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## Centralizers of Anosov diffeomorphisms on tori

# CENTRALIZERS OF ANOSOV DIFFEOMORPHISMS
# ON TORI

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ABSTRACT. — We prove here that the elements of an open and dense subset of Anosov diffeomorphisms on tori have trivial centralizers: they only commute with their own powers.

## 1. Introduction

Let M be a smooth connected compact manifold, and Diff(M) the group of $C^\infty$ diffeomorphisms of M endowed with the $C^\infty$ topology. The diffeomorphisms which satisfy Axiom A and the (strong) transversality condition — every stable manifold intersects transversely every unstable manifold — form an open subset $\mathfrak{A}(M)$ of Diff(M) and are, by Robbin [4] and a recent result of Mañé [2], exactly the $C^1$-structurally stable diffeomorphisms.

We continue here the study, initiated in [3], of centralizers of diffeomorphisms in $\mathcal{G}(M)$; the concepts we just mentioned are detailed there. We now treat the relevant case where M is the torus $\mathbb{T}^n$ and the diffeomorp¹ ⁻ms are such that all of $\mathbb{T}^n$ is a hyperbolic set (Anosov diffeomorphisms). Recall that for $f \in$ Diff (M), its centralizer $Z(f)$ in Diff(M) is defined as the set of elements that commute with $f$. We say that $f$ has trivial centralizer if $Z(f)$ is reduced to the iterates $\{f^n, n \in \mathbb{Z}\}$ of $f$.

CONJECTURE. — *There is an open and dense subset of $f \in \mathfrak{A}(M)$ which have trivial centralizer.*

N. Kopell [1] proved this conjecture for $M = S^1$. In [3], we investigated the conjecture in higher dimensions. We proved it for dim $M = 2$, and that (in any dimension) the centralizer is trivial for a residual set of $f \in \mathfrak{A}(M)$. Actually, one can state the question for any connected component of $\mathfrak{A}(M)$, where the topological dynamics stays the same. We then also proved the conjecture when the topological dynamics exhibit one periodic attractor or repellor [3].

The purpose of this paper is to settle the case of Anosov diffeomorphisms on tori.

THEOREM. — *For an open and dense subset of Anosov diffeomorphisms of the n-dimensional torus $\mathbb{T}^n$, the centralizer is trivial.*

Some basic facts about Anosov diffeomorphisms that we shall use in the sequel are collected in Section 2 and in Section 3 we recall a previous but recent result [3] that is

also needed here. A basic proposition (Proposition 2) is stated in Section 4 and, from it and the result recalled in Section 3, we prove the main theorem. Sections 5 through 7 are dedicated to the proof of this basic proposition. First, in Section 5, we present a statement about the induced linear automorphism (Proposition 3) and show in Section 6 that it implies Proposition 2. Finally, in Section 7, we provide the proof of Proposition 3.

It is worthwhile noting that, while our techniques most probably can be generalized to the case of Anosov diffeomorphisms on infranilmanifolds (which are, conjecturally, the only ones to exist) they do not apply directly to an "abstract" Anosov diffeomorphism: the algebraic structure of the manifold is strongly used. We also point out that there is a version of the above conjecture for flows, which was solved by Sad [5].

## 2. Anosov diffeomorphisms on tori

We recall some basic material on Anosov diffeomorphisms on tori. For $A \in GL_n(\mathbb{Z}) = Aut(H_1(\mathbb{T}^n, \mathbb{Z}))$, we denote by $Homeo_A(\mathbb{T}^n)$ [resp. $Diff_A(\mathbb{T}^n)$, resp. $\mathscr{D}_A(\mathbb{T}^n)$] the set of homeomorphisms (resp. diffeomorphisms, resp. Anosov diffeomorphisms) of M which induce A in homology. Then, $\mathscr{D}_A(\mathbb{T}^n)$ is non empty if and only if A is hyperbolic. We fix from now on some hyperbolic $A \in GL_n(\mathbb{Z})$. The main result about $\mathscr{D}_A(\mathbb{T}^n)$ is best expressed by considering the finite covering $\tilde{\mathscr{D}}_A(\mathbb{T}^n)$ of $\mathscr{D}_A(\mathbb{T}^n)$ formed by pairs $(f, p) \in \mathscr{D}_A(\mathbb{T}^n) \times \mathbb{T}^n$ such that $f(p) = p$. We recall that any $f \in \mathscr{D}_A(\mathbb{T}^n)$ has at least one fixed point.

