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GLOBAL EXISTENCE OF SMOOTH SOLUTIONS FOR THE VLASOV-FOKKER-PLANCK EQUATION IN 1 AND 2 SPACE DIMENSIONS

BY PIERRE DEGOND

ABSTRACT. — In this paper, we propose a deterministic proof of the existence of global in time smooth solutions for the Vlasov-Fokker-Planck equations. The method relies on direct estimates of the decay of the solution when the velocity goes to infinity. It also yields a proof of the convergence of the solutions, towards those of the Vlasov-Poisson equation, when the diffusion coefficient goes to zero.

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I. Introduction

This paper deals with the Vlasov-Fokker-Planck equation. We will consider the (non physical) case of a plasma involving only one specie of particles. However, the method extends easily to the case of several species of particles.

We denote by $f(x, v, t)$ the distribution function of the particles (where $x \in \mathbb{R}^d$ is the position, $v \in \mathbb{R}^d$, the velocity, and $t > 0$, the time. We will denote by d the dimension of the system which will be equal to 1, 2 or 3).

The Vlasov-Fokker-Planck system is written

$$(1)\qquad \left\{\begin{aligned} &\frac{\partial f}{\partial t} + v\,.\,\nabla_x f + \mathrm{E}\,.\,\nabla_v f - \sigma\Delta_v f = 0; \qquad f(x, v, 0) = f_0(x, v)\\ &\mathrm{E}(x, t) = \mathrm{C}(d)\int \frac{x-y}{|x-y|^d}\rho(y, t)\,dy;\\ &\rho(x, t) = \int f(x, v, t)\,dv.\end{aligned}\right.$$

We denote respectively by $\mathrm{E}(x, t)$, $\rho(x, t)$, the electric field, and the electric charge. σ is a diffusion coefficient which, in numerous physical situations, is very small. $\mathrm{C}(d)$ is a constant which only depends on the dimension. Otherwise speciefield, all the integrals will be taken over $\mathbb{R}^d$ (either $\mathbb{R}_x^d$ or $\mathbb{R}_v^d$). All the physical constants are taken equal to unity. We recall the following notations

$$v.\nabla_x=\sum_{i=1}^{d} v_i\frac{\partial}{\partial x_i};\qquad \mathrm{E}.\nabla_v=\sum_{i=1}^{d}\mathrm{E}_i\frac{\partial}{\partial v_i}$$
$$\Delta_x=\sum_{i=1}^{d}\frac{\partial^2}{\partial x_i^2};\qquad \Delta_v=\sum_{i=1}^{d}\frac{\partial^2}{\partial v_i^2}.$$

We recall that the second equation of (1) is simply a restatement of the elliptic problem:

$$\mathrm{E}=-\nabla_x\varphi;\qquad -\Delta_x\varphi=4\pi\rho;\qquad \varphi(x, t)\to 0\quad \text{as } |x|\to\infty.$$

When σ goes to zero, we obtain formally the classical Vlasov-Poisson equations

$$(2)\qquad \left\{\begin{aligned} &\frac{\partial f}{\partial t}+v.\nabla_x f+\mathrm{E}.\nabla_v f=0;\qquad f(x, v, 0)=f_0(x, v)\\ &\mathrm{E}(x, t)=\mathrm{C}(d)\int\frac{x-y}{|x-y|^d}\rho(y, t)\,dy;\\ &\rho(x, t)=\int f(x, v, t)\,dv.\end{aligned}\right.$$

Many authors have worked on the Vlasov-Poisson equations. We recall that the existence of weak solutions is proved in [1], [8]. The existence of smooth solutions is examined in [9], for the dimension 1 in [13], [7], [14] for the dimension 2, and in [3] for the dimension 3.

On the contrary, very few papers have been published on the Vlasov-Fokker-Planck equations. As far as I know, the only one is due to H. Neunzert, M. Pulvirenti, and L. Triolo [11], who have used a probabilistic method to prove the global existence of smooth solutions in 1 and 2 dimensions. In this paper, we propose a fully deterministic proof of this result, and we also show that the solutions of equation (1) converge, when σ goes to zero, towards the solutions of the Vlasov-Poisson equation (2).

The results of [11], and our result have to be compared to Ukai and Okabe's paper [13], where the global existence of smooth solutions for the Vlasov-Poisson system (2) is proved in 1 and 2 dimensions. (Their method also yields a local existence result if $d=3$.) What our result shows is that a singular perturbation term such as a Laplacien in the velocity space does not perturb too much the Vlasov equation. One may also think that such a term should improve the regularity of the solution, and consequently, may also improve the existence results.

This may be a wrong idea. Indeed, the key fact is to obtain an estimate on the L^∞ norm of the charge ρ, so as to control the electric field. However, no direct L^∞ estimate can be obtained, since integrating equations (1) or (2) with respect to v, leads to the fluid equations. It is well known that the fluid equations do not constitute a closed system of equations. And, even with a closure assumption, it is often very difficult to obtain an L^∞ estimate.

In [13], Ukai and Okabe use the characteristics to reduce the problem to the decay of the initial data f_0. In [11], H. Neunzert, M. Pulvirenti and L. Triolo use a probabilistic approach to give a meaning to the notion of characteristic, for equation (1). (Indeed, the $\Delta_v f$ term perturbes the classical characteristics with a Brownian motion. This interpretation is, in some sense, very close to the physics). Then, they can extend the proof of Ukai and Okabe.

In this paper, we deliberately turn our back to the method of characteristics, in order to give a fully deterministic proof. The key idea is that the maximum principle yields an estimate of

$$\operatorname*{Max}_{(x,\, v)}((1+|v|^2)^{\gamma/2} f(x, v, t)).$$

which provides the required estimate of ρ, if γ is sufficiently large. Another tool which becomes classical in this field, is the use of interpolation estimates. The method also provides a straightforward proof of the convergence of the solutions of (1), towards those of (2), when σ goes to zero.

We now state the theorems: Theorem I.1 gives the existence and uniqueness result; Theorem I.2, a regularity result (whose proof will be omitted, *see* remark IV.2), and Theorem I.3, the convergence when σ goes to zero. (For the meaning of the notations, see the end of this paragraph.)

THEOREM 1.1. — *We suppose that the initial data $f_0(x, v)$ (for $x \in \mathbb{R}^d$ and $v \in \mathbb{R}^d$), is nonnegative and satisfies with a real $\gamma > d$.*

$$f_0 \in W^{1,1}(\mathbb{R}^{2d}); \qquad (1+|v|^2)^{\gamma/2}(|f_0|+|Df_0|) \in L^\infty(\mathbb{R}^{2d}) \tag{3}$$

Then, the Vlasov-Fokker-Planck system

$$\begin{cases} \dfrac{\partial f}{\partial t}+v.\nabla_x f+E.\nabla_v f-\sigma\Delta_v f=0; \qquad f(x, v, 0)=f_0(x, v) \\ E(x, t)=C(d)\displaystyle\int \frac{x-y}{|x-y|^d}\rho(y, t)\,dy; \\ \rho(x, t)=\displaystyle\int f(x, v, t)\,dv \end{cases} \tag{4}$$

admits a classical solution, in a time interval $[0, T[$ *such that:* $T=\infty$ *if* $d=1$ *or* 2 *(f is a global solution);* T *is finite and depends on* f_0 *if* $d=3$, *but can be chosen independent of* σ, *if* σ *ranges in a bounded interval* $]0, \sigma_0]$.

This solution is such that

$$f \geqq 0; \qquad f \in L^\infty_{\mathrm{loc}}([0, T[, W^{1,1}(\mathbb{R}^{2d})) \tag{5}$$

$$(1+|v|^2)^{\gamma/2}(|f|+|Df|) \in L^\infty_{\mathrm{loc}}([0, T[, L^\infty(\mathbb{R}^{2d})) \tag{6}$$

$$\Delta_v f \in L^2_{\text{loc}}([0, T[, L^2(\mathbb{R}^{2d})) \tag{7}$$

$$E \in L^\infty_{\text{loc}}([0, T[, W^{1,\infty}(\mathbb{R}^d_x)); \qquad \frac{\partial f}{\partial t} \in L^2_{\text{loc}}([0, T[\times \mathbb{R}^{2d}). \tag{8}$$

Two solutions of equation (4), *which satisfy* (5) *to* (8), *must coincide.*

THEOREM 1.2. — *We assume that f_0 is nonnegative and satisfies*

$$f_0 \in W^{m,1}(\mathbb{R}^{2d}); \qquad (1+|v|^2)^{\gamma/2}(|f_0|+\ldots+|D^m f_0|) \in L^\infty(\mathbb{R}^{2d}) \tag{9}$$

with $m \geqq 1$ and $\gamma > d$. Then the solution obtained in Theorem 1.1 *verifies*

$$f \in L^\infty_{\text{loc}}([0, T[, W^{m,1}(\mathbb{R}^{2d}))$$
$$(1+|v|^2)^{\gamma/2}(|f|+\ldots+|D^m f|) \in L^\infty_{\text{loc}}([0, T[, L^\infty(\mathbb{R}^{2d}))$$
$$\nabla_v(D^m f) \in L^2_{\text{loc}}([0, T[, L^2(\mathbb{R}^{2d}))$$
$$E \in L^\infty_{\text{loc}}([0, T[, W^{m,\infty}(\mathbb{R}^d_x)).$$

