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L-indistinguishability and *R*-groups for quasisplit groups :
unitary groups in even dimension

L-INDISTINGUISHABILITY AND R-GROUPS
FOR QUASI-SPLIT GROUPS:
UNITARY GROUPS IN EVEN DIMENSION

By C. David KEYS [1]

Let G be a quasi-split group over a local field F. In this paper, we determine the structure of L-packets for the minimal unitary principal series representations of G. Let $h: G \to \tilde{G}$ be a homomorphism with abelian kernel and cokernel, between quasi-split groups. Restrictions of representations from $\tilde{G}$ to G are studied, and we show that a type of reciprocity holds.

Reducibility of the minimal principal series is determined by the R-group, which we show in this case is isomorphic to the group $\mathbf{S}_\varphi$ defined in terms of L-group data arising from a Langlands parameter φ ([13], [17]).

Information concerning the components of the Ind λ is obtained by analysis on the finite groups $\mathbf{S}_\varphi$.

Let Π_φ be an L-packet consisting of components of a unitary principal series representation. We show that representations π in Π_φ are parametrized by the irreducible representations $\rho(\pi)$ of the group $\mathbf{S}_\varphi$. Then we may define a pairing

$$\langle \ , \ \rangle : \mathbf{S}_\varphi \times \Pi_\varphi \to \mathbf{C}$$

so that

$$\text{trace } \mathscr{A}(r, \lambda) \, \mathrm{I}(g) = \sum_\pi \langle r, \pi \rangle \text{ trace } \pi(g)$$

for $g \in C_c^\infty(G)$ and r in $R(\lambda) \cong \mathbf{S}_\varphi$, where $\mathrm{I} = \text{Ind } \lambda$. The number of components in an L-packet, and multiplicities, may be determined by a Möbius inversion formula.

The problem of normalizations of intertwining operators arising in the theory of the R-group defines a 2-cocycle

$$\eta : \quad R \times R \to \mathbf{C}^\times$$

for which the commuting algebra $\mathrm{End}\,(\mathrm{Ind}\,\lambda)$ is isomorphic to the group algebra $\mathbf{C}\,[R]_\eta$, with multiplication twisted by η. In section 2, we recall the general theory of the R-group, and show that for λ one-dimensional, the 2-cocycle η is a coboundary. More generally, for $P = MN$ nonminimal and G connected, the cocyle associated to $\mathrm{Ind}\,(P, G; \sigma)$ has trivial cohomology class if the inducing representation σ of M is generic.

Then the commuting algebra is given by the group algebra $\mathbf{C}\,[R]$. The L-packet Π_φ is parametrized by the dual $\mathbf{S}_\varphi^\wedge$, and we define the pairing $\langle\ ,\ \rangle$ of $\mathbf{S}_\varphi$ and Π_φ.

Define $U(n, n)$, $SU(n, n)$, and $GU(n, n)$ to be the quasi-split unitary groups associated to a separable quadratic extension E/F of local fields. The groups $\mathbf{S}_\varphi$ which occur for the minimal principal series are explicitly classified. Non-abelian $\mathbf{S}_\varphi$, more complicated than those known previously, occur for the groups $SU(n, n)$, and multiplicity one fails. - The non-abelian $R \cong \mathbf{S}_\varphi$ are shown to be extensions of a certain reflection group $\tilde{R}$ of type $A_1 \times \ldots \times A_1$ by an abelian Galois group $\mathrm{Gal}\,(L/E)$ with exponent dividing the rank of $\tilde{R}$.

An explicit classification of the R-groups occurring for the minimal principal series of the quasi-split unitary groups is given in section 3. We may construct a list of characters which have non-trivial R-groups, such that any character with non-trivial R is conjugate under the Weyl group to one on the list.

In section 4, we consider the restrictions of representations in an L-packet, and show that a type of reciprocity relates restriction of representations to induction of parameters.

More precisely, let $h: G \to \tilde{G}$ have abelian kernel and cokernel. Consider representations $\mathrm{Ind}\,(\tilde{P}, \tilde{G}; \tilde{\lambda})$ and $\mathrm{Ind}\,(P, G; \lambda)$, where $\tilde{\lambda}$ and $\lambda = \tilde{\lambda} \circ h$ have Langlands' parameters $\tilde{\varphi}$ and $\varphi = h^* \circ \tilde{\varphi}$, respectively. Then $\mathbf{S}_{\tilde{\varphi}}$ may be considered as a subgroup of $\mathbf{S}_\varphi$. Let $\tilde{\pi} \in \Pi_{\tilde{\varphi}}$ be parametrized by an irreducible representation $\tilde{\rho}$ of $\mathbf{S}_{\tilde{\varphi}}$. Then the components in the restriction of $\tilde{\pi}$ to G are parametrized by the components of the induced representation $\mathrm{Ind}\,(\mathbf{S}_{\tilde{\varphi}}, \mathbf{S}_\varphi; \tilde{\rho})$ of $\mathbf{S}_\varphi$. Further, the multiplicity of $\pi \in \Pi_\varphi$ in the restriction of $\tilde{\pi}$ to G equals the multiplicity of $\rho = \rho(\pi)$ in the representation $\mathrm{Ind}\,(\mathbf{S}_{\tilde{\varphi}}, \mathbf{S}_\varphi; \tilde{\rho})$ of $\mathbf{S}_\varphi$.

In section 5, we give typical examples of the non-abelian groups $\mathbf{S}_\varphi$ which occur for $SU(n, n)$. The number of irreducible inequivalent components of $\mathrm{Ind}\,\lambda$, and multiplicites, is given by the number of irreducible representations of the groups $\mathbf{S}_\varphi$, and their degrees. Formulas for these may be found by Möbius inversion.

In section 6, we give an example with R non-abelian, similar in spirit to an example of Vogan, with $\dim \lambda = 2$ and G disconnected (and not quasi-split), for which the cocyle η is not a coboundary. We may compute the cocyle in this case, and use the theorem $\mathrm{End}\,(\mathrm{Ind}\,\lambda) \cong \mathbf{C}\,[R]_\eta$ to determine the number of inequivalent components, their multiplicities, and the action of the intertwining operators.

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1. Notation and Definitions

Let E/F be a separable quadratic extension of local fields. Define the groups

$$GU(n, n) = \{ g \in GL(2n, E) \mid g\, J^t\, \bar{g} = xJ \text{ for some } x \in F^\times \},$$

$$U(n, n) = \{ g \in GL(2n, E) \mid g\, J^t\, \bar{g} = J),$$

and

$$SU(n, n) = \{ g \in SL(2n, E) \mid g\, J^t\, \bar{g} = J \}.$$

Here

$$J = \begin{pmatrix} & & & 1 \\ & & -1 & \\ & \cdot & \cdot & \\ & 1 & & \\ -1 & & & \end{pmatrix}$$

and $x \mapsto \bar{x}$ is the Galois automorphism of E/F.

These groups are F-points of quasi-split algebraic groups defined as E/F-forms of $GL(2n) \times GL(1)$, $GL(2n)$, and $SL(2n)$. For an extension K of E, define the groups of K-points to be $GL(2n, K) \times K^\times$, $GL(2n, K)$, and $SL(2n, K)$.

Let

$$\sigma(g, x) = (\bar{x}\, J^t\, \bar{g}^{-1}\, J^{-1}, \bar{x})$$

for $(g, x) \in GL(2n, E) \times E^\times$, and define the F-points of the unitary group of similitudes $GU(n, n)$ to be the fixed points of σ.

Define the F-points of the unitary group $U(n, n)$, and the special unitary group $SU(n, n)$ to be the fixed points of $\sigma(g) = J^t\, \bar{g}^{-1}\, J^{-1}$, for g in $GL(2n, E)$ or $SL(2n, E)$.

Recall the definition of the L-group from [1], [11], or [13]. Realize the Weil group $W_{E/F}$ as

$$\{ x \times \tau \mid x \in E^\times, \text{ and } \tau \in \mathrm{Gal}(E/F) \}$$

with multiplication defined by

$$(x \times \tau)(x' \times \tau') = x \cdot \tau(x') \cdot a_{\tau, \tau'} \times \tau\tau'$$

where $a_{\tau, \tau'} = 1$ unless $\tau = \tau' = \sigma$ is the non-trivial element of $\mathrm{Gal}(E/F)$, and $a_{\sigma, \sigma} = a$ is a fixed element of $F^\times$ which is not a norm from $E^\times$.

The L-group is an extension ${}^L G = {}^L G^0 \times_s W_{E/F}$. Included in L-group data are complex groups $({}^L G^0, {}^L B^0, {}^L T^0)$ "dual" to the quasi-split group G with Borel subgroup and maximal torus, as well as root vectors $X_{\alpha^\vee}$ for the simple roots of ${}^L T^0$ in ${}^L B^0$.

For $G = SU(n, n)$, $\tilde{G} = U(n, n)$, and $\tilde{\tilde{G}} = GU(n, n)$, we have ${}^L G^0 = PGL(2n, \mathbf{C})$, ${}^L \tilde{G}^0 = GL(2n, \mathbf{C})$, and ${}^L \tilde{\tilde{G}}^0 = GL(2n, \mathbf{C}) \times GL(1, \mathbf{C})$.

The Weil group acts through projection onto Gal (E/F). The action of $1 \times \sigma$ on $({}^{L}G^{0},$ ${}^{L}B^{0},\, {}^{L}T^{0},\, X_{\alpha^{\vee}})$ is the "algebraic dual" of the Galois action on G.

The action of $1 \times \sigma$ on ${}^{L}G^{0}$ and ${}^{L}\widetilde{G}^{0}$ is given by $g \mapsto J^{t} g^{-1} J^{-1}$, and the action on ${}^{L}\widetilde{\widetilde{G}}^{0}$ is given by $(g, x) \mapsto (x J^{t} g^{-1} J^{-1}, x)$.

We assemble some explicit formulas which will be useful later.

Take S to be the maximal F-split torus

$$S = \{ \operatorname{diag}(a_1, a_2, \ldots, a_n, a_{-n}, \ldots, a_{-1}) \,|\, a_{-i} = a_i^{-1} \in F^{\times} \}.$$

Let M, $\widetilde{M}$, and $\widetilde{\widetilde{M}}$ be the centralizers of S in $G = SU(n, n)$, $\widetilde{G} = U(n, n)$, and $\widetilde{\widetilde{G}} = GU(n, n)$, respectively. Then M and $\widetilde{M}$ are diagonal with $\bar{a}_i a_{-i} = 1$, and $\widetilde{\widetilde{M}}$ is diagonal with $\bar{a}_i a_{-i} = k \in F^{\times}$, where $a_i \in E^{\times}$. We may take minimal parabolic subgroups

$$P = MN < \widetilde{P} = \widetilde{M}N < \widetilde{\widetilde{P}} = \widetilde{\widetilde{M}}N.$$

The root system Φ of (G, S) is of type C_n. Extend the coroots $\alpha^{\vee} : F^{\times} \to S$ to $\alpha^{\vee} : E^{\times} \to \widetilde{M}$ as follows.

For $\alpha = e_i - e_j$, and $x \in E^{\times}$, let

$$\alpha^{\vee}(x) = \operatorname{diag}(1, \ldots, x, \ldots, x^{-1}, \ldots, 1, 1, \ldots, \bar{x}, \ldots, \bar{x}^{-1}, \ldots, 1)$$

with x, x^{-1} in the i-th and j-th positions.

For $\alpha = e_i + e_j$, let

$$\alpha^{\vee}(x) = \operatorname{diag}(1, \ldots, x, \ldots, \bar{x}, \ldots, 1, 1, \ldots, x^{-1}, \ldots, \bar{x}^{-1}, \ldots, 1)$$

with $x, \bar{x}$ in the i-th and j-th positions.

For $\alpha = 2 e_i$, let

$$\alpha^{\vee}(x) = \operatorname{diag}(1, \ldots, x, \ldots, 1, 1, \ldots, \bar{x}^{-1}, \ldots, 1)$$

with x in the i-th position. Take $x \in E^{\times}$ to define an element of $\widetilde{M}$, and $x \in F^{\times}$ to define an element of M. These formulas will be used in the explicit classification of reducibility.

The coroots corresponding to the simple roots of Φ determine isomorphisms

$$\widetilde{M} \cong (E^{\times})^{n}$$

and

$$M \cong (E^{\times})^{n-1} \times F^{\times}.$$

Let λ be a quasi-character of $\widetilde{M}$ or M, and define characters λ_{α} of $E^{\times}$, or $F^{\times}$, by

$$\lambda_{\alpha}(x) = \lambda(\alpha^{\vee}(x)).$$

These characters behave under multiplication as the $\alpha^{\vee} \in \Phi^{\vee}$.

The Weyl group $W \cong S_n \times_s \mathbf{Z}_2^n$ acts on M and $\widetilde{M}$ as follows. A permutation (ij), i.e., the reflection corresponding to the root $e_i - e_j$, interchanges the i-th and j-th entries, as

well as the $-i$-th and $-j$-th entries. A sign change c_i, $1 \leq i \leq n$, i.e., the reflection corresponding to the root $2e_i$, interchanges the i-th and $-i$-th entries. The reflection corresponding to the root $e_i + e_j$ is $(ij) \, c_i c_j$.

The Weyl group acts on characters of M and $\tilde{M}$ by

$$w^{-1} \lambda(m) = \lambda(wmw^{-1}).$$

Then W also acts on the λ_α, and $w^{-1} \lambda_\alpha(x) = \lambda_{w\alpha}(x)$. Note that $w\lambda = \lambda$ if and only if $w\lambda_\alpha = \lambda_\alpha$ for all simple roots α.

