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POSITIVE MEASURE SETS OF ERGODIC RATIONAL MAPS

MARY REES

ABSTRACT. — In the set of rational maps of degree d, and in some other families, a positive measure set of ergodic maps is found.

Introduction

In this paper, we study rational maps of the extended complex plane, and the question of their ergodicity. The set of all rational maps of degree d is a complex manifold of complex dimension $2d+1$, and hence admits a natural Lebesgue measure class, as does any submanifold. We show that, in complex submanifolds of rational maps containing at least one map with the forward orbit of each critical point finite and containing an expanding periodic point, and satisfying a simple non-degeneracy condition, the set of maps, which are ergodic with respect to Lebesgue measure on $\mathbb{C} \cup \{\infty\}$, has positive Lebesgue measure. In particular, a positive measure subset of all rational maps of degree d consists of ergodic maps. In contrast, there is a conjecture that the set of maps with zero measure Julia set is open and dense. It will also be clear from the estimates that the positive measure set of ergodic maps which is constructed, consists, in some suitable measure theoretic sense, of expanding maps.

This paper grew, in part, out of an unsubstantiated remark in the first version of the introduction of [R] concerning rational maps with positive exponents. The result obtained in this paper has a resemblance to Jakobson's Theorem [J] for the 1-(real)-parameter family $f_\lambda(x) = \lambda x(1-x)$ ($\lambda \in [0,4]$) of maps of [0,1]: that for a positive measure set of λ, f_λ has an absolutely continuous invariant measure, and is ergodic. Fundamentally, the idea of the present proof is the same, although the details are different. (One-dimensional arguments cannot be used.) The proof can be modified to prove Jakobson's Theorem for a larger class of real polynomial maps.

One reason for understanding how maps of an interval—or analytic maps of the plane—can be ergodic, is that it seems to be relevant to the problem of constructing volume-preserving diffeomorphisms with positive exponents. It would possibly be helpful if it turned out that complex analysis theorems would be used to construct analytic examples. Unfortunately, in the present paper, only Montel's Theorem is used—and only once. However, complex analysis theorems have been used to prove related results. Sullivan used the measurable Riemann Mapping Theorem to prove conservativity of maps with Julia set equal to S^2 [S2]. Also, Herman ([H1], [H2]) has a formula, using the Divergence Theorem, for bounding below exponents in certain examples.

I am heavily indebted to J. C. Yoccoz of this paper who clarified many of the ideas, and entirely reorganized and rewrote most of the paper in an intelligible form. He also introduced ideas such as the use of Schwarz's Lemma, standard univalent function theory, and the method of estimating quantities by realizing them as coefficients in the power series of an analytic function (in 7.1). Many of the proofs—most notably the fundamental estimates of section 5, and 7.1—are his, verbatim.

1.2. PRECISE FORMULATION OF THE RESULTS. — Let f be a rational map such that the orbit of each critical point is finite but no critical point is periodic. Then any periodic point is expanding. This was proved by Thurston by producing a hyperbolic structure on S^2 with ramification points at the critical point forward orbits. Another argument is the following. If $f^n(z_0)=z_0$ and $|(f^n)'(z_0)|=1$, and φ denotes the local inverse of f^n mapping $f^n(z_0)$ to z_0, then $\{\varphi^k : k \geqq 1\}$ form a normal family on a neighbourhood of $f^n(z_0)$ (because critical point forward orbits are finite). $|(\varphi^k)'| \not\to 0$ because $|(f^n)'(z_0)|=1$. So $\{f^{nk} : k \geqq 1\}$ is a normal family in a neighbourhood of z_0, which must therefore be the centre of a Siegel disc. This is impossible because the critical point forward orbits are finite. The possibility that $|(f^n)'(z_0)|<1$ is ruled out for the same reason, together with the fact that no critical point is periodic.

Thus, such a map has Julia set equal to S^2. This is because there are no wandering domains in the complement of the Julia set—which was proved by Fatou in this case [F] and by Sullivan in general [S2]. It also follows from the fact that the map is expanding with respect to Thurston's hyperbolic metric.

Now let $\{f_\lambda : \lambda \in \Lambda\}$ be an analytic family of rational maps of degree d, where for some $\lambda_0 \in \Lambda$, f_{λ_0} has all critical points non-periodic, but critical point forward orbits finite. Let $x_i(\lambda)$ be the critical points of f_λ counted with multiplicity, $1 \leqq i \leqq 2d-2=m_0$. Write $f_n(\lambda, z)=f_\lambda^n(z)$, and suppose $f_{r_i}(\lambda_0, x_i(\lambda_0))$ is periodic with period s_i.

The non-degeneracy condition. — $DF_i(\lambda_0) \neq 0$ for $\leqq i \leqq m_0$, where

$$F_i(\lambda)=f_{r_i+s_i}(\lambda, x_i(\lambda))-f_{r_i}(\lambda, x_i(\lambda)).$$

Note that $DF_i(\lambda_0)$ exists even though x_i may not be differentiable at λ_0. Write $y_i(\lambda)=f_{r_i}(\lambda, x_i(\lambda))$, and let $z_i(\lambda)$ be the periodic point of period s_i with $y_i(\lambda_0)=z_i(\lambda_0)$. Then y_i, z_i are both differentiable functions. Then one sees that:

$$DF_i(\lambda_0)=(f'_{s_i}(z_i(\lambda_0))-1)\left(\frac{Dy_i}{D\lambda}(\lambda_0)-\frac{Dz_i}{D\lambda}(\lambda_0)\right),$$

so that $DF_i(\lambda_0) \neq 0$ if and only if $(D/D\lambda)\,(y_i-z_i)\,(\lambda_0) \neq 0$.

1.3. The main theorem of this paper is:

THEOREM A. — *Suppose* $\{f_\lambda : \lambda \in \Lambda\}$ *is such that* f_{λ_0} *has all critical point forward orbits finite and critical points non-periodic. Suppose the non-degeneracy condition is satisfied at* λ_0. *Then for a positive measure set of* λ, f_λ *is ergodic with respect to Lebesgue measure, and has an invariant probability measure equivalent to Lebesgue.*

1.4. COROLLARY. — *A positive measure set of rational maps of degree d consists of maps which are ergodic with respect to Lebesgue measure, and have invariant probability measures equivalent to Lebesgue.*

Proof. — Write

$$f_\lambda(z) = f(\lambda, z) = \lambda \frac{(z-2)^d}{z^d}, \qquad \lambda > 0.$$

Any rational map near $\{f_\lambda : |\lambda| > 1/2\}$ can be written in the form $(\lambda(z-2)^d + p(z))/(z^d + q(z))$ for some λ, where p, q are polynomials of degree $\leqq d-1$. The critical points of f_λ are 2, 0, and $f_\lambda(2)=0$, $f_\lambda(0)=\infty$, $f_\lambda(\infty)=\lambda$. $f_{\lambda_0}(\lambda_0)=\lambda_0$ if $((\lambda_0-2)/\lambda_0)^d=1$, that is, $\lambda_0 = 2/(1-\zeta)$ for ζ such that $\zeta^d=1$, $\zeta \neq 1$. For such a λ_0, $f'_{\lambda_0}(\lambda_0) = 2d/(\lambda_0-2) = d(1-\zeta)/\zeta$. So λ_0 is an expanding fixed point of f_{λ_0} if $d|1-\zeta|>1$, which is, in fact, always true if $\zeta^d=1$ and $\zeta\neq 1$. To apply Theorem A we need to show:

$$\left(\frac{\partial f}{\partial z}(\lambda_0, \lambda_0) - 1\right)\frac{\partial f_{r_i}}{\partial \lambda}(\lambda_0, x_i) + \frac{\partial f}{\partial \lambda}(\lambda_0, \lambda_0) \neq 0$$

for $i=1, 2$ with $x_1=2$, $r_1=3$, $x_2=0$, $r_2=2$, $f_3(\lambda, 2)=f_2(\lambda, 0)=\lambda$. So the non-degeneracy conditions become:

$$\frac{\partial f}{\partial z}(\lambda_0, \lambda_0) - 1 + 1 \neq 0 \qquad \text{(twice)}$$

which is true. So Theorem A can be applied.

2. Reductions

2.1. It suffices to prove Theorem A for 1-dimensional families with $\lambda_0 = 0 \in \mathbb{C}$. For we may write $\Lambda = \bigcup_{v \in T_{\lambda_0}(\Lambda)} \Lambda_v$ with $\lambda_0 \in \Lambda_v$ for all v, where $\Lambda_v \subseteq \Lambda$ is 1-dimensional having tangent v at λ_0. The non-degeneracy condition at λ_0 is satisfied for the family Λ_v if and only if $DF_i(\lambda_0)\, v \neq 0$. This is true for almost all v if $DF_i(\lambda_0) \neq 0$. So Theorem A for 1-dimensional families and Fubini's Theorem imply Theorem A. For it is easy to see that the set of λ for which f_λ is ergodic is measurable (using the criterion for ergodicity given by Hopf's Theorem, for example.) By changing coordinates we may assume $\lambda_0 = 0 \in \mathbb{C}$.

2.2. Replacing f_λ by f_λ^n, we may assume the $z_i(\lambda)$ are *fixed* expanding points, and that $r_i=1$ or 2 for all i. (Of course, this procedure will increase the number m_0.) For clearly f_λ is ergodic if f_λ^n is, and if γ is an invariant measure for f_λ^n which is equivalent to Lebesgue, then so is $(f_\lambda)^*\gamma$, and $(f_\lambda)^*\gamma=\gamma$ by ergodicity of f_λ^n. So γ is an invariant measure for f_λ.

3. Elementary Estimates

We now show that, for λ near λ_0, we have certain estimates on f_λ. *Throughout*, when talking about a rational map f of S^2, $|f'(x)|$ will denote $\lim_{y\to x} d(fx, fy)/d(x, y)$, and d will denote the usual spherical metric. The measure used on S^2 will be spherical measure.

3.1. ESTIMATES FOR f_0. — For $1\leqq i\leqq m_0$, let φ_i be the local conjugacy between f_0 near $z_i=z_i(0)$ and its linear part; take $U_i=\varphi_i\{|z|<\theta_i\}$ with θ_i small enough; let W_i be the inverse of f_0 defined on U_i such that $W_i(z_i)=z_i$; put $V_i=W_i^3(U_i)$ (for instance), $V=\bigcup_{i\leqq m_0} V_i$, $U=\bigcup_{i\leqq m_0} U_i$; let T_i be the inverse of $f_0^{r_i}$ sending z_i to x_i.

Put $X=S^2-\bigcup_{i=1}^{m_0}\bigcup_{j=0}^{r_i} f_0^j T_i V_i$. Then one has, for suitable a_0:

1. $|W_i'(x)|<e^{-a_0}$, $x\in U_i$;
2. $|(f_0^{-n})'|X|<C_0/2$ if $n\leqq p_0$;
3. $|(f_0^{-n})'|X|<1/4\,C_0$ if $n\geqq p_0$;
4. $|(T_i W_i^n)'(x)|<(1/C_0^2)\,e^{-a_0 n}$ if $x\in U_i-V_i$, $n\geqq p_0$.

Here, f_0^{-n} is any branch of the inverse of f_0^n, and $a_0>0$, $C_0>1$, $p_0\geqq 0$ are fixed. 2,3 follow from Montel's Theorem. 3 also uses the fact that expanding periodic points are dense in S^2. For a more detailed explanation of this see [F].

3.2. ESTIMATES FOR f_λ. — Keeping for W_i, T_i, U_i, V_i, X the same meaning as before (with f_λ instead of f_0), take η_0 sufficiently small to have, for $|\lambda|\leqq\eta_0$ (diminishing a_0 if necessary):

C1.
$$|W_i'(z)|<e^{-a_0}\quad\text{for}\quad z\in U_i;$$
$$W_i^4(U_i)\subseteq V_i\subseteq W_i^2(U_i),\qquad W_i(U_i)\subseteq U_i;$$

C2.
$$|(f_\lambda^{-n})'|X|\leqq C_0\quad\text{for}\quad n\leqq p_0,$$
$$\leqq e^{-a_0 n}<\frac{1}{2C_0}\quad\text{for}\quad p_0\leqq n\leqq 3p_0.$$

Also, given a positive integer n_0, if η_0 is sufficiently small, for $|\lambda|\leqq\eta_0$ we have:

C3. critical points stay at least $2n_0^2$ times in U;

C4. let $t_0\ldots t_n$ be points such that $t_{i+1}=f_\lambda(t_i)$ and there exist $j\in[0, p_0]$, $k\in[n-p_0, n]$, $L\in[1, m_0]$ such that

$$p_0\leqq k-j-r_L\leqq n_0,\qquad t_k\in U_L-V_L,\quad t_j=T_L W_L^{k-j-r_L}(t_k);$$

then for the inverse S of f_λ^n which maps t_n to t_0, we have $|S'(t_n)| < e^{-a_0 n}$.

