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# LOWER CURVATURE BOUNDS, TOPONOGOV'S THEOREM, AND BOUNDED TOPOLOGY, II

BY U. ABRESCH

ABSTRACT. — Extending a result by Gromov, we establish an upper bound on the Betti numbers of asymptotically non-negatively curved manifolds.

## Introduction

In this paper we continue studying asymptotically non-negatively curved manifolds. Our goal is to estimate their Betti numbers from above in terms of the curvature decay and the dimension. In special cases bounds of this type are due to Gromov [G]; he deals with non-negatively curved manifolds and with compact manifolds. Related is also the work of Berard and Gallot [BG] who have applied heat equation methods in order to get bounds for all topological invariants of compact manifolds.

We recall that a complete Riemannian manifold $(M^n, g)$ with base point $o$ is said to be *asymptotically non-negatively curved*, iff there exists a monotone function $\lambda$: $[0, \infty) \to [0, \infty)$ such that

(i) $b_0(\lambda) := \int_0^\infty r \,.\, \lambda(r)\, dr < \infty$

and

(ii) the sectional curvatures at $p \geqq -\lambda(d(o, p))$ for all $p \in M^n$.

A detailed exposition of the analytical impact of the convergence of the integral $b_0(\lambda)$ has been given in chapter II of [A]; for instance there exists a unique non-negative solution of the Riccati equation $u' = u^2 - \lambda$ with the property that $u(r) \to 0$ for $r \to \infty$. This gives rise to another numerical invariant

$$b_1(\lambda) := \int_0^\infty u(r)\, dr.$$

Both $b_0$ and $b_1$ depend on $\lambda$ in a monotone way, and they can be regarded as invariants of the manifold $M^n$ by taking the minimal monotone function $\lambda$ which obeys the conditions (i) and (ii).

In principle $b_0$ and $b_1$ can be regarded as equivalent invariants: $b_1 \leqq b_0 \leqq \exp(b_1) - 1$. However, $b_1$ is better adapted to our problem. A natural family of weighted $L^1$-norms on the Betti numbers of a space X is induced by the Poincaré series

$$P_t(X) := \sum_i t^i . \beta_i(X).$$

MAIN THEOREM. — *For any asymptotically non-negatively curved manifold* $(M^n, g, o)$ *the Betti numbers with respect to an arbitrary coefficient field can be bounded universally in terms of the dimension and the invariant* $b_1$:

$$P_{t(n)^{-1}}(M^n) \leqq C(n) . \exp\left(\frac{15n-13}{4} . b_1(M^n)\right),$$

*where*

$$C(n) := \exp(5n^3 + 8n^2 + 4n + 2)$$

*and*

$$t(n) := 5^{n^2} 8^n \exp\left(\frac{8}{3} . \frac{1}{n+1}\right).$$

*Moreover these manifolds have finitely many ends and the Betti numbers at infinity are bounded as follows:*

$$\sum_{\text{ends } E} P_{t(n)^{-1}}(E) \leqq C(n) . \exp((n-1) . b_1(M^n)).$$

*Remarks.* — (i) By the examples given in chapter IV of part one it is reasonable that the bounds in both the estimates grow exponentially in $n . b_1(M^n)$.

However there is no geometric reason known so far, why the constants $C(n)$ and $t(n)^n$ should grow exponentially in $n^3$.

(ii) Notice that:

$$\# \{\text{ends of } M^n\} \leqq \sum_{\text{ends}} P_1(E) \leqq t(n)^{n-1} . \sum_{\text{ends}} P_{t(n)^{-1}}(E).$$

Thus we have recovered a weaker version of Theorem III. 3 in [A].

(iii) Using the long exact homology sequence, one obtains an estimate on the relative Betti numbers:

$$P_{t(n)^{-1}}(M^n, \bigcup_{\text{ends}} E) \leqq (1 + t(n)^{-1}) . C(n) . \exp\left(\frac{15n-13}{4} . b_1(M^n)\right).$$

(iv) Because of Poincaré duality the inequality

$$\sum_i \beta_i(M^n) \leqq \tilde{C}(n) . \exp\left(\frac{15n-13}{4} . b_1(M^n)\right)$$

holds with

$$\tilde{C}(n) := 3 . t(n)^{n/2} . C(n) \leqq \exp(6n^3 + 9n^2 + 4n + 4).$$

*Special Cases.* — (*a*) $M^n$ has non-negative sectional curvature:

$$P_{t(n)^{-1}}(M^n) \leqq C(n).$$

(*b*) the sectional curvatures of $M^n$ are bounded from below by $-k^2$ and even more are non-negative outside a ball with radius $d$ around the base point $o$: (e. g.: $M^n$ compact with diametre $d$)

$$P_{t(n)^{-1}}(M^n) \leqq C(n) . \exp\left(\frac{15n-13}{4} . k . d\right)$$

*Method of Proof.* — We use a modification of Gromov's direct geometric proof. The basic idea is to combine Morse theory arguments on the distance function and covering arguments. In a first step we do things locally and derive an estimate for small balls (sections 1-3). In a second step we reduce the theorem to these local bounds (sections 4 and 5).

In principle the local result is already contained in Gromov's paper (*c.f.* [G]); however, we shall rearrange the details in a more subtle way. Therefore our constants grow only exponentially in $n^3$; they do not depend doubly exponentially on $n$. The key to this improvement is a non-standard packing lemma (*c.f.* Appendix A).

The way in which we put together the local estimates is essentially new. We use metrical annuli as intermediate objects when extending the estimate from small balls to all of the manifold $M^n$.

## 1. A topological Lemma

In this section we are going to do the topological part of the argument. There are two reasons for avoiding the Betti numbers in the intermediate steps in the proof:

(*a*) Given a point $p \in M^n$ and any number $N > 0$, it is easy to put a bumpy metric on $M^n$ such that $\dim H_1(B(p, 1)) \geqq N$. The idea is to produce a sufficiently complicated intersection pattern of the distance sphere $\mathbb{S}(p, 1) \subset M^n$ with the cut-locus of $p$.

(*b*) For arbitrary subsets $X_1, X_2 \subset M^n$ it is impossible to estimate the dimension of $H_*(X_1 \cup X_2)$ in terms of $\dim H_*(X_1)$ and $\dim H_*(X_2)$ only.

Some pieces of information about $X_1 \cap X_2$ are required in addition. These obstructions towards an "obvious proof" are related, and they both can be circumvented looking at topological pairs (Y, X) where $X \subset Y \subset M^n$ are open subsets. We consider the numbers

1.1 $$\begin{cases} rk_i(Y, X) := \operatorname{rank}(H_i(X) \to H_i(Y)) \\ rk_*^t(Y, X) := \sum_{i \geqq 0} rk_i(Y, X) . t^i. \end{cases}$$

It is worthwhile noticing that under the hypothesis above the numbers $rk_i(Y, X)$ vanish for $i > n$.

1.2 We consider open subsets $B_j^0 \subset B_j^1 \subset \ldots \subset B_j^{n+1}$, $1 \leqq j \leqq N$, such that

$$X \subset \bigcup_{j=1}^{N} B_j^0$$

and

$$Y \supset \bigcup_{j=1}^{N} B_j^{n+1}.$$

LEMMA. — *Let $t > 0$ and suppose that any $B_j^n$ intersects at most $t$ distinct sets $B_{j'}^n$, $j' \neq j$; then there holds the following inequality:*

$$rk_*^{t^{-1}}(Y, X) \leqq rk_*^{t^{-1}}\left(\bigcup_{j=1}^{N} B_j^{n+1}, \bigcup_{j=1}^{N} B_j^0\right)$$
$$\leqq (e-1) . N . \sup\{rk_*^{t^{-1}}(B_{j_0}^{\sigma+1} \cap \ldots \cap B_{i_{n-\sigma}}^{\sigma+1}, B_{j_0}^{\sigma} \cap \ldots \cap B_{j_{n-\sigma}}^{\sigma}) \mid 0 \leqq \sigma \leqq n,\ 1 \leqq j_0 < \ldots < j_{n-\sigma} \leqq N\}.$$

Essentially this lemma is already contained in Gromov's paper (*c.f.* [G]). For the sake of completeness we include an elementary

*Proof.* — Consider open subsets $X_1 \subset X_2 \subset X_3$ and $Y_1 \subset Y_2 \subset Y_3$ in $M^n$. The Mayer-Vietoris sequence gives raise to a commutative diagram with exact rows:

$$\begin{array}{ccccccc}
\to H_\mu(X_1) \oplus H_\mu(Y_1) & \to & H_\mu(X_1 \cup Y_1) & \to & H_{\mu-1}(X_1 \cap Y_1) & \to \\
\downarrow & & \downarrow i_\mu & & \downarrow i'_{\mu-1} & \\
\to H_\mu(X_2) \oplus H_\mu(Y_2) & \to & H_\mu(X_2 \cup Y_2) & \to & H_{\mu-1}(X_2 \cap Y_2) & \to \\
\downarrow j_{\mu,X} \oplus j_{\mu,Y} & & \downarrow j_\mu & & \downarrow & \\
\to H_\mu(X_3) \oplus H_\mu(Y_3) & \to & H_\mu(X_3 \cup Y_3) & \to & H_{\mu-1}(X_3 \cap Y_3) & \to
\end{array}$$

All the vertical homomorphisms are induced by inclusions. The standard diagram chasing technique shows that:

1.3 $$rk(j_\mu \circ i_\mu) \leqq rk(j_{\mu,X} \oplus j_{\mu,Y}) + rk(i'_{\mu-1})$$

or in different terminology:

$$rk_\mu(X_3 \cup Y_3, X_1 \cup Y_1) \leqq rk_\mu(X_3, X_2) + rk_\mu(Y_3, Y_2) + rk_{\mu-1}(X_2 \cap Y_2, X_1 \cap Y_1).$$

This estimate extends as follows to the family $B_j^i$ when we use the Leray spectral sequence instead of the Mayer-Vietoris sequence:

1.4 $$rk_i\left(\bigcup_{j=1}^{N} B_j^{i+1+\nu}, \bigcup_{j=1}^{N} B_j^{\nu}\right) \leqq \sum_{\mu=0}^{i} \sum_{j_0<\ldots<j_{i-\mu}} rk_\mu (B_{j_0}^{\mu+\nu+1}\cap\ldots\cap B_{j_{i-\mu}}^{\mu+\nu+1}, B_{j_0}^{\mu+\nu}\cap\ldots\cap B_{j_{i-\mu}}^{\mu+\nu});$$

here $\nu$ denotes some non-negative integer which does not exceed $n-i$.