PROPOSITION 0. — *For* $(f, p) \in \tilde{\mathscr{D}}_A(\mathbb{T}^n)$, *there is a unique* $h = h_f$ *in* $Homeo_1(\mathbb{T}^n)$ *such that* $hfh^{-1} = A$ *and* $h(p) = 0$. *The map* $f \mapsto h_f$ *is continuous.*

*Proof.* — We only recall briefly the proof of unicity; more precisely, any homeomorphism h of $\mathbb{T}^n$ which commutes with A and fixes 0 belongs to $GL_n(\mathbb{Z})$. Indeed, replacing h by $hB^{-1}$, where B is induced by h in homology and therefore commutes with A, we can assume that $h \in Homeo_1(\mathbb{T}^n)$. Then, for the lift $\tilde{h}$ of h to $\mathbb{R}^n$ which fixes the origin 0, we have

$$\underset{\mathbb{R}^n}{Sup} \| \tilde{h}(y) - y \| < +\infty.$$

From this we get, for any $y \in \mathbb{R}^n$,

$$\underset{n \in \mathbb{Z}}{Sup} \| A^n(\tilde{h}(y) - y) \| = \underset{n \in \mathbb{Z}}{Sup} \| \tilde{h}(A^n(y)) - A^n(y) \| < +\infty.$$

As A is hyperbolic, this implies that $\tilde{h}$ is the identity. ∎

## 3. A previous basic result

The proof of the theorem is based on two results, one of which was proved in [3] and that we recall now.

For $f \in \mathfrak{A}(M)$, consider the following properties:

(i) The spectra of $Df$ at the different fixed points of $f$ are distinct;

(ii) At each fixed point of $f$, the eigenvalues of $Df$ are simple, and there is no resonance neither between the stable eigenvalues nor between the unstable eigenvalues.

These properties are clearly satisfied by the elements of an open and dense subset $\mathfrak{A}^*(M)$ of $\mathfrak{A}(M)$. Let $\mathcal{U}$ be an open, connected, simply connected subset of $\mathfrak{A}^*(M)$ and $p: f \mapsto p(f)$ a continuous map from $\mathcal{U}$ to $M$ such that $f(p(f)) = p(f)$. There exist finite sets $I_j$, $1 \leq j \leq 4$, and, with $I = \bigcup_{j=1}^{4} I_j$, a continuous map $\lambda = (\lambda_i)_{i \in I}$ from $\mathcal{U} \times I$ to $\mathbb{C}$ such that, for $f \in \mathcal{U}$:

— the $\lambda_i(f)$ with $i \in I_1$ (resp. $i \in I_3$) are the real stable (resp. unstable) eigenvalues of $Df$ at $p = p(f)$;

— the $\lambda_i(f)$ with $i \in I_2$ (resp. $i \in I_4$) are the stable (resp. unstable) eigenvalues with strictly positive imaginary part of $Df$ at $p$.

Let $\mathbb{K}_i$ be equal to $\mathbb{R}$ for $i \in I_1 \cup I_3$ and to $\mathbb{C}$ for $i \in I_2 \cup I_4$. By Sternberg's linearization theorem (see [3]), there exists, for $f \in \mathcal{U}$, a $C^\infty$-diffeomorphism $k_s = k_s(f)$ from $\prod_{i \in I_1 \cup I_2} \mathbb{K}_i$ onto $W^s(f, p(f))$ such that $k_s^{-1} f k_s$ is the product of the homotheties of ratio $\lambda_i(f)$ on $\mathbb{K}_i$, for $i \in I_1 \cup I_2$. Furthermore, $k_s$ is unique up to a product of homotheties in $\prod_{i \in I_1 \cup I_2} \mathbb{K}_i$, and we can choose the map $f \mapsto k_s(f)$ to be continuous. Similarly with $I_3 \cup I_4$, $W^u(f, p(f))$ and $k_u$. Then, for $f \in \mathcal{U}$, a diffeomorphism $g \in Z(f)$ must fix $p$ [because of property (i)] and preserve the stable manifold $W^s(f, p(f))$ and the unstable manifold $W^u(f, p(f))$. Moreover $k_s^{-1} g k_s$ and $k_u^{-1} g k_u$ are products of homotheties on their factors $\mathbb{K}_i$, of respective ratios $\mu_i$. We define the *compact part* of $Z(f)$ at $p(f)$ by the condition that all ratios $\text{Log} |\mu_i| / \text{Log} |\lambda_i|$, $i \in I$, are equal; this is a subgroup of $Z(f)$ which contains the iterates of $f$. We then proved in [3]:

PROPOSITION 1. — *There is an open and dense subset $\mathcal{U}_1$ of $\mathcal{U}$ such that for any $f \in \mathcal{U}_1$ the centralizer $Z(f)$ has trivial compact part at $p(f)$.*

*Remark.* — A similar result holds if we consider periodic orbits of a fixed period; here, we only need the case of fixed points.

