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DYNAMICS OF MEROMORPHIC MAPS: MAPS WITH POLYNOMIAL SCHWARZIAN DERIVATIVE

BY ROBERT L. DEVANEY AND LINDA KEEN

ABSTRACT. — We investigate the dynamics of a class of meromorphic functions, namely those with polynomial Schwarzian derivative. We show that certain of these maps have dynamics similar to rational maps while others resemble entire functions. We use classical results of Nevanlinna and Hille to describe certain aspects of the topological structure and dynamics of these maps on their Julia sets. Several examples from the families $z \to \lambda \tan z$, $z \to e^z/(\lambda e^z + e^{-z})$, and $z \to \lambda e^z/(e^z - e^{-z})$ are discussed in detail.

There has been a resurgence of interest in the past decade in the dynamics of complex analytic functions. Most of this work has centered around the dynamics of polynomials or rational maps [B1, DH, Ma, S] or entire transcendental functions [DK, DT, GK, BR]. Our goal is to extend some of this work to the meromorphic case, pointing out along the way some of the similarities and principal differences between this case and the other classes of maps.

One of the principal differences arising in the iteration of meromorphic (non-rational) functions is the fact that, strictly speaking, iteration of these maps does not lead to a dynamical system. Infinity is an essential singularity for such a map, so the map cannot be extended continuously to infinity. Hence the forward orbit of any pole terminates, and, moreover any preimage of a pole also has a finite orbit. All other points have well defined forward orbits.

Despite the fact that certain orbits of a meromorphic map are finite, the iteration of such maps is important. For example, the iterative processes associated to Newton's method applied to entire functions often yields a meromorphic function as the root-finder. *See* [CGS].

In this paper we will deal exclusively with a very special class of meromorphic functions, namely, those whose Schwarzian derivative is a polynomial. This class of maps includes a number of dynamically important families of maps, including $\lambda \tan z$ and $\lambda \exp z$.

The Schwarzian derivative has played a role in the analysis of dynamical systems in other settings. For example, if the Schwarzian of two C^3 maps of the interval is negative, the same is true for their composition. Singer [Si] has shown that the class of functions satisfying these conditions share many of the special properties of complex analytic maps. This has allowed Guckenheimer [G] to classify these maps in a dynamical sense. Our use of the Schwarzian derivative is completely different from these authors' approach.

The main property of maps with polynomial Schwarzian derivative that makes this class special was noted first by Nevanlinna [N]. These maps are precisely the maps that have only finitely many asymptotic values and no critical values. As is well known, the fate of the asymptotic values and critical values under iteration plays a crucial role in determining the dynamics. A further important fact about these maps concerns the covering properties of the map itself. Hille [H] has shown that the plane may be decomposed into exactly p sectors of equal angle (when the Schwarzian derivative has degree $p-2$) each of which is associated to one of the asymptotic values. This structure theorem allows us to say much about the Julia set of the map.

The plan of this paper is as follows. In paragraph 1 we recall the classical results of Nevanlinna and Hille. In paragraph 2-4 we present a series of three illustrative examples drawn from the families $\lambda \tan z$, $e^z/(\lambda e^z + e^{-z})$, and $\lambda e^z/(e^z - e^{-z})$. In paragraph 5 we show that many of the standard properties of the Julia sets of polynomials, rational maps, and entire functions carry over to our case. In particular, we show that these maps never have wandering domains or domains at infinity, so that the Fatou–Sullivan classification of the complement of the Julia set holds for these maps. In paragraph 6 we discuss several theorems which indicate the similarity of the Julia sets of our maps with those of rational functions. For example, if all of the asymptotic values of the map lie in a single immediate basin of attraction of a fixed point, then the Julia set of the map is a Cantor set. We show that the restriction of the map to the Julia set is equivalent to a shift map on infinitely many symbols which is appropriately modified and augmented to account for finite orbits.

Other results show that certain of these maps have several properties in common with entire functions. For example, it is possible for one of the asymptotic values to be a pole. If this is the case, we show that the Julia set of the map contains Cantor bouquets, a typical phenomenon encountered in the study of entire maps. In paragraph 3, we present an example of inhomogenity in Julia sets; in this example, the Julia set is not the entire Riemann sphere: certain of the points lie on analytic curves in it, but others do not.

1. Classical Results

If F (z) is a meromorphic function, its Schwarzian derivative is defined by

$$\{F, z\} = \frac{F'''(z)}{F'(z)} - \frac{3}{2}\left(\frac{F''(z)}{F'(z)}\right)^2.$$

Associated to the Schwarzian differential equation

$$(*) \qquad \{F, z\} = Q(z)$$

is a linear differential equation obtained by setting

$$g(z) = (F'(z))^{-1/2}.$$

The resulting equation is

$$(**) \qquad g'' + \frac{1}{2} Q(z) g = 0.$$

If g_1, g_2 are linearly independent (locally defined) solutions of $(**)$, their Wronskian is a non-zero constant k. Since

$$\left(\frac{g_1}{g_2}\right)' = \frac{k}{g_2^2},$$

it follows that $F(z)=g_1(z)/g_2(z)$ is a solution of $(*)$. Conversely, each solution of $(*)$ may be written locally as a quotient of independent solutions of $(**)$.

If $Q(z)$ is a polynomial, there is a wide class of maps which satisfy $(*)$. These include such maps as $\lambda \tan z$, $\lambda \exp z$, and $\int^z \exp(R(u))\,du$ where R is a polynomial.

Results of Nevanlinna [N] and Hille [H] allow us to describe the asymptotic properties of the solutions to $(**)$ when Q is a polynomial of degree $p-2$. There are exactly p special solutions, $G_0 \dots G_{p-1}$, called *truncated* solutions, which have the following property: in any sector of the form

$$\left|\arg z - \frac{2\pi\nu}{p}\right| < \frac{3\pi}{p} - \varepsilon$$

with $\varepsilon > 0$, $G_\nu(z)$ has the asymptotic development

$$\log G_\nu(z) \sim (-1)^{\nu+1} z^{p/2}.$$

Each G_ν is an entire function of order $p/2$. It follows that each G_ν tends to zero as $z \to \infty$ along each ray in a sector W_ν of the form

$$\left|\arg z - \frac{2\pi\nu}{p}\right| < \frac{\pi}{p}.$$

Moreover, $G_\nu(z) \to \infty$ in the adjacent sectors $W_{\nu+1}$ and $W_{\nu-1}$. Note that G_ν and $G_{\nu+1}$ are necessarily linearly independent. However, G_ν and $G_{\nu+k}$ for $|k| \geqq 2$ need not be independent.

Any solution of the associated Schwarzian equation may therefore be written in the appropriate sector in the form

$$(\dagger) \qquad \frac{AG_\nu(z) + BG_{\nu+1}(z)}{CG_\nu(z) + DG_{\nu+1}(z)} = F(z)$$

with $AD - BC \neq 0$. Note that $F(z)$ tends to A/C along any ray in the interior of $W_{\nu+1}$ and to B/D in W_ν. The values A/C and B/D are called asymptotic values. Recall that an asymptotic (or critical) path for a function $F(z)$ is a curve $\alpha : [0, 1) \to C$ such that

$$\lim_{t \to 1} \alpha(t) = \infty$$

and

$$\lim_{t \to 1} F(\alpha(t)) = \omega.$$

The point ω is called an asymptotic value of F.

Asymptotic values can be classified. Nevanlinna's results show that our assumption that Q is a polynomial implies that F has only finitely many asymptotic values. They are therefore all isolated. Let B be a neighborhood of ω which contains no other asymptotic values. Consider the components of $F^{-1}(B-\omega)$. Since the only points at which F is not a covering of its image are the asymptotic values, F is a covering map on these components. Hence these components are either disks or punctured disks. If some component is a disk, then ω is called a *logarithmic singularity*.

Example. — Let $F(z)=\tan z$. Then i and $-i$ are logarithmic singularities. F maps the half plane $\operatorname{Im} z>y_0>0$ onto a punctured neighborhood of i. The image of any path $\alpha(t)$ such that $\lim_{t\to 1} \operatorname{Im}\alpha(t)=\infty$ is a path $\beta(t)$ such that $\lim_{t\to 1}\beta(t)=i$. Similarly the image of a lower half plane $\operatorname{Im} z<v_1<0$ is mapped onto a punctured neighborhood of $-i$. ∞ is an accumulation point of the poles; it is not an asymptotic value.

Since a map with a polynomial Schwarzian derivative assumes the special form (†) in each sector W_v, it follows that such a map has exactly p asymptotic values. Two or more of these values may coincide, but in this case, non-adjacent sectors of asymptotic paths correspond to this value. Also, F has no critical points since

$$F'(z)=\frac{k}{g_2^2(z)}$$

where k is a constant and g_2 is entire. We summarize these facts in a theorem originally proved by Nevanlinna [N].