We specialize to the case $\nu=n-i$ and compute:

$$rk_*^{t^{-1}}\left(\bigcup_{j=1}^{N} B_j^{n+1}, \bigcup_{j=1}^{N} B_j^{0}\right) \leqq \sum_{i=0}^{n} t^{-i}.rk_i\left(\bigcup_{j=1}^{N} B_j^{n+1}, \bigcup_{j=1}^{N} B_j^{n-i}\right)$$
$$\leqq \sum_{\nu=0}^{n} \sum_{\sigma=\nu}^{n} \sum_{j_0<\ldots<j_{n-\sigma}} t^{\nu-n}.rk_{\sigma-\nu}(B_{j_0}^{\sigma+1}\cap\ldots\cap B_{j_{n-\sigma}}^{\sigma+1}, B_{j_0}^{\sigma}\cap\ldots\cap B_{j_{n-\sigma}}^{\sigma})$$
$$=\sum_{\sigma=0}^{n} \sum_{j_0<\ldots<j_{n-\sigma}} t^{\sigma-n}.\sum_{\nu=0}^{\sigma} t^{\nu-\sigma}.rk_{\sigma-\nu}(\ldots, \ldots),$$

hence:

1.5 $$rk_*^{t^{-1}}\left(\bigcup_{j=1}^{N} B_j^{n+1}, \bigcup_{j=1}^{N} B_j^{0}\right) \leqq \sum_{\sigma=0}^{n} \sum_{j_0<\ldots<j_{n-\sigma}} t^{\sigma-n}.rk_*^{t^{-1}}(B_{j_0}^{\sigma+1}\cap\ldots\cap B_{j_{n-\sigma}}^{\sigma+1}, B_{j_0}^{\sigma}\cap\ldots\cap B_{j_{n-\sigma}}^{\sigma}).$$

To complete the proof, we point out that the number of non-empty intersections

$$B_{j_0}^{\sigma}\cap\ldots\cap B_{j_{n-\sigma}}^{\sigma}$$

does not exceed

$$\frac{N}{n-\sigma+1}.\binom{t}{n-\sigma}, \qquad 0\leqq\sigma\leqq n;$$

therefore the number of non-vanishing terms on the right-hand side of 1.5 is bounded from above by

$$\sum_{\sigma=0}^{n}\frac{N}{n-\sigma+1}.\binom{t}{n-\sigma}.t^{\sigma-n}\leqq N.\sum_{\sigma=0}^{n}\frac{1}{(n-\sigma+1)!}\leqq N.(e-1). \quad \square$$

In most of our applications the sets $B_j^i$ will be open metrical balls. We shall use the notation $\rho.B(p, r):=B(p, \rho.r)$. It is convenient to draw the following

1.6 COROLLARY. — *Let* $\rho>1$, $t\geqq 1$ *and suppose that:*

(i) $X\subset M^n$ *is covered by open metrical balls* $B_j^0$, $1\leqq j\leqq N$,

(ii) $i<n$ *and* $B_j^i\cap B_{j'}^i\neq 0\Rightarrow \rho.B_j^i\subset B_{j'}^{i+1}$,

(iii) $\rho.B_j^n\subset B_j^{n+1}\subset Y$, $1\leqq j\leqq N$, *and*

(iv) *each ball* $B_j^n$ *intersects at most t other balls* $B_{j'}^n$.

*Then the following estimate holds:*

$$rk_*^{t^{-1}}(Y, X)\leqq rk_*^{t^{-1}}\left(\bigcup_{j=1}^{N} B_j^{n+1}, \bigcup_{j=1}^{N} B_j^0\right)$$
$$\leqq (e-1).N.\sup\{rk_*^{t^{-1}}(\rho.B_j^i, B_j^i)\,|\,0\leqq i\leqq n,\ 1\leqq j\leqq N\}.$$

*Remark.* — Condition (ii) is obviously met, if all the balls $B_j^0$ have equal radii and if $B_j^i=(2+\rho)^i.B_j^0$, $0\leqq i\leqq n$, $1\leqq j\leqq N$.

## 2. The Morse theory of the distance function

Any point $p\in M$ gives rise to a function $d_p: M\to\mathbb{R}$ defined by $d_p(\tilde{p}):=d(p, \tilde{p})$. $\tilde{p}\in M$ is called a *critical point of* $d_p$, iff for any $v\in T_{\tilde{p}}M$ there is a minimizing geodesic segment $\gamma$ which joins $\tilde{p}=\gamma(0)$ to $p$, and which obeys $\langle v, \gamma'(0)\rangle\geqq 0$. $\mathfrak{s}_p:=\{$critical points of $d_p\}$ is said to be the *singular set of* $d_p$. Notice that $\tilde{p}$ is *non-critical* for $d_p$, iff the initial vectors of all minimizing geodesics joining $\tilde{p}$ to $p$ lie in an open half-space of $T_{\tilde{p}}M$. Hence for any non-critical point $\tilde{p}$, there is an open neighborhood U and a continuous non-vanishing vector field $v_U$ which is defined on U and has an acute angle with the initial vector of any minimizing geodesic joining a point in U to $p$. As $\mathfrak{s}_p$ is closed, one can reason in the standard way and obtain:

2.1 LEMMA. — *For any* $p\in M$ *there exists a smooth vector field* $v_p: M\to TM$, *which obeys* $\langle v_p|_{\gamma(0)}, \gamma'(0)\rangle>0$ *for all minimizing geodesics* $\gamma$ *which join some point* $\gamma(0)\in M\setminus\mathfrak{s}_p$ *to* $p$.

We shall use the vector field $v_p$ in order to construct retraction maps; we point out that $d_p$ is monotone decreasing along the integral curves of $v_p$. Under some suitable hypothesis this tool even gives raise to isotopies rather than homotopy equivalences (*c.f.* [G], [GS].)

In order to establish a Morse theory on $d_p$, it is moreover necessary to determine how the topology of M changes at the critical points of $d_p$ and to count the strata of the singular set $\mathfrak{s}_p$ in a reasonable manner. As $d_p$ is not differentiable at the cut-locus of $p$, both these problems cannot be tackled in the usual way.

2.2. However, it is possible to bound in some sence the number of critical points of $d_p$: let $L>1$; we consider $p_1, \ldots, p_k\in M$ such that

(i) $$d_p(p_{i-1})\geqq L.d_p(p_i),$$

and

(ii) $$p_i \text{ is critical for } d_p, \qquad 2 \leqq i \leqq k.$$

Such a sequence $(p_i)_{i=1}^k$ will be called a metrical $(k;\ L_1,\ L,\ L_k)$-*frame of* $p$, provided $L_1$ and $L_k$ are positive real numbers which satisfy $l_1 := d_p(p_1) \leqq L_1$ and $l_k := d_p(p_k) \geqq L_k$. We shall say that a subset $X \subset M$ is $(k;\ L_1,\ L,\ L_k)$-*framed*, iff for each $p \in X$ there exists a metrical $(k;\ L_1,\ L,\ L_k)$-frame.

2.3 Lemma. — (i) *In a manifold* $(M^n, g, o)$ *which has asymptotically non-negative curvature any metrical* $(k;\ L_1,\ L,\ L_k)$-*frame of the base point* $o$ *obeys:*

$$k \leqq 2 . \pi^{n-1} . \left(\frac{L}{L-1}\right)^{2n-2} . \exp((2n-2) . b_1).$$

(ii) *Let* $p$ *be any point in an arbitrary Riemannian manifold, and suppose that the sectional curvatures in the ball* $B(p, (1+L^{-1}) . L_1)$ *are bounded from below by* $-\eta^2$, $\eta \geqq 0$. *If, moreover,*

$$3 . (1+\sqrt{2})^{n-1} . \eta . L_1 . \coth(\eta . L_1) \leqq L,$$

*then* $k \leqq 2n$ *holds for any metrical* $(k;\ L_1,\ L,\ L_k)$-*frame of* $p$.

*Remark.* — The lemma relates the parameter $k$ to the dimension $n$ of $M^n$, and thus it justifies the terminology, although the word "frame" might be a little misleading. In a similar context Gromov heuristically speaks of the "number of essential directions in $M^n$".

*Proof.* — We fix minimizing geodesics $\gamma_i$ which join $p$ with $p_i$; their initial vectors in $T_p M$ will be denoted by $v_i$. We head towards a lower bound on the angle between any two of these vectors and then make use of some packing arguments: let $1 \leqq i < j \leqq k$ and study the geodesic triangle $\Delta = (p_i, p, p_j)$ with edges $\gamma_i$, $\gamma_j$, and a minimizing geodesic $\gamma_{ij}$. We observe that $p_j$ is critical for $d_p$, and thus $\gamma_j$ can be replaced by another minimizing geodesic $\tilde{\gamma}_j$ such that the angle at $p_j$ in the modified triangle $\tilde{\Delta}$ does not exceed $\pi/2$. The data on $\Delta$ and $\tilde{\Delta}$ are turned into inequalities by means of the (generalized) Toponogov theorem. We start with $\tilde{\Delta}$:

(i) Proposition III. 1 (ii) applies with $\varepsilon = (L-1)/L$, and in the limit $a \to 1$ it yields:

$$d_p(p_i) \leqq d(p_i, p_j) + d_p(p_j) . \sqrt{1 - \left(\frac{L-1}{L}\right)^2 . \exp(-2 . b_1)}.$$

(ii) Obviously a minimizing geodesic which joins $p_i$ to any point on $\gamma_j$ is not longer than $d_p(p_i) + d_p(p_j)$, and therefore it is contained in the ball

$$B(p,\ d_p(p_i) + d_p(p_j)) \subset B(p,\ (1+L^{-1}) . L_1).$$

Hence the hyperbolic plane with curvature $-\eta^2$ is an admissible model. We deform the Alexandrov triangle such that

$$\bar{l}_i \geqq d_p(p_i), \quad \bar{l}_j = d_p(p_j), \quad \bar{l} = d(p_i, p_j), \qquad \text{and} \qquad \measuredangle \text{ at } \bar{p}_j = \frac{\pi}{2};$$

then the Law of Cosines yields:

$$\cosh \eta . \bar{l} = \frac{\cosh \eta . \bar{l}_i}{\cosh \eta . d_p(p_j)} \geqq \frac{\cosh \eta . d_p(p_i)}{\cosh \eta . d_p(p_j)}.$$

Using the above estimates we can treat the triangle $\Delta$ in a similar way:

(i) Reversing the implication in Proposition III. 1 (i), we obtain:

$$\cos \measuredangle (v_i, v_j) < \sqrt{1 - \left(\frac{L-1}{L}\right)^4 . \exp(-4 . b_1)},$$

or :

$$|\measuredangle (v_i, v_j)| \geqq |\sin \measuredangle (v_i, v_j)| > \left(\frac{L-1}{L}\right)^2 . \exp(-2 . b_1).$$

In this case the claim immediately follows from the standard packing estimate, which has been stated in Lemma III. 3. 1 for instance.