The group $\tilde{M}$ should be considered as the group T_F of F-rational points of an algebraic torus T, which splits over E. A character $\tilde{\lambda}$ of $\tilde{M}$ corresponds [13] to an admissible homorphism

$$\tilde{\varphi}: \quad W_{E/F} \to {}^L\tilde{T},$$

where

$$\tilde{\varphi}(x \times \tau) = \tilde{\varphi}_0(x \times \tau) \times x \times \tau.$$

Recall that $\mathrm{Hom}\,(E^\times, \tilde{M}) \cong \mathrm{Hom}\,({}^L\tilde{T}^0, \mathbf{C}^\times)$. The map

$$\tilde{\varphi}_0: \quad W_{E/F} \to {}^L\tilde{T}^0$$

is determined on $E^\times$ by the requirement that

$$\alpha^\vee \circ \tilde{\varphi}_0 = \tilde{\lambda} \circ \alpha^\vee = \tilde{\lambda}_\alpha,$$

i.e., that for each (simple) root α, the following diagram commutes:

$$
\begin{array}{ccc}
E^\times \hookrightarrow W_{E/F} & \xrightarrow{\;\tilde{\varphi}_0\;} & {}^L\tilde{T}^0 \\
\Big\downarrow{\scriptstyle \alpha^\vee} & & \Big\downarrow{\scriptstyle \alpha^\vee} \\
\tilde{M} \quad = \tilde{T}_F & \xrightarrow{\;\tilde{\lambda}\;} & \mathbf{C}^\times
\end{array}
$$

Let the non-trivial element σ of $\mathrm{Gal}\,(E/F)$ satisfy $\sigma^2 = a$ in the Weil group. Then we also require

$$\tilde{\varphi}_0(1 \times \sigma) \cdot {}^\sigma \tilde{\varphi}_0(1 \times \sigma) = \tilde{\varphi}_0(a \times 1).$$

The Langlands parameter

$$\tilde{\varphi}: \quad W_F \to {}^L\tilde{G}$$

corresponding to $\mathrm{Ind}\,(\tilde{P}, \tilde{G}; \tilde{\lambda})$ is given by (the equivalence class of) the map

$$W_F \to W_{E/F} \to {}^L\tilde{T} \to {}^L\tilde{G}.$$

More generally, we should use the "thickened" Weil group $W_F \times SL\,(2, \mathbf{C})$ to account for special representations [13]. $W_{E/F}$ will suffice for the purposes of this paper.

Define the parameter $\varphi: W_{E/F} \to {}^L T \hookrightarrow {}^L G$ corresponding to an Ind $(P, SU(n, n); \lambda)$ by the same method. If $\lambda \in M^\wedge$ is the restriction of $\tilde{\lambda} \in \tilde{M}^\wedge$, then the diagrams immediately define a lift

$$\tilde{\varphi}_0: \quad W_{E/F} \to {}^L \tilde{G}^0 = GL(2n, \mathbf{C})$$

of the map

$$\varphi_0: \quad W_{E/F} \to {}^L G^0 = PGL(2n, \mathbf{C}).$$

Then defining the lift $\tilde{\varphi} = \tilde{\varphi}_0 \times id$, we easily use the formulas for the $\alpha^\vee$ given above to check that

$$\tau \tilde{\varphi}_0(x) \tau^{-1} = \tilde{\varphi}_0({}^\tau x)$$

for $x \in E^\times$ and $\tau \in \mathrm{Gal}(E/F)$. Then $\tilde{\varphi}$ is in fact a homomorphism, hence is a lift of φ.

In general, for the unitary principal series induced from a minimal parabolic subgroup of a quasi-split group, one uses Theorem 2 (a) of Langlands' "Representations of Abelian Algebraic Groups" to define the parameter $\varphi_T: W_F \to {}^L M$ corresponding to a character λ of M, since M is the group of F-rational points of an algebraic torus T which splits over some Galois extension E/F. See [1]. Then the Langlands' parameter corresponding to Ind $(P, G; \lambda)$ is given by the composition $\varphi: W_F \to {}^L M \hookrightarrow {}^L G$.

For a unitary character λ with parameter $\varphi = \varphi(\lambda)$, define the L-packet Π_φ to be the set of irreducible components of Ind λ.

2. R-group and L-indistinguishability

We first recall the general theory of the R-group, which determines reducibility of the unitary principal series. The commuting algebra End (Ind λ) is given as a group algebra $\mathbf{C}[R]_\eta$ with multiplication twisted by a 2-cocycle η. For λ one-dimensional, we show that η has trivial cohomology class. This allows us to parametrize the L-packet Π_φ by the dual $S_\varphi^\wedge \cong R^\wedge$, and to define a pairing of Π_φ with $S_\varphi^\wedge$,

Standard intertwining operators

$$A(\bar{w}, \lambda_s): \quad \mathrm{Ind}(P, G; \lambda_s) \to \mathrm{Ind}(P, G; w\lambda_s)$$

are defined initially for certain s by an integral formula

$$(A(\bar{w}, \lambda_s) f)(g) = \int_{N \cap w \bar{N} w^{-1}} f(gn\bar{w}) \, dn,$$

where $\bar{w}$ is a fixed representative for the element w of the Weyl group $W = N(S)/M$, and $\bar{P} = M\bar{N}$ is the parabolic opposed to P. The operators are then defined as meromorphic functions of s by analytic continuation [20], and satsify the cocycle relation

$$A(\bar{w}_1 \bar{w}_2, \lambda) = A(\bar{w}_1, w_2 \lambda) A(\bar{w}_2, \lambda),$$

provided $length\,(w_1 w_2) = length\,(w_1) + length\,(w_2)$.

Note that if the coset representative $\bar{w}$ is changed by an element m of M, the operator changes by $\delta^{-1/2} \lambda^{-1} (m)$. Let

$$w = w_{\alpha(1)} \, w_{\alpha(2)} \cdots w_{\alpha(l)}$$

be a reduced expression for w as a product of simple reflections, where $l = length\,(w)$. The expression is not unique. However, we may fix representative $\bar{w}_\alpha$ for each simple reflection so that the representative

$$\bar{w} = \bar{w}_{\alpha(1)} \, \bar{w}_{\alpha(2)} \cdots \bar{w}_{\alpha(l)}$$

for w is independent of the reduced decomposition chosen.

Fix such representatives $\bar{w}$ and write $A\,(w, \lambda)$ for $A\,(\bar{w}, \lambda)$. We may define normalized intertwining operators

$$\mathscr{A}\,(w, \lambda_s) = \gamma\,(w, \lambda, s)^{-1} A\,(w, \lambda_s)$$

for certain meromorphic functions γ of s, the c-functions of Harish-Chandra [20]. The normalization may be chosen so that

$$\mathscr{A}\,(w_\alpha, w_\alpha \lambda_s)\, \mathscr{A}\,(w_\alpha, \lambda_s) = I$$

for all simple roots α.

Let $W\,(\lambda) = \{\, w \in W \mid w\lambda = \lambda \,\}$.

THEOREM 2.1 (*Harish-Chandra*, [20]). — *Suppose λ is unitary. Then the commuting algebra of* $\mathrm{Ind}\,(P, G; \lambda)$ *is spanned by the operators*

$$\{\, \mathscr{A}\,(w, \lambda) \mid w \in W\,(\lambda) \,\}.$$

Plancherel factors μ_α are defined by

$$A\,(w_\alpha, w_\alpha \lambda_s)\, A\,(w_\alpha, \lambda_s) = c^2\, \mu_\alpha\,(\lambda, s)^{-1}\, I,$$

where c^2 is a certain positive constant. Let

$$\Delta' = \{\, \text{roots } \alpha \mid \mu_\alpha\,(\lambda, s) \text{ has a zero at } s = 0 \,\}$$

and let W' be the subgroup of W generated by the reflections w_α, for $\alpha \in \Delta'$. Then [21]

$$W' = \{\, w \mid \mathscr{A}\,(w, \lambda) \text{ is scalar} \,\}.$$

THEOREM 2.2 (*Silberger* [21]). — *Suppose λ is unitary. Then the dimension of the commuting algebra of* $\mathrm{Ind}\,(P, G; \lambda)$ *is* $\mid W\,(\lambda)/W' \mid$.

Define

$$R = R\,(\lambda) = \{\, w \in W\,(\lambda) \mid \alpha > 0 \text{ and } \alpha \in \Delta' \text{ imply that } w\alpha > 0 \,\}.$$

Then R is a complement for W' in $W\,(\lambda)$.

Harish-Chandra's Theorem and the decomposition

$$W(\lambda) = R \times_s W'$$

imply that the normalized operators

$$\{ \mathscr{A}(w, \lambda) \mid w \in R \}$$

span the commuting algebra of Ind $(P, G; \lambda)$, since W' corresponds to scalar operators. By Silberger's Theorem, the operators corresponding to R form a linear basis for the commuting algebra.

By Schur's Lemma, the intertwining operators satisfy the cocycle relation with no condition on the lengths of the Weyl group elements, up to a scalar. Thus, *a priori*, the map $w \to \mathscr{A}(w, \lambda)$ is only a projective homomorphism. We define a 2-cocycle η of the Weyl group by

$$\mathscr{A}(w_1 w_2, \lambda_s) = \eta(w_1, w_2) \, \mathscr{A}(w_1, w_2 \lambda_s) \, \mathscr{A}(w_2, \lambda_s).$$

Then the commuting algebra of Ind $(P, G; \lambda)$ is isomorphic to the group algebra $\mathbf{C}[R]_\eta$, with multiplication twisted by the 2-cocycle η.

We show that the cocycle

$$\eta: \quad W \times W \to \mathbf{C}^\times$$

is a coboundary if λ is one-dimensional. Fix the choices of coset representatives for Weyl group elements as above. The following is in [5].

THEOREM 2.3. — *Suppose λ is one-dimensional and that the intertwining operators corresponding to the simple reflections w_α are normalized so that*

$$\mathscr{A}(w_\alpha, w_\alpha \lambda_s) \, \mathscr{A}(w_\alpha, \lambda_s) = I.$$

Then $\eta \equiv 1$, i.e., the cocycle relation holds with no condition on the lengths of the Weyl group elements.

Since the restriction of the 2-cocycle η to $R \times R$ is trivial, we have the following theorem for the unitary minimal principal series representations of connected reductive p-adic groups.

THEOREM 2.4. — *Suppose λ is a one-dimensional unitary character with $R = R(\lambda)$. Then*

(i) *The commuting algebra of* Ind $(P, G; \lambda)$ *is isomorphic to the group algebra* $\mathbf{C}[R]$,

(ii) Ind λ *decomposes with multiplicity one if and only if R is abelian.*

(iii) *The inequivalent irreducible components π_i of the representation* Ind λ *of* G *are parametrized by the irreducible representations $\rho_i = \rho(\pi_i)$ of* R, *and*

(iv) *The multiplicity with which a component $\pi_i = \pi(\rho_i)$ occurs in* Ind λ *is equal to the dimension of the representation $\rho_i \in R^\wedge$ which parametrizes it.*

Proof. — The result $\eta \equiv 1$ implies (i) and (ii). The parametrization of (iii) is determined as follows. Let ρ be an irreducible representation of R, with dimension dim ρ and

character θ_ρ. Then, summing over $r \in R$, the operators

$$P_\rho = |R|^{-1} \dim \rho \sum_r \overline{\theta_\rho(r)} \mathscr{A}(r, \lambda)$$

form $|R^\wedge|$ orthogonal projections onto invariant subspaces of Ind λ, by the orthogonality relations for the characters θ_ρ. Each is non-zero by Silberger's Theorem.

Further, each P_ρ is central in the commuting algebra, since the characters θ_ρ are class functions on R. Thus, if we let $V_\rho = P_\rho V$, where V is the space of Ind λ, then the V_ρ are pairwise disjoint.

Thus

$$V = V_{\rho(1)} \oplus V_{\rho(2)} \oplus \ldots \oplus V_{\rho(k)}$$

as an orthogonal direct sum, with the $V_{\rho(i)}$ disjoint, $1 \leq i \leq k = |R^\wedge|$. But since End (Ind λ) $\cong \mathbf{C}[R]$, the number of inequivalent components of Ind λ equals the number of conjugacy classes in R, which equals k. Thus the subspaces V_ρ are isotypic. Note that the elementary indempotents of $\mathbf{C}[R]$ further decompose each V_ρ into its irreducible components.

Then the P_ρ are the projections onto the isotypic components of Ind λ, and the component π corresponding to ρ occurs with multiplicity $d(\rho)$, the degree of ρ.

If π corresponds to ρ under the identification $\Pi_\varphi \cong R^\wedge$, we write $\rho = \rho(\pi)$ and $\pi = \pi(\rho)$. Note that $\pi(\rho) \mapsto \pi(\rho \otimes \rho')$ determines an action of the group of one-dimensional characters ρ' of R on the set of components Π_φ. The action is faithful and simply transitive on the subset of the L-packet Π_φ consisting of the components which occur with multiplicity one.

The inverse map to $\rho \mapsto \pi$ is explicitly constructed as follows. Given an irreducible component (π, V_0) of Ind λ, fix a non-zero vector v_0 in V_0. We consider the translates of v_0 under R. Let W be the span of the vectors

$$\{ \mathscr{A}(r, \lambda) v_0 \,|\, r \in R \}.$$

Since End (Ind λ) $\cong \mathbf{C}[R]$, W intersects each irreducible component V_1 of the π-isotypic subspace of Ind λ. In fact, W intersects each irreducible component in a one-dimensional subspace, by Schur's Lemma. Thus the dimension of W equals the multiplicity of π in Ind λ. The representation ρ of R is defined by extending

$$\rho(r) [\mathscr{A}(r', \lambda) v_0] = \mathscr{A}(rr', \lambda) v_0$$

to W by linearity.