THEOREM. — *Maps with degree $p-2$ polynomial Schwarzian derivatives are exactly those functions which have p logarithmic singularities, $a_0, \ldots, a_{p-1}$. The a_i need not be distinct. There are exactly p disjoint sectors $W_0, \ldots, W_{p-1}$, at ∞, each with angle $2\pi/p$ in which F has the following behavior: there is a collection of disks B_i, one around each of the a_i, satisfying $F^{-1}(B_i-a_i)$ contains a unique unbounded component $U_i\subset W_i$ and $F: U_i\to B_i-a_i$ is a universal covering.*

The U_i are called exponential tracts. Since the truncated solutions tend to 0 or ∞ in adjacent sectors, it follows that the boundary curve of each U_i has as asymptotic directions the pair of rays which bound the sectors W_i. Recall that a ray β is called a Julia ray for F if, in any angle about β, F assumes all (but at most one) values infinitely often. See *Fig.* 1. It is an immediate consequence of (†) that the ray β_i bounding a pair of adjacent sectors W_{i-1} and W_i for F is a Julia ray.

Example. — The positive and negative imaginary axes are Julia rays for e^z. Similarly, the rays

$$\arg z=\frac{(2k+1)\pi}{2p}$$

for $k\in\mathbf{Z}$ are Julia rays for $\exp(z^p)$.

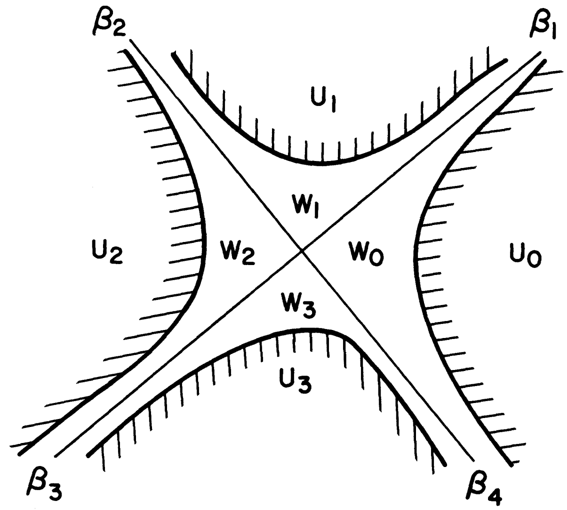

Fig. 1. – Exponential tracts and Julia rays.

When F is meromorphic, the arguments of the poles of F accumulate to the argument of the Julia rays. Therefore, except for finitely many poles, it is possible to associate to a given pole p, that particular Julia ray, $\beta_i(p)$, such that the pole lies in a small angle about $\beta_{i(p)}$. We also associate two asymptotic values to each pole; $v_1(p)$ is the asymptotic value corresponding to $U_{i(p)-1}$ and $v_2(p)$ is the asymptotic value corresponding to $U_{i(p)}$.

Example. – The positive and negative axes are Julia rays for tan z, the poles are contained in the rays and $v_1(p)=i$, $v_2(p)=-i$ for each of the positive poles while $v_1(p)=-i$, $v_2(p)=i$ for each of the negative poles.

2. The tangent family

In this and the next two sections, we will describe the dynamics of some specific and important families of maps with constant Schwarzian derivatives. We will concentrate on the structure of the Julia set of such a map. We call a point z *stable* for F if there is a neighborhood U of z such that the iterates F^n are uniformly bounded on U. The metric here is the standard Euclidean metric. The Julia set of F, denoted by J(F), is the complement of the stable set. We will see in paragraph 5 that the Julia set of a meromorphic function with polynomial Schwarzian derivative has two additional equivalent formulations:

1. J(F) is the closure of the set of repelling periodic orbits.

2. J(F) is the closure of the set consisting of the poles of F^n for $n>0$.

We will use both of these characterizations in the next three sections.

Consider the equation

$$(**) \qquad \{F, z\} = k$$

where $k \in \mathbf{R} - \{0\}$. By the results of paragraph 1, any map with constant Schwarzian derivative is affine conjugate to a map in this class.

The truncated solutions of $(**)$ are given by

$$e^{\pm \sqrt{E - k/2}\, z}$$

and the general solution is

$$\frac{A\, e^{\sqrt{-k/2}\, z} + B\, e^{-\sqrt{-k/2}\, z}}{C\, e^{\sqrt{-k/2}\, z} + D\, e^{-\sqrt{-k/2}\, z}}$$

with $AD - BC \neq 0$. Two of these parameters can be fixed by affine conjugation. We will consider one parameter subfamilies of this family in this and the next two sections.

Let

$$T_\lambda(z) = \lambda \tan z = \frac{\lambda}{i} \frac{e^{iz} - e^{-iz}}{e^{iz} + e^{-iz}}$$

where $\lambda > 0$. We have $\{T_\lambda, z\} = 2$. T_λ has asymptotic values at $\pm \lambda i$, and T_λ preserves the real axis.

PROPOSITION. — *If $\lambda \in \mathbf{R}$, $\lambda > 1$, then $J(T_\lambda)$ is the real line and all other points tend asymptotically to one of two fixed sinks located on the imaginary axis.*

Proof. — Write $T_\lambda(z) = L_\lambda \circ E(z)$ where

$$E(z) = \exp(2iz)$$

$$L_\lambda(z) = -\lambda i \left(\frac{z-1}{z+1}\right).$$

E maps the upper half plane onto the unit disk minus 0 and L_λ maps the disk back to the upper half plane. Both E and L_λ preserve boundaries, so T_λ maps the interior of the upper half plane into itself. Now T_λ also preserves the imaginary axis and we have

$$T_\lambda(iy) = i\lambda \tanh(y)$$

The graph of λ tanh y shows that T_λ has a pair of attracting fixed points located symmetrically about 0 if $\lambda > 1$. See *Fig.* 2. By the Schwarz Lemma, all points in the upper (resp. lower) half-plane tend under iteration to one of these points.

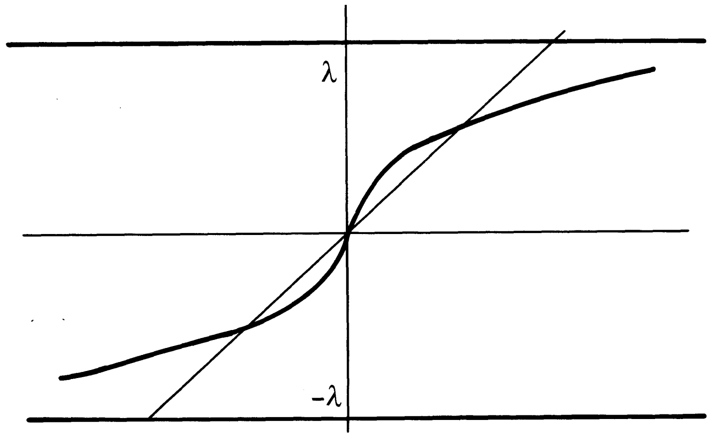

Fig. 2. — The graph of λ tanh x when $\lambda > 1$.

Hence neither the upper nor the lower half plane is in $J(T_\lambda)$. The real line is in $J(T_\lambda)$. This follows from the facts that the real line satisfies $T_\lambda^{-1}(\mathbf{R}) \subset \mathbf{R}$ and $T_\lambda(\mathbf{R}) = \mathbf{R} \cup \infty$, and that $T'_\lambda(x) > 1$ for all $x \in \mathbf{R}$ if $\lambda > 1$ ($T'_\lambda(x) \geqq 1$ if $\lambda = 1$). Each interval of the form

$$\left(\frac{2k-1}{2}\pi, \frac{2k+1}{2}\pi\right)$$

is expanded over all of $\mathbf{R}$. If U is any open interval in $\mathbf{R}$, then there is an integer k such that $T_\lambda^k(U)$ covers one of these intervals of length π. Hence $T_\lambda^{k+1}(U)$ covers U. It follows that there exist repelling fixed points and poles of T_λ^{k+1} in U.

Q.E.D.

Remarks. — 1. If $\lambda = 1$, then $J(T_\lambda) = \mathbf{R}$, and all points with non-zero imaginary parts tend asymptotically to the neutral fixed point at 0.

2. When $\lambda < -1$, the dynamics of T_λ are similar to those for $\lambda > 1$, except that T_λ has an attracting periodic cycle of period two. Points in the upper and lower half-planes hop back and forth as they are attracted to the cycle. Since $|T'_\lambda(x)| > 1$ for $x \in \mathbf{R}$, it follows as above that $J(T_\lambda) = \mathbf{R}$ for $\lambda < -1$.

For $0 < |\lambda| < 1$, 0 is an attracting fixed point for T_λ. In this case, the Julia set of T_λ breaks up into a Cantor set, as we show below. We will employ symbolic dynamics to describe the Julia set in this case. Let Γ denote the set of one-sided sequences whose entries are either integers or the symbol ∞. If ∞ is an entry in a sequence, then we terminate the sequence at this entry, *i.e.*, Γ consists of all infinite sequences

$(s_0, s_1, s_2, \ldots)$ where $s_j \in \mathbf{Z}$ and all finite sequences of the form $(s_0, s_1, \ldots, s_j, \infty)$ where $s_i \in \mathbf{Z}$.

