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ON A GENERALIZATION OF HILBERT'S 21ST PROBLEM

BY RICHARD M. HAIN (1)

1. Introduction

As his 21st problem, Hilbert [12] asked if every linear representation

$$\rho: \quad \pi_1(\mathbb{P}^1-\{t_1, \ldots, t_N\}, t) \to GL(n)$$

of the fundamental group of the punctured Riemann sphere arises as the monodromy representation of a system

$$\mathbf{z}'(t)=\mathbf{z}(t)\,A(t)$$

of n first order linear ordinary differential equations on $\mathbb{P}^1$ with regular singular points at $\{t_1, \ldots, t_N\}$. (i.e., the $n\times n$ matrix $A(t)\,dt$ of 1-forms has only simple poles, and these are contained in $\{t_1, \ldots, t_N\}$.) Birkhoff [2] and Plemelj [20] showed that the answer is yes when ρ is generic, while Lappo-Danilevsky [16] gave a constructive solution for representations in a neighborhood of the trivial representation. If one allows $A(t)\,dt$ to have additional singularities, around which there is no monodromy (so called *apparent singularities*), then all ρ occur as monodromy representations (*cf.* [3], p. 311).

In this paper we consider a generalization of Hilbert's problem (also called the *Riemann-Hilbert* problem) that we now discuss. Suppose that V is a smooth algebraic variety over $\mathbb{C}$ and that X is a smooth compactification of V such that X-V is a divisor D in X with normal crossings. A meromorphic $\mathfrak{gl}(n)$-valued 1-form ω on X, which is holomorphic on V and has logarithmic poles along D, defines a meromorphic connection ∇ on the trivial bundle $\mathbb{C}^n\times X$ by defining

$$\nabla f = df - f\,\omega,$$

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where $f: X \to \mathbb{C}^n$ is a locally defined function. This connection is holomorphic over V and has regular singular points along D in the sense of Deligne [7]. If ω is integrable (i. e., $d\omega + \omega \wedge \omega = 0$), then the connection is flat and we have a monodromy representation

$$\rho: \quad \pi_1(V, x) \to GL(n).$$

1.1. Generalized Riemann-Hilbert problem. — Characterize the monodromy representations $\pi_1(V, x) \to GL(n)$ of integrable 1-forms on V which have logarithmic singularities along D.

Unlike the conjectured situation for V a Zariski open subset of $\mathbb{P}^1$, not every monodromy representation occurs. To see this, consider the case where $\dim \Omega^1(X \log D) = 1$. Here

$$\Omega^{\cdot}(X \log D) = \left\{ \begin{array}{c} \text{global meromorphic forms on X, holomorphic} \\ \text{on V with logarithmic singularities along D} \end{array} \right\}.$$

If $\omega \in \Omega^1(X \log D) \otimes \mathfrak{gl}(n)$ is a matrix of such 1-forms, then $\omega = \eta A$, where A is a constant matrix and $\eta \in \Omega^1(X \log D)$. The monodromy $\pi_1(V, x) \to GL(n)$ is given by $\gamma \mapsto \exp\left(\int_\gamma \eta A\right)$. That is, the monodromy representation factors through the 1-parameter subgroup $\sigma_A: \mathbb{C} \to GL(n)$ generated by A :

(1.2)
$$\begin{array}{ccc} \pi_1(V, x) & \xrightarrow{\rho} & GL(n), \\ & {\scriptstyle\theta} \searrow \quad \nearrow {\scriptstyle\sigma_A} & \\ & \mathbb{C} & \end{array}$$

where $\theta(\gamma) = \int_\gamma \eta$. The converse is also true; if ρ factors through σ_A, then ρ is the monodromy representation of $\omega = \eta A$. Two interesting examples where $\dim \Omega^1(X \log D) = 1$ are the following.

1.3. If $X = V$ is a complex torus $\mathbb{C}/\mathbb{Z} \oplus \mathbb{Z}\tau$, then $\dim \Omega^1(X) = 1$. Since the subgroup

$$\left\{ \begin{pmatrix} 1 & \lambda \\ 0 & 1 \end{pmatrix} : \lambda \in \mathbb{C} \right\}$$

of GL(2) is isomorphic to $\mathbb{C}$ and since dz and $\overline{dz}$ are linearly independent in $H^1(X; \mathbb{C})$, the representation $\pi_1(X) \to GL(2)$ that takes γ to

$$\begin{pmatrix} 1 & \int_\gamma \overline{dz} \\ 0 & 1 \end{pmatrix}$$

does not factor as in (1.2). Consequently, ρ is not a monodromy representation. Here one should note that ρ fails to be a monodromy representation for *Hodge theoretic* reasons: the homomorphism θ is the canonical map

$$\pi_1(V) \to H_1(V; \mathbb{C})/F^0 H_1(V),$$

where $F^0 H_1$ denotes $H^{0,1}(V)^*$.

(1.4) If $V=\mathbb{C}^2-\{(x, y) : x^2=y^3\}$, then $\pi_1(V)\cong\langle a, b : a^2=b^3\rangle$.

Denote the symmetric group on 3 letters by Σ_3. The representation

$$\rho : \quad \pi_1(V) \to \Sigma_3 \subsetneq GL(3)$$

obtained by taking a to (1 2), b to (1 2 3) and then including Σ_3 into GL(3) as permutation matrices is not a monodromy representation as it has non-abelian image. Here one should note that ρ fails to be a monodromy representation for *group theoretic* reasons: the homomorphism θ is the Hurewicz homomorphism

$$\pi_1(V) \to H_1(V; \mathbb{C}).$$

In general the image of a monodromy representation is not abelian. For this reason we need to consider the mixed Hodge structure on $\pi_1(V, x)$: The J-adic completion of the complex group ring $\mathbb{C}\pi_1(V, x)$ of the fundamental group is defined to be

$$\mathbb{C}\pi_1(V, x)\hat{} = \varprojlim \mathbb{C}\pi_1(V, x)/J^n,$$

where J denotes the kernel of the algebra homomorphism $\mathbb{C}\pi_1(V) \to \mathbb{C}$ that takes each element of $\pi_1(V)$ to 1. A theorem, essentially due to Morgan [17] (*cf.* [10], [11]), asserts that $\mathbb{C}\pi_1(V, x)\hat{}$ has a natural mixed Hodge structure and that the Hodge filtration

$$\ldots \supseteq F^{-2} \supseteq F^{-1} \supseteq F^0 \supseteq 0$$

is preserved by the multiplication. Consequently, the subspace

$$I = F^0 \cap J + F^{-1} \cap J^2 + F^{-2} \cap J^3 + \ldots$$

is a closed ideal. Denote the composite

$$\pi_1(V, x) \to \mathbb{C}\pi_1(V, x)\hat{} \to \mathbb{C}\pi_1(V, x)\hat{}/I$$

by θ. Our main result is

THEOREM. — *There exists a topological* $\mathbb{C}$*-algebra* $\mathscr{A} \subseteq \mathbb{C}\pi_1(V, x)\hat{}/I$ *such that*

(*a*) im $\theta \subseteq \mathscr{A}$,

(*b*) $\rho : \pi_1(V, x) \to GL(n)$ *is a monodromy representation of an integrable* 1*-form on* V *with logarithmic singularities along* D *if and only if there exists a continuous* $\mathbb{C}$*-algebra* *homomorphism* $\varphi : \mathscr{A} \to \mathrm{GL}(n)$ *such that*

$$\begin{array}{ccc} \pi_1(\mathrm{V}, x) & \xrightarrow{\rho} & \mathrm{GL}(n) \\ {\scriptstyle\theta}\downarrow & & \downarrow \\ \mathscr{A} & \xrightarrow[\varphi]{} & \mathfrak{gl}(n) \end{array}$$

commutes.