(ii) Here we apply the Law of Cosines directly to the Alexandrov triangle:

$$\bar{l}_i = d_p(p_i), \quad \bar{l}_j = d_p(p_j), \quad \bar{l} = d(p_i, p_j) \qquad \text{and} \qquad \measuredangle \text{ at } \bar{p} \leqq \measuredangle \text{ at } p = \measuredangle (v_i, v_j);$$

we obtain the inequality:

$$\begin{aligned}\cos \measuredangle (v_i, v_j) &\leqq \frac{\cosh(\eta . \bar{l}_i) . \cosh(\eta . \bar{l}_j) - \cosh(\eta . \bar{l})}{\sinh(\eta . \bar{l}_i) . \sinh(\eta . \bar{l}_j)} \\ &\leqq \frac{\coth(\eta . \bar{l}_i)}{\sinh(\eta . \bar{l}_j)} . \left(\cosh(\eta . \bar{l}_j) - \frac{1}{\cosh(\eta . \bar{l}_j)}\right) \\ &= \coth(\eta . \bar{l}_i) . \tanh(\eta . \bar{l}_j) \leqq \frac{\bar{l}_j}{\bar{l}_i} . \eta . \bar{l}_i . \coth(\eta . \bar{l}_i);\end{aligned}$$

hence:

$$\cos \measuredangle (v_i, v_j) \leqq L^{-1} . \eta . L_1 . \coth(\eta . L_1).$$

By assumption the packing argument given in appendix A applies. □

## 3. Morse theory and coverings

We have no idea how to control in which way the topology of $M^n$ changes at the critical strata of $d_p$; thus the gluing arguments have to be eliminated from the Morse theory. We are going to use covering arguments along the lines of section 1 instead. The idea of deformation will be applied to reduce to special covering situations which we know quite a lot about. For this purpose we introduce some more language:

3.1 Definition. — *Given* $\rho > 1$; *a ball* $B = B(p, r)$ *in* $M^n$ *is said to be* $\rho$*-compressible to* $\tilde{B} = B(\tilde{p}, \tilde{r})$, *if and only if:*

(i) $\tilde{r} \leqq (1-\rho^{-1}).r$,

(ii) $\rho.\tilde{B} \subset \rho.B$,

*and*

(iii) $\tilde{B}$ *is a deformation retract of some subset* $X \subset \rho.B$, *which also contains* $B$.

$B$ *is called* $\rho$*-incompressible, iff there does not exist a ball* $\tilde{B}$ *as above.*

The injectivity radius is a continuous positive function on $M^n$, which has a positive lower bound $r_0$ on $\rho.B$. $\rho$-Compressing the given ball $B$ repeatedly, one will therefore arrive at a $\rho$-incompressible ball or at a topological ball within finitely many steps. Thus it is natural to try and reduce to incompressible balls, when bounding the invariant $rk_*^{t^{-1}}(\rho.B, B)$.

3.2 Lemma. — *If* $t > 0$, *and if the ball* $B$ *is* $\rho$*-compressible to* $\tilde{B}$, *then:*

$$rk_*^{t^{-1}}(\rho.B, B) \leqq rk_*^{t^{-1}}(\rho.\tilde{B}, \tilde{B}).$$

*Proof.* — The claim is an immediate consequence of the following commutative diagram, where the graded maps are induced by inclusion:

$$\begin{array}{ccccc} H_*(\tilde{B}) & \xrightarrow[\text{deformation retract}]{\cong} & H_*(X) & \leftarrow & H_*(B) \\ \downarrow & & \downarrow & & \\ H_*(\rho.\tilde{B}) & \rightarrow & H_*(\rho.B) & & \end{array}$$

□

Moreover, incompressible balls allow for some statements about the critical points of several distance functions.

3.3. Lemma. — *If* $B = B(p, r)$ *is* $\rho$*-incompressible, then for any* $\tilde{p} \in (\rho-1)/2.B$ *there is a critical point* $p_c$ *of* $d_{\tilde{p}}$ *such that:*

$$\min\{(1-\rho^{-1}).r,\ r-\rho^{-1}.d(p, \tilde{p})\} \leqq d(\tilde{p}, p_c) \leqq r + d(p, \tilde{p}) \leqq \frac{1+\rho}{2}.r.$$

*Proof.* — Conversely, let us assume that there is no critical point $p_c$ of $d_{\tilde{p}}$ which obeys the above inequalities. We put:

$$\tilde{r} \ := \min\{(1-\rho^{-1}).r,\ r-\rho^{-1}.d(p, \tilde{p})\}$$

and consider the balls $\tilde{B} := B(\tilde{p}, \tilde{r})$ and $X := B(\tilde{p}, r+d(p, \tilde{p}))$. We point out that $r+2.d(p, \tilde{p}) \leqq \rho.r$, hence $B \cup \tilde{B} \subset X \subset B(p, r+2.d(p, \tilde{p})) \subset \rho.B$ and $\rho.\tilde{B} \subset \rho.B$. Lemma 2.1 gives rise to a vector field $v_{\tilde{p}}$ which does not vanish on the closed annulus $\bar{X} \setminus \tilde{B}$. As $d_{\tilde{p}}$ is monotone decreasing along the integral curves of $v_{\tilde{p}}$, we obtain a retraction map, and, in contrast to the hypothesis, B turns out to be $\rho$-compressible to $\tilde{B}$. □

We proceed and consider the covering situation in some more detail.

3.4. Assumptions. — Let $\xi$, $\tilde{\xi}$, L, and $L_1$ be some positive real numbers; define functions $\rho$, $q$, $t_0$, and $N_0$ by

$$\rho := 3+2.L^{-1}$$

$$q := (2+3.L)^{-1}$$

$$t_0 := \left(\frac{2L}{\tilde{\xi}.L_1}.\sinh\frac{\tilde{\xi}.L_1}{2L}\right)^{n-1}.\left(1+8.\left(5+\frac{2}{L}\right)^n\right)^n$$

and

$$N_0 := \left(\frac{L}{\tilde{\xi}.L_1}.\sinh\frac{\tilde{\xi}.L_1}{L}\right)^{n-1}.\left(1+4.(2+3L).\left(5+\frac{2}{L}\right)^n\right)^n.$$

We *suppose* that:

(i) the ball $\rho.B$ associated to $B=B(p, r)$ is $(k; L_1, L, (1+2L).r)$-framed.

(ii) the curvatures in $\rho.B$ are bounded from below by $-\tilde{\xi}^2$.

(iii) the curvatures in $B(p, (1+L^{-1}).L_1)$ are bounded from below by $-\xi^2$.

Furthermore it is useful to introduce the notation:

$$\mathrm{cont}_k^{t^{-1}}(L_1, L, \tilde{\xi}) := \sup\{ rk_*^{t^{-1}}(\rho.B, B) \mid \text{the ball B meets the conditions 3.4 (i) and 3.4 (ii)}\}.$$

3.5 Lemma. — (i) *Suppose that the assumptions* 3.4 *hold; then for any* $t \geqq t_0$ *there is the estimate:*

$$rk_*^{t^{-1}}(\rho.B, B) \leqq (e-1).N_0.\sup\{ rk_*^{t^{-1}}(\rho.\tilde{B}, \tilde{B}) \mid \tilde{B}=B(\tilde{p}, \tilde{r}) \text{ where } \tilde{r} \leqq q.r \text{ and } \tilde{p} \text{ lies in B}\}.$$

(ii) *If moreover* B *is* $\rho$*-incompressible, then all the balls* $\rho.\tilde{B}$ *on the right-hand side of the above estimate are* $(k+1; L_1, L, (1+2L).q.r)$*-framed.*

(iii) *If* $t \geqq t_0(L_1, L, \tilde{\xi})$, *then:*

$$\mathrm{cont}_k^{t^{-1}}(L_1, L, \tilde{\xi}) \leqq \max\{1, (e-1).N_0.\mathrm{cont}_{k+1}^{t^{-1}}(L_1, L, \tilde{\xi})\}.$$

(iv) *If condition* 3.4 (iii) *holds and if* $L \geqq \sqrt{1+\xi^2.L_1^2}.3.(1+\sqrt{2})^{n-1}$, *then:*

$$\mathrm{cont}_{2n}^{t^{-1}}(L_1, L, \tilde{\xi}) = 1.$$

*Proof.* — (i) We put $\rho_i := (2+\rho)^i$ for $0 \leqq i \leqq n$ and $\rho_{n+1} := \rho.\rho_n$. We pick a maximal set of pairwise disjoint metrical balls $B_j^{-1}$, $1 \leqq j \leqq N$, whose centres lie in B and whose radii equal $r_{-1}:=0.5.q.\rho_n^{-1}.r$. Obviously for $0\leqq i\leqq n$ the families $B_j^i:=2.\rho_i.B_j^{-1}$, $1\leqq j\leqq N$, cover B. Since $1+q.\rho=1+L^{-1}<\rho$, we conclude that the balls $B_j^{n+1}$ are contained in $\rho.B$. Therefore the estimate is a consequence of corollary 1.6, provided that (*a*) $N_0\geqq N$ and that (*b*) $t_0$ bounds from above the number of balls $B_{j'}^n$ which intersect any fixed $B_j^n$. In order to verify both the conditions, we point out that the $B_j^{-1}$ are disjoint, and that:

$$(a)\qquad B_j^{-1}\subset\left(1+\frac{q}{2.\rho_n}\right).B\subset(1+4.\rho_n.q^{-1}).B_j^{-1}\subset\left(\frac{3+q}{2.\rho_n}\right).B$$

$$(b)\quad B_{j'}^n\cap B_j^n\neq\varnothing\;\Rightarrow\;B_{j'}^{-1}\subset\left(2+\frac{1}{2.\rho_n}\right).B_j^n\subset\left(4+\frac{1}{2.\rho_n}\right).B_{j'}^n$$
$$\subset\left(1+4.q+\frac{q}{2.\rho_n}\right).B.$$

