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C. PETERS

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MONODROMY AND PICARD-FUCHS EQUATIONS FOR FAMILIES OF K3-SURFACES AND ELLIPTIC CURVES

By C. PETERS

Introduction

Consider the following two problems.

(I) Find the monodromy representation of the variations of Hodge structure coming from a given projective family of compact complex manifolds.

(II) For any of these variations, determine explicitly the Gauss-Manin connection (or, for one-dimensional families — the Picard-Fuchs equations).

Loosely speaking, the relation between (I) and (II) is that the monodromy of the solutions of the Picard-Fuchs equations is the same as the monodromy of a suitable direct factor of the corresponding variation of Hodge structure. Details can be found in paragraph 6.

In general both problems are hard. There are two classes of varieties for which the moduli-space coincides with a suitable period domain, i. e., the geometry is faithfully reflected in geometry. These classes are the (polarized) abelian varieties and the (polarized) K3-surfaces. For polarized abelian varieties the answer to question (I) is a direct consequence of the existence of universal families of polarized abelian varieties with level m-structure ($m \geqq 3$) and I won't treat this in detail. Only some examples of families of elliptic curves are given which play a rôle later on when dealing with question (II). I provide an answer to problem (I) for the class of K3-surfaces. Moreover, it is shown that problem (II) can be solved for several 1-dimensional families of abelian surfaces related to elliptic curves. As an interesting by-product I give an intrinsic characterisation of the rather mysterious family of K3's related to $\zeta(3)$ considered by Beukers and myself in [B-P]. More precisely, the variation of Hodge structure on a certain rank 3 subsystem of the cohomology of this family is explained in terms of a universal construction. It follows that the monodromy in this case is $\Gamma_0^0(6)^*$ (*see* 5.3.2 for notations).

In paragraph 1, resp. paragraph 2 the weight one, resp. weight two Hodge structures are considered and certain universal variations are constructed.

In paragraph 3 the weight two case is related to K3-surfaces of transcendental type T.

In paragraph 4 those weight two Hodge structures are studied that arise as the second exterior power of weight one Hodge structures of genus 2 and a particular case related with elliptic curves is considered in paragraph 5.

Since in existing literature there is no treatment of the theory of Picard-Fuchs equations which is adequate for my purposes, I give one in paragraph 6. One of the examples in paragraph 6 plays a crucial rôle in paragraph 7 where the previously mentioned explanation is presented concerning the family of K3's related to $\zeta(3)$. This requires the use of all of the main results from earlier sections. It illustrates the interplay between geometry and arithmetic aspects of lattice theory on the one hand and Picard-Fuchs equations on the other hand. In particular, the solutions to problems (I) and (II) are intimately related in this case.

Acknowledgements. — I want to thank Gerard van der Geer who posed the question of the universality of the family of K3's from [B-P] and who also suggested where to look for an answer. Both Jan Stienstra and Frits Beukers have been extremely helpful concerning the example of paragraph 7.

0. Preliminaries

0.1. Local systems. — A local system $\mathbf{V}$ of $\mathbf{C}$-vectorspaces on a topological space S is uniquely determined by its monodromy representation

$$\rho: \quad \pi_1(S, s_0) \to \operatorname{Aut}(V), \qquad V = V_{s_0}.$$

It S is a complex manifold, the vector bundle $\mathscr{V} = \mathbf{V} \otimes \mathscr{O}_S$ carries a canonical flat connection

$$\nabla: \quad \mathscr{V} \to \mathscr{V} \otimes \Omega^1_S \qquad (v \otimes f \to v \otimes df)$$

whose sheaf of germs of holomorphic sections is precisely $\mathbf{V}$.

In geometric situations, the monodromy representation actually factors over $\operatorname{Aut}(V_{\mathbf{Z}})$, $V_{\mathbf{Z}}$ a lattice within V with $V = V_{\mathbf{Z}} \otimes \mathbf{C}$ endowed with an integral symmetric of skew-symmetric bilinear form and where only those automorphisms are considered that preserve this form.

0.2. Variations of Hodge structure. — For definitions and elementary properties I refer to [P-S].

The "geometric situation" gives rise to a standard example. More precisely, let $f: X \to S$ be a projective family of smooth connected projective varieties over $\mathbf{C}$. The m-th primitive cohomology groups of the fibres fit together to a locally constant sheaf and the Hodge decomposition provides a variation of weight m Hodge structure on this locally constant sheaf.

For a discussion of period-domains, markings and period maps I refer to [P-S], § 4,5.

0.3. INTEGRAL BILINEAR FORMS.

0.3.1. *Skew-symmetric forms.* — If $\langle\ ,\ \rangle$ is a skew-symmetric integral form on $V_{\mathbf{Z}}$, rank $V_{\mathbf{Z}}=2g$ and there exists a basis $\{e_1, \ldots, e_g, e_{g+1}, \ldots, e_{2g}\}$ for $V_{\mathbf{Z}}$ such that $(\langle e_i, e_j\rangle)$ is the matrix

$$Q_\Delta=\begin{pmatrix} 0 & -\Delta \\ \Delta & 0 \end{pmatrix}, \qquad \Delta=\operatorname{diag}(d_1, d_2, \ldots, d_g), \quad d_i\in\mathbf{Z}; \qquad d_i \mid d_{i+1}.$$

The symplectic group $\operatorname{Sp}(Q_\Delta)\subset \operatorname{SL}(2g, \mathbf{Z})$ acts on the Siegel-upperhalf space

$$\mathfrak{h}_g=\{Z\in \mathbf{C}^{g,g};\ Z=Z^t,\ \operatorname{Im}>0\}$$

in the usual manner:

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}.Z=(AZ+B\Delta^{-1})(\Delta CZ+\Delta D\Delta^{-1})^{-1}$$

and $\operatorname{Sp}(Q_\Delta)/\pm\operatorname{id}$ acts effectively on $\mathfrak{h}_g$.

0.3.2. *Symmetric forms.* — A pair $(V_{\mathbf{Z}}, \langle\ ,\ \rangle)$ consisting of a free $\mathbf{Z}$-module $V_{\mathbf{Z}}$ together with an integral symmetric bilinear form on it is called a *lattice*. Its automorphisms are called *isometries*. They form the group $O(V_{\mathbf{Z}})$.

Standardexamples of lattices

$$U=\mathbf{Z}e_1\oplus\mathbf{Z}e_2 \qquad \text{with} \quad (\langle e_i, e_j\rangle)=\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \text{(hyperbolic plane).}$$

Rootlattices A_k, D_k, E_k.

Rank-1 *lattices* $\langle k\rangle=\mathbf{Z}e$ with $\langle e, e\rangle=k$.

If V_1, V_2 are two lattices, $V_1\perp V_2$ denotes their orthogonal direct sum and

$$\overset{n}{\perp} V=\underbrace{V\perp\ldots\perp V}_{n}.$$

If V is a lattice, $V(k)$ is the lattice with same underlying $\mathbf{Z}$-module, but $\langle x, y\rangle_{V(k)}=k\langle x, y\rangle_V$.

0.3.3. For a sublattice W of $V_{\mathbf{Z}}$ one puts

$$O(V_{\mathbf{Z}}, W)=\{g\in O(V_{\mathbf{Z}});\ g(W)\subseteqq W\},$$

and if moreover $l\in W^\perp$, one puts

$$O(V_{\mathbf{Z}}, W, l)=\{g\in O(V_{\mathbf{Z}}, W);\ g(l)=l\}.$$

A pair $(V_{\mathbf{Z}}, W)$ is said to be "faithfully restrictive" if the restriction homomorphism $O(V_{\mathbf{Z}}, W)\to O(W)$ is injective. If this is the case $O(V_{\mathbf{Z}}, W)$ is identified with its image

$$O_0(\mathrm{W}) = \{ g \mid \mathrm{W};\ g \in O(\mathrm{V_Z}, \mathrm{W}) \}.$$

There is a similar notion of a faithfully restrictive triple $(\mathrm{V_Z}, \mathrm{W}, l)$ and in this case $O(\mathrm{V_Z}, \mathrm{W}, l)$ is identified with

$$O_0(\mathrm{W}, l) = \{ g \mid \mathrm{W};\ g \in O(\mathrm{V_Z}, \mathrm{W}, l) \}.$$

Sometimes one writes $O_0(\mathrm{W})$ instead of $O_0(\mathrm{W}, l)$ (if no confusion is possible).

1. Weight one Hodge structures

In this section $\mathrm{V_Z}$ is a free **Z**-module of rank $2g$ endowed with an integral symplectic form $\langle\ ,\ \rangle$. A basis $\{e_1, \ldots, e_{2g}\}$ for $\mathrm{V_Z}$ has been chosen such that this form has matrix Q_Δ (*cf.* 0.3).

1.1. A polarized weight one Hodge structure on $\mathrm{V} = \mathrm{V_Z} \otimes \mathbf{C}$ is given by a maximal totally isotropic subspace $\mathrm{V}^{1,0}$ of V such that $\sqrt{-1}\langle x, \bar{x} \rangle > 0,\quad \forall x \in \mathrm{V} \setminus \{0\}$. Such Hodge structures are classified by the Siegel upper half space $\mathfrak{h}_g$. In fact $\mathrm{Z} \in \mathfrak{h}_g$ corresponds to the g-dimensional subspace of $\mathrm{V} \cong \mathbf{C}^{2g}$ spanned by the g columns of the matrix $\begin{pmatrix} \mathrm{Z} \\ \Delta^{-1} \end{pmatrix}$.

1.1.1. The universal family

$$q:\ \mathscr{E}_\Delta \to \mathfrak{h}_g$$

of marked polarized abelian varieties is constructed by letting $\mathrm{V_Z}$ act on $\mathfrak{h}_g \times \mathrm{V}$ in the usual manner:

$$u(\mathrm{Z}, v) = \left(\mathrm{Z},\ v + {}^t u \begin{pmatrix} \mathrm{Z} \\ \Delta^{-1} \end{pmatrix}\right), \qquad \forall u \in \mathrm{V_Z},\quad v \in \mathrm{V},\quad \mathrm{Z} \in \mathfrak{h}_g.$$

1.1.2. Over $\mathfrak{h}_g$ one has the tautological variation of weight one Hodge structure

$$\mathscr{V}^{1,0} \subset \mathrm{V} \otimes \mathcal{O}_{\mathfrak{h}_g}$$

which coincides with the geometric variation of weight one Hodge structure on $\mathrm{R}^1 q_* \mathbf{Z} \cong$ constant system on $\mathfrak{h}_g$ with stalk V.

1.2. The group $\mathrm{Sp}(\mathrm{Q}_\Delta)$ acts properly on $\mathfrak{h}_g$, and by [B-B] its quotient

$$\mathrm{A}_g = \mathrm{Sp}(\mathrm{Q}_\Delta)/\mathfrak{h}_g$$

is quasi-projective.