## 4. A basic proposition and proof of the theorem

The other result which is basic to the proof of the theorem relates, for $(f, p) \in \tilde{\mathscr{D}}_A(\mathbb{T}^n)$ and $f \in \mathfrak{A}^*(\mathbb{T}^n)$, the local linearizing conjugacies $k_s$, $k_u$ at $p$, introduced above, and the global conjugacy $h_f$.

Let $\mathscr{D}_1$ be the set of $(f, p) \in \tilde{\mathscr{D}}_A(\mathbb{T}^n)$ such that $f \in \mathfrak{A}^*(\mathbb{T}^n)$, and let $\mathscr{V}$ be a connected component of $\mathscr{D}_1$. We can then define $I_j$, $1 \leq j \leq 4$ as above. For $f \in \mathscr{V}$, let J be a non

trivial subset of $I_1 \cup I_2$ (resp. $I_3 \cup I_4$); that is, J is neither the empty set nor the whole set. We then denote by $W_J(f)$ the image of $\prod_{i \in J} \mathbb{K}_i = \mathbb{K}_J$ under $k_s(f)$ [resp. $k_u(f)$].

PROPOSITION 2. — *There is an open and dense subset $\mathscr{V}_1$ of $\mathscr{V}$ such that for any $f \in \mathscr{V}_1$ and any non trivial subset J of either $I_1 \cup I_2$ or $I_3 \cup I_4$, the dimension of the linear subspace of $\mathbb{R}^n$ generated by $h_f(W_J(f))$ is strictly greater than the dimension of $\mathbb{K}_J$.*

Observe that the openness of $\mathscr{V}_1$ follows from the continuity of the maps $f \to h_f$, $f \to k_s(f), f \to k_u(f)$. Before proving density, we show how to deduce the theorem from Propositions 1 and 2.

*Proof of the theorem.* — By Proposition 1 there is an open and dense set $\mathscr{V}_2$ in $\mathscr{V}_1$ of diffeomorphisms having centralizer with trivial compact part at $p$. When $\mathscr{V}$ varies among the components of $\mathscr{D}_1$, the union of these $\mathscr{V}_2$ is an open and dense subset of $\widetilde{\mathscr{D}}_A(\mathbb{T}^n)$.

Let $f \in \mathscr{V}_2$, $g \in Z(f)$, and assume that $g$ is not an iterate of $f$. Denote by $(\mu_i)_{i \in I}$ the ratios of the homotheties associated to $g$ under the linearizing maps $k_s$, $k_u$. Then, as $g$ does not belong to the compact part of $Z(f)$ at $p$, the ratios $\mathrm{Log}|\mu_i|/\mathrm{Log}|\lambda_i(f)|$ are not all equal. Replacing, if necessary, $g$ by $g^k f^l$, with $k, l \in \mathbb{Z}$, we can assume that $Dg(p)$ is hyperbolic, but the stable manifold $W^s(g,p)$ of $g$ at $p$ is distinct from the stable manifold $W^s(f,p)$ and the unstable manifold $W^u(f,p)$ of $f$ at $p$. Indeed this is true when none of these ratios is zero and they take both signs; and this last condition is satisfied by $g^k f^l$ (for some $k \geq 1$, $l \in \mathbb{Z}$) if $l/k$ is such that the ratios for $g$ take values on both sides of $l/k$, none being equal to $l/k$. Replacing, if necessary, $f$ by $f^{-1}$, we can assume that $W^s(f,p) \cap W^s(g,p) = W_J(f)$ for some non trivial subset J of $I_1 \cup I_2$.

The map $h_f g h_f^{-1} = B$ commutes with A and fixes $O$, hence it belongs to $GL_n(\mathbb{Z})$ (*see* §2). Moreover, as $Dg(p)$ is hyperbolic, no sufficiently small neighbourhood of $p$ contains a full orbit of $g$; the same must be true for B at $O$, which means that B is hyperbolic. Then $h_f(W_J(f))$ is equal to the intersection of the stable subspaces of A and B, a linear subspace with the same dimension as $\mathbb{K}_J$. This contradicts the definition of $\mathscr{V}_1$ and shows that any $f \in \mathscr{V}_2$ has trivial centralizer. ■

## 5. A statement about the induced linear automorphism

To prove that $\mathscr{V}_1$ is dense in $\mathscr{V}$, we consider some $(f,p) \in \mathscr{V}$, and a non trivial invariant subset J of $I_3 \cup I_4$, such that $h_f(W_J(f))$ is equal to a non trivial A-invariant linear subspace of the unstable subspace of A. We will show that there exist arbitrarily small perturbations (in the $C^\infty$ topology) $f'$ of $f$ such that the linear subspace of $\mathbb{R}^n$ generated by $h_{f'}(W_J(f))$ is equal to the unstable subspace of A.

