $\mathfrak{m}_i = \mathfrak{m}_{\alpha_i}$. Let L denote a chain $C = C_1, C_2, \ldots, C_n$ of H stable closed irreducible subvarieties of $\mathfrak{n}$ by taking C_{i+1} to be an irreducible component of $C_i \cap \mathfrak{m}_i$ and let $\mathscr{L}$ denote the set of all such chains. Let n_i^L denote the multiplicity of C_{i+1} in $C_i \cap \mathfrak{m}_i$.

Set

$$\Delta_L^C = \left(\prod_{i=1}^{n-1} n_i^L \right) \prod_{i \mid C_i = C_{i+1}} \alpha_i$$

which is a product of $\operatorname{codim}_{\mathbb{C}} C$ distinct positive roots with a positive integer coefficient. From the definition [19], 2.3 of p_C it follows that $p_{\{0\}}$ is the product of the positive roots. This gives the

COROLLARY. — *For each* H *stable closed, irreducible subvariety* C *of* $\mathfrak{n}$ *one has*

$$p_C = \sum_{L \in \mathscr{L}} \Delta_L^C.$$

In particular p_C is the sum of products of distinct $\operatorname{codim}_{\mathbb{C}} C$ *positive roots with non-negative integer coefficients.*

Notice that the result cannot depend on the ordering of R^+ which is not *a priori* obvious but which is a familiar phenomenon in intersection theory.

8.4. Take $\mathfrak{g} = \mathfrak{sl}(n+1)$ with $S = \{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ the standard choice of simple roots. For all $i < j = \{1, 2, \ldots, n+1\}$ set

$$\alpha_{ij} = \sum_{k=i}^{j-1} \alpha_k \qquad \text{and} \qquad e_{ij} = e_{\alpha_{ij}}.$$

Take $n = 3$ and consider the orbital variety C with ideal of definition $I(\bar{C}) = \langle e_{12}, e_{23}, e_{13} e_{24} - e_{23} e_{14} \rangle$. Then $\bar{C}$ is a complete intersection and by say

[8], 4.16 one has

$$p_C=\alpha_1\alpha_3(\alpha_1+2\alpha_2+\alpha_3).$$

On the other hand, taking the ordering $\{\alpha_1, \alpha_3, \alpha_2, \dots\}$ of R^+ we see that $|\mathscr{L}|=2$ and correspondingly p_C is the sum of the two terms $\alpha_1\alpha_3(\alpha_1+\alpha_2)$, $\alpha_1\alpha_3(\alpha_2+\alpha_3)$. Again C can be viewed as a codimension 1 subvariety $\mathfrak{m}_{\alpha_1}\cap\mathfrak{m}_{\alpha_3}$. Then the integrals occurring in 3.3(*) correspond to taking U a subset of cardinality 1 in $\{\alpha_2, \alpha_1+\alpha_2, \alpha_2+\alpha_3, \alpha_1+\alpha_2+\alpha_3\}$. By the symmetry each of these integrals must be equal, whilst their sum is just $\mathrm{mult}_0(C)=2$. Hence they are all equal to 1/2 and we obtain from 3.3(*) that

$$\begin{aligned}p_C&=\frac{1}{2}\alpha_1\alpha_3\{\alpha_2+\alpha_1+\alpha_2+\alpha_2+\alpha_3+\alpha_1+\alpha_2+\alpha_3\}\\&=\alpha_1\alpha_3(\alpha_1+2\alpha_2+\alpha_3)\end{aligned}$$

as required.

Now take $n=4$ and consider the orbital variety C for which

$$I(\bar{C})=\langle e_{12}, e_{34}, e_{45}, e_{35}, e_{13}e_{24}-e_{23}e_{14}, e_{14}e_{25}-e_{24}e_{15}, e_{13}e_{25}-e_{23}e_{15}\rangle$$

which is not a complete intersection (and so cannot be computed from [8], Prop. 4.15). Applying 8.3 we obtain

$$p_C=\alpha_1\alpha_3\alpha_4(\alpha_3+\alpha_4)[(\alpha_1+\alpha_2)(\alpha_1+2\alpha_2+2\alpha_3+\alpha_4)+(\alpha_2+\alpha_3)(\alpha_2+\alpha_3+\alpha_4)].$$

Up to the scale factor of 1/12 this agrees as it should in $\mathfrak{sl}(5)$ [and conjecturally in $\mathfrak{sl}(n+1)$] with the Goldie rank polynomial occurring in the third line of [17], Thm. 11.4 (iii).