The topology on Γ was described in [Mo]. For completeness, we will recall this topology here. If $(s_0, s_1, s_2, \ldots)$ is an infinite sequence, we choose as a neighborhood basis of this sequence the sets.

$$\mathrm{U}_k = \{ (t_0, t_1, \ldots) \mid t_i = s_i \text{ for } i \leqq k \}.$$

If, on the other hand, the sequence is finite $(s_0, \ldots, s_j, \infty)$, then we choose the U_k as above for $k<j$ as well as sets of the form

$$\mathrm{V}_l = \{ (t_0, t_1, \ldots) \mid t_i = s_i \text{ for } i \leqq j \text{ and } |t_{j+1}| \geqq l \}$$

for a neighborhood basis.

There is a natural map $\sigma : \Gamma \to \Gamma$ called the shift automorphism which is defined by $\sigma(s_0 s_1 s_2 \ldots) = (s_1 s_2 \ldots)$. Note that $\sigma(\infty)$ is not defined. In Moser's topology, σ is continuous and Γ is a Cantor set. The pair Γ, σ is often called the shift on infinitely many symbols. Γ provides a model for many of the Julia sets of maps in our class, and $\sigma \mid \Gamma$ is conjugate to the action of F on J(F). One such instance of this is shown in the following proposition.

PROPOSITION. — *Suppose* $\lambda \in \mathbf{R}$ *and* $0 < |\lambda| < 1$. *Then* $\mathrm{J}(\mathrm{T}_\lambda)$ *is a Cantor set in* $\hat{\mathbf{C}}$ *and* $\mathrm{T}_\lambda \mid \mathrm{J}(\mathrm{T}_\lambda)$ *is topologically conjugate to* $\sigma \mid \Gamma$.

Proof. – Since $0 < |\lambda| < 1$, 0 is an attracting fixed point for T_λ. Let B denote the component of the basin of attraction of 0 in $\mathbf{R}$. B is an open interval of the form $(-p, p)$ where $\mathrm{T}_\lambda(\pm p) = \pm p$. (The points $\pm p$ lie on a periodic orbit of period two if $-1 < \lambda < 0$). The preimages $\mathrm{T}_\lambda^{-1}(\mathrm{B})$ consist of infinitely many disjoint open intervals. Let L_j, $j \in \mathbf{Z}$, denote the complementary intervals, enumerated left to right so that I_0 abuts p. See *Fig.* 3. Then $\mathrm{T}_\lambda : \mathrm{I}_j \to (\mathbf{R} \cup \infty) - \mathrm{B}$ for each j, and $|\mathrm{T}'_\lambda(x)| > 1$ for each $x \in \mathrm{I}_j$. Standard arguments [Mo] then show that

$$\Lambda = \{ x \in \mathbf{R} \cup \{ \infty \} \mid \mathrm{T}_\lambda^j(x) \in \cup I_j \text{ for all } j \}$$

is a Cantor set and $\mathrm{T}_\lambda \mid \Lambda$ is conjugate to $\sigma \mid \Gamma$.

Now Λ is invariant under all branches of the inverse of T_λ. It therefore contains preimages of poles of all orders and is closed. Hence Λ is the Julia set of T_λ. The classification of stable regions tells us that all other points lie in the basin of 0.

Q.E.D.

Remarks. – 1. The basin of 0 is therefore infinitely connected. This contrasts with the situation for polynomial or entire maps in which *finite* attracting fixed points always have a simply connected immediate basin of attraction.

2. The arguments above will be used later to show that Cantor sets appear often as the Julia sets of meromorphic maps with polynomial Schwarzian derivatives.

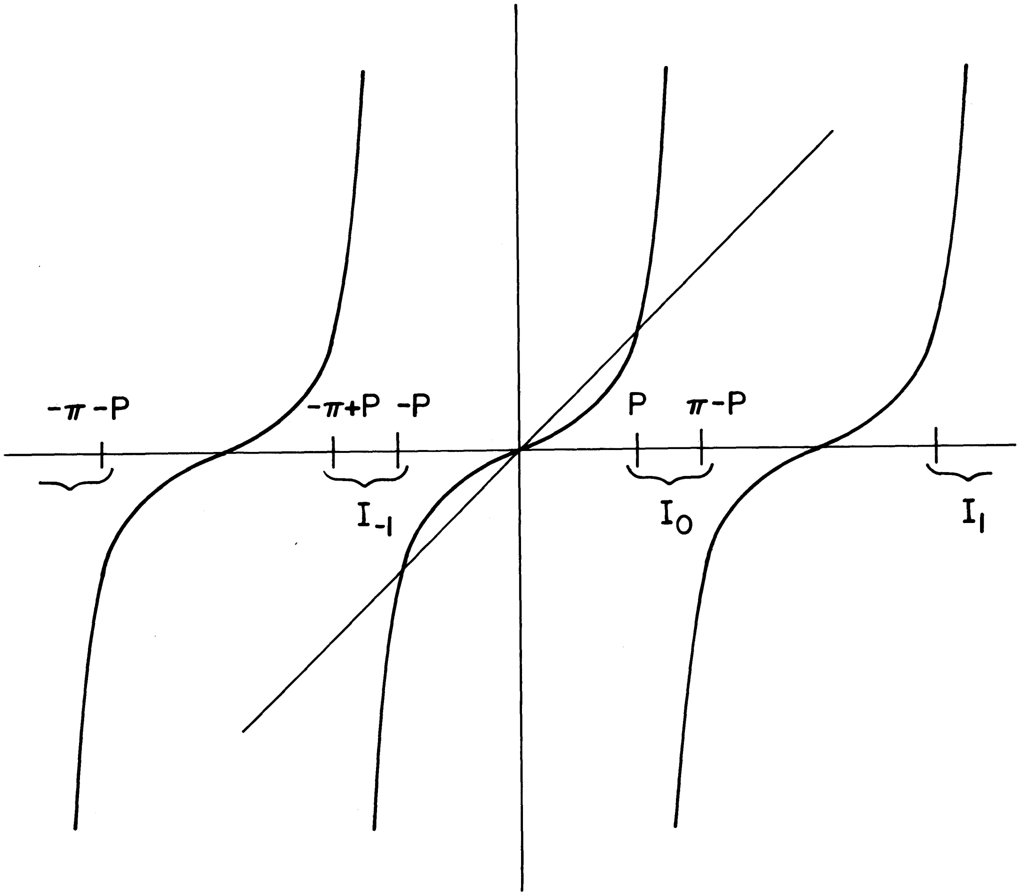

Fig. 3. — The intervals I_j.

The Julia set of T_λ is a similar Cantor set for all λ with $|\lambda|<1$, as shown in the following Proposition.

PROPOSITION. — *Suppose* $0<|\lambda|<1$. *Then* $J(T_\lambda)$ *is a Cantor set and each* T_λ *is quasi-conformally conjugate to* $T_{1/2}$.

Proof. — We use techniques of Teichmüller theory for the proof. The origin is the only attracting fixed point of T_λ, $|\lambda|<1$. There is a holomorphic map φ_λ from the unit disk U to a neighborhood of the origin so that

$$\psi_\lambda = \varphi_\lambda \circ T_\lambda \circ \varphi_\lambda^{-1} \text{ has the form } \zeta \to \lambda\zeta.$$

Therefore we can find an annulus $A_\lambda = \{\zeta \mid r\lambda < |\zeta| < r\}$ which is mapped by ψ_λ onto the annulus $A'_\lambda = \{\zeta \mid (r\lambda)^2 < |\zeta| < r\lambda\}$.

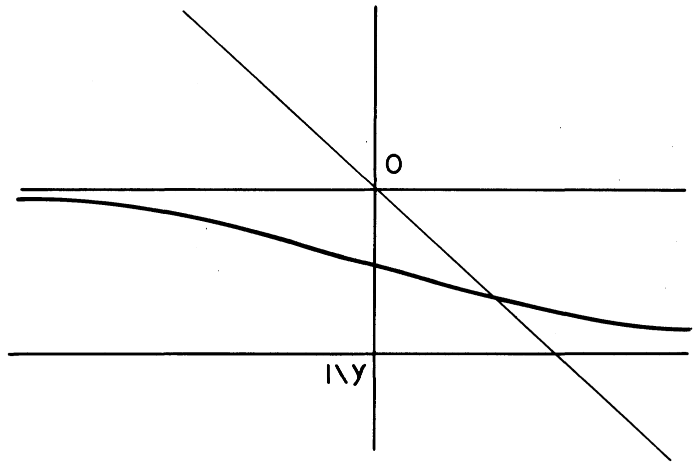

Fig. 4. — The graph of F_λ.