We choose a minimal non zero A-invariant subspace E of $h_f(W_J(f))$; the dimension of E is 1 or 2, and the restriction of A to E is a similitude of ratio $\lambda \in \mathbb{C}$, with $|\lambda| > 1$. We denote by $E^s$ (resp. $E^u$) the stable (resp. unstable) subspace of A in $\mathbb{R}^n$, and also the stable (resp. unstable) manifold $W^s(A,0)$ in $\mathbb{T}^n$ [resp. $W^u(A,0)$]. We choose an inner product

$\langle \, , \, \rangle$ in $\mathbb{R}^n$, with associated norm $\| \; \|$, and a constant $\alpha \in (0, 1)$ such that the following properties hold:

(1) $E^s$ and $E^u$ are orthogonal for $\langle \, , \, \rangle$;

(2) $\| A v \| \leq \alpha \| v \|$, $\| A^{-1} w \| \leq \alpha \| w \|$ for $v \in E^s$, $w \in E^u$;

(3) $\| A v \| = | \lambda | \| v \|$ for $v \in E$;

(4) $\| v \| \geq 1$ for $v \in \mathbb{Z}^n - \{ 0 \}$.

Denote by $d$ the distance in $\mathbb{T}^n$ induced by $\| \; \|$. The restrictions of $\| \; \|$ to $E^s$, $E^u \subset \mathbb{R}^n$ define distances $d_s$, $d_u$ on $E^s$, $E^u$ and we denote by the same letters the images of these distances by the injections of $E^s$, $E^u$ in $\mathbb{T}^n$. The closed ball in $\mathbb{T}^n$ (resp. $E^s \subset \mathbb{T}^n$, resp. $E^u \subset \mathbb{T}^n$) with center $x$ and radius $r$ for the distance $d$ (resp. $d_s$, resp. $d_u$) will be denoted by $B(x, r)$ [resp. $B_s(x, r)$, resp. $B_u(x, r)$].

A main point in the proof of Proposition 2 is to find a certain number of homoclinic points for A near E in $E^u$, with "non interacting" orbits; this is made precise in the following statement.

PROPOSITION 3. — *Let* K *be a neighbourhood of* O *in* $E^u$, *and* m *be the dimension of* $E^u$. *There exist* $\delta_0 > 0$, *and, for any* $\varepsilon_0 > 0$, *points* $z_1, \ldots, z_m$ *in* E, $y_1, \ldots, y_m \in E^u$ *such that the following properties hold:*

  (i) $z_i \in K$, *for* $1 \leq i \leq m$;

  (ii) $d_u(z_i, y_i) < \varepsilon_0$, *for* $1 \leq i \leq m$;

  (iii) $y_1, \ldots, y_m$ *is a basis of* $E^u$;

  (iv) $y_i$ *is homoclinic for* A *in* $\mathbb{T}^n$, *for* $1 \leq i \leq m$;

  (v) $A^n(y_i) \notin B(z_j, \delta_0) = B_j$ *for* $n > 0$, $1 \leq i, j \leq m$;

  (vi) $B_i \cap A^n(B_u(z_j, \delta_0)) = \varnothing$ *for* $n < 0$, $1 \leq i, j \leq m$;

  (vii) $B_i \cap B_j = \varnothing$ *for* $i \neq j$.

We first assume Proposition 3 to be true and based on it we provide a proof of Proposition 2. We then will be left with proving Proposition 3, where only the hyperbolic linear automorphism A is involved.