THEOREM 1.3. — *We suppose that f_0 is nonnegative and satisfies:*

$$f_0 \in W^{2,1}(\mathbb{R}^{2d}); \qquad (1+|v|^2)^{\gamma/2}(|f_0|+|Df_0|+|D^2 f_0|) \in L^\infty(\mathbb{R}^{2d}), \ \gamma > d \tag{10}$$

We denote by f^σ the solution of (1) *and f, the solution of* (2). *Then for any finite time interval* $[0, T^*]$ (*with* $T^* < T$ *if* $d=3$), *f^σ converges to f in the following sense.*

$$\operatorname*{Max}_{[0,\, T^*]}\left(\|(f-f^\sigma)(t)\|_1 + \|(1+|v|^2)^{\gamma/2}(f-f^\sigma)(t)\|_\infty + \|(E-E^\sigma)(t)\|_\infty\right) = O(\sigma).$$

The outline of the paper is the following: Paragraph II presents the iteration method on which the proof is based. Paragraph III gives the fundamental estimates (using the maximum principle) which allow the convergence of the procedure in paragraph IV. Paragraph V is devoted to the approximation result, when σ goes to zero. Then, in a somewhat lengthy appendix we prove some results on the linear Fokker-Planck equation. Indeed, we have been unable to find these results in the litterature; so, we have given them for the sake of completeness. In another short appendix, we prove the interpolation inequalities which are the second main tool of the proof.

We complete this introduction with some notations. For a function $u(y)$, $(y \in \mathbb{R}^k)$, we denote by $\| \ \|_{m,p}$ the usual $W^{m,p}(\mathbb{R}^k)$-norm, and by $\| \ \|_p$, the L^p norm. If u is a function of the pair $(x, v) \in \mathbb{R}^d \times \mathbb{R}^d$, we denote by $\nabla_x u$, $\nabla_v u$ (without dots) the partial derivatives of u with respect to x or v, and by Du the total derivative [with respect to (x, v)]. Consequently, $D^m u$ will be the m-th total derivative of u.

II. Construction of an iterative sequence

The proof will be based on the following approximation scheme: we initiate it with the zero functions. Then if we assume that the electric field $E^n(x, t)$ is known, and belongs to $L^\infty_{loc}([0, \infty[, W^{1,\infty}(\mathbb{R}^d_x))$, we can solve the linear Fokker-Planck equation (*see* appendix A):

$$\left\{\begin{aligned} &\frac{\partial f^{n+1}}{\partial t}+v.\nabla_x f^{n+1}+E^n.\nabla_v f^{n+1}-\sigma\Delta_v f^{n+1}=0;\\ &f^{n+1}(x, v, 0)=f_0(x, v).\end{aligned}\right. \tag{11}$$

Then, we can compute the charge ρ^{n+1} and the electric field E^{n+1} according to

$$\rho^{n+1}(x, t)=\int f^{n+1}(x, v, t)\,dv; \qquad E^{n+1}(x, t)=C(d)\int \frac{x-y}{|x-y|^d}\rho^{n+1}(y, t)\,dy.$$

Now, Appendix A gives the existence and uniqueness of the solution f^{n+1} of equation (11), and the following estimates.

$$f^{n+1}\geqq 0; \qquad \|\rho^{n+1}(t)\|_1=\|f^{n+1}(t)\|_1\leqq\|f_0\|_1; \qquad \|f^{n+1}(t)\|_\infty\leqq\|f_0\|_\infty. \tag{12}$$

Unfortunately, estimates on the derivatives are also needed, to obtain the strong convergence of f^n. Furthermore, the use of interpolation estimates (66) and (67) requires L^∞ estimates on ρ^n and $\nabla_x\rho^n$. As we explained in the introduction, these will be obtained though some estimates on the decay at infinity, in the velocity space, of f^n. This is the aim of the next paragraph.

Remark 2.1. — When $\sigma=0$, (11) becomes a classical linear transport equation, with regular coefficients. It can be solved by means of characteristics, and estimates (12) are obvious.

III. The basic estimates

We define

$$Y^n(x, v, t)=(1+|v|^2)^{\gamma/2}f^n(x, v, t); \qquad Z^n(x, v, t)=(1+|v|^2)^{\gamma/2}\,Df^n(x, v, t)$$

for which we prove the:

LEMMA 3.1. — *We assume that f_0 satisfies the hypotheses of Theorem 1.1, and that σ ranges in a bounded interval $]0, \sigma_0]$. Then, there exist two functions $\alpha(t)$ $\beta(t)$ independent of n and of σ, such that*

$$\left\{\begin{aligned} &\alpha, \beta\in L^\infty_{loc}([0, \infty[) \quad \text{if } d=1 \text{ or } 2\\ &\alpha, \beta\in L^\infty_{loc}([0, T[) \quad \text{if } d=3, \qquad T=T(f_0, \sigma_0, \gamma)\end{aligned}\right. \tag{13}$$

and such that for every n and t, we have

$$\|Y^n(t)\|_\infty\leqq\alpha(t); \qquad \|Z^n(t)\|_\infty\leqq\beta(t). \quad \blacksquare \tag{15}$$

The proof relies on the maximum principle for the linear Fokker-Planck equation, which is proved in appendix A. We apply it to the equations solved by Y^n and Z^n. We also widely use, some interpolation estimates shown in appendix B. (The logarithmic estimate (67) for $\| \nabla_x E \|_\infty$ is very close to that proved by T. Beale, T. Kato and A. Majda [4].)

Proof of Lemma 3.1. — We will omit the superscripts n and $n+1$, when the context will be clear.

STEP 1. — *A Gronwall inequality for* Y_n. — Multiplying equation (1) by $(1+|v|^2)^{\gamma/2}$ we easily get

$$\frac{\partial Y}{\partial t}+v.\nabla_x Y+E.\nabla_v Y-\sigma\Delta_v Y \tag{15}$$
$$=\gamma(1+|v|^2)^{(\gamma-2)/2}(v.E)f-2\sigma\gamma(1+|v|^2)^{(\gamma-2)/2}v.\nabla_v f$$
$$-\sigma\gamma(\gamma-2)(1+|v|^2)^{(\gamma-4)/2}|v|^2 f-3\sigma\gamma(1+|v|^2)^{(\gamma-2)/2}f.$$

But we have:

$$-2\sigma\gamma(1+|v|^2)^{(\gamma-2)/2}v.\nabla_v f=-\frac{2\sigma\gamma}{1+|v|^2}v.\nabla_v Y+2\sigma\gamma^2(1+|v|^2)^{(\gamma-4)/2}f.$$

so that equation (15) can be rewritten

$$\frac{\partial Y}{\partial t}+v.\nabla_x Y+\left(E+2\sigma\gamma\frac{v}{1+|v|^2}\right)\nabla_v Y-\sigma\Delta_v Y=R_1+R_2$$

$$\begin{aligned} R_1&=\gamma(E.v)(1+|v|^2)^{(\gamma-2)/2}f\\ R_2&=\left(\sigma\gamma(\gamma+2)\frac{|v|^2}{1+|v|^2}-3\sigma\gamma\right)(1+|v|^2)^{(\gamma-2)/2}f.\end{aligned} \tag{16}$$

Now, thanks to the hypotheses on the initial data, we can use the L^∞ estimate (45), and we obtain:

$$\| Y^{n+1}(t)\|_\infty \leqq \| Y_0\|_\infty+\int_0^t (\| R_1^{n+1}(s)\|_\infty+\| R_2^{n+1}(s)\|_\infty)\,ds \tag{17}$$

But

$$\| R_2^{n+1}(s)\|_\infty \leqq C(\gamma).\sigma_0 \|(1+|v|^2)^{(\gamma-2)/2}f^{n+1}(s)\|_\infty \leqq C(\gamma)\,\sigma_0\,\| Y^{n+1}(s)\|_\infty.$$

And

$$\| R_1^{n+1}(s)\|_\infty \leqq \gamma \| E^n(s)\|_\infty \|(1+|v|^2)^{(\gamma-1)/2}f^{n+1}(s)\|_\infty.$$

Then we use the interpolation results (64), (65) and (66) of appendix B and the inequalities (12) and we get

$$\begin{aligned}\| E^n(s)\|_\infty &\leqq C(d)\,\|\rho^n(s)\|_1^{1/d}\,\|\rho^n(s)\|_\infty^{(d-1)/d}\\ &\leqq C(d,\gamma)\,\| f^n(s)\|_1^{1/d}\,\| f^n(s)\|_\infty^{(d-1)(\gamma-d)/d\gamma}\,\| Y^n(s)\|_\infty^{(d-1)/\gamma}\end{aligned}$$

$$\leqq C(d, \gamma, f_0) \| Y^n(s) \|_\infty^{(d-1)/\gamma}$$

$$\| (1+|v|^2)^{(\gamma-1)/2} f^{n+1}(s) \|_\infty \leqq C(\gamma) \| f^{n+1}(s) \|_\infty^{1/\gamma} \| Y^{n+1}(s) \|_\infty^{1-1/\gamma}$$
$$\leqq C(\gamma, f_0) \| Y^{n+1}(s) \|_\infty^{1-1/\gamma}.$$

So (17) becomes the expected Gronwall inequality [1]:

$$(18) \quad \| Y^{n+1}(t) \|_\infty \leqq C_1 + C_2 \int_0^t \| Y^{n+1}(s) \|_\infty ds + C_3 \int_0^t \| Y^{n+1}(s) \|_\infty^{1-1/\gamma} \| Y^n(s) \|_\infty^{(d-1)/\gamma} ds.$$

Intuitively, we see that this Gronwall is at most linear if

$$(d-1)/\gamma \leqq 1/\gamma; \text{ i. e.: } d \leqq 2.$$

STEP 2. — *Estimate of* Y_n.