Define a Beltrami differential (a mesurable structure) on $A_{1/2}$ as follows. Let $f: A_{1/2} \to A_\lambda$ be a differentiable map such that

$$\hat{\mu}(\zeta) = \frac{f_{\bar{\zeta}}}{f_\zeta}$$

is a measurable function and ess sup $|\hat{\mu}(\zeta)| \leqq k < 1$. Extend $\hat{\mu}$ to U using the map $\psi_{1/2}$ by

$$\hat{\mu}(\psi_{1/2}(\zeta)) = \hat{\mu}(\zeta).$$

Pull this Beltrami differential back to the neighborhood of 0 by

$$\mu(z) = \hat{\mu} \circ \varphi_{1/2}.$$

Extend $\mu(z)$ to the rest of the orbit of $\varphi_{1/2}^{-1}(A_{1/2})$ as follows:

$$\mu(z) = \mu(T_{1/2}^n(z)) \overline{(T_{1/2}^n)'}(z)/(T_{1/2}^n)'(z), \qquad \text{for} \quad z \in T_{1/2}^{-n}(z).$$

Note that this defines the structure everywhere in the stable set. Set $\mu(z) \equiv 0$ everywhere else in **C**. Use the measurable Riemann mapping theorem [AB] to find a quasiconformal map g of the plane whose Beltrami differential is μ. g is uniquely determined by what it does to two points. We will assume that g fixes the origin and maps the asymptotic values to a pair of points symmetric with respect to the origin. The map $F = g^{-1} \circ T_{1/2} \circ g$ is meromorphic by construction. It has exactly two finite asymptotic values and no critical points so by Nevanlinna's Theorem, since it fixes 0 and its asymptotic values are symmetric, it is of the form T_λ for some λ, hence the λ we began with. Since g is quasiconformal, it conjugates stable points to stable points. The Julia set is a quasiconformal image of the Julia set of $T_{1/2}$ hence is a Cantor set.

Q.E.D.

Remark. — It would be interesting to describe the complete bifurcation diagram for this family.

3. Asymptotic values which are poles

It is known that entire transcendental functions of finite type often have Julia sets which contain analytic curves. Indeed, for a wide class of these maps (*see* [DT]), all repelling periodic orbits lie at the endpoints of invariant curves which connect the orbit to the essential singularity at ∞.

In this section we give an example of a family of maps with constant Schwarzian derivative for which certain of the repelling fixed points lie on analytic curves in the Julia set, but for which many of the other periodic points do not. This lack of homogeneity in the Julia set is caused by the fact that one of the asymptotic values is also a pole.

Consider the family of maps

$$F_\lambda(z)=\frac{\lambda e^z}{e^z-e^{-z}}$$

with $\lambda>0$. Clearly, $\{F_\lambda, z\}=-2$ and F_λ is periodic with period πi. These maps have asymptotic values at 0 and λ, and 0 is also a pole.

The graph of F_λ restricted to $\mathbf{R}$ shows that F_λ has two fixed points on $\mathbf{R}$ at p and q with $p<0<q$. We note that $F_\lambda(z)=L_\lambda \circ E(z)$ where $E(z)=\exp(-2z)$ and L_λ is the linear fractional transformation

$$L_\lambda(z)=\frac{\lambda}{1-z}$$

F_λ has poles at $k\pi i$ where $k\in\mathbf{Z}$. F_λ has the following mapping properties:

1. F_λ preserves $\mathbf{R}^+$ and $\mathbf{R}^-$.

2. F_λ maps the horizontal lines $\operatorname{Im} z=(1/2)(2k+1)\pi$ onto the interval $(0, \lambda)$ in $\mathbf{R}$.

3. F_λ maps the imaginary axis onto the line $\operatorname{Re} z=\lambda/2$, with the points $k\pi i$ mapped to ∞.

4. F_λ maps horizontal lines onto circular arcs passing through both 0 and λ.

5. F_λ maps vertical lines with $\operatorname{Re} z>0$ to a family of circles orthogonal to those in 4 which are contained in the plane $\operatorname{Re} z>\lambda/2$.

6. F_λ maps vertical lines with $\operatorname{Re} z<0$ to a family of circles orthogonal to those in 4 which are contained in the plane $\operatorname{Re} z<\lambda/2$.

As a consequence of these properties, we have

PROPOSITION. — *If* $\lambda > 0$, *then the fixed point* q *is attracting. Moreover, if* $\text{Re } z > 0$, *then* $F_\lambda^n(z) \to q$ *as* $n \to \infty$. *Hence* $J(F_\lambda)$ *is contained in the half plane* $\text{Re } z \leqq 0$.

Proof. — First compute $|F'_\lambda(q)| < 1$. Then use property 5 above and the Schwarz Lemma.

PROPOSITION. — $J(F_\lambda)$ *contains* $\mathbf{R}^- \cup \{0\}$.

Proof. — The fixed point p is repelling. This follows from the fact that F_λ has negative Schwarzian derivative: if $|(F_\lambda)'(p)| \leqq 1$, then it follows that p would have to attract a critical point or asymptotic value of F_λ on $\mathbf{R}$. This does not occur since q attracts λ and 0 is a pole.

Let $x \in (-\infty, p)$. One may check easily that $|(F_\lambda^2)'(x)| > 1$. Moreover, $|(F_\lambda^{2n})'(x)| \to \infty$ as $n \to \infty$. This again follows from the fact that F_λ^2 has negative Schwarzian derivative on $\mathbf{R}^-$. Let U be a neighborhood of x in $\mathbf{C}$. Note that F_λ^{2n} expands U until some image overlaps the horizontal lines $y = \pm\pi/2$. By the above properties, these points are in the basin of q. Hence the family $\{F_\lambda^{2n}\}$ is not normal at x, and so $(-\infty, p) \subset J(F_\lambda)$. The image of this interval under F_λ is $(p, 0)$, so $\mathbf{R}^- \subset J(F_\lambda)$.

Q.E.D.

Thus some points in the Julia set lie on analytic curves; for example, $\mathbf{R}^-$ and all of its preimages. But not all points in the Julia set lie on smooth invariant curves:

PROPOSITION. — *There is a unique repelling fixed point* p_1 *in the half strip*

$$\pi/2 < \text{Im } z < 3\pi/2$$

and this point does not lie on any smooth invariant curve in $J(F_\lambda)$.

Proof. — Let R be the rectangle

$$\pi/2 < \text{Im } z < 3\pi/2$$
$$\nu < \text{Re } z < 0$$

where ν is chosen far enough to the left in $\mathbf{R}$ so that

$$|F_\lambda(\nu + iy)| < \pi/4.$$

Then $F_\lambda(R)$ is a "disk" which covers R and $F_\lambda | R$ is $1-1$. So, F_λ^{-1} has a unique attracting fixed point p_1 in R. Since this argument is independent of ν for ν large enough negative, the first part of the Proposition follows.

Now suppose that p_1 lies on a smooth invariant curve γ in $J(F_\lambda)$. Since $J(F_\lambda)$ is invariant under F_λ^{-1}, we may assume that γ accumulates on the boundary of the strip

$$\pi/2 < \text{Im } z < 3\pi/2$$
$$\text{Re } z < 0$$

by taking iterates of F_λ^{-1} as above. The upper and lower boundaries of the strip are stable by property 2; hence γ cannot meet $y=\pi/2$ or $y=3\pi/2$. Similarly, γ cannot meet the line $x=0$ (except possibly at $i\pi$). So γ can only accumulate at ∞ or $i\pi$. If γ accumulates at ∞, then γ must also accumulate at $i\pi$, since $F_\lambda(i\pi)=\infty$. Since all points on γ leave the strip under iteration, it follows that γ must contain $i\pi$. Now γ cannot have a tangent vector at $i\pi$, for if so, γ would enter the region $\operatorname{Re} z \geqq 0$, $\operatorname{Im} z \neq i\pi$, which lies in the stable set.

Q.E.D.

Remarks. — 1. There is a continuous invariant curve which lies in the Julia set and accumulates on p_1. Indeed, the horizontal line l_0 given by $y=\pi$, $x \leqq 0$ lies in $J(F_\lambda)$ since it is mapped onto $\mathbf{R}^-$ by F_λ. Consider the successive preimages $l_n = F_\lambda^{-n}(l_0)$, where F_λ^{-1} is the branch of the inverse of F_λ whose image is $\pi/2 < \operatorname{Im} z < 3\pi/2$. Then l_1 meets l_0 at $i\pi$, l_2 meets l_1 at $F_\lambda^{-1}(i\pi)$, and so forth. Since p_1 is an attracting fixed point for F_λ^{-1}, the curve l formed by concatenating the l_i is invariant and accumulates on p_i as $i \to \infty$. Note that this curve is considerably different from a dynamical point of view from the invariant curve $\mathbf{R}^-$ through p.

2. We will show in paragraph 6 that F_λ also possesses a collection of invariant curves which are quite different from l. These curves will lie in a Cantor bouquet, and all points (except the endpoints) on these curves will tend toward ∞ or 0 under iteration of F_λ^2.

4. Bifurcation to an entire function

Most maps with polynomial Schwarzian derivatives are bona fide meromorphic functions, but occasionally they are entire functions. In this section we describe an "explosion" in the Julia set which occurs when a meromorphic familly suddenly encounters an entire function. An *explosion* occurs at a parameter value for a family of functions whenever the Julia sets of the functions in the family change suddenly, when the parameter is reached, from a nowhere dense subset of $\hat{\mathbf{C}}$ *to all of* $\hat{\mathbf{C}}$.