## 6. Proof of Proposition 2

6.1. We first define the kind of perturbations of $f$ we will consider.

We define the following subsets of $\mathbb{R}^n$:

$$K_1 = \{ x \in \mathbb{R}^n, | x_i | \leq 1 \text{ for } 1 \leq i \leq n \},$$

$$K_2 = \{ x \in \mathbb{R}^n, | x_i | \leq 1 \text{ for } 1 \leq i \leq m \text{ and } x_i = 0 \text{ for } m < i \leq n \},$$

$$K_3 = \left\{ x \in \mathbb{R}^n, | x_i | \leq \frac{1}{2} \text{ for } 1 \leq i \leq m \text{ and } x_i = 0 \text{ for } m < i \leq n \right\},$$

where $x = (x_1, \ldots, x_n)$. Let $|x| = \left( \sum\limits_{i=1}^{n} x_i^2 \right)^{1/2}$. We fix some $C^\infty$ function $\eta : [0, +\infty) \to [0,1]$, satisfying $\eta(t) = 0$ for $t \geq 1$ and $\eta(t) = 1$ for $t \leq 1/2$.

Let $\delta \in (0, 1/2)$, $q_1, \ldots, q_l \in K_3$, $q_1', \ldots, q_l' \in K_2$ $(l \geq 1)$ be such that:

(1) $$|q_i - q_j| \geq 2\delta \quad \text{for} \quad i \neq j;$$

(2) $$|q_i - q_i'| \leq \frac{\delta}{2} \quad \text{for all} \quad 1 \leq i \leq l.$$

Then, the $C^\infty$ vector field X in $\mathbb{R}^n$ defined by

$$X(y) = \sum_{j=1}^{l} \eta(\delta^{-1}|y - q_j|)(q_j' - q_j)$$

has support contained in the interior of $K_1$; it is tangent to $K_2$ along $K_2$, and for a given $y \in K_1$, by (1) at most one term in the sum defining $X(y)$ is non zero. If $(F_t)_{t \in \mathbb{R}}$ is the flow generated by X, one has, for $1 \leq i \leq l$,

$$F_1(q_i) = q_i'.$$

Finally, given a neighbourhood $\tilde{\mathcal{W}}$ of the identity in $\mathrm{Diff}(\mathbb{R}^n)$, there exists $\varepsilon = \varepsilon(\delta, \tilde{\mathcal{W}}) < \delta/2$ such that if we have the stronger condition

(2') $$|q_i - q_i'| \leq \varepsilon \quad \text{for} \quad 1 \leq i \leq l,$$

then $F_1$ belongs to $\tilde{\mathcal{W}}$.

6.2. We choose a diffeomorphism H of $K_1$ onto a compact neighborhood of $p$ in $\mathbb{T}^n$, such that $H(0) = p$ and $H(K_2) \subset W^u(f, p)$. Let $\delta_0 > 0$ be as in Proposition 3 with $K = H(K_3)$. Let $\delta \in (0, 1/2)$ be such that $|y - z| \leq \delta$, for $y, z \in K_1$, implies that $d(h_f H(y), h_f H(z)) < \delta_0$.

Let $\mathcal{W}$ be a neighbourhood of $f$ in the $C^\infty$ topology. We choose $\varepsilon = \varepsilon(\delta, \mathcal{W}) < \delta/2$ such that if the data in (6.1) satisfy (1) and (2'), then $HF_1 H^{-1} f$ belongs to $\mathcal{W}$. Then, by uniform continuity, there exists $\varepsilon_0 > 0$ such that if $y, z \in E^u$ satisfy $h_f^{-1}(z) \in K$, $d_u(y, z) < \varepsilon_0$, then $h_f^{-1}(y) \in H(K_2)$ and $|H^{-1} h_f^{-1}(y) - H^{-1} h_f^{-1}(z)| < \varepsilon$. For this $\varepsilon_0 > 0$ we apply Proposition 3 and get points $y_1, \ldots, y_m \in E^u$, $z_1, \ldots, z_m \in E$ satisfying properties (i)-(vii). We now construct as in (6.1) a flow $(F_t)_{t \in \mathbb{R}}$ with data $\delta, l = m$, $q_i = H^{-1} h_f^{-1}(z_i)$ and $q_i' = H^{-1} h_f^{-1}(y_i)$. For $0 \leq t \leq 1$, let $f_t = HF_t H^{-1} f \in \mathcal{W}$; we may assume that $\mathcal{W} \subset \mathcal{D}_A(\mathbb{T}^n)$ and denote by $h_t$ the global conjugacy of $f_t$ and A (so that $h_0 = h_f$).

6.3. We finally prove that $f_1$ belongs to $\mathcal{V}_1$, and more precisely that the linear subspace of $\mathbb{R}^n$ generated by $h_1(W_J(f_1))$ is $E^u$. As $\mathcal{W}$ is arbitrary and $f_1 \in \mathcal{W}$, this will show that $\mathcal{V}_1$ is dense in $\mathcal{V}$.