Consistent with our observations in (1.3) and (1.4), θ factors into a groups theoretic piece and a Hodge theoretic piece:

$$\begin{array}{ccc} \pi_1(\mathrm{V}, x) & \xrightarrow{\text{group theory}} & \mathbb{C}\pi_1(\mathrm{V}, x)^{\wedge} \\ & {\scriptstyle\theta}\searrow & \downarrow{\scriptstyle\text{Hodge theory}} \\ & & \mathbb{C}\pi_1(\mathrm{V}, x)^{\wedge}/\mathrm{I} \end{array}.$$

The group theoretic restriction on monodromy representations given by the theorem is that the kernel of each monodromy representation must contain

$$\mathrm{D}^{\infty} := \ker\{\pi_1(\mathrm{V}, x) \to \mathbb{C}\pi_1(\mathrm{V}, x)^{\wedge}\}.$$

In (1.4), D^{∞} is the commutator subgroup of $\pi_1(\mathrm{V}, x)$. Even when D^{∞} is trivial, the Hodge theoretic component may restrict the possible monodromy representations by imposing rigidity conditions on their images such as in (1.3).

Define the *irregularity* $q(\mathrm{V})$ of V to be $h^{1,0}(\mathrm{X})$. This is independent of the compactification X. When $q(\mathrm{V})=0$ (e. g., $\mathrm{V} \subseteq \mathbb{P}^n$) the ideal I is trivial and it appears as though the only restrictions on monodromy representations are group theoretic. In this case we conjecture that if $\pi_1(\mathrm{V}, x)$ satisfies a mild group theoretic condition, then $\rho : \pi_1(\mathrm{V}, x) \to \mathrm{GL}(n)$ is a monodromy representation if and only if it factors through $\pi_1(\mathrm{V}, x)/\mathrm{D}^{\infty}$:

$$\begin{array}{ccc} \pi_1(\mathrm{V}, x) & \xrightarrow{\rho} & \mathrm{GL}(n) \\ & \searrow \quad \nearrow & \\ & \pi_1(\mathrm{V}, x)/\mathrm{D}^{\infty}. & \end{array}$$

(A precise statement is given in 7.1.) When $\mathrm{X} = \mathbb{P}^1$ the conjecture reduces to the classical Riemann-Hilbert problem as free groups satisfy our technical condition and D^{∞} is trivial. In the general case, the techniques of Lappo-Danielevsky [16] and Golubeva [9] can be used to prove the conjecture for representations in a neighborhood of the trivial representation.

As a corollary of our main theorem we are able to characterize unipotent monodromy representations.

THEOREM. — *If* $\rho: \pi_1(V, x) \rightarrow GL(n)$ *is unipotent, then* ρ *is a monodromy representation of a nilpotent* 1*-form if and only if* ρ *factors through*

$$\pi_1(V, x) \overset{\theta}{\rightarrow} [\mathbb{C}\pi_1(V, x)/J^n]/F^0 \cap J + F^{-1} \cap J^2 + \ldots$$

Since the vanishing of the irregularity $q(V)$ is equivalent to the vanishing of I, and since $\pi_1(V, x) \rightarrow GL(n)$ is unipotent if and only if it factors through $\pi_1(V, x) \rightarrow \mathbb{C}\pi_1(V, x)/J^n$, we obtain the next result.

COROLLARY. — *For a smooth variety* V *every unipotent representation* $\pi_1(V, x) \rightarrow GL(n)$ *is a monodromy representation of a nilpotent* 1*-form if and only if* $q(V) = 0$.

A different generalization of Hilbert's 21st problem has been considered by Deligne [7]. He showed that every representation $\rho: \pi_1(V, x) \rightarrow GL(n)$ is the monodromy representation of some holomorphic vector bundle $E \rightarrow X$ which has an integrable connection with regular singular points along D. Here we are attempting to characterize those representations for which we can choose E to be trivial. In the classical case, allowing non-trivial bundles is equivalent to allowing apparent singularities.

The proof of the main theorem combines K.-T. Chen's de Rham theory for the fundamental group [4] with Deligne's mixed Hodge theory for non-singular varieties [8]. The key ingredient from Chen's theory is the formula (2.5) which gives a formula for the monodromy of a flat connection on a trivial bundle, while the principal ingredient from Hodge theory is the fact that each element of $\Omega^{\bullet}(X \log D)$ is closed and that the resulting map of $\Omega^{\bullet}(X \log D)$ into $H^{\bullet}(V)$ is injective. This implies, amongst other things, that the integrability condition for $\omega \in \Omega^1(X \log D) \otimes \mathfrak{gl}(n)$ is $\omega \wedge \omega = 0$. This leads us to consider the algebra

$$R = \mathbb{C}\{X_1, \ldots, X_l\}/(\sum a_{ij}^k [X_i, X_j], k = 1, \ldots, m),$$

where

– $X_1, \ldots, X_l$ is a basis of the dual of $\Omega^1(X \log D)$.

– the a_{ij}^k are complex constants given by the cup product $\Omega^1 \otimes \Omega^1 \rightarrow \Omega^2$.

– $\mathbb{C}\{X_1, \ldots, X_l\}$ denotes the formal power series in the non-commuting indeterminantes $X_1, \ldots, X_l$ that are *universally convergent*. That is, the power series that converge absolutely whenever the X_j are specialized to matrices $A_j \in \mathfrak{gl}(n)$.

There is a universal integrable 1-form $\tilde{\omega} \in \Omega^1(X \log D) \otimes R$. Namely

$$\tilde{\omega} = w_1 X_1 + \ldots + w_l X_l = \mathrm{id} \in \Omega^1 \otimes (\Omega^1)^*.$$

The relations in R guarantee that $\tilde{\omega} \wedge \tilde{\omega} = 0$. Every integrable form $\omega \in \Omega^1(X \log D) \otimes \mathfrak{gl}(n)$ is then obtained from $\tilde{\omega}$ by specializing the X_j to matrices. Chen's monodromy formula yields a universal monodromy representation

$$\theta: \quad \pi_1(V, x) \rightarrow R.$$

These, and the topology on R, are described in sections 3 and 4.