Suppose V_1 is an irreducible component of Ind λ equivalent to V_0, and let v_1 be a non-zero vector in $W \cap V_1$. By Schur's Lemma.

$$\mathscr{A}(r, \lambda) V_1 \subseteq V_1 \quad \text{if an only if} \quad \mathscr{A}(r, \lambda) \text{ acts on } V_1 \text{ as a scalar } c.$$

This happens if and only if $\mathscr{A}(r, \lambda) v_1 = cv_1$, i. e., $\rho(r) v_1 = cv_1$.

Finally, we note that ρ is an irreducible representation of R and P_ρ is the projection onto the π-isotypic subspace of Ind λ. By the Theorem, there exists an irreducible representation τ of R such that P_τ is the projection onto the π-isotypic subspace. In particular,

$$v_0 = P_\tau v_0 = |R|^{-1} \dim \tau \, \Sigma \, \overline{\theta_\tau(r)} \, \mathscr{A}(r, \lambda) \, v_0 = |R|^{-1} \dim \tau \, \Sigma \, \overline{\theta_\tau(r)} \, \rho(r) \, v_0$$

is non-zero. Thus τ is contained in ρ. Then a counting argument as in the proof of the Theorem implies that each ρ arising this way is irreducible, and $\rho = \tau$.

The correspondence between the set of components and the dual $R^\wedge$ depends on the choice of normalizations of intertwining operators giving End (Ind $\lambda) \cong \mathbf{C}[R]$. But any two such normalizations must differ by a one-dimensional character ρ' of R, i. e.,

$$\mathscr{A}'(r, \lambda) = \rho'(r) \mathscr{A}(r, \lambda).$$

Then the parametrization of the L-packet Π_φ by $R^\wedge$ will be shifted by the character ρ'. If π corresponds to $\rho(\pi)$ with the first normalization, it corresponds to $\rho(\pi) \otimes \rho'$ with the second. We write $\pi(\rho) = \pi'(\rho \otimes \rho')$.

For quasi-split G, $M = T(F)$ is the group of F-rational points of a torus, and L-packets of representations of M are singletons. A quasi-character λ of M corresponds to an admissible homomorphism $\varphi_T = \varphi_T(\lambda)$ of the Weil group W_F into $^L T$. Assume that λ is unitary. Then the composition

$$\varphi: \quad W_F \to {}^L T \hookrightarrow {}^L G$$

parametrizes an L-packet Π_φ of representations of G, consisting of the components of Ind λ, and all L-indistinguishability in this case is accounted for by reducibility.

We recall the L-group description ([12], [13]) of the reducibility group and the decomposition $W(\lambda) = R(\lambda) \times_s W'$, and check that $R(\lambda) \cong S_\varphi$ in our situation. The proof is the same as for real groups, modulo knowledge of Plancherel measure. The key point concerns the location of zeros of the Plancherel measure. *See* Lemma 5.3.15 of [18].

Langlands introduces several groups attached to a parameter φ. Let S_φ be the centralizer of $\varphi(W_F) \subseteq {}^L G$ in $^L G^0$, S_φ^0 the connected component of the identity in S_φ, and Z^{W_F} the Weil group invariants in the center of $^L G^0$. Then define

$$\mathbf{S}_\varphi = S_\varphi / Z^{W_F} S_\varphi^0.$$

Let U be the connected component of the identity in the Weil group invariants $(^L T^0)^{W_F}$ of $^L T^0$. Let N be the normalizer of $^L T^0$ in $^L G^0$.

LEMMA 2.5. — *Let* $P = MN$ *be a minimal parabolic subgroup of a quasi-split group* G. *Suppose* $\lambda \in M^\wedge$ *corresponds to the parameter* φ. *Then*

(i) *The centralizer of* U *in* $^L G^0$ *is* $^L T^0$.

(ii) U *is a maximal torus of* S_φ^0.

(iii) $^L T^0 \cap S_\varphi = (^L T^0)^{W_F} = U \cdot Z^{W_F}$.

(iv) $^L T^0 \cap S_\varphi^0 = U$.

(v) $N \cap S_\varphi^0$ *is equal to the normalizer of* U *in* S_φ^0.

Proof. — The centralizer of U in $^L G^0$ equals $^L T^0$, since U contains regular elements of $^L T^0$.

Note that U is contained in S_φ^0. Then (ii) follows from (i). Statement (iv) also follows easily.

For (iii), it is enough to note that $(^L T^0)^{W_F}$ is connected if $^L G^0$ has trivial center.

For (v), first note that an element x of S_φ^0 which normalizes U also normalizes the centralizer $^L T^0$ of U. To prove the reverse inclusion, start with an element x in $N \cap S_\varphi^0$. The inner automorphism $\mathrm{Int}(x)$ carries $^L T^0$ into $^L T^0$ and commutes with the image $\varphi(W_F)$. This implies that $\mathrm{Int}(x)$ commutes with the usual action of the Weil group on $^L T^0$, and hence preserves both the invariants $(^L T^0)^{W_F}$ and the identity component U.

PROPOSITION 2.6. — *The following sequence is exact.*

$$1 \to (N \cap S_\varphi^0)/U \to (N \cap S_\varphi)/U \cdot Z^{W_F} \to S_\varphi/S_\varphi^0 \cdot Z^{W_F} \to 1.$$

Moreover, the middle term in the exact sequence can be identified with the stabilizer $W(\lambda)$ *of* λ *in the Weyl group of* G.

With this identification, $W(S_\varphi^0, U)$ *is the subgroup* $W'(\lambda)$ *of* $W(\lambda)$. *Thus* $S_\varphi \cong R(\lambda)$.

Proof. — The exactness of the sequence is easy. For the surjectivity at the right end one uses the conjugacy of maximal tori in S_φ^0. Let $s \in S_\varphi$. Then s normalizes S_φ^0, and $s U s^{-1}$ is a maximal torus of S_φ^0. Hence there exists an element $t \in S_\varphi^0$ such that $s U s^{-1} = t U t^{-1}$. Then $t^{-1} s$ normalizes U, hence normalizes $^L T^0$. Thus $t^{-1} s \in N \cap S_\varphi^0$ maps to the same class as s.

The identification of $(N \cap S_\varphi)/U \cdot Z^{W_F}$ with $W(\lambda)$ requires some care. For each ω in the Weyl group W, we get $^L \omega : {}^L T \to {}^L T$, and by Langlands' correspondence for tori

$$W(\lambda) = \{ \omega \in W \mid {}^L \omega \circ \varphi \text{ is equivalent to } \varphi \}.$$

Let $\omega \in W$ and let $n \in N$ be a representative for ω, where ω is viewed as a W_F-invariant element of the Weyl group of $^L T^0$. The W_F-invariance of ω implies that n normalizes $^L T$, not just $^L T^0$.

We claim first that $^L \omega \circ \varphi_T$ is equivalent to $\mathrm{Int}(n) \circ \varphi_T$. The truth of the claim is independent of the choice of representative n for ω and is clear if n is invariant under W_F. Therefore the claim follows from

LEMMA. — *Any* $\omega \in W$ *has a* W_F-*invariant representative* $n \in N$.

This is Lemma 6.2 in the survey article of Borel in the Corvallis proceedings if G has a cyclic splitting field. The same proof works in general.

Using the claim, we see that ω fixes λ if and only if $\mathrm{Int}(n) \circ \varphi_T$ is equivalent to φ_T, that is, if and only if there is a $t \in {}^L T^0$ such that $tn \in S_\varphi$. Therefore

$$W(\lambda) = ((N \cap S_\varphi) \cdot {}^L T^0)/{}^L T^0 = (N \cap S_\varphi)/(U \cdot Z^{W_F}).$$

By the exact sequence,

$$S_\varphi = W(\lambda)/W(S_\varphi^0, U), \quad \text{since } W(S_\varphi^0, U) = (N \cap S_\varphi^0)/U.$$

We want to identify S_φ with the R-group $R(\lambda)$. All that remains is to show that $W(S_\varphi^0, U)$ is the subgroup $W'(\lambda)$ of $W(\lambda)$ generated by reflections w_α with respect to reduced roots α for which the Plancherel factor $\mu_\alpha(\lambda, s)$ has a zero at $s = 0$.

Let S be a maximal F-split torus in T and let Σ_r be the set of reduced positive roots of S in G. A root $\alpha \in \Sigma_r$ determines a Levi subgroup M_α of G containing T. The root spaces of T in the Lie algebra $\text{Lie}(M_\alpha)$ are precisely the root spaces of T in $\text{Lie}(G)$ corresponding to roots whose restrictions to S is some multiple of α.

The group ${}^L(M_\alpha)^0$ occurs naturally as a subgroup of ${}^L G^0$ containing ${}^L T^0$. The root spaces of ${}^L T^0$ in $\text{Lie}({}^L(M_\alpha)^0)$ are the root spaces of ${}^L T^0$ in $\text{Lie}({}^L G^0)$ corresponding to the coroots of T in M_α.

We now claim that the action of W_F on ${}^L G^0$ preserves the subgroup ${}^L(M_\alpha)^0$, on which it acts in the usual way (from the definition of the L-group). First, there is an ω in W such that $\omega\alpha$ is simple. From the Lemma, there is a W_F-invariant representative $n \in N$ of ω. Then ${}^L(M_\alpha)^0 = \text{Int}(n^{-1})({}^L(M_{\omega\alpha})^0)$. Since $\text{Int}(n^{-1})$ commutes with the action of the Weil group, we are reduced to the case in which α is simple. Then the claim is obvious.

Recall that U is a maximal torus of S_φ^0. Any root of U in S_φ^0 is the restriction of some root of ${}^L T^0$ in ${}^L G^0$. The group $\varphi(W_F)$ acts on $\text{Lie}({}^L G^0)$ via the adjoint action ${}^L G \to \text{Aut}(\text{Lie}({}^L G^0))$ and preserves the subspaces $\text{Lie}({}^L(M_\alpha)^0)$. We get one positive root of U in S_φ^0 for every $\alpha \in \Sigma_r$ such that $\varphi(W_F)$ fixes a non-zero vector in

$$V_\alpha = \text{Lie}({}^L(M_\alpha)^0) \cap \text{Lie (unipotent radical of } {}^L B^0).$$

We need to show that the following are equivalent:

(A) The image $\varphi(W_F)$ fixes a non-zero vector in V_α.

(B) The Plancherel factor $\mu_\alpha(\lambda, s)$ has a zero at $s = 0$.

Since $\mu_\alpha(\lambda, s)$ is the same for M_α as it is for G, we may assume that the F-rank of G is 1 and that $G = M_\alpha$. Further, the truth of (A) and (B) is unchanged if G is replaced by the simply connected cover of its derived group. The two statements are also compatible with restriction of scalars. Thus we have only two cases to consider.

Case 1. — $G = SL(2, F)$ and λ is a character of $F^\times$.
Then

$$V_\alpha = \begin{pmatrix} 0 & * \\ 0 & 0 \end{pmatrix}$$

with W_F acting on the entry $*$ by the character λ.

Therefore (A) is equivalent to $\lambda \equiv 1$.

Case 2. — G is a quasi-split SU (2, 1) associated to a separable quadratic extension E/F. Then λ is a character of $E^\times$.

Then

$$V_\alpha = \begin{pmatrix} 0 & * & * \\ 0 & 0 & * \\ 0 & 0 & 0 \end{pmatrix} = V_1 \oplus V_2$$

with

$$V_1 = \begin{pmatrix} 0 & * & 0 \\ 0 & 0 & * \\ 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad V_2 = \begin{pmatrix} 0 & 0 & * \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Then W_F acts on V_1 by the representation of W_F induced from the character λ of W_E, and W_F acts on V_2 by the character $\text{sgn}_{E/F} \cdot \lambda|_{F^\times}$.

The fixed points of W_F on V_α are the direct sum of the fixed points on V_1 and V_2.

Therefore (A) holds if and only if $\lambda \equiv 1$ or $\text{sgn}_{E/F} \cdot \lambda|_{F^\times} \equiv 1$.

Then from the explicit formulas for $\mu_\alpha(\lambda, s)$ in [15] and [7], we see that (A) is equivalent to (B) in both of the cases.

Remarks. — From the explicit normalizations of the operators $\mathscr{A}(w_\alpha, \lambda)$ for the groups $SL(2)$ and $SU(2,1)$ over real and non-archimedean fields, one knows more than just the location of the zeros of Plancherel measure. The normalizing factors may be shown to be given in terms of Euler factors, as conjectured by Langlands. Further details and applications will appear in a joint paper with F. Shahidi.

Now that we know $S_\varphi \cong R$, and thus $\Pi_\varphi \cong S_\varphi^\wedge$ (Theorem 2.4), we define a pairing of $S_\varphi \times \Pi_\varphi$ by

$$\langle r, \pi \rangle = \langle r, \rho \rangle = \text{trace } \rho(r)$$

for $\rho = \rho(\pi) \in R^\wedge$.

For $g \in C_c^\infty(G)$, the operators

$$I(g) = \int_G g(x) \, \text{Ind} \, \lambda(x) \, dx$$

and

$$\pi(g) = \int_G g(x) \, \pi(x) \, dx$$

have finite rank, $\pi = \pi(\rho) \in \Pi_\varphi$, and we may take traces.

The parametrization of Π_φ by the dual $R^\wedge = S_\varphi^\wedge$ is defined so that the operators $\mathscr{A}(r, \lambda)$ act on and permute the dim ρ irreducible subspaces of the $\pi(\rho)$-isotopic component as the representation ρ. Thus we get

THEOREM 2.7:

$$\text{trace } \mathscr{A}(r, \lambda) \, \mathrm{I}(g) = \sum_{\rho} \langle r, \rho \rangle \text{ trace } \pi(\rho)(g)$$

for g in $C_c^{\infty}(G)$, *where the sum is over an* L-*packet.*

Suppose we change the normalization of intertwining operators by a character ρ' of R. Recall that the parametrization $\Pi_{\varphi} \cong R^{\wedge}$ is shifted by ρ', i. e., $\pi'(\rho) = \pi(\rho'^{-1} \otimes \rho)$.