With the notation of Proposition 3, the image under $h_f$ of the support of $HF_t H^{-1}$ $(0 \leq t \leq 1)$ is contained in $\bigcup\limits_{i=1}^{l} B_i$ (by the definition of $\delta$); by property (vi), $p$ does not

belong to the support of $HF_t H^{-1}$ and the manifolds $W_J(f)$, $W_J(f_1)$ coincide in a neighbourhood of $p$ in $W^u(f, p)$. By property (v), we also have, for $1 \leq i \leq m$, $0 \leq t \leq 1$ and $n \geq 0$:

$$f_t^n(h_f^{-1}(y_i)) = f^n(h_f^{-1}(y_i)) = h_f^{-1}(A^n(y_i)),$$

hence $h_f^{-1}(y_i)$ belongs to $W^s(f_t, p)$. On the other hand, by relation (vi), we have, for $1 \leq i \leq m$, $0 \leq t \leq 1$, and $n < 0$:

$$(3) \qquad f_t^n(h_f^{-1}(y_i)) = f^n HF_t^{-1} H^{-1} h_f^{-1}(y_i).$$

But $H^{-1} h_f^{-1}(y_i)$ belongs to $K_2$, hence $HF_t^{-1} H^{-1} h_f^{-1}(y_i)$ belongs to $W^u(f, p)$, and we conclude that $h_f^{-1}(y_i)$ belongs to $W^u(f_t, p)$, and indeed it is homoclinic for $f_t$.

As the set of homoclinic points of $A$ is totally discontinuous, and $h_t$ depends continuously on $t$, the map $t \to h_t(h_f^{-1}(y_i))$ is constant and equal to $h_0 h_f^{-1}(y_i) = y_i$. By construction of $F_1$, we have $HF_1 H^{-1}(h_f^{-1}(z_i)) = h_f^{-1}(y_i)$ for $1 \leq i \leq m$, hence by relation (3) we have that $f_1^n(h_f^{-1}(y_i)) = f^n(h_f^{-1}(z_i))$ for $1 \leq i \leq m$, $n < 0$. But $h_f^{-1}(z_i)$ belongs to $W_J(f) \supset h_f^{-1}(E)$, and $W_J(f)$ and $W_J(f_1)$ coincide near $p$ in $W^u(f, p)$, hence $h_f^{-1}(y_i)$ belongs to $W_J(f_1)$ for $1 \leq i \leq m$. As $h_1 h_f^{-1}(y_i) = y_i$, we get the conclusion we were after, by property (iii) of Proposition 3. This concludes the proof of Proposition 2. ∎

It remains to show Proposition 3, which will be done in the next section.

# 7. Proof of Proposition 3

7.1. The lemma we now state will be used at a crucial step in the proof of Proposition 3. Let $\theta > 1$, $N_0 \in \mathbb{N}^* = \mathbb{N} - \{0\}$ be such that $\theta^{N_0} > 6(N_0 + 1)$. Define $l_0 = 2(1/(\theta - 1) + N_0(\theta - 1)/3\theta)$, and denote by $T$ the homothety of ratio $\theta$ in $\mathbb{R}$. Let $k \in \mathbb{N}^*$, and suppose we are given, for each $1 \leq i \leq k N_0$, a family $(L_{i,j})$ of intervals of length 1.

LEMMA. — Let $l \geq l_0$; assume that for each $1 \leq i \leq k N_0$ the centers of the intervals $L_{i,j}$ are mutually distant apart by at least $3 \theta l/(\theta - 1)$. Then, the set:

$$L = T^{k N_0}([0, l]) - \bigcup_{i=1}^{k N_0} \bigcup_j T^{k N_0 - i}(L_{i,j})$$

contains an interval of length $l$.

Proof. — We prove the assertion for $k = 1$; the assertion for $k \geq 2$ follows by iterating $k$-times the process. The number of intervals $L_{i,j}$ (with fixed $i$, $1 \leq i \leq N_0$) contained in the interior of $[0, \theta^i l]$ is at most $[\theta^i(\theta - 1)/3\theta] + 1$, hence the number of components of $L$ is less than:

$$1 + \sum_{i=1}^{N_0} \left( \left[ \frac{\theta - 1}{3\theta} \theta^i \right] + 1 \right) \leq 1 + N_0 + \frac{1}{3} \theta^{N_0} < \frac{1}{2} \theta^{N_0}.$$

On the other hand, the Lebesgue measure of the intersection $[0, \theta^i l] \cap (\bigcup_j L_{i,j})$ is less than $[\theta^i (\theta - 1)/3 \theta] + 1$, hence the measure of $[0, \theta^{N_0} l] - L$ is less than:

$$\sum_{i=1}^{N_0} \theta^{N_0 - i} \left( \left[ \frac{\theta - 1}{3 \theta} \theta^i \right] + 1 \right) \leq \theta^{N_0} \left( \frac{1}{\theta - 1} + N_0 \frac{\theta - 1}{3 \theta} \right) = \frac{1}{2} l_0 \theta^{N_0},$$

For $l \geq l_0$, the measure of L is therefore bigger than or equal to $(1/2) l \theta^{N_0}$ and the result follows.