Using the Hodge theory for π_1 as developed in [10] and described in [11], we show in section 5 that R is the algebra $\mathscr{A}$ of the theorem and that it is canonically associated to the mixed Hodge structure on $\mathbb{C}\pi_1(V, x)^\wedge$. In section 6 we prove the main theorem and its corollaries while in section 7 we discuss the inverse problem and state a conjecture that generalizes Hilbert's 21st problem to Zariski open subsets of $\mathbb{P}^n$.

The complex of C^∞ forms on a manifold will be denoted by $E^\bullet M$.

2. Connections on Trivial Bundles

By a *trivialized bundle* over a manifold M we mean a trivial bundle $\mathbb{C}^n \times M \to M$ with a fixed trivialization. The trivialization, being fixed, gives a 1-1 correspondence between connections ∇ on this bundle and matrix valued 1-forms $\omega \in E^1 M \otimes gl(n)$ according to the rule

$$\nabla f = df - f\,\omega, \tag{2.1}$$

where $f: M \to \mathbb{C}^n$ is smooth. This connection is flat if and only if its curvature vanishes:

$$d\omega + \omega \wedge \omega = 0. \tag{2.2}$$

A 1-form is *integrable* if it satisfies (2.2).

Associated with a connection on a trivialized bundle $\mathbb{C}^n \times M \to M$ is the *transport function*

$$T: \quad PM \to GL(n),$$

where PM denotes the space of piecewise smooth paths $\gamma: [0, 1] \to M$. It is the unique function $PM \to GL(n)$ such that the parallel transport of a vector $v \in \mathbb{C}^m$ along γ is $v\,T(\gamma)$. Equivalently, if $\gamma_t \in PM$ is the path defined by $\gamma_t(s) = \gamma(st)$, then $T(\gamma)$ is the solution at $t=1$ of the equation

$$dT(\gamma_t) = T(\gamma_t)\,\gamma^*\omega, \; T(\gamma_o) = I. \tag{2.3}$$

(*cf.* [5]). It satisfies $T(\alpha\beta) = T(\alpha)\,T(\beta)$ whenever $\alpha(1) = \beta(0)$.

An explicit formula for T can be given in terms of ω. The formula is due to Chen. First recall the definition of an iterated integral.

2.4. Definition. — Suppose that R is an associative algebra and that $w_1, \ldots, w_r \in E^1 M \otimes R$. For $\gamma \in PM$, define

$$\int_\gamma w_1 w_2 \ldots w_r = \int_{0 \leqq t_1 \leqq t_2 \leqq \ldots \leqq t_r \leqq 1} \int A_1(t_1) A_2(t_2) \ldots A_r(t_r)\, dt_1 \ldots dt_r,$$

where $\gamma^* w_j = A_j(t)\,dt$. We regard the iterated integral as a function

$$\int w_1 \ldots w_r: \quad PM \to R.$$

2.5. LEMMA. — *Suppose that* $\omega \in E^1 M \otimes \mathfrak{gl}(n)$. *For each* $\gamma \in PM$, *there exists* $M > 0$ *such that*

$$\left\| \int_\gamma \overbrace{\omega\omega \ldots \omega}^{r} \right\| = 0\left(\frac{M^r}{r!}\right)$$

so that the series

$$I + \int_\gamma \omega + \int_\gamma \omega\omega + \int_\gamma \omega\omega\omega + \ldots$$

converges absolutely. Further, the transport $T : PM \to GL(n)$ *of the connection on* $\mathbb{C}^n \times M \to M$ *given by* (2.1) *is given by*

$$T(\gamma) = I + \int_\gamma \omega + \int_\gamma \omega\omega + \int_\gamma \omega\omega\omega + \ldots \quad \square$$

The result follows by solving (2.3) by Picard iteration (*cf.* [18]) and can be found in [11].

When ω is integrable, the value of T on the path γ depends only on its homotopy class relative to its endpoints. Thus T induces the monodromy representation

$$(2.6) \qquad \begin{cases} \rho : & \pi_1(M, x) \to GL(n) \\ & \{\gamma\} \mapsto T(\gamma) \end{cases}.$$

3. Universally convergent Power series

Suppose that V is a finite dimensional complex vector space. Denote by $\mathbb{C}\langle V \rangle$ the free associative algebra generated by V. The powers of the maximal ideal J generated by V define a topology on $\mathbb{C}\langle V \rangle$. The completion of $\mathbb{C}\langle V \rangle$ in this topology is a ring of formal power series that we shall denote by $\mathbb{C}\langle\langle V \rangle\rangle$: If $X_1, \ldots, X_l$ is a basis of V, then $\mathbb{C}\langle\langle V \rangle\rangle$ is isomorphic to the ring $\mathbb{C}\langle\langle X_1, \ldots, X_l \rangle\rangle$ of formal power series in the non-commuting indeterminates $X_1, \ldots, X_l$. A typical element of this ring will be written $\sum a_I X_I$, where $I = (i_1, \ldots, i_k)$ is a multi index, $a_I \in \mathbb{C}$ and $X_I = X_{i_1} X_{i_2} \ldots X_{i_k}$.

3.1. For the time being fix a basis $\mathscr{X} = \{X_1, \ldots, X_l\}$ of V. The power series $\sum a_I X_I$ is *universally convergent* if

$$\sum |a_I| r^{|I|} < \infty$$

for all $r \in \mathbb{R}^+$. Here $|I|$ denotes the length of the multi index I. Thus $\sum a_I X_I$ is universally convergent if and only if $\sum a_I A_I$ converges absolutely whenever the X_j are specialized to elements $A_1, \ldots, A_l$ of $\mathfrak{gl}(n)$. It is not immediately clear that this notion is independent of the basis chosen for V. This will be proved in (3.4).

3.2. The set of all universally convergent power series in the indeterminates $\mathscr{X}=\{X_1, \ldots, X_l\}$ forms a subalgebra $\mathbb{C}\{\mathscr{X}\}$ of $\mathbb{C}\langle\langle V\rangle\rangle$. Define a topology on $\mathbb{C}\{\mathscr{X}\}$ as follows: For each $\varepsilon>0$ and $r>0$, set

$$U_{r,\varepsilon}=\{\sum a_I X_I \in \mathbb{C}\{\mathscr{X}\}: \sum |a_I| r^{|I|}<\varepsilon\}.$$

The proof of the next proposition is a straightforward exercise.

3.3. Proposition. — *The* $\{U_{r,\varepsilon}: r>0, \varepsilon>0\}$ *form a basis for a topology on* $\mathbb{C}\{\mathscr{X}\}$ *such that:*

(*a*) *the topology induced on* $\mathbb{C}$ *by the inclusion* $\mathbb{C}\to\mathbb{C}\{\mathscr{X}\}$ *is the standard topology;*

(*b*) $\mathbb{C}\{\mathscr{X}\}$ *is a topological* $\mathbb{C}$*-algebra;*

(*c*) *if* $s=\sum a_I X_I \in \mathbb{C}\{\mathscr{X}\}$ *and* $s_n=\sum_{|I|\leqq n} a_I X_I$ $n=0, 1, 2, \ldots$ *is the sequence of partial sums of* s, *then* $s_n\to s$ *in this topology.*

Suppose that $\mathscr{Y}=\{Y_1, Y_2, \ldots, Y_l\}$ is another basis of V.

