Annales scientifiques de l'É.N.S.

Marko Tadić

Classification of unitary representations in irreducible representations of general linear group (non-Archimedean case)

Annales scientifiques de l'É.N.S. 4e série, tome 19, n° 3 (1986), p. 335-382

<http://www.numdam.org/item?id=ASENS_1986_4_19_3_335_0>

Ann. scient. Éc. Norm. Sup.,
4ᵉ série, t. 19, 1986, p. 335 à 382.

CLASSIFICATION OF UNITARY REPRESENTATIONS IN IRREDUCIBLE REPRESENTATIONS OF GENERAL LINEAR GROUP (non-archimedean case)

BY MARKO TADIĆ

Contents

Introduction

A basic problem of the harmonic analysis on a particular group G is to describe the set of all equivalence classes of irreducible unitary representations of G.

Let G be a connected reductive group over a local field F. The set of all equivalence classes of (algebraically) irreducible admissible representations of G is denoted by $\tilde{G}$. The set of all equivalence classes of topologically irreducible unitary representations (on Hilbert spaces) is denoted by $\hat{G}$, and called the unitary dual of G. The unitary dual $\hat{G}$ is in a natural bijection with the subset of all unitarizable classes in $\tilde{G}$ ([6], [32]). In this way we shall identify $\hat{G}$ with the subset of all unitarizable classes in $\tilde{G}$. Thus, a description of the unitary dual can be done in two steps. The first step is to parametrize

$\tilde{G}$, and the second one is to identify all unitarizable classes in $\tilde{G}$. The first step is called the problem of non-unitary dual, and the second one is called the unitarizability problem.

The first problem has been studied much more then the second one. It has been completely solved for the case of archimedean F, by Langlands classification. Langlands classification is done also for the nonarchimedean case, but here remains to classify the Langlands parameters.

The second problem is carried out in the case of non-archimedean F only for the groups SL (2) and closely related group GL (2) (possible reference is [10]) ([1]).

In the case of archimedean F, despite the complete knowledge of the non-unitary dual $\tilde{G}$ the second problem was carried out completely only for a few groups of lower ranks. For a survey of this case one may consult the paper [17] of A. W. Knapp and B. Speh and therefore we are not going into further details (*see* also [31]).

We shall give one more remark about the problem of unitary dual for groups SL $(n, \mathbb{C})$, which are closely related to the groups GL $(n, \mathbb{C})$. In 1950, I. M. Gelfand and M. A. Neumark constructed a family of irreducible unitary representations of SL $(n, \mathbb{C})$ for which they presumed they exhausted the unitary dual [13]. In 1967 E. M. Stein showed that the constructed family of representations was not complete, by constructing a new complementary series [25]. G. Olshanskii generalized in [20] this result. He constructed some complementary series for GL (n) over division algebras (archimedean and non-archimedean). In [2] J. N. Bernstein constructed a much wider family of complementary series for GL (n) over a non-archimedean field. This complementary series will be discussed later. J. N. Bernstein paper contains some very important general results about unitarizability in the case of GL (n) over non-archimedean fields.

In this paper we give a solution of the unitarizability problem for the groups GL (n) over a local non-archimedean field F. More precisely, Zelevinsky parameters and Langlands parameters of all unitarizable classes in GL $(n, F)^{\sim}$ are determined. Moreover, an explicit formula connecting Zelevinsky and Langlands parameters of GL $(n, F)^{\wedge}$ is proved. We prove also the Bernstein conjecture on complementary series from [2].

The results and techniques we need in this paper on non-unitary dual of GL (n) over non-archimedean field, which are characteristic for this case, belong mainly to I. M. Gelfand, D. A. Kazhdan, J. N. Bernstein and A. V. Zelevinsky ([4], [5], [12], [33]).

Concerning the facts about the unitary representations required for this paper, in the first stage of development of ideas of this paper, the results of D. Miličić from [18] and also results of [27] (obtained using [18]) had an important role. In the second stage, the results of J. N. Bernstein in [2] played an important role. F. Rodier pointed out to me possibility of studying the unitary duals of GL (n) over non-archimedean fields using these two groups of ideas.

Now we shall describe the main results of this paper.

([1]) Added in proof: the authors recently learned that H. Jacquet, I. I. Piatetski-Shapiro and J. Shalika carried out the second problem for non-archimedean GL(3) in "Automorphic forms on GL(3) I", Ann. of Math., Vol. 109, 1979, 169-212.

Let Alg GL(n, F) be the category of all smooth representations of finite length of GL(n, F) where F is a local non-archimedean field. If $\pi \in$ Alg GL(n, F) then we write deg $\pi = n$. If $\tau \in$ Alg GL(n_1, F) and $\sigma \in$ Alg GL(n_2, F) then $\tau \times \sigma \in$ Alg GL$(n_1 + n_2, F)$ denotes the representation induced by $\tau \otimes \sigma$.

The set of all equivalence classes of irreducible smooth representations of GL(n, F) for all $n \geqq 0$, is denoted by Irr. The subset of all unitarizable representations in Irr is denoted by Irr^u. Let C be the set of all cuspidal representations in Irr. Set $C^u = \mathrm{Irr}^u \cap C$. Note that C^u is just the set of all representations in C with unitary central character. Set

$$\mathrm{GL}(n, F)\hat{} = \{ \pi \in \mathrm{Irr}^u; \deg \pi = n \}.$$

Let R_n be the Grothendieck group of the category Alg GL(n, F). The induction functor $(\tau, \sigma) \to \tau \times \sigma$ defines a structure of a graded ring on

$$R = \bigoplus_{n \geqq 0} R_n.$$

We consider Irr $\subset$ R and now Irr is a basis of $\mathbb{Z}$-module R.

Let ν be the character $g \to |\det g|_F$ of GL(n, F) where $| \ |_F$ is a natural absolute value on F (*see* 1.3).

For a positive integer d set $\Delta[d] = \{ -(d-1)/2, -(d-3)/2, \ldots, (d-1)/2 \}$. If $\rho \in C$ then we put

$$\Delta[d]^{(\rho)} = \{ \nu^\alpha \rho; \alpha \in \Delta[d] \}.$$

Let d, n be positive integers. Let

$$a(n\, d)^{(\rho)} = (\nu^{(-n+1)/2} \Delta[d]^{(\rho)}, \nu^{(-n+3)/2} \Delta[d]^{(\rho)}, \ldots, \nu^{(n-1)/2} \Delta[d]^{(\rho)})$$

be a multiset (*see* the first section) where

$$\nu^\alpha \Delta[d]^{(\rho)} = \{ \nu^\alpha \sigma; \sigma \in \Delta[d]^{(\rho)} \}.$$

By the Zelevinsky classification we can associate to $a(n, d)^{(\rho)}$ a representation $\langle a(n, d)^{(\rho)} \rangle \in$ Irr.

Fix an integer n. Take a decomposition

$$n = \sum_{i=1}^{s} a_i p_i u_i + 2 \sum_{j=1}^{r} b_j g_j v_j$$

where a_i, p_i, u_i, b_j, q_j, v_j are positive integers. The case of $r = 0$ or $s = 0$ is possible. Let $\alpha_1, \ldots, \alpha_r \in (0, 1/2)$. Take σ_i, $\tau_j \in C^u$ such that deg $\sigma_i = u_i$ and deg $\tau_j = v_j$. Set

$$(*) \quad \pi((a_1, p_1, \sigma_1), \ldots, (a_s, p_s, \sigma_s), (b_1, q_1, \tau_1, \alpha_1), \ldots, (b_r, q_r, \tau_r, \alpha_r))$$
$$= \langle a(a_1, p_1)^{(\sigma_1)} \rangle \times \ldots \times \langle a(a_s, p_s)^{(\sigma_s)} \rangle$$
$$\times [\nu^{\alpha_1} \langle a(b_1, q_1)^{(\tau_1)} \rangle \times \nu^{-\alpha_1} \langle a(b_1, q_1)^{(\tau_1)} \rangle] \times \ldots$$

$$\times [\nu^{\alpha_r} a(b_r, q_r)^{(\tau_r)} \rangle \times \nu^{-\alpha_r} \langle a(b_r, q_r)^{(\tau_r)} \rangle].$$

The main result of this paper is the following:

THEOREM A. — (i) *Representations* $\pi((a_1, p_1, \sigma_1), \ldots, (b_r, q_r, \tau_r, \alpha_r))$ *are in* $\mathrm{GL}(n, \mathrm{F})\hat{}$.

(ii) *Each irreducible unitarizable representation of* $\mathrm{GL}(n, \mathrm{F})$ *can be obtained in this way.*

(iii) *Two representations*

$$\pi((a_1^i, p_1^i, \sigma_1^i), \ldots, (a_{s^i}^i, p_{s^i}^i, \sigma_{s^i}^i), (b_1^i, q_1^i, \tau_1^i, \alpha_1^i), \ldots, (b_{r^i}^i, q_{r^i}^i, \tau_{r^i}^i, \alpha_{r^i}^i)), \qquad i=1, 2,$$

are equal if and only if $s^1 = s^2$, $r^1 = r^2$ *and*

$$((a_1^1, p_1^1, \sigma_1^1), \ldots, (a_s^1, p_s^1, \sigma_s^1)) = ((a_1^2, p_1^2, \sigma_1^2), \ldots, (a_s^2, p_s^2, \sigma_s^2)),$$
$$((b_1^1, q_1^1, \tau_1^1, \alpha_1^1), \ldots, (b_r^1, q_r^1, \tau_r^1, \alpha_r^1)) = ((b_1^2, q_1^2, \tau_1^2, \alpha_1^2), \ldots, (b_r^2, q_r^2, \tau_r^2, \alpha_r^2))$$

as multisets.

By the work of Zelevinsky and Bernstein we have the involution ^{t}: Irr → Irr (*see* 1.1.4).

The importance of the involution ^{t} lies, among the others, in the fact that ^{t} connects Zelevinsky and Langlands classification, what was shown by F. Rodier in [21].

THEOREM B. — *We have*

$$\pi((a_1, p_1, \sigma_1), \ldots, (a_s, p_s, \sigma_s), (b_1, q_1, \tau_1, \alpha_1), \ldots, (b_r, q_r, \tau_r, \alpha_r))^t$$
$$= \pi((p_1, a_1, \sigma_1), \ldots, (p_s, a_s, \sigma_s), (q_1, b_1, \tau_1, \alpha_1), \ldots, (q_r, b_r, \tau_r, \alpha_r)).$$

In this paper we prove the Bernstein conjecture on complementary series from [2]. For $\pi \in \mathrm{Irr}$ let π^+ be the Hermitian contragredient of π. Rigid representations are defined in 4.1.

THEOREM C (Bernstein conjecture on complementary series). — (i) *Suppose that* $\nu^\alpha \sigma \times \nu^{-\alpha} \sigma^+$ *is irreducible and unitarizable for all* $\alpha \in (-1/2, 1/2)$. *Then* σ *is a unitarizable rigid representation.*

(ii) *Suppose that* σ *is a rigid representation such that* $\nu^\alpha \sigma \times \nu^{-\alpha} \sigma^+$, *is an irreducible unitarizable representation for some* $\alpha \in (0, 1/2)$. *Then there exist* $\sigma_1, \sigma_2 \in \mathrm{Irr}^u$ *so that*

$$\sigma = \sigma_1 \times \nu^{-1/2} \sigma_2.$$

Let D^u be the set of all square integrable classes in Irr^u.

Let n be a positive integer, and let $\delta \in \mathrm{D}^u$. The representation

$$\nu^{(n-1)/2} \delta \times \nu^{((n-1)/2)-1} \delta \times \ldots \times \nu^{-(n-1)/2} \delta$$

has a unique irreducible quotient which is denoted by $u(\delta, n)$.

THEOREM D. — *Let B be the set of all*

$$u(\delta, n), \nu^\alpha u(\delta, n) \times \nu^{-\alpha} u(\delta, n)$$

where n *is a positive integer,* $\delta \in \mathrm{D}^u$ *and* $0 < \alpha < 1/2$.

(i) *If* $\pi_1, \ldots, \pi_r \in B$, *then* $\pi_1 \times \ldots \times \pi_r \in \mathrm{Irr}^u$.

(ii) *If* $\sigma \in \mathrm{Irr}^u$, *then there exist* $\pi_1, \ldots, \pi_s \in B$ *so that* $\sigma = \pi_1 \times \ldots \times \pi_s$. *The elements* $\pi_1, \ldots, \pi_s$ *are unique up to a permutation.*

This theorem describes the Langlands parameters of Irr^u.

Note that the statement of Theorem D. makes sense also if the field F is archimedean. In fact, Theorem D. holds also over archimedean F and the ideas of sections 3 and 4 of this paper can be applied as well in the case of archimedean F (*see* [30]).

Note that for GL (n) over archimedean fields, irreducible square integrable representations exist only for

$$GL(1, \mathbb{C}) \cong \mathbb{C}^\times, GL(1, \mathbb{R}) = \mathbb{R}^\times \quad \text{and} \quad GL(2, \mathbb{R}).$$

If δ is an irreducible square integrable representation of GL (1, $\mathbb{R}$) or GL (1, $\mathbb{C}$), i.e. a unitary character of $\mathbb{R}^\times$ or $\mathbb{C}^\times$, then

$$u(\delta, n): \quad g \to \delta(\det g)$$

is an one-dimensional unitary representation of GL (n, $\mathbb{R}$) or GL (n, $\mathbb{C}$). If δ is an irreducible square integrable representation of GL (2, $\mathbb{R}$) than $u(\delta, n)$ were studied by B. Speh [24]. In the non-archimedean case we have much more square integrable representations. Therefore we have much more representations $u(\delta, n)$ (2).

The involution ^{t} on R was defined in [33]. A. V. Zelevinsky conjectured in [33] that ^{t} carries the irreducible representations into irreducible ones, i.e. that $(\mathrm{Irr})^t = \mathrm{Irr}$. Proof of this was announced. The fact $(\mathrm{Irr})^t = \mathrm{Irr}$ was used in this paper. As up to this date there is no written proof of this fact in the full generality, known to this author, we shall write a few words about the role of this fact for this paper.

In [35], J.-L. Waldspurger proved that $\pi^t \in \mathrm{Irr}$ for $\pi \in \mathrm{Irr}$ in a great number of cases.

In the sections 1-6 of this paper we are using neither $(\mathrm{Irr})^t = \mathrm{Irr}$ nor the results depending on this fact. We use $(\mathrm{Irr})^t = \mathrm{Irr}$ in sections 7 and 8 where Theorems A, B, C and D are proved.

As there is no reference for the written proof of $(\mathrm{Irr})^t = \mathrm{Irr}$ in the full generality, we added Apendix in which the main results are proved without assuming $(\mathrm{Irr})^t = \mathrm{Irr}$, when char $F = 0$. Therefore we are not using the result of [28] in that section. Using the idea of B. Speh in [24], we prove the unitarizability of representations $u(\delta, n)$ by global methods.

Now we introduce some basic notation. The field of real numbers is denoted by $\mathbb{R}$, the subring of integers is denoted by $\mathbb{Z}$, the subset of non-negative integers is denoted by $\mathbb{Z}_+$ and the subset of positive integers is denoted by $\mathbb{N}$. If $\alpha, \beta \in \mathbb{R}$ then $(\alpha, \beta) = \{\gamma \in \mathbb{R}; \alpha < \gamma < \beta\}$.

---

(2) Added in proof: another approach to the unitary dual of archimedean GL(n) is done by D. A. Vogan (Invent. Math. Vol. 83, 1986, 449-505).

A part of the results of the present paper was announced in [29].

I would like to thank F. Rodier for the information about G. Olshanskii paper [20] and the J. N. Bernstein paper [2], and for some useful suggestions. I would also like to thank the members of the « Séminaire sur les groupes réductifs et les formes automorphes », Université Paris-VII, for their hospitality during my visit of Paris in November 1983, when the main ideas of this paper arose, and for providing me with a copy of the Bernstein paper [2]. I am grateful to M. Duflo for his advices concerning this paper. I would like to express my appreciation to D. Miličić for the interest he arose in me concerning the topology of the dual spaces and explaining me the basic facts, and to H. Kraljević for encouragements and discussions during my work on the topic of this paper.

Finally, I want to thank the Max-Planck-Institut für Mathematik for their hospitality during the academic year 1984/1985 which enabled me to finish this paper. In particular, I am grateful to the director F. Hirzebruch as well as G. Harder and J. Schwermer who helped a lot to make my stay at the MPI useful.

1. Zelevinsky classification

For more detailed informations and proofs of the results presented in this section one needs to use [33].

1.1. Let F be a locally compact nonarchimedean field. The group $GL(n, F)$ is denoted by G_n. Sometimes we shall identify $GL(1, F)$ with F^x, the multiplicative group of F.

A representation π of G_n on a complex vector space V is called a smooth G_n-module if every vector in V has an open stabilizer in G_n. If for each open compact subgroup K in G_n the vector space of all K-fixed vectors in V is finite dimensional, then V is called an admissible G_n-module. The Grothendieck group of the category $\operatorname{Alg} G_n$ of all smooth G_n-modules of finite length is denoted by R_n. The induction functor $(\pi_1, \pi_2) \rightarrow \pi_1 \times \pi_2$ induces a bilinear morphism $R_n \times R_m \rightarrow R_{n+m}$. Set

$$R = \bigoplus_{n=0}^{\infty} R_n.$$

We have a structure of commutative graded ring on R. If $\pi \in R_n$ put $\deg \pi = n$.

1.2. Denote by $\tilde{G}_n$ the set of all equivalence classes of irreducible smooth representations of G_n. Note that representations in $\tilde{G}_n$ are admissible. Suppose that G_n-module V possesses a G_n-invariant inner product. Then we say that V is unitarizable. For an irreducible G_n-module, a G_n-invariant inner product is unique up to a scalar, if it exists. Denote by $\hat{G}_n$ the set of all classes of unitarizable representations in $\tilde{G}_n$. Let $C(G_n)$ be the subset of all cuspidal representations in $\tilde{G}_n$, i.e. the subset of all representations in $\tilde{G}_n$ whose matrix coefficients are compactly supported functions on G_n modulo the center of G_n. Let $C^u(G_n) \subseteq C(G_n)$ be the subset of all unitarizable representations. We consider

$$\tilde{G}_n \subseteq R_n.$$

Then $\tilde{G}_n$ is a $\mathbb{Z}$-basis of R_n.

Put

$$\mathrm{Irr} = \bigcup_{n=0}^{\infty} \tilde{G}_n,$$

$$\mathrm{Irr}^u = \bigcup_{n=0}^{\infty} \hat{G}_n,$$

$$C = \bigcup_{n=1}^{\infty} C(G_n),$$

$$C^u = \bigcup_{n=1}^{\infty} C^u(G_n).$$

1.3. W shall denote by $\nu \in \tilde{G}_n$ the one-dimensional representation $g \to |\det g|_F$. Here $| \ |_F$ is an absolute value on F which defines topology of F such that if ω_F generates the maximal ideal p_F in the ring of integers 0_F of F then

$$|\omega_F|_F = [\mathrm{card}(0_F/p_F)]^{-1}.$$

For $\alpha \in \mathbb{C}$ put $\nu^\alpha(\pi) = \nu^\alpha \pi$, $\pi \in R_n$. Now we can extend ν^α to the whole R and ν^α is an automorphism of the ring R.