7.2. We choose a unit vector $w$ in E and denote by D the orthogonal subspace (with respect to $\langle \, , \, \rangle$) of $\mathbb{R} w$ in $E^u$, and, for $\varepsilon \geq 0$, by $P_\varepsilon$ the endomorphism of $E^u$ equal to the identity on $\mathbb{R} w$ and to the multiplication by $\varepsilon$ on D. We may assume that $K \cap \mathbb{R} w = \{ tw, |t| \leq a \}$ for some $a > 0$. We pick $m$ distinct real numbers $u_1, \ldots, u_m$ in $((1/2) a |\lambda|^{-1}, (1/2) a)$ (recall that $A/E$ is a similitude of ratio $\lambda \in \mathbb{C}$) and choose a basis $t_1, \ldots, t_m$ of $E^u$ such that $P_0 (t_i) = u_i w$ for $1 \leq i \leq m$.

For $\delta_1 > 0$, $1 \leq i \leq m$, let $J_i$ (resp. $J_i'$) the line segment $\{ u_i w + tw, 0 \leq t \leq \delta_1 \}$ (resp. $\{ t_i + tw, 0 \leq t \leq \delta_1 \}$). Denote by $V_i$ (resp. $V_{i,u}$) the $\delta_1$-neighbourhood of $J_i$ in $\mathbb{T}^n$ (resp. $E^u$) for the distance $d$ (resp. $d_u$). We choose $\delta_1 \in (0, (1/4) a)$ sufficiently small so that the following properties hold:

(4)                        $J_i \subset K$,     for $1 \leq i \leq m$;

(5)                        $\bar{V}_i \cap \bar{V}_j = \varnothing$,     $0 \notin \bar{V}_i$  for $1 \leq i \leq m$;

(6)                        $\bar{V}_i \cap A^n (\bar{V}_{j,u}) = \varnothing$  for all $1 \leq i, j \leq m$,   $n < 0$;

(7)  if $y_i \in J_i'$ for $1 \leq i \leq m$, then $y_1, \ldots, y_m$ is a basis of $E^u$.

Next we choose an open neighbourhood $\mathcal{O}$ of $O$ in $E^s$ such that $A(\mathcal{O}) \subset \mathcal{O}$ and $\mathcal{O} \cap \bar{V}_i = \varnothing$ for $1 \leq i \leq m$; let $\beta > 0$ be such that for any $x \in E^u$, the ball $B_u(x, \beta)$ intersects $\mathcal{O}$. Define $r = 2a + 4\beta$.

7.3. In the context of the Lemma in (7.1), let $\theta = |\lambda|$, and $N_0 \geq 1$ satisfying $\theta^{N_0} > 6(N_0 + 1)$. This defines $l_0$; let $N_1$ be an integer such that $\theta^{N_1} > l_0 r \delta_1^{-1}$. For $\delta_0 > 0$, consider the union $\tilde{B}(\delta_0)$ of the balls $B_u(q, a)$ for $q \in B_s(0, \delta_0) \cap E^u$. We choose $\delta_0 \in (0, \delta_1/2)$ small enough to have:

(8)  $A^n V_i' \cap V_j' = \varnothing$ for $0 < n \leq N_1$, $1 \leq i, j \leq m$, where $V_i'$ is the $\delta_0$-neighbourhood of $J_i$ in $\mathbb{T}^n$;

(9)  $d_u (q, q') \geq (3 \theta/(\theta - 1) l_0 + 1) r$ for distinct $q, q'$ in $B_s(0, \delta_0) \cap E^u$.

7.4. Let $\varepsilon_0 \in (0, \delta_0)$ be given (see Proposition 3). We choose $\varepsilon_1 > 0$ small enough so that we have:

(10)                    $\| P_{\varepsilon_1} (y) - P_0 (y) \| < \frac{1}{2} \varepsilon_0$     for  $y \in \bigcup_{i=1}^m J_i'$.