Consider the family

$$F_\lambda(z) = \frac{e^z}{\lambda e^z + e^{-z}} = \frac{1}{\lambda + e^{-2z}}.$$

When $\lambda=0$, the corresponding element of this family is the entire function $F_0(z)=\exp(2z)$ whose dynamics are well understood. It is known that $J(F_0)=\mathbf{C}$, since the orbit of the asymptotic value 0 tends to ∞. *See* [D] or [Mi].

When $\lambda>0$, $J(F_\lambda) \neq C$. This follows since F_λ has a unique attracting fixed point p_λ on the real line. The graph of F_λ is depicted in Figure 4.

In fact, we can say much more about $J(F_\lambda)$.

Proposition. — *For all* $\lambda>0$, *the Julia set of* F_λ *is a Cantor set in* $\hat{\mathbf{C}}$ *and* $F_\lambda | J(F_\lambda)$ *is the shift map on infinitely many symbols as described in paragraph* 2.

Proof. — First note that the entire real axis lies in the basin of attraction of p_λ. This follows since F_λ has negative Schwarzian derivative and maps $\mathbf{R}$ diffeomorphically onto the interval bounded by the asymptotic values, $(0, 1/\lambda)$. In particular, both asymptotic values lie in the immediate basin of p_λ and so there are disks about these points which lie in the basin. Taking preimages of these disks, it follows that there are half planes of the form $\mathrm{Re}\, z < \nu_1$ and $\mathrm{Re}\, z > \nu_2$ with $\nu_1 < p_\lambda < \nu_2$ which lie in the immediate basin of p_λ.

We may find a strip S_μ surrounding the interval $[\nu_1, \nu_2]$ of the form

$$\{ z \,\|\, \mathrm{Im}\, z \,| < \mu,\ \nu_1 \leqq \mathrm{Re}\, z \leqq \nu_2 \}$$

which is mapped inside itself. Now let B denote the "ladder-shaped" region consisting of the two half planes together with S_μ and all of its πi translates. See *Fig.* 5. Clearly, F_λ maps B inside itself as long as the ν_i are chosen large enough.

Q.E.D.

The complement of B consists of infinitely many congruent rectangles R_j where $j \in \mathbf{Z}$

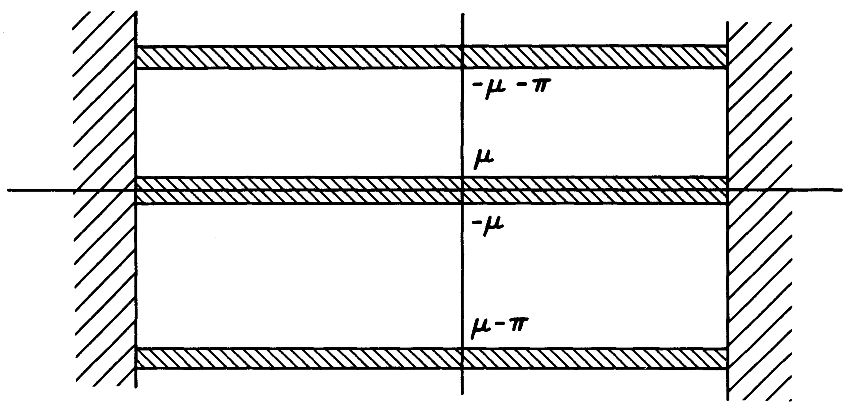

Fig. 5. — The region B.

and the R_j are indexed according to increasing imaginary part. F_λ maps each R_j diffeomorphically onto $\hat{\mathbf{C}} - F_\lambda(B)$. In particular, $F_\lambda(R_j)$ covers each R_k and ∞. It follows that there exists at least one point z corresponding to any sequence $(s_0, s_1, \ldots)$ in the sequence space Γ which has the property that $F_\lambda^n(z) \in R_{s_n}$ for each n. We claim that this point is unique and lies in the Julia set.

To see that the point with the prescribed itinerary $(s_0, s_1, \ldots)$ is in the Julia set, we first note that B is contained in the immediate attracting basin of the attractive fixed point. Moreover, B intersects its preimages, so the immediate basin is completely invariant. Since it contains both asymptotic values, it is the whole stable set. Therefore, the entire Julia set is contained in the rectangles. Points whose orbits remain in the rectangles are in the Julia set, so the points corresponding to sequences are in the Julia set.

In paragraph 6, we will show that $|(F_\lambda)'(z)| \to \infty$ for all $z \in J(F_\lambda)$ *i.e.*, that F_λ is expanding on its Julia set. Since two points corresponding to the same sequence must remain a bounded distance apart, it follows that two points in the Julia set cannot have the same itinerary. $J(F_\lambda)$ therefore is a Cantor set modeled on the sequence space with infinitely many symbols as described in paragraph 2.

Q.E.D.

5. Dynamics

In this section we discuss the dynamics of maps which have Schwarzian derivative a polynomial $P(z)$. Since such maps are meromorphic, and map the finite plane $\mathbf{C}$ onto the Riemann sphere $\hat{\mathbf{C}}$, we must first discuss the basic dynamical concepts in some detail. Since these concepts are well known for the case of entire maps, we will concentrate only on the meromorphic case. An immediate consequence of the Picard Theorem is that any pole has an infinite backward orbit. There are maps with an exceptional pole, *i.e.*, a pole with no preimage, but these maps do not have polynomial Schwarzian derivative.

Recall that a point z is *stable* for F if there is a neighborhood U of z such that the iterates F^n are uniformly bounded on U in the standard Euclidean metric. The Julia set $J=J(F)$ is the complement in $\hat{\mathbf{C}}$ of the stable set. Therefore it is the closure of the set of points at which the F^n fail to form a normal family of functions.

Remark. — Infinity is in the Julia set by the Picard Theorem. The poles are also in the Julia set. The forward iterates F^n, $n \geqq 1$, are not defined at these points. The first iterate has the point at ∞ as its image. The image of ∞ is not defined, and so the Julia set is not forward invariant.

The Julia set for such maps shares many of the properties of Julia sets for rational or entire maps. One of the standard properties of the Julia set is that it is also the closure of the set of repelling periodic orbits. The standard proofs of this fact do not work in the meromorphic case. Nevertheless, we can prove:

PROPOSITION. — *Suppose that* F *admits at least one repelling periodic orbit. Then* $J(F)$ *is the closure of the repelling periodic points for* F.

Remark. — We will show below that a meromorphic function always has infinitely many repelling periodic points, so the hypothesis of this proposition is always fulfilled.

Proof. — Suppose $\{F^n\}$ is not normal at z. Let X be an open neighborhood of z. We will produce a repelling periodic point in X.

Suppose p is a repelling periodic point for F. We may assume that p is a fixed point. Let Y be a disk about p on which F has a single valued inverse F^{-1}. By choosing Y smaller if necessary, we may assume that $F^{-1}: Y \to Y$ is a contraction. So $F^{-n}: Y \to Y$ is well-defined for all n and shrinks Y down toward p. Since $\{F^n\}$ is not normal at p, it follows that there exists an open neighborhood $U \subset Y$ such that $F^k: U \to F^k(U) \subset X$ is one to one and $F^k(U) = W$ is a neighborhood of z. This follows since z is not an exceptional point and F has no critical points.

Since $\{F^n\}$ is not normal at z, we may find an open disk $\tilde{W} \subset W$ and $l \in \mathbf{Z}^+$ such that $F^l \mid \tilde{W}$ is one to one and maps $\tilde{W}$ onto a neighborhood V of p.

Finally, there exists $m \in \mathbf{Z}^+$ such that $F^{-m}(Y) = Y_{-m}$ is an open simply connected region entirely contained in V. We can restrict $\tilde{W}$ so that $V = Y_{-m}$. Note that F^{m+k} maps Y_{-m} over W so that $F^{m+k+l}(V)$ covers V and so contains a repelling periodic point. Therefore, there is a repelling periodic point in U. Note that $F^k(U) \subset W \subset X$, so we are done.

Q.E.D.

As noted above, each pole lies in the Julia set. I. N. Baker pointed out to us that we can say more.

PROPOSITION. — $J(F)$ *is the closure of all of the preimages of the poles of* F.

Proof. – Since F is meromorphic, it has at least one pole. Suppose U is a neighborhood of some $z \in J(F)$. By Montel's Theorem $\bigcup_{n>0} F^n(U)$ misses at most two points in $\bar{\mathbf{C}}$, hence there is a pole in the forward orbit of U.

The proof of the following proposition is the standard proof for a rational or entire function.

PROPOSITION. — *Let* $z \in J(F)$. *Then* $J(F)$ *is the closure of the union of all preimages of* z.

In order to describe the stable set of maps with polynomial Schwarzian, we first recall the classical facts discussed in paragraph 1. By Nevanlinna's Theorem, each such map F has p asymptotic values $a_0, \ldots, a_{p-1}$. To each a_i there corresponds a sector W_i with angles $2\pi/p$ in which F has the following behavior: we may choose a small disk B_i about a_i such that, if U_i is the component of $F^{-1}(B_i)$ meeting W_i, then $F: U_i \to B_i - a_i$ is a universal covering map. The sectors are separated by the Julia rays β_i. Almost all poles p have associated Julia rays, $\beta_{i(p)}$, and asymptotic values $v_1(p)$ and $v_2(p)$. Finally, the a_i corresponding to adjacent sectors must be distinct.