C4 is an analogue of 3.1.4 for f_0 [using (2)].

It is important that all constants in all future estimates do not depend on n_0 (provided it is big enough).

3.3. Recall from the statement of Theorem A that the 1-parameter family $\{f_\lambda : |\lambda| < \eta_0\}$ may have critical points $x_i(\lambda)$ of f_λ emerging from critical points of f_0 of higher order. Thus, it may not be possible to bound below the distance between the critical points of f_λ. The following lemma describes the behaviour of the derivative of f_λ, and of f_λ^r, f_λ^{-1}, f_λ^{-r} for small r, near the critical points.

LEMMA. — *Let* $F : \{z : |z| \leqq B\} \to \mathbb{C}$ *have derivative* $F'(z) = A(z) \prod_{i=1}^{u} (z - z_i)$ *with* $|z_i| < B$, $A \neq 0$ *in* $|z| \leqq B$. *Then for some* C_1 *depending only* u, A, B :

$$|F'(z)| \geqq \frac{1}{C_1} \operatorname{Min}_i |F(z) - F(z_i)|^{u/u+1},$$

and if T *is a multivalued inverse of* F *defined on a ball of radius r with connected image,* ImT *has diameter* $\leqq (C_1/C_0)\, r^{1/u+1}$.

Proof. — Let $|z| < B$ and suppose, for example, that z_1 is the critical point of F nearest to z. Then, with $z_t = (1-t) z + t z_1$, $0 \leqq t \leqq 1$, we have, as $|z_t - z_i| \leqq 2 |z_0 - z_i|$ for $2 \leqq i \leqq u$:

$$|F'(z_t)| \leqq C \prod_{i=1}^{u} |z_t - z_i| \leqq C' |z_t - z_1| \prod_{i=2}^{u} |z - z_i|.$$

Hence

$$|F(z) - F(z_1)| \leqq \frac{1}{2} C' |z - z_1|^2 \prod_{i=2}^{u} |z - z_i|.$$

So

$$|F(z) - F(z_1)|^u \leqq C'' |z - z_1|^{2u} \prod_{i=2}^{u} |z - z_i|^u \leqq C'' \prod_{i=1}^{u} |z - z_i|^{u+1} \leqq C''' |F'(z)|^{u+1},$$

which gives the estimate on $|F'(z)|$. Since $T'(F(z)) = (F'(z))^{-1}$, we have $|T' w| \leqq C_1 (\operatorname{Min}_i |w - F(z_i)|)^{-u/u+1}$ for $w \in \operatorname{ImT}$. If we take any straight line joining points on the boundary of ImT, then its image can be replaced by a path l with the same endpoints, in the same homotopy class relative to the points $F(z_i)$, of length $\leqq O(r)$, and with length $\leqq O(\eta)$ within η of any $F(z_i)$. $T(l)$ then has the same endpoints as the original straight line in ImT. Then the length of T(I) is

$$\int_l |T' w|\,d|w| \leqq \text{Const.} \int_0^r \eta^{-u/u+1}\,d\eta \leqq \frac{C_1}{C_0} r^{1/u+1},$$

enlarging C_1 if necessary.

Note. — It will be convenient to assume $C_1 \geqq C_0$.

4. Inverses

4.1. General definition. — Given $\varepsilon>0$, an inverse is determined by a sequence $t_0 \ldots t_n$ of points such that $t_{i+1}=f_\lambda(t_i)$: it is the multivalued inverse of f_λ^n with connected image defined on the ball with centre t_n, radius ε, which takes t_0 as a value at t_n.

Definition. — Take ε'_0 such that any ball B of radius $2\varepsilon'_0$ cutting V and not contained in V satisfies $B \subseteq U$ and $f_\lambda(B) \subseteq U \setminus V$.

4.2. Lemma. — *Let* $|\lambda| \leqq \eta_0$, *so that* f_λ *satisfies* C1-C4. *There exists* $a_1>0$, *and, given* $\varepsilon'<\varepsilon'_0$, $q_0 \geqq p_0$ *with the following property: let* $\varepsilon=(\varepsilon'/C_1)^{1+u}$ (C_1 *as in* 3.3), $t_0 \ldots t_n$ *be a sequence of points with* $f_\lambda(t_i)=t_{i+1}$ *and no* n_0^2 *consecutive points* U, S *the inverse defined on the ball* $B_\varepsilon(t_n)$ *of centre* t_n, *radius* ε. *Then one has:*

(*a*) $SB_\varepsilon(t_n) \subseteq B_{\varepsilon'}(t_0)$;

(*b*) *if* $n \geqq q_0$, $SB_\varepsilon(t_n) \subseteq B_{\varepsilon_n}(t_0)$ *where* $\varepsilon_n=\varepsilon\exp(-a_1 n)$, $\varepsilon_{q_0}<\varepsilon_1/2$;

(*c*) *if* $B_\varepsilon(t_n) \subseteq X$ *and* $n \geqq p_0$, S *is univalued and* $|S'| < \exp(-a_1 n)$ *on* $B_\varepsilon(t_n)$.

Proof. — If $B_\varepsilon(t_n) \subseteq X$ and $n \leqq 3p_0$, the lemma follows from C2. Next, suppose $B_{2\varepsilon}(t_n) \subseteq U$ and $S=S_1 T_L W_L^m$ with length $S_1=j \leqq 3p_0$. Then $|(W_L^m)'| \leqq e^{-a_0 m}$ on $B_\varepsilon(t_n)$. So by 3.3, $T_L W_L^m B_\varepsilon(t_n) \subseteq B_\alpha(t_j)$ where $\alpha=(C_1/C_0)\,\varepsilon^{1/1+u} e^{-a_0 m/1+u}<\varepsilon'$, and also, since $|S'_1| \leqq C_0$ on $B_{\varepsilon'}(t_j)$, $SB_\varepsilon(t_n) \subseteq B_{\varepsilon'}(t_0)$.

Also, if $B_{2\varepsilon}(t_n) \subseteq U_L \setminus V_L$ and $m \leqq n_0^2$, then $f_\lambda^p(y_j) \notin B_{2\varepsilon}(t_n)$ for $p \leqq n_0^2$ because $f_\lambda^p(y_j) \in U$ for $p \leqq 2n_0^2$. So S is univalued on $B_{2\varepsilon}(t_n)$ and by Schwarz's Lemma

$$|S'| \leqq C_2 \frac{\alpha}{\varepsilon} = C_2 C_1 \varepsilon^{-u/u+1} e^{-a_0 m/1+u}$$

on $B_\varepsilon(t_n)$. So the lemma is proved if $B_{2\varepsilon}(t_n) \subseteq U_L$ and S is as above, provided q_0, a_1 are suitably chosen, and we use C4 to bound $|S'|$ if $p_0 \leqq m \leqq n_0$.

If length $S \geqq 3p_0$, we use the inductive hypothesis. We can write S either as $S_1 S_2$ where $j=$length S_1, $p_0 \leqq j \leqq 3p_0$ and $B_{\varepsilon'}(t_j) \subseteq X$, or as $S_1 T_L W_L^m S_2$ with $j=$length S_1, $j<p_0$, $m \geqq p_0$, length $S_2=n-k$ and $B_{2\varepsilon'}(t_k) \subseteq U_L \setminus V_L$ or $S_2=$identity. This last possibility has already been dealt with. For $S=S_1 S_2$ we have $|S'_1|<e^{-a_0 j}$, completing the proof. For $S=S_1 T_L W_L^m S_2$ we have $|S'| \leqq e^{-a_0 n}$ if length $S_2 \leqq p_0$ and $m \leqq n_0$ by C4, and otherwise from the earlier estimate we have

$$|(S_1 T_L W_L^m)'| < e^{-a_1 (k+p_0)} C_0^{-1},$$

which is $<e^{-a_1 n} C_0^{-1}$ if length $S_2 = n-k \leqq p_0$, for suitable a_1. If length $S_2 \geqq p_0$ we use the inductive estimates on S_2 to complete the proof.

4.3. Now let

$$2\varepsilon_0 = \left(\frac{\varepsilon_0'}{2C_1}\right)^{1+u}, \qquad 2\varepsilon_1 = \left(\frac{\varepsilon_0}{2C_1}\right)^{1+u}$$

Then

$$SB_{2\varepsilon_0}(t_n) \subseteq B_{\varepsilon_0'}(t_0),$$
$$SB_{2\varepsilon_1}(t_n) \subseteq B_{\varepsilon_0/2}(t_0)$$

for *any* n and any inverse S. (If length $S \geqq n_0$, this follows from writing S as a composition of inverses length $\leqq n_0$ and $\geqq q_0$.)

There is q_0 so that

$$SB_{2\varepsilon_0}(t_n) \subseteq B_{\varepsilon_1/2}(t_0) \qquad \text{for} \quad n \geqq q_0,$$

any inverse S, and

$$SB_{2\varepsilon_0}(t_n) \subseteq B_{\varepsilon_1 \exp(-a_1 n)}(t_0) \qquad \text{for} \quad n \geqq q_0,$$

S as in 4.2.

4.4. *Critical Inverses.* — Critical inverses are those with $\varepsilon = \varepsilon_0$, and where t_0 is a critical point.

An interval $[k, k+m]$ (with $k > r_i$) is an *x_i-follower* if the image of the inverse defined by the sequence $f_\lambda^k(x_i) \ldots f_\lambda^{k+m}(x_i)$ (and $\varepsilon = \varepsilon_0$) contains some critical point $x_j (1 \leqq j \leqq m_0)$. Note one necessarily has $k \geqq 2 n_0^2$.

DEFINITION. — Let $|\lambda| \leqq \eta_0$. λ is *(N, i, α)-good* if the number of integers $L \in [0, N]$ which belong to an x_i-follower of length $\geqq n_0$ is less than αN.

4.5. From now on, assume that if $|\lambda| \leqq \eta_0$ then f_λ satisfies C1-C4 and f_0 has all critical points non-periodic, but with finite forward orbits, and that the non-degeneracy condition is satisfied (although this is only needed for Theorem C). Now Theorem A can be replaced by Theorems B and C.

THEOREM B. — *There exist $\alpha_0 > 0$, $\eta_0 > 0$ such that if f_λ is (N, i, α_0)-good for all $N > 0$, $i \leqq m_0$ and if $|\lambda| < \eta_0$ then f_λ is ergodic with respect to Lebesgue measure and there is an f_λ-invariant probability measure equivalent to Lebesgue.*

THEOREM C. — *For any $\alpha_0 > 0$, $\eta_0 > 0$, the set of λ such that $|\lambda| < \eta_0$ and f_λ is (N, i, α_0)-good for all $i \leqq m_0$, $N > 0$ has positive Lebesgue measure.*

4.6. GENERAL INVERSES. — A general inverse is as in 4.1 with $\varepsilon = \varepsilon_1$.

Followers for general inverses. — Let S be a general inverse determined by a sequence $t_0 \ldots t_n$. Then $[k, k+L] \subseteq [0, n]$ is a *follower* (for S) if some $x_i \in \operatorname{Im} \bar{S}$ where $\bar{S}$ is the general inverse determined by $t_k, \dots t_{k+L}$. We shall also call $\bar{S}$ itself a follower. For any $0<p<L$, we shall say $[k, k+p]$ is *at the left end* of the follower and $[k+p, k+L]$ is *at the right end.*

Elementary properties of followers. — 1. Let $t_0 \dots t_n$ be a sequence of points with $t_{i+1}=f_\lambda(t_i)$, S the corresponding inverse and $[k, k+m]$ a follower for S (with critical point x_j). One has:

$$d(f_\lambda^L(x_j), t_{k+L})<\frac{\varepsilon_0}{2}, \qquad 0 \leqq L \leqq m,$$

$$d(f_\lambda^L(x_j), t_{k+L})<\frac{\varepsilon_1}{2}, \qquad 0 \leqq L \leqq m-q_0.$$

2. If S is some general inverse, and $[k, k+L+m]$ is a follower of S (with critical point x_i) and $[k+L, k+L+p]$ is another follower of S (with $L>r_i$) then $[L, L+\inf(m, p)]$ is an x_i-follower. This results from 1.