We put $y_n(t) = \text{Max}(1, \| Y_n(t) \|_\infty)$.

In the case $d \leqq 2$, (18) simplifies into:

$$(19) \qquad y_{n+1}(t) < C_1 + C_2 \int_0^t y_{n+1}(s)\, ds + C_3 \int_0^t y_{n+1}^{1-1/\gamma}(s)\, y_n^{1/\gamma}(s)\, ds.$$

Now, let $\alpha(t)$ be the solution of the linear equation

$$\dot{\alpha}(t) = (C_2 + C_3)\, \alpha(t); \qquad \alpha(0) = C_1.$$

Then we prove by induction on n that we have

$$(20) \qquad y_n(t) \leqq \alpha(t); \qquad \forall t \geqq 0, \quad \forall n \in \mathrm{N}.$$

Indeed, denoting by $\psi_{n+1}(t)$ the right hand side of (19) we prove that an upper bound T for the set

$$\{ t \in \mathbb{R}_+ / \psi_{n+1}(s) \leqq \alpha(s), \qquad \forall s \in [0, t[\}$$

does not exist. If the converse were true, we would have

$$\dot{\psi}_{n+1}(T) = C_2\, y_{n+1}(T) + C_3\, y_{n+1}^{1-1/\gamma}(T)\, y_n^{1/\gamma}(T) < C_2\, \alpha(T) + C_3\, \alpha(T) = \dot{\alpha}(T)$$

which shows a contradiction. (20) is the desired estimate.

In the case $d = 3$, (18) leads to:

$$y_{n+1}(t) < C_2 \int_0^t y_{n+1}^{1+1/\gamma}(s)\, ds + C_3 \int_0^t y_{n+1}^{1-1/\gamma}(s)\, y_n^{2/\gamma}(s)\, ds.$$

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[1] From now on, C_i will denote constants depending only on d, γ, f_0 and σ_0.

And the same reasoning can be applied with a function $\alpha(t)$ solution of

$$\dot{\alpha}(t)=(C_2+C_3)\,\alpha(t)^{1+1/\gamma};\qquad \alpha(0)=C_1.$$

which exists during a time interval [0, T[, where T depends on f_0, γ and σ_0 (through C_1, C_2 and C_3).

STEP 3. — *Estimate on* Z_n.

We differenciate equation (1) with respect to (x, v). Considering $\mathrm{D}f$ as a vector ${}^t(\nabla_x f, \nabla_v f)$, we obtain the vectorial equation:

$$\frac{\partial}{\partial t}(\mathrm{D}f)+v.\nabla_x(\mathrm{D}f)+E.\nabla_v(\mathrm{D}f)-\sigma\Delta_v(\mathrm{D}f)=-\mathrm{A}.\mathrm{D}f \tag{21}$$

where A is the 6×6 matrix decomposed in 3×3 blocks:

$$\mathrm{A}=\begin{pmatrix}0 & \mathrm{Id}\\ \nabla_x \mathrm{E} & 0\end{pmatrix}$$

Now, we multiply equation (21) by $(1+|v|^2)^{\gamma/2}$ and get the following equation for Z:

$$\frac{\partial \mathrm{Z}}{\partial t}+v\nabla_x \mathrm{Z}+\left(\mathrm{E}+2\sigma\gamma\frac{v}{1+|v|^2}\right).\nabla_v \mathrm{Z}-\sigma\Delta_v \mathrm{Z}=\mathrm{S}_1+\mathrm{S}_2+\mathrm{S}_3, \tag{22}$$

where S_1 and S_2 are obtained from (16) by replacing f by $\mathrm{D}f$, and

$$\mathrm{S}_3=-(1+|v|^2)^{\gamma/2}\,\mathrm{A}.\mathrm{D}f.$$

Now from the previous steps, we get:

$$\begin{aligned}\|\mathrm{S}_1^{n+1}(t)\|_\infty &\leqq \mathrm{C}(\gamma)\,\|\mathrm{E}^n(t)\|_\infty\,\|(1+|v|^2)^{(\gamma-1)/2}\,\mathrm{D}f^{n+1}(t)\|_\infty\\ &\leqq \mathrm{C}(\gamma, d, f_0)\,\alpha(t)^{(d-1)/\gamma}\,\|\mathrm{Z}^{n+1}(t)\|_\infty\end{aligned} \tag{23}$$

$$\begin{aligned}\|\mathrm{S}_2^{n+1}(t)\|_\infty &\leqq \sigma_0\,\mathrm{C}(\gamma)\,\|\mathrm{Z}^{n+1}(t)\|_\infty\\ \|\mathrm{S}_3^{n+1}(t)\|_\infty &\leqq \|\mathrm{A}^n(t)\|_\infty\,\|\mathrm{Z}^{n+1}(t)\|_\infty.\end{aligned} \tag{24}$$

In the sequel, we will denote by $\psi_i(t)$ some known functions depending only on d, γ, σ_0 and f_0, and satisfying (13). (ψ_i will generally be obtained from α.)

We apply the logarithmic identity (67), and obtain

$$\begin{aligned}\|\mathrm{A}^n(t)\|_\infty &\leqq \mathrm{C}(d)\,(1+\|\rho^n(t)\|_\infty\,(1+\mathrm{Log}(1+\|\nabla_x\rho^n(t)\|_\infty))+\|\rho^n(t)\|_1)\\ &\leqq \mathrm{C}(d, \gamma, f_0)\,(1+\alpha(t)^{d/\gamma}\,(1+\mathrm{Log}(1+\|\nabla_x\rho^n(t)\|_\infty)))\\ &\leqq \psi_1(t)\,(1+\mathrm{Log}(1+\|\nabla_x\rho^n(t)\|_\infty)).\end{aligned} \tag{25}$$

But we have

$$\|\nabla_x\rho^n(t)\|_\infty \leqq \left\|\int|\nabla_x f^n(t)|\,dv\right\|_\infty \leqq \mathrm{C}(\gamma, d)\,\|\mathrm{Z}^n(t)\|_\infty. \tag{26}$$

So, the maximum principle (45) applied to (22) and the estimates (23) to (26) lead to

$$(27)\quad \|Z^{n+1}(t)\|_\infty \leqq \|Z_0\|_\infty + \int_0^t \psi_2(s)\|Z^{n+1}(s)\|_\infty\,ds + \int_0^t \psi_3(s)(1+\mathrm{Log}(1+\|Z^n(s)\|_\infty))\|Z^{n+1}(s)\|_\infty\,ds.$$

Now we introduce the function

$$z_n(t) = \mathrm{Max}(1, \|Z^n(t)\|_\infty)$$

which satisfies the inequality (from 27)

$$z_{n+1}(t) < C + \int_0^t \psi_4(s)\, z_{n+1}(s)\,\mathrm{Log}\, z_n(s)\,ds.$$

and we denote by $\beta(t)$ the solution of the differential equation

$$\dot\beta(t) = \psi_4(t)\,\beta(t)\,\mathrm{Log}\,\beta(t); \qquad \beta(0) = C$$

whose solution is

$$\beta(t) = \exp\left((\mathrm{Log}\, C)\exp\int_0^t \psi_4(s)\,ds\right).$$

We see that β satisfies (13), and the same argument as for step 2 proves that Z^n satisfies

$$\|Z^n(t)\|_\infty \leqq \beta(t), \qquad \forall n \in \mathrm{N}, \quad \forall t \text{ in} \begin{cases} \mathbb{R}_+ & \text{if } d=1 \text{ or } 2 \\ [0, \mathrm{T}[& \text{if } d=3. \end{cases}$$

So Lemma 3.1 is proved. ■

COROLLARY 3.1. — *There exists a function* $\tilde\alpha(t)$ *depending only on* d, γ, f_0 *and* σ_0, *satisfying* (13), *and such that:*

$$(28)\qquad \|\rho^n(t)\|_\infty + \|\nabla_x \rho^n(t)\|_\infty + \|\mathrm{E}^n(t)\|_\infty + \|\nabla_x \mathrm{E}^n(t)\|_\infty \leqq \tilde\alpha(t)$$

$$(29)\qquad \|\mathrm{D}f^n(t)\|_1 \leqq \tilde\alpha(t) \quad ■$$

Proof. — (29) alone, is not obvious. Going back to equation (21), and applying estimate (46), leads to

$$\|\mathrm{D}f^{n+1}(t)\|_1 \leqq \|\mathrm{D}f_0\|_1 + \int_0^t \|\mathrm{A}^n.\mathrm{D}f^{n+1}(s)\|_1\,ds.$$

But thanks to (28), $\|\mathrm{A}^n(s)\|_\infty$ is bounded by $\tilde\alpha(s)$. So $\|\mathrm{D}f^{n+1}(t)\|_1$ satisfies a linear Gronwall inequality whose coefficients are independent of n. This gives (29). ■

Remark 3.1. — Lemma 3.1 and Corollary 3.1 are also true for $\sigma=0$. (*See* remark 3.2).