3.4. Proposition. — (*a*) *The set of universally convergent power series in* $\mathbb{C}\langle\langle V\rangle\rangle$ *does not depend upon the choice of a basis of* V. *That is,* $\mathbb{C}\{\mathscr{X}\}=\mathbb{C}\{\mathscr{Y}\}$.

(*b*) *The topology on the set of universally convergent power series does not depend upon the choice of basis.*

Proof. — We can write $X_i=\sum c_{ij} Y_j$. Let $C=\max|c_{ij}|$. First observe that if $s=\sum a_I X_I \in \mathbb{C}\{\mathscr{X}\}$ and

$$a_k=\max_{|I|=k}|a_I|,$$

then

$$\sum |a_I| r^{|I|} \leqq \sum a_k (rl)^k,$$

where $l=\dim V$. Now, rewriting s in terms of the Y_j's we have

$$s=\sum b_J Y_J$$

where $|b_J|\leqq a_k l^k C^k$ and $k=|J|$. Therefore

$$\sum_J |b_J| r^{|J|} \leqq \sum_k a_k (lCr)^k<\infty.$$

It follows that $\mathbb{C}\{\mathscr{X}\}\subseteqq\mathbb{C}\{\mathscr{Y}\}$ and, by symmetry, that $\mathscr{C}\{\mathscr{X}\}=\mathbb{C}\{\mathscr{Y}\}$. This proves (*a*).

Denote by

$$V_{u,\varepsilon}=\{\sum b_J Y_J: \sum |b_J| u^{|J|}<\varepsilon\}$$

the basic open sets of the topology on $\mathbb{C}\{\mathscr{Y}\}$ defined by $\mathscr{Y}$. If $s=\sum a_I X_I=\sum b_J Y_J \in U_{r,\varepsilon}$, then, as above,

$$\sum_J |b_J| u^{|J|} \leqq \sum_k a_k (lCu)^k \leqq \sum_I |a_I| (lCu)^{|I|}<\varepsilon$$

when $u \leqq r/l C$. Thus $U_{r,\varepsilon} \subseteqq V_{u,\varepsilon}$ when $u = r/l C$. It follows that the identity $\mathbb{C}\{\mathscr{X}\} \to \mathbb{C}\{\mathscr{Y}\}$ is continuous. By symmetry, $\mathbb{C}\{\mathscr{X}\} \to \mathbb{C}\{\mathscr{Y}\}$ is a homeomorphism. □

We shall denote the ring of universally convergent power series in $\mathbb{C}\langle\langle V \rangle\rangle$ with the topology defined in (3.2) by $\mathbb{C}\{V\}$. It has a nice universal mapping property.

3.5. Proposition. — *There is a* 1-1 *correspondence between continuous* $\mathbb{C}$-*algebra homomorphisms* $\mathbb{C}\{V\} \to \mathfrak{gl}(n)$ *and* $\mathbb{C}$ *linear maps* $V \to \mathfrak{gl}(n)$ *such that*

$$\begin{array}{ccc} V & \searrow & \\ \downarrow & & \mathfrak{gl}(n) \\ \mathbb{C}\{V\} & \nearrow & \end{array}$$

commutes.

Proof. — Given a $\mathbb{C}$-algebra homomorphism $\mathbb{C}\{V\} \to \mathfrak{gl}(n)$, one obtains a $\mathbb{C}$-linear map by restriction. Conversely, a $\mathbb{C}$-linear map $\varphi : V \to \mathfrak{gl}(n)$ extends to a $\mathbb{C}$-algebra homomorphism $\varphi : \mathbb{C}\langle V \rangle \to \mathfrak{gl}(n)$. According to (3.3 *c*), $\mathbb{C}\langle V \rangle$ is dense in $\mathbb{C}\{V\}$. Thus it suffices to show that $\varphi : \mathbb{C}\langle V \rangle \to \mathfrak{gl}(n)$ is continuous.

Choose a basis $X_1, \ldots, X_l$ of V. Let $A_j = \varphi X_j$ and $r = \max \| A_j \|$. If $\{p_m(X_1, \ldots, X_l)\}_{m=1}^{\infty}$ is a sequence of polynomials in $\mathbb{C}\langle V \rangle$ converging to 0, then, for each $\varepsilon > 0$ there exists N such that $p_m(X_1, \ldots, X_l) \in U_{r,\varepsilon}$ whenever $m \geqslant N$. But this implies that

$$\| \varphi\, p_m(X_1, \ldots, X_l) \| = \| p_m(A_1, \ldots, A_l) \| < \varepsilon,$$

which implies that φ is continuous. □

4. The basic construction

Now suppose that V is a smooth complex algebraic variety and that X is a smooth completion of V such that X − V is a divisor D in X with normal crossings. Denote the algebra of global meromorphic differentials on X that are holomorphic on V and have, at worst, logarithmic poles along D by $\Omega^{\bullet}(X \log D)$. By a theorem of Deligne ([8], (3.2.14))

$$(4.1) \qquad \Omega^p(X \log D) = F^p H^p(V; \mathbb{C}) \subseteqq H^p(V; \mathbb{C}),$$

where $F^{\bullet}$ denotes the Hodge filtration associated with the mixed Hodge structure on the cohomology of V. Implicit in (4.1) are the facts:

4.1 (*a*) each element of $\Omega^{\bullet}(X \log D)$ is closed,

4.1 (*b*) no non-zero element of $\Omega^{\bullet}(X \log D)$ is exact.

Denote the dual of $\Omega^p(X \log D)$ by W_p. The dual of the cup product

$$\Omega^1(X \log D) \otimes \Omega^1(X \log D) \to \Omega^2(X \log D)$$

is a map

$$\Delta:\quad W_2 \to W_1 \otimes W_1.$$

Let R be the closed ideal of $\mathbb{C}\langle\langle W_1 \rangle\rangle$ generated by the image of Δ. Set

$$A = \mathbb{C}\langle\langle q\, W_1 \rangle\rangle / R.$$

4.2. *Remarks.* — (*a*) Choose bases $w_1, \ldots, w_l$ of $\Omega^1(X \log D)$, $z_1, \ldots, z_m$ of $\Omega^2(X \log D)$ and a dual basis $X_1, \ldots, X_l$ of W_1. Then

$$\mathbb{C}\langle\langle W_1 \rangle\rangle = \mathbb{C}\langle\langle X_1, \ldots, X_l \rangle\rangle$$

and R is the closed ideal generated by

$$\sum a_{ij}^k [X_i, X_j], \qquad k = 1, \ldots, m,$$

where the complex constants a_{ij}^k are defined by

$$w_i \wedge w_j = \sum a_{ij}^k z_k$$

and $[A, B] = AB - BA$.