Then, if $\mathscr{A}'(r, \lambda) = \rho'(r) \mathscr{A}(r, \lambda)$, the formula (2.7) for the new normalization and parametrization gives

$$\begin{aligned}
\rho'(r) \text{ trace } \mathscr{A}(r, \lambda) \, \mathrm{I}(g) &= \text{trace } \mathscr{A}'(r, \lambda) \, \mathrm{I}(g) \\
&= \sum_{\rho} \theta_{\rho}(r) \text{ trace } \pi'(\rho)(g) \\
&= \sum_{\rho} \theta_{\rho}(r) \text{ trace } \pi(\rho'^{-1} \otimes \rho)(g) \\
&= \sum_{\rho} \theta_{\rho' \otimes \rho}(r) \text{ trace } \pi(\rho)(g) \\
&= \rho'(r) \sum_{\rho} \theta_{\rho}(r) \text{ trace } \pi(\rho)(g).
\end{aligned}$$

Thus a change in normalization amounts to multiplying both sides of (2.7) by the scalar $\rho'(r)$. Note that the possible choices for a "base representation" of Π_{φ}, to be indexed by the trivial character of R, correspond to the one-dimensional characters ρ' of R.

3. Classification of R-groups for
SU (n, n), U (n, n), and GU (n, n)

We give an explicit classification of the reducible (unitary) principal series by constructing a list of characters with non-trivial R-groups, such that any λ with non-trivial $R(\lambda)$ is conjugate under the Weyl group to one on the list. Multiplicity one fails for SU (n, n).

A less explicit classification is easy to obtain, and may be generalized. The R-groups $\tilde{R}$ for U (n, n) are contained in the group $\mathbf{Z}_2^n$ of sign changes in the Weyl group. Further, the $\tilde{R}$ are the reflection groups of type $A_1 \times \ldots \times A_1$ generated by the reflections w_{α} corresponding to the roots $\alpha = 2 e_i$ for which the restriction of λ_{α} to $F^{\times}$ is the character $\text{sgn}_{E/F}$ of order 2 attached to E/F by class field theory. Then the R-groups for the unitary group of similitudes GU (n, n) consist of the subgroups (of index 2) of even sign changes in these $\tilde{R}$.

The R-groups for SU (n, n) are "generalized dihedral" groups, given as extensions

$$1 \to \tilde{R} \to R \to \text{Gal}(L/E) \to 1$$

of the $\tilde{R}$ by the Galois group of an abelian extension L/E. The quotient $R/\tilde{R}$ embeds naturally into the group of characters $(E^{\times})^{\wedge}$, and determines the extension L/E by class field theory.

For $x \in E^{\times}$, let $X_i = (2e_i)^{\vee}(x) \in \tilde{M}$. Fix a character $\lambda \in M^{\wedge}$. Then define a map

$$\Theta: \quad W(\lambda) \to (E^{\times})^{\wedge},$$

by

$$\Theta(w) = \lambda_w, \quad \text{where} \quad \lambda_w(x) = \lambda(X_i w X_i^{-1} w^{-1}).$$

This expression is independent of i, since w fixes λ.

More generally, let $h : G \to \tilde{G}$ be a homomorphism between quasi-split groups with abelian kernel and cokernel. Fix a character λ of M and let $\tilde{\lambda}$ be a character of $\tilde{M}$ for which $\lambda = \tilde{\lambda} \circ h$. Suppose w fixes λ. Then the map

$$\tilde{m} \mapsto w\tilde{\lambda}(\tilde{m}) . \tilde{\lambda}^{-1}(\tilde{m}) = \tilde{\lambda}(w^{-1}\tilde{m}w\tilde{m}^{-1})$$

defines a character $\Theta(w) = \lambda_w$ of the abelian group $\tilde{M}$. Note that λ_w is trivial on $h(M)$, so Θ factors through a map

$$\Theta: \quad W(\lambda) \to (\tilde{M}/h(M))^{\wedge}.$$

Note that $\lambda_w \equiv 1$ if and only if $w \in W(\tilde{\lambda})$, by definition.

LEMMA 3.1. — *The map Θ is a homomorphism, and the kernel of Θ contains $W'(\lambda)$. Further, if $\lambda = \tilde{\lambda} \circ h$, then $\ker \Theta = W(\tilde{\lambda})$, $W'(\lambda) = W'(\tilde{\lambda})$, and $R(\lambda) \cap \ker \Theta = R(\tilde{\lambda})$.*

Proof. — Let w_1 and w_2 be in $W(\lambda)$. Then

$$\begin{aligned}
\Theta(w_1)(\tilde{m})\Theta(w_2)(\tilde{m}) &= \lambda_{w_1}(\tilde{m})\lambda_{w_2}(\tilde{m}) \\
&= \tilde{\lambda}(w_1^{-1}\tilde{m}w_1\tilde{m}^{-1}) . \tilde{\lambda}(w_2^{-1}\tilde{m}w_2\tilde{m}^{-1}) \\
&= \lambda(w_1^{-1}\tilde{m}w_1\tilde{m}^{-1}) . w_2^{-1}\lambda(w_2^{-1}\tilde{m}w_2\tilde{m}^{-1}) \\
&= \lambda(w_1^{-1}\tilde{m}w_1\tilde{m}^{-1}w_2[w_2^{-1}\tilde{m}w_2\tilde{m}^{-1}]w_2^{-1}) \\
&= \lambda(w_1^{-1}\tilde{m}w_1 w_2\tilde{m}^{-1}w_2^{-1}) \\
&= \lambda(w_2^{-1}w_1^{-1}\tilde{m}w_1 w_2\tilde{m}^{-1}) \\
&= \Theta(w_1 w_2)(\tilde{m}).
\end{aligned}$$

We use the facts that a commutator $w_2^{-1}\tilde{m}w_2\tilde{m}^{-1}$ is in the image $h(M)$ and that w fixes the character $\lambda = \tilde{\lambda} \circ h$.

Since the operator $\mathscr{A}(w_{\alpha}, \tilde{\lambda})$ is identified with $\mathscr{A}(w_{\alpha}, \lambda)$ on the space $V(\tilde{\lambda}) \cong V(\lambda)$, one operator is scalar if and only if the other is. It follows that $W'(\lambda) = W'(\tilde{\lambda})$, and the rest of the Lemma follows.

COROLLARY 3.2. — *Let λ be the restriction of $\tilde{\lambda}$ to M. Then the quotient $R(\lambda)/R(\tilde{\lambda})$ is abelian.*

Proof. — The map

$$R(\lambda)/R(\tilde{\lambda}) = R(\lambda)/R(\lambda) \cap \ker \Theta \hookrightarrow (\tilde{M}/M)^{\wedge}$$

induced by Θ is injective, so we may consider the quotient as a subgroup of $(\tilde{M}/M)^{\wedge}$, which is abelian.

We now consider the case of the inclusion of groups $SU(n, n) \hookrightarrow U(n, n)$. Then the quotient $R(\lambda)/R(\tilde{\lambda})$ may be considered as a subgroup of $E^{\times}$. The group of characters given as the image of Θ arises in an interesting fashion, as in [9]. *See* also [3] and [16].

Let

$$L(\tilde{\lambda}) = \{ w \in W \mid w\tilde{\lambda} = \tilde{\lambda} \otimes (\omega \circ \det) \text{ for some one-dimensional } \omega \},$$

and

$$\bar{L}(\tilde{\lambda}) = \{ \omega \mid w\tilde{\lambda} = \tilde{\lambda} \otimes (\omega \circ \det) \text{ for some } w \in W \}.$$

Here ω is a character of $E^1 = \{ x \in E^{\times} \mid \mathrm{Norm}_{E/F}(x) = 1 \}$. By Hilbert's Theorem 90, $E^1 = \{ x\bar{x}^{-1} \mid x \in E^{\times} \}$. Thus we may identify the group of characters ω of E^1 with the characters ν of $E^{\times}$ which are trivial on $F^{\times}$, via the isomorphism determined by $\omega(x/\bar{x}) = \nu(x)$.

First, we show that $L(\tilde{\lambda}) = W(\lambda)$. Restricting $w\tilde{\lambda} = \tilde{\lambda} \otimes (\omega \circ \det)$ to M gives $L(\tilde{\lambda}) \leq W(\lambda)$. If $w \in W(\lambda)$, we may define ω satisfying $w\tilde{\lambda} = \tilde{\lambda} \otimes (\omega \circ \det)$ by

$$\omega(x/\bar{x}) = w\tilde{\lambda} \otimes \tilde{\lambda}^{-1}(\alpha^{\vee}(x)),$$

where $\alpha = 2e_n$. Thus $w \in L(\tilde{\lambda})$, and $W(\lambda) \leq L(\tilde{\lambda})$. Note that the expression defining ω is just $\Theta(w)$.

There is a natural homomorphism $L(\tilde{\lambda}) \to \bar{L}(\tilde{\lambda})$ with kernel $W(\tilde{\lambda})$. With the above identifications, this homomorphism is just Θ.

We first classify the groups $R = R(\tilde{\lambda})$ which occur for the unitary group $U(n, n)$. Let $\lambda = \tilde{\lambda}|_M$, so that

$$R(\tilde{\lambda}) = R(\lambda) \cap W(\tilde{\lambda}) = R(\lambda) \cap \ker \Theta.$$

PROPOSITION 3.3. — *For any* R-*group* $R(\lambda)$ *for* $SU(n, n)$, $R(\lambda) \cap \ker \Theta$ *is contained in the group* $\mathbf{Z}_2^n$ *of sign changes in the Weyl group. In particular, any* $R(\tilde{\lambda})$ *for* $U(n, n)$ *is contained in* $\mathbf{Z}_2^n$.

Proof. — Suppose that $w = sc \in R(\tilde{\lambda}) = R(\lambda) \cap \ker \Theta$, with s in S_n and c in $\mathbf{Z}_2^n$. We must show that $s = 1$.

If $s \neq 1$, we may as well conjugate by an element of S_n to assume that the orbit of n under s is

$$n \mapsto k \mapsto k + 1 \mapsto \ldots \mapsto n - 1 \mapsto n.$$

Then by conjugation by elements of $\mathbf{Z}_2^n$, we may assume that c changes the sign of at most one root $2e_i$, say $2e_n$, in the orbit.

If $c(2e_i) = +2e_i$ for all $k \leq i \leq n$, then $w \in R(\lambda) \leq \ker \Theta$ implies

$$1 \equiv \lambda_w(x^{-1}) = \lambda((2e_n)^\vee(x^{-1}) \, w \, (2e_n)^\vee(x) \, w^{-1})$$
$$= \lambda((2e_n)^\vee(x^{-1})(2e_k)^\vee(x)) = \lambda((e_k - e_n)^\vee(x))$$

for all $x \in E^\times$.

Thus $e_k - e_n \in \Delta'$. But then $e_k - e_n > 0$ and $w(e_k - e_n) = e_{k+1} - e_k < 0$ contradict $w \in R$.

If $c(2e_n) = -2e_n$ and $c(2e_i) = +2e_i$ for $k \leq i \leq n-1$. then $w \in R(\tilde{\lambda}) \leq \ker \Theta$ implies

$$1 \equiv \lambda_w(\bar{x}) = \lambda((2e_n)^\vee(\bar{x}) \, sc \, (2e_n)^\vee(\bar{x}^{-1})(sc)^{-1})$$
$$= \lambda((2e_n)^\vee(\bar{x}) \, s \, (2e_n)^\vee(x) \, s^{-1})$$
$$= \lambda((2e_n)^\vee(\bar{x})(2e_k)^\vee(x)) = \lambda((e_k + e_n)^\vee(x))$$

for all $x \in E^\times$.

Thus $e_k + e_n \in \Delta'$. But $e_k + e_n > 0$ and

$$w(e_k + e_n) = sc(e_k + e_n) = s(e_k - e_n) = e_{k+1} - e_k < 0$$

contradict $w \in R$.

COROLLARY 3.4. — *Suppose $w = sc$ and $w' = s'c'$ are in $R(\lambda)$, with s and s' in S_n, and c and c' in $\mathbf{Z}_2^n$. Then the permutations s and s' commute.*

Proof. — Since the commutator $[w, w']$ is in the coset $[s, s'] \cdot \mathbf{Z}_2^n$ in the Weyl group, and $[R, R] \leq R \cap \ker \Theta \leq \mathbf{Z}_2^n$, the commutator $[s, s']$ must be trivial.

Thus, the R-groups which occur for $\tilde{G} = U(n, n)$ are products of the 2-element group $\mathbf{Z}_2$, with the number of factors bounded by $n = \operatorname{rank} \tilde{G}$. The following gives a more precise description of the groups $R(\tilde{\lambda})$.

LEMMA 3.5. — (i) *Suppose a product of sign changes $c = c_k c_{k+1} \ldots c_n$ is in $R(\tilde{\lambda})$, where $1 \leq k \leq n$. Then each c_i is in $R(\tilde{\lambda})$, for $k \leq i \leq n$.*

(ii) *If $c = c_k c_{k+1} \ldots c_n$ is in an $R(\lambda)$, and $2 \leq k \leq n$, then each c_i is in $R(\lambda)$, for $k \leq i \leq n$. Further, c_i is in $R(\tilde{\lambda})$, where $\lambda = \tilde{\lambda}|_M$.*

Proof. — Let $\alpha = 2e_i$. Recall that $w_\alpha = c_i$ fixes a character $\tilde{\lambda}$ of $\tilde{M}$ if and only if the restriction of $\tilde{\lambda}_\alpha$ to the norm subgroup $N_{E/F}(E^\times)$ is trivial. Then $c_i \in R(\tilde{\lambda})$ if and only if the restriction $\lambda_\alpha = \tilde{\lambda}_\alpha|_{F^\times}$ is the character $\operatorname{sgn}_{E/F}$ of $F^\times$ of order 2 which is trivial on norms from $E^\times$.