3. Let $[k, k+m]$ be a follower for some general inverse S. Then no other follower for S can start in $[k+2, k+\min(m, n_0^2)]$. This follows from 2, and the fact that the forward orbit of x_i stays in U for $2n_0^2$ iterates.

4.7. Canonical decomposition of an inverse. — In the canonical decomposition $S=S_0 T_1 \dots S_r$, the S_i, T_i have the properties explained below.

Let λ be (n, i, α)-good for $n \leqq N$, $i \leqq m_0$. Let S be a general inverse of length $n \leqq N$ determined by $t_0 \dots t_n$. Consider all followers $[k, k+L]$ for S such that

1. $t_{k-1} \in X$, if $k>0$.
2. $[k, k+L]$ is *maximal*, that is, no $[k, k+p]$ is a follower for $p>L$.
3. $n-k-L<\alpha^{1/2} L$.

If $[k_1, k_1+L_1]$, $[k_2, k_2+L_2]$ are two such with $k_1<k_2$, then $k_2-k_1 \geqq 2n_0^2$.

Case 1. — If $k_1+L_1 \leqq k_2+L_2$, then by 4.6.2, $[k_2-k_1, L_1]$ is an x_i-follower for some *i*. So $L_1-(k_2-k_1) \leqq \alpha L_1$, and:

$$n-k_1 \leqq (1+\alpha^{1/2}) L_1 \leqq (1+\alpha^{1/2})(k_2-k_1)/(1-\alpha),$$
$$n-k_2 \leqq (1-(1-\alpha)/(1+\alpha^{1/2}))(n-k_1)=\alpha^{1/2}(n-k_1).$$

Case 2. — If $k_2+L_2 \leqq k_1+L_1$, then $L_2 \leqq \alpha(L_2+k_2-k_1)$.

So

$$n-k_2 \leqq L_2(1+\alpha^{1/2}) \leqq \alpha(1+\alpha^{1/2})((n-k_1)-(n-k_2))/(1-\alpha),$$

and

$$n-k_2 \leqq \alpha(1+\alpha^{1/2})(n-k_1)/(1+\alpha^{3/2}) \leqq 2\alpha(n-k_1).$$

So now, if $[k_i, k_i+L_i]$ $(1 \leqq i \leqq r)$ are all the followers for S satisfying $1-3$ with $k_i<k_{i+1}$, then since $n-k_r \geqq 1$,

$$(r-1)\log(2\alpha^{1/2})+n \geqq 0, \qquad \text{and} \qquad r \leqq 1+2(\log n)/\log(1/2\alpha).$$

Let T_i be the inverse determined by $t_{k_i} \dots t_{k_i+r_i}$ mapping a neighbourhood of some $z_j(\lambda) \in V_j$ to a neighbourhood of $x_j(\lambda) \in X$. Then we can write $S = S_0 T_1 \dots T_r S_r$ (with possibly S_0 or $S_r = \text{identity}$). We may have to consider some S_i, T_i to be defined on balls of radii ε_0, but with this modification the composition $S_0 T_1 \dots S_r$ is well-defined on $B_{\varepsilon_1}(t_n)$ by 4.2, 4.3.

5. Fundamental estimates for inverses

In this section, the estimates work provided α is small enough and n_0 is large enough. Let $|\lambda| \leqq \eta_0$ and λ be (m, i, α) good for $m \leqq n$ and $i \leqq m_0$.

5.1. *There is $a_2 > 0$ so that if* S *is a critical inverse of length $n \geqq q_0$ determined by $t_0 \dots t_n$ and radius ε_0, then* Im S *is contained in the ball radius $e^{-a_2 n} \varepsilon_0$ round t_0.*

This follows from 4.2 if $n < n_0^2$.

Now let $n \geqq n_0^2$. Let J be the union in $[0, n]$ of followers of length $\geqq n_0$. Write

$$J = \bigcup_{i=1}^{k} [c_i, d_i] \qquad \text{with} \quad d_i - c_i \geqq n_0,\ c_{i+1} > d_i + 1.$$

For $1 \leqq i \leqq k$, let J'_i be the union of $[c_i, d_i]$ and of the followers (of length $< n_0$) starting in $[c_i, d_i]$. Put $J' = \bigcup_{i=1}^{k} J'_i$. Then $\# J' \leqq 2 \# J$.

Write $J' = \bigcup_{i=1}^{L} [c'_i, d'_i]$ with $d'_i - c'_i \geqq n_0$, $c'_{i+1} > d'_i + 1$.

Let J″ be the union of J′ and those intervals of $[d'_1 + 1, c'_2], \dots [d'_L + 1, n]$ of length $\leqq q_0$. (Note that $c_1 = c'_1 \geqq 2 n_0^2$.)

Write $J'' = \bigcup_{i=1}^{m} [c''_i, d''_i]$ with $c_1 = c'_1 = c''_1 \geqq 2 n_0^2$, $d''_i - c''_i \geqq n_0$, $c''_{i+1} - d''_i > q_0$, $d''_m = n$ or $n - d''_m \geqq q_0$. Then $\# (J'') \leqq 3 \# (J)$ (if $q_0 \leqq n_0$).

By construction (4.6.3) $d''_i + 1$ is not contained in any follower (for $1 \leqq i < m$, and $i = m$ if $d''_m \neq n$).

If $t_0 \dots t_n$ is the sequence associated to S (with t_0 a critical point) denote by S_1 the inverse associated to the subsequence $t_0, \dots, t_{c''_1}$, by S_{2i-1} $(1 < i < m)$ the inverse associated to $t_{d''_{i-1}+1}, \dots, t_{c''_i}$, by $S_{2i} (1 \leqq i \leqq m)$ the inverse associated to $t_{c''_i} \dots, t_{d''_i+1}$, and, if $d''_m \neq n$, by S_{2m+1} the inverse associated to $t_{d''_m+1}, \dots, t_n$.

So $S = S_1 \dots S_{2m+1}$ (perhaps with $S_{2m+1} = \text{id}$).

For $1 \leqq i < m$ (and $i = m$ if $d''_m \neq n$) S_{2i} is a *univalued* function on the disc of radius ε_0 centred at $t_{d''_i+1}$ (because $d''_i + 1$ does not belong to any follower), with values in the disc of radius $\varepsilon_0/2$ centred at $t_{c''_i}$. By Schwarz's Lemma, a disc of radius $\varepsilon < \varepsilon_0$ centred at $t_{d''_i+1}$ has its image by S_{2i} contained in the disc of radius $\varepsilon/2$ centred at $t_{c''_i}$.

For $1 \leqq i \leqq m$, if we take the domain of S_{2i-1} to be $B_{\varepsilon_0}(t_{c''_i})$, then since c''_i is the starting point of a follower and $c''_i - 1$ is not in a follower, $t_{c''_i}$ is near a critical point in X and

$B_{\varepsilon_0}(t_{c_i''}) \subseteq X$. So by 4.2,

$$|S'_{2i-1}| < \exp(-a_1 n_i) \qquad \text{where} \quad n_i = \text{length}(S_{2i-1}).$$

Writing $S_1 = T\tilde{S}_1$ with length $T = r_i$ if $t_0 = x_i(\lambda)$, one gets $|\tilde{S}'_1| \leqq \exp(-a_1 n_1)$ with $n_1 = \text{length } \tilde{S}_1$.

If $d''_m = n$, the image by S_{2m} of the ε_0-ball centred at t_n is contained in the $\varepsilon_1/2$-ball centred at $t_{c''_m}$. If $d''_m \neq n$, the image by S_{2m+1} of the ε_0-ball centred at t_n is contained in the ball centred at $t_{d''_m+1}$ and radius $\varepsilon_1 \exp(-a_1 n_m)$ where $n_m = \text{length}(S_{2m+1})$.

One concludes that the image by $\tilde{S}_1 S_2 \ldots S_{2m+1}$ of $B_{\varepsilon_0}(t_n)$ is contained in the ball centred at t_j ($j = r_i$ if $t_0 = x_i(\lambda)$) of radius $\varepsilon_1 \exp\left(-a_1 \sum_{i=1}^{m} n_i\right)$. Since $\sum_{i-1}^{m} n_i \geqq (1-3\alpha)n$, applying Lemma 1 to T, we obtain that $SB_{\varepsilon_0}(t_n)$ is contained in the ball of radius $\varepsilon_0 \exp(-a_2 n)$ round t_0 for suitable $a_2 > 0$.

5.2. *There is $a_3 > 0$ such that if S is a general inverse of length $n \geqq q_0$ determined by $t_0 \ldots t_n$ and radius $2\varepsilon_1$, then* Im S *is contained in the ball radius $e^{-a_3 n}\varepsilon_1$, round t_0.*

1. If $n < n_0^2$ this follows from 4.2.

Now let $n \geqq n_0^2$.

2. If there is k in $[p_0, n_0^2 - q_0]$ such that $B_{2\varepsilon_0}(t_k) \subseteq X$, write $S = S_1 S_2$ with S_1 determined by $t_0 \ldots t_k$. Then $S_2 B_{\varepsilon_1}(t_n) \subseteq B_\varepsilon(t_k)$ *where* $\varepsilon = \exp(-a_3(n-k))\varepsilon_1$ by the inductive hypothesis, since $n - k \geqq q_0$, and $|S'_1| \leqq \exp(-a_1 k)$ by 4.2.

3. If $S = S_1 S_2 S_3$ where length $S_1 \leqq p_0$, length $S_2 \geqq (1/2)n$ and S_2 is a follower, then the image of S_3 is contained in the ε_0-ball which is the domain of the critical inverse associated to S_2. So Im $S_2 S_3$ is contained in the ball radius ε round t_k with $\varepsilon = \varepsilon_0 \exp(-1/2\, a_2 n)$, if $k = \text{length } S_1$. We can also have S_2 so that $S_1 = \text{identify}$ or $B_{\varepsilon_0}(t_k) \subseteq X$, since t_k is near a critical point. So $|S'_1| \leqq C_0$ and Im S is contained in the ball radius $C_0 \varepsilon_0 \exp(-(1/2)\, a_2 n)$ round t_0. Since $n \geqq n_0^2$ we can assume this is $\leqq \varepsilon_1 \exp(-a_3 n)$ for suitable a_3 and n_0 large enough.

4. If none of the above is possible then we can write $S = S_1 S_2 S_3$ where length $S_1 \leqq p_0$, S_2 is a maximal follower and $n_0^2 - q_0 - p_0 \leqq \text{length } S_2 < (n/2)$.

Let $m = \text{length } S_2$, $t_k, \ldots t_{k+m}$ the sequence corresponding to S_2.

Put $m' = [\alpha m] + 1$, and let $\bar{S}_3$ be the inverse corresponding to $t_{k+m-m'}, \ldots t_n$. By the inductive hypothesis, Im $\bar{S}_3$ is contained in the ball of radius $\varepsilon_1 \exp - a_3(n-k-m+m')$. As S_2 is maximal and λ is (N, i, α)-good, from 4.7 any follower containing $k+m+1$ has left end point $> k+m-m'$.

Let S_4, S_5 be the general inverses with corresponding sequences $t_k, \ldots t_{k+m-m'}$ and $t_{k+m-m'} \ldots t_{k+m+1}$ respectively. There is b_2 such that $|f'_\lambda| \leqq \exp(b_2)$. Then the image by S_5 of the ε_1-ball B_0 centred at t_{k+m+1} contains the ball B_1 centred at $t_{k+m-m'}$ of radius $(\varepsilon_1 \exp - b_2(m'+1))$. So the restriction of S_4 to B_1 is univalued. By the estimate for critical inverses, the image of B_0 by the inverse corresponding to $t_k, \ldots t_{k+m+1}$ is contained in the ball of centre t_k, radius $\varepsilon_0 \exp(-a_2 m)$, hence this is also the case for the image of B_1 by S_4. By Schwarz's Lemma, we obtain that the image of the ε_1-ball centred at t_n by $S_2 S_3 = S_4 \bar{S}_3$ is contained in the ball centred at t_k of radius

$$\frac{\varepsilon_0 \exp(-a_2 m)}{\varepsilon_1 \exp(-b_2 (m'+1))} . \varepsilon_1 \exp -a_3 (n-(k+m-m'))$$

[provided α is small enough to have $b_2 (m'+1) < a_3 (n-(k+m-m'))$].

If we had $0 < a_3 < a_2 - b_2 \alpha$ (and therefore α small enough) and n_0 big enough, we get the desired estimates (as $|S'_1| < C_0$).