(*b*) For future reference we record the following fact. The graded vectorspace W. is a connected coalgebra with diagonal $\Delta: W. \to W. \otimes W.$ dual to the cup product. We can apply Adam's cobar construction (*cf.* [5]) to get a differential graded algebra $\mathscr{F}(W.)$. It follows from the definition of the cobar construction that A is the J-adic completion of $H_0(\mathscr{F}(W.))$, where J denotes the augmentation ideal. Since W. is commutative, $H_0(\mathscr{F}(W.))$ is a Hopf algebra and A has a natural complete Hopf algebra structure. □

Let $\mathscr{A}$ be the image of $\mathbb{C}\{W_1\}$ in A under the canonical projection $\mathbb{C}\langle\langle W_1 \rangle\rangle k \to A$. Give $\mathscr{A}$ the topology induced by the surjection $\mathbb{C}\{W_1\} \to \mathscr{A}$.

4.3. Proposition: (*a*) *$\mathscr{A}$ is a topological $\mathbb{C}$-algebra.*

(*b*) *There is a* 1-1 *correspondence between continuous $\mathbb{C}$-algebra homomorphisms $\mathscr{A} \to \mathrm{gl}(n)$ and $\mathbb{C}$-linear functions $\varphi: W_1 \to \mathrm{gl}(n)$ for which the composite*

$$W_2 \xrightarrow{\Delta} W_1 \otimes W_1 \xrightarrow{\varphi \otimes \varphi} \mathrm{gl}(n) \otimes \mathrm{gl}(n) \xrightarrow{\text{mult}} \mathrm{gl}(n)$$

is zero.

Proof. — A homomorphism $\mathscr{A} \to \mathrm{gl}(n)$ determines a function $W_1 \to \mathrm{gl}(n)$ with the required property by restriction. Conversely, a $\mathbb{C}$-linear map $\varphi: W_1 \to \mathrm{gl}(n)$ determines a continuous function $\hat{\varphi}: \mathbb{C}\{W_1\} \to \mathrm{gl}(n)$ by (3.5). Let $\mathscr{R} \subseteq \mathbb{C}\langle W_1 \rangle$ be the ideal generated by the image of Δ and $\bar{\mathscr{R}}$ its closure in $\mathbb{C}\{W_1\}$. One can easily check, using 3.3(*c*), that

$$\bar{\mathscr{R}} \supseteq \mathbb{C}\{W_1\} \cap R.$$

If φ satisfies the condition, then $\hat{\varphi}$ vanishes on $\bar{\mathscr{R}}$ and thus induces an algebra homomorphism $\mathscr{A} \to \mathrm{gl}(n)$ which is continuous, as $\mathscr{A}$ has the quotient topology. □

Now suppose that $\mathbb{C}^n \times X \to X$ is a trivial holomorphic vector bundle over X. Since two trivializations differ by a morphism $g: X \to GL(n)$ and since $GL(n)$ is affine, g is constant and each such bundle has an essentially unique trivialization. Connections on this bundle, holomorphic over V and with regular singular points along D, correspond to $\mathfrak{gl}(n)$-valued 1-forms $\omega \in \Omega^1(X \log D) \otimes \mathfrak{gl}(n)$ by (2.1). Fix such a 1-form ω. We can express ω in terms of a basis $w_1, \ldots, w_l$ of $\Omega^1(X \log D)$:

$$\omega = \sum w_j A_j,$$

where each A_j is a constant matrix. Since each w_j is closed (4.1(*a*)), ω is integrable if and only if

$$0 = \omega \wedge \omega = \sum w_i \wedge w_j A_i A_j = \frac{1}{2} \sum a_{ij}^k z_k [A_i, A_j].$$

(Here we are using the notation of 4.2(*a*).) Since the z_k's are linearly independent, we have proved:

4.4. PROPOSITION. — *The* 1-*form* $\omega = \sum w_j A_j \in \Omega^1(X \log D) \otimes \mathfrak{gl}(n)$ *is integrable if and only if*

$$\sum a_{ij}^k [A_i, A_j] = 0$$

for each k. □

Combining this with (4.3(*b*)) we obtain:

4.5. PROPOSITION. — *There is a* 1-1 *correspondence between integrable* 1-*forms* $\omega \in \Omega^1(X \log D) \otimes \mathfrak{gl}(n)$ *and continuous* $\mathbb{C}$-*algebra homomorphisms* $\mathscr{A} \to \mathfrak{gl}(n)$. □

4.6. We now define the universal integrable connection. As in 4.2(*a*), choose a basis $w_1, \ldots, w_l$ of $\Omega^1(X \log D)$ and a dual basis $X_1, \ldots, X_l$ of W_1. Set

$$\tilde{\omega} = w_1 X_1 + \ldots + w_l X_l \in \Omega^1(X \log D) \otimes \mathscr{A}$$

[This corresponds to $\mathrm{id} \in \mathrm{Hom}(\Omega^1, \Omega^1) \approx \Omega^1 \otimes W_1$ and is thus independent of the choice of bais.] As in (4.4), one checks that $\tilde{\omega}$ is integrable. It follows (*cf.* [4]) that the group homomorphism

$$\theta: \quad \pi_1(V, x) \to A$$
$$\{\gamma\} \mapsto 1 + \int_\gamma \tilde{\omega} + \int_\gamma \tilde{\omega}\tilde{\omega} + \int_\gamma \tilde{\omega}\tilde{\omega}\tilde{\omega} + \ldots$$

is well defined. Furthermore, (2.5) implies that $\mathrm{im}\,\theta \subseteq \mathscr{A}$. One should think of

$$\theta: \quad \pi_1(V, x) \to \mathscr{A}$$

as the universal monodromy representation.

Now suppose that $\omega = \sum w_j A_j \in \Omega^1(X \log D) \otimes \mathfrak{gl}(n)$ is an integrable 1-form. By (4.5) this determines a continuous $\mathbb{C}$-algebra homomorphism $\varphi: \mathscr{A} \to \mathfrak{gl}(n)$ such that

$\varphi(X_j)=A_j$. Let $\rho: \pi_1(V, x) \to GL(n)$ be the associated monodromy representation. By (2.6) and (3.3(c)) the diagram

$$\begin{array}{ccc} \pi_1(V, x) & \xrightarrow{\rho} & GL(n) \\ {\scriptstyle\theta}\downarrow & & \downarrow \\ \mathscr{A} & \xrightarrow[\varphi]{} & \mathfrak{gl}(n) \end{array} \tag{4.7}$$

commutes. Conversely, if we are given a continuous $\mathbb{C}$-linear homomorphism $\varphi: \mathscr{A} \to \mathfrak{gl}(n)$ and a representation $\rho: \pi_1(V, x) \to GL(n)$ such that (4.7) commutes, then, by (2.6) again, ρ is the monodromy representation of $\sum w_j \varphi(X_j)$. This completes the proof of the following result.