To show (i), let c be any sign change in $R(\tilde{\lambda})$. For any i with $c(2e_i) = -2e_i$,

$$\tilde{\lambda}((2e_j)^\vee(x) \, c_i \, (2e_j)^\vee(x^{-1}) \, c_i^{-1}) \equiv 1 \qquad \text{for} \quad j \neq i,$$

while

$$\lambda((2e_i)^\vee(x) \, c_i \, (2e_i)^\vee(x^{-1}) \, c_i^{-1}) = \tilde{\lambda}((2e_i)^\vee(x) \, c \, (2e_i)^\vee(x^{-1}) \, c^{-1}) \equiv 1.$$

So $c \in W(\tilde{\lambda})$ and $c(2e_i) = -2e_i$ imply that $c_i \in W(\tilde{\lambda})$. But $\beta > 0$ and $c\beta > 0$ imply that $c_i \beta > 0$ also. Thus $c \in R(\tilde{\lambda})$ implies that $c_i \in R(\tilde{\lambda})$, for all i with $c(2e_i) = -2e_i$.

For part (ii), take $n \geq 2$, and suppose $c \in R(\lambda)$ with $c(2e_1) = 2e_1$ and $c(2e_i) = -2e_i$. We check that $c_i \in W(\lambda)$ by noting that

$$c_i \lambda \left((2 e_1)^{\vee} (y) \right) = \lambda \left((2 e_1)^{\vee} (y) \right),$$

for all $y \in F^{\times}$,

$$c_i \lambda \left((e_1 - e_j)^{\vee} (x) \right) = \lambda \left((e_1 - e_j)^{\vee} (x) \right)$$

for all $x \in E^{\times}$, and $j \neq i$, and

$$c_i \lambda \left((e_1 - e_i)^{\vee} (x) \right) = c \, \lambda \left((e_1 - e_i)^{\vee} (x) \right) = \lambda \left((e_1 - e_i)^{\vee} (x) \right)$$

for all $x \in E^{\times}$.

Then $c_i \in R(\lambda)$ as before. Further,

$$\lambda \left((2 e_1)^{\vee} (x) \, c_i (2 e_1)^{\vee} (x^{-1}) \, c_i^{-1} \right) \equiv 1$$

implies that $c_i \in \ker \Theta$, so $c_i \in R(\lambda) \cap \ker \Theta = R(\tilde{\lambda})$.

Remark. — The hypothesis $2 \leq k \leq n$ in part (ii) is necessary, since it is possible for the longest Weyl element $w_0 = c_1 c_2 \ldots c_n$ to be in an $R(\lambda)$ for $SU(n, n)$ with no other sign changes in $R(\lambda)$. This happens if and only if w_0 fixes λ and Θ sends w_0 to a character of $F^{\times}$ of order 2, other than $\mathrm{sgn}_{E/F}$.

We may now explicitly classify the reducible Ind $\tilde{\lambda}$ which occur for $U(n, n)$. By Proposition 3.3, $R(\tilde{\lambda})$ is contained in the group of sign changes in the Weyl group.

Suppose $R(\tilde{\lambda})$ is non-trivial. Pick an element c in R for which the set

$$I = I_c = \{ i \mid c (2 e_i) = -2 e_i \}$$

is maximal. Then by Lemma 3.5 and the maximality of I, $R(\tilde{\lambda})$ is the reflection group $\langle c_i \mid i \in I \rangle$ of type $A_1 \times A_1 \times \ldots \times A_1$ generated by the c_i, for $i \in I$.

Without loss of generality, we may replace $\tilde{\lambda}$ by a conjugate under W to assume that $I = \{ i \mid k \leq i \leq n \}$ for some k, $(1 \leq k \leq n)$, and then

$$R(\tilde{\lambda}) = \langle c_k, c_{k+1}, \ldots, c_n \rangle.$$

Recall that a character $\tilde{\lambda}$ of $\tilde{M}$ may be defined by the n characters $\tilde{\lambda}_{\alpha}$ of $E^{\times}$, for $\alpha = e_1 - e_2, e_2 - e_3, \ldots, e_{n-1} - e_n$, and $2 e_n$. Equivalently, we may specify the n characters of $E^{\times}$ corresponding to the roots $\alpha = 2 e_1, 2 e_2, \ldots, 2 e_n$.

Finally, the condition $R(\Delta'_+) \subseteq \Delta'_+$ in the definition of R requires that the characters $\tilde{\lambda}_{\alpha}$ are distinct, for $\alpha = 2 e_k, 2 e_{k+1}, \ldots, 2 e_n$, if

$$R = R(\tilde{\lambda}) = \langle c_k, c_{k+1}, \ldots, c_n \rangle.$$

THEOREM 3.6. — *Let* $G = U(n, n)$. *Any character* $\tilde{\lambda}$ *of* $\tilde{M}$ *with non-trivial* R-group *is conjugate under the Weyl group to one of the following.*

For the roots $\alpha = 2 e_k, \ldots, 2 e_n$, *let* $\tilde{\lambda}_{\alpha}$ *be distinct characters of* $E^{\times}$ *with* $\tilde{\lambda}|_{F^{\times}} = \mathrm{sgn}_{E/F}$. *The characters corresponding to* $2 e_1, \ldots, 2 e_{k-1}$ *are arbitrary, subject to the condition that* k *is minimal with respect to this property.*

The corresponding R-group $R(\tilde{\lambda}) \cong Z_2 \times \ldots \times Z_2$ *is generated by the reflections with respect to the roots* $2 e_i$, *where* $k \leq i \leq n$.

Thus Ind $\tilde{\lambda}$ decomposes with multiplicity one, since $R(\tilde{\lambda})$ is abelian, and components $\tilde{\pi}$ of the corresponding L-packet are parametrized by characters $\tilde{\rho}$ of $R(\tilde{\lambda})$.

Then the operator $\mathscr{A}(\tilde{r}, \tilde{\lambda})$ acts on the space of $\tilde{\pi}(\tilde{\rho})$ as multiplication by the scalar

$$\tilde{\rho}(\tilde{r}) = \langle \tilde{r}, \tilde{\rho} \rangle,$$

for $\tilde{r}$ in $R(\tilde{\lambda}) = S_{\tilde{\phi}}$.

The integrated representations corresponding to Ind $\tilde{\lambda}$ and $\tilde{\pi}(\tilde{\rho})$ satisfy

$$\text{trace } \mathscr{A}(\tilde{r}, \tilde{\lambda}) I(g) = \sum_{\tilde{\rho} \in \tilde{R}} \langle \tilde{r}, \tilde{\rho} \rangle \text{ trace } \pi_{\tilde{\rho}}(g)$$

and

$$\text{trace } \pi_{\tilde{\rho}}(g) = |R|^{-1} \sum_{\tilde{\rho} \in \tilde{R}} \overline{\langle \tilde{r}, \tilde{\rho} \rangle} \text{ trace } \mathscr{A}(\tilde{r}, \tilde{\lambda}) I(g)$$

for $g \in C_c^\infty(\tilde{G})$.

The same situation appears in [6] for the unramified unitary principal series of simply-connected, semi-simple p-adic groups. We may define a pairing as above in this case, fixing the K_0-class-1 component as the base representation to be indexed by the trivial character of R. Note that in the case Φ is of type C_n and $q_{\alpha/2} \neq 1$, then R is trivial, and that G has only one conjugacy class of *hyperspecial* maximal compact subgroups in this case.

We now classify the R-groups which occur for $SU(n, n)$. A character λ of M is defined by $n-1$ characters λ_α of $E^\times$, for the short roots $\alpha = e_1 - e_2, \ldots, e_{n-1} - e_n$, and a character λ_β of $F^\times$, for $\beta = 2e_n$.

Recall that Θ induces an injection

$$R/R \cap \ker \Theta \hookrightarrow (E^\times)^\wedge.$$

If $R \cap \ker \Theta$ is trivial, then R is abelian, and by (b) of Lemma 3.5, $R \cap Z_2^n$ is either the group $\langle w_0 \rangle$, or is trivial. Further, $R \cap \ker \Theta$ is trivial implies that R acts on $\{ \pm 2e_i | 1 \leq i \leq n \}$ without fixed points. Thus $|R|$ divides $2n$. Suppose m divides $2n$. Then if R is to have exponent m, R must inject into $(E^\times/F^\times (E^\times)^m)^\wedge$. Hence the order of R divides both $2n$ and the index $[E^\times : F^\times (E^\times)^m]$. In particular, $|R|$ divides both $2n$ and $[E^\times : F^\times (E^\times)^{2n}]$.

Note that the longest Weyl element w_0 is in $R(\lambda)$ if and only if $\lambda_\alpha(x\bar{x}) \equiv 1$ for $\alpha = e_1 - e_2, \ldots, e_{n-1} - e_n$, and λ_β has order 2 for $\beta = 2e_n$, and $\Delta' = \varnothing$. Suppose $w_0 \in R$. Then, $R \cap Z_2^n = \langle w_0 \rangle$ if and only if λ_β has order 2 and $\lambda_\beta \neq sgn_{E/F}$.

Now consider the case $R \cap \ker \Theta$ is non-trivial. Since $R \cap \ker \Theta \leq Z_2^n$, R will contain a non-trivial product of sign changes. Then if

$$I = \{ i | \exists c \in R \text{ with } c(2e_i) = -2e_i \},$$

R contains each reflection c_i by Lemma 3.5. Thus $\tilde{R} = R \cap \ker \Theta$ is the reflection group of type $A_1 \times A_1 \times \ldots \times A_1$ corresponding to the root system

$$\tilde{\Delta} = \{ \pm 2 e_i \,|\, i \text{ in } I \}.$$

If $\tilde{\lambda} \in \tilde{M}^\wedge$ restricts to λ, then

$$\tilde{R} = R(\tilde{\lambda}) \subsetneqq R(\lambda).$$

Let $R^* = \{ r \in R \,|\, \alpha > 0 \text{ and } \alpha \in \tilde{\Delta} \text{ imply that } r\alpha > 0 \}$.

Then standard arguments concerning reflection groups imply that R^* is a complement of $\tilde{R}$ in R.

Since R^* acts on $\tilde{\Delta}$, and $R^* \cap \ker \Theta$ is trivial, any non-identity element of R^* acts on $\{ 2 e_i \,|\, i \in I \}$ without fixed points. Thus, the order of R^* divides $|I| = \operatorname{rank}(\tilde{\Delta}) \leq n$, and the order of any R-group for $SU(n, n)$ is bounded by $n 2^n$. The action of R^* on $\mathbb{C}[\tilde{\Delta}]$ is a multiple of the regular representation.

Note that Θ induces an inclusion $R^* \subsetneqq (E^\times/F^\times)^\wedge \subsetneqq (E^\times)^\wedge$. Since $|R^*|$ divides $|I|$, R^* may be considered as a subgroup of the (finite) group of characters of $E^\times/F^\times$ of order dividing $|I|$. By class field theory, this subgroup determines an abelian extension L of E with $\operatorname{Gal}(L/E) \cong R^*$. This provides an arithmetic condition limiting the complexity of the $R = R^* \times_s \tilde{R}$ which can occur. Suppose m is a divisor of the rank of $\tilde{R}$. Then if R^* is to have exponent m, its order must divide the index $[E^\times : F^\times(E^\times)^m]$, since then R^* embeds into $(E^\times/F^\times(E^\times)^m)^\wedge$.

The exact sequence $1 \to \tilde{R} \to R \to R^* \to 1$ describes R as an extension of the reflection group $\tilde{R} = \mathbb{Z}_2 \times \ldots \times \mathbb{Z}_2$ by an abelian group $R^* \cong \operatorname{Gal}(L/E)$. Further, any non-trivial element of R^* permutes the generators $\{ w_\alpha \,|\, \alpha = 2 e_i \in \tilde{\Delta} \}$ of $\tilde{R}$ with no fixed points. Thus the order of R^* divides the rank of $\tilde{R}$, and R is non-abelian if and only if both $\tilde{R}$ and R^* are non-trivial.

Non-abelian R of order $m 2^k$ will occur, where $k \leq n$ and m divides $k = \operatorname{rank} \tilde{R}$, subject to the arithmetic conditions above.

We examine the conjugacy classes in the Weyl group to determine which ones may contain elements of an R-group. We then find elements which may be used to "build" the R-groups for $SU(n, n)$. We may then construct an explicit list of possible R-groups for $SU(n, n)$ and use the arithmetic conditions on the extension E/F to determine the existence of characters λ for which these groups occur. Every finite abelian group will occur as an R^* for some G.

Conjugacy classes in Weyl groups are parametrized by certain admissible graphs [2]. An element w in a Weyl group $S_n \times_s \mathbb{Z}_2^n$ of type C_n acts on $\{ \pm 2 e_i \,|\, 1 \leq i \leq n \}$ to produce cycles

$$2 e_{j(1)} \mapsto \pm 2 e_{j(2)} \mapsto \ldots \mapsto \pm 2 e_{j(r)} \mapsto \pm 2 e_{j(1)}.$$

If $w^r (2 e_{j(1)}) = +2 e_{j(1)}$, say that the cycle is positive.

If $w^r (2 e_{j(1)}) = -2 e_{j(1)}$, say that the cycle is negative.

Then two elements of W are conjugate if and only if they have the same signed cycle type.

Note that a positive cycle is conjugate in W to one involving no sign changes, and that a negative cycle is conjugate to one involving a single sign change.

Thus conjugacy classes in W are parametrized by graphs of the form

$$A_{i(1)} + A_{i(2)} + \ldots + C_{j(1)} + C_{j(2)} + \ldots,$$

where

$$\sum (i(k) + 1) + \sum j(k) = n.$$

Here, A_i is the class of a positive cycle of length $i+1$, and C_j is the class of a negative cycle of length j.