Note that the a_i need not be distinct, but it follows from the linearindependence of the truncated solutions G_v and G_{v+1} that a_i corresponding to adjacent W_i are distinct. Thus the basic mapping properties of F are as depicted in Figure 1.

Let us assume that all of the a_i are finite. We may choose R sufficiently large so that $D_R = \{z \mid |z| \leqq R\}$ contains all of the B_i in its interior. Let Γ_R denote the disk in $\hat{\mathbf{C}}$ which in the complement of D_R. Let

$$\tilde{\Gamma}_R = \Gamma_R - \bigcup_{i=0}^{p-1} U_i.$$

If R is large enough, $\tilde{\Gamma}_R$ consists of exactly p "arms" which extend to ∞ in Γ_R and which separate the U_i. Let A_i denote the arm between U_i and U_{i+1}. A_i contains the Julia ray β_i. See *Fig.* 6.

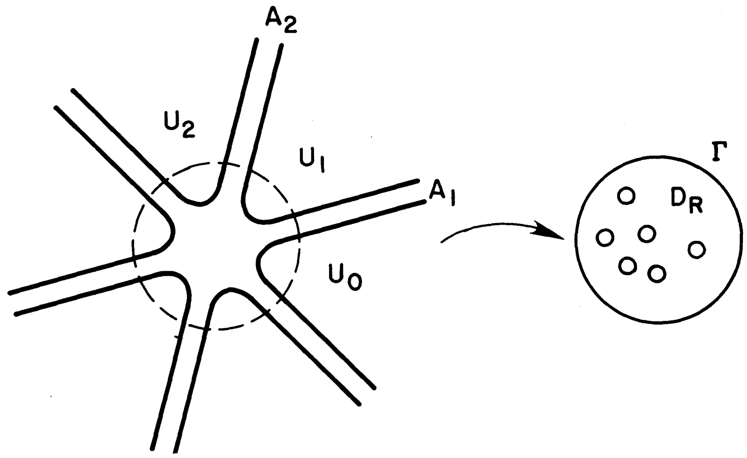

Fig. 6. — Arms in Γ_R.

Since $F|A_i$ is a covering map which covers $\hat{\mathbf{C}}-(B_i \cup B_{i+1})$ infinitely often, it follows that $F^{-1}(\Gamma_R) \cap A_i$ consists of infinitely many disks, each of which is mapped by F in a one-to-one fashion over Γ_R. These disks accumulate only at ∞.

At this point we can fill the gap left in our proof that the Julia set is also the closure of the set of repelling periodic points. In that proof we assumed that F had at least one repelling point. Since F maps infinitely many of the disks above completely over themselves in a one to one manner, each disk must contain a repelling point. This argument holds even if some of the $a_i = \infty$, for we still find disks in the arms which are covered by the map.

We now turn to a discussion of the dynamics of F on its stable set. The set of stable points comprise the largest completely invariant open set. As in the classical Fatou theory, the eventually periodic components of the stable set can be classified as follows: each is either an attracting cycle, a parabolic domain, a Siegel disk, a Herman ring or an "essentially singular cycle" (domain at ∞); superattractive domains cannot occur since there are no critical points. We prove below that all stable components of F are eventually periodic. In the next section we will prove that domains at ∞ cannot occur either, at least in the case where all of the asymptotic values of the map lie in the finite plane.

To prove that all of the components of the stable set are eventually periodic, we need the following preliminary ideas. For a given map F with polynomial Schwarzian derivative, we consider the space T(F) of functions G for which $\{G, z\}$ is a polynomial and there exists a quasiconformal homeomorphism h of $\hat{\mathbf{C}}$ such that $h \circ F = G \circ h$. We define the *Moduli Space* of F, $\mathscr{M}(F)$, to be T(F) modulo affine conjugation.

THEOREM. — *If* $\{F, z\} = Q(z)$ *where* Q *is a polynomial of degree* $p-2$, *then* $\dim \mathscr{M}(F) \leq p$.

Proof. — Let $G \in \mathcal{M}(F)$. G is topologically conjugate to F, so it also has exactly p asymptotic values and no critical values. By Nevanlinna's Theorem, we know that $\{G, z\} = P(z)$ where $P(z)$ is also a polynomial of degree $p-2$. Let g_1 and g_2 be linearly independent solutions of the associated linear differential equation

$$g'' + \frac{1}{2} P(z) g = 0.$$

Since

$$G(z) = \frac{A g_1 + B g_2}{C g_1 + D g_2}$$

where $AD - BC = 1$, there are three more parameters. Two of these can be normalized by affine conjugation so there are p independent parameters for the space $\mathcal{M}(F)$.

Q.E.D.

Remark. — As we saw in paragraph 3, in the case of the tangent family, the parameter space $\mathbf{C}^p$ contains many different moduli spaces corresponding to dynamically different functions.

Now we can prove:

THEOREM. — *Every component of the stable set is eventually periodic.*

Proof. — The crux of the proof is the finite dimensionality of the space $\mathcal{M}(F)$. As in the case of rational functions, the proof breaks into three parts, depending on the connectivity of the components in a stable orbit.

Assume there is a domain D such that the iterates, $D^n = F^n(D)$ are mutually disjoint (D^n is actually the component containing the image of the n-th iterate of D). The maps $F : D^n \to D^{n+1}$ are all unramified since there are no critical points so there is an induced injection on the respective fundamental groups. The three cases to consider are (1) the D^n are all simply connected, (2) the fundamental groups are eventually non-abelian, and (3) the D^n are all annuli. In cases 1 and 2 the proof proceeds as in [Be, S] mutatis mutandi with "rational" replaced by "meromorphic with polynomial Schwarzian." The point is that the existence of such domains implies that the space $\mathcal{M}(F)$ cannot be finite domensional which is a contradiction. We need a new argument in case 3.

If F is entire, the theorem follows from [GK]. Therefore assume F is meromorphic and has infinitely many poles. Suppose now that the domains D^n are all annuli; *i. e.* their boundaries consist of two components of the Julia set, neither of which is a point. If one component were a point, it would have to be a pole p and hence a preimage of ∞. Since F has infinitely many poles which accumulate at ∞, all of which belong to the Julia set, there is an infinite set of preimages of these poles which accumulate at p. It follows that p is not an isolated point of the Julia set.

Therefore there is a conformal map φ^n of D^n onto the annulus $\{z \mid 0 < r_n < |z| < 1\}$, where the number r_n is uniquely determined. There are "natural" foliations of D^n by the inverse images of the circles, $|z| = c$, $r_n < c < 1$. Since F has no critical points, the map

$$z \to \varphi^{n+1} \circ \mathrm{F} \circ (\varphi^n)^{-1}(z)$$

is either of the form

$$z \to e^{i\alpha} z, \qquad \alpha \in \mathrm{R},$$

or

$$z \to e^{i\alpha}(r/z).$$

We conclude that $r_n = r_{n+1} = r$ for all n. Further, F takes the leaves of the natural foliation of D^n onto those of D^{n+1}. Let γ be such a leaf in D and $\gamma^n = \mathrm{F}^n(\gamma)$. The γ^n are all simple since there are no critical points.

There is another natural conformal invariant besides r for these annuli. It is the extremal length of the set of curves homologous to γ. It is well known [A] that the extremal length in this case is:

$$(1) \qquad 2\pi/\log(1/r) = \sup\left\{ (\inf \int_c \rho(z)\,|dz|)^2 \Big/ \iint_{\mathrm{D}^n} \rho(z)^2\, dx\,dy \right\}$$

where the supremum is taken over all Borel measurable non-negative functions $\rho(z)$ for which the double integral is finite and positive, and the infimum is taken over all rectifiable curves C homologous to γ.

For ρ the spherical metric, it is clear that the areas of the areas of the disjoint annuli must tend to zero. From (1) we have

$$(2) \qquad 2\pi/\log(1/r) \geqq \inf\left(\int_{\mathrm{C}} \rho(z)\,|dz| \right)^2 \Big/ \iint_{\mathrm{D}^n} \rho(z)^2\, dx\,dy$$

with the denominators going to zero as n goes to infinity. It follows that

$$(3) \qquad \inf \int_{\mathrm{C}} \rho(z)\,|dz| \to 0 \quad \text{as } n \to \infty.$$

Let I^n be the bounded, and O^n the unbounded component of the complement of D^n. We claim that the diameters of the I^n must go to zero in the spherical metric. The infimum in (3) is at least twice the diameter of the smaller of I^n or O^n. If the spherical diameters of a subsequence of O^n's goes to zero, there is a sequence of γ^n's each of which bounds a neighborhood Γ^n of infinity, and so passes through each of the sectors. The Γ^n nest. For the exponential tract in each sector, U_j, $\mathrm{F}\,|\,\mathrm{U}_j \cap \Gamma^n$ is an infinite to one covering map onto a punctured neighborhood of an asymptotic value a_j. Therefore, given $\mathrm{M} > 0$, $\Gamma^n \subset \{ z \,|\, |z| > \mathrm{M} \}$, for all $n > \mathrm{N} = \mathrm{N}(\mathrm{M})$. It follows that for each j, $\mathrm{F}\,|\,\gamma^n \cap \mathrm{U}_j$ is k to one onto its image for $k = k(n) > 1$, which contradicts the fact that $\mathrm{F}(\gamma^n)$ is simple. We conclude that since the diameters of the I^n go to zero, if an infinite subsequence of O^n are nested, they must nest down to a point.