4.8. LEMMA. — *A representation $\rho: \pi_1(V, x) \to GL(n)$ is the monodromy representation of an integrable 1-form $\omega \in \Omega^1(X \log D) \otimes \mathfrak{gl}(n)$ if and only if there exists a continuous $\mathbb{C}$-linear algebra homomorphism $\varphi: \mathscr{A} \to \mathfrak{gl}(n)$ such that (4.7) commutes.* □

5. The mixed Hodge structure on π_1

Let $V=X-D$ be as in section 4. Denote the complex group ring of $\pi_1(V, x)$ by $\mathbb{C}\pi_1(V, x)$ and its augmentation ideal (i.e., the kernel of the algebra homomorphism $\mathbb{C}\pi_1 \to \mathbb{C}$ that takes each $g \in \pi_1$ to 1) by J. The J-adic completion

$$\mathbb{C}\pi_1(V, x)^\wedge = \varprojlim \mathbb{C}\pi_1(V, x)/J^s$$

has a natural complete Hopf algebra structure (*cf.* [21], Appendix A). The following theorem, essentially due to Morgan [17], is proved in [10] (*see* also [11]).

5.1. THEOREM. — *There are filtrations $W_\bullet$, $F^\bullet$ on $\mathbb{C}\pi_1(V, x)^\wedge$ by closed subspaces such that*

(*a*) *$W_\bullet$ is the complexification of an increasing filtration of $\mathbb{Q}\pi_1(V, x)^\wedge$;*

(*b*) *on each truncation $\mathbb{C}\pi_1(V, x)/J^{s+1}$ of $\mathbb{C}\pi_1(V, x)^\wedge$, the filtrations induce a mixed Hodge structure;*

(*c*) *the filtrations are preserved by the product and coproduct of $\mathbb{C}\pi_1(V, x)^\wedge$;*

(*d*) $J=F^{-1}\cap J+F^{-2}\cap J^2+F^{-3}\cap J^3+\ldots$ □

From (*d*) above it follows that the closed subspace

$$I=F^0\cap J+F^{-1}\cap J^2+F^{-2}\cap J^3+\ldots$$

is an ideal (in fact, a Hopf ideal) of $\mathbb{C}\pi_1(V, x)^\wedge$.

The next result relates the algebra A of section 4 to the mixed Hodge theory of $\pi_1(V)$.

5.2. LEMMA. — *There is a natural isomorphism of complete Hopf algebras*

$$\mathbb{C}\pi_1(V, x)^\wedge/I \to A$$

that is natural with respect to base point preserving morphisms of smooth algebraic varieties.

Proof. — Denote the complex of C^∞ forms on X with logarithmic singularities along D by $L^\bullet$ and let $F^\bullet$ be the usual Hodge filtration of $L^\bullet$ obtained by counting dz's. The base point $x \in V$ induces an augmentation $L^\bullet \to \mathbb{C}$ so that we can apply the reduced bar construction (*see* [5]) *to obtain a d.g.* Hopf algebra $\bar{B}(L^\bullet)$. This is naturally isomorphic to the complex of iterated integrals on the space $P_{x,x}V$ of piecewise smooth loops in V based at x: The element $[w_1|\ldots|w_r]$ of $\bar{B}(L^\bullet)$ corresponds to the iterated integral $\int w_1 \ldots w_r$ restricted to $P_{x,x}V$.

Let $\mathbf{B}_\bullet$ be the increasing filtration of $\bar{B}(L^\bullet)$ by length. That is, $\mathbf{B}_s$ is the linear span of the iterated integrals $[w_1|\ldots|w_r]$ where $r \leqslant s$. Chen's π_1 de Rham theorem ([5], *cf.* [11]) asserts that, for each $s \geqslant 0$, integration induces a natural isomorphism

$$\mathbf{B}_s H^0(\bar{B}(L^\bullet)) \overset{\approx}{\to} \operatorname{Hom}(\mathbb{C}\pi_1(V, x)/J^{s+1}, \mathbb{C}).$$

Since $\pi_1(V, x)$ is finitely generated, each $\mathbb{C}\pi_1(V, x)/J^{s+1}$ is finite dimensional. Therefore

$$\begin{aligned}\operatorname{Hom}(H^0(\bar{B}(L^\bullet)), \mathbb{C}) &\cong \varprojlim \operatorname{Hom}(\mathbf{B}_s H^0(\bar{B}(L^\bullet)), \mathbb{C}) \\ &\cong \varprojlim \mathbb{C}\pi_1(V, x)/J^{s+1} \cong \mathbb{C}\pi_1(V, x)\hat{}\,.\end{aligned}$$

This is easily seen to be an isomorphism of complete Hopf algebras.

The Hodge filtration $F^\bullet$ of $L^\bullet$ extends to one of $\bar{B}(L^\bullet)$ by defining $F^p\bar{B}(L^\bullet)$ to be the linear span of the $[w_1|\ldots|w_r]$ such that $w_j \in F^{p_j}L^\bullet$ and $p_1+\ldots+p_r \geqq p$. It follows from the proof of (5.1) given in [10] (*see* also [11]) that

$$(5.3) \qquad F^p \operatorname{Hom}(\mathbb{C}\pi_1(V, x)/J^{s+1}, \mathbb{C}) \cong \mathbf{B}_s H^0(F^p\bar{B}(L^\bullet)).$$

The holomorphic log complex $\Omega^\bullet = \Omega^\bullet(X \log D)$ is a sub d.g. algebra of $L^\bullet$. We therefore have a Hopf algebra homomorphism

$$H^0(\bar{B}(\Omega^\bullet)) \to H^0(\bar{B}(L^\bullet)).$$

It follows from (4.1), by examining the E_1 term of the Eilenberg-Moore spectral sequence (i.e., the spectral sequence associated with the filtration $\mathbf{B}_\bullet$), that this map is an inclusion and that

$$H^0(\bar{B}(\Omega^\bullet)) \cong \mathbb{C} + F^1 \cap \mathbf{B}_1 + F^2 \cap \mathbf{B}_2 + \ldots$$

Dualizing, we see from (5.3) that the kernel of the surjection

$$\mathbb{C}\pi_1(V, x)\hat{} \to \operatorname{Hom}(H^0(B(\Omega^\bullet)), \mathbb{C})$$

is $I = F^0 \cap J + F^{-1} \cap J^2 + \ldots$

Finally, the duality between bar and cobar and 4.2(*b*) yield the following isomorphisms of complete Hopf algebras:

$$\text{(5.4)}\quad \operatorname{Hom}(H^0(\bar{B}(\Omega^{\cdot})), \mathbb{C}) \cong \varprojlim \operatorname{Hom}(\mathbf{B}_s H^0(B(\Omega^{\cdot})), \mathbb{C})$$

$$\cong \varprojlim H_0(\mathscr{F}(W.))/J^{s+1}$$

$$\cong H_0(\mathscr{F}(W.))^{\wedge} \cong A. \quad \square$$

Our final task is to show that the natural map

$$\pi_1(V, x) \to \pi_1(V, x)^{\wedge}/I$$

and the map $\theta: \pi_1(V, x) \to A$ correspond under the isomorphism of (5.2).

5.5. PROPOSITION. — *The diagram*

$$\begin{array}{ccc} \pi_1(V, x) & \to & \mathbb{C}\pi_1(V, x)^{\wedge} \\ {\scriptstyle\theta}\downarrow & & \downarrow \\ A & \cong & \mathbb{C}\pi_1(V, x)^{\wedge}/I \end{array}$$

commutes, where θ *is the map constructed in* (4.6).