IV. Convergence of iteration scheme end of the proof of theorem 1.1

In this paragraph, we complete the proof of Theorem 1.1 by successively showing the convergence of the iterations towards a weak solution, the regularity of this solution, and its uniqueness.

END OF THE PROOF OF THEOREM 1.1.

STEP 1. — *Convergence of the iterative sequence.*

We introduce an arbitrary finite time T^* ($T^* < T$ if $d=3$). Thanks to estimates (14), we get the following convergences (of subsequences) in the weak star topology of $L^\infty([0, T^*] \times \mathbb{R}^{2d})$:

$$\begin{cases} f^n \rightharpoonup f;\ (1+|v|^2)^{\gamma/2} f^n \rightharpoonup (1+|v|^2)^{\gamma/2} f \\ (1+|v|^2)^{\gamma/2} \mathrm{D} f^n \rightharpoonup (1+|v|^2)^{\gamma/2} \mathrm{D} f \end{cases} \tag{30}$$

and, in $L^\infty([0, T^*] \times \mathbb{R}_x^d)$ weak star:

$$\mathrm{E}^n \rightharpoonup \mathrm{E}; \qquad \nabla_x \mathrm{E}^n \rightharpoonup \nabla_x \mathrm{E}. \tag{31}$$

To pass to the limit in the non linear term of equation (1), we need a strong convergence. We will prove that f^n converges to f in the norm of $L^\infty([0, T^*], L^1(\mathbb{R}^{2d}))$. Indeed, $(f^{n+1}-f^n)$ solves the equation:

$$\frac{\partial}{\partial t}(f^{n+1}-f^n) + v.\nabla_x(f^{n+1}-f^n) + \mathrm{E}^n.\nabla_v(f^{n+1}-f^n) - \sigma\Delta_v(f^{n+1}-f^n) = -(\mathrm{E}^n-\mathrm{E}^{n-1}).\nabla_v f^n.$$

Now, thanks to the L^1 estimate (46) we obtain

$$\begin{aligned} \iint |(f^{n+1}-f^n)(t)|\,dx\,dv & \leqq \int_0^t \iint |\nabla_v f^n(x, v, s)|\,|(\mathrm{E}^n-\mathrm{E}^{n-1})(x, s)|\,dx\,dv\,ds \\ & \leqq \mathrm{C} \int_0^t \iiint \frac{1}{|x-y|^{d-1}} |\nabla_v f^n(x, v, s)|\,|\rho^n(y, s)-\rho^{n-1}(y, s)|\,dx\,dy\,dv\,ds. \end{aligned} \tag{32}$$

But, using inequality (66), Lemma 3.1 and Corollary 3.1 we obtain:

$$\operatorname{Max}_y \left(\iint \frac{1}{|x-y|^{d-1}} |\nabla_v f^n(x, v, s)|\,dx\,dv \right) \tag{33}$$

$$\leqq \left(\iint |\nabla_v f^n(x, v, s)|\, dx\, dv \right)^{1/d} \left(\operatorname*{Max}_x \int |\nabla_v f^n(x, v, s)|\, dv \right)^{(d-1)/d}$$

$$\leqq C(\gamma, d) \| D f^n(s) \|_1^{1/d} \| Z^n(s) \|_\infty^{(d-1)/d} \leqq C(\gamma, d, T^*)$$

So (32) leads to

$$(34) \quad \|(f^{n+1}-f^n)(t)\|_1 \leqq C \int_0^t \int |\rho^n(y, s) - \rho^{n-1}(y, s)|\, dy\, ds \leqq C \int_0^t \|(f^n - f^{n-1})(s)\|_1\, ds.$$

where C is independent of n. So $(f^{n+1}-f^n)$ satisfies:

$$\|(f^{n+1}-f^n)(t)\|_1 \leqq \frac{C^n t^n}{n!} \operatorname*{Max}_{t \in [0, T^*]} \|(f^1 - f^0)(t)\|_1$$

which proves that f^n converges in $L^\infty([0, T^*], L^1(\mathbb{R}^{2d}))$ to a unique limit which coincides with the function f found in (30). Furthermore, since T^* is arbitrary, f exists in $L^\infty_{loc}([0, \infty[, L^1(\mathbb{R}^{2d}))$ for $d=1$ or 2, and in $L^\infty_{loc}([0, T[, L^1(\mathbb{R}^{2d}))$ for $d=3$. It is then easy to prove that f is a weak solution of equation (1).

In the remaining part of the proof, we shall consider the case $d=1$ or 2. For $d=3$, the only change is that $[0, \infty[$ must be replaced by $[0, T[$.

STEP 2. — *Regularity of the solution.*

Thanks to the preceding step, we have:

$$(35) \quad \begin{cases} f \in L^\infty_{loc}([0, \infty[, L^1(\mathbb{R}^{2d}) \cap L^\infty(\mathbb{R}^{2d})) \\ (1+|v|^2)^{\gamma/2}(|f| + |Df|) \in L^\infty_{loc}([0, \infty[, L^\infty(\mathbb{R}^{2d})). \end{cases}$$

Furthermore, estimate (29) shows that Df^n is bounded in $L^\infty_{loc}([0, \infty[, L^1(\mathbb{R}^{2d}))$. So, for almost every t, $Df(t)$ is a bounded measure of $\mathbb{R}^{2d}$, and since it is a function, we get:

$$(36) \quad Df \in L^\infty_{loc}([0, \infty[, L^1(\mathbb{R}^{2d})).$$

Then, the charge $\rho(x, t) = \int f(x, v, t)\, dv$ satisfies

$$\rho \in L^\infty_{loc}([0, \infty[, W^{1,1}(\mathbb{R}^d_x) \cap W^{1,\infty}(\mathbb{R}^d_x)).$$

So, the electric field E found in (31) solves the second equation (1) in a classical sense, and satisfies

$$(37) \quad E \in L^\infty_{loc}([0, \infty[, W^{1,\infty}(\mathbb{R}^d_x)).$$

At last, f is nonnegative, since f^n is nonnegative for every n.

Now, we prove that $\Delta_v f$, (which is, up to now, a distribution) is actually a function such that

$$(38) \quad \Delta_v f \in L^2_{loc}([0, \infty[, L^2(\mathbb{R}^{2d}))$$

For that purpose, we compute the (vectorial) equation solved by the vector $\nabla_v f$:

$$\frac{\partial}{\partial t}(\nabla_v f)+v.\nabla_x(\nabla_v f)+\mathrm{E}.\nabla_v(\nabla_v f)-\sigma\Delta_v(\nabla_v f)=-\nabla_x f$$

$$\nabla_v f(x, v, 0)=\nabla_v f_0(x, v)$$

and we use the regularity proposition A 2. We easily check, thanks to hypothesis (7) and to (35), (36) and (37), that the hypotheses of proposition A.2 are verified. So the L^2 function $\nabla_v f$ actually belongs to $\mathrm{L}^2([0, \mathrm{T}]\times\mathbb{R}^3_x, \mathrm{H}^1(\mathbb{R}^3_v))$ for any arbitrary T. In particular, this proves (38).

Then, applying equation (1) proves that $\partial f/\partial t$ belongs to $\mathrm{L}^2_{\mathrm{loc}}([0, \infty[\times\mathbb{R}^d_x\times\mathbb{R}^d_v)$, and that the equation (1) is satisfied almost everywhere. This shows the regularity of the solution.

STEP 3. — *Uniqueness.*

Let (f, E) and $(\tilde{f}, \tilde{\mathrm{E}})$ be two solutions satisfying (35) and (36), with the same initial data f_0. The proof of estimate (34) can be adapted in a straightforward way to prove that

$$\|(f-\tilde{f})(t)\|_1 \leqq \mathrm{C}\int_0^t \|(f-\tilde{f})(s)\|_1\, ds.$$

Thus, $\|(f-\tilde{f})(t)\|_1=0$ for every t.

Remark 4.1. — All this proof can be achieved for the Vlasov-Poisson equation (i.e. $\sigma=0$) since the required estimates are true. (*See* remarks 2.1 and 3.1). The only property which is not true in this case is the regularity of $\Delta_v f$ [*see* (38)], but it is not needed to give a classical meaning to the Vlasov equation solved by f. Furthermore, a regularity theorem, such as Theorem 1.2 is also true.

Remark 4.2. — We will not give the proof of the regularity Theorem 1.2. The ideas are obvious but lead to somewhat tedious calculations. We must differentiate equation (1) a certain number of times, and proceed by induction to give a linear Gronwall inequality for the derivatives. This type of proof is chieved in [5] for the Vlasov-Maxwell equation. [The hypotheses that f_0 should be compactly supported can be easily replaced by the assumptions (9).]