1.4. For a set X, M(X) will denote the set of all finite multisets in X. They are functions $a : X \to \mathbb{Z}_+$ with finite support. We shall write a in the following way. Let $\{x_1, \ldots, x_n\}$ be the set of all $x \in X$ such that $a(x) \neq 0$. Then we write a as

$$(\underbrace{x_1, \ldots, x_1}_{a(x_1)\text{times}}, \underbrace{x_2, \ldots, x_2}_{a(x_2)\text{times}}, \ldots, \underbrace{x_m, \ldots, x_m}_{a(x_m)\text{times}})$$

We shall consider $X \subseteq M(X)$. Clearly, M(X) has a structure of a commutative semigroup with identity.

For $a \in M(X)$, the number $\sum_{x \in X} a(x)$ is called the cardinal number of a and denoted by $|a|$.

1.5. For $i, j \in \mathbb{Z}$, $i \leq j$ set

$$[i, j] = \{k \in \mathbb{Z};\ i \leq k \leq j\},$$

and call $[i, j]$ a segment in $\mathbb{Z}$. The set of all segments in $\mathbb{Z}$ is denoted by $S(\mathbb{Z})$.

Let $\Delta = [i, j] \in S(\mathbb{Z})$ and $\alpha \in \mathbb{R}$. Then we call

$$[i+\alpha, j+\alpha] = \{\alpha + k;\ k \in \Delta\}$$

a segment in $\mathbb{R}$ and denote it by Δ_α. The set of all segments in $\mathbb{R}$ is denoted by $S(\mathbb{R})$. We consider $S(\mathbb{Z}) \subseteqq S(\mathbb{R})$. Now for $\Delta \in S(\mathbb{R})$ and $\alpha \in \mathbb{R}$ we define Δ_α analogously. In this way we get action of $\mathbb{R}$ on $S(\mathbb{R})$.

1.6. For $\Delta \in S(\mathbb{R})$ and $\rho \in \operatorname{Irr}$ set

$$\Delta^{(\rho)} = \{ \nu^y \rho;\ y \in \Delta \}.$$

Let $\rho \in C$. Then we call $\Delta^{(\rho)}$ a segment in C. The set of all segments in C is denoted by $S(C)$. If $\alpha \in \mathbb{R}$ and $\Delta \in S(\mathbb{R})$ then

$$(\Delta_\alpha)^{(\rho)} = \nu^\alpha (\Delta^{(\rho)}).$$

For $\Delta \in S(C)$ set $\Delta_\alpha = \nu^\alpha \Delta$. Now for $\Delta \in S(\mathbb{R})$ and $\alpha \in \mathbb{R}$ we have

$$(\Delta_\alpha)^{(\rho)} = (\Delta^{(\rho)})_\alpha.$$

This is an action of $\mathbb{R}$ on S (C).

For a multiset $a = (\Delta_1, \ldots, \Delta_m) \in M(S(\mathbb{R}))$, $\alpha \in \mathbb{R}$ and $\rho \in C$ we put

$$a_\alpha = ((\Delta_1)_\alpha, \ldots, (\Delta_m)_\alpha) \in M(S(\mathbb{R})), \qquad a^{(\rho)} = (\Delta_1^{(\rho)}, \ldots, \Delta_m^{(\rho)}) \in M(S(C)).$$

For $b = (\Delta_1, \ldots, \Delta_m) \in M(S(C))$ set $b_\alpha = (\nu^\alpha \Delta_1, \ldots, \nu^\alpha \Delta_m)$. This is an action of $\mathbb{R}$ on $M(S(C))$.

Let $\pi \in \operatorname{Alg} G_n$ and $\alpha \in \mathbb{R}$. Then π_α denotes the representation $\nu^\alpha \pi$. In this way $\mathbb{R}$ acts on $\operatorname{Alg} G_n$.

1.7. For $\Delta_1, \Delta_2 \in S(C)$ we shall say that Δ_1 and Δ_2 are linked if $\Delta_1 \cup \Delta_2$ is a segment and

$$\Delta_1 \cap \Delta_2 \notin \{ \Delta_1, \Delta_2 \}.$$

Let $\Delta_1 = [\nu^i \rho, \nu^j \rho]$, $\Delta_2 = [\nu^k \rho, \nu^l \rho]$ be two linked segments. If $i < k$ then we say that Δ_1 precedes Δ_2 and write $\Delta_1 \rightarrow \Delta_2$.

1.8. For $\Delta \in S(C)$, $\Delta = [\rho, \nu^i \rho]$, $i \in \mathbb{Z}_+$, the representation

$$\rho \times \nu\rho \times \ldots \times \nu^i \rho$$

has a unique irreducible subrepresentation which we denote by $\langle \Delta \rangle$. Let

$$a = (\Delta_1, \ldots, \Delta_m) \in M(S(C)).$$

Choose an ordering on a such that:

$$\Delta_i \text{ precedes } \Delta_j \quad \Rightarrow \quad i > j.$$

Set

$$\pi(a) = \langle \Delta_1 \rangle \times \langle \Delta_2 \rangle \times \ldots \times \langle \Delta_m \rangle.$$

Now $\pi(a)$ has a unique irreducible subrepresentation which is denoted by $\langle a \rangle$. The mapping

$$a \to \langle a \rangle,$$
$$M(S(C)) \to \mathrm{Irr}$$

is a bijection. This is Zelevinsky classification.

For $\alpha \in \mathbb{R}$, $a \in M(S(\mathbb{R}))$, $\rho \in C$ and $b \in M(S(C))$ we have

$$\langle (a^{(\rho)})_\alpha \rangle = \langle (a_\alpha)^{(\rho)} \rangle = \langle a^{(\rho)} \rangle_\alpha$$

and

$$\langle b_\alpha \rangle = \langle b \rangle_\alpha$$

since the representations $\pi(b_\alpha)$ and $\pi(b)_\alpha$ are isomorphic.

1.9. For $a = (\Delta_1, \ldots, \Delta_n) \in M(S(C))$ denote supp $a \in M(C)$,

$$(\mathrm{supp}\, a)(\rho) = \mathrm{card}\{ i;\ \rho \in \Delta_i \}.$$

This is called the support of the multisegment a. In the same way define the support of $a \in M(S(\mathbb{R}))$.

Now we have

$$\mathrm{supp} \langle a \rangle = \mathrm{supp}\, a,\ a \in M(S(C)).$$

The support of a representation is defined in 1.10 of [33].

For $\omega \in M(C)$ set

$$\mathrm{Irr}_\omega = \{ \pi \in \mathrm{Irr};\ \mathrm{supp}\, \pi = \omega \},$$
$$R_\omega = \sum_{\pi \in \mathrm{Irr}_\omega} \mathbb{Z} \pi.$$

Now $\{ R_\omega;\ \omega \in M(C) \}$ is a graduation of the ring R.

1.10. Let $\rho \in C$. We denote by $\rho_{\mathbb{Z}}$ (resp. $\rho_{\mathbb{R}}$) the orbit of ρ under the action of $\mathbb{Z}$ (resp. $\mathbb{R}$).

For $X \subseteq C$ set

$$\mathrm{Irr}(X) = \bigcup_{\omega \in M(X)} \mathrm{Irr}_\omega,$$
$$\mathrm{Irr}^u(X) = \mathrm{Irr}(X) \cap \mathrm{Irr}^u.$$

Let $\rho^1, \ldots, \rho^n \in C$ belong to different $\mathbb{Z}$-orbits. We know from Proposition 1.5.2 of [34] that the multiplication in the Grotendieck group

$$\mathrm{Irr}(\rho^1_{\mathbb{Z}}) \times \ldots \times \mathrm{Irr}(\rho^n_{\mathbb{Z}}) \to \mathrm{Irr}(\rho^1_{\mathbb{Z}} \cup \ldots \cup \rho^n_{\mathbb{Z}})$$

gives us a bijection.

In the same way one obtains: if $\rho^1, \ldots, \rho^n$ belong to different $\mathbb{R}$-orbits then we have a bijection

$$\mathrm{Irr}(\rho^1_{\mathbb{R}}) \times \ldots \times \mathrm{Irr}(\rho^n_{\mathbb{R}}) \to \mathrm{Irr}(\rho^1_{\mathbb{R}} \cup \ldots \cup \rho^n_{\mathbb{R}}).$$

1.11. For a smooth representation π of finite length we associate a multiset $\mathrm{JH}(\pi) \in \mathrm{M}(\mathrm{Irr})$ as follows: $\mathrm{JH}(\pi)(\sigma)$ is the multiplicity of σ in a composition series of π. This multiset is called the Jordan-Hölder series of π.

1.12. Let $a=(\Delta_1, \ldots, \Delta_m) \in \mathrm{M}(\mathrm{S}(\mathrm{C}))$. If Δ_i and Δ_j are linked and $i<j$ then

$$b=(\Delta_1, \ldots, \Delta_{i-1}, \Delta_i \cap \Delta_j, \Delta_{i+1}, \ldots, \Delta_{j-1}, \Delta_i \cup \Delta_j, \Delta_{j+1}, \ldots, \Delta_m)$$

is again a multiset in $\mathrm{S}(\mathrm{C})$. Set $b<a$. For $a, b \in \mathrm{M}(\mathrm{S}(\mathrm{C}))$ we shall write $b \leqq a$ if there exist $a_1, \ldots, a_m \in \mathrm{M}(\mathrm{S}(\mathrm{C}))$ such that

$$b=a_1<a_2<\ldots<a_m=a.$$

Now $\leqq$ is a partial ordering on $\mathrm{M}(\mathrm{S}(\mathrm{C}))$.

1.13. Let $a \in \mathrm{M}(\mathrm{S}(\mathrm{C}))$. Then we know from [33] that

$$\mathrm{JH}(\pi(a))(\langle b \rangle) \neq 0 \quad \Leftrightarrow \quad b \leqq a,$$

and

$$\mathrm{JH}(\pi(a))(\langle a \rangle)=1.$$

1.14. For $\Delta=[\rho, \nu^i \rho] \in \mathrm{S}(\mathrm{C})$ denote by

$$\Delta^t=(\{\rho\}, \{\nu\rho\}, \ldots, \{\nu^i \rho\}) \in \mathrm{M}(\mathrm{S}(\mathrm{C})).$$

The ring R is polynomial over $\{\langle \Delta \rangle; \Delta \in \mathrm{S}(\mathrm{C})\}$ and

$$\langle \Delta \rangle \to \langle \Delta^t \rangle$$

extends to an involutive automorphism ${}^t : \mathrm{R} \to \mathrm{R}$ of the graded ring R. It is announced in [33], [34] and [2] that $\mathrm{Irr}^t \subseteq \mathrm{Irr}$.

1.15. If π is a smooth representation of G_n then $\tilde{\pi}$ denotes the contragredient representation of π. For $\Delta \in \mathrm{S}(\mathrm{C})$ the set $\{\tilde{\rho}; \rho \in \Delta\}$ is again a segment in C which we denote by $\Delta^{\sim}$. For $a=(\Delta_1, \ldots, \Delta_m) \in \mathrm{M}(\mathrm{S}(\mathrm{C}))$ let $\tilde{a}=(\Delta_1^{\sim}, \ldots, \Delta_m^{\sim}) \in \mathrm{M}(\mathrm{S}(\mathrm{C}))$. Then

$$\langle a \rangle^{\sim}=\langle \tilde{a} \rangle.$$

1.16. Let $\Delta=[\rho, \nu^i \rho] \in \mathrm{S}(\mathrm{C})$. Set

$$\Delta^- = \begin{cases} [\rho, \nu^{i-1} \rho] & \text{if } i \geqq 1 \\ \varnothing & \text{if } i=0 \end{cases}$$

$$\Delta' = \nu^{1/2} \Delta^-.$$

For $a=(\Delta_1, \ldots, \Delta_m) \in \mathrm{M}(\mathrm{S}(\mathrm{C}))$ let

$$a^-=(\Delta_1^-, \ldots, \Delta_m^-),$$

$$a' = (\Delta'_1, \ldots, \Delta'_m).$$

Now $\langle a^- \rangle$ is called the highest derivative of $\langle a \rangle$. Set $\pi = \langle a \rangle$. Then $\langle a' \rangle$ is called the highest shifted derivative of π. We put $\pi' = \langle a' \rangle$.

2. Some general facts about unitarizable representations

2.1. Let $(\pi, V) \in \operatorname{Alg} G_n$. We define on V a new multiplication by scalars

$$(\lambda, v) \to \bar{\lambda} v.$$

This vector space is denoted by $\bar{V}$. On $\bar{V}$ we have action of G_n and this representation is denoted by $(\bar{\pi}, \bar{V})$. We say that $\bar{\pi}$ is conjugated to π.

For $\pi \in \operatorname{Alg} G_n$ set

$$\pi^+ = \bar{\tilde{\pi}}.$$

We say that π^+ is Hermitian contragredient of π. Now $^+ : R \to R$ is a homomorphism of the ring. If $\pi \cong \pi^+$ we say that π is a Hermitian representation. It is well known that each unitarizable admissible representation is Hermitian.

2.2. For $a = (\Delta_1, \ldots, \Delta_m) \in M(S(C))$ set

$$a^+ = (\Delta_1^+, \ldots, \Delta_m^+) \in M(S(C))$$

where

$$\Delta_i^+ = \{ \rho^+; \rho \in \Delta_i \}.$$

By Statement 7.8 of [2] we have

$$\langle a \rangle^+ = \langle a^+ \rangle.$$

2.3. If $\rho \in C(G_n)$ then there exists $\alpha \in \mathbb{R}$ such that

$$\rho = \nu^\alpha \rho_0$$

where ρ_0 is unitarizable. Now

$$\rho^+ = \nu^{-\alpha} \rho_0.$$

This means that an irreducible cuspidal representation is unitarizable if and only if it is Hermitian. Also, an irreducible cuspidal representation is unitarizable if and only if it has unitary central character.

This implies that it is easy to characterize the subset C^u of all unitarizable representations in C.

2.4. Let $a = (\Delta_1, \ldots, \Delta_n) \in M(S(\mathbb{R}))$. Set

$$-a = (-\Delta_1, \ldots, -\Delta_n)$$

where

$$-\Delta_i=\{-x;\ x\in\Delta_i\}.$$

Note that $-\Delta_i$ are segments again.

Let $\rho\in C^u$. Then

$$(a^{(\rho)})^+=(-a)^{(\rho)}$$

and

$$\langle a^{(\rho)}\rangle^+=\langle(-a)^{(\rho)}\rangle.$$

This implies that

$$(\mathrm{Irr}(\rho_{\mathbb{Z}}))^+=\mathrm{Irr}(\rho_{\mathbb{Z}})\quad \text{for}\quad \rho\in C^u \quad\text{or}\quad \rho_{1/2}\in C^u,$$
$$(\mathrm{Irr}(\rho_{\mathbb{R}}))^+=\mathrm{Irr}(\rho_{\mathbb{R}})\quad \text{for}\quad \rho\in C.$$

2.5. THEOREM (J. N. Bernstein, [2]). — (i) *If* $\pi_1,\ \pi_2\in\mathrm{Irr}^u$ *then* $\pi_1\times\pi_2\in\mathrm{Irr}^u$.

(ii) *If* $\pi_1,\ \pi_2\in\mathrm{Irr}$ *are Hermitian and* $\pi_1\times\pi_2$ *is unitarizable, then* π_1 *and* π_2 *are unitarizable.*

(iii) *If* $\pi\in\mathrm{Irr}^u$ *then* $\pi'\in\mathrm{Irr}^u$.

In this paper we are using only the following form of (ii): If $\pi_1,\ \pi_2\in\mathrm{Irr}$ are Hermitian and $\pi_1\times\pi_2$ is irreducible unitarizable, then π_1 and π_2 are unitarizable. This fact can be proved in a much more simple manner then (ii) is proved in [2].

The last fact can be generalized to any reductive group. The form of generalization was suggested by F. Rodier (3).

2.6. Let G be the group of F-rational points of a connected reductive linear algebraic group defined over F.

Let H(G) be the Hecke algebra G. If K is an open compact subgroup then H(G, K) will denote the subalgebra of all K-biinvariant functions in H(G).

The definition of $\hat{G}$ and $\tilde{G}$ is analogous to the definition of $\hat{G}_n$ and $\tilde{G}_n$.

The character of a representation π is denoted by ch_π.

2.7. THEOREM. — (i) *Let* (π_n) *be a sequence in* $\hat{G}$ *such that for each* $f\in H(G)$ *there exists a finite limit* $\lim_n \mathrm{ch}_{\pi_n}(f)$ *in the complex numbers. Then there exist* $n_\sigma\in\mathbb{Z}_+$, $\sigma\in\hat{G}$, *such that*

$$\lim_n \mathrm{ch}_{\pi_n}(f)=\sum_{\sigma\in\hat{G}} n_\sigma\,\mathrm{ch}_\sigma(f)$$

for each fixed $f\in H(G)$. *The set* $\{\sigma;\ n_\sigma\,\mathrm{ch}_\sigma(f)\neq 0\}$ *for a given f, is always finite.*

---

(3) Added in proof: this generalization has been obtained earlier by B. Speh.

If we have numbers $m_\sigma \in \mathbb{Z}_+$, $\sigma \in \tilde{G}$, *such that*

$$\sum_{\sigma \in \tilde{G}} m_\sigma \operatorname{ch}_\sigma(f) = \sum_{\sigma \in \hat{G}} n_\sigma \operatorname{ch}_\sigma(f)$$

for all $f \in H(G)$ *then* $m_\sigma = 0$ *for* $\sigma \in \tilde{G} \setminus \hat{G}$, *and* $n_\sigma = m_\sigma$ *for* $\sigma \in \hat{G}$.

(ii) *If* (π_n) *is a sequence in* $\hat{G}$ *then there exists a subsequence* (π_{n_k}) *of* (π_n) *such that* $(\operatorname{ch}_{\pi_{n_k}}(f))$ *is convergent for each fixed* $f \in H(G)$.

(iii) *In the situation of* (i), *the set*

$$\{\sigma;\ n_\sigma \neq 0\}$$

is finite.

Remark that in the situation of (i) the sequence (π_n) in the topology of the dual $\hat{G}$ converges to σ if and only if $n_\sigma \neq 0$. The first two statements of the above theorem are consequences of [18] and [1]. All three statements may be obtained directly from [27] (*see* particulary Remark 5.8 of [27]) where the topology of the unitary dual of reductive p-adic groups was studied. In this paper we shall use only the statement (i). Here we shall give another proof of (i) and (ii) which is not using the general facts about dual spaces. We shall prove (i) and (ii) simultaneously. In the cases where we shall apply (i) of this theorem, it will be obvious that the finiteness property of (iii) holds. The property (iii) is expressed in the terms of the topology of unitary dual in the following way: a sequence in $\hat{G}$ can have only a finite set of limits. The statement (iii) can be proved using [3].

Proof. — Let (π_n, V_n) be a sequence in $\hat{G}$. Let $\{K_m, m \in \mathbb{N}\}$ be a descending basis of neighbourhoods of identity in G as in [1], consisting of open compact subgroups of G. Let $\pi_n^{K_m}$ be the representation of the Hecke algebra $H(G, K_m)$ on the space of all K_m-fixed vectors $V_n^{K_m}$ in V_n. Now $V_n^{K_m}$ are irreducible $H(G, K_m)$-modules ([7], Proposition 2.2.4) and there exist $c_m \geqq 0$ such that

$$(*) \qquad \dim_{\mathbb{C}} V_n^{K_m} \leqq c_m$$

for all $n \in \mathbb{N}$. The last inequality is a simple consequence of the Statement (A) of [1] and the fact that every irreducible smooth representation of G is admissible.

Let V be a countable dimensional complex vector space with an inner product.