If we wish to apply 3.3(*) we have to compute 15 integrals which by symmetry occur in 3 sets for which each member of a given set has the same value. View C as a subvariety C′ of $\mathfrak{m}_{\alpha_1}\cap\mathfrak{m}_{\alpha_3}\cap\mathfrak{m}_{\alpha_4}\cap\mathfrak{m}_{\alpha_3+\alpha_4}$. Then the 15 possible subsets U of $R^+\setminus\{\alpha_1, \alpha_3, \alpha_4, \alpha_3+\alpha_4\}$ break up into a corresponding collection of three sets. These have, for example, members $\{\alpha_2, \alpha_1+\alpha_3\}$, $\{\alpha_2, \alpha_1+\alpha_2+\alpha_3\}$, $\{\alpha_2, \alpha_2+\alpha_3\}$ respectively. If the corresponding integrals are denoted by I_1, I_2, I_3 then comparison of 3.3(*) with our previous expression for p_C gives $I_1=0$, $I_2=1/6$, $I_3=1/3$, that is for example

$$\lim_{\varepsilon\to 0}\frac{1}{\mathrm{Vol}\,B_\varepsilon^2}\int_{C'\cap B_\varepsilon}(dx_{\alpha_2}\wedge dy_{\alpha_2})(dx_{\alpha_2+\alpha_3}\wedge dy_{\alpha_2+\alpha_3})=\frac{1}{3}.$$

However, we did not attempt to verify this directly.

8.5 One can ask if it is possible to verify directly that p_C as defined by 4.8(*) satisfies 8.2 by direct computation. Ironically, this is extremely difficult, though it is intuitively plausible. The trouble is that it fails in general. Namely if C is a complex irreducible locally closed algebraic subvariety $\mathbb{A}^n(\mathbb{C})$ and P is a hyperplane, then the integral $J_{P\cap C}$ over the algebraic cycle $P\cap C$ can be quite unrelated to J_C (consider the case when C is defined by the equation $z_1^2=z_2^3$ and intersect with the hyperplane $z_2=0$). One needs, for example, that C and P are stable under a non-trivial action of $\mathbb{C}^*$ (as in 8.2) so one can apply 5.5.

One may also ask if it is possible to show directly that p_C as defined by 4.8(*) satisfies [19], 2.6. Here we recall that this and [19], 2.9, led to the action of W on the space spanned by characteristic polynomials of orbital varieties.

In fact the analogue of [19], 2.6, can be proved by an analysis along the lines of Sect. 4 where G is replaced by the parabolic subgroup $P_\alpha \supset B$ defined by $\alpha \in S$. We leave the details to the reader; but remark that to make a precise comparison with [19], 2.6, one needs to check that the factor z occurring in [19], 2.6, is just the degree of the moment map $(p, \nu) \mapsto \nu$ of $P_\alpha \times_B \mathfrak{m}_\alpha$ onto $P_\alpha \mathfrak{m}_\alpha$.

8.6. By 8.3 and [19], 5.2, every Goldie rank polynomial $\tilde{p}_w$ can be expressed as a positive linear combination of products of distinct positive roots. Let $\operatorname{Supp} \tilde{p}_w \subset B$ denote the simple roots which occur in such an expression for $\tilde{p}_w$. We show that this positivity has an important consequence for when a primitive ideal corresponding to a *regular* integral central character can be completely prime.

Let $L(\lambda) : \lambda \in \mathfrak{h}^*$ denote the simple highest weight module with highest weight $\lambda-\rho$ and set $J(\lambda)=\operatorname{Ann}_{U(\mathfrak{g})} L(\lambda)$ which is primitive ideal. Now fix λ regular [*i. e.* $(\lambda, \alpha) \neq 0$, $\forall \alpha \in R$] and integral [*i. e.* $\lambda \in P(R)$]. For each $B' \subset B$, let $w_{B'}$ denote the unique longest element of the Weyl group $W_{B'}$ generated by the $s_\alpha : \alpha \in B'$.

THEOREM. — *Assume that* $J(w\lambda) : w \in W$ *is completely prime. Set* $B' = \operatorname{Supp} \tilde{p}_w$. *Then* $2(\alpha, \lambda)/(\alpha, \alpha)=1$ *for all* $\alpha \in B'$ *and* $J(w\lambda)=J(w_B w_{B'} \lambda)$.