V. Convergence of the Vlasov-Fokker-Planck equation towards the Vlasov-Poisson equation when σ goes to zero

Proof of Theorem 1-3. — We consider a finite time interval [0, T*], with $\mathrm{T}^*<\mathrm{T}$ in the case $d=3$. We denote by $(f^\sigma, \mathrm{E}^\sigma)$ the solution of the Vlasov-Fokker-Planck equation (1), with the same initial data f_0, satisfying (10).

$X^\sigma = f - f^\sigma$ solves the equation:

$$\frac{\partial X^\sigma}{\partial t} + v.\nabla_x X^\sigma + E.\nabla_x X^\sigma - \sigma\Delta_v X^\sigma = -\sigma\Delta_v f - (E - E^\sigma).\nabla_v f^\sigma. \tag{39}$$

Since f_0 satisfies (10), we have

$$\Delta_v f \in L^\infty([0, T^*], L^1(\mathbb{R}^{2d})); \ (1+|v|^2)^{\gamma/2}\Delta_v f \in L^\infty([0, T^*], L^\infty(\mathbb{R}^{2d}))$$

Now, using the same method as for inequality (33), and the fact that estimates (14) and (29) are independent of σ in a range $]0, \sigma_0]$, we obtain:

$$\begin{aligned}
&\|(E-E^\sigma).\nabla_v f^\sigma(t)\|_1 \\
&\qquad \leqq C(\gamma, d)\|(\rho-\rho^\sigma)(t)\|_1 \|\nabla_v f^\sigma(t)\|_1^{1/d} \|(1+|v|^2)^{\gamma/2}\nabla_v f^\sigma\|_\infty^{(d-1)/d} \\
&\qquad \leqq C(\gamma, d, T^*, f_0)\|(\rho-\rho^\sigma)(t)\|_1.
\end{aligned}$$

So estimate (46) applied to (39) gives:

$$\|(f-f^\sigma)(t)\|_1 \leqq \sigma\int_0^t \|\Delta_v f^\sigma(s)\|_1\, ds + C\int_0^t \|(f-f^\sigma)(s)\|_1\, ds$$

which proves that

$$\operatorname*{Max}_{t\in[0,\,T^*]} \|(f-f^\sigma)(t)\|_1 \leqq C(\gamma, d, T^*, f_0, \sigma_0)\,\sigma.$$

Now, we consider $Y^\sigma = (1+|v|^2)^{\gamma/2} X^\sigma$. Y^σ is a solution of the equation:

$$\frac{\partial Y^\sigma}{\partial t} + v.\nabla_x Y^\sigma + \left(E + 2\sigma\gamma\frac{v}{(1+|v|^2)}\right).\nabla_v Y^\sigma - \sigma\Delta_v Y^\sigma = \sum_{i=1}^4 R_i$$

where

$$\begin{aligned}
R_1 &= -\sigma((1+|v|^2)^{\gamma/2}\Delta_v f); && \|R_1(t)\|_\infty \leqq C.\sigma. \\
R_2 &= \gamma(E.v)(1+|v|^2)^{(\gamma-2)/2}(f-f^\sigma); && \|R_2(t)\|_\infty \leqq C\|Y^\sigma(t)\|_\infty. \\
R_3 &= \sigma\left(\gamma(\gamma+2)\frac{|v|^2}{1+|v|^2} - 3\gamma\right)(1+|v|^2)^{(\gamma-2)/2} X^\sigma; && \|R_3(t)\|_\infty \leqq C.\sigma. \\
R_4 &= -(E-E^\sigma).(1+|v|^2)^{\gamma/2}\nabla_v f^\sigma; && \|R_4(t)\|_\infty \leqq C\|(E-E^\sigma)(t)\|_\infty.
\end{aligned}$$

Besides, we have according to (66)

$$\begin{aligned}
\|(E-E^\sigma)(t)\|_\infty &\leqq C\|(\rho-\rho^\sigma)(t)\|_1^{1/d}\|(\rho-\rho^\sigma)(t)\|_\infty^{(d-1)/d} \\
&\leqq C\sigma^{1/d}\|Y^\sigma(t)\|_\infty^{(d-1)/d} \leqq C(\sigma + \|Y^\sigma(t)\|_\infty).
\end{aligned}$$

Now the maximum principle (45) yields

$$\|Y^\sigma(t)\|_\infty \leqq C_1\sigma + C_2\int_0^t \|Y^\sigma(s)\|_\infty\, ds$$

proving that

$$\operatorname*{Max}_{[0,\, T^*]} \| Y^\sigma(t) \|_\infty \leqq C(\gamma, d, f_0, T^*, \sigma_0) \,.\, \sigma$$

This ends the proof.

Remark. — We could also show the convergence of the derivatives, if f_0 is more regular. The method would be exactly the same. So, such a theorem and its proof is omitted.

APPENDIX A

The linear Fokker-Planck equation

This appendix is attended to provide a rigorous treatment of the linear Fokker-Planck equation:

$$(40) \qquad \frac{\partial u}{\partial t} + v \,.\, \nabla_x u + a \,.\, \nabla_v u - \sigma \Delta_v u = U; \qquad u(x, v, 0) = u_0(x, v)$$

where

$$a(x, v, t) = (a_i(x, v, t))_{i=1}^d$$

is a given vector field, and $u_0(x, v)$ and $U(x, v, t)$ are given functions.

It seems that such an equation is not a classical example, and cannot be found in the litterature about linear evolution equations. M. S. Baouendi and P. Grisvard [2] have proved an existence theorem for the stationary problem associated with (40), but in a bounded domain, and in the case $d=1$. However, their idea of using a theorem of Lions [10], remains, as far as I know, the most powerful tool for the study of degenerate problems such as equation (40). A great part of what follows is an adaptation of their paper.

Proposition A.1 gives an existence and uniqueness theorem for equation (40), in an L^2 setting. A regularity result is stated in proposition A.2, while proposition A.3 is devoted to a maximum principle, and an L^∞ estimate. Proposition A.4 provides an L^1 estimate for an equation (40) involving a divergence free field a. All these propositions are gathered here for greater convenience.

PROPOSITION A.1. — *We assume that*

$$(41) \qquad u_0 \in L^2(\mathbb{R}^{2d}); \qquad U \in L^2([0, T] \times \mathbb{R}_x^d, H^{-1}(\mathbb{R}_v^d))$$

$$a \in L^\infty([0, T] \times \mathbb{R}^{2d}); \qquad \nabla_v . a = \sum_{i=1}^{d} \frac{\partial a_i}{\partial v_i} \in L^\infty([0, T] \times \mathbb{R}^{2d}). \tag{42}$$

Then, equation (40) *has a unique solution u in the class of functions* Y *defined according to*

$$Y = \left\{ u \in L^2([0, T] \times \mathbb{R}^d_x, H^1(\mathbb{R}^d_v)), \frac{\partial u}{\partial t} + v . \nabla_x u \in L^2([0, T] \times \mathbb{R}^d_x, H^{-1}(\mathbb{R}^d_v)) \right\}$$

and satisfying the initial condition in the sense of Lemma A.1.

PROPOSITION A.2. — *We assume that* (41) *and* (42) *are fulfilled and we suppose that u is a weak solution of* (40), *belonging to* $L^2([0, T] \times \mathbb{R}^{2d})$. *Then u also belongs to* Y *and coincides with the unique solution provided by proposition* A.1.

PROPOSITION A.3. — *We assume that* (41) *and* (42) *are fulfilled. Then, the solution u provided by proposition* A.1 *satisfies:*

(43) (i) $u_0 \geqq 0$ *and* $U \geqq 0 \Rightarrow u \geqq 0$,

(44) (ii) $u_0 \in L^\infty(\mathbb{R}^{2d})$ *and* $U \in L^1([0, T], L^\infty(\mathbb{R}^{2d})) \Rightarrow u \in L^\infty([0, T] \times \mathbb{R}^{2d})$

and we have

$$\| u(t) \|_\infty \leqq \| u_0 \|_\infty + \int_0^t \| U(s) \|_\infty \, ds. \tag{45}$$

PROPOSITION A.4. — *We assume that* (41) *and* (42) *are fulfilled, and in addition we suppose that a is divergence free:* $(\nabla_v . a = 0)$, *and that*

$$u_0 \in L^1(\mathbb{R}^{2d}); \qquad U \in L^1([0, T] \times \mathbb{R}^{2d}).$$

Then the solution u of proposition A.1 *belongs to* $L^\infty([0, T], L^1(\mathbb{R}^{2d}))$ *and satisfies:*

$$\| u(t) \|_1 \leqq \| u_0 \|_1 + \int_0^t \| U(s) \|_1 \, ds \tag{46}$$

Proof of proposition A.1. — The change of unknown

$$\tilde{u}(x, t) = \exp(-\lambda t) u(x, t)$$

leads to the equation

$$\begin{cases} \dfrac{\partial \tilde{u}}{\partial t} + v . \nabla_x \tilde{u} + a . \nabla_v \tilde{u} - \sigma \Delta_v \tilde{u} + \lambda \tilde{u} = \tilde{U} = e^{-\lambda t} U; \\ \tilde{u}(x, v, 0) = u_0(x, v) \end{cases} \tag{47}$$

Assuming that λ satisfies

(48) $$\lambda > \frac{1}{2} \| \nabla_v . a \|_\infty$$

we will prove existence and uniqueness for u. In the remaining of the proof, we will drop the tildes.