These inclusions yield:

$$(a)\qquad N\leqq\sup_j\frac{\operatorname{vol}(1+0{,}5.q.\rho_n^{-1}).B}{\operatorname{vol}B_j^{-1}}\leqq\sup_j\frac{\operatorname{vol}(1+4.\rho_n.q^{-1}).B_j^{-1}}{\operatorname{vol}B_j^{-1}}$$

$$(b)\qquad \#\{B_{j'}^n\,|\,B_{j'}^n\cap B_j^n\neq\varnothing\}\leqq\sup_{j'}\frac{\operatorname{vol}(1+8.\rho_n).B_{j'}^{-1}}{\operatorname{vol}B_{j'}^{-1}}.$$

Since all the balls are contained in $\rho.B$, the right-hand sides of these inequalities can be evaluated by means of the volume comparison theorem for concentric metrical balls (*c. f.* [BC]); we may use model spaces with constant curvature $-\xi^2$, and we compute:

$$(a)\quad N\leqq\frac{\displaystyle\int_0^{1+4\,\rho_n/q}\sinh(\xi.\sigma.r_{-1})^{n-1}\,d\sigma}{\displaystyle\int_0^1\sinh(\xi.\sigma.r_{-1})^{n-1}\,d\sigma}$$
$$\leqq\left(1+\frac{4}{q}.\rho_n\right).\sup\left\{\frac{\sinh\xi.\sigma.(1+(4/q).\rho_n).r_{-1}}{\sinh\xi.\sigma.r_{-1}}\,\Big|\,0\leqq\sigma\leqq1\right\}^{n-1}$$
$$\leqq\left(1+\frac{4}{q}.\rho_n\right)^n.\left(\frac{\sinh\xi.(1+(4/q).\rho_n).r_{-1}}{\xi.(1+(4/q).\rho_n).r_{-1}}\right)^{n-1}\leqq N_0.$$

The last step is due to the fact that:

$$\left(1+\frac{4}{q}.\rho_n\right).r_{-1}=\left(2+\frac{q}{2.\rho_n}\right).r\leqq(2+L^{-1}).r\leqq L_1/L$$

and $\rho_n . q^{-1}=(2+3\,L).(5+(2/L))^n$.

$$(b)\quad \#\{B^n_{j'} \mid B^n_{j'} \cap B^n_j \neq \varnothing\} \leqq \frac{\int_0^{1+8\rho_n} \sinh(\xi.\sigma.r_{-1})^{n-1}\,d\sigma}{\int_0^{1} \sinh(\xi.\sigma.r_{-1})^{n-1}\,d\sigma}$$

$$\leqq (1+8.\rho_n)^n.\left(\frac{\sinh \xi.(1+8\rho_n).r_{-1}}{\xi.(1+8\rho_n).r_{-1}}\right)^{n-1} \leqq t_0.$$

This time the last estimate is due to the fact that:

$$(1+8.\rho_n).r_{-1}=\left(4+\frac{1}{2.\rho_n}\right).q.r \leqq \frac{9}{2}.(2+3\,L)^{-1}.(1+2\,L)^{-1}.L_1 \leqq \frac{L_1}{2.L}.$$

(ii) It follows from lemma 3.3 that for any point $\tilde{p} \in (1+q.\rho).B$ there exists a critical point $p_c$ of $d_{\tilde{p}}$, which obeys:

$$L.d(p_c, \tilde{p}) \leqq L.(2+q.\rho).r=(1+2\,L).r$$

and

$$d(p_c, \tilde{p}) \geqq (1-\rho^{-1}.(1+q.\rho)).r=q.(\rho.L-L-1).r=q.(1+2\,L).r;$$

therefore the set $(1+q.\rho).B$ as well as the subballs $\rho.\tilde{B}$ are $(k+1;$ $L_1$, L, $(1+2\,L).q.r)$-framed.

(iii) Obviously $rk_*^{t^{-1}}(\rho.B, B)=1$, if the metrical ball B is a topological ball as well. Therefore lemma 3.2 reduces the proof to the case where B is a $\rho$-incompressible ball. The estimate given in (i) holds, and the property (ii) allows to bound the right-hand side as desired.

(iv) Suppose B were a $\rho$-incompressible ball which obeyed the conditions 3.4 (i), (ii) and (ii) with $k=2n$; then by means of (ii) there would exist $(2n+1;\ L_1,\ L,\ 0)$-framed balls. As by hypothesis

$$L \geqq \sqrt{1+\xi^2.L_1^2}.3.(1+\sqrt{2})^{n-1} \geqq \xi.L_1.\coth(\xi.L_1).3.(1+\sqrt{2})^{n-1},$$

the above conclusion contradicts to lemma 2.3 (ii). □

3.6 Proposition. — *Suppose that the ball* $B=B(p, r)$ *obeys the conditions* 3.4 (ii) *and* (iii) *with* $L \geqq \sqrt{1+\xi^2.L_1^2}.3.(1+\sqrt{2})^{n-1}$; *moreover assume that the boundary of* $B(p, L_1)$ *in* $M^n$ *is non-empty and that* $L_1 \geqq 2r.(L+2+L^{-1})$. *Then for any* $t \geqq t_0(L_1, L, \xi)$, *one has:*

$$rk_*^{t^{-1}}(\rho.B, B) \leqq (e-1)^{2n-1}.N_0(L_1, L, \xi)^{2n-1}.$$

*Proof.* — We fix a point $p_1$ on the boundary of $B(p, L_1)$; it is easy to verify that $\rho.B=(3+2.L^{-1}).B$ is $(1;\ L_1,\ L,\ (1+2\,L).r)$-framed. We apply lemma 3.5 (iii) inductively and use 3.5 (iv) in order to stop at $k=2n$. □

Heuristically speaking, the proposition bounds the topology of small metrical balls B in a Riemannian manifold $M^n$. All the conditions can be formulated in terms of the curvature, the diametre of $M^n$, and the radius of the ball B. No assumption on the injectivity radius is required.

3.7. Corollary (c.f. Gromov). — *If* $M^n$ *is non-negatively curved, non-compact, and connected, and if* $t \geqq 2^{1/n}.8^n.5^{n^2}$, *then for any ball* $B=B(p, r)$ *in* $M^n$ *the following estimate holds:*

$$rk_*^{t^{-1}}(M^n, B) \leqq rk_*^{t^{-1}}(3.3.B, B) \leqq \exp(5.n^3+3.5.n^2).$$

*Proof.* — We put $\xi := \tilde{\xi} := 0$, $L := 3.(1+\sqrt{2})^{n-1}$, and $L_1 := 2r.L.(1+L^{-1})^2$.

Then the proposition applies; to make things more explicit, we make use of the following computations:

$$t_0 = \left(1+8.5^n.\left(1+\frac{2}{5L}\right)^n\right)^n \leqq 8^n.5^{n^2}.\exp\left(\frac{2n^2}{5.L}+\frac{n}{8}.5^{-n}\right) \leqq 2^{1/n}.8^n.5^{n^2}$$

and

$$N_0 = \left(1+12.L.5^n.\left(1+\frac{2}{5L}\right)^n.\left(1+\frac{2}{3L}\right)\right)^n \leqq (12.L)^n.5^{n^2}.\exp\left(\frac{2n^2}{5L}+\frac{2n}{3L}+\frac{n.5^{-n}}{12.L}\right)$$

i. e.

$$\begin{aligned}((e-1).N_0)^{2n-1} &\leqq (5.(1+\sqrt{2}))^{n^2.(2n-1)}.\left(\frac{36.\sqrt{e-1}}{1+\sqrt{2}}\right)^{n.(2n-1)} \\ &\quad .\exp\left(\frac{n.(2n-1)}{L}.\left(\frac{2n}{5}+\frac{2}{3}+\frac{1}{12}.5^{-n}\right)\right) \\ &\leqq (5.(1+\sqrt{2}))^{(n^2+1.2.n).(2n-1)}.5 \leqq 5.\exp(5.n^3+3.5.n^2-3.n). \quad \square\end{aligned}$$

## 4. Metrical annuli in asymptotically non-negatively curved manifolds

Most of the preceding results are valid for arbitrary Riemannian manifolds; especially proposition 3.6 holds in general. In this section we are going to specialize to asymptotically non-negatively curved manifolds $(M^n, g, o)$. Our goal is to get rid of the assumption on the diametre of $M^n$. Towards this purpose it is natural to consider metrical annuli

$$A(R_1, R_2) := \overline{B(o, R_2)} \setminus B(o, R_1)$$

around the base point $o$ of $M^n$. We want to bound from above

$$rk_*^{t^{-1}}(A(1-\varepsilon).R_1, (1+\varepsilon).R_2), A(R_1, R_2)),$$

provided $t$ and $\varepsilon$ are sufficiently large. The idea is to cover the annuli by balls of a very special type: a metrical ball $B=B(p, r)$ in $M^n$ is said to be $\delta$-*small* ($\delta>0$), iff $r=\delta.d_0(p)$.

We recall that by lemma II.1.1 the curvatures at a point $p \in M^n$ are bounded from below by

$$-2.b_0.f(d_0(p)).d_0(p)^{-2};$$

here $r \mapsto f(r)$ denotes a monotone non-increasing error function which takes values in [0, 1] and converges to 0 for $r \to \infty$.