Suppose again that $\tilde{R} = R \cap \ker \Theta$ is non-trivial. Then $\tilde{R}$ is the reflection group associated to

$$\tilde{\Delta} = \{ \pm 2 e_i \mid \exists r \in R \text{ with } r(2e_i) = -2e_i \}.$$

If $R = \tilde{R}$, then we are done, since $R \cong \mathbf{Z}_2 \times \ldots \times \mathbf{Z}_2$ and the classification is like that for $U(n, n)$. Otherwise, R^* is non-trivial.

Suppose that R^* is non-trivial. We want to determine the conjugacy classes of W which may contain elements of an R^*. Suppose R contains an element $w = sc$, where $s \in S_n$ and $c \in \mathbf{Z}_2^n$, with $s \neq 1$. By Proposition 3.3, w has no fixed points in $\{ \pm 2 e_i \mid 1 \leq i \leq n \}$. Suppose the shortest cycle in s has length k. Then $w^k = s^k c'$, with $c' \in \mathbf{Z}_2^n$.

If $s^k = 1$, then all cycles of s have length k, so w has type

$$A_{k-1} + A_{k-1} + \ldots + C_k + C_k + \ldots,$$

and further, $c' \in R$. By Lemma 3.5 (if $c' \neq w_0$ or if we assume $\tilde{R}$ non-trivial), R will contain each sign change occurring in c'. Thus, to determine generators of R, we may multiply w (if necessary) by some of these sign changes to assume that each cycle is positive. So if $s^k = 1$, we may assume without loss of generality that w is of type $A_{k-1} + \ldots + A_{k-1}$. In the case $\tilde{R}$ is trivial, w may be a product of negative cycles, of type $C_k + \ldots + C_k$, with $w^k = w_0 =$ the longest Weyl element. In this case $R \cap \mathbf{Z}_2^n = \langle w_0 \rangle$.

If instead $s^k \neq 1$, then $w^k = s^k c'$ again has no fixed points. Since s contains a cycle of length k, s^k does have fixed points, so there exists an index i such that

$$w^k(2e_i) = c'(2e_i) = -2e_i.$$

In fact, $c'(2e_i) = -2e_i$ for each i fixed by s^k, by Proposition 3.3. This implies that each cycle of length k of s must be negative, since the action of c' on the roots $2e_i$ in an orbit associated to such a cycle is the product of the orbits of all the c_i associated with the cycle.

Finally, $w^{2k} = s^{2k} c''$ does fix some root $2e_i$, so $s^{2k} = 1$ by Proposition 3.3. Thus s is a product of cycles of lengths k and $2k$. Further, any sign change occurring in $w^{2k} = c''$ will be in R by Lemma 3.5, so to determine generators of R, we may assume without loss of generality that each cycle of w of length $2k$ is positive. Then w has conjugacy type $A_{2k-1} + \ldots + A_{2k-1} + C_k + \ldots + C_k$, and w^k has type $A_1 + \ldots + A_1 + C_1 + \ldots + C_1$.

The action of R^* on $\tilde{R}$ is given as k copies of the regular representation of R^*, where $k = \operatorname{rank} \tilde{R} / |R^*|$.

THEOREM 3.7. — *Any R-group for* SU (n, n) *fits into a split sequence*

$$1 \to \tilde{R} \to R \to R^* \to 1,$$

with R non-abelian if any only if both $\tilde{R}$ and R^* are non-trivial.

$$\tilde{R} = R \cap \ker \theta \cong \mathbf{Z}_2^d$$

is a reflection group of type $A_1 \times \ldots \times A_1$ *associated to a root system*

$$\tilde{\Delta} = \{ \pm 2 e_i \mid \exists c \in R \text{ with } c(2e_i) = -2e_i \}$$

and is an R-group for U (n, n).

Since $\tilde{R}$ is a reflection group, the group

$$R^* = \{ r \in R \mid \alpha \in \tilde{\Delta} \text{ and } \alpha > 0 \text{ imply } r\alpha > 0 \}$$

is a complement of $\tilde{R}$ in R, with the order of R^* dividing the rank d of $\tilde{\Delta}$. Further, R^* injects naturally into the finite group of characters $(E^\times / F^\times (E^\times)^d)^\wedge$. Thus R^* may be identified with an abelian Galois group Gal (L/E).

If $\tilde{R}$ is non-trivial, it is generated by sign changes of type $A_0 + \ldots + A_0 + C_1$, and R^* may be built from elements of types $A_m + \ldots + A_m$ and $A_{2k-1} + \ldots + A_{2k-1} + C_k + \ldots + C_k$.

If $\tilde{R}$ is trivial, then $R = R^*$ may be built from elements of types $A_m + \ldots + A_m$, and $C_k + \ldots + C_k$ (or w_0, in case $R \cap \mathbf{Z}_2^n = \langle w_0 \rangle$), or from elements of types $A_m + \ldots + A_m$ and $A_{2k-1} + A_{2k-1} + C_k + \ldots + C_k$.

4. Restriction of Representations and Reciprocity

We will study restrictions of representations between

$$G = \text{SU}(n, n) \leqq \tilde{G} = \text{U}(n, n) \leqq \tilde{\tilde{G}} = \text{GU}(n, n).$$

More generally, we consider restrictions determined by a homomorphism

$$h: \quad G \to \tilde{G}$$

with abelian kernel and cokernel, between quasi-split groups. We may assume that $h(P) \leqq \tilde{P}$. If $\tilde{\lambda}|_M = \lambda$, i.e., $\tilde{\lambda} \circ h = \lambda$, then Ind $(\tilde{P}, \tilde{G}; \tilde{\lambda})|_G \cong$ Ind $(P, G; \lambda)$. Further,

$R(\tilde{\lambda}) \leqq R(\lambda)$. The homomorphism h induces a map $h^*: {}^L\tilde{G} \to {}^LG$, and $S_{\tilde{\varphi}} \leqq S_{\varphi}$ if $\varphi = h^* \circ \tilde{\varphi}$.

Recall from section 2 that the components of an $\mathrm{Ind}(\tilde{\lambda})$ are parametrized by the dual of the group $R(\tilde{\lambda}) \cong S_{\tilde{\varphi}}$.

We show that the following type of reciprocity holds. If a component $\tilde{\pi}$ of $\mathrm{Ind}\,\tilde{\lambda}$ is parametrized by an irreducible representation $\tilde{\rho}$ of $R(\tilde{\lambda})$, then the components of the restriction of $\tilde{\pi}$ to G are parametrized by the irreducible components of the induced representation $\mathrm{Ind}\,(R(\tilde{\lambda}), R(\lambda); \tilde{\rho})$ of $R(\lambda)$.

The reciprocity result follows from a simple algebraic lemma. Let $A \to B$ be a ring homomorphism. Let W_A be an A-module, and let W_B and V be B-modules. Let $E = \mathrm{End}_B(V)$. Then $\mathrm{Hom}_B(W_B, V)$ and $\mathrm{Hom}_A(W_A, V)$ have natural E-module structures, using composition of homomorphisms to define the multiplication. There is a natural homomorphism.

$$\mathrm{Hom}_A(W_A, W_B) \to \mathrm{Hom}_E(\mathrm{Hom}_B(W_B, V), \mathrm{Hom}_A(W_A, V)).$$

LEMMA. — *Suppose that W_B is a direct summand of V. Then the above homomorphism is an isomorphism.*

Proof. — Choose B-module maps $i: W_B \to V$ and $p: V \to W_B$ with $p \circ i = id$ on W_B. Then the inverse map is given by

$$\psi \mapsto (g_\psi: w_A \mapsto p(\psi(i)(w_A)))$$

where $\psi \in \mathrm{Hom}_E(\mathrm{Hom}_B(W_B, V), \mathrm{Hom}_A(W_A, V))$ and $w_A \in W_A$.

Fix Langlands parameters φ and $\tilde{\varphi}$ corresponding to (unitary) principal series $\mathrm{Ind}(P, G; \lambda)$ and $\mathrm{Ind}(\tilde{P}, \tilde{G}; \tilde{\lambda})$, with $\tilde{\varphi}$ a lift of φ. Then we have $\lambda = \tilde{\lambda}|_M$, $R(\tilde{\lambda})$ is normal in $R(\lambda)$, and the quotient $R(\lambda)/R(\tilde{\lambda})$ is abelian. The commuting algebras of $\mathrm{Ind}\,\lambda$ and $\mathrm{Ind}\,\tilde{\lambda}$ are given by the group algebras $\mathbf{C}[R(\lambda)]$ and $\mathbf{C}[R(\tilde{\lambda})]$, respectively. Further, the restriction of a standard intertwining operator for $\tilde{G}$ is a standard intertwining operator for G, i.e.,

$$\mathscr{A}(w, \tilde{\lambda})|_G = \mathscr{A}(w, \tilde{\lambda}|_M) = \mathscr{A}(w, \lambda).$$

THEOREM 4.1. — *Let $\tilde{\pi}$ be an irreducible component of $\mathrm{Ind}(\tilde{P}, \tilde{G}; \tilde{\lambda})$, corresponding to an irreducible representation $\tilde{\rho}$ of $S_{\tilde{\varphi}}$. Let π be an irreducible component of $\mathrm{Ind}(P, G; \lambda)$, corresponding to an irreducible representation ρ of S_φ. Then the multiplicity of π in the restriction of $\tilde{\pi}$ from $\tilde{G}$ to G is equal to the multiplicity of $\tilde{\rho}$ in the restriction of ρ to $S_{\tilde{\varphi}}$, which equals the multiplicity of the representation ρ in $\mathrm{Ind}(S_{\tilde{\varphi}}, S_\varphi; \tilde{\rho})$.*

In particular, the irreducible components of the restriction of $\tilde{\pi}$ to G are parametrized by the irreducible components ρ of the induced representation $\mathrm{Ind}(S_{\tilde{\varphi}}, S_\varphi; \tilde{\rho})$ of S_φ.

Proof. — It is enough to show that $\mathrm{Hom}_G(\pi, \tilde{\pi}) \cong \mathrm{Hom}_{S_{\tilde{\varphi}}}(\tilde{\rho}, \rho)$. Apply the Lemma with $A = \mathbf{C}[G]$, $B = \mathbf{C}[\tilde{G}]$, $V = \mathrm{Ind}(\tilde{P}, \tilde{G}; \tilde{\lambda})$, $W_A = \pi$ and $W_B = \tilde{\pi}$.

Note that $E = \mathrm{End}_G(V) = \mathbf{C}[S_{\tilde{\varphi}}]$ and that

$$\tilde{\rho} \cong \mathrm{Hom}_B(W_B, V) \qquad \text{and} \qquad \rho \cong \mathrm{Hom}_A(W_A, V).$$

Consider now the inclusion $SU(n, n) \hookrightarrow U(n, n)$. Recall that $R(\tilde{\lambda})$ is then a reflection group, and thus has a complement R^* in $R(\lambda)$.

Let $\tilde{\rho}$ be an irreducible representation of $R(\tilde{\lambda})$ and let $Z(\tilde{\rho})$ be the centralizer of $\tilde{\rho}$ in $R(\lambda)$. The subgroup $Z^* = Z(\tilde{\rho}) \cap R^* \leq R^* \cong \mathrm{Gal}(L/E)$ coming from the inclusion

$$R(\tilde{\lambda}) \leq Z(\tilde{\rho}) \leq R(\lambda)$$

corresponds to an intermediate field extension

$$E \subseteq K \subseteq L,$$

with $Z^* \cong \mathrm{Gal}(L/K)$ and

$$R(\lambda)/Z(\tilde{\rho}) \cong R^*/Z^* \cong \mathrm{Gal}(K/E).$$

The intersections of the kernels of the characters for each of the subgroups

$$Z^* \hookrightarrow (E^\times/F^\times)^\wedge \hookrightarrow (E^\times)^\wedge$$

give a lattice of norm subgroups $N(K^\times)$ of $E^\times$.

Define a lattice of intermediate subgroups G^K between $SU(n, n)$ and $U(n, n)$ by

$$G^K = \{ g \in U(n, n) \,|\, \det(g) \in N(K^\times) \},$$

for $E \subseteq K \subseteq L$. Then the restriction of $\mathrm{Ind}(\tilde{P}, U(n, n); \tilde{\lambda})$ to G^K has commuting algebra $\mathbf{C}[Z(\tilde{\rho})] = \mathbf{C}[\mathrm{Gal}(L/K) \times_s \tilde{R}]$, and restrictions among the various G^K satisfy the reciprocity result.

In the next section we count the number of characters fixed by subgroups Z between R and $\tilde{R}$ and use Möbius inversion to calculate the number of components, and multiplicities, in an L-packet.

5. Structure of L-packets

THE NUMBER OF COMPONENTS AND MULTIPLICITIES. — We give examples of typical non-abelian groups S_φ which occur for $SU(n, n)$, and derive formulas for the number of components in a general L-packet, and multiplicities, using Möbius inversion. The examples illustrate the reciprocity describing restriction of representations between the groups $U(n, n)$ and $SU(n, n)$, and also the parametrization $\Pi_\varphi \cong S_\varphi^\wedge$. The multiplicities are just the dimensions of the corresponding irreducible representations of the group S_φ.

The simplest non-abelian $S_\varphi \cong R(\lambda)$ occurs for the rank 2 group $SU(2, 2)$. Define $\lambda \in M^\wedge$ by requiring $\lambda_\alpha = \lambda \circ \alpha^\vee = \alpha^\vee \circ \varphi$ to have order 2 in $(E^\times/F^\times)^\wedge$ for $\alpha = e_1 - e_2$, and

$\lambda_\beta = sgn_{E/F}$, the character of $F^\times/N(E^\times)$ of order 2, for $\beta = 2e_2$. Then $\Delta' = \emptyset$ and

$$R(\lambda) = W(\lambda) = W.$$

The isomorphism $R(\lambda) \cong S_\varphi$ is transparent. The subgroup

$$R^* = \langle w_\alpha \rangle = \langle (12) \rangle$$

maps onto the subgroup $\langle \lambda_\alpha \rangle$ of $(E^\times)^\wedge$. The subgroup

$$\tilde{R} = \langle w_\alpha w_\beta w_\alpha, w_\beta \rangle = \langle c_1, c_2 \rangle$$

of sign changes is the R-group for a representation Ind $\tilde{\lambda}$ of $U(2, 2)$, where $\lambda = \tilde{\lambda}|_M$.