We first recall the theorem of Lions, that we will use further.

THEOREM [10]. — *Let* F *be a Hilbert space, provided with a norm* $|.|$, *and an inner product* $(,)$. *Let* Φ *be a subspace of* F, *provided with a prehilbertian norm* $\|.\|$, *such that the injection* $\Phi \subsetneq \mathrm{F}$, *is continuous. We consider a bilinear form* E:

$$\mathrm{E}: \quad \mathrm{F} \times \Phi \ni (u, \varphi) \to \mathrm{E}(u, \varphi) \in \mathbb{R}$$

such that $\mathrm{E}(., \varphi)$ *is continuous on* F, *for any fixed* φ *in* Φ, *and such that*

$$|\mathrm{E}(\varphi, \varphi)| \geqq \alpha \| \varphi \|^2, \qquad \forall \varphi \in \Phi, \quad \text{with } \alpha > 0.$$

Then, given a linear form L *in* Φ', *there exists a solution* u *in* F *of the problem*

(49) $$\mathrm{E}(u, \varphi) = \mathrm{L}(\varphi), \qquad \forall \varphi \in \Phi.$$

Now, let F be equal to the space:

$$\mathrm{X} = \mathrm{L}^2([0, \mathrm{T}] \times \mathbb{R}_x^d, \mathrm{H}^1(\mathbb{R}_v^d))$$

and let Φ be the space $\mathscr{D}([0, \mathrm{T}[\times \mathbb{R}_x^d \times \mathbb{R}_v^d)$ of infinitely differentiable functions, with compact support in $[0, \mathrm{T}[\times \mathbb{R}_x^d \times \mathbb{R}_v^d$. Φ is provided with

$$\| \varphi \|_\Phi^2 = \| \varphi \|_\mathrm{X}^2 + \frac{1}{2} \iint | \varphi(x, v, 0) |^2 \, dx \, dv, \qquad \forall \varphi \in \Phi.$$

The bilinear form E, and the linear form L, are defined according to

$$\mathrm{E}(u, \varphi) = \iiint \left[u . \left(-\frac{\partial \varphi}{\partial t} - v . \nabla_x \varphi + \lambda \varphi \right) + \nabla_v u . (a \varphi + \sigma \nabla_v \varphi) \right] dx \, dv \, dt$$

$$\mathrm{L}(\varphi) = \langle \mathrm{U}, \varphi \rangle_{\mathrm{X}', \mathrm{X}} + \iint u_0(x, v) \varphi(x, v, 0) \, dx \, dv.$$

Thanks to these definitions, Lions'theorem applies and the variational equation (49) admits a solution u in X. u satisfies equation (47) in the sense of distributions, and in particular, we deduce that

$$\frac{\partial u}{\partial t} + v . \nabla_x u = \mathrm{U} - a . \nabla_v u + \sigma \Delta_v u - \lambda u \in \mathrm{X}'$$

so that u belongs to Y. In order to give a meaning to the initial condition, and also, to show the uniqueness, we have to prove a trace theorem, and a Green formula for the functions of Y. Consequently, we temporarily admit the following Lemma:

LEMMA A.1. — (i) *If u belongs to* Y, *u admits* (*continuous*) *trace values* $u(x, v, 0)$, $u(x, v, \mathrm{T})$ *in* $\mathrm{L}^2(\mathbb{R}^{2d})$.

(ii) *For u and $\tilde{u}$ in* Y, *we have*

$$(50)\quad \langle \frac{\partial u}{\partial t}+v.\nabla_x u, \tilde{u}\rangle_{\mathrm{X}', \mathrm{X}}+\langle \frac{\partial \tilde{u}}{\partial t}+v.\nabla_x \tilde{u}, u\rangle_{\mathrm{X}', \mathrm{X}} = \iint u(x, v, \mathrm{T})\,\tilde{u}(x, v, \mathrm{T})\,dx\,dv-\iint u(x, v, 0)\,\tilde{u}(x, v, 0)\,dx\,dv.$$

So, using the equation (47) and the Green formula (50), we deduce that the solution u of the variational equality (49) satisfies

$$\iint [u(x, v, 0)-u_0(x, v)]\,\varphi(x, v, 0)\,dx\,dv=0, \qquad \forall\,\varphi\in\Phi$$

Consequently, the initial condition is satisfied in $\mathrm{L}^2(\mathbb{R}^{2d})$.

Now, for uniqueness, we suppose that u is a solution of (47) with $\mathrm{U}=0$ and $u_0=0$, which belongs to Y. Applying (50) we obtain

$$\begin{aligned}(51)\quad 0&=\langle \frac{\partial u}{\partial t}+v.\nabla_x u, u\rangle_{\mathrm{X}', \mathrm{X}}+(a.\nabla_v u, u)_{\mathrm{L}^2}\\ &\quad+\sigma(\nabla_v u, \nabla_v u)_{\mathrm{L}^2}+\lambda(u, u)_{\mathrm{L}^2}\\ &\geqq\frac{1}{2}\iint |u(x, v, \mathrm{T})|^2\,dx\,dv+\left(\lambda-\frac{1}{2}\|\nabla_v.a\|_\infty\right)(u, u)_{\mathrm{L}^2}\\ &\geqq\left(\lambda-\frac{1}{2}\|\nabla_v.a\|_\infty\right)(u, u)_{\mathrm{L}^2}.\end{aligned}$$

Thanks to (48) we get $u=0$, which proves uniqueness.

Proof of Lemma A.1. — The proof will be devided in 2 parts. The first is devoted to a density theorem, and the second, to the Lemma itself.

Part 1. — In this part, we prove that the set $\tilde{\mathrm{Y}}$ of C^∞ functions of (x, t) in $\mathbb{R}^d_x\times[0, \mathrm{T}]$, with values in $\mathrm{H}^1(\mathbb{R}^d_v)$, which are compactly supported in $\mathbb{R}^d_x\times\mathbb{R}^d_v\times[0, \mathrm{T}]$, is dense in Y. The proof is devided into 3 steps.

FIRST STEP. — *Truncation with respect to the velocity.*

Let u be a function in Y, and let $\chi_{\mathrm{R}}(v)$ be a sequence of functions such that:

$$(52)\quad \begin{cases}\chi_{\mathrm{R}}(v)=1 \quad \text{for } |v|\leqq \mathrm{R}; \quad \chi_{\mathrm{R}}(v)=0 \quad \text{for } |v|\geqq 2\,\mathrm{R}\\ \|\nabla_v\chi_{\mathrm{R}}\|_\infty=0\left(\dfrac{1}{\mathrm{R}}\right).\end{cases}$$

and put $u_R = \chi_R u$. Then classically we have

$$u_R \to u \text{ in X}$$

$$\left\| \left(\frac{\partial}{\partial t} + v . \nabla_x \right) u_R \right\|_{X'} \leqq \left\| \left(\frac{\partial}{\partial t} + v . \nabla_x \right) u \right\|_{X'}.$$

So, u_R converges to u in Y weak, and a classical argument about weak and strong topologies shows that u can be approximated by a sequence of functions in Y, which are compactly supported with respect to the velocity.

SECOND STEP. — *Truncation with respect to x.*

Let u be a function of Y, compactly supported in velocity, and let $\chi_R(x)$ be a truncation sequence verifying (52). Then:

$$u_R = \chi_R(x) u \to u \text{ in X}$$

and

$$\left(\frac{\partial}{\partial t} + v . \nabla_x \right) u_R = \chi_R(x) \left(\frac{\partial}{\partial t} + v . \nabla_x \right) u + v . u \nabla_x u_R$$

so that

$$\left\| \left(\frac{\partial}{\partial t} + v . \nabla_x \right) u_R \right\|_{X'} \leqq \left\| \left(\frac{\partial}{\partial t} + v . \nabla_x \right) u \right\|_{X'} + \frac{C(u)}{R}$$

and we conclude as in the first step.

THIRD STEP. — *Regularization.*

Let u be a function of Y, which is compactly supported with respect to both (x, v) in $\mathbb{R}_x^d \times \mathbb{R}_v^d$.

We first consider the case where u is compactly supported in $]0, T[\times \mathbb{R}_x^d \times \mathbb{R}_v^d$.

Let $\zeta_\varepsilon(x, t)$ be a regularizing sequence. Then we have classically:

$$u^\varepsilon = \zeta_\varepsilon * u \to u \quad \text{in X}$$

and

$$\left(\frac{\partial}{\partial t} + v . \nabla_x \right) u^\varepsilon = \zeta^\varepsilon * \left(\frac{\partial}{\partial t} + v . \nabla_x \right) u \to \left(\frac{\partial}{\partial t} + v . \nabla_x \right) u \quad \text{in X}'.$$

since $((\partial/\partial t) / + v . \nabla_x) u$, as a function of (x, t), is in L^2.