Passing to a subsequence we may suppose that $\dim_{\mathbb{C}} V_n^{K_1} = d_1$ for all $n \in \mathbb{N}$. Then choose a d_1-dimensional subspace V^1 of V. Let

$$(**) \qquad \tilde{\varphi}_n^1 :\ V_n^{K_1} \to V^1$$

be an isomorphism of unitary spaces. Now we have a unique representation $(\pi_n^{K_1})^\#$ on V^1 such that $\tilde{\varphi}_n^1$ is an isomorphism of $H(G, K_1)$-modules.

Let dg be a Haar measure on G. For $f\in H(G, K_1)$, $v_1, v_2\in V_n^{K_1}$ unit vectors we have

$$|(\pi_n^{K_1}(f)v_1, v_2)| = \left|\int_G f(g)(\pi_n(g)v_1, v_2)\,dg\right| \leqq \int_G |f(g)|\,dg.$$

Thus, the operator norm $|\pi_n^{K_1}(f)|$ on $V_n^{K_1}$ is bounded by $\int_G |f(g)|\,dg$, so $\{(\pi_n^{K_1})^\#(f);\ n\in\mathbb{N}\}$ is a bounded family of operators on V^1 for each fixed $f\in H(G)$.

The algebra $H(G, K_1)$ is finitely generated (in fact the proof can be done without this fact but then it would be a little longer). Let $f_1, \ldots, f_k\in H(G, K_1)$ be generators. Passing to a subsequence of (π_{n_p}) we may suppose that $((\pi_{n_p}^{K_1})^\#(f_i))_p$ is a convergent sequence for $1\leqq i\leqq k$. Now each $((\pi_{n_p}^{K_1})^\#(f))_{n_p}$ converges to some $\pi^1(f)$ and clearly π^1 is a representation of $H(G, K_1)$. We have

$$(***)\qquad \lim_n \operatorname{trace}(\pi_{n_p}^{K_1})(f)=\operatorname{trace}\pi^1(f),\qquad f\in H(G, K_1).$$

Denote by $({}^1\pi_p, {}^1V_p)$ the sequence (π_{n_p}, V_{n_p}) and denote $\tilde{\varphi}_{n_p}^1$ by φ_p^1.

Passing to a subsequence of $({}^1\pi_n)$ we may suppose that $\dim {}^1V_n^{K_2}=d_2$ for all n. Now $d_2\geqq d_1$ and we can find in V a d_2-dimensional subspace V^2 containing V^1. Let $\tilde{\varphi}_n^2 : {}^1V_n^{K_2}\to V^2$ be an isomorphism of unitary spaces such that $\tilde{\varphi}_n^2$ restricted to ${}^1V_n^{K_1}$ is $(**)$. We have a unique representation $({}^1\pi_n^{K_2})^\#$ of $H(G, K_2)$ on V^2 such that $\tilde{\varphi}_n^2$ is an isomorphism of $H(G, K_2)$-modules. Passing to a subsequence $({}^2\pi_n, {}^2V_n)$ we may suppose that $(({}^2\pi_n^{K_2})^\#(f))_n$ converges for all $f\in H(G, K_2)$ to some $\pi^2(f)$. Now $({}^2\pi_n^{K_2})^\#(\chi_{K_1})$ is an orthogonal projection on V^1. Here χ_{K_1} denotes the characteristic function of K_1. The restriction of the representation π^2 to the subalgebra $H(G, K_1)$ and the subspace V^1 is just π^1. Corresponding isomorphism $\tilde{\varphi}_m^2 : {}^2V_n^{K_2}\to V^2$, with suitable m, is denoted by φ_n^2.

Now we define recursively sequences $(({}^m\pi_n, {}^mV_n))_n$, $d_m\in\mathbb{Z}_+$, subspaces V^m of V, unitary space isomorphisms $\varphi_n^m : {}^mV_n^{K_m}\to V^m$, representations $({}^m\pi_n^{K_m})^\#$ and π^m of $H(G, K_m)$ on V^m in the following way.

We have defined these objects for $m=2$. Suppose that $m\geqq 2$ and that we have defined the above objects for that m. Passing to a subsequence of $(({}^m\pi_n, {}^mV_n))_n$ we may suppose that $\dim {}^mV_n^{K_{m+1}}=d_{m+1}$ for all n. Since $d_{m+1}\geqq d_m$, we may find a d_{m+1}-dimensional subspace V^{m+1} containing V^m. Let $\tilde{\varphi}_n^{m+1} : {}^1V_n^{K_{m+1}}\to V^{m+1}$ be an isomorphism of unitary spaces such that $\tilde{\varphi}_n^{m+1}$ restricted to ${}^mV_n^{K_m}$ is just $\varphi_n^m : {}^mV_n^{K_m}\to V^m$. We denote by $({}^m\pi_n^{K_{m+1}})^\#$ a unique representation of $H(G, K_{m+1})$ on V^{m+1} such that $\tilde{\varphi}_n^{m+1}$ in an isomorphism of representations. Passing to a subsequence $(({}^{m+1}\pi_n, {}^{m+1}V_n))_n$ of $(({}^m\pi_n, {}^mV_n))_n$ we may suppose that $(({}^{m+1}\pi_n^{K_{m+1}})^\#(f))_n$ converges for each fixed $f\in H(G, K_{m+1})$ to some $\pi^{m+1}(f)$. The restriction of the representation π^{m+1} to the subalgebra $H(G, K_m)$ and the subspace V^m is π^m. Corresponding isomorphism $\tilde{\varphi}_k^{m+1} : {}^{m+1}V_n^{K_{m+1}}\to V^{m+1}$, with suitable k, is denoted by φ_n^{m+1}.

We have constructed sequences $({}^k\pi_n)_n$, $k \in \mathbb{N}$, where $({}^{k+1}\pi_n)_n$ is a subsequence of $({}^k\pi_n)_n$. To each $k \in \mathbb{N}$ corresponds V^k and a representation π^k of $H(G, K_k)$. Denote

$$V_0 = \bigcup_{k \geqq 1} V^k.$$

On V_0 we have a representation π_0 of $H(G)$ because $H(G) = \bigcup_{k \geqq 1} H(G, K_k)$. To this representation corresponds a smooth representation of G, denoted by π_0 again. We have $\pi_0^{K_k} = \pi^k$. Thus π_0 is admissible.

For $f \in H(G)$ set $f^* \in H(G)$

$$f^*(g) = \overline{f(g^{-1})}.$$

The representations (π_n, V_n) are unitary so

$$(\pi_n(f) v_1, v_2) = (v_1, \pi_n(f^*) v_2).$$

Therefore,

$$(\pi_0(f) v, w) = (v, \pi_0(f^*) w)$$

for $f \in H(G)$, $v, w \in V_0$. From this one obtains directly that the inner product on V_0 is G-invariant.

Let $\sigma_k = {}^k\pi_k$. Then $(\sigma_k)_k$ is a subsequence of $(\pi_n)_n$ and

$$\lim_k \mathrm{ch}_{\sigma_k}(f) = \mathrm{ch}_{\pi_0}(f).$$

Since an admissible unitarizable representation is completely reducible, it is a direct sum of unitarizable irreducible representations, with finite multiplicities ([7], 2.1.14). Thus, we have proved (ii) and part of (i).

Suppose that

$$\sum_{\sigma \in \tilde{G}} m_\sigma \mathrm{ch}_\sigma(f) = \sum_{\sigma \in \hat{G}} n_\sigma \mathrm{ch}_\sigma(f)$$

for all $f \in H(G)$. We define n_σ to be 0 for $\sigma \in \tilde{G} \setminus \hat{G}$. Let m_{σ_0} or n_{σ_0} be nonzero. Choose an open compact subgroup K of G such that σ_0 has a nonzero vector invariant for K. Let $\tilde{G}^K$ be the subset of $\tilde{G}$ consisting of all classes of representations with nonzero vector invariant for K. Now

$$\sum_{\sigma \in \tilde{G}^K} m_\sigma \mathrm{ch}_\sigma(f) = \sum_{\sigma \in \tilde{G}^K} n_\sigma \mathrm{ch}_\sigma(f)$$

for all $f \in H(G, K)$. Note that the sets

$$\{\sigma \in \tilde{G}^K;\ m_\sigma \neq 0\},$$
$$\{\sigma \in \tilde{G}^K;\ n_\sigma \neq 0\}$$

are finite. Since for $\sigma_1, \sigma_2 \in \tilde{G}^K$, $\sigma_1 \neq \sigma_2$, corresponding representations of H(G, K) are unequivalent ([7], Proposition 2.2.2), the linear independence of characters of representations of H(G, K) implies $m_\sigma = n_\sigma$ for $\sigma \in \tilde{G}^K$. Thus $m_{\sigma_0} = n_{\sigma_0}$. This proves the rest of (i).

Remark. — The D. Miličić description of the topology of the dual spaces of C*-algebras with bounded trace in [18] gives that irreducible subquotients of ends of complementary series of a reductive group over any local field are unitarizable (after using some general fact from the representation theory of such groups). As this author knows, this fact was noticed first by D. Miličić.

The topology of the dual spaces of real semi-simple Lie groups was studied by D. Miličić in his Ph. D. thesis (University of Zagreb, 1973). These results have not been published. Note that there exists also a considerable difference between the topology of the dual spaces of reductive groups over archimedean fields and over non-archimedean fields. For example, no one of the three main results of [27] mentioned in the introduction of that paper, holds for real reductive groups (counterexamples are either SL(2, $\mathbb{R}$) or SL(2, $\mathbb{C}$)).

2.8. Suppose that P is a parabolic subgroup in G. Let P = MN be a Levi decomposition of P. Denote by ^{0}M the subgroup of all $m \in M$ such that $|\omega(m)|_F = 1$ for all rational characters ω of M. A character χ of M is called unramified if it is trivial on ^{0}M.

The group of all unramified characters Unr(M) of M has a topology of convergence over compacts. In fact, Unr(M) has a canonical structure of a complex Lie group.

For a smooth representation σ of M, $\mathrm{Ind}(\sigma | P, G)$ denotes the representation of G induced by σ from P.

The next fact can be easily obtained from the formula for the character of an induced representation (computed in [8]).

LEMMA. — *For $\chi \in \mathrm{Unr}(M)$ let $\pi_\chi = \mathrm{Ind}(\chi\sigma | P, G)$. Then the mapping*

$$\chi \to \mathrm{ch}_{\pi_\chi}(f)$$

is continuous, for fixed $f \in H(G)$.

2.9. We are again in the case of GL(n, F).

PROPOSITION. — *Let $\sigma \in \hat{G}_n$. Suppose that $\sigma_\alpha \times \sigma_{-\alpha}$ is irreducible for $\alpha \in (-\alpha_0, \alpha_0)$, $\alpha_0 > 0$. Then $\sigma_\alpha \times \sigma_{-\alpha}$ is unitarizable for $\alpha \in (-\alpha_0, \alpha_0)$ and all composition factors of $\sigma_{\alpha_0} \times \sigma_{-\alpha_0}$ are unitarizable.*

This is a consequence of the Proposition 8.3 of [2], the last theorem and the last lemma.

2.10. The following proposition may be useful for applying the last proposition.

PROPOSITION. — (i) *If $\sigma \in \hat{G}_n$ then there exist $\alpha_0 > 0$ such that $\sigma_\alpha \times \sigma_{-\alpha}$ is irreducible for $\alpha \in (-\alpha_0, \alpha_0)$.*

(ii) *Let $\sigma \in \tilde{G}_n$ with $n \geqq 1$. Suppose that*

$$\sigma_\alpha \times (\sigma^+)_{-\alpha}$$

is unitarizable and irreducible for $\alpha \in (\beta, \gamma)$, $\beta < \gamma$. *Then*

$$\gamma - \beta \leqq 1.$$

If $\gamma - \beta = 1$ *then* $\sigma_\beta \times (\sigma^+)_{-\beta}$ *and* $\sigma_\gamma \times (\sigma^+)_{-\gamma}$ *reduces.*

Proof. — The fact that the set of all $\alpha \in \mathbb{R}$ such that $\sigma_\alpha \times \sigma_{-\alpha}$ is reducible, is finite and (ii) of Theorem 2.5 implies (i).

Proposition and Remark in 8.3 of [2] implies that $\gamma - \beta \leqq 1$. The reducibility of $\sigma_\beta \times (\sigma^+)_{-\beta}$ and $\sigma_\gamma \times (\sigma^+)_{-\gamma}$, when $\gamma - \beta = 1$, follows from proposition 2.9, the fact that the set

$$\{ \alpha \in \mathbb{R};\ \sigma_\alpha \times (\sigma^+)_{-\alpha} \text{ is reducible} \}$$

is closed, and the first part of (ii).

2.11. Let for a moment S be an abelian multiplicative semigroup with identity.

Let B be a subset of S such that B generates S as semigroup. Suppose that B satisfies the following property: if $s \in S$ and $s = b_1 b_2 \ldots b_i = c_1 c_2 \ldots c_j$ with b_i, $c_j \in B$, then $i = j$ and there exists a permutation σ of $\{1, 2, \ldots, i\}$ such that $b_k = c_{\sigma(k)}$, $1 \leqq k \leqq i$. In this case we shall say that S is a free abelian semigroup over B.

Let S_i, $i \in I$, be a family of subsemigroups of S such that each S_i contains identity of S. Suppose that the union of all S_i, $i \in I$, generates S as semigroup and suppose that the following condition holds: if s_i, $r_i \in S_i$, $i \in I$, are such that s_i and r_i are different from identity only for finitely many i and if

$$\prod_{i \in I} s_i = \prod_{i \in I} r_i,$$

then $s_i = r_i$ for all $i \in I$ (we take $\prod_{i \in I} s_i$ to be identity if all s_i are identity, otherwise we take $\prod_{i \in I} s_i$ to be the product of all s_i which are different from identity). In this case we say that S is a direct sum of semigroups S_i, $i \in I$, and we write $S = \bigoplus_{i \in I} S_i$.

If S is a direct sum of S_i, $i \in I$, and if S_i are free abelian semigroups, then one can see directly that S is a free abelian semigroup.

If A is a subset of S then $X(A)$ will denote the subsemigroup of S, with identity, generated by A.

2.12. The Theorem 2.5 implies that Irr^u is multiplicatively closed in R. Thus Irr^u is a semigroup. The group $\mathbb{R}$ acts in C and a set of representatives of $\mathbb{R}$-orbits in C is C^u.

PROPOSITION:

$$\mathrm{Irr}^u = \bigoplus_{\rho \in C^u} \mathrm{Irr}^u(\rho_{\mathbb{R}}).$$

Proof. — Let $\rho^1, \ldots, \rho^n \in C^u$ be different elements. Then we have an injection

$$\mathrm{Irr}^u(\rho^1_{\mathbb{R}}) \times \ldots \times \mathrm{Irr}^u(\rho^u_{\mathbb{R}}) \to \mathrm{Irr}^u(\rho^1_{\mathbb{R}} \cup \ldots \cup \rho^n_{\mathbb{R}})$$

by 1.10. We shall prove that this is a surjection. Let

$$\pi \in \mathrm{Irr}^u(\rho^1_{\mathbb{R}} \cup \ldots \cup \rho^n_{\mathbb{R}}).$$

Then $\pi = \pi_1 \times \ldots \times \pi_n$, $\pi_i \in \mathrm{Irr}(\rho^i_{\mathbb{R}})$. Now $\pi = \pi^+$ implies $\pi_i = \pi_i^+$ by 1.10. The Theorem 2.5 (ii) implies $\pi_i \in \mathrm{Irr}^u(\rho^i_{\mathbb{R}})$.

Now it is easy to see that $\mathrm{Irr}^u(\rho_{\mathbb{R}})$, $\rho \in C^u$, generates the whole Irr^u.

2.13. As in 2.12 we get, for $\rho \in C^u$,

$$\mathrm{Irr}^u(\rho_{\mathbb{R}}) = [\bigoplus_{\alpha \in (0,\, 1/2)} \mathrm{Irr}^u((\rho_\alpha)_{\mathbb{Z}} \cup (\rho_{-\alpha})_{\mathbb{Z}})] \oplus [\sum_{\alpha \in \{0,\, 1/2\}} \mathrm{Irr}^u(\rho_\alpha)_{\mathbb{Z}}].$$

It is now a question to classify the unitarizable representations in $\mathrm{Irr}((\rho_\alpha)_{\mathbb{Z}} \cup (\rho_{-\alpha})_{\mathbb{Z}})$. The case $\alpha \in \{0, 1/2\}$ is called the rigid case and $\alpha \in (0, 1/2)$ the nonrigid case ([2], 8.4).

2.14. Take $\Delta \in S(\mathbb{R})$ such that $\Delta = -\Delta$ and $\rho \in C^u$. Then

$$\langle \Delta^{(\rho)} \rangle \quad \text{and} \quad \langle (\Delta^t)^{(\rho)} \rangle$$

are unitarizable representations (Lemmas 8.8 and 8.9 of [2]).

The above statement may be proved using Proposition 2.9 and (ii) of Theorem 2.5, by induction.

2.15. In [27], the following theorem is proved.

THEOREM. — *Let* G *be the group of rational points of a connected linear reductive* F-*group and* P = MN *a parabolic subgroup. Let* σ *be an irreducible cuspidal representation of* M. *Then the set of all* $\chi \in \mathrm{Unr}(M)$ *such that* $\mathrm{Ind}(\chi\sigma \mid P, G)$ *contains a unitarizable composition factor, is a compact subset of* Unr (M).

This can be easily proved from (ii) of the Theorem 2.7 using Bernstein center [3]. We shall not use this theorem in the paper.

3. Certain primes in R

3.1. For a positive integer m we put

$$\Delta[m] = \left[\frac{-m+1}{2}, \frac{m-1}{2}\right] \in S(\mathbb{R}).$$

If $d, n \in \mathbb{N}$ then we set

$$a(n, d) = (\Delta[d]_{(-n+1)/2}, \Delta[d]_{(-n+3)/2}, \ldots, \Delta[d]_{(n-1)/2}) \in M(S(\mathbb{R}))).$$

If $n=1$ then $a(1, d) = \Delta[d]$. If $d=1$ then $a(n, 1) = \Delta[n]^t$.

3.2. By 2.14 we have that $\langle a(1, n)^{(\rho)} \rangle$, $\langle a(n, 1)^{(\rho)} \rangle$ are unitarizable for $\rho \in C^u$. Also we have, by the definition of the involution ^{t},

$$\langle a(1, n)^{(\rho)} \rangle^t = \langle a(n, 1)^{(\rho)} \rangle.$$

3.3. The ring R is a factorial ring ([33], Corollary 7.5). The ring R is a polynomial ring in indeterminates $\langle \Delta \rangle$ over $\mathbb{Z}$, $\Delta \in S(C)$. Thus $\langle \Delta \rangle$ are prime elements of R. Since $t : R \to R$ is a ring isomorphism, $\langle \Delta \rangle^t$ are prime elements. It means that $\langle a(n, 1)^{(\rho)} \rangle$ and $\langle a(1, n)^{(\rho)} \rangle$ are prime elements. We shall prove generally that $\langle a(n, d)^{(\rho)} \rangle$ are prime elements of the factorial ring R.

We shall suppose in the rest that $d>1$, $n>1$. We fix $\rho \in C$.

3.4. Let $a(n, d)^{(\rho)} = (\Delta_1, \Delta_2, \ldots, \Delta_n)$,

$$\Delta_1 \to \Delta_2 \to \Delta_3 \to \ldots \to \Delta_n.$$

LEMMA. — *Let $b \in M(S(C))$ such that $b \leqq a(n, d)^{(\rho)}$. Then*

$$b(\Delta_i) \in \{0, 1\}$$

for $1 \leqq i \leqq n$.