1.2.1. Since $-\mathrm{id}$ does not act freely on the constant system V over $\mathfrak{h}_g$, there is no univeral weight one Hodge structure on A_g. A standard way to remedy this is to pass to a subgroup of finite index in $\mathrm{Sp}(\mathrm{Q}_\Delta)$ that does not contain $-\mathrm{id}$, e. g.,

$$\Gamma_g(m) = \{ \gamma \in \mathrm{Sp}(Q_\Delta);\ \gamma \equiv \mathrm{id} \bmod m \}$$

provided, of course, $m \geq 3$.

1.2.2. Examples for $g=1$.

In this case we have a group Γ of finite index in $\mathrm{SL}(2, \mathbf{Z})$ such that Γ acts freely on $\mathfrak{h} = \mathfrak{h}_1$ and on $\mathscr{E}_\Gamma \to \mathfrak{h}$, where $\Delta = 1$. The quotient elliptic fibration $\mathscr{E}^0_\Gamma \to Y^0_\Gamma = \Gamma/\mathfrak{h}$ can be extended in a natural way to an elliptic surface $\mathscr{E}_\Gamma \to Y_\Gamma$ where Y_Γ is obtained from Y^0_Γ by adding the cusps of Γ (*cf.* [S1]). Later on I'll come back to three specific examples:

(i) $$\Gamma = \Gamma_1(3).$$

The modular family $\mathscr{E}_\Gamma \to Y_\Gamma$ is explicitly given as the pencil of cubics in $\mathbf{P}_2$ with equation

$$x^3 + y^3 + z^3 - 3\,txyz$$

with singular fibres above the cusps $t = 1,\ \infty,\ -(1/2) \pm (1/2)\sqrt{-3}$.

(ii) $$\Gamma = \Gamma^0_0(5) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2, \mathbf{Z});\ c \equiv 0,\ a \equiv 1 \bmod 5 \right\}.$$

The modular family in this case is the following pencil of cubics in $\mathbf{P}_2$

$$x(y-z)(z-x) - t(x-y)yz = 0,$$

with singular fibres above $t = 0$, ∞ and the roots of $t^2 - 11t - 1 = 0$.

(iii) $$\Gamma = \Gamma^0_0(6).$$

The modular family is given by the pencil of cubics.

$$xyz - t(x+y+z)(xy+yz+zx) = 0$$

in $\mathbf{P}_2$ with singular fibres above $t = 0,\ \infty,\ 1,\ 1/9$.

These examples (with minor modifications) are from [B] (*cf.* also [S-B]).

2. Weight two Hodge structures

In this section $V_{\mathbf{Z}}$ is a free $\mathbf{Z}$-module with an integral bilinear form $\langle\ ,\ \rangle$ on it.

2.1. A polarized weight two Hodge structure (*) on $V = V_{\mathbf{Z}} \otimes \mathbf{C}$ consists of a Hodge decomposition

$$V = V^{2,0} \oplus V^{1,1} \oplus V^{0,2}, \qquad \overline{V^{2,0}} = V^{0,2} \quad \text{and} \quad \overline{V^{1,1}} = V^{1,1}$$

(*) This definition deviates slightly from the usual one, but is better adapted to the present situation.

such that $(V^{2,0})^{\perp}=V^{2,0}\oplus V^{1,1}$ and $\langle x, \bar{x}\rangle>0$ for all $x\in V^{2,0}\setminus\{0\}$. Those polarized weight two Hodge structures for which $\dim V^{2,0}=1$ thus are classified by

$$D(V)=\{[x]\in \mathbf{P}(V) \mid \langle x, x\rangle=0, \langle x, \bar{x}\rangle>0\}.$$

Over D(V) one has the tautological variation of weight 2 Hodge structure

$$V\otimes\mathcal{O}_{D(V)}\supset(\mathcal{V}^{2,0})^{\perp}\supset\mathcal{V}^{2,0}$$

and $O(V_{\mathbf{Z}})$ acts on the constant system $V_{\mathbf{Z}}$ over D(V) preserving the Hodge bundle $\mathcal{V}^{2,0}$.

2.1.1. If $\langle\ ,\ \rangle$ has signature $(2, n)$, the domain D(V) consists of two connected components, each of which is a bounded domain in $\mathbf{C}^n$ of type IV ([P], Chapitre 2, § 8). For later reference I need to know how to distinguish between the components. Let $(e_1, \ldots, e_{n+2}\}$ be a basis of $V_{\mathbf{Z}}\otimes\mathbf{R}$ such that $(\langle e_i, e_j\rangle)=\mathrm{diag}(1, 1, -1, -1, \ldots, -1)$. If $[x]\in D(V)$ has homogeneous coordinates $(x_1, x_2, \ldots, x_{x+2})$ with respect to this basis $\mathrm{Im}(x_1 x_2^{-1})\neq 0$ and the sign distinguishes between the two components.

2.2. In this subsection $V_{\mathbf{Z}}$ is a fixed unimodular lattice of signature $(3, m)$ $(m\geqq 3)$, $l\in V_{\mathbf{Z}}$ a fixed vector with $\langle l, l\rangle=2d>0$ and T is a fixed primitive sublattice of $V_{\mathbf{Z}}$ contained in $[l]^{\perp}$ and such that $\langle\ ,\ \rangle \mid T$ has signature $(2, n)$ $(n\geqq 1)$. The domain D(T) can be identified with $\mathbf{P}(T\otimes_{\mathbf{Z}}\mathbf{C})\cap D(V)$ and one considers those polarized weight 2 Hodge structures on T that are restrictions from such on V.

2.2.1. The projective orthogonal group $O(T)/\pm\mathrm{id}$ acts properly and effectively on D(T) and it acts on the constant local system T over D(T). Since one considers only Hodge structures on T coming from $V_{\mathbf{Z}}$ by restriction, one should instead consider the action of the group $O(V_{\mathbf{Z}}, T, l)$ (*cf.* 0.3) on the constant system $V_{\mathbf{Z}}$. For expository reasons I assume that the triple $(V_{\mathbf{Z}}, T, l)$ is faithfully restrictive (*cf.* 0.3) and so the group $O(V_{\mathbf{Z}}, T, l)$ can be identified with a subgroup $O_0(T)$ of $O(T)$. Now $O_0(T)/\pm\mathrm{id}$ acts properly and effectively on D(T) and by [B-B] the quotient is quasi-projective. To get a free action one has to take away the fixed point loci of the elements of $O_0(T)/\pm\mathrm{id}$:

$$D^0(T, l)=D(T)\setminus\{\text{fixed point loci of non-trivial elements in } O_0(T)/\pm\mathrm{id}\}.$$

2.2.2. Proposition. — *The group* $O_0(T)/\pm\mathrm{id}$ *acts freely on* $D^0(T, l)$ *and the quotient is a smooth quasi-projective variety of dimension n.*

Proof. — Since $O_0(T)$ acts properly on D(T) the set $O_0(T)/D(T)\setminus\mathring{D}(T, l)$ consists of finitely many (smooth) analytic subvarieties in $O_0(T)/D(T)$. Since the codimension of the boundary of the Satake compactification is at least 2, by [R-S] these varieties extend to the (projective) Satake compactification. So $O_0(T)/\mathring{D}(T, l)$ is quasi-projective.

2.3. Let me first consider the case that $-\mathrm{id}\notin O_0(T)$. Then the pair $(V_{\mathbf{Z}}, T)$ of constant systems on $D(T, l)$ descends to a pair $(\mathbf{V}_{\mathbf{Z}}, \mathbf{T})$ of locally constant systems on

$$\mathring{M}(T, l)=O_0(T)/\mathring{D}(T, l)$$

and the variation of weight two Hodge structure on V descends to $\mathbf{V_Z}\otimes\mathcal{O}_{\mathring{M}(T,l)}$ and restricts to a variation of weight two Hodge structure on $\mathbf{T}\otimes\mathcal{O}_{\mathring{M}(T,l)}$.

2.3.1. If however $-\mathrm{id}\in O_0(\mathrm{T})$ one could (as in 1.2.1) pass to a suitable subgroup, the most obvious one being $\mathrm{SO}_0(\mathrm{T})$. It leads to the introduction of the notion of an *oriented lattice:* a lattice together with a choice of one of the two isomorphism of the maximal exterior power with **Z**.

2.3.2. The next proposition immediately follows from the previous considerations.

PROPOSITION. — *Suppose* $-\mathrm{id}\notin \mathrm{SO}_0(\mathrm{T})$ *(automatic for n odd). Then* $\mathrm{SO}_0(\mathrm{T})$ *acts freely on* $(\mathrm{V_Z}, \mathrm{T})$ *yielding a pair* $(\mathbf{V_Z}, \mathbf{T})$ *of local systems over* $\mathrm{SO}_0(\mathrm{T})/\mathring{\mathrm{D}}(\mathrm{T}, l)$. *An orientation for* T *induces a unique orientation for all stalks of* **T**. *The variation of weight two Hodge structure on* **T** *over* $\mathrm{SO}_0(\mathrm{T})/\mathring{\mathrm{D}}(\mathrm{T}, l)$ *is universal for variations of Hodge structure of the type considered.*

3. (T, *l*)-Marked K3-surfaces

Here I put

$$\mathrm{L}=\overset{2}{\perp}\mathrm{E}_8(-1)\overset{3}{\perp}\mathrm{U}$$

and I fix

$$l\in\mathrm{L},\qquad \langle l, l\rangle=2d>0.$$

3.1. Points of D(L) correspond to marked K3-surfaces. I refer to ([B-P-V], Chapter VIII) and [M] for details regarding the following facts. If $(\mathrm{X}, \gamma: \mathrm{H}^2(\mathrm{X}, \mathbf{Z})\xrightarrow{\sim}\mathrm{L})$ is a marked K3-surface, the complexification of γ transports the usual Hodge structure on $\mathrm{H}^2(\mathrm{X}, \mathbf{C})$ to one on $\mathrm{L}\otimes\mathbf{C}$ and the corresponding point in D(L) is called the period point of (X, γ). Conversely, for every point $[\omega]\in\mathrm{D}(\mathrm{L})$ there exists a marked K3-surface whose period point is the given point $[\omega]$. The isomorphism class of X is uniquely determined by $[\omega]$, but in general several markings are possible, preventing the existence of a universal marked family of K3's over D(L).

3.1.1. The situation improves if one considers only algebraic K3's X with a marking γ such that $\gamma^{-1}(l)\in\mathrm{H}^2(\mathrm{X}, \mathbf{Z})$ is the class of an ample divisor.