Now, the case where u is compactly supported in $[0, T[\times \mathbb{R}^{2d}$ or in $]0, T] \times \mathbb{R}^{2d}$, can be treated analogously by introducing a translation. This ends part 1.

Part 2. — We prove that the mapping:

(53) $$\tilde{Y} \ni u \to (u(.,., 0), u(.,., T)) \in L^2(\mathbb{R}^{2d})$$

can be continuously extended to Y. This shows (i) of Lemma A.1. Statement (ii) is then straightforward, thanks to the denseness of $\tilde{Y}$ in Y.

Using a partition of unity, we can consider the case of u in $\tilde{Y}$, compactly supported in $[0, T[\times \mathbb{R}^{2d}$. Thus we have

$$\iint |u(x, v, 0)|^2\, dx\, dv = -\int_0^T \frac{d}{dt}\left(\iint |u(x, v, t)|^2\, dx\, dv\right) dt = -2\int_0^T \iint u \,.\, \frac{\partial u}{\partial t}(x, v, t)\, dx\, dv\, dt$$

On the other hand, we have

$$|u(x_i = +\infty, \hat{x}_i, v, t)|^2 - |u(x_i = -\infty, \hat{x}_i, v, t)|^2 = 0 = 2\int_{-\infty}^{+\infty} \frac{\partial u}{\partial x_i} u(x_i, \hat{x}_i, v, t)\, dx_i$$

where $\hat{x}_i = (x_i, \ldots, x_{i-1}, x_{i+1}, \ldots, x_d)$.

So

$$0 = 2\iiint v_i \frac{\partial u}{\partial x_i} u(x, v, t)\, dx\, dv$$

Then, we obtain

$$\iint |u(x, v, 0)|^2\, dx\, dv = -2\iiint \left(\frac{\partial u}{\partial t} + v \,.\, \nabla_x u\right) u(x, v, t)\, dx\, dv\, dt$$

$$\leqq 2\left\| \frac{\partial u}{\partial t} + v \,.\, \nabla_x u \right\|_{X'} \| u \|_X \leqq C \| u \|_Y^2.$$

Thus, the mapping defined in (53) can be continuously extended to Y. This ends the proof of Lemma A.1, and of proposition A.1. ■

Remark. — At the expense of some technical difficulties (arising because of the change of type when $v_i = 0$), we can also prove trace theorems with respect to x.

We omit the proof since it is unuseful for our purpose. The proof of such a result in the case $d = 1$ can be found in P. Degond and S. Gallic [6]:

LEMMA A.2. — (i) *The mapping* $\tilde{Y} \ni u \to u(x_i, \hat{x}_i, v, t)$ *can be extended into a continuous linear mapping from* Y *into* $L^2(\mathbb{R}_{x_i}^{d-1} \times \mathbb{R}_v^d \times [0, T], d\hat{x}_i \otimes |v_i|\, dv \otimes dt)$.

(ii) *more generally, if* Ω *is an open set of* $\mathbb{R}_x^d$ *whose boundary is regular and compact, then the mapping*

$$u \ni \tilde{Y} \to u(x, v, t)|_{\partial\Omega \times \mathbb{R}_v^d \times [0, T]}$$

can be extended into a continuous mapping from Y *into* $L^2(\partial\Omega \times \mathbb{R}_v^d \times [0, T]$, $|v \,.\, n(x)|\, d\sigma(x) \otimes dv \otimes dt)$ *where* $d\sigma(x)$ *is the superficial measure of* $\partial\Omega$ *and* $n(x)$ *is the outward normal of* $\partial\Omega$ *at point* x.

Furthermore for u and $\tilde{u}$ in Y, *we get:*

$$\int_0^T \int_\Omega \langle \frac{\partial u}{\partial t} + v\,.\,\nabla_x u,\ \tilde{u} \rangle_{H^{-1},\,H^1}\, dx\, dt$$

$$= -\int_0^T \int_\Omega \langle \frac{\partial \tilde{u}}{dt} + v\,.\,\nabla_x \tilde{u},\ u \rangle_{H^{-1},\,H^1}\, dx\, dt$$

$$+ \int_0^T \int_{\partial\Omega} \int u(x, v, t)\, \tilde{u}(x, v, t)\,.\,(v\,.\,n(x))\, d\sigma(x)\, dv\, dt.$$

Proof of proposition A.2. — Thanks to the hypotheses, $a\,.\,\nabla_v\, u$ belongs to $X' = L^2([0, T] \times \mathbb{R}_x^d,\ H^{-1}(\mathbb{R}_v^d))$. So, u is also solution of the equation

$$\frac{\partial u}{\partial t} + v\,.\,\nabla_x u - \sigma \Delta_v u = g; \qquad u\,|_{t=0} = u_0. \tag{54}$$

with $g = U - a\,.\,\nabla_v u$ is given in X′. Now, Proposition A.1 provides a solution $\tilde{u}$ of (54) which belongs to X. The problem reduces to prove that $u = \tilde{u}$.

We denote by φ the difference $u - \tilde{u}$. φ is a weak solution in $L^2([0, T] \times \mathbb{R}_x^d \times \mathbb{R}_v^d)$ of equation

$$\frac{\partial \varphi}{\partial t} + v\,.\,\nabla_x \varphi - \sigma \Delta_v \varphi = 0; \qquad \varphi\,|_{t=0} = 0. \tag{55}$$

Now the solution of equation (55) can be found explicitely by taking the Fourier transform $\hat{\varphi}(\xi, \omega, t)$ of φ with respect to (x, v). $\hat{\varphi}$ is a solution of

$$\frac{\partial \hat{\varphi}}{\partial t} - \xi\,.\,\nabla_\omega \hat{\varphi} + \sigma\,|\,\omega\,|^2\, \hat{\varphi} = 0; \qquad \hat{\varphi}\,|_{t=0} = 0$$

which is a classical linear transport equation, with regular coefficients. It is well known that the unique L^2 solution of such an equation is zero. Thus φ is identically zero, which proves Proposition A.2. ■

Proof of proposition A.3. — (45) is an easy consequence of (43). The proof of (43) is an application of the variational method. (*See* e.g. Tartar [12]): we have the

LEMMA A.3. — *Let* $u \in Y$ *then* u^+ *and* u^- *defined according to*

$$u^+(x, v) = \mathrm{Max}(u(x, v), 0); \qquad u^-(x, v) = \mathrm{Max}(-u(x, v), 0)$$

belong to X, *and we have*

$$\langle \frac{\partial u}{\partial t} + v\,.\,\nabla_x u,\ u^- \rangle = \frac{1}{2}\left(\iint |\,u^-(x, v, 0)\,|^2\, dx\, dv - \iint |\,u^-(x, v, T)\,|^2\, dx\, dv \right). \tag{56}$$

We postpone the proof of this Lemma. Then, if u is the solution of (47) given by proposition A.1, for u_0 and U positive, we get

$$\begin{aligned}(57)\quad \langle \mathrm{U}, u^- \rangle &= \langle \frac{\partial u}{\partial t} + v.\nabla_x u, u^- \rangle_{\mathrm{X}', \mathrm{X}} + (a.\nabla_v u, u^-)_{\mathrm{L}^2} \\ &\qquad + \sigma(\nabla_v u, \nabla_v u^-)_{\mathrm{L}^2} + \lambda(u, u^-)_{\mathrm{L}^2} \\ &\leqq -\left(\iint |u^-(x, v, \mathrm{T})|^2\, dx\, dv + (a.\nabla_v u^-, u^-)_{\mathrm{L}^2} + \lambda(u^-, u^-)_{\mathrm{L}^2}\right. \\ &\leqq -\left(\lambda - \frac{\|\nabla_v . a\|_\infty}{2}\right)(u^-, u^-)_{\mathrm{L}^2}.\end{aligned}$$

Since $\langle \mathrm{U}, u^- \rangle$ is positive, each side of (57) must be equal to zero, and we conclude, as for (51), that $u^- = 0$.

Proof of Lemma A.3. — Thanks to the denseness of $\tilde{\mathrm{Y}}$ into Y, it is sufficient to prove (56) for u in $\tilde{\mathrm{Y}}$. (For the definition of $\tilde{\mathrm{Y}}$, see the proof of Lemma A.1, part 1). It is even sufficient to prove that for u in $\tilde{\mathrm{Y}}$, we have

$$(57)\quad \int_0^{\mathrm{T}} \int_{\mathbb{R}_x^d} \left(\frac{\partial u}{\partial t}, u^-\right)_{\mathrm{L}^2} dx\, dt = \frac{1}{2}\left(\iint |u^-(x, v, 0)|^2\, dx\, dv - \iint |u^-(x, v, \mathrm{T})|^2\, dx\, dv\right)$$

$$(58)\qquad \int_0^{\mathrm{T}} \int_{\mathbb{R}_x^d} \left(v_i \frac{\partial u}{\partial x_i}, u^-\right)_{\mathrm{L}^2} dx\, dt = 0, \qquad \forall i \in [1, d].$$

For that purpose, we apply a theorem stated in [12], that we recall here.