4.1. ASSUMPTIONS. — Let $L_0 := 3.(1+\sqrt{2})^{n-1}$ and let $\eta < (1+\sqrt{b_0.f(R_1/2L_0)})^{-1}$ be some positive number; we put:

$$L := L_0.\sqrt{1+2.b_0.f\left(\frac{R_1}{2L_0}\right).\left(\frac{\eta}{1-\eta}\right)^2}$$

$$\rho := 3+2.L^{-1}$$

and

$$\varepsilon_n := \frac{1}{2}.L^3.(1+L)^{-4}.\eta.$$

4.2 LEMMA

(i)
$$L_0 \leqq L \leqq \sqrt{3}.L_0 \leqq 2.L_0 - 1$$

$$\varepsilon_n \leqq \frac{\eta}{2.(L+4)} \leqq \frac{\eta}{22}$$

$$\rho.\varepsilon_n \leqq \frac{9}{2}.\frac{\eta}{3L+10} \leqq \frac{\eta}{7}.$$

(ii) *If* $0 < \delta \leqq \varepsilon_n$ *and* $B = B(p, r)$ *is any* $\delta$*-small ball in* $M^n$ *with centre* $p$ *in* $A(R_1, R_2)$, *then the estimate*

$$rk_*^{t^{-1}}((\rho.B, B) \leqq (e-1)^{2n-1}.\tilde{N}_0^{2n-1}$$

*holds for*

$$\tilde{N}_0 := (12.L)^n.5^{n^2}.\exp\left(\frac{n}{L}.\left(\frac{2n}{5}+\frac{2}{3}+\frac{1}{12}.5^{-n}\right)+\frac{\sqrt{2}.(n-1)}{L+2}\right)$$

*and for all*

$$t \geqq \tilde{t}_0(n) := 8^n.5^{n^2}.\exp\left(\frac{2n^2}{5L}+\frac{n-1}{\sqrt{2}.(L+2)}+\frac{n}{8}.5^{-n}\right).$$

*Proof.* — (i) These estimates are obvious consequences of the definitions.

(ii) We define

$$L_1 := 2L.(1+L^{-1})^2.\varepsilon_n.d_0(p) = \left(\frac{L}{L+1}\right)^2.\eta.d_0(p).$$

It is easy to verify that:

(*a*) the curvatures in $B(p, (1+L^{-1}).L_1) \subset B(p, (L.\eta/(L+1)).d_0(p))$ are bounded from below by $-\xi^2$, where

$$\xi := \sqrt{2.b_0.f\left(\frac{R_1}{L+1}\right)}.\left(1-\frac{L.\eta}{L+1}\right)^{-1}.d_0(p)^{-1}.$$

(*b*)
$$\xi.L_1 \leqq \sqrt{2.b_0.f\left(\frac{R_1}{L+1}\right)}.\frac{\eta}{1-\eta},$$

i. e. :

$$L \geqq \sqrt{1+\xi^2.L_1^2}.L_0.$$

(*c*) the curvatures in the ball $\rho.B \subset B(p, \rho.\varepsilon_n.d_0(p)) \subset B(p, (\eta/7).d_0(p))$ are bounded from below by $-\tilde{\xi}^2$, where

$$\tilde{\xi} := \sqrt{2.b_0.f\left(\frac{R_1}{L+1}\right)}.\left(1-\frac{\eta}{7}\right)^{-1}.d_0(p)^{-1}.$$

Therefore proposition 3.6 applies; it remains to compute $N_0(L_1, L, \tilde{\xi})$ and $t_0(L_1, L, \tilde{\xi})$. Since

$$\frac{2.(1+L^{-1})^2.\varepsilon_n}{1-(\eta/7)} = \frac{L}{(1+L)^2}.\frac{\eta}{1-(\eta/7)} \leqq \frac{1}{L+2}.\frac{\eta}{1-\eta},$$

we see that:

$$\frac{\tilde{\xi}.L_1}{L} = \frac{1}{L+2}.\frac{\eta}{1-\eta}.\sqrt{2.b_0.f\left(\frac{R_1}{L+1}\right)} \leqq \frac{\sqrt{2}}{L+2}.$$

Because of the inequality

$$\frac{\sinh(x)}{x} \leqq \cosh(x) \leqq \exp(|x|)$$

we obtain that:

$$t_0 \leqq \exp\left(\frac{n-1}{\sqrt{2}.(L+2)}\right).\left(1+8.\left(5+\frac{2}{L}\right)^n\right)^n \leqq \tilde{t}_0(n),$$

and

$$N_0 \leqq \exp\left(\frac{\sqrt{2}.(n-1)}{L+2}\right).\left(1+12.L.\left(1+\frac{2}{3L}\right).\left(5+\frac{2}{L}\right)^n\right)^n \leqq \tilde{N}_0. \quad \square$$

4.3. Construction. — There is a sequence of numbers

$$0<\varepsilon_{-1}<\varepsilon_0<\ldots<\varepsilon_n<\varepsilon_{n+1}<1,$$

uniquely determined by the following conditions:

$$\varepsilon_n \text{ is the number given in 4.1,}$$

$$\varepsilon_{n+1} := \rho.\varepsilon_n,$$

$$\varepsilon_{i+1}=(2+\rho.(1+\varepsilon_i)).(1-\varepsilon_i)^{-1}.\varepsilon_i, \qquad 0\leqq i<n,$$

and

$$\varepsilon_0 := 2.\varepsilon_{-1}.(1-\varepsilon_{-1})^{-1}.$$

(i. e. $\varepsilon_{-1}=\varepsilon_0.(2+\varepsilon_0)^{-1}.$)

We put $\rho_i := \varepsilon_i.\varepsilon_0^{-1}$ for $-1\leqq i\leqq n+1$.

We pick a maximal family of disjoint $\varepsilon_{-1}$-small balls $B_j^{-1}$, $1\leqq j\leqq N$, whose centres $p_j$ lie in $A(R_1, R_2)$. Moreover, we shall consider all balls

$$B_j^i := \rho_i.(2+\varepsilon_0).B_j^{-1}, \qquad 1\leqq j\leqq N, \qquad 0\leqq i\leqq n+1.$$

4.4. Immediate consequences:

(i) $p\in B_j^i \Rightarrow (1-\varepsilon_i).d_0(p_j)\leqq d_0(p)\leqq(1+\varepsilon_i).d_0(p_j)$.

(ii) $B_{j'}^i\cap B_j^i\neq\varnothing \Rightarrow \begin{cases} p_{j'}\in 2.(1-\varepsilon_i)^{-1}.B_j^i \text{ and} \\ \sigma.B_{j'}^i\subset(2+\sigma.(1+\varepsilon_i)).(1-\varepsilon_i)^{-1}.B_j^i \text{ for all } \sigma>0. \end{cases}$

(iii) the balls $B_j^0$, $1\leqq j\leqq N$, cover the annulus $A(R_1, R_2)$.

In order to prove (iii), let us assume that there is some point $p_{N+1}\in A(R_1, R_2)\setminus\bigcup\{B_j^0\mid 1\leqq j\leqq N\}$; it is a consequence of (ii) that the ball $B_{N+1}^{-1}:=B(p_{N+1}, \varepsilon_{-1}.d_0(p_{N+1}))$ is disjoint from the other $B_j^{-1}$. This conclusion contradicts the maximality of the family $(B_j^{-1})_{j=1}^N$.

We point out that for sufficiently large $t$ the corollary 1.6 gives rise to an upper bound on

$$rk_*^{t-1}(A((1-\varepsilon_{n+1}).R_1, (1+\varepsilon_{n+1}).R_2), A(R_1, R_2))$$

in terms of the quantities $rk_*^{t-1}(\rho.B_j^i, B_j^i)$, $0\leqq i\leqq n$, $1\leqq j\leqq N$, which in turn have already been controlled in lemma 4.3.

We continue listing some inequalities which will be used in the subsequent computations.

(iv) $5^i\leqq\rho_i\leqq 5^i.\exp\left(\dfrac{2i}{5L}+\dfrac{1}{4.(L+4)}\right), \qquad 0\leqq i\leqq n.$

$$2.5^n\leqq\varepsilon_n.\varepsilon_{-1}^{-1}=\rho_n.(2+\varepsilon_0)=2.\rho_n+\varepsilon_n\leqq 2.5^n.\exp\left(\frac{2n}{5L}+\frac{1+5^{-n}}{4.(L+4)}\right).$$

(c.f. Appendix B.)

(v) $\varepsilon_{-1}^{-1} \leqq 4.5^n.\frac{L}{\eta}.(1+L^{-1})^4.\exp\left(\frac{2n}{5L}+\frac{1+5^{-n}}{4.(L+4)}\right) \leqq 4.5^n.\frac{L}{\eta}.\exp\left(\frac{2n}{5L}+\frac{17}{4L}\right).$

(vi) $B_{j'}^n \cap B_j^n \neq \varnothing \quad \Rightarrow \quad B_{j'}^{-1} \subset (2+\varepsilon_{-1}.\varepsilon_n^{-1}.(1+\varepsilon_n)).(1-\varepsilon_n)^{-1}.B_j^n \subset \tau_n.B_{j'}^n,$

where

$$\tau_n := \left(\frac{2}{1-\varepsilon_n}\right)^2 + \varepsilon_{-1}.\varepsilon_n^{-1}.\left(\frac{1+\varepsilon_n}{1-\varepsilon_n}\right)^2.$$

(vii) $\left(\frac{1}{1-\varepsilon_n}\right)^2 \leqq \left(1+\frac{1}{2L+7}\right)^2 = 1+\frac{4L+15}{(2L+7)^2} \leqq \frac{4L+17}{4L+13},$

$\left(\frac{1+\varepsilon_n}{1-\varepsilon_n}\right)^2 \leqq \left(1+\frac{2}{2L+7}\right)^2 = 1+4.\frac{2L+8}{(2L+7)^2} \leqq \frac{L+5}{L+3}.$