Then, Ind λ decomposes in this case into 4 components of multiplicity 1, and 1 component of multiplicity 2. The group $\tilde{R}$ determines reducibility under $U(2,2)$, and Ind $\tilde{\lambda}$ decomposes into 4 components of multiplicity 1. These 4 components are indexed by the characters $\tilde{\rho}$ of $\tilde{R}$, and the intertwining operator $\mathscr{A}(\tilde{r}, \lambda)$ acts on the space $V(\tilde{\rho})$ by the scalar $\tilde{\rho}(\tilde{r})$, for $\tilde{r}$ in $\tilde{R}$.

The group R^* governs behavior of restriction to $SU(2, 2)$. The two components of Ind $\tilde{\lambda}$ indexed by the trivial character $\tilde{\rho}_1$, and the character $\tilde{\rho}_2$ defined by

$$\tilde{\rho}_2(w_\beta) = \tilde{\rho}_2(w_\alpha w_\beta w_\alpha) = -1,$$

each decompose into two irreducible subspaces on restriction to $SU(2, 2)$. These are the intersections of $V(\tilde{\rho}_1)$ and $V(\tilde{\rho}_2)$ with the $+1$ and -1 eigenspaces of the operator $\mathscr{A}(w_\alpha, \lambda)$, since $R^* = \langle w_\alpha \rangle$ stabilizes $\tilde{\rho}_1$ and $\tilde{\rho}_2$. This accounts for the 4 components of multiplicity 1 in Ind λ, indexed by the one-dimensional representations ρ of R. For w in R, the operator $\mathscr{A}(w, \lambda)$ acts on these $V(\rho)$ by the scalar $\rho(w)$.

The other two components of Ind $\tilde{\lambda}$ in this example, indexed by the characters defined by

$$\tilde{\rho}_3(w_\beta) = \tilde{\rho}_4(w_\alpha w_\beta w_\alpha) = 1$$

and

$$\tilde{\rho}_3(w_\alpha w_\beta w_\alpha) = \tilde{\rho}_4(w_\beta) = -1,$$

remain irreducible, but become equivalent, upon restriction to $SU(2, 2)$. The characters $\tilde{\rho}_3$ and $\tilde{\rho}_4$ are conjugate by w_α, and the operator $\mathscr{A}(w_\alpha, \lambda)$ gives the equivalence under $SU(2, 2)$. This accounts for the component $V(\rho_0)$ of Ind λ occurring with multiplicity 2, where

$$\rho_0 = \text{Ind}(\tilde{R}, R; \tilde{\rho}_3) \cong \text{Ind}(\tilde{R}, R; \tilde{\rho}_4).$$

Note that

$$\rho_0(c_1) = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix},$$

$$\rho_0(c_2) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

and

$$\rho_0(w_\alpha) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

give the action of the operators $\mathscr{A}(c_1, \lambda)$, $\mathscr{A}(c_2, \lambda)$, and $\mathscr{A}(w_\alpha, \lambda)$, respectively, on the subspaces $V(\tilde{\rho}_3) \oplus V(\tilde{\rho}_4)$.

Thus we know how each intertwining operator in the commuting algebra acts on the SU(2,2)-irreducible subspaces. In particular, if $\mathscr{A}(w, \lambda)$ acts as a scalar on an irreducible component, we know the scalar. It is the value $\rho(w)$ if the component is indexed by a one-dimensional character ρ. It is the (non-zero) diagonal entry corresponding to the space $V(\tilde{\rho}_3)$ or $V(\tilde{\rho}_4)$ in the matrices defining the two-dimensional ρ_0 above.

It follows that the pairing of $S_\varphi \times \Pi_\varphi$ given by

$$\langle r, \rho \rangle = \theta_\rho(r) = \text{trace } \rho(r)$$

for r in $R \cong S_\varphi$ and ρ in $R^\wedge \cong \Pi_\varphi$ satisfies

$(\bigstar)$ $$\text{trace } \mathscr{A}(r, \lambda) I(g) = \sum_{\pi(\rho)} \langle r, \rho \rangle \text{ trace } \pi(\rho)(g)$$

for $g \in C_c^\infty(G)$, where $I = \text{Ind}(\lambda)$, and the sum is over $\pi(\rho) \in \Pi_\varphi \cong S_\varphi^\wedge$.

If a different normalization of intertwining operators is used, the parametrization $\Pi_\varphi \cong S_\varphi^\wedge$ will be shifted by a one-dimensional $\rho' \in S_\varphi^\wedge$ and each side of $(\bigstar)$ will be multiplied by $\rho'(r)$. We may consider the component parametrized by the trivial character as a base representation in Π_φ. Then such a shift corresponds to the choice of the component $V(\rho')$ as the base.

To study the general non-abelian S_φ which occur for SU(n, n), recall from section 4 that the component parametrized by

$$\rho = \text{Ind}(Z(\tilde{\rho}), R; \xi)$$

occurs in the L-packet with multiplicity $|R/Z(\tilde{\rho})|$. Here, $Z(\tilde{\rho})$ is the centralizer of an irreducible representation $\tilde{\rho}$ of $S_{\tilde{\varphi}}$, and ξ is an irreducible component of $\rho = \text{Ind}(S_{\tilde{\varphi}}, Z(\tilde{\rho}); \tilde{\rho})$. Consider the formula

$$|R| = \sum_{\{\tilde{\rho}\}} |Z^*(\tilde{\rho})| \cdot |R/Z(\tilde{\rho})|^2$$

with a different point of view. Instead of summing over orbits $\{\tilde{\rho}\}$ of R in $\tilde{R}^\wedge$, sum over subgroups Z with $\tilde{R} \leq Z \leq R$ and count the number of representations of R induced irreducibly from Z.

Setting $Z = Z^* \times_s \tilde{R}$, let $c(Z^*, R^*)$ be the number of characters of $\tilde{R}$ with stabilizer Z^* in R^*. Then there are $c(Z^*, R^*) \cdot |Z^*|$ characters ξ of Z to consider, which

form $c(Z^*, R^*) |Z^*| . |R^*/Z^*|^{-1}$ orbits. For each orbit there is a component which occurs with multiplicity $|R^*/Z^*|$.

Thus, in general, there are

$$\sum c(Z^*, R^*) . |Z^*| . |R^*/Z^*|^{-1} = m^{-2} |R^*| \sum c(Z^*, R^*)$$

components of multiplicity $m = |R^*/Z^*|$, where the sums are over all subgroups Z^* of index m in R^*.

Since $R^* = \text{Gal}(L/E)$, $Z^* = \text{Gal}(L/K)$, and $R^*/Z^* = \text{Gal}(K/E)$, the sum may be considered as over all field extensions K/E of degree m lying in L.

Note that $|\tilde{R}| = \sum c(Z^*, R^*)$, summing over all $Z^* \leq R^*$. The correct formula for the dimension of the commuting algebra

$$|R| = |R^*| . |\tilde{R}| = \sum (c(Z^*, R^*) . |Z^*| . |R^*/Z^*|^{-1}) |R^*/Z^*|^2$$

follows immediately.

Then the number of inequivalent components in the L-packet is given by the sum, over $Z^* \leq R^*$,

$$\sum c(Z^*, R^*) |Z^*| . |R^*/Z^*|^{-1} = \sum |R^*|^{-1} c(Z^*, R^*) . |Z^*|^2.$$

Next, the structure theorem describing the extension

$$1 \to \tilde{R} \to R \to R^* \to 1$$

gives an inductive method to define the coefficients $c(Z^*, R^*)$. This will allow us to give explicit formulas for the $c(Z^*, R^*)$ in terms of Möbius inversion.

If $N = \text{rank } \tilde{R} = |\tilde{\Delta}|$, then $|R^*|$ divides N. The N-dimensional representation of $R^* = \text{Gal}(L/E)$ on the space $\mathbf{C}[\tilde{\Delta}]$ is the multiple k times the regular representation of R^*, where $k = N/|R^*|$. The action of R^*/Z^* on $\mathbf{C}[\tilde{\Delta}(\text{mod } Z^*)]$ is the same multiple k of the regular representation of R^*/Z^*.

It follows that $c(R^*/Z^*) = c(Z^*, R^*)$ depends only on the quotient R^*/Z^*.

We give formulas for the number of characters

$$c(Z^*, R^*) = c_1(Z^*, R^*)$$

with stabilizer Z^* in the case that $k = 1$, i.e., $N = |R^*|$. Then, if $N = k . |R^*|$, with $k > 1$, the number of characters $c_k(Z^*, R^*)$ with stabilizer Z^* may be found from the $c_1(Z^*, R^*)$ by similar combinatorial arguments.

We remark that

$$c_k(Z, R^*) = \sum c_1(Z_1, R^*) c_1(Z_2, R^*) \ldots c_1(Z_k, R^*),$$

where the sum is over k-tuples $(Z_1, Z_2, \ldots, Z_k)$ of subgroups with

$$Z_1 \cap Z_2 \cap \ldots \cap Z_k = Z.$$

Define the number $c(H) = c_1(H)$ inductively for any finite abelian group H by

$$2^{|H|} = \sum_{Z \leq H} c(Z, H) = \sum_{Z \leq H} c(H/Z).$$

The sum is over the lattice of subgroups of H, partially ordered by inclusion. Then by the Möbius inversion formula [14],

$$c(H) = \sum_{Z \leq H} \mu(Z, H) 2^{|Z|},$$

where μ is the generalized Möbius function defined inductively by

$$\mu(H, H) = 1$$

and

$$\mu(Z, H) = - \sum_{Z \leq X \leq H} \mu(Z, X).$$

For this lattice, $\mu(Z, H)$ depends only on the quotient H/Z. The quotients of $R^* \cong \mathrm{Gal}(L/E)$ correspond to the lattice of fields lying between L and E.

If R^* is cyclic, then so are the quotients R^*/Z^*, and we write $c(m)$ for $c(R^*/Z^*)$ if $|R^*/Z^*| = m$. In this case, we have the inversion formula

$$2^n = \sum_{m \mid n} c(m) \quad \Leftrightarrow \quad c(n) = \sum_{m \mid n} \mu(n/m) 2^m$$

where μ is the usual Möbius function defined by

$\mu(1) = 1$,

$\mu(n) = (-1)^k$ if n is a product of k distinct primes, and

$\mu(n) = 0$ otherwise.

Example. — Define the character λ of M for SU (n, n) by

$$\lambda_\alpha = \lambda \circ \alpha^\vee = \alpha^\vee \circ \varphi = \psi$$

of order n, for the roots $\alpha = e_1 - e_2$, $e_2 - e_3, \ldots, e_{n-1} - e_n$, and $\lambda_\beta = sgn_{E/F}$ for $\beta = 2e_n$. Then $\Delta' = \varnothing$ and $R(\lambda) = R^* \times_s \tilde{R}$, where $R^* = \langle (12 \ldots n) \rangle \cong Z_n$ and $\tilde{R} \cong Z_2^n$ is the group of all sign changes in the Weyl group. Since R^* is cyclic in this example,

$$c(m) = c(Z_m) = \sum_{d \mid m} \mu(m/d) 2^d.$$

There are

$$c(m) m^{-2} n = n m^{-2} \sum_{d \mid m} \mu(m/d) 2^d$$

components in the L-packet which occur with multiplicity m, for each m dividing n.

There are a total of

$$\sum_{m \mid n} c(m) m^{-2} n = n \sum_{m \mid n} m^{-2} \sum_{d \mid m} \mu(m/d) 2^d$$

inequivalent components in the L-packet.

Note that the number of components of multiplicity one is $2n$.

For example, if $n = 12$, $R^* \cong Z_{12}$, and $\tilde{R} \cong Z_2^{12}$, then Π_φ contains 24 components of multiplicity 1, 6 components of multiplicity 2, 8 compents of multiplicity 3, 9 components of multiplicity 4, 18 components of multiplicity 6, and 335 components of multiplicity 12.

As another example, let $n = 77$, $R^* \cong Z_{77}$, and $\tilde{R} \cong Z_2^{77}$. Then Π_φ contains 154 components of multiplicity 1, 198 components of multiplicity 7, 1,302 components of multiplicity 11, and 1,962,541,914,958,813,595,274 components of multiplicity 77.

Now, we give some examples to illustrate the case of a non-cyclic R^*.

Consider the group SU (4,4), with rank $n = 4$. Suppose the characters $\lambda_\alpha = \psi$ and $\lambda_\nu \neq \psi$ all have order 2 in $(E^\times / F^\times)^\wedge$, for the roots $\alpha = e_1 - e_2$ and $e_3 - e_4$, and the root $\nu = e_2 - e_3$. Let the character $\lambda_\beta = sgn_{E/F}$ for $\beta = 2e_4$. Then $R^* = \langle (12)(34),$ $(13)(24) \rangle \cong Z_2 \times Z_2 \cong (\lambda_\alpha, \lambda_\nu)$ and $\tilde{R} \cong Z_2^4$ is the group of sign changes in the Weyl group.

We may then use the values $c(1) = 2$, $c(Z_2) = 2$, $c(Z_2 \times Z_2) = 8$ in the formulas derived above to find that the L-packet for this example contains 8 components of multiplicity 1, 6 components of multiplicity 2, and 2 components of multiplicity 4.

For $n = 4$, the example above with $R^* \cong Z_4$ cyclic would give instead 8 components of multiplicity 1, 2 components of multiplicity 2, and 3 components of multiplicity 4.