Proof. — Since $b \leqq a(n, d)^{(\rho)}$ then, for $\sigma \in C$, the number of segments in b which begin in σ is less then or equal to the number of segments in $a(n, d)^{(\rho)}$ which begin in σ. The same conclusion is true for the numbers of segments which end in σ. This implies our lemma.

3.5. For $a \in M(S(C))$ we have

$$\pi(a) = \sum_{b \leqq a} m(b; a) \langle b \rangle, \qquad m(b, a) \geqq 1$$

([33], Theorem 7.1). We fix a and consider the system of all equations

$$\pi(c) = \sum_{b \leqq c} m(b, c) \langle b \rangle$$

with $c \leqq a$. This is a triangular unipotent system in indetermines $\langle b \rangle$, with $b \leqq a$, for a suitable order of indetermines $\langle b \rangle$. We can solve this system. In particular, we shall obtain

$$\langle a \rangle = \sum_{b \leqq a} m_{a, b} \pi(b).$$

Here $m_{a, a} = 1$.

3.6. Suppose that $b_0 < a$ but there is no $b \in M(S(C))$ such that

$$b_0 < b < a.$$

Then 3.5 implies $m_{a, b_0} \neq 0$ because $m_{a, b_0} = -m(b_0, a)$and $m(b_0, a) \neq 0$.

3.7. We return to the notation of 3.4. Set

$$\Delta_i^b = \Delta_i \cup \Delta_{i+1}$$
$$\Delta_i^s = \Delta_{i-1} \cap \Delta_i$$

Denote

$$a_{1,2} = (\Delta_1^b, \Delta_2^s, \Delta_3, \ldots, \Delta_n)$$
$$a_{2,3} = (\Delta_1, \Delta_2^b, \Delta_3^s, \Delta_4, \ldots, \Delta_n)$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$a_{n-1,\,n} = (\Delta_1, \Delta_2, \ldots, \Delta_{n-2}, \Delta_{n-1}^b, \Delta_n^s).$$

Now $a_{i,\,i+1}$ satisfy 3.6 with respect to $a(n, d)^{(\rho)}$. We shall denote $a(n, d)^{(\rho)}$ by a_0.

3.8. PROPOSITION. — *Let $d, n \in \mathbb{N}$, and $\rho \in \mathrm{C}$. Then $\langle a\,(n, d)^{(\rho)} \rangle$ is a prime element of* R.

Proof. — It is enough to consider the case $d \geqq 2$, $n \geqq 2$. Set $a_0 = a(n, d)^{(\rho)}$. We shall suppose that $\langle a\,(n, d)^{(\rho)} \rangle$ is not a prime element. Let $\langle a\,(n, d)^{(\rho)} \rangle = \mathrm{P}_1 \times \mathrm{P}_2$ be some nontrivial decomposition. Since $\langle a\,(n, d)^{(\rho)} \rangle$ is a homogeneous element of the graded ring R, P_1 and P_2 are homogeneous elements and $\deg \mathrm{P}_1 > 0$, $\deg \mathrm{P}_2 > 0$.

We shall look at R as a polynomial algebra over indeterminates $\langle \Delta \rangle$, $\Delta \in \mathrm{S}(\mathrm{C})$.

Now 3.5, 3.6 and 3.7 imply that we can decompose $\langle a\,(n, d)^{(\rho)} \rangle$ in the basis $\pi(b)$.

$$(*)\quad \langle a\,(n, d)^{(\rho)} \rangle = \langle \Delta_1 \rangle \times \ldots \times \langle \Delta_n \rangle$$
$$+ m_{a_0,\, a_{1,2}} \langle \Delta_1^b \rangle \times \langle \Delta_2^s \rangle \times \langle \Delta_3 \rangle \times \ldots \times \langle \Delta_n \rangle + \ldots$$
$$+ \ldots + m_{a_0,\, a_{n-1,\,n}} \langle \Delta_1 \rangle \times \ldots \times \langle \Delta_{n-2} \rangle \times \langle \Delta_{n-1}^b \rangle \times \langle \Delta_n^s \rangle + \textit{other terms}$$

where $m_{a_0,\, a_{i,\,i+1}} \neq 0$, for $1 \leqq i \leqq n$.

Let

$$\mathrm{P}_i = \sum_{b \in \mathrm{M}(\mathrm{S}(\mathrm{C}))} m_b^i\, \pi(b), \qquad i = 1, 2,$$

where $m_b^i \in \mathbb{Z}$ are uniquely determined. Since

$$\langle a\,(n, d)^{(\rho)} \rangle = \mathrm{P}_1 \times \mathrm{P}_2,$$

the formula (*) implies that there exist $b_1, b_2 \in \mathrm{M}(\mathrm{S}(\mathrm{C}))$ such that

$$b_1 + b_2 = (\Delta_1, \Delta_2, \ldots, \Delta_n), \text{ i.e.}$$
$$\pi(b_1) \times \pi(b_2) = \langle \Delta_1 \rangle \times \ldots \times \langle \Delta_n \rangle$$

and $m_{b_i}^i \neq 0$ for $i = 1$ and 2. This implies that there exists a partition of $\{1, 2, \ldots, n\}$ into two parts

$$\{1, 2, \ldots, n\} = \{p(1), \ldots, p(r)\} \cup \{q(1), \ldots, q(s)\},$$

$r+s=n$, such that

$$b_1=(\Delta_{p(1)}, \ldots, \Delta_{p(r)})$$

and

$$b_2=(\Delta_{q(1)}, \ldots, \Delta_{q(s)}).$$

These parts are nonempty because P_1 and P_2 are homogeneous elements of R of positive degrees.

There exist $p(i)$ and $q(j)$ which are consecutive. We fix such i and j. Clearly $1 \leqq i \leqq r$ and $1 \leqq j \leqq s$. Let

$$\{p(i), q(j)\}=\{k, k+1\} \qquad (1 \leqq k \leqq n-1).$$

As before, from (*) one obtains that there exist $c_1, c_2 \in M(S(C))$ such that

$$c_1+c_2=a_{k,\ k+1}$$

and $m^1_{c_1} \neq 0$, $m^2_{c_2} \neq 0$. Thus

$$c_1+c_2=(\Delta_1, \Delta_2, \ldots, \Delta_{k-1}, \Delta^b_k, \Delta^s_{k+1}, \Delta_{k+2}, \ldots, \Delta_n).$$

This means that c_1 and c_2 determine a partition of the set $\{\Delta_1, \ldots, \Delta_{k-1}, \Delta^b_k, \Delta^s_{k+1}, \Delta_{k+2}, \ldots, \Delta_n\}$ and we consider c_1, c_2 as subsets of this set. Without loss of generality we may suppose that $\Delta^b_k \in c_1$. Suppose that $\Delta^s_{k+1} \notin c_1$. Let $\deg \rho = z \in \mathbb{N}$. Now

$$\deg \langle \Delta_m \rangle = dz,$$
$$\deg P_1 = rdz,$$
$$\deg P_2 = sdz,$$
$$\deg \Delta^b_m = (d+1)z,$$
$$\deg \Delta^s_m = (d-1)z.$$

Since $\Delta^b_k \in c_1$ and $\Delta^s_{k+1} \notin c_1$, there exist $m \in \mathbb{Z}_+$ such that

$$\deg \pi(c_1) = m\,dz + (d+1)z = z(d(m+1)+1).$$

The element P_1 is homogeneous, so $\deg \pi(c_1) = \deg P_1$ i.e.

$$r\,dz = z(d(m+1)+1).$$

This implies that d divides 1 i.e. $d=1$. This is a contradiction since we consider the case of $d \geqq 2$.

Thus $\Delta^b_k, \Delta^s_{k+1} \in c_1$. Now degrees of c_1 and c_2 implies that there exist a partition

$$\{1, 2, \ldots, k-2, k-1, k+2, k+3, \ldots, n\}$$
$$=\{u(1), \ldots, u(r-2)\} \cup \{v(1), \ldots, v(s)\}$$

such that

$$c_1=(\Delta_k^b, \Delta_{k+1}^s, \Delta_{u(1)}, \ldots, \Delta_{u(r-2)})$$
$$c_2=(\Delta_{v(1)}, \ldots, \Delta_{v(s)}).$$

In particular, $u(f)\notin\{k, k+1\}$ for $1\leqq f\leqq r-2$ and $v(f)\notin\{k, k+1\}$ for $1\leqq f\leqq s$. Thus $q(j)\notin\{v(1), v(2), \ldots, v(s)\}$ i.e.

$$\{q(1), \ldots, q(s)\}\nsubseteq\{v(1), \ldots, v(s)\}.$$

Therefore $\{v(1), \ldots, v(s)\}$ has a nonempty intersection with the complement of $\{q(1), \ldots, q(s)\}$, i.e.

$$\{v(1), \ldots, v(s)\}\cap\{p(1), \ldots, p(r)\}\neq\varnothing.$$

Let $g=p(t_1)=v(t_2)$ be in the intersection $(1\leqq t_1\leqq r, 1\leqq t_2\leqq s, 1\leqq g\leqq n)$.

Now, the polynomial P_1 has a degree in the indeterminate $\langle\Delta_{p(t_1)}\rangle=\langle\Delta_g\rangle$ greater than or equal to 1, and the polynomial P_2 has a degree in the indeterminate $\langle\Delta_{v(t_2)}\rangle=\langle\Delta_g\rangle$ greater then or equal to 1. Thus, the polynomial $\langle a(n, d)^{(\rho)}\rangle=P_1\times P_2$ has a degree in the indeterminate $\langle\Delta_g\rangle$ greater than or equal to 2. This contradicts Lemma 3.4 and 3.5.

4. On completeness argument

4.1. Let $\pi\in\text{Irr}$. If

$$\text{supp}\,\pi\in M\Big(\bigcup_{\rho\in C^u}\rho_{(1/2)\mathbb{Z}}\Big),$$

then we say that π is a rigid representation. If

$$\text{supp}\,\pi\in M\Big(\bigcup_{\substack{\rho\in C^u\\ \alpha\in(0,\,1/2)}}((\rho_\alpha)_{\mathbb{Z}}\cup(\rho_{-\alpha})_{\mathbb{Z}})\Big),$$

then we say that π is a nonrigid representation.

Each $\pi\in\text{Irr}$ can be uniquely decomposed

$$\pi=\pi(r)\times\pi(n)$$

where $\pi(r)$ is a rigid and $\pi(n)$ a nonrigid representation.

Now π is unitarizable if and only if $\pi(r)$ and $\pi(n)$ are unitarizable.

We have corresponding decomposition of unitarizable representations

$$\text{Irr}^u=\Big[\bigoplus_{\substack{\alpha\in(0,\,1/2)\\ \rho\in C^u}}\text{Irr}^u((\rho_\alpha)_{\mathbb{Z}}\cup(\rho_{-\alpha})_{\mathbb{Z}}\Big]\oplus\Big[\bigoplus_{\substack{\alpha\in\{0,\,1/2\}\\ \rho\in C^u}}\text{Irr}^u((\rho_\alpha)_{\mathbb{Z}})\Big].$$

4.2. PROPOSITION. — *Let* $\sigma \in \mathrm{Irr}^u$ *be a rigid representation. Then*

$$\pi(\sigma, \alpha) = \sigma_\alpha \times (\sigma_\alpha)^+ = \sigma_\alpha \times \sigma_{-\alpha}$$

is irreducible and unitarizable for

$$\alpha \in (-1/2, 1/2).$$

The representation $\pi(\sigma, 1/2) = \pi(\sigma, -1/2)$ *is reducible when* $\deg \sigma > 0$.

Proof. — Suppose that $\sigma = \sigma^1 \times \sigma^2$ where $\sigma^1, \sigma^2 \in \mathrm{Irr}^u$. Let $\pi(\sigma_1, \alpha)$, $\pi(\sigma_2, \alpha)$ be irreducible unitarizable representations for $\alpha \in (-1/2, 1/2)$. Then $\pi(\sigma, \alpha) = \pi(\sigma_1, \alpha) \times \pi(\sigma_2, \alpha)$ is unitarizable and irreducible for $\alpha \in (-1/2, 1/2)$ by Theorem 2.5. We can decompose $\sigma = \sigma^1 \times \ldots \times \sigma^m$ where each σ^1 is supported in one $\mathbb{Z}$-orbit in C and different σ^i are supported in distinct $\mathbb{Z}$-orbits. Since σ is unitarizable σ^i are unitarizable because σ is rigid. Now $\sigma^i_\alpha \times \sigma^i_{-\alpha}$ is irreducible for $\alpha \in (-1/2, 1/2)$ since σ^i is supported in $\rho_{\mathbb{Z}}$ or $\rho_{1/2+\mathbb{Z}}$, $\rho \in C^u$. Proposition 2.9 implies that $\pi(\sigma^i, \alpha)$ are unitarizable for $\alpha \in (-1/2, 1/2)$. The reducibility of $\pi(\sigma, 1/2)$ follows from (ii) of Proposition 2.10.

4.3. PROPOSITION. — Suppose that $\pi^1, \pi^2 \in \mathrm{Irr}^u$ are rigid representations. Then

$$\pi^1 \times (\pi^2)_\varepsilon$$

is irreducible for $\varepsilon \in \{-1/2, 0, 1/2\}$.

Proof. — By Theorem 2.5 it is enough to consider $\varepsilon \in \{-1/2, 1/2\}$. We shall prove the proposition for $\varepsilon = 1/2$. Now

$$\pi(\pi^1 \times \pi^2_{1/2}, \alpha) = (\pi^1_\alpha \times \pi^2_{1/2+\alpha}) \times (\pi^1_{-\alpha} \times \pi^2_{-1/2-\alpha}) = (\pi^1_\alpha \times \pi^1_{-\alpha}) \times (\pi^2_{1/2+\alpha} \times \pi^2_{-1/2-\alpha}).$$

Take $\alpha = -1/4$, Now $\pi(\pi^1 \times \pi^2_{1/2}, -1/4) = \pi(\pi^1, -1/4) \times \pi(\pi^2, 1/4)$ is irreducible by Theorem 2.5. Thus

$$(\pi^1 \times \pi^2_{1/2})$$

is irreducible. This implies the proposition.

COROLLARY. — *Let* $\pi^1, \ldots, \pi^k \in \mathrm{Irr}^u$ *be rigid representations. Then*

$$\pi^1_{\varepsilon_1} \times \pi^2_{\varepsilon_2} \times \ldots \times \pi^k_{\varepsilon_k}$$

is irreducible when all $\varepsilon_i \in \{0, 1/2\}$ (*or when all* $\varepsilon_i \varepsilon \{-1/2, 0\}$).

4.4. PROPOSITION. — *Let* σ *be an irreducible representation of* G_n *where* $n \geqq 1$. *Suppose that*

$$\pi(\sigma, \alpha) = \sigma_\alpha \times (\sigma^+)_{-\alpha}$$

is irreducible and unitarizable for all $\alpha \in (-1/2, 1/2)$. *Then* σ *is a rigid representation.*

Proof. — Let $\sigma=\sigma_1\times\ldots\times\sigma_s$ be a decomposition into product of (irreducible) representations. Now

$$\pi(\sigma,\alpha)=[(\sigma_1)_\alpha\times(\sigma_1^+)_{-\alpha}]\times\ldots\times[(\sigma_s)_\alpha\times(\sigma_s^+)_{-\alpha}].$$

Theorem 2.5 (ii) reduces the proposition to the case when σ is in $\mathrm{Irr}((\rho_x)_{\mathbb{Z}})$.

Take $a\in M(S(\mathbb{Z}))$ such that $\sigma=\langle a_x^{(\rho)}\rangle$. Then

$$\pi(\sigma,\alpha)=\langle a_{x+\alpha}^{(\rho)}\rangle\times\langle -(a_x)^{(\rho)}\rangle_{-\alpha}=\langle a_{x+\alpha}^{(\rho)}\rangle\times\langle(-a)_{-x-\alpha}^{(\rho)}\rangle=\langle a_{x+\alpha}^{(\rho)}+(-a)_{-x-\alpha}^{(\rho)}\rangle.$$

Suppose that $x\in(0,1/2)$. Now $\pi(\sigma,\alpha)$ is irreducible and unitary for $\alpha\in(-1/2,1/2)$. By (ii) of Proposition 2.10 we have that $\pi(\sigma,1/2)=\langle a_{x+1/2}^{(\rho)}\rangle\times\langle(-a)_{-x-1/2}^{(\rho)}\rangle$ reduces so that $2x\in\mathbb{Z}$. This means that σ is a rigid representation.

4.5. LEMMA. — *Let* $\varphi\in M(S(\mathbb{R}))$ *consist of one-point segments (i.e.* $\varphi\in M(\mathbb{R})$*). Suppose that* sup $\varphi\in M(\mathbb{Z})$ (*resp.* supp $\varphi\in M(1/2+\mathbb{Z})$). *Then there exist* $n_1,\ldots,n_s,m_1,\ldots,m_r\in\mathbb{N}$ *and* $\alpha_1,\ldots,\alpha_s,\beta_1,\ldots,\beta_r\in\{0,-1/2\}$ *such that*

$$\varphi+(\Delta[n_1]_{\alpha_1},\ldots,\Delta[n_s]_{\alpha_s})=(\Delta[m_1]_{\beta_1},\ldots,\Delta[m_r]_{\beta_r}),$$

and supp $\Delta[n_i]_{\alpha_i}\in M(\mathbb{Z})$, (resp. supp $\Delta[n_i]_{\alpha_i}\in M(1/2+\mathbb{Z})$), $i=1,\ldots,s$).

Proof. — We shall prove the case supp $\varphi\in M(\mathbb{Z})$. The proof of the other case is analogous.

If supp $\varphi\in M(\{0\})$ then the lemma is evident. Suppose that the statement of the lemma is true when supp $\varphi\in M([-(k-1),k-1])$ for some $k\in\mathbb{N}$.

Take φ such that supp $\varphi\in M([-k,k])$. *Now*

$$\varphi+(\underbrace{\Delta[2k]_{-1/2},\ldots,\Delta[2k]_{-1/2}}_{\varphi(k)-\text{times}})+(\underbrace{\Delta[2k-1],\ldots,\Delta[2k-1]}_{\varphi(-k)-\text{times}})$$

$$=(\underbrace{\Delta[2k+1],\ldots,\Delta[2k+1]}_{\varphi(k)-\text{times}})+(\underbrace{\Delta[2k]_{-1/2},\ldots,\Delta[2k]_{-1/2}}_{\varphi(-k)-\text{times}})+\varphi_1$$

where supp $\varphi_1\in M([-(k-1),k-1])$. We can apply inductive assumption on φ_1.

4.6. For $n,d\in\mathbb{N}$ and $\rho\in C$ set

$$a(n,d)^{(\rho)}=(\nu^{-(n-1)/2}\Delta[d]^{(\rho)},\nu^{1-((n-1)/2)}\Delta[d]^{(\rho)},\ldots,\nu^{(n-1)/2}\Delta[d]^{(\rho)}).$$

This is a multisegment in C.

Denote by (U^m) the following statement

(U^m)*:* if $n,d\in\mathbb{N}$ and $\rho\in C^u$ such that

$$(nd)\deg\rho\leqq m$$

then $\langle a(n,d)^{(\rho)}\rangle$ is unitarizable.