(viii) $\tau_n.\varepsilon_n \leqq \left(4.\frac{4L+17}{4L+13}+\frac{1}{2}.5^{-n}.\frac{L+5}{L+3}\right).\frac{\eta}{2.(L+4)}$

$$\leqq \frac{2\eta}{L+3}.\left(1-\frac{1}{(L+4).(4L+13)}+\frac{1}{8}.5^{-n}.\left(1+\frac{1}{L+4}\right)\right)$$

$$\leqq \frac{2\eta}{L+3}.\left(1+\frac{1}{8}.5^{-n}\right).$$

(ix) $\frac{\tau_n.\varepsilon_n}{\varepsilon_{-1}} \leqq 8.\left(1+\frac{1}{2L+7}\right)^2.5^n.\exp\left(\frac{2n}{5L}+\frac{1+5^{-n}}{4.(L+4)}\right)+\left(1+\frac{2}{2L+7}\right)^2$

$$\leqq (1+8.5^n).\exp\left(\frac{2n}{5L}+\frac{5+5^{-n}}{2.(2L+7)}\right).$$

4.5. LEMMA. — *The number of balls $B_{j'}^n$ which intersect a given ball $B_j^n$ is bounded from above by*

$$t_1(n) = 5^{n^2}.8^n.\exp\left(\frac{8}{3.(n+1)}\right).$$

*Proof.* — Since by construction the balls $B_{j'}^{-1}$ are disjoint, we can deduce from 4.4 (vi) that:

$$(\bigstar) \qquad \#\{B_{j'}^n \mid B_{j'}^n \cap B_j^n \neq \varnothing\} \leqq \sup_{j'} \frac{\operatorname{vol} \tau_n.B_{j'}^n}{\operatorname{vol} B_{j'}^{-1}}.$$

The curvatures in the ball $\tau_n.B_{j'}^n$ are bounded from below by $-\xi_{j'}^2$ where

$$\xi_{j'} := \sqrt{2.b_0.f\left(\frac{R_1}{L+1}\right)}.(1-\tau_n.\varepsilon_n)^{-1}.d_0(p_{j'})^{-1}.$$

Since

$$\frac{\tau_n . \varepsilon_n}{1-\tau_n . \varepsilon_n} \leqq \frac{2}{L+3} . \frac{\eta}{1-\eta} . \left(1+\frac{1}{8} . 5^{-n}\right)$$

and since $p_{j'} \in A(R_1, R_2)$, it follows that:

$$\xi_{j'} . \tau_n . \varepsilon_n . d_0(p_{j'}) \leqq \frac{2\sqrt{2}}{L+3} . \left(1+\frac{1}{8} . 5^{-n}\right).$$

Therefore the volume comparison for concentric metrical balls yields:

$$\frac{\operatorname{vol} \tau_n . B_{j'}^n}{\operatorname{vol} B_{j'}^{-1}} \leqq \frac{\displaystyle\int_0^{\tau_n . \varepsilon_n . d_0(p_{j'})} \sinh(\xi_{j'} . \sigma)^{n-1} d\sigma}{\displaystyle\int_0^{\varepsilon_{-1} . d_0(p_{j'})} \sinh(\xi_{j'} . \sigma)^{n-1} d\sigma} \leqq \left(\frac{\tau_n . \varepsilon_n}{\varepsilon_{-1}}\right)^n . \left(\frac{\sinh(\xi_{j'} . \tau_n . \varepsilon_n . d_0(p_{j'}))}{\xi^{j'} . \tau_n . \varepsilon_n . d_0(p_{j'})}\right)^{n-1}.$$

We plug this estimate into (★) and obtain by means of 4.4 (ix) that:

$$\# \{ B_{j'}^n \mid B_{j'}^n \cap B_j^n \neq \varnothing \}$$
$$\leqq 8^n . 5^{n^2} . \exp\left(\frac{2n^2}{5L} + n . \frac{5+5^{-n}}{2.(2L+7)} + \frac{n}{8} . 5^{-n} + \frac{2\sqrt{2}.(n-1)}{L+3} . \left(1+\frac{5^{-n}}{8}\right)\right) \leqq t_1(n). \quad \square$$

In order to bound the number N of balls $B_j^0$ in the covering, we look at the pulled back situation under $\exp_0 : T_0 M \to M$. We pick a family of vectors $v_j \in T_0 M$ such that

(i) $\exp_0(v_j)$ is the centre $p_j$ of $B_j^0$;

(ii) $\| v_j \| = d_0(p_j)$, i. e. the curve $t \mapsto \exp_0(t . v_j)$, $t \in [0, 1]$, is minimizing.

Given numbers $0 < \zeta_1, \zeta_2, \zeta_3 < 1$ and a non-zero vector $w \in T_0 M$, we consider the sets

$$AC(w; \zeta_1, \zeta_2, \zeta_3) := \{ v \in T_0 M \| \measuredangle(v, w) | \leqq \zeta_1 . \zeta_3 . \exp(-b_1) \text{ and }$$
$$(1-\zeta_1).(1-\zeta_2).|w| \leqq |v| \leqq (1-\zeta_1).(1+\zeta_2).|w| \}.$$

4.6. Lemma. — (i) *If* $1-\varepsilon_{-1} \leqq (1-\zeta_1).(\sqrt{1-\zeta_3^2}-\zeta_2)$, *then* $\exp_0$ *maps the set* $AC(v_j; \zeta_1, \zeta_2, \zeta_3)$ *into* $B_j^{-1}$ *for all j.*

(ii) *The number of disjoint sets* $AC(w; \zeta_1, \zeta_2, \zeta_3)$ *in the annulus*

$$A := \{ v \in T_0 M \mid (1-\zeta_1).(1-\zeta_2).R_1 \leqq |v| \leqq (1-\zeta_1).(1+\zeta_2).R_2 \}$$

*is bounded from above by*

$$\left(2+\zeta_2^{-1} . \ln\frac{R_2}{R_1}\right) . \pi^{n-1} . (2.\zeta_1 . \zeta_3)^{1-n} . \exp((n-1).b_1)$$

(iii) *The number* N *of distincts balls* $B_j^{-1}$ *in* $M^n$ *is bounded from above by*

$$N_1 := \left(\frac{5^{-n}}{8L+32} + \ln\frac{R_2}{R_1}\right).16.(16.\pi)^{n-1}.\left(\frac{1}{\eta}.L.5^n\right)^{(3n-1)/2}$$
$$\times \exp\left(\frac{3n-1}{L}.\left(\frac{n}{5}+\frac{17}{8}\right)\right).\exp((n-1).b_1).$$

*Proof.* — (i) It is sufficient to show that $\exp_0$ maps the sets $AC(v_j; \zeta_1, 0, \zeta_3)$ into the $(\varepsilon_{-1}-\zeta_2.(1-\zeta_1))$-small balls $(1-\zeta_2.(1-\zeta_1).\varepsilon_{-1}^{-1}).B_j^{-1}$. This amounts to studying the generalized geodesic triangle $\Delta=(p_j, p, o)$, where $p$ is the image under $\exp_0$ of any vector $v\in AC(v_j; \zeta_1, 0, \zeta_3)$.

Thus, in the notation of proposition III.1, we have:

$$l_1 = d_0(p_j),$$
$$l_0 = d_0(p) \leqq (1-\zeta_1).l_1,$$

and

$$\cos(\measuredangle\ at\ o) \geqq \sqrt{1-\zeta_1^2.\zeta_3^2.\exp(-2.b_1)}$$

Hence, the proposition applies and yields the desired inequality:

$$d(p, p_j) \leqq l_1 - l_0.\sqrt{1-\zeta_3^2} \leqq (1-(1-\zeta_1).\sqrt{1-\zeta_3^2}).d_0(p_j) \leqq (\varepsilon_{-1}-\zeta_2.(1-\zeta_1)).d_0(p_j).$$

(ii) We make use of the diffeomorphism

$$\Phi : T_0 M\setminus\{0\} \to \mathbb{S}^{n-1}\times\mathbb{R},$$

$v\mapsto(|v|^{-1}.v, \ln|v|)$ and the canonical volume form on $\mathbb{S}^{n-1}\times\mathbb{R}$, and we compute $-B^S(r)$ will denote a ball of radius $r$ in $\mathbb{S}^{n-1}$—:

# disjoint sets $AC(w; \zeta_1, \zeta_2, \zeta_3)$ in A

$$\leqq \sup_w \frac{\operatorname{vol}\Phi(A)}{\operatorname{vol}\Phi(AC(w; \zeta_1, \zeta_2, \zeta_3))}$$
$$\leqq \ln\frac{(1+\zeta_2).R_2}{(1-\zeta_2).R_1}.\left(\ln\frac{1+\zeta_2}{1-\zeta_2}\right)^{-1}.\frac{\operatorname{vol}\mathbb{S}^{n-1}}{\operatorname{vol}B^S(\zeta_1.\zeta_3.\exp(-b_1))}$$
$$\leqq\left(1+\left(\ln\frac{1+\zeta_2}{1-\zeta_2}\right)^{-1}.\ln\frac{R_2}{R_1}\right).2.\pi^{n-1}.(2.\zeta_1.\zeta_3.\exp(-b_1))^{1-n}.$$

This inequality immediately yields the claimed bound, as it is known that $\ln(1+\zeta_2)-\ln(1-\zeta_2)\geqq 2\zeta_2$ for all $\zeta_2\geqq 0$.

(iii) If $\zeta_1+\zeta_2+\zeta_3^2=\varepsilon_{-1}$, then the hypothesis of (i) are meet, the sets $AC(v_j; \zeta_1, \zeta_2, \zeta_3)$ are disjoint, and part (ii) yields the required control on N. We put

$$\zeta_1=2.\zeta_2=2.\zeta_3^2=0.5.\varepsilon_{-1}$$

and obtain the estimate:

$$N\leqq\left(2.\varepsilon_{-1}+4.\ln\frac{R_2}{R_1}\right).(2.\pi)^{n-1}.\sqrt{\varepsilon_{-1}}^{-(3n-1)}.\exp((n-1).b_1).$$

Using the estimates given in 4.4, one easily verifies that the right-hand side is dominated by $N_1$. □

4.7. PROPOSITION. — *Assume like in* 4.1 *that* $0<R_1\leqq R_2$ *and that*

$$\eta\leqq(1+\sqrt{b_0.f(R_1/2L_0)})^{-1}.$$

*If moreover*

$$t\geqq t_1(n):=5^{n^2}.8^n.\exp\left(\frac{8}{3.(n+1)}\right)$$

*the following estimates hold:*

(i) $$rk_*^{t^{-1}}\left(A\left(\frac{6}{7}.R_1,\frac{8}{7}.R_2\right), A(R_1,R_2)\right)$$
$$\leqq C_a(n).\left(\frac{1}{2000}+\ln\frac{R_2}{R_1}\right).\sqrt{\eta.\sqrt{2}}^{-(3n-1)}.\exp((n-1).b_1),$$