Proof. — Choose a basis $w_1, \ldots, w_l$ of $\Omega^1(X \log D)$ and a dual basis $X_1, \ldots, X_l$ of W_1. As in 4.2(*a*), we can identify A with

$$\mathbb{C}\langle\langle X_1, \ldots, X_l \rangle\rangle / (\sum a_{ij}^k [X_i, X_j]),$$

where the a_{ij}^k are complex constants. The universal connection form is

$$\tilde{\omega} = w_1 X_1 + \ldots + w_l X_l$$

and θ takes a loop γ to

$$1 + \int_\gamma \tilde{\omega} + \int_\gamma \tilde{\omega}\tilde{\omega} + \ldots = 1 + \sum \int_\gamma w_i X_i + \sum \int_\gamma w_i w_j X_i X_j + \sum \int_\gamma w_i w_j w_k X_i X_j X_k + \ldots$$

On the other hand, the isomorphism (5.4) takes $X_{i_1} \ldots X_{i_r}$ to the linear functional on $H^0(\bar{B}(\Omega^{\cdot}))$ induced by the functional on $\bar{B}(\Omega^{\cdot})$ that takes $[w_{i_1} | \ldots | w_{i_r}]$ to 1 and all other $[w_{j_1} | \ldots | w_{j_s}]$ to 0.

Consequently, the composite

$$\pi_1(V, x) \xrightarrow{\theta} A \approx \operatorname{Hom}(H^0(\bar{B}(\Omega^{\cdot})), \mathbb{C})$$

takes γ to the functional induced by the functional on $\bar{B}(\Omega^{\cdot})$ defined by

$$[w_{i_1} | \ldots | w_{i_r}] \to \int_\gamma w_{i_1} \ldots w_{i_r}.$$

It follows that the diagram commutes. $\square$

6. Main results

Let $V=X-D$ be as in section 3. Combining (4.8), (5.2) and (5.5), we obtain our main theorem.

6.1. THEOREM. — *There is a topological* $\mathbb{C}$*-algebra* $\mathscr{A}$ *contained in*

$$\mathbb{C}\,\pi_1(V, x)\hat{}/F^0 \cap J + F^{-1} \cap J^2 + \dots$$

such that

(*a*) *the image of the natural homomorphism*

$$\theta:\quad \pi_1(V, x) \to \mathbb{C}\,\pi_1(V, x)/F^0 \cap J + F^{-1} \cap J^2 + \dots$$

is contained in $\mathscr{A}$;

(*b*) *a representation* $\rho: \pi_1(V, x) \to GL(n)$ *is the monodromy representation of an integrable* 1*-form* $\omega \in \Omega^1(X \log D) \otimes \mathfrak{gl}(n)$ *if and only if there is a continuous* $\mathbb{C}$*-algebra homomorphism* $\varphi: \mathscr{A} \to \mathfrak{gl}(n)$ *such that the diagram*

$$\begin{array}{ccc} \pi_1(V, x) & \xrightarrow{\rho} & GL(n) \\ {\scriptstyle\theta}\downarrow & & \downarrow \\ \mathscr{A} & \to & \mathfrak{gl}(n) \end{array}$$

commutes. □

The theorem imposes obvious conditions on monodromy representations. Let $R=\ker\theta$ and

$$D^\infty = \ker\{\pi_1(V, x) \to \mathbb{C}\,\pi_1(V, x)\hat{}/I\}.$$

6.2. COROLLARY. — *If* $\rho: \pi_1(V, x) \to GL(n)$ *is a monodromy representation, then*

$$\ker\rho \geqq R \geqq D^\infty. \quad \square$$

6.3. *Remark.* — Using the Hopf algebra structure of $A=\mathbb{C}\,\pi_1(V)\hat{}/I$, we can get slightly more information. Let

$$\mathfrak{g} = \{X \in A : \Delta X = 1 \otimes X + X \otimes 1\}, \qquad \mathscr{G} = \{X \in 1 + J : \Delta X = X \otimes X\}$$

be the set of primitive elements of A and group-like elements of A, respectively. Here $\Delta: \to A \hat{\otimes} A$ denotes the coproduct. If $\mathfrak{g}$ is finite dimensional, then $\mathfrak{g} \subseteqq \mathscr{A}$. Since $\mathscr{G} = \exp\mathfrak{g}$, $\mathscr{G}$ is also in $\mathscr{A}$. Finally, the fact that $\operatorname{im}\theta \subseteqq \mathscr{G}$ implies that if $\mathfrak{g}$ is finite dimensional, then $\rho: \pi_1(V, x) \to GL(n)$ is a monodromy representation if and only if it factors through a homomorphism $\mathscr{G} \to GL(n)$ of complex Lie groups:

$$\begin{array}{ccc} \pi_1(V, x) & \to & \mathscr{G} \\ & {\scriptstyle\rho}\searrow \quad \swarrow & \\ & GL(n) & \end{array}.$$

For example, if $\dim \Omega^1 (X \log D) = 1$, then $\mathcal{G} = \mathbb{C}$. In this way we obtain the characterization of monodromy representations given in (1.2). Unfortunately, $\mathfrak{g}$ is seldom finite dimensional.

We now consider the unipotent case. By the Kolchin-Engel theorem [13], a representation $\rho: G \to GL(n)$ is unipotent if and only if the induced map $\mathbb{C}G \to \mathfrak{gl}(n)$ induces a map

$$\bar{\rho}: \quad \mathbb{C}G/J^n \to \mathfrak{gl}(n).$$

A matrix valued 1-form ω is said to be *nilpotent* if there exists a nilpotent Lie subalgeba $\mathfrak{n}$ of $\mathfrak{gl}(n)$ such that $\omega \in \Omega^1 (X \log D) \otimes \mathfrak{n}$. Monodromy representations of integrable nilpotent 1-forms are always unipotent.

6.4. THEOREM. — *For a unipotent representation, the following are equivalent:*

(*a*) *There exists a* $\mathbb{C}$*-linear algebra homomorphism*

$$\mathbb{C}\pi_1 (V, x)\hat{}/J^n + I \to \mathfrak{gl}(n)$$

such that

$$\begin{array}{ccc} \pi_1 (V, x) & \xrightarrow{\rho} & GL(n) \\ \downarrow & & \downarrow \\ \pi_1 (V, x)/J^n + 1 & \xrightarrow[\varphi]{} & \mathfrak{gl}(n) \end{array}$$

commutes.

(*b*) ρ *is the monodromy representation of an integrable, nilpotent form.*

Proof. — As in (5.2), we have an algebra isomorphism

$$\mathbb{C}\pi_1 (V, x)\hat{}/J^n + I \cong \mathbb{C}\langle\langle X_1, \ldots, X_l \rangle\rangle / (\textstyle\sum a_{ij}^k [X_i, X_j]) + J^n.$$

From (2.5) and (4.6) we conclude that if we are given φ as in 6.4(*a*), then ρ is the monodromy representation of $\omega = \sum w_j \varphi(X_j)$. Since φ is an algebra homomorphism and each X_j is nilpotent in $\mathbb{C}\pi_1 (V, x)/J^n + I$, ω is a nilpotent connection. Thus (*a*) implies (*b*).