Now we prove, in a series of steps, that any such infinite nested sequence must contain at least two points in the intersection of their bounded complements.

First, we remark that since the preimages of the poles are dense in the Julia set, given D^n, there is a $k > 0$ such that I^{n+k} contains at least one pole p_k. If I^{n+k} contained more than one pole, the bounded region defined by γ^{n+k} would be mapped at least two to one onto a neighborhood of infinity, and $\mathrm{F}(\gamma^{n+k})$ could not be simple.

Next, let n be so large that the pole $p=p_k$ has an associated Julia ray, $\beta=\beta_{i(p)}$. The winding number of γ^{n+k} with respect to p is one. Under the mapping F, the interior of γ^{n+k} is mapped to the exterior of $F(\gamma^{n+k})$, and the exterior is mapped to the interior. It follows that if $\nu_1(p)$ and $\nu_2(p)$ are the asymptotic values associated to pole p, then the winding umber of $\delta=F(\gamma^{n+k})$ is one, with respect to both ν_1 and ν_2.

Finally, choose an integer l so that $F^l(\delta)$ contains exactly one pole q in its interior and so that $\beta_{i(q)}=\beta$ also. $\delta^1=F^{l+1}(\delta)$ also has winding number one with respect to ν_1 and ν_2. Iterate this process and obtain a sequence of curves δ^s which nest and all contain both the asymptotic values ν_1 and ν_2 in their interiors. (3) implies these curves eventually nest in, and nest down to a single point; hence $\nu_1=\nu_2$. However, ν_1 and ν_2 are asymptotic values corresponding to adjacent sectors and must be distinct. Therefore we have arrived at a contradiction and there no wandering annuli.

Q.E.D.

6. Topology of the Julia sets

Our goal in this section is to describe some of the topological properties of the Julia set of a map F with polynomial Schwarzian derivative. Recall that $\Gamma_R=\{z \mid |z|>R\} \cup (\infty)$ and that A_i are the arms containing Julia rays.

PROPOSITION. — *Suppose all of the asymptotic values of* F *are finite. Let* $\Lambda_R=\{z \mid F^j(z)\in\Gamma_R \text{ for all } j\}$. *If* R *is chosen large enough, then* Λ_R *is a closed, forward invariant subset of* J(F). *Moreover,* Λ_R *is homeomorphic to a Cantor set which is modeled on the shift space with infinitely many symbols.*

Proof. — As there are only countably many disks in $F^{-1}(\Gamma_R)\cap A_i$ for each arm A_i, we may choose an indexing of these disks by the natural numbers. Say $F^{-1}(\Gamma_R)\cap(\bigcup A_i)=\bigcup_{j=0}^{\infty} D_j$. Thus each $D_j\subset\Gamma_R$ and F maps each D_j onto Γ_R. In particular, $F \mid D_j$ covers each other D_k and ∞. Standard arguments as described in paragraph 2 then yield the result.

Q.E.D.

Thus the set of points whose orbits remain in a neighborhood of ∞ form a closed forward invariant subset of the Julia set which is homeomorphic to a Cantor set. We may apply these ideas on a global level if we can guarantee that all of the asymptotic values lie in a single immediate attracting basin of a fixed point.

COROLLARY. — *Suppose each of the* a_i *lie in the immediate attracting basin of an attracting fixed point. Then* J(*F*) *is a Cantor set and* $F \mid J(F)$ *is conjugate to the shift map on infinitely many symbols.*

Proof. — Our assumption allows us to choose a simple closed curve in **C** which bounds an open set in the immediate attractive basin, and which contains all of the B_i. The Julia set is contained in the complement of this set. Applying the above argument to this curve instead of Γ_R yields the result.

Q.E.D.

Recall that the Julia set of a rational map is also a Cantor set under this hypothesis so that these meromorphic maps are dynamically similar to rational maps. By contrast, there are no entire transcendental functions whose Julia sets are Cantor sets [Ba].

This proposition also allows us to complete the classification of the components of the stable set begun in the previous section, by showing that domains at ∞ do not exist in general.

COROLLARY. — *Suppose* F *has polynomial Schwarzian derivative and all of the asymptotic values of* F *are finite. Then* F *has no domains at* ∞.

Proof. — Suppose F admits a domain at ∞, U. Let $z \in U$. By definition, there is an open neighborhood $V \subset U$ containing z and on which the F^n converge uniformly to ∞. Therefore, given R large, there exists $k = k(R)$ such that, for $n > k$, $F^n(V)$ is completely contained in Γ_R. By the previous proposition, $F^n(V)$ is completely contained in the set Λ_R and hence in the Julia set.

Q.E.D.

When the asymptotic values are alternately finite and infinite as we move around the sectors at ∞, it is easy to see that the meromorphic map is actually entire. The Julia sets of such maps were studied in [DT] where it was shown that they contain "Cantor bouquets."

Suppose now that F is meromorphic and that one of the asymptotic values, say a_0, is also a pole. Then F maps U_0 onto $B_0 - a_0$ as a universal cover, and, provided B_0 is chosen small enough F maps B_0 biholomorphically onto a neighborhood of ∞ in $\hat{\mathbf{C}}$. We now show that when this occurs, the Julia set of F again contains Cantor bouquets. Recall that in paragraph 3 we discussed a special case of this phenomenon.

Recall the following terminology from [DT]. Let Σ_N be the set of sequences $(s_0, s_1, s_2, \ldots)$ where the s_i are integers of absolute value less than N. Let U be open and connected. Let $F : U \to \mathbf{C}$ be analytic. An invariant subset C of $J(F) \subset U$ is called a Cantor N-bouquet for F if

1. There is a homeomorphism $h : \Sigma_N \times [0, \infty) \to C$.
2. $\pi \circ h^{-1} \circ F \circ h(s, t) = \sigma(s)$ where $\pi : \Sigma_N \times [0, \infty) \to \Sigma_N$ is the projection map.
3. $\lim_{t \to \infty} h(s, t) = \infty$.
4. $\lim_{n \to \infty} F^n \circ h(s, t) = \infty$ if $t \neq 0$.

An N-bouquet includes naturally in an (N+1)-bouquet by considering only sequences with entries less than or equal to N in absolute value.

The invariance of C requires that $F(h(s, 0)) = h(\sigma(s), 0)$. Hence the set of points $\Lambda = h(s, 0)$ is an invariant set on which F is topologically conjugate to the shift. We call Λ the *crown* of C. The curve $h_s(t)$ for $t > 0$ is called the *tail* associated to s. In analogy with real dynamical systems, we think of the tails associated to s as the strong unstable manifold associated to $h(s, 0)$, although this is by no means correct. Alternatively, we may view the tails as comprising a piece of the "stable manifold" of infinity.

Let C_n be an n-bouquet and suppose $C_n \subset C_{n+1} \subset \ldots$ is an increasing sequence of bouquets with the natural inclusion maps. The set

$$C = \overline{\bigcup_{n \geq 0} C_n}$$

is then called a *Cantor bouquet.*

Our main result is:

THEOREM. — *Suppose* $F(z)$ *has polynomial Schwarzian derivative with degree* $p-2$. *Suppose that* F *has an asymptotic value* a_i *which is also a pole. Let* W_i *be the sector containing the exponential tract corresponding to* a_i. *Then for each* $N>0$, $J(F)$ *contains a Cantor* N-*bouquet in* W_i *which is invariant under* F^2.

To show that the Julia set of a meromorphic map with polynomial Schwarzian derivative contains a Cantor bouquet if one of the asymptotic values is a pole, we need a result which is proved in [DT] and a Lemma which follows from it. Let G be an analytic map. In the proof of the theorem, G will be F^2 restricted to the exponential tract in W_i. As above, let Γ_R denote the complement of D_R in $\hat{C}$ and suppose U is a disk which satisfies

1. U is contained in a sector W
2. $G|U$ is a universal covering map onto $\Gamma_R - \infty$.

Let η be a ray in $\mathbf{C} - W$; the preimages of η under G determine infinitely many fundamental domains in U. Let $\mathscr{W}$ be a collection of N of these fundamental domains in U which have the additional property that each is completely contained in Γ_R as well. See *Fig.* 7.