So one is led to introduce

$$\mathrm{D}_l=\{[\omega]\in\mathrm{D}(\mathrm{L});\ \langle\omega, l\rangle=0\}.$$

It turns out that not all points in D_l can correspond to algebraic K3's with a marking γ such that $\gamma^{-1}(l)$ is ample. One should leave out all "nodal" hyperplanes

$$\mathrm{H}_r=\{[\omega]\in\mathrm{D},\ \langle\omega, r\rangle=0\},\qquad r\in[l]^{\perp},\quad \langle r, r\rangle=-2.$$

Indeed, over

$$\mathrm{D}_l^{\mathrm{pol}}=\mathrm{D}_l\setminus\bigcup_r\mathrm{H}_r$$

there is a universal family of marked K3-surfaces (X, γ) such that $\gamma^{-1}(l)$ is ample (i. e. gives an algebraic "polarisation" explaining the superscript "pol")

$$p: \quad \mathscr{X}_l \to D_l^{pol}, \qquad \gamma: R^2 p_* \mathbf{Z} \xrightarrow{\sim} L.$$

3.1.2. *Remark.* — It is possible—and indeed more natural—to consider also points on H_r, but then one has to allow marked "generalised" K3's. Briefly, those consist of pairs (X, γ) where X is a surface having at most rational double points, whose minimal resolution $\rho: Y \to X$ is a K3-surface and where $\gamma: H^2(Y, \mathbf{Z}) \xrightarrow{\sim} L$ is a marking. Points in D_l correspond to pairs (X, γ) where in addition $\gamma^{-1}(l)$ is the class of a divisor $\rho^* \mathscr{L}$ with $\mathscr{L}$ ample on X (*cf.* [M], section 5 and 6). The notion of marking for families of generalised K3's is slightly involved and in order to avoid some cumbersome technicalities I'll restrict my attention to honest K3's.

3.2. Let me apply the situation of paragraph 2 to the lattice $V_{\mathbf{Z}} = L$. Points of D(T) correspond to marked K3's (X, γ) such that γ sends the transcendental lattice T_X into T. Since $\gamma(T_X) = T$ if and only if the period point of (X, γ) does not belong to a proper **Q**-subspace of $\mathbf{P}(T \otimes \mathbf{C})$, for generic points in D(T) the transcendental lattice of the corresponding K3 is isometric to T. This motivates

3.2.1. DEFINITION. — *A* K3-*surface is of* transcendental type T *if for some marking* γ *the transcendental lattice is mapped isometrically into* T. *The marking* γ *is called a* T-*marking. If moreover* $\gamma^{-1}(l)$ *is ample on* X, γ *is called a* (T, l)-marking *and* X *is said to be of* type (T, l).

3.3. If one would apply the construction of (3.1) directly to D(T) one encounters the problem that D(T) might entirely be contained in a hyperplane H_r for some root $r \in [l]^\perp$. In order to avoid situations like that, one needs "admissible" pairs (T, l):

3.3.1. DEFINITION. — (T, l) *is admissible if* $(l, r) \neq 0$ *for all roots* $r \in T^\perp$.

3.3.2. LEMMA. — *Given a non-degenerate sublattice* T *of* L *of signature* (2, n) *there exists* $l \in T^\perp$, $\langle l, l \rangle > 0$ *such that* (T, l) *is admissible.*

Proof. — The signature of $\langle\ ,\ \rangle | T^\perp$ is $(1, 19-n)$ and so $\{x \in T^\perp \otimes \mathbf{R}; \langle x, x \rangle > 0\}$ consists of two half cones and leaving out the hyperplanes H_r, r a root in T one gets (in general infinitely many) open convex polyhedral subcones. So in particular at least one ray $\mathbf{R}.l$, $l \in T^\perp$ is in the interior of a subcone. Then (T, l) is admissible. □

3.3.3. Now it makes sense to introduce for admissible pairs (T, l)

$$D^{pol}(T, l) = D(T) \cap D_l^{pol}.$$

Over it, the family p from 3.1.1 restricts to a universal (T, l)-marked family of K3's:

$$p^{pol}: \mathscr{X}(T, l) \to D^{pol}(T, l).$$

3.4. Every $\sigma \in O(L)$ with $\sigma(T) \subset T$, $\sigma(l) = l$ acts unambiguously on $\mathscr{X}(T, l)$. Indeed, if the fibre of p over $[\omega]$ is the marked K3 $(X_{[\omega]}, \gamma)$, there is a unique isomorphism

$(X_{[\omega]}, \gamma) \xrightarrow{\sim} (X_{\sigma[\omega]}, \sigma\circ\gamma)$ by the Global Torelli theorem for K3's [B-P-V]. In case (L, T, l) is faithfully restrictive [*cf.* (0.3)], there is an unambiguous action of $O_0(T)$ on $\mathscr{X}(T, l)$ and the action on the base is free when restricted to $\mathring{D}(T, l)^{pol} = \mathring{D}(T, l) \cap D^{pol}(T, l)$. Exactly as in [Ba-P], section 2.4 it follows that $O_0(T)/D^{pol}(T, l)$ and hence $O_0(T)/D^{pol}(T, l)$ is quasi-projective. Comparing this situation with Proposition 2.3.4 one finds.

3.4.1. THEOREM. — *Assume that* (L, T, l) *is faithfully restrictive* (0.3) *and that* $-\mathrm{id} \notin SO_0(T)$. *The group* $SO_0(T)$ *acts freely on the restriction to* $D^{pol}(T, l)$ *of the family* p^{pol} *from* 3.3.3, *yielding a projective family of* K3-*surfaces of transcendental type* T *over a smooth quasi-projective base:*

$$\bar{p}: \quad \mathscr{Y}(T, l) \to SO_0(T)/\mathring{D}^{pol}(T, l).$$

The local system $R^2\bar{p}_*\mathbf{Z}$ *is isometric to* $\mathbf{L} = L/SO_0(T)$. *The* T-*marking yields a subsystem* $\mathbf{T} = T/SO_0(T)$ *of* $\mathbf{L}$ *and the variation of Hodge structure on* $R^2\bar{p}_*\mathbf{Z}$, *when transported to* $\mathbf{L}$ *restricts to the variation of Hodge structure on* $\mathbf{T}$ *over* $SO_0(T)/D^{pol}(T, l)$ *together with an orientation for one from Proposition* 2.3.4.

3.4.2. COROLLARY. — *Under the assumptions of Theorem* 3.4.1 *the family* $\bar{p}$ *is a universal family of* K3's *of type* (T, l) *together with an orientation for* $R^2\bar{p}_*\mathbf{Z}$. *Its monodromygroup is* $SO_0(T)$. □

4. The second exterior power of Hodge structures of weight one on a rank 4-lattice

In this section I use the following notation

H: $= \mathbf{Z}e_1 \oplus \mathbf{Z}e_2 \oplus \mathbf{Z}e_3 \oplus \mathbf{Z}e_4$.

det: $\Lambda^4 H \xrightarrow{\sim} \mathbf{Z}$ (one of two choices is fixed).

$\langle\ ,\ \rangle$: the bilinear form on $\Lambda^2 H$ given by $\det(u \wedge v)$.

$$K: = \overset{3}{\perp} U = (\Lambda^2 H, \langle\ ,\ \rangle) = \underbrace{(\mathbf{Z}f_1 + \mathbf{Z}g_1)}_{U} \perp \underbrace{(\mathbf{Z}f_2 + \mathbf{Z}g_2)}_{U} \perp \underbrace{(\mathbf{Z}f_3 + \mathbf{Z}g_3)}_{U}, \quad \text{with}$$

$f_1 = e_1 \wedge e_2$, $g_1 = e_3 \wedge e_4$, $f_2 = e_1 \wedge e_3$, $g_2 = e_4 \wedge e_2$, $f_3 = e_1 \wedge e_4$, $g_3 = e_2 \wedge e_3$.

l: the primitive vector $df_2 + g_2 \in K$ of norm $2d > 0$ (2).

K_l: the orthogonal complement of l in K.

4.1. *The Plücker map* $\pi : \mathrm{Gr}(2, H\otimes\mathbf{C}) \to \mathbf{P}(K\otimes\mathbf{C})$. — Since $\langle\ ,\ \rangle$ is unimodular, the evaluation map $x \to \langle x, - \rangle$ gives an isomorphism between $\Lambda^2 H$ and its dual. In particular, to $l \in \Lambda^2 H$ there corresponds a symplectic form $Q_l(u, v) = \langle u \wedge v, l \rangle$ with

(2) Any two primitive vectors of the same norm are isometric ([B-P-V], Theorem 2.9) hence one may assume that l is the given vector.

matrix

$$(Q_l(e_i, e_j)) = Q_\Delta, \qquad \Delta = \begin{pmatrix} 1 & 0 \\ 0 & -d \end{pmatrix}.$$

Since by 1.1 the 2-plane in $H \otimes \mathbf{C}$ corresponding to $Z = \begin{pmatrix} x & z \\ z & y \end{pmatrix} \in \mathfrak{h}_2$ is spanned by $\{xe_1 + ze_2 + e_3,\ ze_1 + ye_2 - d^{-1} e_4\}$ one can easily compute the coordinates of $\pi \mid \mathfrak{h}_2$ with respect to the basis $\{f_1, g_1, f_2, g_2, f_3, g_3\}$:

$$\pi \begin{pmatrix} x & z \\ z & y \end{pmatrix} = (xy - z^2,\ -d^{-1},\ -z,\ d^{-1} z,\ -d^{-1} x,\ -y).$$

4.1.1. LEMMA. — (i) *π maps $\mathfrak{h}_2$ isomorphically onto a connected component of* $D(K_l)$;

(ii) Λ^2 *induces a homomorphism* $Sp(Q_l) \to SO(K, l)$ *and* $\pi(M . Z) = (\Lambda^2 M) . \pi(Z)$ *for all* $M \in Sp(Q_l)$, $Z \in \mathfrak{h}_2$.

Proof. — (i) From the coordinates of $\pi \mid \mathfrak{h}_2$ it follows that $\mathfrak{h}_2$ goes injectively to $D(K_l)$. Conversely, if $[\omega] \in D(K_l)$, $\pi^{-1}[\omega]$ is a Q_l-isotropic subspace of $H \otimes \mathbf{C}$ and the local system $\bigcup \pi^{-1}[\omega]$ on $D(K_l)$ is oriented since $\langle \omega, \bar{\omega} \rangle > 0$, $\forall [\omega] \in D(K_l)$. So $\mathfrak{h}_2$ maps entirely onto a connected component of $D(K_l)$.

(ii) This is a routine computation.