5.3. *There is a constant* $C>0$ *such that if* $S = S_0 T_1 S_1 \dots S_r$ *is the canonical decomposition, then* $\sum_{i=0}^{r} \operatorname{Var} \operatorname{Log} |S'_i|_0 T_{i+1} \dots \leqq C$. C *depends on* α *but not on* n_0. (The earlier constants a_2, a_3 did *not* depend on α.)

Let S correspond to the sequence $t_0 \dots t_n$.

1. Assume first that there exists $k < n$ such that :

(i) $[0, k]$ is a maximal follower,

(ii) $n-k \geqq \alpha^{1/2} k$ [equivalently $k \leqq n/(1+\alpha^{1/2})$].

S is then of the forms $T\tilde{S}$ with length $T \leqq 2$ and we want to estimate the variation of $\operatorname{Log} |T'|$ on the image $\tilde{B}$ by $\tilde{S}$ of $B_{\varepsilon_1}(t_n)$.

(*a*) $k \leqq n_0$. Let S_1 be the inverse corresponding to $t_0 \dots t_k$, S_2 the inverse corresponding to $t_k \dots t_n$, and write $S_1 = T\tilde{S}_1 = T_j W_j^{k-r_j}$. Let $B = B_{\varepsilon_0}(t_k)$. Then $W_j^{k-r_j}$ and $T_j W_j^{k-r_j}$ are *univalent* functions on B. The image B′ by S_2 of $B_{\varepsilon_1}(t_n)$ is contained in $B_{(1/2)\varepsilon_0}(t_k)$ and is contained in the ball centred at t_k of radius $\varepsilon_1 \exp(-a_3(n-k))$ if $n-k \geqq q_0$. So using the distortion theorem for univalent functions (*cf.* [D] for instance) one gets

$$\operatorname{Var}_{B'} \operatorname{Log} |(W_j^{k-r_j})'| \leqq C',$$

$$\operatorname{Var}_{B'} \operatorname{Log} |(T_j W_j^{k-r_j})'| \leqq C'$$

and if $n-k \geqq q_0$,

$$\operatorname{Var}_{B'} \operatorname{Log} |(W_j^{k-r_j})'| \leqq C' \exp -a_3 (n-k),$$

$$\operatorname{Var}_{B'} \operatorname{Log} |(T_j W_j^{k-r_j})'| \leqq C' \exp -a_3 (n-k).$$

Hence

$$\operatorname{Var}_{\tilde{B}} \operatorname{Log} |(T_j)'| \leqq 2 C',$$

$$\operatorname{Var}_{\tilde{B}} \operatorname{Log} |(T_j)'| \leqq 2 C' \exp\left(-a_3 \frac{\alpha^{1/2} n}{1+\alpha^{1/2}}\right)$$

if $n-k \geqq q_0$, which occurs as soon as $n \geqq ((1+\alpha^{1/2})/\alpha^{1/2}) q_0$.

(*b*) $k \geqq n_0$. Let S_1, S_2 have the same meaning as in (*a*). Put $k' = [\alpha k] + 1$. Let $S_4 = T\tilde{S}_4$ be the inverse corresponding to $t_0, \dots t_{k-k'}$. By arguments similar to 5.2.4., $\tilde{S}_4$, S_4 are *univalent* functions on the disc centred at $t_{k-k'}$ of radius $\varepsilon_1 \exp -b_2 (k'+1)$. On the other hand, if S_3 is the inverse corresponding to $t_{k-k'} \dots t_n$, the image B by S_3

of the ε_1 ball centred at t_n is contained in the ball centred at $t_{k-k'}$ of radius $\varepsilon_1 \exp(-a_3(n-k+k'))$ (provided n_0 is big enough).

Now one has:

$$a_3(n-k+k')-b_2(k'+1) \geqq a_3(n-k)-2b_2(k'-1)$$
$$\geqq \frac{a_3}{2}(n-k)+k\left(a_3\frac{\alpha^{1/2}}{2}-2b_2\alpha\right) \geqq \frac{a_3}{2}\,\frac{\alpha^{1/2}}{1+\alpha^{1/2}}n.$$

if α is small enough.

Using the distortion theorem we get:

$$\mathrm{Var}_{\mathrm{B}}(\mathrm{Log}\,|S_4'|) \leqq C' \exp\left(-\frac{a_3}{2}\,\frac{\alpha^{1/2}}{1+\alpha^{1/2}}n\right),$$
$$\mathrm{Var}_{\mathrm{B}}(\mathrm{Log}\,|\tilde{S}_4'| \leqq C' \exp\left(\frac{-a_3}{2}\,\frac{\alpha^{1/2}}{1+\alpha^{1/2}}n\right),$$

hence

$$\mathrm{Var}_{\tilde{\mathrm{B}}}(\mathrm{Log}\,|T'|) \leqq 2C' \exp\left(\frac{-a_3}{2}\,\frac{\alpha^{1/2}}{1+\alpha^{1/2}}n\right).$$

2. In the general case, if $S=S_0 T_1 S_1 \dots$ is the canonical decomposition of S, one obtains from 1 the estimate

$$\sum_{i=0}^{r} \mathrm{Var}\,\mathrm{Log}\,|S_i'| \circ T_{i+1} \circ \dots S_r \leqq C$$

for suitable C.

For write $S_i = U_i^1 \dots U_i^{n_i}$, where U_i^{2j} has length 1 or 2 and is the end of a follower, and U_i^{2j+1} are univalent inverses of f_λ. Then 1 gives an estimate on $\mathrm{Var}\,\mathrm{Log}\,|(U_i^{2j})'|$, and $\mathrm{Var}\,\mathrm{Log}\,|(U_i^{2j+1})'| \leqq C' \exp(-a_3(n-m))$ by univalent function theory, if t_m is the centre of domain (U_i^{2j+1}).

6. Proof of Theorem B

6.1. Lemma. — *Let* $F'(z)=A(z)\prod_{i=1}^{u}(z-z_i)$ *for* $|z| \leqq r$, *where* $|z_i|<r$ *and* $|A(z)|>0$ *for* $|z| \leqq r$. *Let* B *be a ball of radius* δ *in the image of* F, *and let* T *be a multivalued inverse of* F *on* B *with connected image. Then:*

1. *If* $w, w' \in B$ *with* $\min_i |w-F(z_i)| = C_2\delta$ *and* $\mathrm{Min}_i |w'-F(z_i)| = \eta\delta$, *then* $|T'w'/T'w| \leqq C_3 \eta^{-u/u+1}$, *where* C_3 *depends only on* C_2, r, A.

2. TB *has diameter between* $C_3 \mathrm{Min}_{z\in B} |T'z|\,\delta$ *and* $(1/C_3)\,\mathrm{Min}_{z\in B} |T'z|\,\delta$.

Proof. — We may assume all $F(z_i)$ are distance $O(\delta)$ from the boundary of B [by restricting B to a smaller ball if necessary, but still of radius $O(\delta)$]. We know that $|T'|$ varies by a bounded proportion on a set of points in B which are distance $O(\delta)$ from all $F(z_i)$ (since this set is contained in a union of larger balls on each of which T is univalued), in particular on a neighbourhood of the boundary of B of width $O(\delta)$.

Let ε be the diameter of Im T. Let $F(z_0)=w$, $F(z'_0)=w'$. We first consider $|F'(z_0)|/|F'(z'_0)|$ for z'_0 at a distance $O(\varepsilon)$ from the boundary of Im T. In fact, we shall see that this is sufficient.

(A) Let $z, z' \in \mathrm{Im}\,T$. Let $|y-z'|$ be $\geqq O(|z-z'|)$ for any $y \in F^{-1}F(z) \cap \mathrm{Im}\,T$. Then $|F(z)-F(z')| \geqq O(|F'(z)||z-z'|)$.

For let I be a connected path with one endpoint at z' and mapping onto the straight line segment between $F(z)$ and $F(z')$. Then the endpoints of I are $\geqq O(|z-z'|)$ apart, and $|F(z)-F(z')| = \int_I |F'(y)||dy|$. But $|F'(y)| \geqq O(|F'(z)|)$ if $|y-z_i| \geqq O(|z-z_i|)$ for all i, and this is true on a segment of I of length $\geqq O(|z-z'|)$, giving (A).

Thus in particular:

(B) if $z' \in \mathrm{Im}\,T$ is at a distance $O(\varepsilon)$ from $\partial\, \mathrm{Im}\,T$, then

$$\delta \geqq |F(z')-\partial B| \geqq O(\underset{w \in \partial B}{\mathrm{Max}}|F'(w)||z-z'| = O(\underset{w \in B}{\mathrm{Max}}|F'(w)|)\,\varepsilon \geqq O(\delta).$$

This proves 2. If there is no $z_i \in \mathrm{Im}\,T$ distance $O(\varepsilon)$ from the boundary then $|F'(z_0)|/|F'(z'_0)|$ is automatically bounded. If there is at least one z_i in Im T, let z_1 be nearest to z'_0. If some z''_0 in $\mathrm{Im}\,T \cap F^{-1}F(z'_0)$ has $|z''_0-z_1| \ll |z'_0-z_1|$ then $|F'(z''_0)| \leqq O(|F'(z'_0)|)$ and we replace z'_0 by z''_0. We can then still assume z'_0 is at a distance $O(\varepsilon)$ from ∂ Im T. Now $|z'_0-z_1| \leqq O(|z'_0-z_i|)$ for *all* z_i (not just z_i in Im T). We may also assume z_0 is the nearest element in $F^{-1}F(z_0) \cap \mathrm{Im}\,T$ to z_1, since $F(z_0)$ is at a distance $O(\delta)$ from all $F(z_i)$, and hence $|F'(z_0)|/|F'(z)|$ is bounded above and below for $z \in F^{-1}F(z_0)$. This gives:

$$|F(z_1)-F(z_0)| \geqq O(|F'(z_0)|)|z_1-z_0| \quad \text{by (A)}.$$

Also $|z_1-z_0| \geqq O(|z_1-z'_0|)$. For

$$|F'(z'_0)||z_1-z'_0| \leqq O(|F(z_1)-F(z'_0)|) \quad [\text{by (A)}] \leqq O(|F(z_1)-F(z_0)|)$$
$$\leqq O(|F'(z_0)|)|z_1-z_0| \quad [\text{since } |F'(z_0)| = O(\underset{z \in \mathrm{Im}\,T}{\mathrm{Max}}|F'(z)|)].$$

So if

$$|z_1-z_0| \leqq |z_1-z'_0| \quad \text{then } |z_i-z_0| \leqq |z_i-z'_0|+|z'_0-z_1|+|z_1-z_0| \leqq O(|z_i-z'_0|),$$

and $|F'(z_0)| \leqq O(F'(z'_0)|$, which implies

$$|F'(z'_0)||z_1-z'_0| \leqq O(|F'(z'_0)|)|z_1-z_0|.$$

So $|z_1-z_0| \geqq O(|z_1-z'_0|)$ as required.

Then

$$\frac{|z_i-z_0|}{|z_i-z'_0|}\leqq O\left(\frac{|z_1-z_0|}{|z_1-z'_0|}\right) \quad \text{for all } i,$$

since

$$|z_i-z_0|\leqq|z_i-z'_0|+|z'_0-z_1|+|z_1-z_0|\leqq 2|z'_0-z_i|+|z_1-z_0|.$$

Then

$$\frac{|F'(z_0)|}{|F'(z'_0)|}\leqq O\left(\prod_{i=1}^{u}\frac{|z_0-z_i|}{|z'_0-z_i|}\right)$$

$$\leqq O\left(\left(\left(\prod_{i=1}^{u}\left|\frac{z_0-z_i}{z'_0-z_i}\right|\right)\frac{|z_0-z_1|}{|z'_0-z_1|}\right)^{u/u+1}\right)$$

$$\leqq O\left(\left(\frac{|F'(z_0)|}{|F'(z'_0)|}\,\frac{|z_0-z_1|}{|z'_0-z_1|}\right)^{u/u+1}\right)\leqq O\left(\left(\frac{|F(z_0)-F(z_1)|}{|F(z'_0)-F(z_1)|}\right)^{u/(u+1)}\right)$$

(using that $|F'(z'_0)|$ is proportional to the maximum of $|F'(z)|$ on the line segment from z'_0 to z_1 for the denominator)

$$\leqq O\left(\left(\frac{\delta}{\eta\delta}\right)^{u/(u+1)}\right)=O(\eta^{-u/(u+1)}).$$

Thus, a set of points at a distance $O(\delta)$ from ∂B in B has image under T at a distance $O(\varepsilon)$ from $\partial \operatorname{Im} T$, so the proof is completed.