Set

$$X_{m-1}=\{\langle a(n,d)^{(\rho)}\rangle,\pi,(\langle a(n,d)^{(\rho)}\rangle,\alpha);n,d\in\mathbb{N},\rho\in C^u,(nd)\deg\rho\leqq m-1,0<\alpha<1/2\}.$$

4.7. LEMMA. — *Suppose that* (U^m) *holds. Let* $\sigma \in \tilde{G}_m$ *be rigid. If*

$$\pi(\sigma, \alpha) = \sigma_\alpha \times (\sigma^+)_{-\alpha}$$

is irreducible and unitarizable for some $\alpha \in (0, 1/2)$, *then there* $\sigma^1, \ldots, \sigma^r \in X_m$ *and* $\varepsilon_i \in \{-1/2, 0\}$ *such that*

$$\sigma = \sigma^1_{\varepsilon_1} \times \ldots \times \sigma^r_{\varepsilon_r}.$$

Proof. — We shall prove the lemma by induction. Let $\sigma = \rho_\beta$, $\rho \in C^u$, $\beta \in (1/2)\mathbb{Z}$. Now

$$\pi(\sigma, \alpha) = \sigma_\alpha \times (\sigma^+)_{-\alpha} = \rho_{\alpha+\beta} \times \rho_{-\alpha-\beta}$$

is irreducible and unitarizable if and only if $\alpha + \beta \in (-1/2, 1/2)$. This implies that $\beta \in \{0, -1/2\}$.

Let $m > 1$. Suppose that σ satisfies the assumption of the lemma. If $\sigma = \sigma_1 \times \sigma_2$ is nontrivial decomposition then

$$\pi(\sigma, \alpha) = \pi(\sigma_1, \alpha) \times \pi(\sigma_2, \alpha).$$

Theorem 2.5 implies that $\pi(\sigma_1, \alpha)$ and $\pi(\sigma_2, \alpha)$ are unitarizable so we can apply the inductive assumption. Therefore, we may restrict ourselves to the case when σ is supported in one $\mathbb{Z}$-orbit. Take $a \in M(S(\mathbb{R}))$ and $\rho \in C^u$ such that $\sigma = \langle a^{(\rho)} \rangle$. Here $\operatorname{supp} a \in M(\mathbb{Z})$ or $\operatorname{supp} a \in M(1/2 + \mathbb{Z})$.

Let a^0 denote the multisegment obtained from a by removing all one point segments. Then $a = a_0 + \varphi$ where φ consists of one point segments. Theorem 2.5 implies that $\pi(\sigma, \alpha)' = \pi(\sigma', \alpha)$ is unitarizable. The inductive assumption implies

$$\sigma' = \langle a(n_1, d_1)^{(\rho)}_{\varepsilon_1} + \ldots + a(n_k, d_k)^{(\rho)}_{\varepsilon_k} \rangle, \qquad \varepsilon_i \in \{0, -1/2\}.$$

This implies

$$a_0 = a(n_1, d_1 + 1)_{\varepsilon_1} + \ldots + a(n_k, d_k + 1)_{\varepsilon_k}.$$

We set $\tilde{d}_i = d_i + 1$, $\tilde{n}_i = n_i$. Now $\langle a(\tilde{n}_i, \tilde{d}_i)^{(\rho)} \rangle$ are unitarizable by (U^m).

For the multisegment φ take $\Delta[n_i]_{\alpha_i}$, $\Delta[m_j]_{\beta_j}$ from Lemma 4.5. Now $\pi(\Delta[n_i]^{(\rho)}_{\alpha_i}, \alpha)$ are unitarizable and

$$\pi(\sigma, \alpha) \times \pi(\langle \Delta[n_1]^{(\rho)}_{\alpha_1} \rangle, \alpha) \times \ldots \times \pi(\langle \Delta[n_s]^{(\rho)}_{\alpha_s} \rangle, \alpha)$$
$$= \pi(\langle a(\tilde{n}_1, \tilde{d}_1)^{(\rho)}_{\varepsilon_1} \rangle, \alpha) \times \ldots \times \pi(\langle a(\tilde{n}_k, \tilde{d}_k)^{(\rho)}_{\varepsilon_k} \rangle, \alpha)$$
$$\times \pi(\langle \Delta[m_1]^{(\rho)}_{\beta_1} \rangle, \alpha) \times \ldots \times \pi(\langle \Delta[m_r]^{(\rho)}_{\beta_r} \rangle, \alpha).$$

This implies that σ_α is dividing the right side of the equality. Thus σ_α is a product of some $\langle a(\tilde{n}_i, \tilde{d}_i)^{(\rho)}_{\varepsilon_i} \rangle_\alpha$, $(\langle a(\tilde{n}_i, \tilde{d}_i)^{(\rho)}_{\varepsilon_i} \rangle^+)_{-\alpha}$, $\langle \Delta[m_j]^{(\rho)}_{\beta_j} \rangle_\alpha$ and $(\langle \Delta[m_j]^{(\rho)}_{\beta_j} \rangle^+)_{-\alpha}$ since these are prime elements of the ring R. Considering the support of σ_α we obtain that σ_α is a product of some $\langle a(\tilde{n}_i, \tilde{d}_i)_{\varepsilon_i}^{(\rho)} \rangle_\alpha$, $\langle \Delta[m_j]_{\varepsilon_j}^{(\rho)} \rangle_\alpha$ i.e. σ is a product of some $\langle a(\tilde{n}_i, \tilde{d}_i)^{(\rho)} \rangle_{\alpha_i}$, $\langle \Delta[m_j]^{(\rho)} \rangle_{\alpha_j}$ with ε_i, $\beta_j \in \{0, 1/2\}$.

4.8. LEMMA. — *Let $m \geqq 1$. Suppose that (U^{m-1}) holds. Let*

$$X_{m-1} = \{\langle a(n, d)^{(\rho)} \rangle, \pi(\langle a(n, d)^{(\rho)} \rangle, \alpha); \quad n, d \in \mathbb{N}, \rho \in C^u, (nd) \deg \rho \leqq m-1, 0 < \alpha < 1/2\}.$$

Then:

(i) *If $\sigma_1, \ldots, \sigma_k \in X_{m-1}$, then $\sigma_1 \times \ldots \times \sigma_k \in \operatorname{Irr}^u$. In particular, if*

$$\deg \sigma_1 + \ldots + \deg \sigma_k = m, \quad \text{then } \sigma_1 \times \ldots \times \sigma_k \in \hat{G}_m.$$

(ii) *Set*

$$I(\hat{G}_m) = \hat{G}_m \diagdown \{\langle a(n, d)^{(\rho)} \rangle; n, d \in \mathbb{N}, \rho \in C^u, (nd) \deg \rho = m\}.$$

If $\pi \in I(\hat{G}_m)$, then there exist $\sigma_1, \ldots, \sigma_i \in X_{m-1}$ such that

$$\pi = \sigma_1 \times \ldots \times \sigma_i.$$

Representations $\sigma_1, \ldots, \sigma_i$ are determined uniquely up to a permutation.

Proof. — By (U^{m-1}) and Proposition 2.9, $X_{m-1} \subseteqq \operatorname{Irr}^u$. Now (i) is a consequence of the fact that Irr^u is multiplicatively closed.

The uniqueness of a factorization of π in (ii) is a direct consequence of Proposition 3.8 (it can be obtained also without use of that proposition, but then argument would be longer).

We shall prove existence of a factorization of π in (ii) by induction. For $m=1$ there is nothing to prove.

Let $m \geqq 2$. Take $\pi \in I(\hat{G}_m)$. We can decompose

$$\pi = \tau_1 \times \ldots \times \tau_k$$

such that $\tau_i \in \operatorname{Irr}^u$, and such that there exist $\rho_i \in C^u$, $0 \leqq \alpha_i \leqq 1/2$ so that

$$\operatorname{supp} \tau_i \in M(\{\nu^n(\nu^{\alpha_i}\rho_i), \nu^n(\nu^{-\alpha_i}\rho_i); n \in \mathbb{Z}\})$$

$i=1, \ldots, k$ (4.1). If the factorization $\pi = \tau_1 \times \ldots \times \tau_k$ is non-trivial, then the inductive assumption and (U^{m-1}) implies existence of the factorization.

Thus, we may suppose that

$$\operatorname{supp} \pi \in M(\{\nu^n(\nu^\alpha \rho), \nu^n(\nu^{-\alpha}\rho); n \in \mathbb{Z}\}).$$

Let $\pi = \langle a \rangle$, $a \in M(S(C))$.

We proceed now in the same way as in the proof of Lemma 4.7.

We shall consider first the case $\alpha \in \{0, 1/2\}$. Since the highest shifted derivative π' of π is irreducible and unitarizable, we obtain by inductive assumption, considering the support of π, that

$$\pi' = \langle a(n_1, d_1)^{(\rho)} + \ldots + a(n_k, d_k)^{(\rho)} \rangle.$$

This implies that

$$\pi = \langle a(n_1, d_1+1)^{(\rho)} + \ldots + a(n_k, d_k+1)^{(\rho)} + \varphi \rangle$$
$$= \langle a(n_1, d_1^*)^{(\rho)} + \ldots + a(n_k, d_k^*)^{(\rho)} + \varphi \rangle, \qquad d_i^* = d_i + 1$$

where $\varphi \in M(\{\nu^n(\nu^\alpha \rho);\ n \in \mathbb{Z}\})$. The assumption $\pi \in I(\hat{G}_m)$ implies

$$(n_i d_i^*) \deg \rho < m.$$

Thus $\langle a(n_i, d_i^*)^{(\rho)} \rangle \in X_{m-1}$. Since π is unitary we have $\pi = \pi^+$, and thus

$$\varphi = \{\nu^{p_1+\alpha}\rho,\ \nu^{-(p_1+\alpha)}\rho,\ \ldots,\ \nu^{p_j+\alpha}\rho,\ \nu^{-(p_j+\alpha)}\rho\}$$

where $p_i \in \mathbb{Z}_+$. Now

$$\pi \times \langle \Delta[2p_1+2\alpha-1]^{(\rho)} \rangle \times \ldots \times \langle \Delta[2p_j+2\alpha-1]^{(\rho)} \rangle$$
$$= \langle a(n_1, d_1^*)^{(\rho)} \rangle \times \ldots \times \langle a(n_k, d_k^*)^{(\rho)} \rangle$$
$$\times \langle \Delta[2p_1+2\alpha+1]^{(\rho)} \rangle \times \ldots \times \langle \Delta[2p_j+2\alpha+1]^{(\rho)} \rangle,$$

where $\Delta[s]^{(\rho)} = \varnothing$ if $s \leqq 0$. Proposition 3.8 implies that π is a subproduct of the right hand side and this implies existence of the factorization.

Suppose that $0 < \alpha < 1/2$. Using Lemma 4.7 we obtain a factorization.

5. Langlands classification

5.1. Let $\Delta = \{\rho, \nu\rho, \ldots, \nu^{m-1}\rho\} \in S(C)$. Then the representation

$$\rho \times \nu\rho \times \ldots \times \nu^{m-1}\rho$$

possesses the unique irreducible quotient which is denoted by $L(\Delta)$.

5.2. Let $a = (\Delta_1, \ldots, \Delta_n) \in M(S(C))$. Choose a permutation σ of $\{1, \ldots, n\}$, so that

$$\Delta_{\sigma(i)} \to \Delta_{\sigma(j)} \quad \Rightarrow \quad \sigma(i) > \sigma(j), \qquad 1 \leqq i, j \leqq n.$$

The representation

$$L(\Delta_{\sigma(1)}) \times L(\Delta_{\sigma(2)}) \times \ldots \times L(\Delta_{\sigma(n)})$$

does not depend on σ as above. This can be proved in the same way like Proposition 6.4 of [33], using Theorem 9.7, (*a*) of [33]. We denote this representation by $\lambda(a)$. The representation $\lambda(a)$ has a unique irreducible quotient, which is denoted by $L(a)$.

The mapping

$$a \to L(a),$$
$$M(S(C)) \to \text{Irr}$$

is a bijection. This is another parametrization of Irr and it is a version of Langlands classification of Irr. As presented here, this classification was presented by F. Rodier in [21] (*see* also [15]).

5.3. Let D^u denote the set of all classes of square-integrable representations in Irr^u. Set

$$D=\{\nu^\alpha \pi;\ \pi\in D^u,\ \alpha\in\mathbb{R}\}.$$

Elements of D are called essentially square-integrable representations. For $\delta=\nu^\alpha\pi\in D$, $\pi\in D^u$, $\alpha\in\mathbb{R}$ we define δ^u and $e(\delta)$ by

$$\delta^u=\pi \quad \text{and} \quad e(\delta)=\alpha.$$

5.4. By Theorem 9.3 of [33],

$$\Delta\to L(\Delta),\qquad S(C)\to D$$

is a bijection. Denote this bijection by φ. This bijection lifts to a bijection of $M(S(C))$ and $M(D)$ which is again denoted by φ. Now

$$d\to L(\varphi^{-1}(d)),$$
$$M(D)\to Irr$$

is a bijection which will be again denoted by L. This is a parametrization of Irr and can be described directly, without going to $M(S(C))$, as follows. Let $d=(\delta_1, \ldots, \delta_n)\in M(D)$. Suppose that the ordering of δ_i satisfies

$$i<j \quad\Rightarrow\quad e(\delta_i)\geqq e(\delta j).$$

Then $\lambda(d)=\delta_1\times\ldots\times\delta_n$ possesses a unique irreducible quotient, and it is equal to $L(d)$. This classification $d\to L(d)$, $M(D)\to Irr$ is directly related to the Langlands classification of [15] in a simple manner.

5.5. For $d=(\delta_1, \ldots, \delta_n)\in M(D)$, $\alpha\in\mathbb{R}$, set, as before:

$$\tilde{d}=(\tilde{\delta}_1, \ldots, \tilde{\delta}_n),$$
$$\bar{d}=(\bar{\delta}_1, \ldots, \bar{\delta}_n),$$
$$d^+=\tilde{\bar{d}}=(\delta_1^+, \ldots, \delta_n^+),$$
$$\nu^\alpha d=(\nu^\alpha\delta_1, \ldots, \nu^\alpha\delta_n).$$

Let $\delta\in D$, $\delta=\nu^{e(\delta)}\delta^u$ and $\alpha\in\mathbb{R}$. Now

$$\tilde{\delta}=\nu^{-e(\delta)}(\delta^u)^\sim \quad \text{i.e.}\quad e(\tilde{\delta})=-e(\delta) \ \text{and}\ (\tilde{\delta})^u=(\delta^u)^\sim;$$
$$\bar{\delta}=\nu^{e(\delta)}(\delta^u)^- \quad \text{i.e.}\quad e(\bar{\delta})=e(\delta) \ \text{and}\ (\bar{\delta})^u=(\delta^u)^-;$$
$$\delta^+=\nu^{-e(\delta)}\delta^u \quad \text{i.e.}\quad e(\delta^+)=-e(\delta) \ \text{and}\ (\delta^+)^u=\delta^u,$$
$$\nu^\alpha\delta=\nu^{\alpha+e(\delta)}\delta^u \quad \text{i.e.}\quad e(\nu^\alpha\delta)=\alpha+e(\delta) \ \text{and}\ (\nu^\alpha\delta)^u=\delta^u.$$

Now we shall recall some very well known facts about the classification.

5.6. PROPOSITION. — *For* $\alpha \in \mathbb{R}$, $a \in M(S(C))$ *and* $d \in M(D)$ *we have:*

(i) $$\begin{cases} \nu^\alpha \lambda(a) \cong \lambda(\nu^\alpha a), & \overline{\lambda(a)} \cong \lambda(\bar{a}); \\ \nu^\alpha \lambda(d) \cong \lambda(\nu^\alpha d), & \overline{\lambda(d)} \cong \lambda(\bar{d}). \end{cases}$$

(ii) $$\begin{cases} \nu^\alpha L(a) = L(\nu^\alpha a), & \overline{L(a)} = L(\bar{a}); \\ \nu^\alpha L(d) = L(\nu^\alpha d), & \overline{L(d)} = L(\bar{d}). \end{cases}$$

(iii) $$\begin{cases} L(a)^\sim = L(\tilde{a}), & L(a)^+ = L(a^+); \\ L(d)^\sim = L(\tilde{d}), & L(d)^+ = L(d^+). \end{cases}$$

Proof. — For (i) one constructs desired isomorphisms directly. Clearly (i) implies (ii). The relation $L(d)^\sim = L(\tilde{d})$ is another expression of the relation (3.3.13) of [15]. Now (ii) implies $L(d)^+ = L(d^+)$. We obtain $L(a)^\sim = L(\tilde{a})$ from the previous case $L(d)^\sim = L(\tilde{d})$ and Proposition 9.5 of [33] which states that $L(\Delta)^\sim = L(\Delta^\sim)$.

5.7. Let $\sigma \in \mathrm{Irr}$. Take a, $b \in M(S(C))$ such that

$$\sigma = \langle a \rangle, \qquad \sigma = L(b).$$

Then $\operatorname{supp} a = \operatorname{supp} b$ (Proposition 1.10 of [33]). Consider $\operatorname{supp} \sigma$ as an element of $M(S(C))$ in a natural way. Then the set of all representations in Irr whose support is equal to $\operatorname{supp} \sigma$ is just the set of all composition factors of

$$\pi(\operatorname{supp} \sigma) = \lambda(\operatorname{supp} \sigma).$$

Suppose that π_1, $\pi_2 \in \mathrm{Irr}$ and σ is a composition factor of $\pi_1 \times \pi_2$, then

$$\operatorname{supp} \sigma = \operatorname{supp} \pi_1 + \operatorname{supp} \pi_2.$$

5.8. We introduce, like in [21], an additive endomorphism t of R defined by

$$t(\langle a \rangle) = L(a), \qquad a \in M(S(C)).$$

There exists a unique mapping

$$t: \quad M(S(C)) \to M(S(C))$$

such that

$$t(\langle a \rangle) = \langle t(a) \rangle, \qquad a \in M(S(C)),$$

i. e.

$$L(a) = \langle t(a) \rangle.$$

This implies

$$t(L(a)) = L(t(a)), \qquad a \in M(S(C)).$$

Formally, we have

$$t^{-1}(\mathrm{L}(a))=\langle a\rangle,$$
$$t^{-1}(\mathrm{L}(a))=\mathrm{L}(t^{-1}(a)),$$
$$t^{-1}(\langle a\rangle)=\langle t^{-1}(a)\rangle.$$

The homorphism t contains all informations about connection of Zelevinsky and Langlands classification.

We have

$$\operatorname{supp} t(\pi)=\operatorname{supp}\pi, \qquad \pi\in \mathrm{Irr}.$$

6. Technical lemmas

6.1. LEMMA. — *Suppose that* $2\leqq k\leqq d$ *and* $\rho\varepsilon \mathrm{C}$. *Then*

$$\langle a(k, d)^{(\rho)}\rangle\times\langle a((k-2), d)^{(\rho)}\rangle$$

is irreducible if and only if

$$\langle a(k, d-1)^{(\rho)}\rangle\times\langle a(k-2, d-1)^{(\rho)}\rangle$$

is irreducible.

Proof. — Let $k\leqq d$ and $\rho\varepsilon \mathrm{C}$. The representation

$$\pi_1=\langle a(k, d)^{(\rho)}\rangle\times\langle a((k-2), d)^{(\rho)}\rangle$$

is a subquotient of the representation

$$\pi_2=\pi(a(k, d)^{(\rho)})\times\pi(a((k-2), d)^{(\rho)})=\pi(a(k, d)^{(\rho)}+a((k-2), d)^{(\rho)}).$$

It means that each composition factor of π_1 is also a composition factor of π_2.