Next, to illustrate the case $k = 2$, consider a similar example for SU (8,8), with $R^* \cong Z_2 \times Z_2$ and $\tilde{R} \cong Z_2^8$ with rank 8, instead of rank 4 as above. Considering the lattice of subgroups of R^*, and using the values for the c_1 above, we count $c_2(1, Z_2 \times Z_2) = 216$ characters of $\tilde{R}$ with trivial stabilizer in R^*, $c_2(Z_2, Z_2 \times Z_2) = 12$ for each of the three subgroups of R^* isomorphic to Z_2, and $c_2(Z_2 \times Z_2, Z_2 \times Z_2) = 4$. Then our formulas above give 16 components of multiplicity 1, 36 components of multiplicity 2, and 54 components of multiplicity 4.

Finally, return to the case SU (12,12). Given a subgroup

$$Z_2 \times Z_2 \times Z_3 \text{ of } (E^\times / F^\times)^\wedge,$$

we may define λ with $R^* \cong Z_2 \times Z_2 \times Z_3$ and $\tilde{R} \cong Z_2^{12}$ as follows. Let $\lambda_\beta = sgn_{E/F}$ for $\beta = 2e_{12}$. Take as generators for R^* the (commuting) permutations

$$(1\ 2\ 3)(4\ 5\ 6)(7\ 8\ 9)(10\ 11\ 12),$$
$$(1\ 4)(2\ 5)(3\ 6)(7\ 10)(8\ 11)(9\ 12),$$

and

$$(1\ 7)(2\ 8)(3\ 9)(4\ 10)(5\ 11)(6\ 12).$$

Then use the definition of the embedding

$$R^* \hookrightarrow (E^\times / F^\times)^\wedge$$

to determine the characters λ_α. The situation is reminiscent of the explicit classification of R-groups for SL (n) [5].

We may then use the values $c(1) = 2$, $c(\mathbf{Z}_2) = 2$, $c(\mathbf{Z}_2 \times \mathbf{Z}_2) = 8$, $c(\mathbf{Z}_3) = 6$, $c(\mathbf{Z}_2 \times \mathbf{Z}_3) = c(\mathbf{Z}_6) = 54$, and $c(\mathbf{Z}_2 \times \mathbf{Z}_2 \times \mathbf{Z}_3) = 3{,}912$ to find that the L-packet in this case contains 24 components of multiplicity 1, 18 of multiplicity 2, 8 of multiplicity 3, 6 of multiplicity 4, 54 of multiplicity 6, and 326 of multiplicity 12.

The reader may check as an exercise that the extension

$$1 \to R \sim\, \cong \mathbf{Z}_2^8 \to R \to R^* \cong \mathbf{Z}_2 \times \mathbf{Z}_2 \times \mathbf{Z}_2 \to 1$$

gives an L-packet for SU $(8,8)$ containing 16 components of multiplicity 1, 28 of multiplicity 2, 28 of multiplicity 4, and 23 of multiplicity 8. Note first that $c(\mathbf{Z}_2 \times \mathbf{Z}_2 \times \mathbf{Z}_2) = 184$.

Then a similar example for SU $(16,16)$ given by an extension

$$1 \to \tilde{R} \simeq \mathbf{Z}_2^{16} \to R \to R^* \cong \mathbf{Z}_2 \times \mathbf{Z}_2 \times \mathbf{Z}_2 \to 1,$$

which has $k = \operatorname{rank} \tilde{R}/|R^*| = 2$, would give 32 components of multiplicity 1, 168 of multiplicity 2, 756 of multiplicity 4, and 7,992 of multiplicity 8.

6. An example with non-trivial cocycle

We present an example of a minimal principal series representation with non-abelian R-group, for which the 2-cocycle η arising in the problem of normalization of intertwining operators in the theory of the R-group, has non-trivial cohomology class. Then the commuting algebra is not the group algebra $\mathbf{C}[R]$, but instead the twisted group algebra $\mathbf{C}[R]_\eta$. The example is similar to an example of Vogan's.

Let G be an extension of SU $(2,2) \times$ SU $(2,2)$ by the dihedral group D of order 8. Let D be generated by X and Y, with relations X and Y have order 2, and XY has order 4.

Let X act on the first factor of SU $(2,2) \times$ SU $(2,2)$ as conjugation by the diagonal element diag $(1, 1, x_0, x_0)$, where $x_0 \in F^\times$ is not a norm from $E^\times$, and act trivially on the second factor.

Let Y act trivially on the first factor and act on the second factor as conjugation by $diag(a_0, 1, 1, \bar{a}_0^{-1})$, where a_0 is fixed below.

Take $P_0 = M_0 N_0$ a minimal parabolic in SU $(2,2)$. Then

$$P = (P_0 \times P_0) \times_s D = MN$$

is a minimal parabolic subgroup of G, with

$$M = (M_0 \times M_0) \times_s D.$$

Let σ_0 be the irreducible 2-dimensional representation of the dihedral group D, and let λ be the character of $M_0 \cong E^\times \times F^\times$ defined by the conditions that λ_α is a character of $E^\times$ of order 2, for $\alpha = e_1 - e_2$, such that the restriction $\lambda_\alpha|_{F^\times}$ is trivial, and that $\lambda_\beta = sgn_{E/F}$ is the character of $F^\times$ of order 2 attached to E/F by local class field theory, for $\beta = 2e_2$.

We choose the element a_0 above and the character λ_α so that

$$\lambda_\alpha(a_0) = -1.$$

Note that we have $\lambda_\beta(x_0) = -1$. We will consider the unitarily induced representation $\mathrm{Ind}(P, G; \sigma)$, where $\sigma = (\lambda, \lambda, \sigma_0)$ is irreducible and 2-dimensional. The stabilizer $W(\sigma)$ of σ is the entire Weyl group $W = W_0 \times W_0$ of G, where W_0 is the Weyl group for $SU(2,2)$.

In general, if $w\sigma \cong \sigma$, consider the normalized integral intertwining operator

$$\mathscr{A}(w, \sigma) : \mathrm{Ind}(P, G; \sigma) \to \mathrm{Ind}(P, G; w\sigma).$$

If

$$T: \quad V_{w\sigma} \to V_\sigma$$

gives the equivalence $w\sigma \cong \sigma$, we may define

$$\sigma(w) = T$$

to extend the representation σ of M on $V_\sigma = V_{w\sigma}$ to a representation σ of the group $\langle w \rangle \times_s M$.

Then $\sigma(w)$ will be unique up to a scalar, which must be an m-th root of unity, where m is the order of w, since we want

$$\sigma(w)^m = \sigma(w^m) = I.$$

Thus

$$\sigma(w)\mathscr{A}(w, \sigma): \quad \mathrm{Ind}(P, G; \sigma) \to \mathrm{Ind}(P, G; \sigma)$$

is a self-intertwining operator. The operators

$$\{ \sigma(w)\mathscr{A}(w, \sigma) \,|\, w \in R = R(\sigma) \}$$

form a linear basis for the commuting algebra of $\mathrm{Ind}(P, G; \sigma)$.

Define a 2-cocycle $\eta: R \times R \to \mathbf{C}^\times$ by

$$\sigma(w_1 w_2)\,\mathscr{A}(w_1 w_2, \sigma) = \eta(w_1, \omega_2)\sigma(w_1)\,\mathscr{A}(w_1, \sigma)\sigma(w_2)\,\mathscr{A}(w_2, \sigma)$$

$$= \eta(w_1, w_2)\sigma(w_1)\sigma(\omega_2)\,\mathscr{A}(w_1, \sigma)\,\mathscr{A}(w_2, \sigma).$$

Then the commuting algebra is given as the group algebra $\mathbf{C}[R]_\eta$, with multiplication twisted by η.

The coset representatives for elements of W and the normalizations of the integral intertwining operators $\mathscr{A}(w, \sigma)$ are chosen as in section 2, so that the cocycle relation

$$\mathscr{A}(w_1 w_2, \sigma) = \mathscr{A}(w_1, w_2 \sigma)\,\mathscr{A}(w_2, \sigma)$$

holds with no condition on the lengths of the Weyl group elements.   Then the 2-cocycle is determined by the conditions

$$\sigma(w_1\, w_2) = \eta(w_1,\, w_2)\, \sigma(w_1)\, \sigma(w_2).$$

Thus the possible obstruction arises from a problem involving finite groups.

For the case of 1-dimensional σ, the condition $w\sigma \cong \sigma$ means $w\sigma = \sigma$, and we can always take $\sigma(w) = 1$ to extend σ.   Thus $\eta \equiv 1$, and $\mathrm{End}(\mathrm{Ind}\,\sigma) \cong C[R]$, as in section 2.   We hope that these remarks help to clarify the nature of the results of [5].

One general result for arbitrary parabolics of p-adic groups may be obtained as in the case of groups over finite fields.   *See* § 6 of [4].   Suppose G connected.   If there is a component of Ind σ occurring with multiplicity one, then $C[R]_\eta$ has a one-dimensional representation, which implies that η is a coboundary.   We may then absorb it in our normalizations, so in fact $\mathrm{End}(\mathrm{Ind}\,\sigma) \cong C[R]$.   In particular, this is the case if the supercuspidal representation σ of M has a Whittaker model.

We show that in the present example, the 2-cocycle η is not a coboundary, i. e., the projective homomorphism

$$w \mapsto \sigma(w)\, \mathscr{A}(w, \sigma)$$

or equivalently.

$$w \mapsto \sigma(w),$$

can not be made into a homomorphism.

Write $(w_\alpha, 1)$, $(w_\beta, 1)$, $(1, w_\alpha)$, and $(1, w_\beta)$, for $\alpha = e_1 - e_2$ and $\beta = 2e_2$, for the simple reflections generating $W = W_0 \times W_0$.   Since the standard intertwining operators are essentially SL_2 operators, we check immediately that $\Delta' = \varnothing$ in our example, so $R = W(\sigma) = W$.

Try to extend the representation σ consistently to a representation of R.

Since $(w_\alpha, 1)\, X\, (w_\alpha, 1)^{-1} = (w_\alpha, 1)\, X\, (w_\alpha, 1)^{-1}\, X^{-1} . X = X$, we must define $\sigma(w_\alpha, 1)$ to satisfy

$$(1) \qquad\qquad \sigma(w_\alpha, 1)\, \sigma(X)\, \sigma(w_\alpha, 1)^{-1} = \sigma(X).$$

Since

$$(w_\beta, 1)\, X\, (w_\beta, 1)^{-1} = (w_\beta, 1)\, (X\, w_\beta^{-1}\, X^{-1},\, 1)\, X = (\mathrm{diag}(1,\, x_0,\, x_0^{-1},\, 1),\, 1)\, X,$$

we must define $\sigma(w_\beta, 1)$ so that

$$(2) \qquad\qquad \sigma(w_\beta, 1)\, \sigma(X)\, \sigma(w_\beta, 1)^{-1} = sgn(x_0)\, \sigma(X) = -\sigma(X).$$

Since Y acts trivially on the first factor of $SU(2,2) \times SU(2,2)$, we get

$$(3) \qquad\qquad \sigma(w_\alpha, 1)\, \sigma(Y)\, \sigma(w_\alpha, 1)^{-1} = \sigma(Y),$$

and

$$(4) \qquad\qquad \sigma(w_\beta, 1)\, \sigma(Y)\, \sigma(w_\beta, 1)^{-1} = \sigma(Y).$$

Similarly, considering the actions of X and Y on the second factor, we require the extension of σ to satisfy

(5)
$$\sigma(1, w_\alpha) \sigma(X) \sigma(1, w_\alpha)^{-1} = \sigma(X),$$

(6)
$$\sigma(1, w_\beta) \sigma(X) \sigma(1, w_\beta)^{-1} = \sigma(X)$$

(7)
$$\sigma(1, w_\alpha) \sigma(Y) \sigma(1, w_\alpha)^{-1} = -\sigma(Y),$$

and

(8)
$$\sigma(1, w_\beta) \sigma(Y) \sigma(1, w_\beta)^{-1} = \sigma(Y).$$

Since $\sigma(w_\alpha, 1)$ must commute with $\sigma(X)$ and $\sigma(Y)$, it must be a scalar, and $(w_\alpha, 1)^2 = 1$ implies

$$\sigma(w_\alpha, 1) = \pm I.$$

Similarly,

$$\sigma(1, \omega_\beta) = \pm I.$$

Further, since the operators $\sigma(X)$ and $\sigma(Y)$ anticommute, the operator $\sigma(1, w_\alpha) \sigma(X)^{-1}$ must commute with both $\sigma(X)$ and $\sigma(Y)$, and then

$$\sigma(1, w_\alpha) \sigma(X)^{-1} = \pm I = \varepsilon_\alpha I.$$

Similarly,

$$\sigma(w_\beta, 1) \sigma(Y)^{-1} = \pm I = \varepsilon_\beta I.$$

But then

$$\sigma(1, w_\alpha) = \varepsilon_\alpha \sigma(X)$$

and

$$\sigma(w_\beta, 1) = \varepsilon_\beta \sigma(Y)$$

must also anticommute;

$$\sigma(1, w_\alpha) \sigma(w_\beta, 1) = -\sigma(w_\beta, 1) \sigma(1, w_\alpha),$$

although $(1, w_\alpha)$ and $(w_\beta, 1)$ commute in W. Thus σ can only be extended to a projective representation of R, and the 2-cocycle η gives non-trivial cohomology.

Finally, we can compute the 2-cocycle η and examine

$$\mathrm{End}(\mathrm{Ind}\ \sigma) \cong \mathbf{C}[W]_\eta$$

to show that Ind σ has 8 components occurring with multiplicity 2, and 2 components with multiplicity 4.

If η were trivial, we would instead get 16 components of multiplicity 1, 8 components with multiplicity 2, and 1 component with multiplicity 4.

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