6.2. Lemma. — *Let* $S=S_0T_1S_1\ldots S_r$ *be defined on a ball* B. Let X *be any subset of* B. *For a constant* D,

$$\operatorname{Meas}(SX)\leqq D^r \operatorname{Meas}(\operatorname{Im} S)(\operatorname{Meas} X)^{(1+u)^{-r}}.$$

Proof. — Prove inductively on r that $\operatorname{Im}(S_0T_1\ldots S_r)$ contains a ball of radius δ, is contained in a ball of radius $D_1^r\delta$, and that

$$\operatorname{Meas}(S_0\ldots S_rX)\leqq E_r\delta^2(\operatorname{Meas}(X))^{(1+u)^{-r}},$$

where $E_r=D_2^r E_{r-1}^{1/u+1}$, so that $E_r\leqq E^r$ where

$$\operatorname{Log} E\geqq\left(1+\frac{1}{1+u}+\ldots+\frac{1}{(1+u)^{r-1}}\right)\log D_2.$$

So assume meas $(S_1\ldots S_rX)\leqq E_{r-1}\delta_1^2(\operatorname{Meas}(X))^{(1+u)^{-r+1}}$, where $S_1\ldots S_rB$ contains a ball of radius δ_1 and is contained in a ball of radius $D_1^{r-1}\delta_1$. Since $|S'_0|$ varies by a bounded proportion on $T_1\ldots S_rB$, it suffices to prove the inductive result for $T_1\ldots S_r$. The result will then be true for $S_0\ldots S_r$ for D_1, D_2 sufficiently large independent of r.

Write $X_1 = S_1 \dots S_r X$ and let B_1, B_2 denote respectively the ball of radius $D_1^{r-1} \delta_1$ containing $S_1 \dots S_r B$ and the ball of radius δ_1 contained in $S_1 \dots S_r B$.

Take any $z_0 \in B_1$ with $\operatorname{Min}_i |z_0 - z_i| = O(D_1^{r-1} \delta_1)$, where $z_i (i \leqq u)$ are the singularities of T_1. We know $|T_1'(z_0)|$ is boundedly proportional to $|T_1'(z)|$ for any $z \in B_1$ with $\operatorname{Min}_i |z - z_i| = O(D_1^{r-1} \delta_1)$.

By 6.1 $T_1 B_1$ has diameter $\leqq \text{Const.}\, D_1^{r-1} \delta_1 |T_1'(z_0)|$, and $T_1 B_2$ contains a ball of radius $\delta = \text{Const.}\, \delta_1 |T_1'(z_0)|$.

Then

$$\operatorname{meas}(T_1 X_1) = \int \chi_{X_1}(T_1^{-1} z)\, d|z|^2 = \int \chi_{X_1}(z) |T_1'(z)|^2 d|z|^2$$

$$\leqq \text{Const.} \int \chi_{X_2}(z) |T_1'(z_0)|^2 \left((\operatorname{Min}_i |z - z_i|)^{-1} \operatorname{Min}_i |z_0 - z_i| \right)^{2u/u+1} d|z|^2$$

(by 6.1) where, for some ρ, $X_2 = \{ z : \operatorname{Min}_i |z - z_i| < \rho \}$ and $\operatorname{meas}(X_2) = \operatorname{meas}(X_1)$.

So

$$\operatorname{meas}(T_1 X_1) \leqq \text{Const.}\, |T_1'(z_0)|^2 (D_1^{r-1} \delta_1)^{2u/(u+1)} \sum_i \int_{|z - z_i| < \rho} |z - z_i|^{-2u/(u+1)} d|z|^2$$

$$= \text{Const.}\, |T_1'(z_0)|^2 (D_1^{r-1} \delta_1)^{2u/(u+1)} \rho^{2/u+1}.$$

But $\rho^2 = \text{Const.}\, \operatorname{meas}(X_1) \leqq \text{Const.}\, E_{r-1} \delta_1^2 (\operatorname{meas}(X))^{(1+u)^{-r+1}}$.

So $\operatorname{meas}(T_1 X_1) \leqq \text{Const.}\, (|T_1'(z_0)| \delta_1)^2 D_1^{(r-1)2u/(u+1)} (E_{r-1})^{1/(u+1)} (\operatorname{meas} X)^{(1+u)^{-r}}$.

Then since $\delta = \text{Const.}\, |T_1'(z_0)| \delta_1$ we obtain the result.

6.3. Proposition. — *Let $|\lambda| \leqq \eta_0$ and let λ be (m, i, α)— good for $m \leqq n$, $i \leqq m_0$. Let $X_{s,n}$ be the union of all* Im S *with* S *of length n and such that in the canonical decomposition* $S = S_0 T_1 \dots S_r$, $r > s$. *Then* $\operatorname{meas}(X_{s,n}) < \eta_s$ *where η_s is independent of n and $\eta_s \to 0$ as $s \to \infty$.*

Proof. — Let S be such that there are $> s\,T's$ in the canonical decomposition of S. Write $S = W_1 W_2$, where W_2 is determined by $t_{n-m} \dots t_n$ if S is determined by $t_0 \dots t_n$, and $n-m$ is the smallest integer such that some $[n-m, n-m+L]$ is a follower with $m - L < 2\alpha^{1/2} L$. Then $m > (1/3\alpha)^{s/2}$ by the same argument as in 4.7. Also, if $[p, p+q]$ is a follower in W_1 with $n - m - (p+q) < \alpha^{1/2} q$, then by minimality of $n-m$ we know $n - (p+q) > 2\alpha^{1/2} q$. So $2\alpha^{1/2} q < \alpha^{1/2} q + m$ and $q < \alpha^{-1/2} m$. So in the canonical decomposition of W_1 there are $t\,T's$ where

$$t \leqq 1 + \frac{2 \log(\alpha^{-1/2} m)}{\log((1/2)\alpha)}.$$

So if we now take all S whose union is $X_{s,n}$ and consider the decomposition $S=W_1 W_2$, then we may as well assume W_1 has radius $2\varepsilon_1$ centred on some x_i. If we fix i, m, L and look at all possible W_2 with [0, L] an x_i-follower for W_2, we see that $\operatorname{Im} W_2 \subseteq \operatorname{Im} \bar{W}_2$ where $\bar{W}_2$ is the inverse of length L with domain radius ε_0 determined by $x_i \ldots f_\lambda^L x_i$.

So $$X_{s,n} \subseteq \bigcup_{\substack{i \leqq m_0 \\ \alpha^{1/2} L \leqq m-L \\ n \geqq m > (1/3\,\alpha)^{s/2}}} \{\operatorname{Im} W_1 \bar{W}_2 : W_1 \text{ is inverse length } n-m \text{ with domain } B_{2\varepsilon_1}(x_i) \text{ and } \bar{W}_2 \text{ is determined by } x_i \ldots f_\lambda^L(x_i) \text{ and radius } \varepsilon_0\}.$$

Then by 6.2,

$$\operatorname{meas}(\operatorname{Im} W_1 \bar{W}_2) \leqq \operatorname{meas}(\operatorname{Im} W_1)\, D^t (e^{-2a_2 L} \varepsilon_0^2)^{(1+u)^{-t}} \leqq e^{-m^\gamma} \operatorname{meas}(\operatorname{Im} W_1) \quad \text{for} \quad \gamma > 0$$

such that

$$t \log(1+u) < (1-\gamma') \operatorname{Log} L \qquad \text{for a} \quad \gamma' > \gamma.$$

Then summing over L, W_1, i, m, we obtain

$$\operatorname{meas}(X_{s,n}) \leqq \sum \operatorname{meas}(\operatorname{Im} W_1) \sum_{m > (1/3\,\alpha)^{s/2}} m^2 e^{-m^\gamma} < \sum_{m > (1/3\,\alpha)^{s/2}} m^2 e^{-m^\gamma} \to 0 \text{ as } s \to \infty.$$

6.4. Proof of theorem B. — Let λ denote spherical measure, transferred to $\mathbb{C} \cong$, with $\lambda(\bar{\mathbb{C}})=1$. Let $(f^n)_* \lambda$ be defined by $(f^n)_* \lambda(A) = \lambda(f^{-n} A)$. Then 6.2, 6.3 show the measures $(f^n)_* \lambda$ are uniformly absolutely continuous with respect to λ. For given $\varepsilon > 0$, choose $\eta_s < \varepsilon/2$. Choose a cover u of $\bar{\mathbb{C}}$ of index p, by balls of radius ε_1. Then {Im S: domain S ∈ u, length $S=n$} is also a cover of index p for each n. Now choose δ so that $\delta^{(1+u)^{-s}} < \varepsilon/2p$. Then if $\lambda(A) < \delta$,

$$(f^n)_* \lambda(A) \leqq \sum \{\lambda(SA) : \text{domain } S \in u, \text{ length } S = n\}$$
$$< \eta_s + \sum \{\lambda(SA) : \text{domain } S \in u, \text{ length } S = n, \text{ S has } \leqq s\ T's \text{ in its canonical decomposition}\}.$$

Then for S in the sum, $\lambda(SA) \leqq \lambda(\operatorname{Im} S)\, \delta^{(1+u)^{-s}}$ (6.2) $< \lambda(\operatorname{Im} S)\, \varepsilon/2p$. So

$$(f^n)_* \lambda(A) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2p} \sum \{\lambda(\operatorname{Im} S) : \text{domain } S \in u, \text{ length } S \leqq n\} < \varepsilon,$$

proving uniform absolute continuity.

So then any weak limit of $(1/(n+1)) \sum_{i=0}^{n} (f^i)_* \lambda$ is an invariant measure absolutely continous with respect to λ. (This type of argument comes from [SZ].)

To show ergodicity of λ, let A be any set satisfying $f_\lambda(A) \subseteq A$, $\lambda(A) > 0$. Since $\operatorname{diam}(\operatorname{Im} S) \to 0$ as length $S \to \infty$ (uniformly in S), for sufficiently large n A can be approximated to within $(\varepsilon/p)\,\lambda(A)$ in λ-measure by a union of sets Im S with S of length

n and domain $S \varepsilon u$. That is, we can have:

$$\lambda(\cup \{\operatorname{Im} S: \operatorname{Im} S \cap A \neq \emptyset, \text{ domain } S \in u, \text{ length } S = n\}) < \lambda(A)\left(1 + \frac{\varepsilon}{p}\right).$$

Then $\sum \lambda(\operatorname{Im} S \setminus A) < \varepsilon\lambda(A) \leqq \varepsilon \sum \lambda(\operatorname{Im} S)$, where both summations are taken over S with length $S = n$, domain $S \in u$, $\operatorname{Im} S \cap A \neq \emptyset$. So for a set of S with $\sum \lambda(\operatorname{Im} S) > \lambda(A)/2$, $\lambda(\operatorname{Im} S \setminus A) < 2\varepsilon\lambda(\operatorname{Im} S)$. Suppose $\eta_s < \lambda(A)/4p$. Then there is an S with $\leqq s$ T′s in its canonical decomposition with $\lambda(\operatorname{Im} S \setminus A) < 2\varepsilon\lambda(\operatorname{Im} S)$. Then applying f_λ^n to $\operatorname{Im} S \setminus A$ by 5.3, since $f_\lambda(A) \subseteq A$, $\lambda(\text{domain } S \setminus A) < \delta(\varepsilon, s)\lambda$ (domain S) where $\delta(\varepsilon, s) \to 0$ as $\varepsilon \to 0$. Letting $\varepsilon \to 0$, there is at least one ball B of radius ε_1 which is contained a. e. in A. We can also take B to have non-null intersection with the Julia set of f_λ-for the positive measure A can be reduced if necessary (but keeping the property $f_\lambda(A) \subseteq A$) so that it does not contain any ball radius ε_1 not intersecting the Julia set. So $f_\lambda^n(B) = \overline{\mathbb{C}}$ for some N, by the expanding property of the Julia set. So $\lambda(\overline{\mathbb{C}} - A) = 0$ and λ is ergodic. By ergodicity, we obtain that any invariant measure absolutely continuous with respect to λ is unique (up to a scalar), and equivalent to λ.

7. Proof of theorem C

The following proposition is the key idea in the proof of Theorem C. As mentioned before, the idea of using the power series of an analytic function is due to the referee.