Let $m=\deg\rho$. Let

$$a(k, d)^{(\rho)}+a((k-2), d)^{(\rho)}=(\Delta_1, \ldots, \Delta_{2k-2}).$$

Suppose that $b\leqq a(k, d)^{(\rho)}+a((k-2), d)^{(\rho)}$. The inequality $k\leqq d$ implies that

$$\Delta_i\cap\Delta_j\neq\varnothing$$

for any $1\leqq i$, $j\leqq 2k-2$. This implies that the cardinal number of the multiset b is $2k-2$. Therefore, the highest derivative of b has degree

$$m(2(k-1)(d-1)).$$

This degree does not depend on b.

Suppose that π_1 is irreducible. Then the highest derivative of π_1 is irreducible and equal to

$$\langle a(k, d-1)^{(\rho-1/2)}\rangle \times \langle a(k-2, d-1)^{(\rho-1/2)}\rangle.$$

This implies that

$$\langle a(k, d-1)^{(\rho)}\rangle \times \langle a(k-2, d-1)^{(\rho)}\rangle$$

is irreducible.

Suppose that π_1 is not irreducible. Then the derivative $D(\pi_1)$ is

$$D(\pi_1)=\langle a(k, d-1)^{(\rho-1/2)}\rangle \times \langle a(k-2, d-1)^{(\rho-1/2)}\rangle + (\textit{terms of higher degree}).$$

Let $\pi_1=\sigma_1+\ldots+\sigma_n$, $\sigma_i \in \mathrm{Irr}$. Here $n \geqq 2$. Let σ_i^- be the highest derivative of σ_i. Then

$$D(\pi_1)=\sigma_1^- + \ldots + \sigma_n^- + (\text{terms of higher degree}).$$

The above discussion about degrees of highest derivatives implies that

$$\langle a(k, d-1)^{(\rho-1/2)}\rangle \times \langle a(k-2, d-1)^{(\rho-1/2)}\rangle = \sigma_1^- + \ldots + \sigma_n^-.$$

Thus $\langle a(k, d-1)^{(\rho-1/2)}\rangle \times \langle a(k-2, d-1)^{(\rho-1/2)}\rangle$ is not irreducible. Therefore $\langle a(k, d-1)^{(\rho)}\rangle \times \langle a(k-2, d-1)^{(\rho)}\rangle$ is not irreducible.

6.2. LEMMA. — *Let $m \geqq 1$. Suppose that (U^{m-1}) holds. Let $n, d \in \mathbb{N}$, $\rho \in C^u$ such that*

$$(nd)\deg\rho = m \qquad \text{and} \qquad n \leqq d.$$

Then $\langle a(n, d)^{(\rho)}\rangle$ is unitarizable.

Proof. — If $n=1$, then $\langle a(1, d)^{(\rho)}\rangle = \langle \Delta[d]^{(\rho)}\rangle$ and this is unitarizable by 2.14. Suppose $n \geqq 2$. By (U^{m-1})

$$\langle a(n-1, d)^{(\rho)}\rangle$$

is unitarizable. Proposition 2.9 implies that all composition factors of

$$\nu^{1/2}\langle a(n-1, d)^{(\rho)}\rangle \times \nu^{-1/2}\langle a(n-1, d)^{(\rho)}\rangle$$

are unitarizable. Thus

$$\langle \nu^{1/2} a(n-1, d)^{(\rho)} + \nu^{-1/2} a(n-1, d)^{(\rho)}\rangle = \langle a(n, d)^{(\rho)} + a(n-2, d)^{(\rho)}\rangle$$

is unitarizable. Now Lemma 6.1 implies that

$$\langle a(n, d)^{(\rho)}\rangle \times \langle a(n-2, d)^{(\rho)}\rangle$$

is irreducible, since

$$\langle a(n, d-1)^{(\rho)}\rangle \times \langle a(n-2, d-1)^{(\rho)}\rangle$$

is irreducible. By (ii) of Theorem 2.5 $\langle a(n, d)^{(\rho)} \rangle$ is unitarizable.

6.3. LEMMA. — *Let* $n, d \in \mathbb{N}$ *and* $n < d$. *Then*

$$t(\langle a(n, d)^{(\rho)} \rangle) \neq \langle a(n, d)^{(\rho)} \rangle, \qquad \rho \in C,$$

i. e.

$$L(a(n, d)^{(\rho)}) \neq \langle a(n, d)^{(\rho)} \rangle.$$

Proof. — Let $\rho \in C(G_m)$, i. e. $\deg \rho = m$. Suppose that

$$L(a(n, d)^{(\rho)}) = \langle a(n, d)^{(\rho)} \rangle.$$

Thus $\lambda(a(n, d)^{(\rho)})$ and $\pi(a(n, d)^{(\rho)})$ have a common composition factor, by the definition of $L(a(n, d)^{(\rho)})$ and $\langle a(n, d)^{(\rho)} \rangle$.

Let P_0 (resp. P_1) be the unique standard parabolic subgroup of $GL(mdn, F)$ whose Levi factor M_0 (resp. M_1) is naturally isomorphic to $GL(m, F)^{dn}$ (resp. $GL(md, F)^n$). Let N_i be the unipotent radical of P_i, $i = 0,1$.

In the rest of the proof we shall use freely notation of § 1 of [33].

First of all, note that there is no standard parabolic subgroup of $GL(mnd, F)$ associated to P_0 different from P_0. Since $\langle a(n, d)^{(\rho)} \rangle$ is a composition factor of an induced representation from P_0 by a cuspidal irreducible representation of M_0, by § 6 of [7]

$$r_{(m, m, \ldots, m), (nmd)}(\langle a(n, d)^{(\rho)} \rangle) \neq 0.$$

Now $\langle a(n, d)^{(\rho)} \rangle = L(a(n, d)^{(\rho)})$ implies that representations

$$r_{(m, \ldots, m), (nmd)}(\pi(a(n, d)^{(\rho)}))$$

and

$$r_{(m, \ldots, m), (nmd)}(\lambda(a(n, d)^{(\rho)}))$$

have a common non-trival irreducible composition factor.

Let $a(n, d)^{(\rho)} = (\Delta_1, \ldots, \Delta_n)$ where $\Delta_1 \to \Delta_2 \to \ldots \to \Delta_n$. Choose $\sigma \in C(G_m)$ so that

$$\Delta_1 = [\sigma, \nu^{d-1}\sigma],$$
$$\Delta_2 = [\nu\sigma, \nu^{d}\sigma],$$
$$\ldots\ldots\ldots\ldots\ldots$$
$$\Delta_n = [\nu^{n-1}\sigma, \nu^{n-1+d-1}\sigma].$$

Now

$$(*) \quad (\sigma \otimes \nu\sigma \otimes \ldots \otimes \nu^{d-1}\sigma) \otimes (\nu\sigma \otimes \ldots \otimes \nu^{d}\sigma) \otimes \ldots \otimes (\nu^{n-1}\sigma \otimes \ldots \otimes \nu^{n-1+d-1}\sigma)$$

is a composition factor of $r_{(m, \ldots, m), (nmd)}(\pi(a(n, d)^{(\rho)}))$ and each other irreducible composition factor is obtained from $(*)$ after a permutation of factors of $(*)$ with a permutation which preserves the order of elements of each bracket (*see* (3) of Proof of Proposition 6.9 of [33]).

In the similar way

$$(**)\quad (\nu^{d-1}\sigma \otimes \nu^{d-2}\sigma \otimes \ldots \otimes \sigma) \otimes (\nu^{d}\sigma \otimes \ldots \otimes \nu\sigma) \otimes \ldots \otimes (\nu^{n-1+d-1}\sigma \otimes \ldots \otimes \nu^{n-1}\sigma)$$

is a composition factor of $r_{(m, \ldots,), (nmd)}(\lambda(a(n, d)^{(\rho)}))$ and each other irreducible composition factor is obtained from $(**)$ after a permutation of $(**)$ with a permutation which preserves the order of elements of each bracket.

Let τ be an irreducible factor of $r_{(m, \ldots, m), (nmd)}(\pi(a(n, d)^{(\rho)}))$. Let $\tau = \nu^{\alpha_1}\sigma \otimes \ldots \otimes \nu^{\alpha_{mnd}}\sigma$. Then there exist $1 \leqq p_1 < \ldots < p_d \leqq mnd$ so that

$$(***)\qquad \alpha_{p_i} = i-1, \qquad i = 1, \ldots, d.$$

Let ω be an irreducible composition factor of $r_{(m, \ldots, m), (nmd)}(\lambda(a(n, d)^{(\rho)}))$. Let $\omega = \nu^{\beta_1}\sigma \otimes \ldots \otimes \nu^{\beta_{mnd}}\sigma$. Now simple combinatorial observation implies that if $\beta_{g_1} < \beta_{g_2} < \ldots < \beta_{g_r}$, for some $1 \leqq g_1 < g_2 < \ldots < g_r \leqq mnd$, then $r \leqq n$.

Therefore

$$r_{(m, \ldots, m), (nmd)}(\pi(a(n, d)^{(\rho)})) \quad \text{and} \quad r_{(m, \ldots, m), (nmd)}(\lambda(a(n, d)^{(\rho)}))$$

can not have a common non-trival composition factor.

We obtained a contradiction. This proves the lemma.

6.4. Let $n, d \in \mathbb{N}$, $\rho \in C$. One sees directly that

$$\operatorname{supp}\langle a(n, d)^{(\rho)}\rangle = \operatorname{supp}\langle a(n, d)^{(\rho)}\rangle.$$

Suppose that $n \leqq d$. Then $\operatorname{supp}\langle a(n, d)^{(\rho)}\rangle(\sigma) = 0$ for $\sigma \in C$ and $\sigma \notin \{\nu^{\alpha}\sigma;\ \alpha \in [(n+d)/2 + \mathbb{Z}]\}$.

Let $\sigma = \nu^{\alpha}\rho$, $\alpha \in [(n+d)/2 + \mathbb{Z}]$. Then

$$\operatorname{supp}\langle a(n, d)^{(\rho)}\rangle(\sigma) = \begin{cases} 0 & \alpha \leqq -(n+d)/2, \\ \alpha + (n+d)/2, & 1-(n+d)/2 \leqq \alpha \leqq (n-d)/2, \\ n & (n-d)/2 \leqq \alpha \leqq (d-n)/2, \\ -\alpha + (n+d)/2, & (d-n)/2 \leqq \alpha \leqq -1 + (n+d)/2, \\ 0 & (n+d)/2 \leqq \alpha. \end{cases}$$

7. Unitary dual

In 1.14 we have defined ${}^t: R \to R$ and in 5.8 we have defined $t: R \to R$. F. Rodier showed that $(\mathrm{Irr})^t = \mathrm{Irr}$ implies $t = {}^t$ [21]. In this section and the following one we assume $(\mathrm{Irr})^t = \mathrm{Irr}$ and thus we do not make difference between t and ^{t}.

7.1. The following theorem completely solves the unitarizability problem for $\mathrm{GL}(n)$ over non-archimedean field, and also presents explicit connection between Zelevinsky and Langlands classifications in the unitary case. The Bernstein Conjecture 8.10 of [2] was stating that $t(\mathrm{Irr}^u) \subseteq \mathrm{Irr}^u$. The following theorem describes completely $t: \mathrm{Irr}^u \to \mathrm{Irr}^u$.

THEOREM. — *Let*

$$\mathrm{B} = \{ \langle a(n, d)^{(\rho)} \rangle, \pi(\langle a(n, d)^{(\rho)} \rangle, \alpha); n, d \in \mathbb{N}, \rho \in \mathrm{C}^u, 0 < \alpha < 1/2 \}.$$

Fix $m \in \mathbb{N}$. *Then*

(i) *If* $\sigma_1, \ldots, \sigma_k \in \mathrm{B}$ *are such that*

$$\deg \sigma_1 + \ldots + \deg \sigma_k = m,$$

then $\sigma_1 \times \ldots \times \sigma_k \in \hat{\mathrm{G}}_m$.

(ii) *If* $\pi \in \hat{\mathrm{G}}_m$, *then there exist* $\tau_1, \ldots, \tau_j \in \mathrm{B}$ *so that*

$$\pi = \pi_1 \times \ldots \times \tau_j.$$

Such $\tau_1, \ldots, \tau_j$ *are unique up to a permutation and*

$$\deg \tau_1 + \ldots + \deg \tau_j = m.$$

(iii) *The following formulas hold*

$$t(\langle a(n, d)^{(\rho)} \rangle) = \langle a(d, n)^{(\rho)} \rangle$$
$$t(\pi(\langle a(n, d)^{(\rho)} \rangle, \alpha)) = \pi(\langle a(d, n)^{(\rho)} \rangle, \alpha)$$

for elements of B.

7.2. *Remark.* — Note that by (i), $\mathrm{B} \subseteq \mathrm{Irr}^u$. The statement (iii), together with (i) and (ii) describes explicitly $t(\langle a \rangle)$ in terms of Zelevinsky classification, when $\langle a \rangle$ is unitarizable. The same description is valid for Langlands classification.

Proof. — We shall prove (i), (ii) and (iii) by induction on m (in (iii), $m = (nd) \deg \rho$). Define X_{m-1} as in Lemma 4.8.

Suppose that $m = 1$. Then (i), (ii) and (iii) hold. Here the only possible $\langle a(n, d)^{(\rho)} \rangle$ is for $n = d = 1$ and ρ a unitary character of G_1, t is here identity.

Suppose that (i), (ii) and (iii) hold for $k \leq m-1$. Then (U^{m-1}) holds. Now Lemma 6.2 implies that $\langle a(n, d)^{(\rho)} \rangle$ is unitarizable, for $n \leq d$ and (nd) *deg* $\rho = m$. By [28], $t(\langle a(n, d)^{(\rho)} \rangle)$ is unitarizable, i. e.

$$t(\langle a(n, d)^{(\rho)} \rangle) \in \hat{\mathrm{G}}_m.$$

From the inductive assumption one sees that

$$t(\langle a(n, d)^{(\rho)} \rangle) \notin \mathrm{I}(\hat{\mathrm{G}}_m),$$

where $I(\hat{G}_m)$ is defined in (ii) of Lemma 4.8 (by the inductive assumption and Lemma 4.8 we know how t acts on $I(\hat{G}_m)$). One can obtain that also from Proposition 3.8 ($t(\langle a(n, d)^{(\rho)}\rangle)$ is a prime element of R since it is the image of a prime element under an automorphism of R, and elements in $I(\hat{G}_m)$ are composite by Lemma 4.8).

The above discussion and Lemma 4.8 imply

$$t(\langle a(n, d)^{(\rho)}\rangle)\in\{\langle a(n_1, d_1)^{(\rho_1)}\rangle;\ n_1, d_1\in\mathbb{N},\ \rho_1\in C^u,\ (n_1 d_1)\deg\rho_1=m\}.$$

Thus $t(\langle a(n, d)^{(\rho)}\rangle)=\langle a(n_1, d_1)^{(\rho_1)}\rangle$ for some n_1, d_1, ρ_1 as above. The fact

$$\operatorname{supp}\langle a(n, d)^{(\rho)}\rangle=\operatorname{supp}\langle a(n_1, d_1)^{(\rho_1)}\rangle$$

implies

$$\rho=\rho_1$$
$$\{n, d\} = \{n_1, d_1\}$$

(support of $\langle a(n, d)^{(\rho)}\rangle$ is computed in 6.4).

Therefore,

$$t(\langle a(n, d)^{(\rho)}\rangle)\in\{\langle a(n, d)^{(\rho)}\rangle, \langle a(n, d)^{(\rho)}\rangle\}.$$

If $n=d$ then

$$t(\langle a(n, d)^{(\rho)}\rangle)=\langle a(n, d)^{(\rho)}\rangle.$$

If $n<d$, then Lemma 6.3 implies

$$t(\langle a(n, d)^{(\rho)}\rangle)=\langle a(d, n)^{(\rho)}\rangle.$$

Thus $\langle a(d, n)^{(\rho)}\rangle$ is unitarizable. This means that $X_m\subseteq \mathrm{Irr}^u$. Since t is an involution, we have that

$$t(\langle a(n, d)^{(\rho)}\rangle)=\langle a(n, d)^{(\rho)}\rangle.$$

Thus, (iii) holds. Clearly (i) holds because if $\sigma_1, \ldots, \sigma_k\in B$ and $\deg\sigma_1+\ldots+\deg\sigma_k=m$, then then $\sigma_1, \ldots, \sigma_k\in X_m$. Lemma 4.8 implies (ii).

7.3. The above theorem can be expressed in the following form:

THEOREM. — *Let*

$$B=\{a(n, d)^{(\rho)}, (\nu^\alpha a(n, d)^{(\rho)}+\nu^{-\alpha} a(n, d)^{(\rho)});\ n, d\in\mathbb{N},\ \rho\in C^u,\ 0<\alpha<1/2\}.$$

Let $X(B)$ *be the additive subsemigroup of* $M(S(C))$ *generated by* B. *Then*

$$a\to\langle a\rangle$$

and

$$a\to L(a)$$

are bijections from X(B) *onto* Irr^u.

The mapping

$$t:\quad \mathrm{B}\to\mathrm{B},$$
$$t:\quad a(n,\, d)^{(\rho)}\to a(d,\, n)^{(\rho)};$$
$$(\nu^{\alpha} a(n,\, d)^{(\rho)}+\nu^{-\alpha} a(n,\, d)^{(\rho)})\to(\nu^{\alpha} a(d,\, n)^{(\rho)}+\nu^{-\alpha} a(d,\, n)^{(\rho)})$$

extends uniquely to a morphism of semigroups ${}^t: \mathrm{X}(\mathrm{B})\to\mathrm{X}(\mathrm{B})$. *Now*

$$\langle a\rangle=\mathrm{L}(t(a))$$

and

$$t(\langle a\rangle)=\langle t(a)\rangle.$$

7.4. For $\delta\in\mathrm{D}^u$ and $n\in\mathbb{N}$ set

$$u(\delta,\, n)=\mathrm{L}(\nu^{-(n-1)/2}\,\delta,\ \nu^{1-((n-1)/2)}\,\delta,\ \ldots,\ \nu^{(n-1)/2}\,\delta).$$

We can characterize $u(\delta,\, n)$ as the unique irreducible quotient of

$$\nu^{(n-1)/2}\,\delta\times\nu^{((n-1)/2)-1}\,\delta\times\ldots\times\nu^{-(n-1)/2}\,\delta$$

Clearly $\mathrm{L}(a(n,\, d)^{(\rho)})=u(\mathrm{L}(\Delta[d]^{(\rho)}),\, n)$ and this implies

$$\{\mathrm{L}(a(n,\, d)^{(\rho)});\ n,\ d\in\mathbb{N},\ \rho\in\mathrm{C}^u\}=\{u(\delta,\, n);\ \delta\in\mathrm{D}^u,\ n\in\mathbb{N}\}.$$

7.5. Theorem. — *Set*

$$\mathrm{B}=\{u(\delta,\, n),\ \nu^{\alpha} u(\delta,\, n)\times\nu^{-\alpha} u(\delta,\, n);\ \delta\in\mathrm{D}^u;\ n\in\mathbb{N},\ 0<\alpha<1/2\}.$$

(i) *If* $\pi_1, \ldots, \pi_r\in\mathrm{B}$, *then* $\pi_1\times\ldots\times\pi_r\in\mathrm{Irr}^u$.

(ii) *Let* $\sigma\in\mathrm{Irr}^u$. *Then there exist* $\pi_1, \ldots, \pi_r\in\mathrm{B}$ *so that* $\sigma=\pi_1\times\ldots\times\pi_r$. *The multiset* $(\pi_1, \ldots, \pi_r)$ *is uniquely determined by* σ.