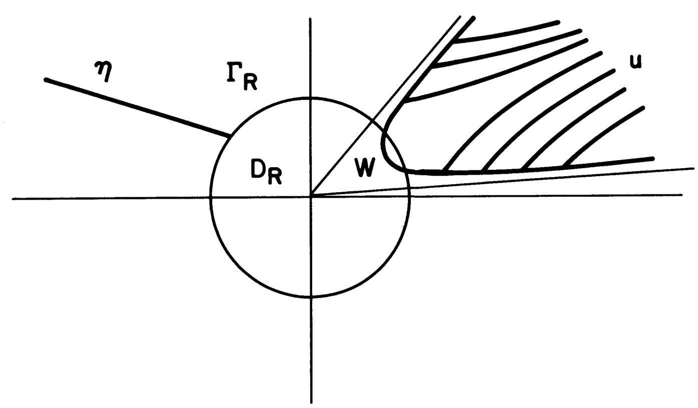

Fig. 7. — Fundamental domains in U.

PROPOSITION. — *Suppose there are constants* R_1, α, $C>0$ *such that if* z, $G(z) \in \mathscr{W}$ *with* $|z| = r > R_1$,

(i) $|G(z)| > C e^{r^\alpha}$

(ii) $G'(z)|>C e^{r^\alpha}$

(iii) $|\arg G'(z)|<C e^{-r^\alpha}$

Then $\{z \mid G^j(z) \in \mathscr{W}$ *for all* $j\}$ *forms a Cantor* N-*bouquet which lies in* J(G).

LEMMA. — *Suppose* G *is analytic in the sector* W

$$|\operatorname{Arg} z|<\frac{\pi}{2q}, \qquad |z|>r$$

and that

$$\log F(z) \sim z^q$$

in W. *Let* $\Lambda=\{z \in W \mid G^i(z) \in W\}$ *for all* $i \geqq 0$. *Then, for each* $N>0$, Λ *contains a forward invariant Cantor* N-*bouquet.*

Proof. — We invoke the above proposition. Let Γ_R be given as above and set

$$\tilde{\Gamma}_R=\{z \in W \mid G(z) \in \Gamma_R\}.$$

If R is chosen large enough, then $F: \tilde{\Gamma}_R \rightarrow \Gamma_R$ is a universal covering. Clearly, G satisfies (i) and (ii) in the proposition. So it suffices to verify (iii).

Let γ be a ray in the complement of W. The preimages of γ in Γ_R are infinitely many curves which bound fundamental domains for $G \mid \tilde{\Gamma}_R$. If $q \geqq 2$, all of these curves are asymptotic to the positive real axis, whereas, if $q=1$, they all have bounded imaginary parts. In any event, let us select N of the fundamental domains determined by these preimages. Call this set of strips $\mathscr{W}_R$. If both z, $F(z)$ lie in $\mathscr{W}_R$, it follows that $|\operatorname{Im} z|$ and $|\operatorname{Im} G(z)|$ are bounded, say by ν, and we have

$$\tan |\operatorname{Arg} G'(z)|=\frac{|\operatorname{Im} G'(z)|}{|\operatorname{Re} G'(z)|} \leqq \frac{c_1 |z|^q \nu}{c_2 |G(z)|}$$

for some constants c_1 and c_2. Hence $|\operatorname{Arg} G'(z)|<c e^{-r^\alpha}$ for some constants c, α as required.

Q.E.D.

We now complete the proof of the Theorem.

Proof. — We may assume that W_i is given by

$$|\operatorname{Arg} F(z)|<\frac{\pi}{p}.$$

In this sector, we may use (†) from paragraph 1 to show that

$$\log (F(z)-a_i) \sim -2 z^{p/2}$$

Since a_i is a pole for F, it follows that

$$\log F^2(z) \sim 2 z^{p/2}$$

in this sector. Hence the lemma applies to F^2.

Q.E.D.

We conclude this section with a brief discussion of the Lebesgue measure of the Julia sets of these meromorphic functions. Following Sullivan [S], we say that a map is expanding on its Julia set if all of the asymptotic values are attracted to attracting periodic orbits. The reason for this terminology is the following.

PROPOSITION. — *Suppose that a meromorphic map* F *with polynomial Schwarzian derivative is expanding on its Julia set. Then*

$$|(F^n)'(z)| \to \infty$$

as $n \to \infty$ *for all* $z \in J(F)$.

Proof. — The proof is exactly the same as in [Mc], Prop. 6.1. It uses the Poincaré metric as in Douady's proof for rational functions [Do]. The only difference occurs at poles, but there $|F'(z)| = \infty$ already.

Q.E.D.

The following result is an immediate consequence.

THEOREM. — *If* $\{F, z\}$ *is a polynomial and* F *is expanding on its Julia set, then* J(F) *has two dimensional Lebesgue measure zero.*

Proof. — Following McMullen, we note that the Julia set is "thin at infinity". *See* [Mc], § 7. That is, if $B(z, R)$ is a ball of radius R centered at z, there exists an ε and an R such that for all z we have

$$\text{Area}(J(F) \cap B(z, R)) < (1-\varepsilon)\pi R^2.$$

This follows since the exponential tracts are asymptotically contained in the basins of attraction of the attracting orbits. Now invoke Proposition 7.3 in [Mc], using the backward invariance of the Julia set. The idea is that, since F is expanding the balls $B(z, R)$, for $z \in J(F)$, are contracted down onto the Julia set with bounded distortion, and that therefore J(F) cannot contain points with positive density.

Q.E.D.

BIBLIOGRAPHIE

[A] L. V. AHLFORS, *Lectures on Quasiconformal Mapping*, Van Nostrand, 1966.

[AB] L. AHLFORS and L. BERS, *Riemann's Mapping Theorem for Variable Metrics* (*Ann. Math.*, Vol. 72, 1960, pp. 385-404).

[Be] L. BERS, *On Sullivan's Proof of the Finiteness Theorem and the Eventual Periodicity Theorem*, Preprint.

[Bl] P. BLANCHARD, *Complex Analytic Dynamics on the Riemann Sphere*, (*B.A.M.S.*, 1984, pp. 85-141).

[Ba] I. N. BAKER, *Wandering Domains in the Iteration of Entire Functions* (*Proc. London Math. Soc.*, Vol. 49, 1984, pp. 563-576).

[BR] I. N. BAKER and P. RIPPON, *Iteration of Exponential Functions* (*Ann. Acad. Sci. Fenn.*, Series 1A Math., Vol. 9, 1984, pp. 49-77).

[Br] H. BROLIN, *Invariant Sets under Iteration of Rational Functions* (*Ark. für Matematik.*, Vol. 6, 1965, pp, 103-144).

[CGS] J. CURRY, L. GARNETT and D. SULLIVAN, *On the Iteration of a Rational function* (*Comm. Math. Phys.*, Vol. 91, 1983, pp. 267-277).

[Do] A. DOUADY, *Systèmes dynamiques holomorphes* (*Astérisque*, Vol. 105, 1983, pp. 39-63). *See* also *Erg. Theory and Dyn. Syst.*, Vol. 4, 1984, pp. 35-52

[DH] A. DOUADY and J. H. HUBBARD, *Itération des polynômes quadratiques complexes* (*C.R. Acad. Sci. Paris*, t. 29, série I, 1982, pp. 123-126).

[DK] R. L. DEVANEY and M. KRYCH, *Dynamics of* Exp (z) (Ergodic Theory and Dynamical Systems, Vol. 4, 1984, pp. 35-52).

[DT] R. DEVANEY and F. TANGERMAN,*Dynamics of Entire Functions Near the Essential Singularity* (*Ergodic Theory and Dynamical Systems*, Vol. 6, 1986, pp. 496-503).

[GK] L. R. GOLDBERG and L. KEEN, *A Finiteness Theorem For A Dynamical Class of Entire Functions* (*Ergodic Theory and Dynamical Systems*, Vol. 6, 1986, pp. 183-193).

[G] J. GUCKENHEIMER, *Monotone Equivalence and Topological Equivalence*, Preprint.

[H] E. HILLE, *On the Zeroes of the Functions of the Parabolic Cylinder* (*Ark. Mat. Astron. Fys.*, Vol. 18, No. 26, 1924).

[Ma] B. MANDELBROT, *The Fractal Geometry of Nature*, San Francisco, Freeman, 1982.

[Mc] C. MCMULLEN, Area and Hausdorff Dimension of Julia Sets of Entire Functions (*Trans. A.M.S.*, 300, No. 1, 1987, pp. 329-342).

[Mi] M. MISIUREWICZ, *On Iterates of* e^z (*Ergodic Theory and Dynam. Sys.*, Vol. 1, 1981, pp. 103-106).

[Mo] J. K. MOSER, *Stable and Random Motions in Dynamical Systems*, (Princeton: Princeton University Press, 1973).

[N] R. NEVANLINNA, *Über Riemannsche Flächen mit endlich vielen Windungspunkten* (*Acta Math.*, Vol. 58, 1932).

[S] D. SULLIVAN, *Quasiconformal Homeomorphisms and Dynamics I* (*Acta Math.* 1985).

[Si] D. SINGER, *Stable Orbits and Bifurcations of Maps of the Interval* (*S.I.A.M. J. Appl. Math.*, Vol. 35, 1978, pp. 260-267).

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