7.1. Proposition. — *Let* $N \geqq n_0^{3/2}$, *let* $|\lambda| \leqq \eta_0$ *and* λ *be* (n, i, α)*-good for all* $n \leqq N$, $i \leqq m_0$. *The union of images of inverses* S *of length* N *such that more than* αN *points in* $[0, N]$ *are in a follower for* S *of length* $\geqq n_0$ *has measure* $\leqq \exp(-a_4 N)$, *where* a_4 *depends on* α *but not on* n_0.

Note. — Later we shall need to apply this with follower length $\geqq (1/4) n_0$, which is easily deduced by changing variable names. We use n_0 here for easier writing.

Proof. — (*a*) Let S be an inverse of length N. Let $c'_1 \ldots c'_r$ be the starting points of followers of S of length $\geqq n_0$. Put $c_i = c'_i - 1$ or c'_i depending on whether there exists a follower starting at $c'_i - 1$ or not. Reindex to have no repetitions amongst the c_i so that $c_1 < \ldots c_s$. Let d'_i be the biggest integer contained in a follower starting at c_i or $c_i + 1$: if $d'_i \geqq c_{i+1}$ (respectively $d'_s = N$) put $d_i = c_{i+1}$ (respectively $d_s = N$); if $d'_i < c_{i+1}$ and there is a maximal follower (of length $< n_0$) starting in $[c_i, d'_i]$ with right endpoint bigger than d'_i, then call this right endpoint d_i. In all cases one has $d_i - c_i \geqq n_0$, $1 \leqq i \leqq s$.

Define

$$u_i = d_i - c_i \quad \text{for} \quad 1 \leqq i \leqq s,$$
$$v_i = c_{i+1} - d_i \quad \text{for} \quad 1 \leqq i < s,$$
$$v_0 = c_1,$$
$$v_s = N - d_s.$$

Then $\sum u_i + \sum v_i = \mathrm{N}$ *and* $u_i \geqq n_0$ for $1 \leqq i \leqq s$.

(*b*) Estimate the union of images of S which have fixed s, u_i, v_i. Write $\mathrm{S} = \mathrm{V}_0 \mathrm{U}_1 \mathrm{V}_1 \ldots \mathrm{V}_s$ where V_i has length v_i and U_i has length u_i.

We may assume U_i is determined by one of the following:

(i) a sequence $x_j \ldots f_\lambda^{u_i} x_j$, and U_i has domain $\mathrm{B}_{\varepsilon_0}(f_\lambda^{u_i}(x_j))$;

(ii) a sequence a, $x_j \ldots f_\lambda^{u_i - 1}(x_j)$, where $f_\lambda(a) = x_j$, and the domain has radius ε_0 centre $f_\lambda^{u_i - 1}(x_j)$;

(iii) $\mathrm{U}_i = \mathrm{W}_i \mathrm{X}_i$ where W_i is determined by a sequence as in (*a*) or (*b*) [but with u_i replaced by length (W_i)] and X_i is determined by $x_k \ldots f_\lambda^m(x_k)$ where $m = \text{length } \mathrm{X}_i < n_0$ and the domain has radius ε_0, centre $f_\lambda^m(x_k)$.

Then in all cases, if $\mathrm{V}_i \neq$ identity and V_i is determined by $t_0 \ldots t_p$, U_i is univalued on $\mathrm{B}_{2\varepsilon_1}(t_1)$. If $\mathrm{V}_i \neq$ identity but $\mathrm{V}_k =$ identity for $j \leqq k \leqq i$ then $\mathrm{U}_j \ldots \mathrm{U}_i$ is univalued on $\mathrm{B}_{2\varepsilon_1}(t_1)$. Then the distortion theorem for univalent functions [D] gives $|(\mathrm{U}_j \ldots \mathrm{U}_i)'|^2 \leqq \mathrm{C}^2$ meas $(\mathrm{Im}(\mathrm{U}_j \ldots \mathrm{U}_i))$ on $\mathrm{Im}\ \mathrm{V}_i \ldots \mathrm{V}_s$ if $i < s$ or if length $(\mathrm{V}_i \ldots \mathrm{V}_s) \geqq q_0$.

By (i)$-$(iii) there are $\leqq m_0^2 d n_0$ possibilities for each U_i. We can also assume domain (V_{i-1}) is the ball radius ε_1 round the starting point of U_i, $i \leqq s$, so there are $m_0(d+1)$ possibilities for domain (V_{i-1}). V_{i-1} is univalued on this ball, so we have, by the distortion theorem

$$|(\mathrm{V}_{i-1})'|^2 \leqq \mathrm{C}\ \mathrm{Meas}(\mathrm{Im}(\mathrm{V}_{i-1})) \quad \text{on } \mathrm{Im}\,\mathrm{U}_i \ldots \mathrm{V}_s, \quad i \leqq s.$$

We can also assume domain (V_s) is one of a finite number of balls $\mathrm{B}(w_i, 2\varepsilon_1)$ by taking a finite cover of S^2 by balls $\mathrm{B}(w_i, \varepsilon_1)$: every ball of radius ε_1 will then be in some $\mathrm{B}(w_i, 2\varepsilon_1)$. So for fixed s, u_i, v_i we obtain a bound for the measure of the union of all possible $\mathrm{Im}\,\mathrm{V}_0 \mathrm{U}_1 \ldots \mathrm{U}_s$ of the form $(\mathrm{K} n_0)^s \exp(-2 a_3 \sum u_i)$ for some constant K.

(*c*) So to prove the proposition, it is sufficient to show that

$$\sum_{s \geqq 1} \sum_{\alpha \mathrm{N} \leqq k \leqq \mathrm{N}} (\mathrm{K} n_0)^s \exp(-2 a_3 k) \sum_{\substack{u_1 + \ldots + u_s = k \\ v_0 + \ldots + v_s = \mathrm{N} - k \\ u_i \geqq n_0 \\ v_i \geqq 0}} 1 \leqq \exp(-a_4 \mathrm{N}).$$

The last sum is less than the coefficient of

$$\mathrm{X}^k \mathrm{Y}^{\mathrm{N}-k} \quad \text{in} \left(\frac{\mathrm{X}^{n_0}}{1-\mathrm{X}}\right)^2 \left(\frac{1}{1-\mathrm{Y}}\right)^s.$$

We have (as a formal series)

$$\sum_{s \geqq 0} (\mathrm{K} n_0)^s \left(\frac{\mathrm{X}^{n_0}}{1-\mathrm{X}}\right)^s \left(\frac{1}{1-\mathrm{Y}}\right)^2 = \frac{1}{1 - (n_0 \mathrm{K} \mathrm{X}^{n_0}/((1-\mathrm{X})(1-\mathrm{Y})))}.$$

For n_0 big enough, this function is holomorphic and bounded for $|\mathrm{X}| \leqq \exp(-a_3 \alpha)$, $|\mathrm{Y}| \leqq \exp(-a_3 \alpha)$; hence the coefficient of $\mathrm{X}^k \mathrm{Y}^{\mathrm{N}-k}$ is less than $\mathrm{K}' \exp(a_3 \alpha \mathrm{N})$. So the triple summation is bounded by

$$\sum_{\alpha N \leqq k \leqq N} K' \exp(a_3 \alpha N) \exp(-2 a_3 k) \leqq K' N \exp(-a_3 \alpha N).$$

This gives the result.

7.2. THE FUNCTION $g_n^i(\lambda)$: FIRST ESTIMATE. — Now define $g_n^i(\lambda) = f_n(\lambda, y_i(\lambda))$. Let N_i be the largest integer such that g_n^i maps $\{\lambda : |\lambda| \leqq \eta_0\}$ into U_i for all $n \leqq N_i$, so that $N_i \geqq 2 n_0^2$.

PROPOSITION. — *There is a constant* K_1 *such that*

$$\frac{1}{K_1} |(f_0^n)'(z_i(0))| \leqq \left| \frac{dg_n^i(\lambda)}{d\lambda} \right| \leqq K_1 |(f_0^n)'(z_i(0))| \quad \textit{for all } n \leqq N_i, \ |\lambda| \leqq \eta_0,$$

and such that the image of $\{\lambda : |\lambda| \leqq \eta_0\}$ *under* $g_{N_i}^i$ *contains the ball radius* $2\varepsilon_0$ *round* $z_i(0)$, *assuming* η_0, ε_0 *and* U_i *are small enough* (*independently of* n_0).

Proof. — Denote by φ_λ the conjugacy with image U_i, $\varphi_\lambda(0) = z_i$, $\varphi_\lambda'(0) = 1$, such that $f_\lambda(\varphi_\lambda(z)) = \varphi_\lambda(\mu(\lambda) z)$ for $z \in \varphi_\lambda^{-1}(U_i)$, where $\mu(\lambda) = f_\lambda'(z_i)$. Put $t(\lambda) = \varphi_\lambda^{-1}(y_i(\lambda))$, and write $\varphi_\lambda(z) = \varphi(\lambda, z)$. Then $|\partial\varphi/\partial\lambda(\lambda, z)|$, $|\partial\varphi/\partial z(\lambda, z)|^{\pm 1}$ are bounded for $\lambda \leqq \eta_0$, $z \in U_i$, and $\varphi_\lambda^{-1}(U_i)$ is bounded. Then for $n \leqq N_i$, $g_n^i(\lambda) = \varphi(\lambda, (\mu(\lambda))^n t(\lambda))$, and $(\mu(\lambda))^n t(\lambda) \in \varphi_\lambda^{-1}(U_i)$. Then:

$$\frac{dg_n^i}{d\lambda}(\lambda) = \frac{\partial\varphi}{\partial\lambda}(\lambda, (\mu(\lambda))^n t(\lambda)) + \frac{\partial\varphi}{\partial z}(\lambda, (\mu(\lambda))^n t(\lambda)) B,$$

where $B = (\mu(\lambda))^{n-1} n \mu'(\lambda) t(\lambda) + (\mu(\lambda))^n t'(\lambda)$.

So $|dg_n^i/d\lambda| |B|^{-1}$ is bounded above and below. $|\mu'(\lambda)|$ is bounded for $|\lambda| \leqq \eta_0$, and we have seen that $(\mu(\lambda))^n t(\lambda)$ is too, for $|\lambda| \leqq \eta_0$, $n \leqq N_i$. Then $t'(\lambda)$ is near $t'(0)$, and since $y_i(\lambda) - z_i(\lambda) = \varphi(\lambda, t(\lambda)) - \varphi(\lambda, 0)$, and $t(0) = 0$, $t'(0) = y_i'(0) - z_i'(0) \neq 0$.

So $|B|^{-1} |\mu(\lambda)|^n$ is bounded above and below. So the proposition follows, provided we can show that, for $|\lambda| \leqq \eta_0$ and $n \leqq N_i$, $|\mu(\lambda)|^n |\mu(0)|^{-n}$ is bounded above and below. Suppose $|\lambda| \leqq |\mu(0)|^{-N_i/2}$. Then, since $|\mu'|$ is bounded,

$$\log|\mu(\lambda)|^n - \log|\mu(0)|^n \leqq O(n |\mu(0)|^{-N_i/2}) \leqq 1 \qquad \text{for} \quad n \leqq N_i.$$

So for such λ, $|\mu(\lambda)|^n |\mu(0)|^{-n}$ is bounded above and below. We deduce that $\eta_0 < |\mu(0)|^{-N_i/2}$. For otherwise, $|dg_n^i(\lambda)/d\lambda| |\mu(0)|^{-n}$ is bounded for $|\lambda| \leqq |\mu(0)^{-N_i/2}$, $n \leqq N_i$, and $g_{N_i}^i$ maps the set $\{|\lambda| \leqq |\mu(0)|^{-N_i/2}\}$ to a set of diameter $O(|\mu(0)|^{N_i/2})$. So $\eta_0 < |\mu(0)|^{-N_i/2}$, and $|dg_n^i/d\lambda(\lambda)| |\mu(0)|^{-n}$ is bounded above and below for $|\lambda| \leqq \eta_0$, $n \leqq N_i$, as required.

7.3. THE FUNCTION $g_n^i(\lambda)$: FULL ESTIMATE.

DEFINITION. — Let $C_{N,i}(\lambda)$ be the connected component of $(g_N^i)^{-1}(B(g_N^i(\lambda), \varepsilon/4)$ which contains λ. From now on, choose α_0 such that with $\alpha = \alpha_0$, the estimates of paragraph 5 work.