4.1.2. Concerning the homomorphism induced by Λ^2 in 4.1.1 (i) there is the following result.

LEMMA. — Λ^2 *induces an embedding* $\lambda : Sp(Q_l)/\pm \mathrm{id} \to SO(K, l)$ *onto a subgroup of index* 2. *The cokernel of* λ *is generated by the involution* $\tau = -\mathrm{id} \perp \mathrm{id} \perp \mathrm{id} \in O(K)$.

Proof. — By ([S2], Lemma 2) $\Lambda^2 : SL(H) \to SO(K)$ has kernel $\pm \mathrm{id}_H$ and the cokernel is generated by $-\mathrm{id}_K$. It follows that λ is an embedding and since $\Lambda^2 \, Sp \, Q_l = (\mathrm{Im} \, \Lambda^2) \cap SO(K, l)$ it follows that Coker λ has order $\leqq 2$. It is not difficult to check that $\tau \in SO(K, l)$ and $\tau \notin \mathrm{Im} \, \Lambda^2$, hence Coker λ is generated by τ.

4.1.3. Clearly, the restriction homomorphism $O(K, l) \to O(K_l)$ is injective, i.e. the pair (K, l) is faithfully restrictive in the sense of (0.3). So $O(K, l)$ can be identified with a subgroup $O_0(K_l)$ of $O(K_l)$ and similarly for $SO(K, l)$. The group $SO(K, l)$ operates on the domain $D(K_l)$.

LEMMA. — *The involution* τ (4.1.2) *interchanges the two connected components of* $D(K_l)$.

Proof. — Recall from 2.1.1 how to distinguish between the two components. Use the $\mathbf{Q}$-basis $\{f_1 + g_1, f_3 + g_3, f_1 - g_1, f_3 - g_3, df_2 - g_2\}$. Then $\langle\ ,\ \rangle \mid K_l$ has matrix $\mathrm{diag}(2, 2, -2, -2, -2d)$ and since $\tau(x_1, x_2, x_3, x_4, x_5) = (-x_1, x_2, -x_3, x_4, x_5)$ in this basis, τ interchanges the components.

4.1.4. Combining 4.1.1, 4.1.2 and 4.1.3 one obtains

PROPOSITION. — *The Plücker embedding induces a biholomorphic map*

$$\mathrm{Sp}(Q_l)/\mathfrak{h}_2 \xrightarrow{\sim} \mathrm{SO}_0(K_l)/D(K_l).$$

4.2. Let me now interprete the results of 4.1 in terms of Hodge structures.

4.2.1. Over $\mathfrak{h}_2$ the tautological variation of weight one Hodge structure

$$H \otimes \mathcal{O}_{\mathfrak{h}_2} \supset \mathcal{H}^{1,0}$$

upon taking exterior powers yields a variation of weight two Hodge structure

$$K \otimes \mathcal{O}_{\mathfrak{h}_2} \supset (\Lambda^2 \mathcal{H}^{1,0})^{\perp} \supset \Lambda^2 \mathcal{H}^{1,0}.$$

The Plücker map identifies $\mathfrak{h}_2$ with a connected component of $D(K_l)$ and the above variation of weight two Hodge structure is nothing but the tautological one over $D(K_l)$ (restricted to a component).

4.2.2. The group $\mathrm{Sp}(Q_l)$ acts freely on $\mathfrak{h}_2^0 = \mathfrak{h}_2 \setminus \{$fixed loci of non-trivial elements of $\mathrm{Sp}(Q_l)/\pm \mathrm{id}\}$, but *not* on the constant system H over $\mathfrak{h}_2^0$. However $\mathrm{Sp}(Q_l)/\pm\mathrm{id}$ *does* act freely on the constant systems $K = \Lambda^2 H$ and K_l over $\mathfrak{h}_2^0$. The quotient system of the latter, $\mathbf{K}_l$, carries a variation of weight two Hodge structure. The isomorphism of 4.1.4 identifies this variation with the universal one over $\mathrm{SO}_0(K_l)\setminus D(K_l)$ constructed in 2.3.2.

4.3. *Remark.* — As already observed in 1.2.1, there is no possibility to construct a universal family of polarized abelian varieties (of dimension 2) over $\mathrm{Sp}(Q_l)/\mathfrak{h}_2^0$. However, $\mathrm{Sp}(Q_l)$ acts on the universal family of marked polarized abelian surfaces $q : \mathcal{E}_\Delta \to \mathfrak{h}_2$ (*cf.* 1.1.1). The element $-\mathrm{id}$ has 16 fixed points on every fibre, forming submanifolds in $\mathcal{E}_\Delta$.

Over $\mathrm{Sp}(Q_l)/\mathfrak{h}_2^0$ this yields a family of Kummer surfaces in the classical sense; in modern terminology "generalised K3's" (*cf.* 3.1.2) with polarisation induced by l. They are all of transcendental type $K_l(2)$. The variation of weight two Hodge structure on the corresponding local system $\mathbf{K}_l(2)$ is different from the one considered in 4.2.2, but coincides with it if one is willing to identify $\mathbf{K}_l$ and $\mathbf{K}_l(2)$ as local systems. (Only the bilinear form is different!) For a direct comparison of this variation and the variation on $\mathbf{K}_l$ coming from a suitable (T, l)-marked family of generalised K3's ($T = \mathbf{K}_l$) one needs the concept of a Shioda-Inose structure (*see* e. g. [M2]). I won't given the details here, since it is too remote from the application I have in mind here.

4.4. Given an admissible pair (T, l), where T is a primitive sublattice of K of signature (2, n) one puts

$$\mathfrak{h}_T = \pi^{-1}(D(T)) \subset \mathfrak{h}_2,$$
$$\Gamma_T = \lambda^{-1}(\mathrm{SO}(K, T, l)) \subset \mathrm{Sp}(Q_l)/\pm\mathrm{id}.$$

(Here π is the Plücker map from 4.1 and λ is the map from 4.1.2.)

4.4.1. Concerning $\Gamma_T \backslash \mathfrak{h}_T$ one has two possibilities:

(i) SO(K, T, l) does *not* contain elements interchanging the connected components of D(T). Then

$$\Gamma_T \backslash \mathfrak{h}_T \xrightarrow{\sim} \text{connected component of } SO(K, T, l) \backslash D(T).$$

(ii) In the remaining case one has

$$\Gamma_T \backslash \mathfrak{h}_T \xrightarrow{\sim} SO(K, T, l) \backslash D(T).$$

4.4.2. The group Γ_T acts freely on $\mathfrak{h}_T^0 = \pi^{-1}(\mathring{D}(T))$ and over $\Gamma_T \backslash \mathfrak{h}_T^0$ one has local systems $\mathbf{T} \subset \mathbf{K}_l \subset \mathbf{K}$ underlying a variation of weight two Hodge structures over $\Gamma_T \backslash \mathfrak{h}_T^0$. They are obtained by pulling back the universal one over $SO_0(T) \backslash D(T)$ by means of the isomorphisms in 4.4.1. [Here it is assumed that the triple (K, T, l) is faithfully restrictive, so that $SO_0(T) \cong SO(K, T, l)$.]

4.4.3. Remark 4.3 can be repeated with T instead of K_l. The reader can make the necessary changes for himself.

5. Products of isogeneous elliptic curves

In this section I specialize the considerations of paragraph 4.4 to the situation where (*cf.* beginning of paragraph 4 for notations)

$$T = \mathbf{Z}\text{-span of } f_1 + m g_1, f_3, g_3 \cong \langle 2m \rangle \perp U,$$
$$l = f_2 + g_2 \quad (\text{hence } d = 1).$$

5.1. I shall in this subsection determine $\mathfrak{h}_T$ in this case.

LEMMA. — *Define an embedding* $j : \mathfrak{h} \to \mathfrak{h}_2$ *by* $j(\tau) = \begin{pmatrix} \tau & 0 \\ 0 & -(m\tau)^{-1} \end{pmatrix}$. *The composition* $\pi \circ j$ *maps* $\mathfrak{h}$ isomorphically onto a connected component of D(T), *hence* $\mathfrak{h}_T = j(\mathfrak{h})$.

Proof. — $\pi \circ j(\tau) = (\tau, m\tau, 0, 0, m\tau^2, -1) \in D(T)$ and conversely every point of D(T) has homogeneous coordinates $(u, mu, 0, 0, v, w)$ with $mu^2 = -vw$ and $2mu\bar{u} > v\bar{w} + \bar{v}w$. I may normalize w by putting $w = -1$, so $v = mu^2$ and $\pm u \in \mathfrak{h}$. It follows that $\pi \circ j$ maps $\mathfrak{h}$ and $-\mathfrak{h}$ onto separate connected components of D(T).

5.2. In this subsection Γ_T will be determined. First some notations.

Recall that

$$\Gamma_1(m) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbf{Z});\ c \equiv 0 \bmod m \right\}$$

$$\Gamma_1^*(m) = \text{subgroup of } GL_2^+ \mathbf{Q} \text{ generated by } \Gamma_1(m) \text{ and } i_m = \begin{pmatrix} 0 & -1 \\ m & 0 \end{pmatrix}$$

$$P\Gamma_1^*(m)=\Gamma_1^*(m)/\text{centre}.$$

Clearly, i_m defines an involution in $P\Gamma_1^*(m)$ and $[P\Gamma_1^*(m):\Gamma_1(m)/\pm\text{id}]=2$. The group $P\Gamma_1^*(m)$ acts effectively on $\mathfrak{h}$.

The group $Sp(Q_\Delta)$, $\Delta=\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$ acts on $\mathfrak{h}_2$ and so does the corresponding group with $\mathbf{Q}$-coefficients $Sp(Q_\Delta, \mathbf{Q})$ and the positive multiples $\mathbf{Q}^+ . Sp(Q_\Delta, \mathbf{Q})$. Define

$$j^*:\quad GL^+(2, \mathbf{Q})\to \mathbf{Q}^+ . Sp(Q_\Delta, \mathbf{Q})$$

by

$$\begin{pmatrix}a & b\\ c & d\end{pmatrix}\to\begin{pmatrix}a & 0 & b & 0\\ 0 & d & 0 & c/m\\ c & 0 & d & 0\\ 0 & mb & 0 & a\end{pmatrix}$$

$$k:\quad GL^+(2, \mathbf{Q})\to SO(T\otimes\mathbf{C})$$

by

$$M\to(\det M)^{-2}(\Lambda^2 j^*(M)\,|\,T\otimes\mathbf{Q}).$$

5.2.1. LEMMA. — (i) $j^*\begin{pmatrix}a & b\\ c & d\end{pmatrix}.j(\tau)=j((a\tau+b)(c\tau+d)^{-1})$.

(ii) *For* $m\geqq 2$ *the restriction* $SO(K, T, l)\to SO(T)$ *is injective, i. e.* (K, T, l) *is a faithfully restrictive triple.*

(iii) k *induces an isomorphism*

$$P\Gamma_1^*(m)\xrightarrow{\sim}\{\gamma\in SO(K, T, l)\,|\,\gamma \textit{ preserves the components of } D(T)\}.$$

Proof. — (i) A routine computation.