7.6. Corollary. — *Let* $a\in\mathrm{M}(\mathrm{S}(\mathbb{R}))$, ρ_1, $\rho_2\in\mathrm{C}^u$. *Then* $\langle a^{(\rho_1)}\rangle$ *is unitarizable if and only if* $\langle a^{(\rho_2)}\rangle$ *is unitarizable.*

7.7. *Remark.* — The formula for the Zelevinsky involution in Theorem 7.1 can be obtained using C. Moeglin and J.-L. Waldspurger results in [19] where they proved Zelevinsky conjecture on ${}^t: \mathrm{Irr}\to\mathrm{Irr}$ from [34]. It is also possible to prove this conjecture for representations $\langle a\,(n,\, d)^{(\rho)}\rangle$ using Theorem 7.1 as it was done in the previous draft of this paper.

8. Bernstein conjecture on complementary series

8.1. In 4.1, we introduced the notion of a rigid representation.

THEOREM. — *Let $\sigma \in \mathrm{Irr}$. Suppose that σ is a rigid representation such that $\pi(\sigma, \alpha)$ is an irreducible and unitarizable representation for some $\alpha \in (0, 1/2)$. Then there exist unitarizable representations σ_1 and σ_2 so that*

$$\sigma = \sigma_1 \times \nu^{-1/2} \sigma_2.$$

Proof. — The theorem is a consequence of Theorem 7.1 and Lemma 4.7.

8.2. THEOREM. — *Let $\sigma \in \mathrm{Irr}$. Suppose that*

$$\pi(\sigma, \alpha) = \sigma_\alpha \times (\sigma^+)_{-\alpha}$$

is irreducible and unitarizable for all $\alpha \in (-1/2, 1/2)$. Then σ is a unitarizable rigid representation.

Proof. — Let $\sigma \in \mathrm{Irr}$. Suppose that $\pi(\sigma, \alpha) = \sigma_\alpha \times (\sigma^+)_{-\alpha}$ is irreducible and unitarizable for $\alpha \in (-1/2, 1/2)$. Then 4.4 and 7.1 implies $\sigma = \sigma^1 \times (\sigma^2)_{-1/2}$ where $\sigma^1, \sigma^2 \in \mathrm{Irr}^u$. Clearly, σ^1 and σ^2 are rigid.

Suppose that σ is not unitarizable. This implies that $\deg \sigma^2 \geqq 1$. Proposition 4.2 implies that

$$\pi(\sigma^2, 1/2) = (\sigma^2)_{1/2} \times (\sigma^2)_{-1/2}$$

is reducible. Thus, in the ring R we have

$$\pi(\sigma, 0) = \sigma \times \sigma^+ = (\sigma^1 \times (\sigma^2)_{-1/2}) \times (\sigma_1 \times (\sigma^2)_{1/2}) = (\sigma^1 \times \sigma^1) \times ((\sigma^2)_{-1/2} \times (\sigma^2)_{1/2}).$$

This implies that $\pi(\sigma, 0)$ reduces which is a contradiction.

8.3. *Remark.* — These theorems prove Bernstein Conjecture 8.6 in [2] on complementary series.

APPENDIX

In this appendix we shall prove all the statements of the seventh and the eighth section without using the result that $(\mathrm{Irr})^t \subseteqq \mathrm{Irr}$, if characteristic of F is zero. This means that we shall not use the result of [28].

We consider here the additive homomorphism

$$t: \quad \mathrm{R} \rightarrow \mathrm{R}$$

defined by $t(\langle a \rangle) = \mathrm{L}(a)$, $a \in \mathrm{M}(\mathrm{S}(\mathrm{C})$. This is all we assume in this section about t (we do not assume that t is involutive, and also that t is multiplicative).

A.1. We shall first prove one result about classification L.

PROPOSITION. — *Let $a = (\Delta_1, \ldots, \Delta_n)$, $b = (\Gamma_1, \ldots, \Gamma_m) \in \mathrm{M}(\mathrm{S}(\mathrm{C}))$.* Suppose that Δ_i *and Γ_j are not linked for all $1 \leqq i \leqq n$, $1 \leqq j \leqq m$. Then:*

(i) $$\lambda(a) \times \lambda(b) \cong \lambda(a+b).$$

(ii) $$\langle a \rangle \times \langle b \rangle = \langle a+b \rangle.$$

Proof. — The definition of λ implies (i). Proposition 8.5 of [33] gives $\langle a \rangle \times \langle b \rangle = \langle a+b \rangle$.

A.2. *Remark.* — One can obtain that $L(a) \times L(b) = L(a+b)$ when a, b are as in the above proposition, using multiplicity one of the Langlands representation in $\lambda(a+b)$. For our purposes, the following irreducibility result will be sufficient.

A.3. COROLLARY. — *Let* $r_i = (\rho_1^i, \ldots, \rho_{n_i}^i) \in M(C)$, $i=1,2$. *Suppose that*

(*) $$e(\rho_p^1) - e(\rho_q^2) \notin \{-1, 1\}$$

for $1 \leqq p \leqq n_1$, $1 \leqq q \leqq n_2$. *Let* a, $b \in M(S(C))$ *so that*

$$\operatorname{supp} \langle a \rangle = r_1 \quad \textit{and} \quad \operatorname{supp} \langle b \rangle = r_2.$$

Then $L(a) \times L(b)$ *and* $\langle a \rangle \times \langle b \rangle$ *are irreducible and*

$$L(a) \times L(b) = L(a+b),$$
$$\langle a \rangle \times \langle b \rangle = \langle a+b \rangle.$$

Proof. — The assumption (*) implies that a and b are as in Proposition A.1, and (ii) of Proposition A.1 implies our statement for Zelevinsky classification.

Choose a^*, b^* so that

$$L(a) = \langle a^* \rangle \quad \text{and} \quad L(b) = \langle b^* \rangle.$$

Now $\operatorname{supp} \langle a \rangle = \operatorname{supp} \langle a^* \rangle$ and $\operatorname{supp} \langle b \rangle = \operatorname{supp} \langle b^* \rangle$. The first part of our proof implies that

$$L(a) \times L(b) = \langle a^* \rangle \times \langle b^* \rangle$$

is irreductible.

Now. $L(a) \times L(b)$ is an irreducible quotient of $\lambda(a) \times \lambda(b)$. By (i) of Proposition A.1 we have

$$\lambda(a) \times \lambda(b) \cong \lambda(a+b)$$

since a and b satisfy the assumption of Proposition A.1. The representation $\lambda(a+b)$ has a unique irreducible quotient which is $L(a+b)$. Thus $L(a) \times L(b) = L(a+b)$.

A.4. PROPOSITION. — (i) *Suppose that* a, $b \in M(S(C))$. *Then* $L(a+b)$ *is a composition factor of* $L(a) \times L(b)$.

(ii) *If* c, $d \in M(D)$, *then* $L(c+d)$ *is a composition factor of* $L(c) \times L(d)$.

Proof. — First of all, note that it is enough to prove the proposition for $M(S(C))$.

Let a, $b \in M(S(C))$. Set

$$\operatorname{supp} \langle a \rangle = (\rho_1, \ldots, \rho_u),\ \operatorname{supp} \langle b \rangle = (\sigma_1, \ldots, \sigma_v).$$

Let ε be the minimum of all

$$|1+e(\rho_i)-e(\sigma_j))| \text{ with } 1+(e(\rho_i)-e(\sigma_j))\neq 0,$$
$$|1-(e(\rho_i)-e(\sigma_j))| \text{ with } 1-(e(\rho_i)-e(\sigma_j))\neq 0,$$

when $1\leqq i\leqq u$, $1\leqq j\leqq v$. Then $\varepsilon>0$. Let $0<\alpha<\varepsilon$.. By the choice of α, a and $(\nu^\alpha b)$ satisfy the assumption of Corollary A.3. Thus $L(a)\times L(\nu^\alpha b)$ is irreducible and equals to $L(a+\nu^\alpha b)$. Proposition A.1 implies $\lambda(a)\times\lambda(\nu^\alpha b)\cong\lambda(a+\nu^\alpha b)$.

Suppose that a consists of n segments and b of m segments. We can denote those segments in the following way

$$a=(\Delta_{i(1)}, \ldots, \Delta_{i(n)}), \qquad i(1)<\ldots<i(n),$$
$$b=(\Delta_{j(1)}, \ldots, \Delta_{j(m)}), \qquad j(1)<\ldots<j(m)$$

with $\{i(1), \ldots, i(n)\}\cup\{j(1), \ldots, j(m)\}=\{1, 2, \ldots, n+m\}$, such that

$$\Delta_u\to\Delta_v \quad\Rightarrow\quad v<u.$$

For $0\leqq\alpha<\varepsilon$ set $\Delta^\alpha_{i(k)}=\Delta_{i(k)}$, $\Delta^\alpha_{j(k)}=\nu^\alpha\Delta_{j(k)}$.

Now

$$a+\nu^\alpha b=(\Delta^\alpha_1, \ldots, \Delta^\alpha_{n+m}).$$

By construction

$$\lambda(a+\nu^\alpha b)\cong L(\Delta^\alpha_1)\times\ldots\times L(\Delta^\alpha_{n+m})$$

for $0\leqq\alpha<\varepsilon$. The representation $\lambda(a+\nu^\alpha b)$ possesses a unique irreducible quotient, which is equal to $L(a+\nu^\alpha b)=L(a)\times\nu^\alpha L(b)$ for $0<\alpha<\varepsilon$, and $L(a+b)$ for $\alpha=0$.

Suppose that $L(a+b)$ is a representation of a group G_p. Let $H(G_p)$ be the Hecke algebra of G_p. For an admissible smooth representation π of G_p, ch_π will denote the character of π. With a fixed $f\in H(G_p)$

$$(*) \qquad \alpha\to ch_{L(a)\times\nu^\alpha L(b)}(f)$$

is a continuous function. One can see this from the formula for the character of an induced representation in [8] (*see* also Lemma 2.1 of [26]).

Let (α_n) be a sequence of real numbers converging to 0 such that $0<\alpha_n<\varepsilon$ for all n. Suppose that we have proved that there exist a quotient π of $\lambda(a+b)$ and a subsequence $(\alpha_{n(k)})$ of (α_n) such that

$$(**) \qquad \lim_k ch_{L(a+\nu^{\alpha_{n(k)}}b)}(f)=ch_\pi(f)$$

for all $f\in H(G_p)$.

Now $L(a+b)$ is the unique irreducible quotient of $\lambda(a+b)$, so it is a (unique irreducible) quotient of π. The relations (*) and (**) imply

$$\lim_k \mathrm{ch}_{L(a+\nu^{\alpha_{n(k)}}b)}(f)=\lim_k \mathrm{ch}_{L(a)\times\nu^{\alpha_{n(k)}}L(b)}(f)=\mathrm{ch}_{L(a)\times L(b)}(f)=\mathrm{ch}_\pi(f)$$

for all $f\in H(G_p)$. Since $L(a+b)$ is a subquotient of π, the last equality implies that $L(a+b)$ is the subquotient of $L(a)\times L(b)$.

Thus, for a proof of the proposition we need to contruct π as above.

Roughly speaking, such π is contructed as follows. One can realize all representations

$$L(\Delta_1^\alpha)\times\ldots\times L(\Delta_{n+m}^\alpha)$$

on the same vector space (by restriction to the standard maximal compact subgroup). In this way one obtains a continuous family of representations on the same vector space (for a precise formulation of "continuous family" see Lemma 3.5 of [27]). Now using the compactness of the Grassmanian manifold of a finite dimensional vector space, and the diagonal procedure (several times), one constructs a subsequence $(\alpha_{n(k)})$ and π as above.

For a formal proof, to avoid the whole construction, we pass to the contragredient representations. Now

$$L(a+\nu^{\alpha_n}b)^{\sim}=L(a)^{\sim}\times\nu^{-\alpha_n}L(b)^{\sim}$$

is a subrepresentation of $L(\Delta_1^\alpha)^{\sim}\times\ldots\times L(\Delta_{n+m}^\alpha)^{\sim}$, and we can as in the proof of Lemma 3.6 of [27], construct a subsequence $(\alpha_{n(k)})$ of (α_n) and a subrepresentation π_0 of $L(\Delta_1)^{\sim}\times\ldots\times L(\Delta_{n+m})^{\sim}$ such that

$$\lim_k \mathrm{ch}_{L(a+\nu^{\alpha_{n(k)}}b)\sim}(f)=\mathrm{ch}_{\pi_0}(f)$$

for all $f\in H(G_p)$. Lemma 3.6 of [27] deals with induced representations by cuspidals, but the fact that inducing is by cuspidals, is not used in the part of the proof of the lemma that we need (this fact is used at the end of the proof to reduce the lemma to the case of subrepresentations). Now $\tilde{\pi}_0$ is in a natural way a quotient of

$$L(\Delta_1)\times\ldots\times L(\Delta_{n+m}).$$

Using the fact that

$$\mathrm{ch}_{\tilde{\sigma}}(f)=\mathrm{ch}_\sigma(\tilde{f})$$

where $\tilde{f}$ is defined by $\tilde{f}(g)=f(g^{-1})$, one obtains that

$$\lim_k \mathrm{ch}_{L(a+\nu^{\alpha_{n(k)}}b)}(f)=\mathrm{ch}_{\tilde{\pi}_0}(f)$$

for all $f\in H(G_p)$. Thus, we can take $\pi=\tilde{\pi}_0$. This finishes the proof.

For another possible proof of the preceding proposition see (iii) of A.12.

A.5. *Remark.* — Since the multiplicity of the Langlands representation in $\lambda(a+b)$ is one, then $L(c+d)$ is a composition factor of $L(c)\times L(d)$ whose multiplicity is one.

A.6. COROLLARY. — (i) *Let* $c, d\in M(D)$. *If* $L(c)\times L(d)$ *is irreducible, then*

$$L(c)\times L(d)=L(c+d).$$

(ii) *Let* $a, b\in M(S(C))$. *If* $L(a)\times L(b)$ *is irreducible then*

$$L(a)\times L(b)=L(a+b).$$

A.7. COROLLARY. — (i) *Let* $c, d\in M(D)$. *Suppose that* $L(c)$ *and* $L(d)$ *are unitarizable. Then* $L(c+d)$ *is unitarizable and*

$$L(c+d)=L(c)\times L(d).$$

(ii) *Let* $a, b\in M(S(C))$. *If* $L(a)$ *and* $L(b)$ *are unitarizable, then*

$$L(a+b)=L(a)\times L(b).$$

In the rest of this section we suppose that the *characteristic of the field F is zero.*

Let $\delta\in GL(m, F)^{\sim}$ be a square-integrable representation and $n\in\mathbb{N}$. Then the induced representation

$$(\nu^{(n-1)/2}\delta)\times(\nu^{((n-1)/2)-1}\delta)\times\ldots\times(\nu^{-(n-1)/2}\delta)$$

has a unique irreducible quotient. This quotient was denoted by $u(\delta, n)$.

A.8. THEOREM. — *Suppose that* $\operatorname{char} F=0$. *Let* $\delta\in\operatorname{Irr}$ *be a square-integrable representation and let* $n\in\mathbb{N}$. *Then*

$$u(\delta, n)$$

is a unitarizable representation.

Proof. — The first part of the proof uses a result of [3] or [22], and the second part uses a result of [16].

Let $\delta\in\tilde{G}_m=GL(m, F)^{\sim}$ be a square integrable representation and $n\in\mathbb{N}$.

There exists a division algebra H central over F with dimension m^2 over F. We choose, like in paragraph 5 of [22], a number field k, a place w of k, and a group G defined by a division algebra D over k such that: F is isomorphic to the completion k_w of k at w, the group $G(k_w)$ of k_w-rational points of G is isomorphic to the multiplicative group of H, G satisfies assumptions of paragraph 5 of [22]. Let S_0 be the set of all places v such that $G(k_v)$ is ramified. Clearly $w\in S_0$. Let $\mathbb{A}$ be the Adele ring of k.

Since δ is an irreducible square-integrable representation of $GL(m, F)\cong GL(m, k_w)$, the proof of Proposition 5.15 of [22] implies that there exists an irreducible cuspidal automorphic representation σ of $GL(m, \mathbb{A})$ such that, in the factorisation

$$\sigma=\bigotimes_v \sigma_v$$

which corresponds to the factorisation of $GL(m, \mathbb{A})$ into the restricted product of all $GL(m, k_v)$ (*see* [9]), we have

$$\sigma_v \cong \delta.$$

Let Z^m be the center of the algebraic group $GL(m)$. Then Z^m is isomorphic to $GL(1)$. Now $Z^m(\mathbb{A})$ is naturally isomorphic to the restricted product of $Z^m(k_v)$. Let Z^m_+ be the group of all $z=(z_v)\in Z^m(\mathbb{A})$ such that $z_v=1$ for all finite places, and z_v is a positive real number independent of v infinite.

Let η be the central character of the cuspidal automorphic representation σ. Suppose that η is trivial on Z^m_+.

Let P be the standard parabolic subgroup of $GL(nm)$ whose Levi factor M is naturally isomorphic to $GL(m)^n$. We identify elements of $M(\mathbb{A})$ with n-tuples $(g_1, \ldots, g_m)$, $g_i\in GL(m, \mathbb{A})$. Let π be the representation

$$(g_1, \ldots, g_n) \to \sigma(g_1)\,|\det g_1|^{(n-1)/2} \otimes\sigma(g_2)\,|\det g_2|^{((n-1)/2)-1} \otimes \ldots \otimes \sigma(g_n)\,|\det g_n|^{-(n-1)/2}.$$

Let $\pi=\otimes\pi_v$ be the decomposition of π into the restrict product of representations of $M(k_v)$. The induced representation from $P(\mathbb{A})$ to $GL(nm, \mathbb{A})$ (resp. $P(k_v)$ to $GL(nm, k_v)$) by π (resp. π_v) is denoted by $\mathrm{Ind}(\pi)$ (resp. $\mathrm{Ind}(\pi_v)$).

Since the center Z^{mn} of $GL(n, m)$ is isomorphic to $GL(1)$, we may consider η like a character of $Z^{mn}(\mathbb{A})$. Set $\omega=\eta^m$. Let $L^2(\omega, GL(mn, \mathbb{A}))$ be the space of (classes of) functions on $GL(mn, \mathbb{A})$ such that

$$f(\gamma zg)=\omega(z)f(g)$$

for all $\gamma\in GL(mn, k)$, $z\in Z^{mn}(\mathbb{A})$, $g\in GL(mn, \mathbb{A})$; and $|f|^2$ is integrable function on

$$GL(mn, k)\, Z^{mn}(\mathbb{A})\backslash GL(mn, \mathbb{A})$$

with respect to a non-trivial right-invariant measure. Action of $GL(mn, \mathbb{A})$ on $L^2(\omega, GL(mn, \mathbb{A}))$ by right shifts defines a unitary representation of $GL(mn, \mathbb{A})$.

In paragraph 2 of [16] it is proved that there exists an intertwining operator

$$E:\quad \mathrm{Ind}(\pi) \to L^2(\omega, GL(mn, \mathbb{A}))$$

whose image is an irreducible representation. Let τ be the image of E. Decompose τ into restricted tensor product $\tau=\otimes\tau_v$.