As $P_{\varepsilon_1}$ is a linear automorphism of $E^u$, any family $y_1, \ldots, y_m$ with $y_i \in P_{\varepsilon_1}(J_i')$ is a basis of $E^u$ [see property (7) above]. Therefore we can choose $0 < \varepsilon_2 < (1/2)\varepsilon_0$ such that:

(11)   any family $y_1, \ldots, y_m$, with $d_u(y_i, P_{\varepsilon_1}(J_i')) < \varepsilon_2$ for $1 \leq i \leq m$, is a basis of $E^u$.

We choose $k \in \mathbb{N}^*$ and define $N_2 = N_1 + k N_0$ such that $\alpha^{N_2}\beta < \varepsilon_2$ [where $\alpha$ is defined in property (2) of Section 5].

7.5. Fix $1 \leq i \leq m$; let $D_1$ be the linear subspace of $E^u$ orthogonal to $A^{N_2}w$ and let $B = B_{D_1}(0, \beta)$ be the closed ball in $D_1$ with center $O$ and radius $\beta$.

For $N_1 < j \leq N_2$, let $L_j$ be the set of points $x$ in $A^j(P_{\varepsilon_1}(J_i'))$ such that $x + A^{j-N_2}(B)$ intersects $\tilde{B}(\delta_0)$. The diameter of $A^{-N_2+j}(B)$ in $E^u$ is at most $2\beta$. If two points $x, x' \in L_j$ are such that $x + A^{j-N_2}(B)$ and $x' + A^{j-N_2}(B)$ intersect the same component of $\tilde{B}(\delta_0)$, we have:

(12)   $$d_u(x, x') \leq 2a + 4\beta = r.$$

Otherwise, by (9), we have:

(13)   $$d_u(x, x') \geq \frac{3\theta}{\theta-1} l_0 r.$$

By the definition of $N_1$, the length of $A^{N_1}(P_{\varepsilon_1}(J_i'))$ is at least $l_0 r$; relations (12) and (13) show that we can apply the Lemma in (7.1) and conclude that there exists $J_i''$ of length bigger than or equal to $l_0 r$ contained in $A^{N_2}(P_{\varepsilon_1}(J_i')) - \bigcup_{j=N_1+1}^{N_2} A^{N_2-j}(L_j)$. Observe that $l_0 \geq 1$, hence $l_0 r \geq 2\beta$. By definition of $\beta$, there exists some $y_i' \in J_i''$ such that $y_i' + B$ intersects $\mathcal{O}$ in some point $y_i''$. Let $y_i = A^{-N_2}(y_i'')$, $y_i''' = A^{-N_2}(y_i)$ and $z_i = P_0(y_i'')$.

7.6. We finally check for $y_i$, $z_i$ $(1 \leq i \leq m)$ the conclusions of Proposition 3. Clearly $y_i \in E^u$ and $z_i \in E$. Recall that $\mathcal{O} \subset E^s$.

PROPERTY (i). — We have $y_i''' \in P_{\varepsilon_1}(J_i')$, $P_0(P_{\varepsilon_1}(J_i')) = P_0(J_i') = J_i$, hence (i) is a consequence of (4).

PROPERTY (ii). — We have that

$$d_u(y_i, y_i'') < \alpha^{N_2}\beta < \varepsilon_2 < \frac{1}{2}\varepsilon_0,$$

$$d_u(y_i''', z_i) < \frac{\varepsilon_0}{2} \quad [\textit{see property (10) above}].$$

PROPERTY (iii). — This results from $d_u(y_i, y_i'') < \varepsilon_2$ and (11).

PROPERTY (iv). — Indeed $y_i'' \in E^u \cap \mathcal{O}$, and $y_i = A^{-N_2}(y_i'')$.

PROPERTY (v). — One has that $y_i \in B_i \subset V_i'$ for $1 \leq i \leq m$.

Let $1 \leq i, j \leq m$; for $1 \leq n \leq N_1$, $A^n(y_i)$ does not belong to $V'_j$ according to (8) above. The same holds for $n \geq N_2$ because then $A^n(y_i) = A^{n-N_2}(y''_i) \in \mathcal{O}$ and $\mathcal{O} \cap V'_j = \varnothing$. For $N_1 < n < N_2$, $A^n(y_i)$ does not belong to $\tilde{B}(\delta_0)$ by the construction of $J''_i$. As $\tilde{B}(\delta_0)$ contains $B_j \cap E''$, we have again $A^n(y_i) \notin B_j$.

PROPERTIES (vi), (vii). — These follow immediately from (5), (6) and $\delta_0 < \delta_1$.   $\square$

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