(ii) $T^\perp=\mathbf{Z}\,f_1-mg_1+\mathbf{Z}\,f_2+\mathbf{Z}g_2$, $l=f_2+g_2$. *It follows that*

$$SO(T^\perp, l)=\left\{\text{id}, \begin{pmatrix}-1 & 0 & 0\\ 0 & 0 & 1\\ 0 & 1 & 0\end{pmatrix}\right\}.$$

From [N], paragraph 1.5 it easily follows that there exists no isometry of K which acts as the identity on T and as the second element of $SO(T^\perp, l)$ on $T^\perp$. So the restriction $SO(K, T, l)\to SO(T)$ is injective.

(iii) $\operatorname{Im} j^*$ is the stabilizor of the 2-planes $e_1\wedge e_3$, $e_2\wedge e_4$ and $(e_1+e_4)\wedge(e_2+me_3)$ acting with the same positive determinant on these planes. Since those base $T^\perp\otimes\mathbf{Q}$, the restriction of $\Lambda^2 j^*(M)$ to $T^\perp\otimes\mathbf{Q}$ is multiplication by $\det M$. Since $\operatorname{Im} j^*\cap Sp(Q_\Delta)=j^*\Gamma_1(m)$ it follows from 4.1.2 that

$$\{\sigma\in SO(K)\,|\,\sigma\,|\,T^\perp=\text{id}\}\text{ contains }\Lambda^2 j^*(\Gamma_1(m))$$

as a subgroup of index 2 with τ a generator for the cokernel.

or equivalently

$$\Lambda^2 j^*(\Gamma_1(m)) = \{\sigma\in \mathrm{SO}(\mathrm{K})\,|\,\sigma\,|\,\mathrm{T}^{\perp}=\mathrm{id},\ \sigma \text{ preserves the components of } \mathrm{D}(\mathrm{T})\}.$$

From [N], Corollary 1.5.2 it follows that there exists an automorphism $\gamma\in \mathrm{SO}(\mathrm{K}, \mathrm{T}, l)$ such that

$$\gamma\,|\,\mathrm{T}=\begin{pmatrix}-1 & 0 & 0\\ 0 & 0 & -1\\ 0 & -1 & 0\end{pmatrix},\qquad \gamma\,|\,\mathrm{T}^{\perp}=\begin{pmatrix}-1 & 0 & 0\\ 0 & 0 & 1\\ 0 & 1 & 0\end{pmatrix}.$$

Using e. g. 2.1.1 it follows that γ preserves the components of $\mathrm{D}(\mathrm{T})$. Since $\#\mathrm{SO}(\mathrm{T}^{\perp}, l)=2$ one can conclude that the group $\{\sigma\in\mathrm{SO}(\mathrm{K})\,|\,\sigma$ preserves the components of $\mathrm{D}(\mathrm{T})\}$ contains $\Lambda^2 j^*\Gamma_1(m)$ as a subgroup of indes 2 and that the entire group is generated by $\Lambda^2 j^*\Gamma_1(m)$ and γ. Since $\gamma\,|\,\mathrm{T}=k(i_m)$ and since $\mathrm{SO}(\mathrm{K}, \mathrm{T}, l)\to\mathrm{SO}(\mathrm{T})$ is injective it follows that

$$\mathrm{SO}_0(\mathrm{T})=k(\Gamma_1^*(m))$$

and hence (iii) follows.

5.2.2. Corollary. — *If one identifies* $\mathfrak{h}_\mathrm{T}$ *sith* $\mathfrak{h}$ *by means of* j *the group* Γ_T *becomes identified with* $\mathrm{P}\Gamma_1^*(m)$, *hence there is a biholomorphic map*

$$\Gamma_1^*(m)/\mathfrak{h}\xrightarrow{\sim}\mathrm{SO}(\mathrm{K}, \mathrm{T}, l)/\mathrm{D}(\mathrm{T}).$$

Proof. — A direct consequence of 5.2.1 taking into account that $\tau\in\mathrm{SO}(\mathrm{K}, \mathrm{T}, l)$ interchanges the components of $\mathrm{D}(\mathrm{T})$ and applying 4.4.1 (ii).

5.2.3. Corollary. — *Setting* $\mathfrak{h}^0=\mathfrak{h}\setminus\{\Gamma_1^*(m)$ *orbit of* $(-m)^{1/2}\}$, $\mathrm{D}^0(\mathrm{T})=\mathrm{D}(\mathrm{T})\setminus\{$*fixed point locus of non-trivial elements in* $\mathrm{SO}(\mathrm{K}, \mathrm{T}, l)\}$ *the map of* 5.2.2 *induces an isomorphism*

$$\Gamma_1^*(m)/\mathfrak{h}^0\to\mathrm{SO}(\mathrm{K}, \mathrm{T}, l)/\mathrm{D}^0(\mathrm{T})\quad (m\geqq 3).$$

Proof. — For $m\geqq 3$ the only fixed points of elements in $\Gamma_1^*(m)$ are the orbits of $(-m)^{1/2}$ (=fixed point of i_m).

5.3. In this section I want to give a geometric description of $\mathrm{D}(\mathrm{T})$ connected with produces of isogeneous elliptic curves.

5.3.1. I recall that $\mathrm{D}(\mathrm{T})$ parametrizes polarized weight Hodge structures on $\mathrm{T}\otimes\mathbf{C}$ with $\dim\mathrm{T}^{2,0}=1$. To construct a universal family over a component of $\mathrm{D}(\mathrm{T})$, i. e. the upper half plane, one forms the family

$$\mathscr{A}_\mathrm{T}\to\mathfrak{h}_\mathrm{T}=\mathfrak{h}$$

whose fibre over $\tau\in\mathfrak{h}$ is the product of the elliptic curves E_τ, resp. $\mathrm{E}_{i_m(\tau)}$ with periods τ, resp. $i_m(\tau)$.

Clearly, there is a degree m-isogeny

$$j_m(\tau): \quad E_\tau \to E_{i_m(\tau)}$$

defined by $j_m(\tau)[z \bmod \mathbf{Z}+\mathbf{Z}\tau]=[-\tau^{-1} z \bmod \mathbf{Z}+\mathbf{Z}\, i_m(\tau)]$. The Néron-Severi group of $E_{i_m(\tau)} \times E_\tau$ contains the classes of the fibres and the graph of $j_m(\tau)$. Those classes span a sublattice of $H^2(E_\tau \times E_{i_m(\tau)}, \mathbf{Z})$ isometric to $U \perp \langle -2m \rangle$ and the orthogonal complement T_τ is isometric to $T=U \perp \langle 2m \rangle$. Hence the T_τ fit together to a rank 3 constant local system T over $\mathfrak{h}$ carrying a weight two Hodge structure. This gives precisely the tautological variation of Hodge structure over a component of D(T).

5.3.2. The group $\Gamma_1^*(m)$ contains $-$id, hence does not operate freely on the family $\mathscr{A}_T \to \mathfrak{h}_T$. Therefore one considers

$$\Gamma_0^0(m)=\left\{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbf{Z});\ a \equiv 1 \bmod m,\ c \equiv 0 \bmod m\right\}$$

$\Gamma_0^0(m)^*$: subgroup of PGL(2, $\mathbf{Q}$) generated by $\Gamma_0^0(m)$ and i_m.

The group $\Gamma_0^0(m)$ acts freely on the family $\mathscr{A}_T \to \mathfrak{h}_T$ and $\Gamma_0^0(m)^*$ acts freely on the restriction of this family to $\mathfrak{h}^0$ (*cf.* 5.2.2 for this notation). Defining

$$Y(m)=\Gamma_0^0(m)/\mathfrak{h},$$
$$Y^0(m)=\Gamma_0^0(m)/\mathfrak{h}^0,$$
$$Z(m)=\Gamma_0^0(m)^*/\mathfrak{h}^0,$$

one obtaines the following families of products of elliptic curves

$$\mathscr{A}(m)=\Gamma_0^0(m)/\mathscr{A}_T \to Y(m),$$
$$\mathscr{B}(m)=\Gamma_0^0(m)^*/\mathscr{A}_T \to Z(m).$$

5.3.3. An alternative description of the preceding two families goes as follows.

The modular family $f: \mathscr{E}_{\Gamma_0^0(m)} \to Y(m)$ (*cf.* 1.2.2) of elliptic curves pulls back under the involution $i_m: Y(m) \to Y(m)$ to a family, say f' of elliptic curves. The family $\mathscr{A}(m) \to Y(m)$ is the fibre product of f and f' over $Y(m)$.

If $y \in Y(m)$ and E_y is the corresponding elliptic curve, $E_{i_m(y)} \times E_y$ is the fibre of $\mathscr{A}(m) \to Y(m)$ over y. Interchanging the two factors gives an isomorphism $\tilde{i}_m(y): E_{i_m(y)} \times E_y \to E_y \times E_{i_m(y)}$ lifting i_m. Over $Y^0(m)$ this is a fixed point free involution. The quotient of $\mathscr{A}(m)$ by $\tilde{i}_m$ is $\mathscr{B}(m)$.

5.3.4. From the description given in 5.3.2 one derives

LEMMA. — *The monodromy of the family $\mathscr{B}(m) \to Z(m)$ is $\Gamma_0^0(m)^*$ $(m \geqq 3)$.*

5.3.5. The tautological variation of Hodge structure on D(T) descends under the action of $\Gamma_0^0(m)$ to a variation of Hodge structure on $\Gamma_0^0(m)/D(T)$. If $\mathscr{A}(m)_y$ is the fibre of the family $\mathscr{A}(m) \to Z(m)$, this variation is of course the same as the one induced on the local system $\mathbf{T}$ whose fibre $\mathbf{T}_y$ is the orthogonal complement inside $H^2(\mathscr{A}(m)_y, \mathbf{Z})$

of the lattice spanned by fibres of $E_{i_m(y)} \times E_y$ and the graph of $j_m(y)$ [*cf.* 5.3.1]. In fact for generic y this is precisely the transcendental lattice of $\mathscr{A}(m)_y$.

Similar remarks can be made for $\mathscr{B}(m) \to Z(m)$.