Since $\mathrm{Ind}(\pi)\cong\bigotimes_v \mathrm{Ind}(\pi_v)$ we have the epimorphism

$$E:\quad \bigotimes_v \mathrm{Ind}(\pi_v) \to \bigotimes_v \tau_v.$$

Now $\otimes\mathrm{Ind}(\pi_v)$ is, like a representation of $GL(mn, k_w)$, isomorphic to a direct sum of copies of $\mathrm{Ind}(\pi_w)$ (we need to fix a basis in each $\mathrm{Ind}(\pi_v)$, $v\neq w$, and use the fact that the local Hecke algebras are idempotended algebras). Since $\otimes\tau_v$ is also a direct sum of copies of τ_w, by the same reasons, we obtain directly that there exists a surjective intertwining operator

$$e: \quad \mathrm{Ind}(\pi_w) \to \tau_w.$$

Now

$$\mathrm{Ind}(\pi_w) \cong (\nu^{(n-1)/2}\delta) \times \ldots \times (\nu^{-(n-1)/2}\delta).$$

Thus

$$\tau_w \cong u(\delta, n).$$

Since τ is a subrepresentation of $L^2(\omega, GL(mn, \mathbb{A}))$, τ_w is unitarizable and therefore $u(\delta, n)$ is unitarizable.

It remains to consider the general case (without assumption $\eta \,|\, Z_+^m = 1$).

This case reduces to the case of $\eta \,|\, Z_+^m = 1$ by twisting δ with a suitable character.

The following theorem is a direct consequence of the preceding theorem.

A.9. THEOREM. — *Representations*

$$L(a(n, d)^{(\rho)}), \qquad n, d \in \mathbb{N}, \quad \rho \in C^u$$

are unitarizable.

A.10. THEOREM. — *Let* char $F=0$. *Set*

$$B = \{ \langle a\ (n, d)^{(\rho)} \rangle, \pi\ (\langle a\ (n, d)^{(\rho)} \rangle, \alpha);\ n, d \in \mathbb{N},\ \rho \in C^u,\ 0 < \alpha < 1/2 \}.$$

Fix $m \in \mathbb{N}$. *Then*

(i) *If* $\sigma_1, \ldots, \sigma_k \in B$ *such that*

$$\deg \sigma_1 + \ldots + \deg \sigma_k = m,$$

then $\sigma_1 \times \ldots \times \sigma_k \in \hat{G}_m$.

(ii) *If* $\pi \in \hat{G}_m$, *then there exist* $\tau_1, \ldots, \tau_j \in B$ *so that*

$$\pi = \tau_j \times \ldots \times \tau_j.$$

(iii) *If* $n, d \in \mathbb{N}$, $\rho \in C^u$ *so that* $nd\ (\deg \rho) \leqq m$, *then*

$$\langle a\ (n, d)^{(\rho)} \rangle = L\,(a\,(d, n)^{(\rho)}).$$

(iv) *Let* $n_i, d_i, m_j, e_j \in \mathbb{N}$, $\rho_i, \sigma_j \in C^u$, $0 < \alpha_j < 1/2$ *for* $1 \leqq i \leqq p$, $1 \leqq j \leqq q$ *where* $p, q \in \mathbb{Z}_+$. *Suppose that*

$$\sum_{i=1}^{p} (n_i d_i) \deg \rho_i + 2 \sum_{j=1}^{2} (m_j e_j) \deg \sigma_j = m.$$

Then

$$L\Big(\sum_{i=1}^{p} a(n_i, d_i)^{(\rho_i)} + \sum_{j=1}^{q} [\nu^{\alpha_j} a(m_j, e_j)^{(\sigma_j)} + \nu^{-\alpha_j} a(m_j, e_j)^{(\sigma_j)}]\Big)$$

$$= \Big[\prod_{i=1}^{p} L(a(n_i, d_i)^{(\rho_i)})\Big] \times \Big[\prod_{j=1}^{q} \pi(L(a(m_j, e_j)^{(\sigma_j)}), \alpha_j)\Big]$$

$$= \Big[\prod_{i=1}^{p} \langle a(d_i, n_i)^{(\rho_i)} \rangle\Big] \times \Big[\prod_{j=1}^{q} \pi(\langle a(e_j, m_j)^{(\sigma_j)} \rangle, \alpha_j)\Big]$$

$$= \langle \sum_{i=1}^{p} a(d_i, n_i)^{(\rho_i)} + \sum_{j=1}^{q} [\nu^{\alpha_j} a(e_j, m_j)^{(\sigma_j)} + \nu^{-\alpha_j} a(e_j, m_j)^{(\sigma_j)}] \rangle.$$

Proof. — We shall prove (i), (ii), (iii) and (iv) by induction on m. The proof is similar with the proof of Theorem 7.1.

For $m=1$ there is nothing to prove. Let $m \geqq 2$. Suppose that the theorem holds for $k \leqq m-1$. Let X_{m-1} be defined as in Lemma 4.8. By our inductive assumption (U^{m-1}) holds ((U^m) is defined at 4.6). Thus we can apply Lemma 4.8. Each element of $I(\hat{G}_m)$ is some product of elements of X_{m-1} ($I(\hat{G}_m)$ is defined in (ii) of Lemma 4.8). By definition

$$I(\hat{G}_m) \subseteq \hat{G}_m$$

and

$$\hat{G}_m \setminus I(\hat{G}_m) \subseteq \{ \langle a(n, d)^{(\rho)} \rangle,\ n, d \in \mathbb{N},\ \rho \in C^u \text{ and } (nd) \deg \rho = m \}.$$

Let $\tau \in I(\hat{G}_m)$. Then

$$\tau = \prod_{i=1}^{p} \langle a(n_i, d_i)^{(\rho_i)} \rangle \times \prod_{j=1}^{q} \pi(\langle a(m_j, e_j)^{(\sigma_j)} \rangle, \alpha_j)$$

for some n_i, $d_i \in \mathbb{N}$, ρ_i, $\sigma_j \in C^u$, $0 < \alpha_j < 1/2$, p, $q \in \mathbb{Z}_+$, by Lemma 4.8. By inductive assumption, we have

$$\langle a(n_i, d_i)^{(\rho_i)} \rangle = L(a(d_i, n_i)^{(\rho_i)}).$$

Also

$$\pi(L(a(e_j, m_j)^{(\sigma_j)}), \alpha_j) = \nu^{\alpha_j} L(a(e_j, m_j)^{(\sigma_j)}) \times \nu^{-\alpha_j} L(a(e_j, m_j)^{(\sigma_j)}) = \pi(\langle a(m_j, e_j)^{(\rho_j)} \rangle, \alpha_j).$$

Thus $\pi(L(a(e_j, m_j)^{(\sigma_j)}, \alpha_j)$ is unitarizable. Using Corollary A.7 we obtain

$$\tau = \prod_{i=1}^{p} L(a(d_i, n_i)^{(\rho_i)}) \times \prod_{j=1}^{q} \pi(L(a(e_j, m_j)^{(\sigma_j)}, \alpha_j).$$

This implies that (iv) holds for representations in $I(\hat{G}_m)$.

Let now $n, d \in \mathbb{N}$, $\rho \in C^u$ so that

$$(nd) \deg \rho = m.$$

Now $L(a(n, d)^{(\rho)})$ is unitarizable, by Theorem A.9. By the preceding considerations

$$L(a(n, d)^{(\rho)}) \notin I(\hat{G}_m).$$

Thus

$$L(a(n, d)^{(\rho)}) \in \{\langle a(u, v)^{(\sigma)} \rangle;\ u, v \in \mathbb{N},\ \sigma \in C^u \text{ and } (uv) \deg \sigma = m\}.$$

Therefore, $L(a(n, d)^{(\rho)}) = \langle a(u, v)^{(\sigma)} \rangle$ for some u, v and σ as above. The fact

$$\text{supp} \langle a(n, d)^{(\rho)} \rangle = \langle \text{supp}\, a(u, v)^{(\sigma)} \rangle$$

implies

$$L(a(n, d)^{(\rho)}) \in \{\langle a(n, d)^{(\rho)} \rangle, \langle a(d, n)^{(\rho)} \rangle\}.$$

By Lemma 6.3

$$L(a(n, d)^{(\rho)}) = \langle a(d, n)^{(\rho)} \rangle.$$

Thus $\langle a(d, n)^{(\rho)} \rangle$ is unitarizable. This implies (i), (ii), (iii) and the rest of (iv).

Let R^u be the additive subgroup of R generated by Irr^u. Then Irr^u is a $\mathbb{Z}$-basis of R^u, and R^u is a subring of r.

The following theorem is a direct consequences of the preceding one.

A.11. THEOREM. — (i) *Let* $a \in M(S(C))$. *The representation* $\langle a \rangle$ *is unitarizable if and only if*

$$t(\langle a \rangle) = L(a)$$

is unitarizable.

(ii) *The mapping*

$$t:\ \text{Irr}^u \to \text{Irr}^u$$
$$\langle a \rangle \to L(a), \qquad \langle a \rangle \in \text{Irr}^u,$$

is an involutive automorphism of the multiplicative semigroup Irr^u.

(iii) *The homomorphism in* (ii) *satisfies*

$$t(\langle a(n, d)^{(\rho)} \rangle) = \langle a(d, n)^{(\rho)} \rangle,$$
$$t(\pi(\langle a(n, d)^{(\rho)} \rangle, \alpha)) = \pi(\langle a(d, n)^{(\rho)} \rangle, \alpha),$$
$$n, d \in \mathbb{N},\ \rho \in C^u, \qquad 0 < \alpha < 1/2.$$

(iv) *The mapping* $t|R_u$ *is an involutive ring automorphism.*

A.12. *Remark.* — (i) Lemma 6.3 can not be omitted in our proof of Theorem 7.1, while we can prove (i) and (ii) of Theorem A.10, and also (i), (ii), (iv) of Theorem A.11 without using Lemma 2.3.

(ii) The statement (i) of the preceding theorem is a new proof of Conjecture 8.10 of [2] stated by J. N. Bernstein (in the zero characteristic case).

(iii) Now we shall give an outline of another possible proof of Proposition A.4. If $d \in \mathrm{M}(\mathrm{D})$ then we have in R

$$\lambda(d) = \sum_{x \in \mathrm{M}(\mathrm{D})} m_x^d \mathrm{L}(x), \qquad m_x^d \in \mathbb{Z}_+, m_d^d = 1,$$

$$\mathrm{L}(d) = \sum_{x \in \mathrm{M}(\mathrm{D})} m(d, x) \lambda(x), \qquad m(d, x) \in \mathbb{Z},\ m(d, d) = 1.$$

Take $d_1, d_2 \in \mathrm{M}(\mathrm{D})$. In the ring R we have

$$\begin{aligned}
\mathrm{L}(d_1) \times \mathrm{L}(d_2) &= (\sum m(d_1, x_1) \lambda(x_1)) \times (\sum m(d_2, x_2) \lambda(x_2)) \\
&= \lambda(d_1 + d_2) + \sum_{\substack{x_1 \neq d_1, \\ \text{or } x_2 \neq d_2}} m(d_1, x_1) m(d_2, x_2) \lambda(x_1 + x_2) \\
&= \mathrm{L}(d_1 + d_2) + m(d_1, d_2) m(d_2, d_2) \sum_{y \neq d_1 + d_2} m_y^{d_1 + d_2} \mathrm{L}(y) \\
&\qquad + \sum_{\substack{x_1 \neq d_1, \\ \text{or } x_2 \neq d_2}} (m(d_1, x_1) m(d_2, x_2) \sum_y m_y^{x_1 + x_2} \mathrm{L}(y)).
\end{aligned}$$

For a proof of Proposition A.4 it is enough to show that if

$$m(d_1, x_1) m(d_2, x_2) m_y^{x_1 + x_2} \neq 0$$

where $x_1 \neq d_1$ or $x_2 \neq d_2$, then $y \neq d_1 + d_2$. This can be obtained using relation which exists between a and b when $m_b^a \neq 0$ (i. a. when $\mathrm{L}(b)$ is a composition factor of $\lambda(a)$). For this relation one can consult A.4. *f* of [3].

For details of a proof of such type one can consult proofs of Propositions 3.5 and 5.6 in [30].

REFERENCES

[1] J.N. BERNSTEIN, *All Reductive p-adic Groups Are Tame* (*Funct. Anal. Appl.*, Vol. 8, 1974, pp. 91-93).

[2] J. N. BERNSTEIN, P-*Invariant Distributions on* GL(N) *and the Classification of Unitary Representations of* GL(N) (*Non-Archimedean Case*), *in Lie Group Representations II* (*Proceedings*, University of Maryland, 1982-1983, *Lecture Notes in Math.*, Vol. 1041, Springer-Verlag, Berlin, 1984, pp. 50-102).

[3] J. N. BERNSTEIN, P. DELIGNE, D. KAZHDAN and M. F. VIGNERAS, *Représentations des groupes réductifs sur un corps local*, Hermann, Paris, 1984.

[4] J. N. BERNSTEIN and A. V. ZELEVINSKY, *Representations of the Group* GL(*n*, F), *where* F *Is a Local Non-Archimedean Field* [*Uspekhi Mat. Nauk*, Vol. 31, No. 3, 1976, pp. 5-70 (= *Russian Math. Surveys*, Vol. 31, No. 3, 1976, pp. 1-68)].

[5] J. N. BERNSTEIN and A. V. ZELEVINSKY, *Induced Representations of Reductive p-adic Groups, I* (*Ann. Scient. Éc. Norm. Sup.*, Vol. 10, 1977, pp. 441-472).

[6] P. CARTIER, *Representations of p-adic Groups: a Survey*, in *Proc. Sympos. Pure Math.*, Vol. XXXIII, part 1, Amer. Math. Soc., Providence, R. I., 1979, pp. 111-155.

[7] W. CASSELMAN, *Introduction to the Theory of Admissible Representations of p-adic Reductive Groups*, preprint.

[8] G. VAN DIJK, *Computation of Certain Induced Characters of p-adic Groups* (*Math. Ann.*, Vol. 199, 1972, pp. 229-240).

[9] D. FLATH, *Decomposition of Representations into Tensor Products*, in *Proc. Sympos. Pure Math.*, Vol. XXXIII, part I, Amer. Math. Soc., Providence, R. I., 1979, pp. 179-183.

[10] I. M. GELFAND and M. I. GRAEV, *Representations of a Group of the Second Order with Elements from a Locally Compact Field* (*Russian Math. Surveys*, Vol. 18, 1963, pp. 29-100).

[11] I. M. GELFAND, M. GRAEV and I. I. PIATETSKI-SHAPIRO, *Automorphic Functions and Representation Theory*, W. B. Sannders Co., Philadelphia, 1969.

[12] I. M. GELFAND and D. A. KAZHDAN, *Representations of* GL (n, K), *in Lie Groups and Their Representations*, Akademiai Kiado, Budapest, 1974, pp. 95-118.

[13] I. M. GELFAND and M. A. NEUMARK, *Unitare Darstellungen der Klassichen Gruppen*, German translation, Akademie Verlag, Berlin, 1957.

[14] H. JACQUET, *Generic Representations*, in *Non-Commutative Harmonic Analysis* (*Lecture Notes in Math.*, Vol. 587, Springer-Verlag, Berlin, 1977, pp. 91-101).

[15] H. JACQUET, *Principal* L-*functions of the linear group*, in *Proc. Sympos. Pure Math.*, Vol. XXXIII, part 2, Amer. Math. Soc., Providence, R. I., 1979, pp. 63-86.

[16] H. JACQUET, *On the Residual Spectrum of* GL (n), in *Lie Group Representations II* (*Proceedings*, University of Maryland, 1982-1983, *Lecture Notes in Math.*, Vol. 1041, Springer-Verlag, Berlin, 1984, pp. 185-208).

[17] A. KNAPP and B. SPEH, *Status of Classification of Irreducible Unitary Representations*, in *Harmonic Analysis* (*Proceedings*, Minneapolis, 1981, *Lecture Notes in Math.*, Vol. 908, Springer-Verlag, Berlin, 1982).

[18] D. MILIČIĆ, *On* C*-*Algebras with Bounded Trace* (*Glasnik Mat.*, Vol. 8, (28), 1973, pp. 7-22).

[19] C. MOEGLIN and J.-L. WALDSPURGER, *Sur l'involution de Zelevinsky*, preprint, Paris, 1985.

[20] G. I. OLSHANSKY, *Intertwining Operators and Complementary Series in the Class of Representations of the General Group of Matrices Over a Locally Compact Division Algebra, Induced from Parabolic Subgroups* (*Mat. Sb.*, Vol. 93, No. 2, 1974, pp. 218-253).

[21] F. RODIER, *Représentations de* GL (n, k) *où k est un corps p-adique* [*Séminaire Bourbaki*, n° 587, 1982, (*Astérisque*, 92-93, 1982, pp. 201-218)].

[22] J. ROGAWSKI, *Representations of* GL (n) *and Division Algebras Over a p-adic Field* (Duke Math. J., Vol. 50, 1983, pp. 161-196).

[23] B. SPEH, *The Unitary Dual of* GL (3, $\mathbb{R}$) *and* GL (4, $\mathbb{R}$) (*Math. Ann.*, Vol. 258, 1981, pp. 113-133).

[24] B. SPEH, *Unitary Representations of* GL (n, $\mathbb{R}$) *with Non-Trivial* ($\mathfrak{g}$, K) *Cohomology* (*Invent. Math.*, Vol. 71, 1983, pp. 443-465).

[25] E. M. STEIN, *Analysis in Matrix Spaces and Some New Representations of* SL(N, $\mathbb{C}$) (*Ann. of Math.*, Vol. 86, 1967, pp. 461-490).

[26] M. TADIĆ, *The* C*-*algebra of* SL(2, k) (*Glasnik Mat.*, Vol. 17, (37), 1982, pp. 249-263).

[27] M. TADIĆ, *The Topology of the Dual Space of a Reductive Group Over a Local Field* (*Glasnik Mat.*, 18, (38), 1983, pp. 259-279).

[28] M. TADIĆ, *Proof of a Conjecture of Bernstein*, (*Math. Ann.*, Vol. 272, 1985, pp. 11-16).

[29] M. TADIĆ, *Unitary Dual of p-Adic* GL (n). *Proof of Bernstein Conjectures*, (*Bull. Amer. Math. Soc.*, Vol. 13, No. 1, 1985, pp. 39-42).

[30] M. TADIĆ, *Unitary Representations of General Linear Group Over Real and Complex Field*, preprint, Bonn, 1985.

[31] D. A. VOGAN, Jr., *Understanding the Unitary Dual, in Lie Group Representations I* (*Proceedings*, University of Maryland, 1982-1983, *Lecture Notes in Math.*, Vol. 1024, Springer-Verlag, Berlin 1983, pp. 264-286).

[32] N. R. WALLACH, *Representations of Reductive Lie Groups*, in *Proc. Sympos. Pure Math.*, Vol. XXXIII, part 1, Amer. Math. Soc., Providence, R. I., 1979, pp. 71-86).

[33] A. V. ZELEVINSKY, *Induced representations of reductive p-adic groups II* (*Ann. Scient. Éc. Norm. Sup.*, Vol. 13, 1980, pp. 165-210).

[34] A. V. ZELEVINSKY, *p-adic analogue of the Kazhdan-Lusztig conjecture* (*Funct. Anal. Appl.*, Vol. 15, 1981, pp. 83-92).

[35] J.-L. WALDSPURGER, *Algèbres de Hecke et induites de représentations cuspidales, pour* GL(N) (*Journal für die Reine und angewandte Matematik*, No. 370, 1986, pp. 127-191).

(Manuscrit reçu le 22 janvier 1984,
révisé le 19 septembre 1985.)

Marko TADIĆ,
Max-Planck-Institut
für Mathematik,
Gottfried-Claren-Str. 26,
D-5300 Bonn 3
and
Department of Mathematics,
University of Zagreb,
p. o. box 187,
41001 Zagreb,
Yugoslavia.