Let $P(R)^+$ denote the dominant elements of $P(R)$ and set

$$\tilde{p}_w(\mu)=\operatorname{rk} U(\mathfrak{g})/J(w\mu) : \mu \in P(R)^+,$$

where rk denotes Goldie rank. Now set $n_\alpha^\mu := 2(\alpha, \mu)/(\alpha, \alpha) : \alpha \in B$ which is a non-negative integer for all $\mu \in P(R)^+$. The above positivity property of $\tilde{p}_w$ implies that $\tilde{p}_w(\mu) \leqq \tilde{p}_w(\mu')$ if $n_\alpha^\mu \leqq n_\alpha^{\mu'}$, $\forall \alpha \in B$ with a strict inequality if $n_\alpha^\mu < n_\alpha^{\mu'}$ for some $\alpha \in B'$. We conclude that $n_\alpha^\lambda=1$, for all $\alpha \in B'$ and that $\tilde{p}_w(\mu)=0$ if $n_\alpha^\mu=0$ for some $\alpha \in B'$. Set $R'=\mathbb{Z}B' \cap R$, $R'^+=R' \cap R^+$ and let $p_{B'}$ denote the product of roots in R'^+. It is well-known that the second conclusion above implies that $p_{B'}$ divides $\tilde{p}_w$. (By the polynomial character of Goldie rank, the above vanishing corresponds to the Borho-Jantzen degeneration to the α-wall and then from the reflection functor across the α-wall we obtain $s_\alpha \tilde{p}_w = -\tilde{p}_w$, for all $\alpha \in B'$ — hence the required assertion.) Yet $\tilde{p}_w$ is a W harmonic polynomial and $p_{B'}$ is the highest degree W harmonic polynomial satisfying $\operatorname{Supp} p_{B'}=B'$. Hence $\tilde{p}_w=p_{B'}$ up to a non-zero scalar. Again from [18], 5.1, and well-known properties of the Jantzen matrix (or Kazhdan-Lusztig polynomials) one has $p_{B'}(\mu)=\operatorname{rk}(U(\mathfrak{g})/J(w_B w_{B'} \mu)) : \mu \in P(R)^+$ up to an overall non-zero scalar. By [18], 5.5, this implies the assertion of the theorem.

Remarks. — The second assertion of the theorem should extend to the non-integral (regular) case; but the arguments run into difficulties because one cannot translate in an arbitrary fashion to the walls. Indeed if we let B_λ denote the corresponding simple root system, then it is false that the $2(\mu, \alpha)/(\alpha, \alpha) : \alpha \in B_\lambda$ take arbitrary integer values on $\lambda + P(R)$. In particular the first assertion of the theorem is always false if Supp $\tilde{p}_w$ cannot be conjugated into B. The result fails miserably for λ non-regular. The difficulty in the proof is that Supp $\tilde{p}_w$ contains not only those $\alpha \in B$ for which $\tilde{p}_w(\mu) = 0$ when $n_\alpha^\mu = 0$, but also those $\alpha \in B$ for which $n_\alpha^\lambda = 0$.

8.7. Retain the above notation. Then $L(w_B w_{B'} \lambda)$ is just the module induced from the one dimensional representation $\mathbb{C}_{w_B w_{B'} \lambda - \rho}$ of the parabolic subalgebra $p_{B''}$ defined by $B'' := -w_B B'$. We obtain the

COROLLARY. — *Assume* $\lambda \in P(R)^{++}$. *If* $J(w\lambda) : w \in W$ *is completely prime, then* $J(w\lambda)$ *is an induced ideal.*

APPENDIX

Index of notation

Symbols appearing frequently are listed below in order of appearance (or where they are first defined).

1.1. $\mathbb{C}$, $\mathbb{R}$.

1.2. G, $\mathfrak{g}$, $\mathfrak{n}$, $\mathfrak{h}$, $\mathfrak{n}^-$, $\mathfrak{b}$, B, N, H.

1.4. W.

1.9. $\mathscr{B}$, $\mathscr{B}_u$.

2.1. R, R^+, S, n, P(R), $P(R)^+$, $P(R)^{++}$, e_α, h_α, θ, $\mathfrak{k}$, $\mathfrak{p}$, $\mathfrak{t}$, $\mathfrak{a}$, K, A, T, $\mathfrak{t}^\perp$, $\mathfrak{b}^\perp$.

2.2. z_α, x_α, y_α.

2.3. X, $T^*(X)$, Θ, π, π'.

2.4. $X(w)$, $Y(w)$, $O(w)$, $C(w)$.

2.5. $\mathscr{D}_*$, d, $c(x)$, d_x.

2.6. σ_λ, $f^{(1)}_{\lambda, x}$.

2.7. τ_h, $f^{(2)}$.

2.8. $\Sigma_{\lambda, h}$, $f_{\lambda, h}$.

3.1. $I_y^\lambda(h)$.

3.2. $J_C(h)$, B_ε.

3.4. $C^0(w)$, $Z_u^0(w)$, $Z_u(w)$, A_u.

3.6. $\mathscr{H}$, I_+, c_Z, c_w.

4.1. V_λ, v_λ.

4.6. $Eu_y(Y)$.

4.7. $A(w, y)$.

4.8. p_C.

5.3. $\mathrm{mult}_w(C)$.

5.4. $\text{mult}_0^k(\mathrm{C})$.

6.1. q_Z.

6.2. τ_0, $\mathcal{O}(\lambda)$, $\mathcal{O}_\mathrm{X}$.

6.3. $\mathcal{O}_\mathrm{Z}(\lambda)$.

6.4. $\mathrm{K}(\mathrm{X})$, $\mathrm{K}_\mathrm{G}(\mathrm{X})$, $\mathrm{R}(\mathrm{B})$.

6.5. $\chi(\mathscr{F})$, Π, n_μ^Z, $\mathrm{I}(\mathrm{Z})$.

6.6. c_1, η, ch.

7.1. $\mathrm{D}_u(w)$.

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