5.3.6. If $m=3$, 4, 6 the groups $\Gamma_0^0(m)$ and $\Gamma_1(m)/\pm\mathrm{id}$ are isomorphic and therefore also $\Gamma_0^0(m)^* \cong P\Gamma_1(m)^* \cong \{\gamma \in SO(K, T, l)/\gamma$ preserves components of $D(T)\}$, hence

LEMMA. — *Suppose* $m=3$, 4 *or* 6. *Then*

$$Z(m)=\Gamma_0^0(m)^*\backslash\mathfrak{h} \xrightarrow{\sim} SO(K, T, l)\backslash D(T)$$

and the monodromy of the family $\mathscr{B}(m) \to Z(m)$ *is therefore*

$$\Gamma_0^0(m)^* \cong \{\gamma \in SO(K, T, l) \mid \gamma \textit{ preserves components of } D(T)\}.$$

5.4. For later use I need to compare the local system $\mathbf{T}$ of 5.3.5 and the local system $\mathbf{T}_1$ constructed similarly for the product family $\mathscr{E}_\Gamma^0 \times \mathscr{E}_\Gamma^0 \to Y(m)$ $(\Gamma=\Gamma_0^0(m))$.

Since $H^2(E_y \times E_y, \mathbf{Z}) \cong H^2(E_y) \oplus H^1(E_y) \otimes H^1(E_y)$ the lattice $(\mathbf{T}_1)_y$ orthogonal to the classes of the fibres and the diagonal is seen to be the symmetric subspace $S^2 H^1(E_y)$ of $H^1(E_y) \otimes H^1(E_y)$. This implies the following.

If

$$f: \quad \mathscr{E}_{\Gamma_0^0(m)}^0 \to Y(m)$$

as before denotes the modular family and $\mathbf{V}_{\mathbf{Z}}^\vee = R^1 f_* \mathbf{Z}$, the system $\mathbf{T}_1$ is nothing but the system $S^2 \mathbf{V}_{\mathbf{Z}}^\vee$. The morphism

$$(j_m, \mathrm{id}): \quad \mathscr{E}_\Gamma^0 \times \mathscr{E}_\Gamma^0 \to \mathscr{A}(m) \qquad (\Gamma=\Gamma_0^0(m))$$

induces a homomorphism

$$(j_m, 1)^*: \quad \mathbf{T} \to S^2 \mathbf{V}_{\mathbf{Z}}^\vee$$

which—after tensoring with $\mathbf{Q}$ becomes an isomorphism.

5.5. In this subsection I compare the weight two Hodge structure for the family $\mathscr{B}(m) \to Z(m)$ and the weight two Hodge structure on $\mathscr{B}(m)$ arizing from a suitably polarized family of K3-surfaces of transcendental type T.

5.5.1. The lattice $T \subset K$ can also be seen as a primitive sublattice of $L=K \overset{2}{\perp} E_8(-1)$. The element $l \in K$, when considered as an element of L does not yield an admissible pair (T, l), since l is orthogonal to all roots inside $E_8(-1)$. However, the same argument as in 3.3.2 yields an admissible element of the form $l'=\alpha l+e$ with $\alpha \in \mathbf{N}$, $e \in \overset{2}{\perp} E_8(-1)$.

Over $D^{\mathrm{pol}}(T, l')$ (*cf.* 3.3.3) one has a universal family of (T, l')-marked K3's:

$$\mathscr{X}(T, l') \to D^{\mathrm{pol}}(T, l').$$

5.5.2. Every automorphism of K extends uniquely to an automorphism of L by setting it equal to the identity on $K^{\perp} = \overset{2}{\perp} E_8(-1)$. This gives an embedding

$$O(K) \subset O(L)$$

and hence an embedding

$$O(K, T, l) \subset O(L, T, l').$$

However (L, T, l') is general won't be faithfully restrictive any more. This does not matter since one considers only the action of the subgroup $O(K, T, l)$ of $O(L, T, l')$. This subgroup can be identified with the subgroup $O_0(T)$ of $O(T)$ obtained by restricting elements of $O(K, T, l)$ to T. So—passing to $SO_0(T)$—the situation is slightly different from 3.4.1. However one still obtains a projective family of K3-surfaces of transcendental type T over a smooth quasi-projective base, which—in this case—is a curve:

$$\bar{q}: \quad \mathscr{Y}(T, l') \to SO_0(T)/\mathring{D}^{pol}(T, l').$$

5.5.3. Over $\Gamma_1^*(m)/\mathfrak{h}^0$ there is a variation of weight two Hodge structure obtained as follows. One takes the second exterior power of the tautological weight one Hodge structure on $\mathfrak{h}_2$, restricts is to $j(\mathfrak{h}^0) \subset \mathfrak{h}_2$ and divides out by the action of $\Gamma_1^*(m)$. The underlying local system is

$$\Gamma_1^*(m)/K = SO_0(T)/K = \mathbf{K},$$

where K is the constant system on $\mathfrak{h}^0$. The variation of weight two Hodge structure on $\mathbf{K}$ (and $\mathbf{T}$) is the same as the variation of Hodge structure on the corresponding subsystems of the local system $R^2 \bar{q}_* \mathbf{Z}$ with $\bar{q}$ as in 5.5.2.

5.5.4. If $m=3$, 4 or 6 the variation of Hodge structure from 5.5.3 has a geometric meaning. Now $P\Gamma_1^*(m) = \Gamma_0^0(m)^*$ and hence

$$Z^{pol}(m): \quad = SO_0(T)/\mathring{D}^{pol}(T, l') \subset SO_0(T)/\mathring{D}(T) = Z(m)$$

and the second cohomology of the family $\mathscr{B}(m) \to Z(m)$ gives a variation of Hodge structure over $Z(m)$ which over $Z^{pol}(m)$ is the same as the one from 5.5.3 on $\mathbf{K}$ and hence the same as the one on the corresponding local subsystem of $R^2 \bar{q}_* \mathbf{Z}$ with q as in 5.5.2.

6. Picard-Fuchs equations

In this section $\bar{C}$ is a smooth complex projective curve, t a local parameter at $c_0 \in \bar{C}$ regular and non-zero on C.

6.1. I let $\mathbf{V}$ be a local system of n-dimensional $\mathbf{C}$-vector spaces on C with monodromy-representation $\rho : \pi_1(C, c_0) \to \mathrm{Aut}\,V$, $V = \mathbf{V}_{c_0}$. I shall first describe how to associate to any global holomorphic section α of $\mathscr{V}^\vee = \mathbf{V}^\vee \otimes \mathcal{O}_C$ a unique differential equation $D_\alpha = 0$ of order $\leqq n$ in the variable t. Let $\nabla^\vee$ be the usual integrable connection on $\mathscr{V}^\vee$ and let $\nabla^\vee_{\partial/\partial t}$ be $\nabla^\vee$ followed by contraction with $\partial/\partial t$, so $\nabla^\vee_{\partial/\partial t} \in \mathrm{End}\,(\mathscr{V}^\vee)$. Since $\alpha, \nabla^\vee_{\partial/\partial t}\alpha, \ldots, (\nabla^\vee_{\partial/\partial t})^n \alpha$ must be linearly dependent over $\mathcal{O}_C$, there is a linear relation:

$$[a_n (\nabla^\vee_{\partial/\partial t})^n + \ldots + a_1 (\nabla^\vee_{\partial/\partial t}) + a_0]\,\alpha = 0.$$

The associated differential equation

$$D_\alpha = a_n\, d^n/dt^n + \ldots + a_1\, d/dt + a_0 = 0$$

is the differential equation associated to α alluded to before. The subsheaf $\mathrm{Sol}(D_\alpha)$ of $\mathcal{O}_C$ of local solutions for $D_\alpha = 0$ forms a locally constant sheaf of $\mathbf{C}$-vectorspaces and for every local section v of $\mathbf{V}$, the holomorphic function $\alpha(v)$ is a local solution for $D_\alpha = 0$. This yields a homomorphism

$$\mathbf{V} \to \mathrm{Sol}(D_\alpha).$$

If α is *cyclic*, i.e. if $\alpha, \nabla^\vee_{\partial/\partial t}\alpha, \ldots, (\nabla^\vee_{\partial/\partial t})^{n-1}\alpha$ is a *basis* for $\mathscr{V}$, the coefficient a_n is non-zero and can be *normalized* to 1. Moreover the preceding homomorphism is an isomorphism.

Remark. — From a more global point of view one should consider bundle homomorphisms $\alpha : \mathscr{V} \to \mathscr{L}$, $\mathscr{L}$ a line bundle, instead of the more restrictive case $\mathscr{L} = \mathcal{O}$. If $\mathscr{P}^k(\mathscr{L})$ is the bundle of k-jets one says that α is cyclic if for $k = n-1$ the map $\alpha^k : \mathscr{V} \to \mathscr{P}^k(\mathscr{L})$, $v \to k$-jet of v is an isomorphism. The notion of an n-th *order normalized differential equation on* $\mathscr{L}$ can be introduced, it is a bundle homomorphism $D : \mathscr{P}^n(\mathscr{L}) \to \Omega_C^{\otimes n} \otimes \mathscr{L}$ inducing the identity on the subbundle $\Omega_C^{\otimes n} \otimes \mathscr{L}$. A local section s of $\mathscr{L}$ is called a *local solution* to $D = 0$, if $D(n\text{-jet of } s) = 0$. If α is cyclic, there is a unique n-the order operator D_α on $\mathscr{L}$ such that $D_\alpha \circ \alpha^n = 0$ and so every local section v of $\mathbf{V}$ yield a solution $\alpha(v)$ for $D_\alpha = 0$ and as before one gets an isomorphism $\mathbf{V} \to \mathrm{Sol}(D_\alpha)$. See [D], I, § 4 for details.

6.2. Since C is quasi-projective there exists a canonical extension of $\mathscr{V}$ to an algebraic bundle $\bar{\mathscr{V}}$ such that ∇ extends to a connection $\bar{\nabla}$ on $\bar{\mathscr{V}}$ having at most logarithmic poles in $\bar{C} \setminus C$ ([D], Chap. II, § 5). If $\mathscr{L}$ extends to an algebraic bundle $\bar{\mathscr{L}}$ and α to $\bar{\alpha} : \bar{\mathscr{V}} \to \bar{\mathscr{L}}$ this implies that the points $\bar{C} \setminus C$ are regular singular points for the differential operator D_α.

An important special case arizes when $\mathscr{V}^\vee$ carries a variation of Hodge subbundles $\mathscr{F}^p$ extend to algebraic subbundles $\bar{\mathscr{F}}^p$ of $\bar{\mathscr{V}}^\vee$. If the variation of Hodge structure comes from the cohomology of the fibres of an algebraic family of polarized algebraic manifolds, these algebraic structures on $\bar{\mathscr{V}}^\vee$ and $\bar{\mathscr{F}}^p$ coincide with the intrinsic algebraic structures. In particular the connection ∇, which now is called the *Gauss-Manin connection* has regular singular points ([S], p. 234). The differential equation $D_\alpha = 0$ thus also has regular points. It is called the *Picard-Fuchs equation* for α.