Proposition. — *Let* $\mathrm{V} \subset \mathrm{H} \subset \mathrm{V}'$ *be a canonical triple of Hilbert spaces. We suppose that the mapping* $u \to u^-$ *is a contraction on* V. *Let* u *belong to* $\mathrm{L}^2(0, \mathrm{T}, \mathrm{V}) \cap \mathrm{C}^0$ $(0, \mathrm{T}, \mathrm{H})$ *such that* $du/dt \in \mathrm{L}^2(0, \mathrm{T}, \mathrm{V}')$ *then*

$$(59)\qquad \int_0^{\mathrm{T}} \left(\frac{du}{dt}, u^-(t)\right)_{\mathrm{V}'\,\mathrm{V}} dt = \frac{1}{2}(|u^-(0)|_{\mathrm{H}}^2 - |u^-(\mathrm{T})|_{\mathrm{H}}^2).$$

Applying this proposition with $\mathrm{V} = \mathrm{H} = \mathrm{L}^2(\mathbb{R}_x^d \times \mathbb{R}_v^d)$ leads to (57).

To prove (58) we define

$$\mathrm{H} = \mathrm{V} = \mathrm{L}^2([0, \mathrm{T}] \times \mathbb{R}_{\hat{x}_i}^{d-1} \times \mathbb{R}_{\hat{v}_i}^{d-1} \times \mathbb{R}_{v_i}, dt \otimes d\hat{x}_i \otimes d\hat{v}_i \otimes |v_i|\, dv_i)$$

where

$$\hat{x}_i = (x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_d); \qquad \hat{v}_i = (v_1, \ldots, v_{i-1}, v_{i+1}, \ldots, v_d).$$

Let $\chi_i(v)$ be the characteristic function of the half space $\{(x, v, t), v_i > 0\}$.

Then the function $\tilde{u} = \chi_i(v)\, u(x, v, t)$ satisfies:

$$\tilde{u} \in \mathrm{L}^2(]-\infty, +\infty[, \mathrm{H}); \qquad \frac{\partial \tilde{u}}{\partial x_i} \in \mathrm{L}^2(]-\infty, +\infty[, \mathrm{H}).$$

Since u is compactly supported, the use of formula (59) with ξ_i large enough, leads to:

$$0=\int_{-\xi_i}^{\xi_i}\left(\frac{\partial \tilde{u}}{\partial x_i}, \tilde{u}^-\right)_{\mathrm{H}} dx_i=\iiint_{v_i>0} v_i \frac{\partial u}{\partial x_i} u^- \, dx\, dv\, dt.$$

The same method applies for $v_i<0$ and leads to (58).

This ends the proof of Lemma A.3 and of proposition A.3. ■

Proof of proposition A.4. — We consider a C^∞ function of u: $\psi_\varepsilon(u)$, such that

$$\psi_\varepsilon(u)=0 \quad \text{if } u \leqq 0; \qquad \psi_\varepsilon(u)=1 \quad \text{if } u \geqq \varepsilon;$$

$\psi_\varepsilon(u)$ increases in $[0, \varepsilon]$.

Let $\varphi_\varepsilon(u)$ be a primitive of $\psi_\varepsilon(u)$:

$$\varphi_\varepsilon(u)=\int_{-\infty}^{u} \psi_\varepsilon(v)\, dv.$$

Of course, $\varphi_\varepsilon(u)$ converges, in the distribution sense, towards u^+.

First, it is easy to sketch that if u belongs to Y, then $\varphi_\varepsilon(u)$ and $\psi_\varepsilon(u)$ belongs to Y. Then we prove that for u in Y

(60) $$\left\langle \frac{\partial u}{\partial t}+v.\nabla_x u, \psi_\varepsilon(u) \right\rangle_{(\mathrm{X}', \mathrm{X})}=\iint \varphi_\varepsilon(u)(x, v, \mathrm{T})\, dx\, dv - \iint \varphi_\varepsilon(u)(x, v, 0)\, dx\, dv$$

(61) $$\langle a.\nabla_v u, \psi_\varepsilon(u) \rangle_{(\mathrm{X}', \mathrm{X})}=0.$$

(62) $$-\sigma \langle \Delta_v u, \psi_\varepsilon(u) \rangle_{(\mathrm{X}', \mathrm{X})} \geqq 0.$$

(61) and (62) are easy. We just give the ideas for (60). As ψ_ε is the derivative of φ_ε, (60) is trivial for the functions of $\tilde{\mathrm{Y}}$. It remains to prove that each side of equality (60) is defined and continuous for u in Y. We have

$$\varphi_\varepsilon(u)=u\,\Phi_\varepsilon(u) \qquad \text{with} \quad \Phi_\varepsilon(u)=\int_0^1 \psi_\varepsilon(\theta u)\, d\theta.$$

So, we get

$$|\varphi_\varepsilon(u)| \leqq \mathrm{C}(\varepsilon)\, |u|^2$$

And thanks to the trace Lemma A.1

$$\|\varphi_\varepsilon(u)(.,., \mathrm{T})\|_{\mathrm{L}^1} \leqq \mathrm{C}(\varepsilon)\, \|u(\mathrm{T})\|_{\mathrm{L}^2}^2 \leqq \mathrm{C}(\varepsilon)\, \|u\|_{\mathrm{Y}}^2.$$

The continuity is obvious.

So integrating equation (40) against $\psi_\varepsilon(u)$ leads to

(63) $$\iint \varphi_\varepsilon(u)(x, v, \mathrm{T})\, dx\, dv \leqq \|u_0\|_1 + \int_0^{\mathrm{T}} \|f(s)\|_1\, ds.$$

If we let ε go to zero, (63) tells us that $u^+(\mathrm{T})$ is a bounded measure in $\mathbb{R}^{2d}$.
But u^+ is absolutely continuous, so that it is an L^1 function such that:

$$\|u^+\|_1 \leqq \|u_0\|_1 + \int_0^{\mathrm{T}} \|f(s)\|_1\, ds.$$

The same proof applied to u^- leads to the result. ■

APPENDIX B

Interpolation inequalities

LEMMA B.1. — *For a function $f(v)$: $\mathbb{R}^d \to \mathbb{R}$, we have:*

(64) $$\|(1+|v|^2)^{(\gamma-1)/2} f\|_\infty \leqq \mathrm{C}(\gamma)\, \|f\|_\infty^{1/\gamma}\, \|(1+|v|^2)^{\gamma/2} f\|_\infty^{1-1/\gamma}$$

(65) $$\int |f(v)|\, dv = \|f\|_1 \leqq \mathrm{C}(\gamma, d)\, \|f\|_\infty^{(\gamma-d)/\gamma}\, \|(1+|v|^2)^{\gamma/2} f\|_\infty^{d/\gamma} \qquad \textit{for} \quad \gamma > d$$

Proof. — We have

$$(1+|v|^2)^{(\gamma-1)/2} f(v) \leqq \begin{cases} (1+\mathrm{R}^2)^{(\gamma-1)/2} \|f\|_\infty & \text{if} \quad |v| \leqq \mathrm{R} \\ \dfrac{1}{(1+\mathrm{R}^2)^{1/2}} \|(1+|v|^2)^{\gamma/2} f\|_\infty & \text{if} \quad |v| \geqq \mathrm{R}. \end{cases}$$

So we get

$$\|(1+|v|^2)^{(\gamma-1)/2} f\|_\infty \leqq (1+\mathrm{R}^2)^{(\gamma-1)/2} \|f\|_\infty + \frac{1}{(1+\mathrm{R}^2)^{1/2}} \|(1+|v|^2)^{\gamma/2} f\|_\infty.$$

If we minimize with respect to R, we obtain (64).

Now, for (65) we can write for $\gamma > d$.

$$\begin{aligned} \int |f(v)|\, dv &= \int_{|v| \leqq \mathrm{R}} + \int_{|v| \geqq \mathrm{R}} |f(v)|\, dv \\ &\leqq \mathrm{C}(d)\, \mathrm{R}^d \|f\|_\infty + \int_{|v| \geqq \mathrm{R}} \frac{dv}{(1+|v|^2)^{\gamma/2}} \|(1+|v|^2)^{\gamma/2} f\|_\infty \\ &\leqq \mathrm{C}(d)\, \mathrm{R}^d \|f\|_\infty + \mathrm{C}(d)\, \|(1+|v|^2)^{\gamma/2} f\|_\infty\, \mathrm{R}^{d-\gamma}. \end{aligned}$$

and, by optimization, we get (65). ■

LEMMA B.2. — *Let* $\rho(x)$ *be a function which belongs to* $L^1(\mathbb{R}^d) \cap W^{1,\infty}(\mathbb{R}^d)$ *and let* $E(x)$ *be such that*

$$E(x) = \int \frac{x-y}{|x-y|^d} \rho(y)\, dy.$$

Then we have:

$$\|E\|_\infty \leqq C(d) \|\rho\|_1^{1/d} \|\rho\|_\infty^{(d-1)/d}. \tag{66}$$

$$\|\nabla_x E\|_\infty \leqq C(d)(1 + \|\rho\|_\infty (1 + \mathrm{Log}(1 + \|\nabla_x \rho\|_\infty)) + \|\rho\|_1). \tag{67}$$

(66) is classical and is proved in [13] or in [3]. (67) is a slightly different version of a logarithmic identity of T. Beale, T. Kato and A. Majda [4], but the proof is exactly the same.

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