*where*

$$C_a(n):=(e-1)^{2n}.(5.(1+\sqrt{2}))^{\tilde{c}_a(n)},$$

*and*

$$\tilde{c}_a(n):=2.n^3+\frac{19}{6}.n^2+\frac{5}{12}.n+\frac{3}{4}.$$

(ii) $$\lim_{R_1\to\infty} rk_*^{t^{-1}}\left(A\left(\frac{6}{7}.R_1,\frac{8}{7}.R_1\right), A(R_1,R_1)\right)\leqq C_e(n).\exp((n-1).b_1),$$

*where*

$$C_e(n):=(5.(1+\sqrt{2}))^{\tilde{c}_e(n)},$$

*and*

$$\tilde{c}_e(n):=2.n^3+\frac{19}{6}.n^2-\frac{1}{3}.n-\frac{1}{3}.$$

*Remark.* — We point out that

$$\frac{1}{\sqrt{2}}\cdot\left(1+\sqrt{b_0\cdot f\left(\frac{R_2}{R_1}\right)}\right)\leqq\frac{1}{\sqrt{2}}\cdot(1+\sqrt{b_0})\leqq\sqrt{1+b_0}.$$

Since by remark II.2.4 the last term does not exceed $\exp((1/2).b_1)$, it is *admissible* to pick:

$$\frac{1}{\sqrt{2}.\eta}:=\exp\left(\frac{1}{2}.b_1\right).$$

*Proof.* — By continuity it is sufficient to treat the case $\eta<(1+\sqrt{b_0.f(R_1/2L_0)})^{-1}$. We are going to consider the covering constructed in 4.3. One easily verifies that $t_1(n)\geqq\tilde{t}_0(n)$; hence it is possible to apply corollary 1.6 and lemma 4.2 (ii):

$$\begin{aligned}
rk_*^{t_1^{-1}}\left(A\left(\frac{6}{7}.R_1,\frac{8}{7}.R_2\right),A(R_1,R_2)\right)
&\leqq(e-1).N_1.\sup\{rk_*^{t_1^{-1}}(\rho.B,B)\,|\,B\text{ is }\delta\text{-small with }\delta\leqq\varepsilon_n\\
&\qquad\text{and has centre in the set }A(R_1,R_2)\}\\
&\leqq(e-1)^{2n}.\tilde{N}_0^{2n-1}.N_1\\
&=(e-1)^{2n}.c_1.c_2.\left(\frac{5^{-n}}{8L+32}+\ln\frac{R_2}{R_1}\right).\sqrt{\eta}^{1-3n}.\exp((n-1).b_1).
\end{aligned}$$

Here we have used the abbreviations:

$$c_1:=((12.L)^n.5^{n^2})^{2n-1}.\sqrt{5^n.L^{3n-1}}.(16.\pi)^{n-1}.16$$

and

$$c_2:=\exp\left(\frac{2n^2-n}{L}.\left(\frac{2n}{5}+\frac{8+5^{-n}}{12}\right)+\frac{3n-1}{L}.\left(\frac{n}{5}+\frac{17}{8}\right)+\frac{(2n-1).(n-1).\sqrt{2}}{L+2}\right).$$

Since $L\geqq3.(1+\sqrt{2})^{n-1}$, we have $c_2\leqq36$ for all $n\geqq2$.

(i) Observing that $5^{-n}/(8L+32)\leqq1/2000$, it is sufficient to show that

$$(e-1)^{2n}.2^{(3n-1)/4}.c_1.c_2\leqq C_a(n).$$

Obviously

$$2^{(3n-1)/4}.c_1\leqq(5.(1+\sqrt{2}))^{n^2.(2n-1)+n.(3n-1)/2}.\left(\frac{36.\sqrt{3}}{1+\sqrt{2}}\right)^{n.(2n-1)}$$

$$\times\left(\frac{3.\sqrt{6}}{1+\sqrt{2}}\right)^{(3n-1)/2}.(16.\pi)^{n-1}.16,$$

and the result is due to the inequalities:

$$\frac{36.\sqrt{3}}{1+\sqrt{2}} \leqq (5.(1+\sqrt{2}))^{4/3}, \qquad \left(\frac{3.\sqrt{6}}{1+\sqrt{2}}\right)^{3/2}.16.\pi \leqq (5.(1+\sqrt{2}))^{9/4},$$

and

$$\frac{3.\sqrt{6}}{1+\sqrt{2}}.16.c_2 \leqq (5.(1+\sqrt{2}))^3.$$

(ii) Since the error function $f$ converges to zero for $R_1 \to \infty$, it is possible to pick a function $\eta(R_1)$ which converges to 1 for $R_1 \to \infty$.

Thus one is reduced to checking that

$$(e-1)^{2n}.c_1.c_2.\frac{5^{-n}}{8.L} \leqq C_e(n).$$

This can be done calculating in a similar way as above. □

## 5. The global estimates

The special case of non-negative curvature has been treated in corollary 3.7, and the Betti numbers of the ends of $M^n$ have been bounded in proposition 4.7 (ii). It remains to consider the general case and piece together the estimates on metrical annuli in an arbitrary asymptotically non-negatively curved manifold.

We look at a sequence of critical points $p_1, \ldots, p_k$ of the distance function $d_0$ such that:

$$d_0(p_i) \geqq e.d_0(p_{i+1}), \qquad 1 \leqq i < k,$$

and that its length $k$ is maximal. In the terminology of 2.2 this is a metrical $(k;\, e,$ $0)$-frame of the base point $o$. It is useful to consider the annuli

$$A_i := A(e^{-1}.d_0(p_i),\, e.d_0(p_i)), \qquad 1 \leqq i \leqq k.$$

5.1. IMMEDIATE CONSEQUENCES. — (i) $k \leqq 2.\pi^{n-1}.(e/(e-1))^{2n-2}.\exp((2n-2).b_1)$ [lemma 2.3].

(ii) $M^n \setminus \bigcup_i A_i$ does not contain a critical point of $d_0$; therefore lemma 2.1 gives rise to a vector field $v_0$ which does not vanish on this set.

There are numbers $0 < x_{\tilde{k}} < y_{\tilde{k}} < \ldots < x_2 < y_2 < x_1 < y_1 < x_0 := \infty$ such that:

(iii) $$\bigcup_{i=1}^{k} A_i \subset \bigcup_{i=1}^{k} A\left(\frac{6}{7}.e^{-1}.d_0(p_i),\, \frac{8}{7}.e.d_0(p_i)\right) = \bigcup_{j=1}^{\tilde{k}} A\left(\frac{6}{7}.x_j,\, \frac{8}{7}.y_j\right),$$

(iv) the annuli $A((6/7).x_j,\, (8/7).y_j)$ are disjoint.

It is convenient to also introduce the annuli $\tilde{A}_j$ which are defined as follows:

$$\tilde{A}_j := A\left(\frac{3}{4}.x_j, x_{j-1}\right), \qquad 1 \leqq j < \tilde{k},$$

$$\tilde{A}_{\tilde{k}} := \overline{B(0, x_{\tilde{k}-1})} \setminus \{0\}$$

5.2. OBSERVATIONS

(i) $\sum_{j=1}^{\tilde{k}} \left(\ln\frac{4}{3} + \ln\frac{y_j}{x_j}\right) \leqq k.\left(2+\ln\frac{4}{3}\right).$

(ii) $M^n \setminus \{0\} = \bigcup_{j=1}^{\tilde{k}} \tilde{A}_j.$

(iii) $\tilde{A}_j \cap \tilde{A}_{j+1} = A\left(\frac{3}{4}.x_j, x_j\right), \qquad 1 \leqq j < \tilde{k},$

and

$$\tilde{A}_j \cap \tilde{A}_{j'} = \varnothing \qquad \text{if} \quad |j-j'| \geqq 2.$$

(iv) the Mayer-Vietoris sequence yields estimates on the values of the Poincaré series (for $0 < t \leqq 1$):

$$P_t(M^n \setminus \{0\}) \leqq \sum_{j=1}^{\tilde{k}} P_t(\tilde{A}_j) + \sum_{j=1}^{\tilde{k}-1} P_t\left(A\left(\frac{3}{4}.x_j, x_j\right)\right)$$

$$P_t(M^n) \leqq \sum_{j=1}^{\tilde{k}} \left(P_t(\tilde{A}_j) + P_t\left(A\left(\frac{3}{4}.x_j, x_j\right)\right)\right).$$

(v) the inclusions

$$A(x_j, y_j) \subsetneq A\left(\frac{6}{7}.x_j, \frac{8}{7}.y_j\right) \subsetneq \tilde{A}_j$$

and

$$A\left(\frac{7}{8}.x_j, \frac{7}{8}.x_j\right) \subsetneq A\left(\frac{3}{4}.x_j, x_j\right)$$

allow for deformation retracts along the vector field $v_0$ as has been mentioned above.

(vi) $$P_{t^{-1}}(M^n) \leqq \sum_{j=1}^{\tilde{k}} \left(rk_*^{t^{-1}}\left(A\left(\frac{6}{7}.x_j, \frac{8}{7}.y_j\right), A(x_j, y_j)\right)\right.$$
$$\left.+ rk_*^{t^{-1}}\left(A\left(\frac{3}{4}.x_j, x_j\right), A\left(\frac{7}{8}.x_j, \frac{7}{8}.x_j\right)\right)\right).$$

(vii) if $t \geqq t_1(n)$ the right hand side of the previous formula can be estimated by means of proposition 4.7:

$$P_{t^{-1}}(M^n) \leqq \sum_{j=1}^{\tilde{k}} \left(\frac{1}{1000} + \ln \frac{y_j}{x_j}\right) . C_a(n) . \exp\left(\frac{7n-5}{4} . b_1\right).$$

We use 5.1 (i) and 5.2 (i) in order to compute the bound. This gives us the following result:

5.3 Proposition. — *Let* $M^n$ *be an asymptotically non-negatively curved manifold,* $b_1$ *be its curvature invariant (as above), and let* $t$ *be some number greater than:*

$$t_1(n) := 5^{n^2} . 8^n . \exp\left(\frac{8}{3} . \frac{1}{n+1}\right);$$

*then:*

$$P_{t^{-1}}(M^n) \leqq (5.(1+\sqrt{2}))^{c(n)} . \exp\left(\frac{15n-13}{4} . b_1(M^n)\right),$$

*where:*

$$c(n) := 2.n^3 + \frac{19}{6}.n^2 + \frac{7}{4}.n + 1.$$

*Remarks.* — (i) Up to this point all the numerical estimates done in order to get a simple explicit bound have been chosen in such a way that they do not spoil the leading order terms of $c(n)$ and $t_1(n)$. The factors exponentiated by $c(n)$ are explained as follows: the Fibonacci number 5 reflects the geometry in the local covering argument (*c.f.* corollary 1.6), and the number $1+\sqrt{2}$ is due to the packing argument in appendix A.