6.3. In the sequel I shall only look at the following special case.

Given is a family $f: X \to C$ of n-dimensional polarized algebraic manifolds with $\dim H^{n,0}(X_t)=1$ ($X_t = f^{-1}(t)$). The homology groups $H_n(X_t, \mathbf{Z})$/torsion fit together to a locally constant sheaf $\mathbf{V_Z}$ and the integration map $\gamma \to \int_\gamma$ yields an explicit isomorphism $\mathbf{V_Z} \xrightarrow{\sim} (R^n f \mathbf{Z})^\vee$. The Hodge bundle $\mathscr{F}^n \subset \mathscr{V}^\vee$ is a line-bundle and the morphism $\alpha: \mathscr{V} \to (\mathscr{F}^n)^\vee$ is the dual of the inclusion.

Over a Zariski-open $C_0 \subset C$ the line bundle $\mathscr{F}^n$ is trivial and any section $\omega \in \Gamma(C_0, \mathscr{F}^n)$ spanning $\mathscr{F}^n | C_0$ can be considered as a holomorphic n-form ω_t on the fibre X_t varying holomorphically with t. By the preceding remarks the integrals $f_\gamma(t) = \int_\gamma \omega_t$, γ a local section of $\mathbf{V_Z}$, yield local solutions for the Picard Fuchs equation $D_\alpha = 0$ and if α is cyclic, one has

$$\mathbf{V} = \mathbf{V_Z} \otimes \mathbf{C} \xrightarrow{\sim} \mathrm{Sol}(D_\alpha), \qquad \gamma \mapsto f_\gamma(t)$$

and hence one obtains the full set of local solutions in this way.

6.4. From the preceding considerations one derives.

LEMMA. — *If α is cyclic, the monodromy-representation of $D_\alpha = 0$ on the vector space* V *of local solutions around $c_0 \in C$ (obtained by analytic continuation of the local solutions to* C) *is the monodromy representation*

$$\rho: \quad \pi_1(C, c_0) \to \operatorname{Aut} V_\mathbf{Z} \textit{ defining } \mathbf{V_Z}.$$

6.5. *Examples.*

Example 6.5.1. — Let me reconsider Example 1.2.2.

(i) $\bar{C} = \mathbf{P}_1$, $C = \mathbf{P}_1 \setminus \{0, \infty, \text{roots of } t^2+t+1=0\}$ and the family $f: X \to C$ is the modular family for the group $\Gamma_1(3)$. The Picard-Fuchs equation (for the holomorphic 1-forms on X_t) is

$$(t-1)(t^2+t+1)\, d^2/dt^2 + 3t^2\, d/dt + t = 0.$$

(ii) $\bar{C} = \mathbf{P}_1$, $C = \mathbf{P}_1 \setminus \{0, \infty, \text{roots of } t^2 - 11t - 1 = 0\}$.

The Picard-Fuchs equation for the modular family for the group $\Gamma_0^0(5)$ is

$$t(t^2-11t-1)\, d^2/dt^2 + (3t^2-22t-1)\, d/dt + (t-3) = 0.$$

(iii) $\bar{C} = \mathbf{P}_1$, $C = \mathbf{P}_1 \setminus \{0, 1, 1/9, \infty\}$.

The Picard-Fuchs equation for the modular group $\Gamma_0^0(6)$ is

$$t(t-1)(9t-1)\, d^2/dt^2 + (27t^2-20t+1)\, d/dt + (9t-3) = 0.$$

For the derivation of the Picard-Fuchs equations I refer to [S-B], § 11, where (ii), (iii) are given explicitly, (i) can be derived similarly.

Example 6.5.2. — *The k-th symmetric power of a differential equation.*

Suppose that $D=0$ is a differential equation on C with regular singular points in points of $\bar{C} \setminus C$. The $(k+1)$-th order differential equation $S^k D=0$ on $\bar{C}$ is defined as the differential equation qhose local system of local solutions on C is just $S^k(\mathrm{Sol}\, D)$. If **V** is a local system on C of 2-dimensional **C**-vectorspaces and $\alpha : \mathscr{V}=\mathbf{V}\otimes\mathscr{O} \to \mathscr{L}$ a cyclic homomorphism, $\beta=S^k(\alpha) : S^k \mathscr{V} \to \mathscr{L}^{\otimes k}$ is also cyclic and $D_\beta = S^k(D_\alpha)$.

Lemma. — *If* $\Lambda = d^2/dz^2 + P\, d/dz + Q$, *then*

$$S^2 \Lambda = d^3/dz^3 + 3P\, d^2/dz^2 + (4Q + 2P^2 + P')\, d/dz + (4PQ + 2Q').$$

Proof. — $\Lambda=0$ is equivalent to the system of first order equations

$$\begin{pmatrix} f \\ g \end{pmatrix}' = \begin{pmatrix} 0 & 1 \\ -Q & P \end{pmatrix} \begin{pmatrix} f \\ g \end{pmatrix},$$

where f is a (local) solution for $\Lambda=0$.

It follows that

$$\begin{pmatrix} f\otimes f \\ f\otimes g + g\otimes f \\ g\otimes g \end{pmatrix}' = \begin{pmatrix} 0 & 0 & 1 \\ 0 & -2P & -Q \\ -2Q & 2 & -P \end{pmatrix} \begin{pmatrix} f\otimes f \\ f\otimes g + g\otimes f \\ g\otimes g \end{pmatrix}$$

and since $f\otimes f$ is a local solution for $S^2\Lambda=0$ this third order differential equation is readily computed from the preceding system to coincide with the right hand side of the expression for $S^2\Lambda$ in the statement of the lemma.

Example 6.5.3. — Let $f: X \to C$ be any family of elliptic curves and form the fibre product $g : X \times_C X \to C$. If ω is a global section spanning the Hodge bundle $\mathscr{F}^1$ for f, $\omega\otimes\omega$ spans the Hodge bundle $\mathscr{F}^2$ for g. Exactly as in 5.4 it follows that $S^2 R^1 f_* \mathbf{Z} \subset R^2 g_* \mathbf{Z}$ contains the transcendental lattices for all fibres $X_t \times X_t$ of g (and coincides with the transcendental lattice for generic t).

It follows that $S^2 D_\omega = 0$ is the Picard-Fuchs equation for and the corresponding system of local solutions is $S^2 (R^1 f_* \mathbf{Z})^\vee \otimes \mathbf{C}$. More geometrically, a complete set of solutions is obtained by integrating the holomorphis 2-forms $\omega_t \otimes \omega_t$ over the cycles in the 3-dimensional sublattice $S^2 H_1(X_t, \mathbf{Z}) \subset H_2(X_t \times X_t, \mathbf{Z})$.

Example 6.5.4. — Let me look at the families $\mathscr{A}(m) \to Y(m)$ and $\mathscr{B}(m) \to Z(m)$ from 5.3.2. Let ω be a global section spanning the Hodge bundle $\mathscr{F}^1$ for $f : \mathscr{E}^0_{\Gamma 8(m)} \to Y(m)$.

Then $i_m^* \omega$ spans the Hodge bundle $\mathscr{F}^1$ for f' and $i_m^* \omega\otimes\omega$ spans $\mathscr{F}^2$ for $\mathscr{A}(m) \to Y(m)$.

From 5.4 and the previous example it follows that if

$$\alpha : \quad (R^1 f_* \mathbf{Z})^\vee \otimes \mathscr{O}_{Y(m)} \to (\mathscr{F}^1)^\vee$$

is the cyclic homomorphism corresponding to ω, then

$$(s^2\alpha)(j_m, 1)^* : \quad \mathbf{T}^\vee \otimes \mathcal{O}_{Y(m)} \to (\mathcal{F}^2)^\vee$$

is the cyclic homomorphism corresponding to $j_m^* \omega \otimes \omega = m (i_m^* \omega) \otimes \omega$ and hence

If $D_m = 0$ *is the Picard-Fuchs equation for* ω, *then*
$S^2 D_m = 0$ *is the Picard-Fuchs equation for* $i_m^* \omega \otimes \omega$.

A similar result is true for the family $\mathcal{B}(m) \to Z(m)$. To describe it, let

$$\varphi : \quad Y_0^0(m) \to Z(m)$$

be the unramified double cover (with involution i_m) and let $D_m^* = 0$ be the differential equation on $Z(m)$ which has the property that φ^* (local solution of $D_m^* = 0$) = local solution of $D_m = 0$. It is not difficult to show that the following holds.

The Picard-Fuchs equation for the image of $i_m^* \omega \otimes \omega$ *on* $Z(m)$ *is* $S^2 D_m^* = 0$.

6.6. Special case: $m = 6$. — The Picard-Fuchs equation for the modular family belonging to $\Gamma_0^0(6)$ has been given in 6.5.1. So D_6 is known now. I want to compute D_6^*.

So first I need to compute the double cover

$$\varphi : \quad Y_0^0(6) = \mathbf{P}_1 \setminus \{0, \infty, 1, 1/9\} \to Z(6).$$

Lemma. — *In the notation of* [R], p. 166 *the t-parameter on* $Y_0^0(6)$ *is nothing but* $\theta_2(0 \mid 3\tau)^4/\theta_2(0 \mid \tau)^4$ *and the involution* i_6 *covering* φ *is given by* $i_6(t) = (t - (1/9))/(t - 1)$. *So* $u = t(t - (1/9))/(t - 1)$ *uniformizes* $Z(6)$.

Proof (suggested by F. Beukers). — Using [R], p. 182-183 one verifies that $t^*(\tau) = \theta_2(0 \mid 3\tau)^4/\theta_2(0 \mid \tau)^4$ is a modular function for $\Gamma_0^0(6)$. From [K-F], II, p. 391 it follows that $t^*(\tau)$ equals their function $1/y(\tau)^2$ and since they have determined the values of $y(\tau)$ at the cusps [K-F], I, p. 685 it follows that $t^*(0) = 1/9$, $t^*(\infty) = 0$, $t^*(1/2) = \infty$ and $t^*(1/3) = 1$. So $t^* = t$ uniformizes $Y_0^0(6)$. Since $\tau \to -(6\tau)^{-1}$ permutes the cusps in the same manner as $(t - (1/9))/(t - 1)$ it follows that the involution i_6 is as stated.

It follows that φ is given as in Figure 1.