PROPOSITION. — *Let* $N \geqq N_i + q_0$, *let* λ *be* (n, j, α_0)*—good for all* $n \leqq N$, $j \leqq m_0$. *Suppose* N *is not in any* x_j*-follower for* λ. *Then* g_N^i *maps* $C_{N,\,i}(\lambda)$ $1-1$ *to* $B(g_N^i(\lambda), \varepsilon_1/4)$ *and*

$$\frac{1}{K_2}\left|(f_\lambda^N)'(y_i(\lambda))\right| \leqq \left|(g_N^i)'(\mu)\right| \leqq K_2\left|(f_\lambda^N)'(y_i(\lambda))\right|$$

for all $\mu \in C_{N,\,i}(\lambda)$, *and suitable* K_2.

Hence (since the relevant local inverse of f_λ^N *is univalued on* $B_{\varepsilon_1}(f_\lambda^N(y_i(\lambda)))$,

$$\frac{1}{K} e^{a_3 N} \leqq \left|(g_N^i)'(\mu)\right| \quad \text{for some } K.$$

Proof. — We shall make use of 7.2, which, in particular, proves the proposition for N_i, with K_2 replaced by K_1. To prove the proposition for N, choose $n < N$ such that:

(*a*) $N - 2bN < n < N - bN$, where b is small, depending on a_3;

(*b*) n is not in any x_j-follower for λ.

Then either by an inductive hypothesis of by 7.2, whenever $\mu \in C_{n,\,i}(\lambda)$ (or $|\mu| \leqq \eta_0$ if $n \leqq N_i$) we can assume for suitable K_2 that:

(i) $K_2^{-1/2} \leqq \left|(g_n^i)'(\lambda)\right| / \left|(g_n^i)'(\mu)\right| \leqq K_2^{1/2}$, and,

(ii) $A_n^{-1}\left|(f_\lambda^n)'(y_i(\lambda))\right| \leqq \left|(g_n^i)'(\lambda)\right| \leqq A_n \left|f_\lambda^n)'(y_i(\lambda))\right|$

for A_n such that $\prod_{m \geqq n} (1 + e^{-a_3 m/2}) A_n < K_2^{1/2}$.

From (i), there is a map $\tilde{F}$ defined on $B(g_n^i(\lambda), \varepsilon_1/4)$ which is a local inverse of g_n^i. On the other hand, as N is not in any follower for λ, there is a univalent inverse G of f_λ^{N-n} defined on $\tilde{B} = B(g_N^i(\lambda), \varepsilon_1)$: by 5.2 $G(\tilde{B})$ is contained in $B(g_n^i(\lambda), \varepsilon_1 \exp - a_3(N-n))$. Let C be the boundary of $G(\tilde{B})$. By the inductive hypothesis on g_n^i, $|\lambda - \mu| < K\varepsilon_1 \exp(-a_3 N)$.

So

$$d(f_\lambda^{N-n}(g_n^i(\mu)), g_N^i(\mu)) < e^{b_2(N-n)} e^{-a_3 N} K\varepsilon_1 (N-n) < K\varepsilon_1 e^{(2 b_2 b - a_3) N}(N-n) < \frac{\varepsilon_1}{4}$$

if b is small enough, where b_2 is such that

$$|f'_\mu|, \left|\frac{\partial^2 f}{\partial\mu\,\partial z}(\mu, z)\right| \leqq e^{b_2}.$$

So each point in the ball radius $3\varepsilon_1/4$ round $g_N^i(\lambda)$ must be $g_N^i(\mu)$ for exactly one μ with $g_n^i(\mu) \in G(\tilde{B})$. So g_N^i has a univalued inverse on $B(g_N^i(\lambda), \varepsilon_1/4)$ and

$$K_2^{-1/2} \leqq \left|\frac{(g_N^i)'(\lambda)}{(g_N^i)'(\mu)}\right| \leqq K_2^{1/2}$$

for all $\mu \in C_{N,\,i}(\lambda)$, for suitable K_2.

Also since

$$(g_{N}^{i})'(\lambda)=(f_{\lambda}^{N-n})'(g_{n}^{i}(\lambda))(g_{n}^{i})'(\lambda)$$
$$+\frac{\partial}{\partial\lambda}f_{N-n}(\lambda, g_{n}^{i}(\lambda))=(f_{\lambda}^{N-n})'(g_{n}^{i}(\lambda))(g_{n}^{i})'(\lambda)(1+O(e^{-a_3 N/2}))$$

for b sufficiently small, we have :

$$A_{N}^{-1}|(f_{\lambda}^{N})'(y_i(\lambda))|\leqq|(g_{N}^{i})'(\lambda)|\leqq A_{N}|(f_{\lambda}^{N})'(y_i(\lambda)|,$$

using the inductive hypothesis (ii).

Good Inverses and Good Parameter Values. — The following lemmas 7.4. — 7.7 are for showing that good inverses (which were shown in Proposition 7.1 to be a large proportion of the whole) give rise to good parameter values.

Let λ be (n, i, α_0)—good for $n\leqq N$ and $i\leqq m_0$. The constant b_2(as in § 5) is such that $|f'_{\mu}|\leqq e^{b_2}$ for all $|\mu|\leqq\eta_0$, and also such that second derivatives of $f(\mu, z)$ are bounded by e^{b_2}.

7.4. LEMMA. — *If* $\mu\in C_{N,\,i}(\lambda)$, $r+k\leqq bN$ *and* $x_j(\mu)\in \mathrm{Im}\,\bar{S}$, *where* $\bar{S}$ *is the inverse of* f_{μ}^{k} *with domain radius* $\varepsilon/2$ $(\varepsilon_1/2\leqq\varepsilon\leqq 2\varepsilon_0)$ *defined by* $g_{N+r}^{i}(\mu)\ldots g_{N+r+k}^{i}(\mu)$, *then* $x_j(\lambda)\in\mathrm{Im}\,S'$ *where* S′ *is the inverse of* f_{λ}^{k} *with domain radius* ε *and defined by* $f_{\lambda}^{r}(g_{N}^{i}(\mu))\ldots f_{\lambda}^{r+k}(g_{N}^{i}(\mu))$.

Proof. — By 7.3, $|\lambda-\mu|\leqq K\varepsilon_1 e^{-a_3 N}$.

So $d(x_j(\mu), x_j(\lambda))<e^{-a_{12}N}\varepsilon_1$ for $a_{12}>0$. So it suffices to show the $e^{-a_{12}N}\varepsilon_1$—neighbourhood of $\mathrm{Im}\,\bar{S}$ is contained in $\mathrm{Im}\,S'$.

$$d(g_{N+r}^{i}(\mu), f_{\lambda}^{r}(g_{N}^{i}(\mu))<re^{b_2 r}|\lambda-\mu|<e^{-(1/2)a_3 N}\varepsilon_1/4<e^{-b_2 k}\varepsilon_1/4 \quad \text{if } b \text{ is small enough.}$$

So $\mathrm{Im}\,\bar{S}\cap\mathrm{Im}\,S'\neq\varnothing$, because $\mathrm{Im}\,S'$ contains a ball radius $e^{-b_2 k(\varepsilon_1/2)}$ round $f_{\lambda}^{r}(g_{N}^{i}(\mu))$.

So it suffices to show, if Y denotes the $e^{-a_{12}N}\varepsilon_1$— neighbourhood of the boundary of $\mathrm{Im}\,\bar{S}$, $Y\cap\partial(\mathrm{Im}\,S')=\varnothing$. Now $f_{\mu}^{k}(Y)$ is contained in the $ke^{b_2 k}e^{-a_{12}N}\varepsilon_1$ neighbourhood of the boundary of $B(g_{N+r+k}^{i}(\mu), \varepsilon/2)$, and $f_{\mu}^{k}(\partial(\mathrm{Im}\,S'))$ is contained in the $ke^{b_2 k}|\lambda-\mu|$ neighbourhood of the boundary of $B(g_{N+r+k}^{i}(\mu), \varepsilon)$, and $ke^{b_2 k}|\lambda-\mu|<e^{-(a_3/2)N}\varepsilon_1$ for b sufficiently small. So these neighbourhoods are clearly disjoint for b small enough, and $Y\cap\partial(\mathrm{Im}\,S')=\varnothing$ as required.

7.5. Continue to assume λ is (n, i, α_0)—good for $n\leqq N$, $i\leqq m_0$. Now fix i, and (unless $N\leqq N_i$, when there are no conditions) assume that $\mu\in C_{N,\,i}(\lambda)$ satisfies:

I. $g_{N}^{i}(\mu)\in\mathrm{Im}\,S$ for S an inverse of length $[bN]$ for λ, where S is $(n, i, \alpha_0/4, n_0/2)$—good for $(\alpha_0/4)\,N\leqq n\leqq bN$, that is, at most $(\alpha_0/4)\,n$ integers are contained is followers for S_1 of length $\geqq n_0/2$ if S_1 is determined by $t_0,\ldots t_n$ and S by $t_0\ldots t_{[bN]}$;

II. $B(g_{N}^{i}(\mu), e^{-a_6 N}\varepsilon_1)\subseteqq B(g_{N}^{i}(\lambda), \varepsilon_1/4)$, where a_6 is a fixed positive number $<a_3 b/10\,m_0$.

(Actually, condition II is not needed yet, but it seems best to give both conditions together.)

LEMMA. — *For* μ *as above,* $[N+r,\ N+r+L]\subseteqq[N,\ N+n]$ *is an* x_i*-follower for* μ *with* $L\geqq r_0$ *only if* $[r,\ r+L-k,]$ *is a follower for* S *for all* $k\geqq r_0$, *where* S *is as in* I *and* $r_0>q_0$ *is such that* $e^{-a_1 r_0}<1/4$. *Thus, if* $\alpha_0 N/4\leqq n\leqq b N$, *The number of points in* $[N,\ N+n]$ *which are in* x_i*-followers for* μ *of length* $\geqq n_0$ *is* $<(\alpha_0/2)\,n$.

Proof. — If $[N+r,\ N+r+L]$ is an x_i-follower for μ, and $L\geqq r_0$, then by the definition of a_1 (and 4.2) there is $x_j(\mu)\in \operatorname{Im}\bar{S}$, where $\bar{S}$ is the inverse of f_μ^{L-k} defined by $f_\mu^r(g_N^i(\mu))\ldots f_\mu^{r+L-k}(g_N^i(\mu))$ and radius $\varepsilon_1/4$ for all $k\geqq r_0$.

Then by 7.4, $x_j(\lambda)\in \operatorname{Im} S'$, where S' is the inverse of f_λ^{L-k} defined by $f_\lambda^r(g_N^i(\mu))\ldots f_\lambda^{r+L-k}(g_N^i(\mu))$ with domain radius $\varepsilon_1/2$. Now by I, the union of $[r,\ r+L-r_0]$ in $[0,\ n]$ with $L-r_0\geqq n_0/2$ contains $<(\alpha_0/4)\,n$ points if $(\alpha_0/4)\,N\leqq n\leqq b\,N$. So the number of points in followers $[N+r,\ N+r+L]\subseteqq[N,\ N+n]$ with $L\geqq n_0$ and $(\alpha_0/4)\,N\leqq n\leqq b\,N$ is $\leqq(\alpha_0/4)\,n\,(1+(2\,r_0/n_0))<(\alpha_0/2)\,n$ if n_0 is large enough.

7.6. Suppose μ, λ are as in 7.5 and suppose that for $N'\in(N,\ N(1+b)]$, $C_{N',\,i}(\mu)\subseteqq C_{N,\,i}(\lambda)$. Let $\nu\,\varepsilon\, C_{N',\,i}(\mu)$.

LEMMA. — *If* $\bar{\bar{S}}$ *is the inverse determined by* $f_\lambda^r(g_N^i(\nu))\ldots f_\lambda^{r+k}(g_N^i(\nu))$ *with radius* $\varepsilon_1/2$, *and* $\bar{S}$ *is determined by* $f_\lambda^r(g_N^i(\mu))\ldots f_\lambda^{r+k}(g_N^i(\mu))$ *with radius* ε *and either* $N'-(r+k)-N>q_0$ with $\varepsilon=\varepsilon_1$, *or* $\varepsilon\geqq\varepsilon_0$ *then* $\operatorname{Im}\bar{\bar{S}}\subseteqq\operatorname{Im}\bar{S}$.