(ii) The lower order terms have not been treated that carefully; they could even be improved easily by changing the geometric details in the argument. For instance one could make use of the fact that the critical points of the distance function $d_p$ cannot lie everywhere in an incompressible ball $B(p, r)$; they are contained in a rather small subset, and lemma 3.5 could be modified accordingly.

## APPENDIX A

### A packing problem in $\mathbb{S}^{n-1} \subset \mathbb{R}^n$

We define sequences $(a_n)_{n \geqq 1}$ and $(\alpha_n)_{n \geqq 1}$ of real numbers in (0, 1] resp. (0, $\pi/2$] by:

$$a_1 := 1,$$

$$a_n = a_{n-1} . (1-a_n) . \left(1 + \sqrt{\frac{2}{1+a_n}}\right)^{-1}, \qquad n \geqq 2.$$

and

$$\alpha_n := \arcsin(a_n), \qquad n \geqq 1.$$

PROPOSITION. — *Let* $A_n$ *be the collection of all subsets* $A \subset \mathbb{S}^{n-1}$ *satisfying*

$$(\bigstar) \qquad p, q \in A, \qquad p \neq q \quad \Rightarrow \quad d(p, q) > \frac{\pi}{2} - \alpha_n.$$

*Then:*

$$\max\{\# A \mid A \in A_n\} = 2n.$$

*Remarks.* — (i) We point out that $a_2 = \sin(\pi/10)$, and $(\pi/2) - \alpha_2 = 2\pi/5$, which is the centriangle of the regular 5-gon.

(ii) As explained in example 1 in appendix B, there is the estimate:

$$\frac{1}{3} \cdot (1+\sqrt{2})^{1-n} \leqq a_n \leqq (1+\sqrt{2})^{1-n}, \qquad n \geqq 2.$$

(iii) We may view $\alpha_n$ as a lower bound on the angle $\tilde{\alpha}_n$ defined by

$$\frac{\pi}{2} - \tilde{\alpha}_n = 2 . \sup\{\rho \mid \text{there exist } 2n+1 \text{ disjoint balls of radius } \rho \text{ in } \mathbb{S}^{n-1}\}.$$

In principal such a bound could also have been obtained computing the packing densities of $n$ balls of radius $\rho$ mutually touching each other with respect to the simplex spanned by their centres (*c.f.* [Bö]).

*Proof.* — The vertices of the generalized octahedron in $\mathbb{R}^n$ define a set $A \in A_n$. This proves "$\geqq$". The opposite inequality is shown by induction:

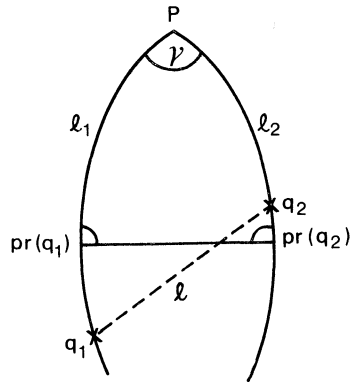
We take some $A \in A_n$. If A contains no more than 2 elements, we are done; else we pick 2 points $p, \tilde{p} \in A$ such that their distance $d(p, \tilde{p})$ is maximal.

To finish the proof, we construct a projection of $A' := A \setminus \{p, \tilde{p}\}$ onto some set in $A_{n-1}$:

(i) *The estimate* $(\pi/2) - \alpha_n < d(p, q) < (3\pi/4) + (1/2) . \alpha_n$ *holds for all* $q \in A'$.

Assuming the converse, one concludes that $q$ and $\tilde{p}$ both lie in the ball of radius $(\pi/4) - (1/2) . \alpha_n$ around the antipodal point of $p$. This observation immediately yields a contradiction to property $(\bigstar)$.

(ii) *The projection* pr: $A' \to \mathbb{S}^{n-2} = \{x \in \mathbb{S}^{n-1} \mid d(p, x) = \pi/2\}$ *along the great circles through p can be controlled by means of spherical trigonometry:*




Suppose that $q_1$, $q_2$ are two distinct points in A′; put:

$$\gamma := d(\mathrm{pr}(q_1), \mathrm{pr}(q_2)),$$
$$l := d(q_1, q_2),$$
$$l_i := d(p, q_i), \qquad i=1, 2.$$

The Law of Cosines may be written as follows:

$$(\bigstar\bigstar) \qquad \cos\gamma = \frac{\cos l - \cos l_1 . \cos l_2}{\sin l_1 . \sin l_2}.$$

It is elementary analysis to verify that under the constraints

$$\frac{\pi}{2} - \alpha_n \leqq l_2 \leqq l_1 \leqq \frac{3\pi}{4} + \frac{1}{2}.\alpha_n, \qquad \frac{\pi}{2} - \alpha_n \leqq l$$

the right-hand side of ($\bigstar\bigstar$) has a unique maximum, which is achieved at:

$$l = l_2 = \frac{\pi}{2} - \alpha_n, \qquad l_1 = \frac{3\pi}{4} + \frac{1}{2}.\alpha_n.$$

Therefore, if $q_1, q_2 \in A'$, we conclude:

$$\begin{aligned}\cos\gamma > \cot\left(\frac{\pi}{2} - \alpha_n\right).\ & \frac{1-\cos((3\pi/4)+(1/2).\alpha_n)}{\sin((3\pi/4)+(1/2).\alpha_n)} \\ &= \tan\alpha_n . \frac{1+\sqrt{1/2.(1-\cos((\pi/2)+\alpha_n))}}{\sqrt{1/2.(1+\cos((\pi/2)+\alpha_n))}} \\ &= \frac{a_n}{\sqrt{1-a_n^2}} . \frac{\sqrt{2}+\sqrt{1+a_n}}{\sqrt{1-a_n}} \\ &= \frac{a_n}{1-a_n} . \left(1+\sqrt{\frac{2}{1+a_n}}\right) = a_{n-1},\end{aligned}$$

hence:

$$\cos\gamma > \cos\left(\frac{\pi}{2}-\alpha_{n-1}\right) \quad \square.$$

# APPENDIX B

## A Lemma on recursively defined sequences

Let $\varphi: [0, 1) \to [a, \infty)$ be a monotone function such that $\varphi(0)=a>1$.

Then $f: [0, 1) \to [0, \infty)$, $x \mapsto x.\varphi(x)$ is invertible. For any $x_0 \in (0, 1]$ there is a unique sequence of real numbers $(x_n)_{n\in\mathrm{N}}$ defined by:

$$x_n = f(x_{n+1}), \qquad x_{n+1}\in(0, 1].$$

LEMMA. — (i) $x_{n+1} \leqq a^{-1}.x_n \leqq a^{-n}.x_1$.

*If moreover* $\varphi(x) \leqq a.(1+x)/(1-x)$ *for all* $x\in[0, 1)$, *there are also lower bounds for the* $x_n$:

(ii) $x_{n+1}/x_n = \varphi(x_{n+1})^{-1} \geqq \varphi(a^{-n}.x_1)^{-1}$,

(iii) $x_n/x_0 \geqq a^{-n}.\left(\dfrac{1-x_1}{1+x_1}\right)^{a/(a-1)} \geqq a^{-n}.\left(\dfrac{a-x_0}{a+x_0}\right)^{a/(a-1)}; \qquad n \geqq 1.$

*Proof.* – (i) and (ii) are obvious. In order to prove the last claim, notice that by induction:

$$(\bigstar) \qquad x_n/x_0 \geqq \prod_{j=0}^{n-1} \varphi(a^{-j}.x_1)^{-1} \geqq a^{-n}.\prod_{j=0}^{n-1}\frac{1-a^{-j}.x_1}{1+a^{-j}.x_1}.$$

We compute:

$$\sum_{j=0}^{n-1}\ln\left(\frac{1-a^{-j}.x_1}{1+a^{-j}.x_1}\right) = -2.\sum_{j=0}^{n-1}\sum_{k=0}^{\infty}\frac{1}{2k+1}.a^{-j.(2k+1)}.x_1^{2k+1}$$

$$\geqq -2.\sum_{k=0}^{\infty}\frac{1}{2k+1}.\frac{1}{1-a^{-2k-1}}.x_1^{2k+1} \geqq \frac{a}{a-1}.\ln\left(\frac{1-x_1}{1+x_1}\right).$$

Combining this inequality with $(\bigstar)$ gives the required estimate.

*Examples :*

1. $\varphi(x) = \left(1+\sqrt{\dfrac{2}{1+x}}\right)/(1-x); \qquad x_0 = 1:$

Clearly $a = 1+\sqrt{2}$ and $x_1 = \sin(\pi/10)$; therefore one has:

$$x_n \geqq \frac{1}{3}.(1+\sqrt{2})^{-n}, \quad \text{provided } n \geqq 1.$$

2. $\varphi(x)=(2+\rho.(1+x))/(1-x)$,

where

$$2+\rho=5+2.L^{-1}, \quad L>0; \qquad x_0 \leqq \frac{1}{2.(L+4)}.$$

Clearly $a=2+\rho$, and one easily computes that:

$$\left(\frac{a+x_0}{a-x_0}\right)^{a/a-1} \leqq \left(\frac{5+(2/L)+(1/2.(L+4))}{5+(2/L)-(1/2.(L+4))}\right)^{5/4} \leqq \left(1+\frac{1}{5.(L+4)}\right)^{5/4} \leqq \exp\left(\frac{1}{4.(L+4)}\right).$$

Therefore

$$5^n \leqq \frac{x_0}{x_n} \leqq 5^n.\exp\left(\frac{2n}{5L}+\frac{1}{4.(L+4)}\right).$$

*Note added in proof.* — In the meantime a nice account on the results obtained by heat equation methods has appeared in [B].

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