Conversely, given an integrable, nilpotent 1-form $\omega = \sum w_i A_i$ on V, define

$$\hat{\varphi}: \quad \mathbb{C}\langle X_1, \ldots, X_l \rangle \to \mathfrak{gl}(n)$$

by $\hat{\varphi}(X_j) = A_j$. Since the A_j lie in a nilpotent sub Lie algebra of $\mathfrak{gl}(n)$, it follows from Engel's theorem that $\hat{\varphi}$ induces a homomorphism

$$\bar{\varphi}: \quad \mathbb{C}\langle\langle X_1, \ldots, X_l \rangle\rangle / J^n \to \mathfrak{gl}(n).$$

The integrability of ω implies that $\bar{\varphi}$ induces a homomorphism

$$\varphi: \quad \mathbb{C}\pi_1 (V, x)\hat{}/J^n + I \cong \mathbb{C}\langle\langle X_1, \ldots, X_l \rangle\rangle / J^n + (\textstyle\sum a_{ij}^k [X_i, X_j]) \to \mathfrak{gl}(n)$$

That the diagram in 6.4(*a*) commutes follows from (2.6) and (4.6). □

As in the introduction, we define the *irregularity* $q(V)$ of $V=X-D$ to be $h^{1,0}(X)$. This is independent of the choice of a smooth completion X of V.

6.5. COROLLARY. — *The following statements are equivalent for a smooth variety* V:

(*a*) *Every unipotent representation* $\rho: \pi_1(V, x) \to GL(n)$ *is the monodromy of a nilpotent integrable* 1-*form* $\omega \in \Omega^1(X \log D) \otimes \mathfrak{gl}(n)$.

(*b*) $q(V)=0$.

(*c*) $W_1 H^1(V; \mathbb{C})=0$.

Proof. — The equivalence of (*b*) and (*c*) follows from [8]: (3.2.14). If $W_1 H^1(V; \mathbb{C})=0$, then one can easily check that $I=0$. Applying (6.4), we see that (*c*) implies (*a*).

Suppose that $W_1 H^1(V) \neq 0$. Consider the unipotent representation

$$\pi_1(V, x) \to GL(\mathbb{C}\pi_1(V, x)/J^2)$$
$$g \mapsto \{U \mapsto Ug\}$$

given by right multiplication. This representation is unipotent and factors through the Hurewicz homomorphism

$$\pi_1(V, x) \to H_1(V) \subsetneq GL(\mathbb{C}\pi_1(V, x)/J^2)$$

as there is a ring isomorphism

$$\mathbb{C}\pi_1(V, x)/J^2 \cong \mathbb{C} \oplus H_1(V; \mathbb{C})$$
$$g \mapsto (1, [g])$$

where $(\lambda, a)^{\cdot}(\mu, b)=(\lambda\mu, \mu a+\lambda b)$ is the multiplication on the right hand side. Since $W_1 H^1(V) \neq 0$, $F^0 H_1(V) \neq 0$. Consequently this representation does not factor through

$$\mathbb{C}\pi_1(V, x)/F^0 \cap J+J^2 \cong \mathbb{C} \oplus H_1(V)/F^0.$$

That is, (*c*) implies (*a*). □

In the case when $X=\mathbb{P}^n$, we recover a result of Aomoto [1].

6.6. THEOREM (Aomoto). — *If* V *is a Zariski open subset of* $\mathbb{P}^n$, *then every unipotent representation of* $\pi_1(V, x)$ *is the monodromy representation of an integrable* 1-*form* ω *on* V *with logarithmic singularities at infinity.* □

7. The inverse problem

When V is a Zariski open subset of $\mathbb{P}^m$, the ideal

$$I=F^0 \cap J+F^{-1} \cap J^2+\ldots=0.$$

Thus, in some sense, only group theory and not Hodge theory imposes conditions on monodromy representations. Let D^∞ be the kernel of the natural map $\pi_1(V, x) \to \mathbb{C}\pi_1(V, x)$. According to (6.2), the kernel of every monodromy representation has to contain D^∞.

7.1. CONJECTURE. — *Assume that V is a Zariski open subset of* $\mathbb{P}^m$ *[or, more generally, that* $W_1 H^1(V)=0$.*] Suppose that there exist* $x_1, \ldots, x_l \in \pi_1(V, x)$ *such that*

(*a*) $[x_1], \ldots, [x_l]$ *are linearly independent in* $H_1(V; \mathbb{Z})$,

(*b*) $x_1, \ldots, x_l$ *generate* $\pi_1(V, x)/D^\infty$.

Then a representation $\rho: \pi_1(V, x) \to GL(n)$ *is the monodromy representation of an integrable* 1*-form* $\omega \in \Omega^1(X \log D) \otimes \mathfrak{gl}(n)$ *if and only if* $\ker \rho \geqq D^\infty$.

When $X=\mathbb{P}^1$, then $\pi_1(V, x)$ is free and $D^\infty=1$. In this case the conjecture reduces to the classical Riemann-Hilbert problem.

The fundamental groups of many Zariski open subsets of $\mathbb{P}^m$ satisfy the conditions in (7.1). For example, it holds when V is the complement of a union of hyperplanes. It would be interesting to know a larger class of examples of open subsets of $\mathbb{P}^n$ for which it holds as well as an example where it fails.

The condition arises as follows. Suppose that V satisfies the hypotheses of (7.1) and that $\rho: \pi_1(V, x) \to GL(n)$ satisfies $\ker \rho \geqq D^\infty$. Set

$$W_j = \log \rho(x_j).$$

If ρ is the monodromy representation of $\omega = w_1 A_1 + \ldots + w_l A_l$, then by taking logarithms of (2.6) we have formal power series expansions

$$(7.2) \qquad W_j = A_j + \sum_{|I| \geqq 2} a_I A_{i_1} \ldots A_{i_s}, \qquad j=1, \ldots, l.$$

One can formally invert the power series (7.2) to find power series

$$(7.3) \qquad A_j = W_j + \sum b_I W_{i_1} \ldots W_{i_s}.$$

Golubeva [9] has shown that, when each $\|\rho(x_j) - I\|$ is small enough, the series (7.3) converge absolutely and that ρ is the monodromy representation of the connection $\omega = \sum w_j A_j$. Thus we have:

7.4. THEOREM. — *Suppose that the Zariski open subset* V *of* $\mathbb{P}^m$ *satisfies the hypotheses of* (7.1). *If* $\rho: \pi_1(V, x) \to GL(n)$ *satisfies* $\ker \rho \geqq D^\infty$ *and if each* $\|\rho(x_j) - I\|$ *is sufficiently small, then* ρ *is the monodromy representation of an integrable* 1*-form with logarithmic singularities at infinity.* □

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