Proof. — It suffices to show $\operatorname{Im}\bar{\bar{S}}\subseteqq f_\lambda^{-k}$ (domain $\bar{S}$) for all $\nu\in C_{N',\,i}(\mu)$. For then the set of ν with $\operatorname{Im}\bar{\bar{S}}\subseteqq\operatorname{Im}\bar{S}$ is open and closed and contains μ, and must be all of $C_{N',\,i}(\mu)$ since this is connected. Thus is suffices to show

$$d\,(f_\lambda^{r+k}(g_N^i(\nu)),\ f_\lambda^{r+k}(g_N^i(\mu)))<\frac{\varepsilon}{2}\qquad\text{for}\quad \nu\in C_{N',\,i}(\mu).$$

Now $\operatorname{Im} S'\subseteqq B\,(f_\lambda^{r+k}(g_N^i(\mu)),\ \varepsilon/2)$ if S' is the inverse of $f_\lambda^{N'-(N+r+k)}$ determined by $f_\lambda^{r+k}(g_N^i(\mu))\ldots f_\lambda^{N'-N}(g_N^i(\mu))$ with radius ε_1, and either $N'-(r+k)-N>q_0$, $\varepsilon=\varepsilon_1$, or $\varepsilon\geqq\varepsilon_0$.

But since $C_{N',\,i}(\mu)$ is connected, $f_\lambda^{r+k}g_N^i(C_{N',\,i}(\mu))\subseteqq\operatorname{Im} S'$ if

$$f_\lambda^{r+k}(g_N^i(C_{N',\,i}(\mu)))\subseteqq f_\lambda^{-N'+r+k+N}\quad(\text{domain } S'),$$

that is, if $d\,(f_\lambda^{N'-N}(g_N^i(\nu)),\ f_\lambda^{N'-N}(g_N^i(\mu))<\varepsilon_1$. But this true because $|\lambda-\mu|,\ |\lambda-\nu|<K\,e^{-a_3 N}\varepsilon_1$ by assumption and $N'-N<b\,N$, so $d\,(f_\lambda^{N'-N}(g_N^i(\mu)),\ g_{N'}^i(\mu))\ll\varepsilon_1$ and similarly for ν, and $d\,(g_{N'}^i(\mu), g_{N'}^i(\nu))<\varepsilon_1/4$.

7.7. We now extend the result of 7.5 to $\nu\in C_{N',\,i}(\mu)$.

LEMMA. — Under the same conditions as in 7.6, $[N+r,\ N+r+L]\subseteqq[N,\ N+n]$ *is an* x_i*-follower for* ν *with* $L\geqq r_0$ *only if* $[r,\ r+L-k]$ *is a follower for* S *for all* $k\geqq r_0$, *where* S *is as in* I (*and* $g_N^i(\mu)\in\operatorname{Im} S$). *Then if* $\alpha_0\,N/4\leqq n\leqq b\,N$, *the number of points in* $[N,\ N+n]$ *which are in* x_i*—followers for* ν *of length* $\geqq n_0$ *is* $<(\alpha_0/2)\,n$.

Proof. — This is very similar to 7.5. If $[N+r,\ N+r+L]\subseteqq[N,\ N+n]$ is a follower for ν, then $[r,\ r+L-r_0]$ is such that the inverse $\bar{\bar{S}}$ of $f_\lambda^{L-r_0}$ determined by $f_\lambda^r(g_N^i(\nu))\ldots f_\lambda^{r+L-r_0}(g_N^i(\nu))$ with domain radius $\varepsilon_1/2$ has some $x_j(\lambda)\in\operatorname{Im}\bar{\bar{S}}$. Then by

7.6, the inverse $\bar{S}$ of $f_\lambda^{L-r_0}$ determined by $f_\lambda^r(g_N^i(\mu))\ldots f_\lambda^{r+L-r_0}(g_N^i(\mu))$ with domain radius ε_1 has $x_j(\lambda)\in \operatorname{Im}\bar{S}$, since $\operatorname{Im}\bar{\bar{S}}\subseteq \operatorname{Im}\bar{S}$. Then the proof is completed as in 7.5.

7.8. We now consider extra conditions on $N'\in(N, N(1+b)]$ where N' is as 7.6-7.7. We continue with the assumptions of 7.4-7.7 on μ, ν.

III. $N'=N+n$ is such that $[n-r_0, n]$ does not intersect any follower for S of length $\geqq(1/2)\,n_0$ and radius ε_1, where S is as in I.

By 7.5, 7.7, III implies N' does not intersect any follower of length $\geqq n_0$ for μ, or ν.

IV. $x_j(\lambda)\notin \operatorname{Im}\bar{S}$, where $\bar{S}$ is determined by $f_\lambda^{N'-N-r}(g_N^i(\mu))\ldots f_\lambda^{N'-N}(g_N^i(\mu))$ for $r\leqq n_0$ and radius $4\,\varepsilon_0$.

Then by 7.4, N' is not in a follower of length $r\leqq n_0$ for μ. Also, by 7.6, $x_j(\lambda)\notin \operatorname{Im} S'$ where S' is determined by $f_\lambda^{N'-N-r}(g_N^i(\nu))\ldots f_\lambda^{N'-N} g_N^i(\nu))$ and radius $2\,\varepsilon_0$. Then by 7.4, N' is not in a follower of length $\leqq n_0$ for ν.

7.9. A SET OF INTERVALS. — Recall from 7.2 that N_i is the largest integer such that g_n^i maps $\{\lambda:|\lambda|\leqq\eta_0\}$ into U_i for all $n\leqq N_i$.

Define $R_0=2\,n_0^2$. We can find disjoint intervals I_k of integers, for $k\geqq 1$, such that if R_k is the least element of I_k, then:

(*a*) $(1+(b/2))\,R_k\leqq N\leqq(1+(3\,b/4))\,R_k$ for all $N\in I_{k+1}$.

(*b*) there is no $N_i\in I_k$, and if $N_i<R_k$, then

$$|N_i-R_k|>\frac{b}{10\,m_0}R_k.$$

(*c*) I_k has width $>(b/10\,m_0)\,R_k$ and $<(b/4)\,R_k$.

Proof of Theorem C. — We shall define sets $\mathfrak{g}_k\subseteq\{\lambda:|\lambda|\leqq\eta_0\}$ such that if $\lambda\in\limsup \mathfrak{g}_k$, then λ is $(n, i, \alpha_0)-$ good for all n, all $i\leqq m_0$, thus proving Theorem C.

Let

$$\mathfrak{g}_0=\left\{\lambda:|\lambda|\leqq\frac{1}{2}\eta_0\right\}.$$

As in 7.2, let N_i be the largest integer such that g_n^i maps $\{\lambda:|\lambda\leqq\eta_0\}$ into U_i for all $n\leqq N_i$.

Suppose inductively that $\mathfrak{g}_k$ has been defined so that:

V for each $\lambda\in\mathfrak{g}_k$, there exists an integer $N(k, \lambda)\in I_k$ such that if $D_k^i(\lambda)=C_{N,\,i}(\lambda)$ for $N=N(k, \lambda)$, then for all $\mu\in D_k^i(\lambda)$, μ is $(n, i, \alpha_0)-$ good for $n\leqq N(k, \lambda)$ and $(n, i, \alpha_0/2)-$ good for $n=N(k, \lambda)$, and $N(k, \lambda)$ is not in a follower for μ. In particular, if $\lambda=\mu$, these things are true for all $i\leqq m_0$.

Then define $\mathfrak{g}_{k+1}=\bigcup\{\bigcap_{i\leqq m_0} E_k^i(\lambda_i):\lambda_i\in\mathfrak{g}_k\}$ where $E_k^i(\lambda)$ is

(*a*) $\{\mu\in D_k^i(\lambda):\mu$ satisfies I and II for $N=N(k, \lambda)\}$ if $N_i<N(k, \lambda)$;

(*b*) $\{\mu\in D_k^i(\lambda):\mu$ satisfies I and II for $N=N_i\}$ if $N(k, \lambda)<N_i<R_{k+1}$;

(*c*) $E_k^i(\lambda)=D_k^i(\lambda)=\{|\lambda|\leqq(1/2)\,\eta_0\}$ if $R_{k+1}<N_i$.

Then if $\mu \in E_k(\lambda_i)$, $\lambda_i \in \mathfrak{g}_k$ and $N = N(k, \lambda_i)$, μ is $(n, i, \alpha_0)-$ good for $n \leqq N$, and $(N, i, \alpha_0/2)-$ good, hence $(n, i, \alpha_0)-$ good for $n \leqq N(1+(\alpha_0/2))$. Then by 7.5, μ is $(n, i, \alpha_0/2)-$ good for $N(1+(\alpha_0/4)) \leqq n \leqq N(1+b)$.

Now (unless $R_{k+1} < N_i$, in which case there is no condition for i) choose $N' = N(k+1, \mu) \in I_{k+1}$ such that III and IV are satisfied for all i (where for i, N in III and IV is taken to be $N(k, \lambda_i)$ if $\mu \in \bigcap_{i \leqq m_0} E_k^i(\lambda_i)$, $\lambda_i \in \mathfrak{g}_k$). This is possible since the proportion of integers which must be avoided for each i is $< n_0/2 n_0^2$ (for IV) and for III: the proportion of integers which must be avoided for followers of length $> n_0/2$ is $< (\alpha_0/4)(1+(2 r_0/n_0))$.

By 7.8, N' is not in an x_i-follower for μ (for any i), nor for ν, provided that the condition $C_{N', i}(\mu) \subseteqq C_{N, i}(\lambda)$ is satisfied [if $\lambda = \lambda_i$ and $N = N(k, \lambda_i)$, where $\mu \in E_k^i(\lambda_i)$].

But since μ is $(n, i, \alpha_0)-$ good for $n \leqq N'$ and all i, and N' is not in an x_i-follower for μ,

$$\left| \frac{d g^i_{N'/d\nu}}{d g^i_{N/d\nu}} \right|$$

is minorized on $C_{N', i}(\mu)$ by $O(e^{-a_3(N'-N)})$, and thus, II and $N' - N > a_3 b/10 m_0$ (from the conditions on the intervals I_k in 7.9) imply $C_{N', i}(\mu) \subseteqq C_{N, i}(\lambda)$ as required.

We can then deduce from 7.7 that ν is $(n, i, \alpha_0)-$ good for $n \leqq N'$, and $(N', i, \alpha_0/2)-$ good, if $\nu \in C_{N', i}(\mu)$, and from 7.8 that N' is in an x_i-follower for ν. Thus $\mathfrak{g}_{k+1}$ satisfies condition V as required.

Finally, we have to bound meas $(\mathfrak{g}_k \setminus \mathfrak{g}_{k+1})$. Now

$$\mathfrak{g}_k \setminus \mathfrak{g}_{k+1} \subseteqq \bigcup_i \bigcup_{\lambda \in J_i} (F_k^i(\lambda) \setminus E_k^i(\lambda)).$$

Here, $F_k^i(\lambda)$ is the largest ball centred on λ and contained in $D_k^i(\lambda)$ (which is a ball up to bounded distorsion) and J_i is a countable subset of $\mathfrak{g}_k$ with $\mathfrak{g}_k \subseteqq \bigcup_{\lambda \in J_i} F_k^i(\lambda)$, chosen so that no point in $\{|\lambda| \leqq \eta_0\}$ lies in more than 20 of the balls $F_k^i(\lambda)$.

Now, by 7.3 the following inequalities are equivalent.

$$\operatorname{meas}(D_k^i(\lambda) \setminus E_k^i(\lambda)) < \text{Const. } e^{-a_9 R_k} \operatorname{meas}(D_k^i(\lambda)),$$

$$\operatorname{meas}(\{g_N^i(\mu) : \mu \in D_k^i(\lambda) \setminus E_k^i(\lambda)\}) < \text{Const. } e^{-a_9 R_k} \operatorname{meas}\{g_N^i(\mu) : \mu \in D_k^i(\lambda)\}.$$

But the latter is true by conditions I, II and 7.1.

So

$$\operatorname{meas}(F_k^i(\lambda) \setminus E_k^i(\lambda)) < \text{const. } e^{-a_9 R_k} \operatorname{meas}(F_k^i(\lambda)).$$

So

$$\operatorname{meas}(\mathfrak{g}_k \setminus \mathfrak{g}_{k+1}) < \text{Const. } e^{-a_9 R_k} \sum_{i \leqq m_0} \sum_{\lambda \in J_i} \operatorname{meas}(F_k^i(\lambda)) < \text{Const. } e^{-a_9 R_k} \times \operatorname{meas}(\{|\lambda| \leqq \eta_0\}).$$

So

$$\sum_k \operatorname{meas}(\mathfrak{g}_k \setminus \mathfrak{g}_{k+1}) < \operatorname{meas}(\mathfrak{g}_0)$$

for n_0 sufficiently large such that $\sum_k$ const. $e^{-a_9 R_k} < 1$, and thus meas (lim sup $\mathfrak{g}_k$) > 0. The proof is completed.

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