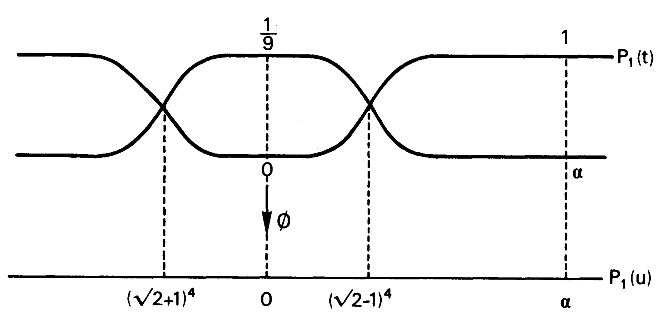


Fig. 1

6.6.2. COROLLARY. — $D_6^* = (u^2 - 34u + 1)u\, d^2/du^2 + (2u^2 - 5u + 1)\, d/du + 1/4(u - 10)$ *and*

$$S^2 D_6^* = (u^2 - 34u + 1)u^2 d^3/du^3 + (6u^2 - 153u + 3)u\, d^2/du^2 + (7u^2 - 112u + 1)\, d/du + (u - 5).$$

6.6.3. From 5.5.4 it now follows that.

The Picard-Fuchs equation for the global section of the Hodge bundle $\mathscr{F}^2$ for the projective family of K3-surfaces of transcendental type T from 5.5.2 is $S^2 D_6^* = 0$.

7. The family of K3-surfaces related to $\zeta(3)$

I use the notation from [B-P], p. 51-52. So X_t is the minimal resolution of singularities of the double cover of $\mathbf{P}_2$ branched along a certain sextic C_t having at most A-D-E singularities for $t \neq 0, 1, \infty, (\sqrt{2} \pm 1)^4$. The surface X_t is a K3-surface.

7.1. In this subsection the transcendental lattice of X_t for generic t will be determined.

7.1.1. PROPOSITION. — *For generic* $t \in \mathbf{P}_1$, *the Picard group of* X_t *is generated by the following* 19 *curves:*

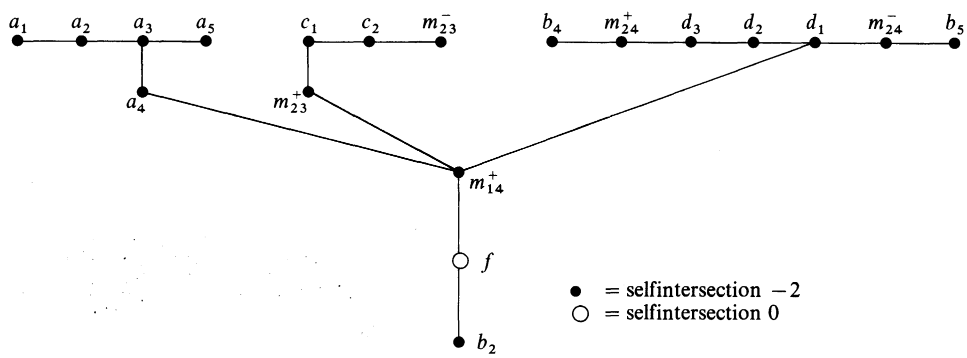
— *a generic member* F *of the elliptic pencil on* X_t *coming from the lines through* P_2*;*

— *the sections* M_{14}^+, B_2*;*

— *the components of the reducible members of the elliptic pencil that do not meet the zero-section* B_2.

The lattice generated by the curves is isometric to $\overset{2}{\perp} E_8(-1) \perp U \perp \langle -12 \rangle$.

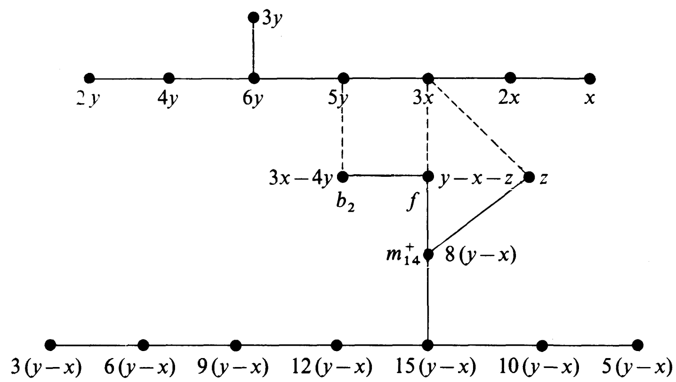
Proof. — Using small letters to denote the cohomology classes of the divisors with corresponding capital letters, the intersection graph of the 19 curves is as follows



The following classes form two disjoint E_8-configurations

$$\{b_4, m_{24}^+, d_3, d_2, d_1, m_{24}^-, b_5, m_{14}^+\} \quad \text{and}$$
$$\{a_1, a_2, a_3, a_5, a_5 - f, m_{23}^+ - f + c_1 - b_2, c_2, m_{23}^-\}.$$

Together with $\{b_2, f, m_{23}^+\}$ they span the same lattice as the curves I started with. They have the following intersection graph



Here dotted lines mean (-1)-intersections. The coefficient placed at the vertex are those that occur for the sublattice orthogonal to both E_8-configurations. These coefficients enter in the intersection matrix of this rank 3-lattice.

One finds that the intersection matrix is

$$(x, y, z)\begin{pmatrix} -4 & 8 & -3 \\ 8 & -12 & 4 \\ -3 & 4 & -2 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix}.$$

Since

$$\begin{pmatrix} -4 & -5 & -4 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}\begin{pmatrix} -4 & 8 & -3 \\ 8 & -12 & 4 \\ -3 & 4 & -2 \end{pmatrix}\begin{pmatrix} -4 & 1 & 1 \\ -5 & 1 & 1 \\ -4 & 0 & 1 \end{pmatrix}=\begin{pmatrix} -12 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

one sees that this lattice is isometric to $\langle -12 \rangle \perp U$. So the Picard lattice of X_t contains $\overset{2}{\perp} E_8(-1) \perp U \perp \langle -12 \rangle$. If the Picard number is 19, which is the case for generic t, this must be the entire Picard lattice. Indeed, since the lattice $\overset{2}{\perp} E_8(-1) \perp U \perp \langle -12 \rangle$ has discriminant 12, if $\operatorname{Pic} X_t \supsetneqq \overset{2}{\perp} E_8(-1) \perp U \perp \langle -12 \rangle$ but still rank $\operatorname{Pic} X_t = 19$, it would follow that $\operatorname{discr}(\operatorname{Pic} X_t) = 3$ and the discriminant form would be $q_{\pm 1}^{(3)}(3)$ (notation [N], p. 113). By [N], Proposition 1.11.2 such a form has signature $-2 \pm 4 \bmod 8$, whereas the signature of the Picard lattice is $1 - 18 \equiv -1 \bmod 8$, which is a contradiction. □

7.1.2. LEMMA. — *If* $\operatorname{Pic} X_t \cong \overset{2}{\perp} E_8(-1) \perp U \perp \langle -12 \rangle$, *the transcendental lattice is isometric to* $U \perp \langle 12 \rangle$.

Proof. — The discriminant form of $\mathrm{Pic}\, X_t$ is just $\langle -1/12 \rangle$, so the discriminant form of the transcendental lattice is $\langle 1/12 \rangle$. Since its signature is (2, 1), by [N], Corollary 1.13.3 there is only one possibility: the transcendental lattice is isometric to $U \perp \langle 12 \rangle$.

7.1.3. COROLLARY. — *For generic t, the transcendental lattice of* X_t *is isometric to* $U \perp \langle 12 \rangle$. □

7.1.4. COROLLARY. — *The family* $\{X_t\}$ *of* K3-*surfaces over* $\mathbf{P}_1(t) \setminus \{0, 1, \infty, (\sqrt{2} \pm 1)^4\}$ *is a family of* K3-*surfaces of transcendental type* (T, l'), $T = U \perp \langle 12 \rangle$, l' *some admissible vector in* $T^\perp$.

Proof. — In the notation of [B-P], p. 52 X_t is the double cover of a rational surfaces $P(= \mathbf{P}_2$ blown up in certain points) branched along a smooth curve $C_t^{(3)}$. Any ample line bundle on P lifts to an ample linebundle $\mathscr{L}_t$ on X_t and $c_1(\mathscr{L}_t) = l_t \in \mathrm{Pic}\, X_t$ is invariant under monodromy. Therefore, if $\gamma(t) : H^2(X_t, \mathbf{Z}) \xrightarrow{\sim} L$ is a marking sending the transcendental lattice into T the class $\gamma(t)\, l(t) = l' \in L$ is independent of t and (T, l') is admissible. By definition $\gamma(t)$ is a (T, l')-marking. □

7.2. Since $\{X_t\}$ has a (T, l')-marking one has a local system $\mathbf{T}'$ of rank-3 modules (with fibres $\cong T$) over $\mathbf{P}_1 \setminus \{0, 1, \infty, (\sqrt{2} \pm 1)^4\}$. The holomorphic 2-form on X_t gives a variation of Hodge structure on $\mathbf{T}'$. Its Picard-Fuchs equation is the one given as equation (3) in [B-P].

7.2.1. THEOREM. — *The variation of Hodge structure on* $\mathbf{T}'$ *just described is the same as the variation of Hodge structure for* $\mathscr{B}(6) \to Z(6)$, *restricted to* $Z(6) \setminus \{1\}$.

The monodromy group of this variation is therefore $\Gamma_0^0(6)^* \cong SO(T)$.

Proof. — The period map $\tau : \mathbf{P}_1 \setminus \{0, 1, \infty, (\sqrt{2} \pm 1)^4\} \to Z(6) = \mathbf{P}_1 \setminus \{0, \infty, (\sqrt{2} \pm 1)^4\}$ is the embedding and the restriction of the variation of Hodge structure from $\mathscr{B}(6) \to Z(6)$ to $Z(6) \setminus \{1\}$ is just the given variation on $\mathbf{T}$.

To see this, observe that the Picard-Fuchs equation for the variation of Hodge structure on $\mathbf{T}$ is just $S^2 D_6 = 0$, which in turn by 6.5.4 is the Picard-Fuchs equation for the family considered in 5.3.4. This family induces a variation of Hodge structure on the oriented local system $\mathbf{T}$ which is universal for variations of Hodge structure of type T on oriented local systems. From the observation it follows that $\mathbf{T}'$ is oriented and from universality it follows that $\mathbf{T}' = \tau^* \mathbf{T}$ and the variation of Hodge structure on $\mathbf{T}$ pulls back to the given one.

But then $\tau^* S^2 D_6^* = 0$ is the Picard-Fuchs equation for the variation on $\mathbf{T}'$. Since $\tau^* S^2 D_6^* = S^2 D_6^*$ it follows that τ must be the standard embedding.

7.2.2. *Remark.* — The missing point $1 \in Z(6)$ can be explained as follows. The polarizing class $\mathscr{L}_t$ degenerates for $t = 1$ to a "quasi-polarization", the corresponding double plane acquires a note and X_1 is a "generalized K3". This shows once more that it is more natural to consider generalized K3's instead of only smooth ones.

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C. A. M. PETERS,
Department of Mathematics,
University of Leiden,
P.O. Box 9512,
2300 RA Leiden,
The Netherlands.